Mathematical Market Microstructure – An Optimization Approach

Lecture IV – Stochastic programming and its applications to trading algorithms and market making (part I)

FINM 37601 – Fall, 2023

By Hongsong Chou, Ph.D.

SCHEDULE OF THIS COURSE

- Lecture I (6pm – 9pm, Oct. 27 Chicago time): An introduction to market design and algorithmic trading within a global context;
- Lecture II (6pm – 9pm, Nov. 3 Chicago time): Practical aspects of trading algorithm optimization and an overview of market microstructure theory;
- Lecture III (6pm – 9pm, Nov. 10 Chicago time): On the optimal design of execution algorithms;
- Lecture IV (6pm – 9pm, Nov. 17 Chicago time): Stochastic programming and its applications to trading algorithms and market making (part I);

(*No class on Thanksgiving week*)

- Lecture V (6pm – 9pm, Dec. 1 Chicago time): Stochastic programming and its applications to trading algorithms and market making (part II), and a discussion on recent developments in market microstructure research and applications of machine learning techniques in trading research and strategy decision.

AGENDA

- Execution Algorithms:
  - An introduction to stochastic programming;
  - Casting optimal execution problem into a dynamic programming problem;
  - Dynamic programming approach vs. quadratic optimization approach in optimal execution algorithm design;

- Market Making Strategies:
  - A review of market makers and their market making strategies;
  - Casting market making into a dynamic programming problem;
  - Simultaneous market making for both option contracts and underlying assets.

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STOCHASTIC OPTIMIZATION INTRODUCTION (I)

- We follow the notations as stated in “Continuous-time Stochastic Control and Optimization with Financial Applications” by Huyen Pham (2010);
- A dynamic system is characterized by its state at any time and evolves in a stochastic environment formulated by a probability space $(\Omega, \mathcal{F}, P)$, where $\Omega, \mathcal{F}, P$ are the sample space, set of events and probability mapped to each event;
- We denote $X_t(\omega)$ the state of the dynamic system at time t realized via the scenario $\omega \in \Omega$;
- Note that the mapping $t \longmapsto X_t(\omega)$ for all ω indicates the (continuous time) dynamics of the system via a stochastic process which is often characterized by a stochastic differential equation;
- A dynamic system is often influenced by a control process, α_t, whose value is decided at any time t in a function of available information by then; we use $\mathcal{A}$ to denote the set of all admissible controls.

STOCHASTIC OPTIMIZATION INTRODUCTION (II)

- The objective is to maximize (or minimize) a functional $J(X, \alpha)$, over all admissible controls;
- Typical objective functional can be in the forms of:

$J(X, \alpha, T) = E\left[\int_0^T f(X_t, \alpha_t)dt + g(X_T)\right]$, on a finite horizon $T < 0$,

or

$J(X, \alpha) = E\left[\int_0^\infty e^{-\beta t} f(X_t, \alpha_t)dt\right]$, on an infinite horizon.

- Here, f is often called a running profit function, g a terminal reward function, and β a discount factor;
- The maximum value, called a value function, is written as

$v = sup_{\alpha,\tau} J(X, \alpha, T)$.

