Chapter 2

Kinematics

2.1 Position and Orientation

Goal

To specify attributes (position and orientation) of various objects (parts, tools, …) with which a manipulator system deals.

2.1.1 Description of A Position

Every point (P) in the universe can be located by a ***position vector*** $\hat{p}$ originating from a known point (A) to that point (P). In order to compute and manipulate a position vector, we translate it into a set of 3 scalars (a 3×1 matrix). We assign three mutually perpendicular unit vectors (axes) coinciding at the known point (A) as a ***coordinate system*** (frame **{A}**):

$$\{A\} \equiv \hat{A} \equiv \begin{bmatrix} \hat{x}_A & \hat{y}_A & \hat{z}_A \end{bmatrix}^T,$$

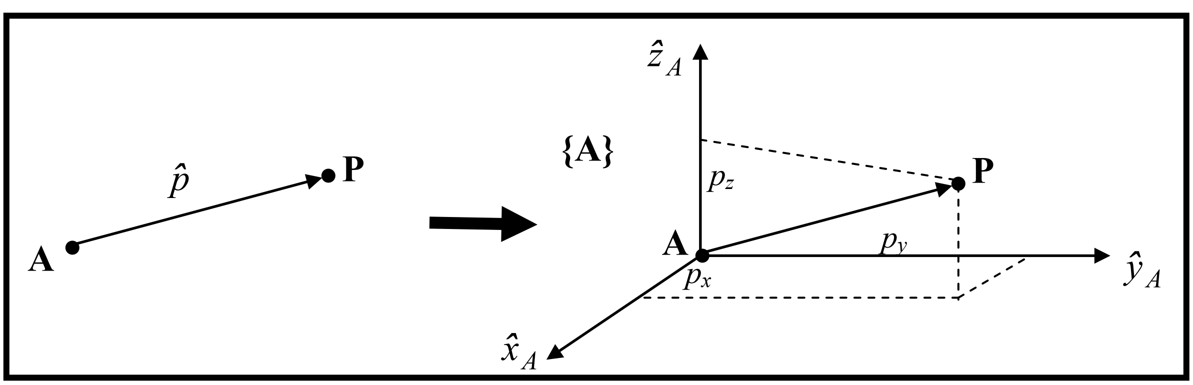
and then express the position vector by the magnitude of its projections on the three axes.

(Vector) $\hat{p} = \hat{A}^T\,{}^A p = p_x\hat{x}_A + p_y\hat{y}_A + p_z\hat{z}_A \;\longrightarrow\; {}^A p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \hat{p}\cdot\hat{x}_A \\ \hat{p}\cdot\hat{y}_A \\ \hat{p}\cdot\hat{z}_A \end{bmatrix}$ **(3×1 Matrix)**

(Vector Inner Product) $\hat{p}_1 \cdot \hat{p}_2 \;\longrightarrow\; \alpha = \left({}^A p_1\right)^T \left({}^A p_2\right)$ **(Scalar)**

(Vector Cross Product) $\hat{p}_1 \times \hat{p}_2 \;\longrightarrow$

$${}^A r = \left({}^A \tilde{p}_1\right)\left({}^A p_2\right) = \begin{bmatrix} 0 & -p_{1z} & p_{1y} \\ p_{1z} & 0 & -p_{1x} \\ -p_{1y} & p_{1x} & 0 \end{bmatrix} \begin{bmatrix} p_{2x} \\ p_{2y} \\ p_{2z} \end{bmatrix}$$ **(3×1 Matrix)**



2.1.2 Description of An Orientation

In addition to representing a point in the space, we also need to describe the orientation of a body in space. We attach a coordinate system to the body (**{B}**), and then give a description of this coordinate system relative to a known coordinate system (**{A}**).

Thus, *positions of points are described with vectors and orientations of bodies are described with an attached coordinate system*.

In order to describe the body coordinate system, the unit vectors of its three principal axes (vectors) are described in terms of the known coordinate system (**{A}**).

