FINM37601: Mathematical Market Microstructure

Assignment 2 – Market Microstructure Variables and Characteristic Time Scale (II)

Due Date: 11:59pm, Nov. 18, 2023

Introduction: This assignment is Part II of our discussion on market microstructure variables and characteristics time scale. In assignment 1, we derived the mean and variance of transaction price assuming a "bid-ask bounce" process for the price. In this assignment, we focus on an execution strategy design based on the optimal balance between market impact and execution risk. Unless necessary, symbols used in the following discussion assume the same meanings as assignment 1 and will not be given new introductions.

Consider that the mid-quote, $(a+b)/2$, is a valid price level between bid1, b, and ask1, a. That is, there are at least two "ticks" between bid1 price and ask1 price where a "tick" is the minimum increment in stock price. For a market participant who wants to buy a unit share of a stock (for example), he/she has a fixed time window, $0 \le t \le T_s$, to fully execute the order. When T_s is reached and the order is not filled, the market participant will cancel the order and submit an aggressive limit order that will be filled at the ask price at T_s. T_s is also often called "execution horizon" which can be determined dynamically by the market participant for different orders.

Throughout the time window T_s, we assume that the ask1 price stays at a. For a market participant who is planning to place a buy order, he/she has three price levels to consider: b, a and $(a+b)/2$. For orders placed at b or $(a+b)/2$, the market participant may need to wait for some time before the order is filled (if at all), while the waiting time for orders placed at a is zero. We also assume that the market participant's order can be filled at bid1, b, and ask1, a, and the mid-quote price level, $(a+b)/2$, with unconditional probabilities of ρ_l, ρ_u, ρ_m, respectively. By definition, $\rho_l + \rho_u + \rho_m = 1$. These three probabilities are the same as those defined in assignment 1[1].

Let's consider the following execution strategy by the market participant: At any time t, the market participant will calculate the probability of his/her order being executed at the mid-quote, $(a+b)/2$. He/she will <u>not</u> consider placing the order at $(a+b)/2$ <u>unless</u> the probability of being filled at $(a+b)/2$ is above a threshold value. Let's call such an execution strategy the "**<u>WaitAndAct</u>**" strategy.

Note that such a "WaitAndAct" strategy depends strongly on the value of ρ_m. Although ρ_m changes with time as liquidity situation in the market evolves, to further simplify the situation, we can assume a "step function" for ρ_m across time: 0 at some time periods, and a constant

[1] This is essentially assuming that the probability distribution of transaction price in the market is the same as that of market participant's trade price. In other words, we are assuming that all market participants in the market are "homogeneous" in their trading behaviors for both buying and selling decisions which are made statistically independently and stationarily.

$\rho_{m\_threshold}$ at others. Furthermore, we assume that price processes in all such periods are stochastically independent from one another. Let T_m be the total duration of $\rho_m = 0$ between $t = 0$ and $t = T_s$; therefore, the total duration for $\rho = \rho_{m\_threshold}$ is $T_s - T_m$.

It is also worth noting that, while the market participant on the buying side has the above execution strategy and related calculations in mind, those on the selling side may think in a like-minded, statistically equivalent way.

<u>Question 1 (15 pts)</u>: If the market participant uses the mid-quote price level, $(a + b)/2$, as the benchmark to measure the market impact of his/her execution within time window T_s, find the expected value of his/her market impact defined as $I = p - (a + b)/2$ where p is the execution price and follows the same "bid-ask bound process" as we discussed in assignment 1. Note that here we always assume that the market participant is buying. Denote this expected value of I as $\mathbb{E}(I)$.

<u>Question 2 (15 pts)</u>: Find the variance of I given in Question 1. Denote this variance of I as $Var(I)$.

<u>Question 3 (15 pts)</u>: Define a utility function as follows: $u = \mathbb{E}(I) + \lambda Var(I)$. Here λ is an "aversion" factor that is constant during optimization. Find an explicit expression for u. Note that the market participant should try to minimize this utility function during execution. Comment on how the values of bid-ask spread may affect the value of u.

<u>Question 4 (15 pts)</u>: In the following, let's assume that $\rho_l \sim \rho_u$; that is, the market participant's order has equal probability of being filled on bid1 price and ask1 price and independent from the value of ρ_m except the constraint of $\rho_l + \rho_u + \rho_m = 1$. Re-evaluate your comments in Question 3 on the dependence of u on bid-ask spread and ρ_m.

<u>Question 5 (20 pts)</u>: With the "step function" assumption in the "WaitAndAct" execution strategy discussion above, derive the upper bound of the "total holding risk" for the market participant within the execution horizon T_s with the following assumptions: (1) $\rho_l \sim \rho_u$; (2) the variance of the impact, I, can be assumed to be $\propto \frac{T}{3}$ where T is the time lapse for an arithmetic Brown process.

<u>Hint for Question 5</u>: the "total holding risk" that you derive should depend on some of the following parameters: $a - b$, T_m, T_s, *and* $\rho_{m\_threshold}$. Also, it is OK that you define the "risk" as "variance", not necessarily standard deviation, to make the derivation easy to handle. The "upper bound" refers to the fact that waiting times for orders on b and $(a + b)/2$ may be less than T_s in practice.

<u>Question 6 (20 pts)</u>: With results derived in Question 5, by assuming the following relation: $\rho_{m\_threshold} \sim 1 - e^{-\eta T_m}$ (here η can be regarded as the arrival rate of transactions happening at the mid-quote price of $(a + b)/2$ and assumed constant), find an optimal T_m^* that helps achieve a minimum of the upper bound of the total holding risk for the market participant.

<u>Hint for Question 6</u>: No need to find an explicit answer to T_m^*; an equation with T_m^*, η and T_s will suffice.

<u>Question 7 (20 bonus pts)</u>: Relax the assumption of $\rho_l \sim \rho_u$ and re-evaluate results from Questions 5 and 6.

End of assignment 2.