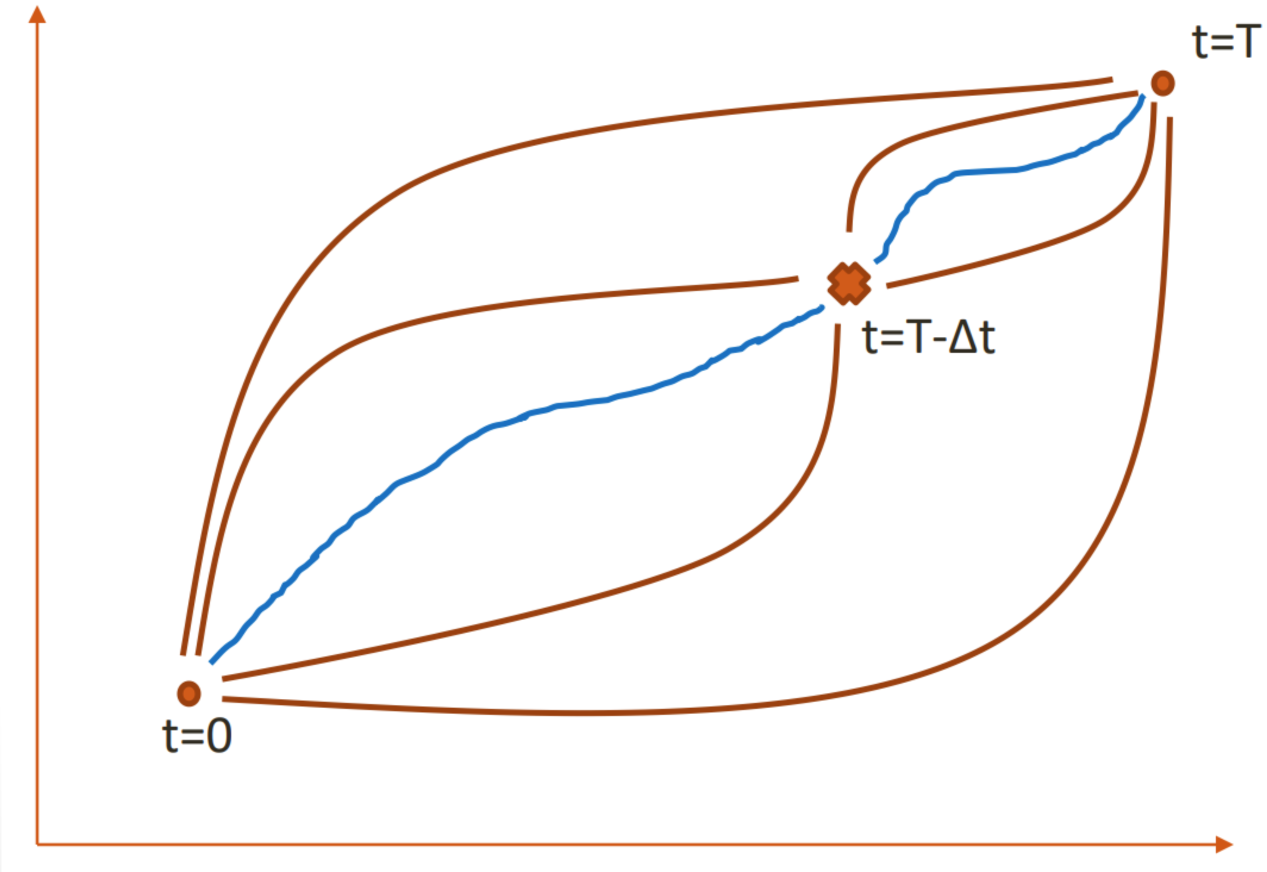
- The main goal of a stochastic optimization problem is to find the maximizing control process and/or stopping time attaining the value function to be determined.

STOCHASTIC OPTIMIZATION INTRODUCTION (III)

- Now let's introduce the stochastic processes that govern certain state variables in the dynamic control problem; the model that we want to use to describe the state variables, X_t, can be written as
$$dX_t = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t.$$
- Here W is a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathrm{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, and $\alpha = (\alpha_t)$ is a progressively measurable (with respect to F) process.
- Dynamic Programming Principle (DPP) for the finite horizon case: assume that $(t,x) \in [0,T] \times \mathbb{R}^n$; then the DPP can be written as
$$v(t,x) = sup_{\alpha\in\mathcal{A}(t,x)} sup_{\theta\in\mathcal{T}(t,T)} E\left[\int_t^\theta f(s, X_s{}^{t,x}, \alpha_s)ds + v(\theta, X_\theta{}^{t,x})\right]$$
$$= sup_{\alpha\in\mathcal{A}(t,x)} inf_{\theta\in\mathcal{T}(t,T)} E\left[\int_t^\theta f(s, X_s{}^{t,x}, \alpha_s)ds + v(\theta, X_\theta{}^{t,x})\right]$$
- The DPP states that the optimal control process should also be optimal for the remaining program at every intermediate time point.

STOCHASTIC OPTIMIZATION INTRODUCTION (IV)

- DPP explained, esp. in solving real life problems:



STOCHASTIC OPTIMIZATION INTRODUCTION (V)

- Hamilton-Jacobi-Bellman Equation: consider the below inequality

$$v(t,x) \geq E\left[\int_t^{t+h} f\left(s, X_s{}^{t,x}, \alpha_s\right) ds + v\left(t+h, X_{t+h}{}^{t,x}\right)\right]. \qquad (1.1)$$

- Next, we apply Ito's Lemma to the right-hand side of the above inequality (assuming enough smoothness of the value function, $v(t,x)$) to obtain the below inequality:

$$0 \geq E\left[\int_t^{t+h} f\left(s, X_s{}^{t,x}, \alpha_s\right) ds + \left(\frac{\partial v}{\partial t} + \mathcal{L}^\alpha v\right)\left(s, X_s{}^{t,x}\right)\right]. \qquad (1.2)$$

- Here

$$\mathcal{L}^\alpha v = \sum_{i=1}^n b_i(t,x,\alpha_t)\frac{\partial f}{\partial x_i} + \sum_{i,k=1}^n \left(\sum_{l=1}^d \sigma_{il}(t,x)\sigma_{lk}(t,x)\right)\frac{\partial^2 f}{\partial x_i \partial x_k}. \qquad (1.3)$$