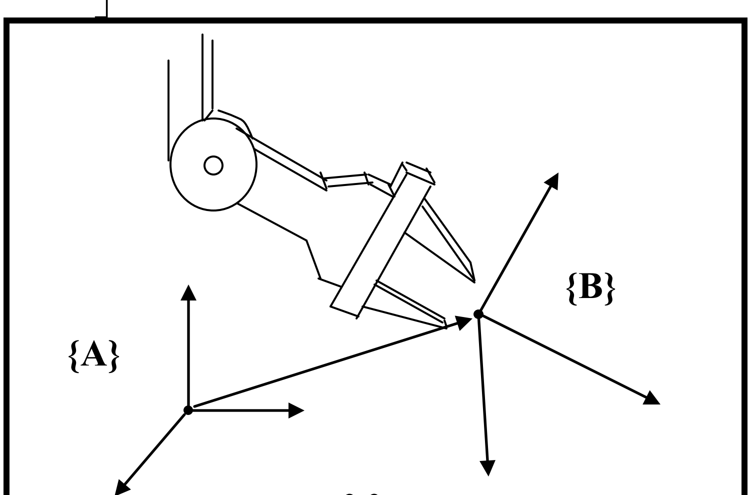
$$^{A}x_B = \begin{bmatrix} \hat{x}_B \cdot \hat{x}_A \\ \hat{x}_B \cdot \hat{y}_A \\ \hat{x}_B \cdot \hat{z}_A \end{bmatrix} \qquad ^{A}y_B = \begin{bmatrix} \hat{y}_B \cdot \hat{x}_A \\ \hat{y}_B \cdot \hat{y}_A \\ \hat{y}_B \cdot \hat{z}_A \end{bmatrix} \qquad ^{A}z_B = \begin{bmatrix} \hat{z}_B \cdot \hat{x}_A \\ \hat{z}_B \cdot \hat{y}_A \\ \hat{z}_B \cdot \hat{z}_A \end{bmatrix}$$

For convenience, we construct a 3×3 matrix that has the above three matrices (called *direction of cosines*) as its columns, and call it ***rotation matrix*** of **{B}** w.r.t. **{A}**.

$$^{A}R_B = \begin{bmatrix} ^{A}x_B & ^{A}y_B & ^{A}z_B \end{bmatrix} = \begin{bmatrix} \left(^{B}x_A\right)^T \\ \left(^{B}y_A\right)^T \\ \left(^{B}z_A\right)^T \end{bmatrix} = \left(^{B}R_A\right)^T$$

NOTE:

$$\left(^{A}R_B\right)^T\,^{A}R_B = \begin{bmatrix} \left(^{A}x_B\right)^T \\ \left(^{A}y_B\right)^T \\ \left(^{A}z_B\right)^T \end{bmatrix} \begin{bmatrix} ^{A}x_B & ^{A}y_B & ^{A}z_B \end{bmatrix} = I_3 \quad \textbf{OR} \quad ^{A}R_B = \left(^{B}R_A\right)^{-1} = \left(^{B}R_A\right)^T$$



2.1.3 Properties of the Rotation Matrix

1) All columns of a rotation matrix R are mutually orthogonal, and have unit magnitude.

