FINM37601: Mathematical Market Microstructure

Assignment 1 – Market Microstructure Variables and Characteristic Time Scale (I)

Due Date: 11:59pm, Nov. 11, 2023

Introduction: This assignment is Part I of our discussion on market microstructure variables and characteristics time scale. In this assignment, we will derive the mean and variance of transaction price assuming a "bid-ask bounce" process for the price. In the next assignment, we will focus on an execution strategy design based on the optimal balance between market impact and execution risk.

Consider a stock that has a bid-ask spread of $s = |b - a| > 0$. For each individual transaction that happens to this stock, the probabilities of hitting the bid, the ask, and the mid-quote of $(a + b)/2$ are ρ_l, ρ_u, ρ_m, respectively. By definition, $\rho_l + \rho_u + \rho_m = 1$. Note that all variables here can be functions of time, t.

Question 1 (20 pts): Derive the formulas for the mean $\mathbb{E}[p]$ and the variance $Var[p]$ of the transaction price p with the variables mentioned above.

Question 2 (15 pts): Simplify your results in Question 1 by assuming that $|\rho_l - \rho_u| = \delta \ll 1$ so that higher than 1 order terms of $|\rho_l - \rho_u|$ can be omitted. In practice, this is the situation when order flow has no significant directional upward or downward movements.

Question 3 (15 pts): Simplify your results in Question 1 by assuming that $\rho_m \ll 1$ so that higher than 1 order terms of ρ_m can be omitted. In practice, this is the situation when "off exchange" trades (such as those in "dark pools") are rare.

Question 4 (20 pts): Let T_* be the "market microstructure characteristic time scale" for this stock. It can be derived by relating the variance calculated from Questions 1-3 with the variance of a continuous arithmetic Brownian price process with a constant volatility of σ; that is,

$$\sqrt{Var[p]} \sim p_0 \sigma (T_*)^{\gamma}. \qquad \text{(Equation 1)}$$

Here p_0 is a base price so that the dimensions on both sides of Equation 1 become the same. In this assignment, let's assume $\gamma = 1/2$. Based on the simplified versions of Questions 2 and 3 respectively, give the formulas of T_* as a function of the following variables: $b, a, \rho_l, \rho_u, \rho_m, \sigma, p_0$.

Question 5 (20 pts): Consider three scenarios of $\rho_m = 0, 0.1, 0.2$, respectively. For each scenario, plot $T_*(\rho_l)$ for $\rho_l =$ 0 to 1.0 (for $\rho_m = 0$) or 0.9 (for $\rho_m = 0.1$) or 0.8 (for $\rho_m = 0.2$) with each step of 0.1. Here we assume $b = 62.00$, $a = 62.05$, $\sigma = 20\%$ annualized, $p_0 = 62.025$.

Question 6 (10 pts): "Dark pool" trading supporters often argue that market quality can be improved with more usage of dark pools, especially from reducing the level of price volatility. Based on the above calculations, provide your own views on whether you agree with such an argument.

End of assignment 1.