- By dividing h on both sides of (**) and setting it to zero, we obtain the below inequality:

$$0 \geq \frac{\partial v}{\partial t} + \mathcal{L}^\alpha v + f(t,x,\alpha). \qquad (1.4)$$

- Because (1.4) holds for all α, we have

$$-\frac{\partial v}{\partial t}(x,t) - sup_{\alpha \in \mathcal{A}}\left[\mathcal{L}^\alpha v + f(t,x,\alpha)\right] \geq 0. \qquad (1.5)$$

STOCHASTIC OPTIMIZATION INTRODUCTION (VI)

- (Continued from previous page) We know that, if we use α_* to indicate the control process that allows us to achieve the optimal value function, we have the following relationship:

$$v(t,x) = E\left[\int_t^{t+h} f\left(s, X_{*,s}{}^{t,x}, \alpha_{*,s}\right) ds + v\left(t+h, X_{*,t+h}{}^{t,x}\right)\right].$$

- Therefore, we have from the inequality (1.4) that

$$-\frac{\partial v}{\partial t}(x,t) - sup_{\alpha\in\mathcal{A}}[\mathcal{L}^\alpha v + f(t,x,\alpha)] = 0.$$

- Let's define the "Hamiltonian" of the control problem as:

$$\mathrm{H} = sup_{\alpha\in\mathcal{A}}[\mathcal{L}^\alpha v + f(t,x,\alpha)],$$

where $\mathcal{L}^\alpha v = \sum_{i=1}^n b_i(t,x,\alpha_t)\frac{\partial f}{\partial x_i} + \sum_{i,k=1}^n\left(\sum_{l=1}^d \sigma_{il}(t,x)\sigma_{lk}(t,x)\right)\frac{\partial^2 f}{\partial x_i \partial x_k}$.

- Then the HJB equation can be written as:

$$-\frac{\partial v}{\partial t}(x,t) - H(t,x,\mathcal{L}^\alpha v) = 0, \text{ for } \forall(t,x) \in [0,T) \times \mathbb{R}^n.$$

- The terminal payoff for the value function should be: $v(T,x) = g(x)$.

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ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (I)

- There are numerous studies in the literature on designing optimal execution strategies using stochastic dynamic programming approach;
- Two references that I'd like to give are:
  - Dynamic Trading Policies with Price Impact, by He and Mamaysky, 2001;
  - Optimal Control of Execution Costs, by Bertsimas and Lo, 1998;
- In this section, we will mostly follow the example set in the second paper mentioned above.

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (II)

- To simplify problem, we assume that we have X shares of one stock that we need to buy from $t = 0$ to T;
- This is the "acquisition" problem, the reverse of the "liquidation" problem mentioned in Lecture 3;
- Because these two cases are (almost) symmetric in terms of obtaining our optimal execution profile, we focus on the acquisition problem below;
- The dynamic system that we will focus on involves the slices of the order that we will trade and the price of the stock;
- Let's further assume the following:
  - We have to finish the execution by time T;
  - We only buy; no sells along the way;
  - The sum of all buy shares will be exactly X;
  - The execution pace is at constant; that is, inter-trade time is constant.

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (IIIA)

- The concept of “volume time”:

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (IIIB)

- Why it is hard to forecast intraday volumes:

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (IV)

- Control process:

$$x_t : \{x_1, x_2, x_3, \dots, x_T\}$$

where $x_i (i = 1,2, \dots, T)$ are sizes of individual execution slices;

- Stochastic process of the price can be written in a discretized fashion as:

$$P_t = P_{t-1} + \varepsilon_t + \mu_t \qquad (2.1)$$

where ε_t is a zero-mean random process with a standard deviation of σ_ε, and μ_t is an impact function, the form of which will be discussed later;

- Note that we are using discretized time points as well; in practice, time intervals can be set at 5 minutes, 10 minutes, etc., or in volume time;
- The objective function can be written as:

$$v(t = 1, P_0) = \min_{\{x_t\}} E\left[\sum_{t=1}^{T} P_t x_t\right]; \qquad (2.2)$$

- The constraint is:

$$\sum_{t=1}^{T} x_t = X; \qquad (2.3)$$

- For later use, we also want to introduce W_t as $W_t = \sum_{i=t}^{T} x_i$. Note that $W_T = x_T$ and $W_1 = X$.