$$R = \begin{bmatrix} X & Y & Z \end{bmatrix}$$

$$X^T X = 1 \qquad X^T Y = 0$$

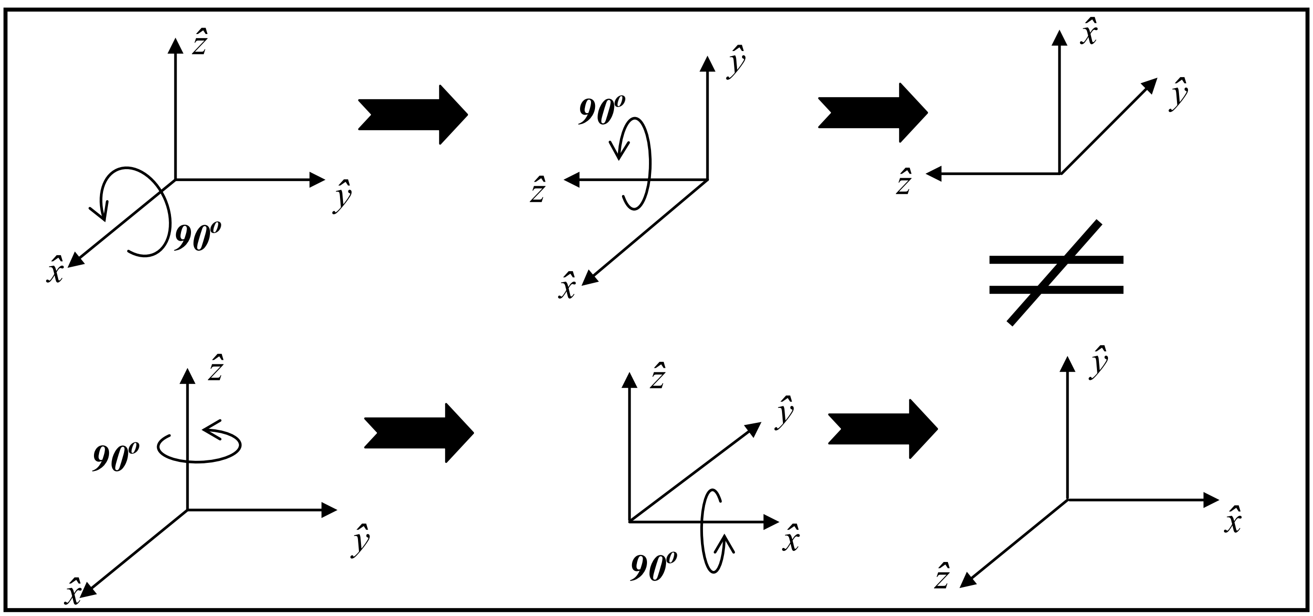
$$Y^T Y = 1 \qquad X^T Z = 0$$

$$Z^T Z = 1 \qquad Y^T Z = 0$$

2) The determinant of a rotation matrix is +1.

3) Rotations do not generally commute. (As multiplication of matrices is not commutative.)

$${}^A R_B \, {}^B R_C \neq {}^B R_C \, {}^A R_B$$



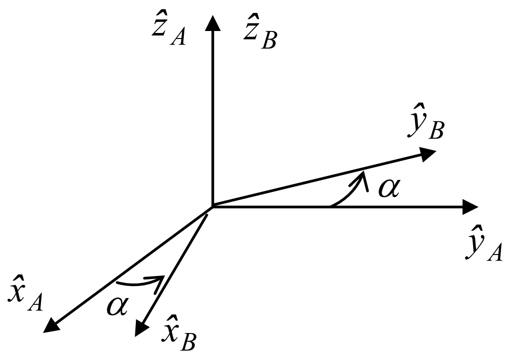
4) The inverse transformation of a rotation matrix is equal to its transpose.

$${}^B R_A = \left({}^A R_B\right)^{-1} = \left({}^A R_B\right)^T$$

5) The orientation of a body can be identified by 3 independent parameters. From Property 1, there are six constraints on the nine elements of the rotation matrix. Hence, only 3 elements can be selected independently.

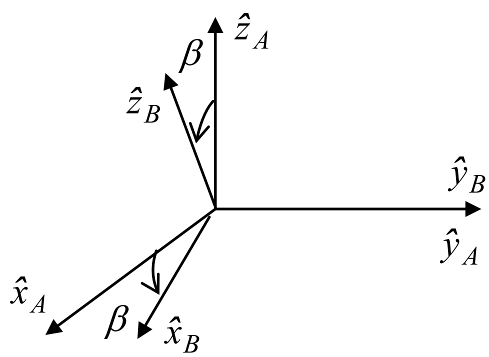
2.1.4 Basic (Principal) Rotation Matrices

About z axis:



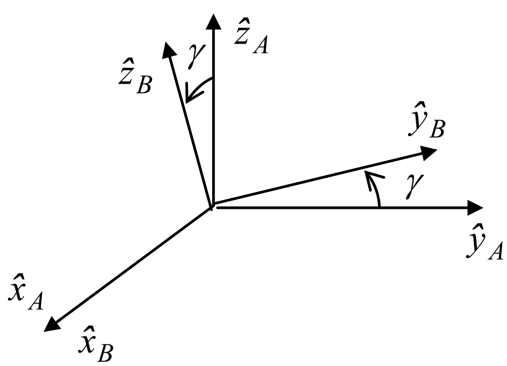
$$^{A}R_B = R_z(\alpha) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