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (V)

- Consider the following "permanent impact" case:

$$\mu_t = \theta x_t, \tag{2.4}$$

which means that the impact is permanent and is proportional to the size of each trade;

- The price process becomes:

$$P_t = P_{t-1} + \theta x_t + \varepsilon_t, \text{ with } \theta > 0. \tag{2.5}$$

- Now we try to solve equation (2.2) to obtain the optimal control process, which is the same as the optimal execution profile; recall that the DPP of dynamic programming states that the optimal control process should also be optimal for the remaining program at every intermediate time point; this means that the following is true for our problem:

$$v(t, P_{t-1}) = \min_{\{x_t\}} E\left[P_t x_t + v(t+1, P_t)\right]. \tag{2.6}$$

- Let the optimal control solution to (2.6) be

$$\bar{x}_t \ :\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_T\}.$$

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (VI)

- Note that for the last slice, because all shares has to be bought, we have

$$v(T, P_{T-1}) = \min_{\{x_T\}} E[P_T x_T]\,; \qquad (2.7)$$

- Because of (2.5), we can re-write (2.7) as

$$v(T, P_{T-1}) = \min_{\{x_T\}} E[(P_{T-1} + \theta x_T)x_T]; \qquad (2.8)$$

- Following Bertsimas and Lo, we re-write (2.8) as

$$v(T, P_{T-1}) = \min_{\{x_T\}} E[(P_{T-1} + \theta W_T)W_T] = (P_{T-1} + \theta W_T)W_T.$$

- Next, we want to solve for the previous step from the last slice, with the value function as

$$v(T-1, P_{T-2}) = \min_{\{x_{T-1}\}} E[(P_{T-2} + \theta x_{T-1} + \varepsilon_t)x_{T-1} + v(T, P_{T-1})]; \qquad (2.9)$$

- Plugging $v(T, P_{T-1}) = (P_{T-2} + \theta x_{T-1} + \varepsilon_t + \theta(W_{T-1} - x_{T-1}))(W_{T-1} - x_{T-1})$

into (2.9) and minimize with respect to x_{T-1} gives the following answers:

$$\bar{x}_{T-1} = W_{T-1}/2, \qquad (2.10)$$

$$v(T-1, P_{T-2}) = W_{T-1}\left(P_{T-2} + \frac{3}{4}\theta W_{T-1}\right). \qquad (2.11)$$

- Because W_{T-1}=$x_{T-1} + x_T$, the above also means that $\bar{x}_T = W_{T-1}/2$.

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (VII)

- Following the above procedure and by way of induction, Bertsimas and Lo shows that

$$\bar{x}_t = X/T, \tag{2.12}$$

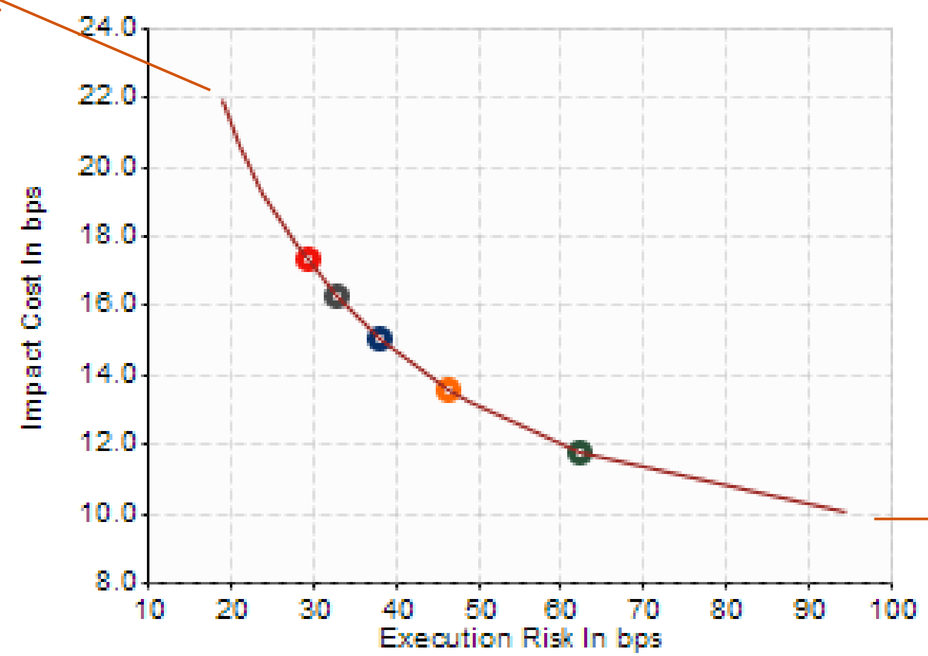
$$v(T-t, P_{T-t-1}) = W_{T-t}\left(P_{T-t-1} + \frac{t+2}{2(t+1)}\theta W_{T-t}\right). \tag{2.13}$$

- Note that this result has an important implication that VWAP (or TWAP) is the algorithm that has the minimum impact on the market assuming a permanent market impact that is linearly proportional to the side of each trade (see next page).

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (VIII)

- Let's look at the case of a stock:

In limit, market order is the one theoretically minimizes risk.



In limit, VWAP algo is the one theoretically minimizes impact.

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (VIIII)

- Consider the case that a drift is added to the price process in addition to the permanent impact:

$$P_t = P_{t-1} + \theta x_t + \gamma \alpha_t + \varepsilon_t, \text{ with} \qquad (2.15)$$

$$\alpha_t = \rho\alpha_{t-1} + \eta_t, \rho \in (-1,1). \qquad (2.16)$$

where η_t is a white noise;

- Note that (2.16) indicates an auto-regressive process with information built in; the alpha will surely change the optimal execution profiles, in the following way:

$$\bar{x}_{T-t} = \delta_{1,t} W_{T-t} + \delta_{2,t}\alpha_{T-t}, \qquad (2.17)$$

$$v(T-t, P_{T-t-1}) = P_{T-t-1}W_{T-t} + c_{1,t}{W_{T-t}}^2 + c_{2,t}\alpha_{T-t}W_{T-t} + c_{3,t}{W_{T-t}}^2 + c_{4,t}. \qquad (2.18)$$

- Here $\delta_{1,t} = {}^{1}/_{k+1}, \delta_{2,t} = {}^{\rho c_{2(t-1)}}/_{2c_{1(t-1)}}$;

with $c_{1,t}, c_{2,t}, c_{3,t}, c_{4,t}$ expressed as:

$$c_{1,t} = \frac{\theta}{2}\left(1 + \frac{1}{t+1}\right), c_{1,0}=\theta;$$

$$c_{2,t} = \gamma + \frac{\theta\rho c_{2,t-1}}{2c_{1,t-1}}, c_{2,0}=\gamma;$$

$$c_{3,t} = \rho^2 c_{3,t-1} - \frac{\rho^2 {c_{2,t-1}}^2}{4c_{1,t-1}}, c_{3,0}=0;$$

$$c_{4,t} = c_{4,t-1} + c_{3,t-1}{\sigma_\eta}^2, c_{4,0}=0.$$

ALGORITHM DESIGN WITH STOCHASTIC CONTROL APPROACH (X)

- Let’s try to understand the meaning of the solution:

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THE KEY PROBLEM IS MARKET IMPACT

- It is obvious that the core problem to both approach is market impact modeling;

- The quadratic optimization approach has to adopt a deterministic market impact model in order to solve the problem, while in principle the dynamic programming approach leaves more room for traders to contemplate various stochastic models for market impact, such as order flow-driven price impact, which is often omitted in the quadratic optimization approach (or, over-simplified).

RISK CAN BE INCORPORATED IN BOTH CASES

- In terms of risk calculation and incorporation in the optimization process, both approaches can get the job done but the implementation details can be quite different.

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THE ROLE OF MARKET MAKERS (I)

- As discussed in lecture 1, the most important role of an exchange (or any other trading venue) is to determine the price of an instrument; before a price is “realized” via trading on an exchange, it is anybody’s “guess”;
- “Information trader” is a group of market participants that, at least according to market microstructure theory, has more accurate prior knowledge about the true price of an instrument than anybody else in the market; the one and only goal of an “information trader” is to make money based on her proprietary knowledge of the true price of the instrument; however, whether or not she can make money depends on many factors, such as the accuracy of her guess of the true price, the strategic maneuver of other market participants (i.e., trading-related liquidity fluctuations, etc.);
- “Liquidity (or, noise) trader”, on the other hand, participate in trading activities not solely on price information; in fact, they may have just random guess on the true price of an instrument;
- “Market maker” sits in between the information trader and the liquidity trader, who usually does not assume the correct knowledge of the true price of an instrument, but always tries to adjust her quoted bid and offer prices in order to facilitate the price discovery process of an exchange; as a result, it has a special role for the well being of the market; in theory, market maker does not assume the goal of “making money”, but she does not want to lose money either.

THE ROLE OF MARKET MAKERS (II)

- The existence of a market maker is often required by an exchange; if the market is composed of only information traders, there can be times that no trade can be formed and no information can be revealed; if the market is composed of only noise traders, no economically beneficial capital allocation can be done;
- A market can achieve an equilibrium when there are only information traders and noise traders (i.e., no market makers); however, such equilibrium can be rather delicate and instable; this is particularly true for instruments that are not liquid;
- A market also can achieve an equilibrium when there are only information traders and market makers (i.e., no noise traders); however, if the information traders always systematically “adverse select” the market makers, such market may eventually breakdown as the market makers will constantly lose money thus the economic incentive to be market makers will disappear completely;
- Therefore, a well-functioning market requires the existence of all three types of market participants: information traders, noise traders and market makers; for market makers, it is of vital importance to maintain a good forecast of price movement so as to protect themselves at the same time of facilitating the price discovery process of an exchange.