About y axis:



$$^{A}R_B = R_y(\beta) = \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}$$

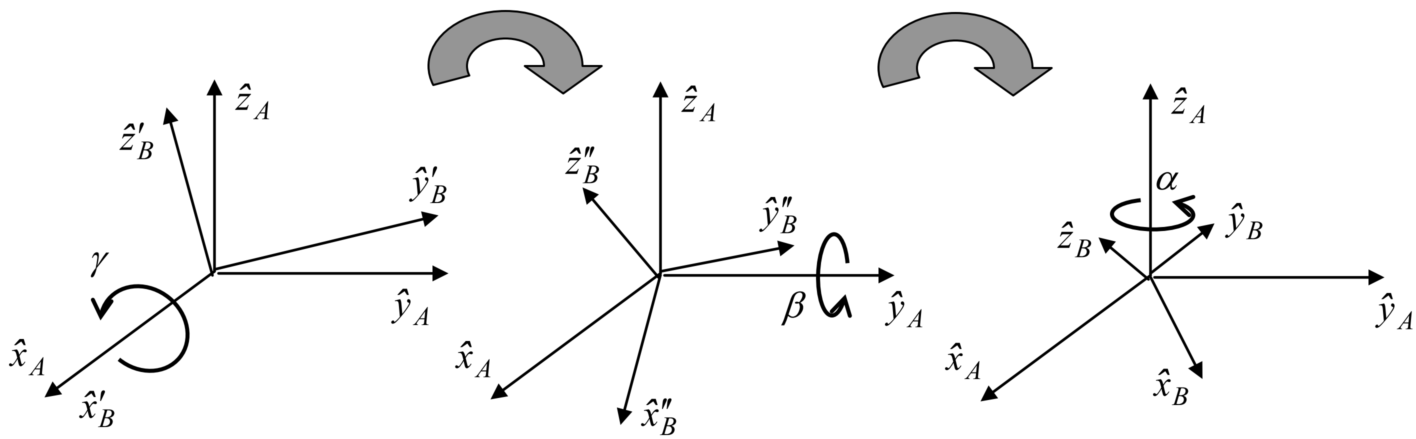
About x axis:



$$^{A}R_B = R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

2.1.5 Cascade Rotations

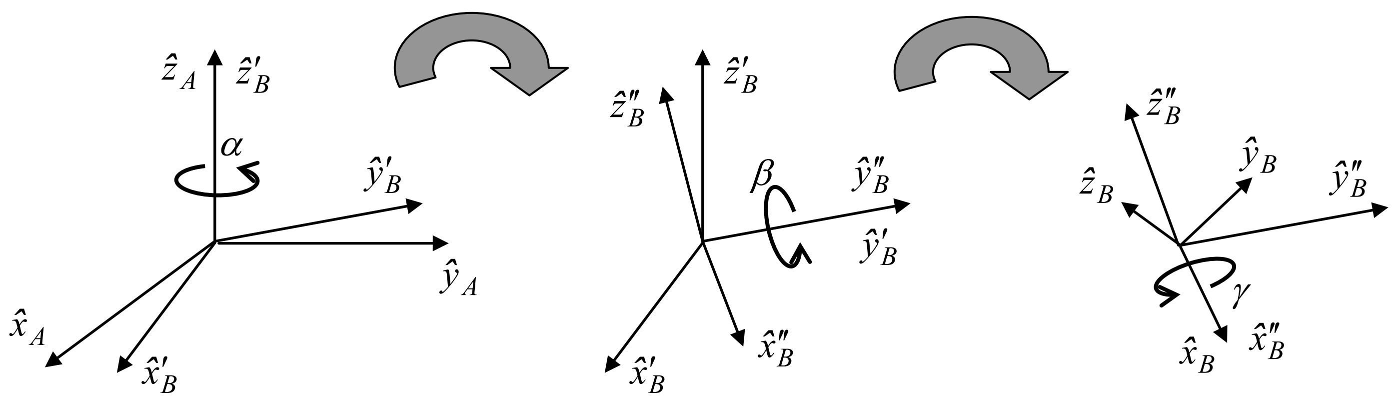
About the Fixed Axes:



Order of rotations

$$\left({}^{A}R_{B}\right)_{f}(\gamma,\beta,\alpha)=R_{z}(\alpha)\cdot R_{y}(\beta)\cdot R_{x}(\gamma)$$

About the Moving Axes:



Order of rotations

$$\left({}^{A}R_{B}\right)_{m}(\alpha,\beta,\gamma)=R_{z}(\alpha)\cdot R_{y'}(\beta)\cdot R_{x''}(\gamma)$$

General Rule: *Three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.*

$$\boxed{\left({}^{A}R_{B}\right)_{f}(\gamma,\beta,\alpha)=\left({}^{A}R_{B}\right)_{m}(\alpha,\beta,\gamma)}$$

2.1.6 Alternative Representations of An Orientation

Only three parameters are needed to specify the orientation of a body (coordinate frame). Two major methods are used to define the orientation parameters:

X-Y-Z-Fixed Angles (Roll-Pitch-Yaw Angles)

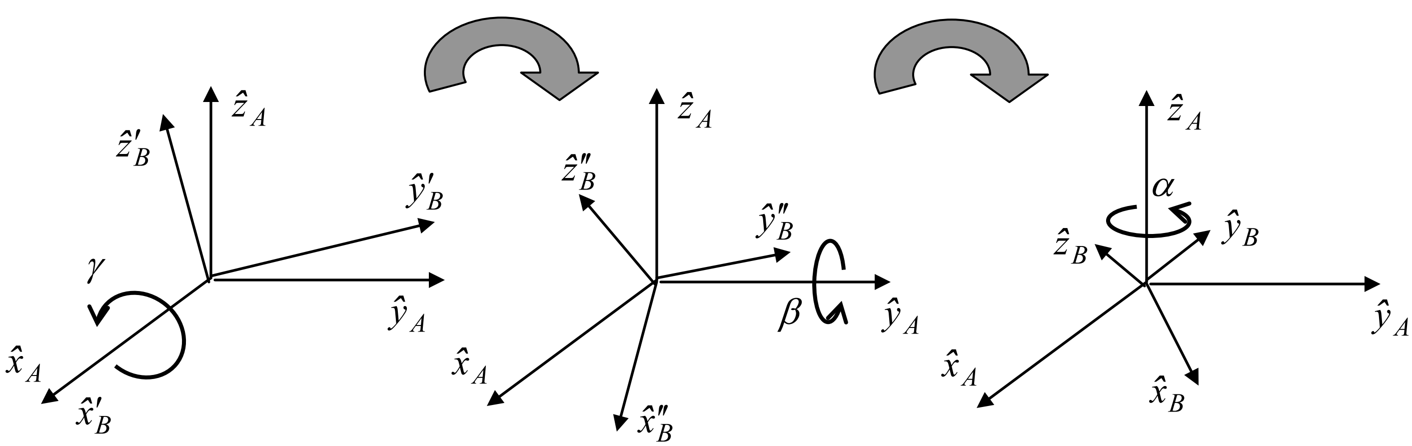
To describe the orientation of a frame (**{B}**):

*Start with the frame coincident with a known frame **{A}**. First rotate **{B}** about $\hat{x}_A$ by an angle γ, then rotate about $\hat{y}_A$ by an angle β, and then rotate about $\hat{z}_A$ by an angle α.*

$$\left({}^{A}R_B\right)_{XYZ}(\gamma,\beta,\alpha)=\overleftarrow{R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)}$$

$$=\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

$$=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$



Knowing the rotation matrix as:

$$\left({}^{A}R_{B}\right)_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

the rotation angles can be computed as follows:

$$\beta = Atan2\left(-r_{31}, \pm\sqrt{r_{11}^2 + r_{21}^2}\right),$$
$$\alpha = Atan2\left(r_{21}/c\beta, r_{11}/c\beta\right),$$
$$\gamma = Atan2\left(r_{32}/c\beta, r_{33}/c\beta\right).$$

Note 1: single solution for which $-90^o \le \beta \le +90^o$ is computed.