KEY RISK ASPECTS IN MARKET MAKING

- A market maker has to balance two kinds of risk at the same time, constantly:
  - Adverse selection risk: when trading with information trader, almost by definition, a market maker will always be on the wrong side; the key task here for the market maker is to reduce the loss as much as possible; adverse selection risk has a “direction”; that is, only when the market maker is buying from (or selling to) the information trader will she bear a loss on her inventory;
  - Market (or volatility) risk: this kind of risk is solely due to the random walk nature of the price process, and has to be managed as a kind of “uncertainty”; the key concept here in terms of managing market risk is hedging.

BALANCING RISK, PROFIT AND COST

- The first task of a market maker is to make sure she lose less than anybody else; the second task is to make sure she can capture as much profit as possible, while task one is well taken care of;
- To balance risk and profit, the market maker has to determine several “time scales” upon which she perform risk management;
- In back-testing and paper-trading market making strategies, cost has to be considered in near-reality exchange simulators; otherwise, the results from research and practice can be drastically different.

COMPENSATING VOL RISK WITH RETURN (I)

- If we assume a simple arithmetic Brownian process of price and a linear inventory liquidation process, the risk of holding a unit inventory can be written as

  $$\lambda\sigma\sqrt{T/_3} \qquad ; \qquad (1.1)$$

  where λ is a risk aversion factor, σ a volatility measure of the stock, T is a risk time scale;
- To entice a market maker to take on such a risk, the market maker has to be compensated with the following return:

$$\rho \cdot Spd + \Theta \cdot C + s \cdot \mu \cdot T_{hl} - \Upsilon \cdot \alpha p + R_{ex}. \qquad (1.2)$$

COMPENSATING VOL RISK WITH RETURN (II)

- Here, Spd is the bid-ask spread; C is the transaction cost; μ is the drift of the mid-quote of the stock; s is a sign indicator for Buy or Sell; αp is the alpha due to "in the money" or "out of money" holdings in the inventory; R_{ex} is an excess return term determined explicitly by the market maker to compensate herself for risk taking;
- Therefore, from a trading strategy point of view, to manage inventory, a market maker should balance equation (1.1) and (1.2) so as to obtain an optimal T_* for her inventory holding time.

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CASTING INVENT. MGMT. AS A CONTROL PROBLEM (I)

- Here, we follow some discussions by Forda and Labadie (arxiv:1206.4810v1);
- Let S(t) be the mid-quote price process that follows an Ito diffusion:

$$dS(t) = b\big(t, S(t)\big)dt + \sigma\big(t, S(t)\big)dW(t) \qquad (2.1)$$

- Instead of focusing on the bid and offer price, let's consider the related bid-ask spread:

$$\delta^{+}(t) \triangleq p^{+}(t) - S(t), \delta^{-}(t) \triangleq S(t) - p^{-}(t), \qquad (2.2)$$

- Further define the inventory process as Q(t), the cash process as X(t).

CASTING INVENT. MGMT. AS A CONTROL PROBLEM (II)

- Then Q(t) and X(t) follow the below processes:

$$dQ(t) = dN^{-}(t) - dN^{+}(t), \tag{2.3}$$

and

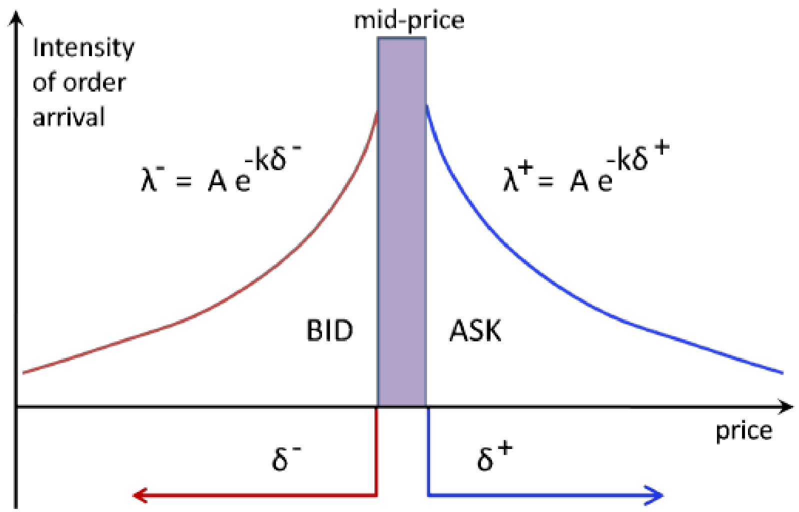
$$dX(t) = [S(t) + \delta^{+}]dN^{+}(t) - [S(t) - \delta^{-}]dN^{-}(t). \tag{2.4}$$

- Here $N^{\pm}(t)$ are two independent processes if bought/sold shares (or, seller/buyer-initiated orders from the counterparties of the market maker);
- Then, the market maker's inventory management problem can be casted into an optimization problem for a pre-assumed utility function of

$$u(t, s, q, x) := \max_{(\delta^{+}, \delta^{-})} \mathbb{E}_{t,s,q,x}[\phi(S(T), Q(T), X(T))] \tag{2.5}$$

ORDER BOOK LIQUIDITY DISTRIBUTION

- The assumption that is often made about order book liquidity distribution is exponential decay from market inside:



Source: Forda and Labadie (arxiv:1206.4810v1)

- A simplified version of this assumption is a linear expansion for $\delta^{+/-}$: $\lambda^{+/-} \approx A - Ak \cdot \delta^{+/-}$; this format is also often used.

SOLVING THE CONTROL PROBLEM (I)

- Now consider the following HJB equation:

$$\begin{aligned}(\partial_t + \mathcal{L})\,u + \max_{\delta+} \lambda^+(\delta^+)\left[u(t,s,q-1,x+(s+\delta^+)) - u(t,s,q,x)\right] \\ + \max_{\delta-} \lambda^-(\delta^-)\left[u(t,s,q+1,x-(s-\delta^-)) - u(t,s,q,x)\right] &= 0\,, \\ u(T,s,q,x) &= \phi(s,q,x)\,. \end{aligned} \tag{2.6}$$

- Here $\mathcal{L}$ is the infinitesimal generator defined as $\mathcal{L} \triangleq \partial_x(t,s)b(s,t) + \frac{1}{2}\partial_{ss}(t,s)\sigma^2(s,t)$; the third "max" terms on the left-hand side of equation (2.6) indicate the increments on the utility function once a trade happens; (2.6) as an equation holds because the optimal control variables, denoted as $\delta_*^{+/-}$, should satisfy $du = 0$.

SOLVING THE CONTROL PROBLEM (II)

- Discussions so far have not taken into account several factors that are important to the trading and inventory management problems of the market maker:
  - Price impact: this has to be taken into account by the price process; it is important to the market maker because it incorporates adverse selection risk; people often assume a parametric model of such a market impact adjustment to the price process; it is worth noting that price impact from trading cost point of view is not the same as drift (the $b(t, S(t))$ in equation (2.1)), as price impact should include order flow information;
  - Fixed cost: items of fixed cost (such as exchange fees, broker commissions, etc.) can be included after the optimization problem in equations (2.6); what can be a dynamic factor in the fixed cost consideration, is in fact variable costs due to real-time margining.

SOLVING THE CONTROL PROBLEM (III)

- Modeling order flow:

  $\lambda^{\pm}=A\cdot\exp(-k\delta^{\pm})$; (2.7)

  that is, order arrival rates depend on the price distance from the market insides;

  note the symmetry between buy and sell order flows;

- Formulizing the utility function:
  - How to deal with cash:

    $u(t,x,q,s) = x + v(t,x,q)$; (2.8)

  - Linear function without inventory penalty:

    $$u(t,s,q,x) = \max_{(\delta+,\delta-)} \mathbb{E}_{t,s,q,x}[X(T)+Q(T)S(T)]\,, \quad (2.9)$$

  - Linear function with inventory penalty:

    $$u(t,s,q,x) = \max_{(\delta+,\delta-)} \mathbb{E}_{t,s,q,x}[X(T)+Q(T)S(T)-\eta Q^2(T)] \quad (2.10)$$

  - Expotential function:

$$u(t,s,q,x) = \max_{(\delta+,\delta-)} \mathbb{E}_{t,s,q,x}\left[-\exp\left\{-\gamma\Big(X(T)+Q(T)S(T)\Big)\right\}\right] \quad (2.11)$$

SOLVING THE CONTROL PROBLEM (IV)

- Consider the market maker's bid-ask spread derived from the stochastic optimization problem with a linear utility function with inventory penalty:
  - $\delta^{\pm} = 1/k + \eta \pm (E_{t,s}[S(T)] - s - 2q\eta);$ (2.12)
  - Note that if the price follows an arithmetic Brownian process, we have $E_{t,s}[S(T)] = s + bT$; thus equation (2.12) can be written as:
    $\delta^{\pm} = 1/k + \eta \pm (bT - 2q\eta);$ (2.13)
  - Then the bid-ask spread can be calculated as:
    $\delta^{+} + \delta^{-} = 2/k + 2\eta.$ (2.14)
  - For k↑, the "elasticity" of order arrival increases; as a result, orders tend to arrive more on the "market inside", thus the bid-ask spread should be tight; "sensitivity" to inventory is "built in" the bid-ask spread formula;
  - Note that the mid-quote of the market maker's orders will be at:
    $midQ = s + ½(\delta^{+} - \delta^{-}) = s + bT - 2q\eta.$ (2.15)

AGENDA

- Part I – Execution Algorithms:
  - An introduction to stochastic programming;
  - Casting optimal execution problem into a dynamic programming problem;
  - Dynamic programming approach vs. quadratic optimization approach in optimal execution algorithm design;
- Part II – Market Making Strategies:
  - A review of market makers and their market making strategies;
  - Casting market making into a dynamic programming problem;
  - Simultaneous market making for both option contracts and underlying assets.

DEFINING THE OPTION MARKET MAKING PROBLEM (I)

- The first and foremost task for the market maker is to post bid and offer quotes in the market; as a result, the market maker has to determine the mid-quote and the bid-ask spread;
- For a general case, consider the following aspects:
  - First of all, the volatility needs to be assumed to be stochastic;
  - Secondly, delta hedging with the underlying asset is not perfect; that is, risks that come from other Greeks have to be managed carefully.

DEFINING THE OPTION MARKET MAKING PROBLEM (II)

- To formulate the problem, let's assume that the underlying asset's price follows a Brownian process described as:

  $$dS_t = \sigma_t S_t dW_t \tag{3.1}$$

  where we have omitted the drift term and W_t is a Brownian process;

- Secondly, we assume that the implied volatility, $\hat{\sigma}_t$, follows its own Brownian process as described as:

  $$d\hat{\sigma}_t = \varphi dU_t \tag{3.2}$$

  where U_t is another Brownian process that is independent from W_t and φ is assumed to be a constant.

DEFINING THE OPTION MARKET MAKING PROBLEM (III)

- Consider that the market making is done on a European option contract at strike K with expiration time of T; keeping the terms up to dt and omitting the second order derivative with respect to realized volatility, we get the stochastic partial differential equation for the mid-quote of the option as

$$dC_t = \Delta_t \sigma_t S_t dW_t + \frac{1}{2}\Gamma_t S_t^2 \sigma_t^2 (u^2 - 1)dt + \nu_t \varphi dU_t \qquad (3.3)$$

where u is a random variable that has a standard normal distribution;

- Note that when achieving (3.3) we made use of no-arbitrage relations that related Θ_t and Γ_t, which essentially means that implied volatility and realized volatility are mean-reverting to each other.

DEFINING THE OPTION MARKET MAKING PROBLEM (IV)

- Because $\nu_t = \Gamma_t S_t^2 \sigma_t \ (T - t)$, we can re-write (3.3) in a discretized fashion as:

$$\Delta C_t = \Delta_t \sigma_t S_t u \sqrt{\Delta t} + \frac{1}{2} \Gamma_t S_t^2 \sigma_t^2 (u^2 - 1) \Delta t + \Gamma_t \sigma_t \varphi \eta (T - t) \sqrt{\Delta t}, \qquad (3.4)$$

where η is a random variable that has a standard normal distribution;

- Next, let's define the objective value function of the market maker; first of all, define the bid and offer spread from the mid-quote of the option contract as $\delta^{b,o}$ and $\delta^{a,o}$; correspondingly, the bid and offer spread from the mid-quote of the underlying asset are defined as $\delta^{b,s}$ and $\delta^{a,s}$; then the P&L and inventory risk can be expressed as:

DEFINING THE OPTION MARKET MAKING PROBLEM (V)

$$\Delta Z_t = \delta^{b,o}\Delta N_t^{+,o} - \delta^{a,o}\,\Delta N_t^{-,o} + \delta^{b,s}\Delta N_t^{+,s} - \delta^{a,s}\,\Delta N_t^{-,s}, \quad (3.5)$$

- The change in the total wealth from the market maker’s holdings (i.e., inventory) can be expressed as:

$$\Delta I_t = \Delta C_t(N_t^{+,o} - N_t^{-,o}) + \Delta S_t(N_t^{+,s} - N_t^{-,s}). \quad (3.6)$$

- Then the objective function can be written as:

$$v(Z_0, S_0, I_0, t_0) = sup_{\delta^{b,o},\delta^{a,o},\delta^{b,s},\delta^{a,s}}\left(E\left(\int_0^{\tau} dZ_t\right) - \gamma \int_0^{\tau} Var(dI_t)\right)$$

(3.7)

- If we plug (3.4) into (3.6) and assuming that the stocks position change is solely due to delta hedging, we obtain

$$\Delta I_t = (N_t^{+,o} - N_t^{-,o})\left(\frac{1}{2}\Gamma_t S_t^2 \sigma_t^2 (u^2 - 1)\Delta t + \Gamma_t \sigma_t \varphi \eta (T - t)\sqrt{\Delta t}\right). \quad (3.8)$$

DISCUSSION ON PRACTICAL MATTERS OF OPTIMAL OPTION MARKET MAKING

- Incorporation of alphas of option contracts:

- Portfolio risk of underlying instruments:

- Transaction cost considerations:

- Multi-period trading:

THAT'S ALL FOR THIS LECTURE.

THANK YOU!