Note 2: *Atan2(y,x)* computes $tan^{-1}(y/x)$, but uses the signs of both *x* and *y* to determine the quadrant in which the resulting angle lies. For example, $Atan2(-2.0,-2.0)=-135^o$; whereas $Atan2(2.0,2.0)=45^o$.

If $\beta = \pm 90^o$, degeneration happens, and only the sum or the difference of α and γ may be computed.

Convention:

If $\beta = +90^o$, set $\alpha = 0$, and calculate $\gamma = Atan2(r_{12}, r_{22})$.

If $\beta = -90^o$, set $\alpha = 0$, and calculate $\gamma = -Atan2(r_{12}, r_{22})$.

Z-Y-Z Euler Angles

To describe the orientation of a frame (**{B}**):

Start with the frame coincident with a known frame **{A}**. *First rotate* **{B}** *about* $\hat{z}_A$ *by an angle* α, *then rotate about* ***the new*** $\hat{y}'_B$ *by an angle* β, *and then rotate about* ***the new*** $\hat{z}''_B$ *by an angle* γ.

$$\left({}^{A}R_{B}\right)_{ZY'Z''}(\alpha,\beta,\gamma)=\overrightarrow{R_{Z}(\alpha)\cdot R_{Y'}(\beta)\cdot R_{Z''}(\gamma)}$$

$$=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

The solution for Z-Y-Z Euler angles from a rotation matrix

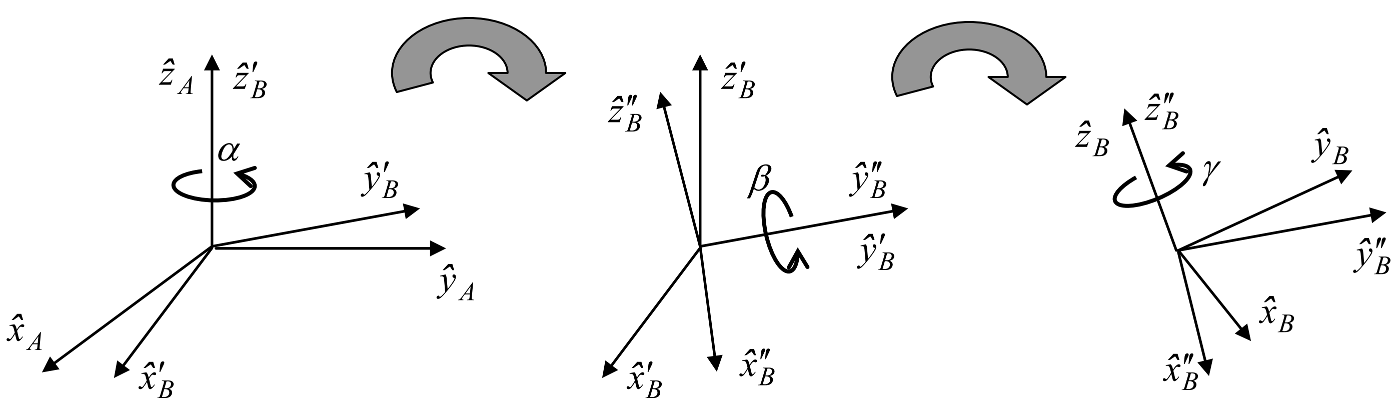
$$\left({}^{A}R_{B}\right)_{ZY'Z''}(\alpha,\beta,\gamma)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

is as follows:

$$\beta = Atan2\left(\pm\sqrt{r_{31}^{2}+r_{32}^{2}},\, r_{33}\right),$$
$$\alpha = Atan2\left(r_{23}/s\beta,\, r_{13}/s\beta\right),$$
$$\gamma = Atan2\left(r_{32}/s\beta,\, -r_{31}/s\beta\right).$$

If $\beta = 0^{o}$, set $\alpha = 0$, and calculate $\gamma = Atan2\left(-r_{12}, r_{11}\right)$.

If $\beta = 180^{o}$, set $\alpha = 0$, and calculate $\gamma = Atan2\left(r_{12}, -r_{11}\right)$.



Screw Axis Representation

THEOREM: For any rotation (of a coordinate frame), a unique axis can be found on which all points remain fixed during the rotation. This axis is called *axis of rotation* $\hat{e}$, and ϕ is the angle (scalar) with which the initial frame rotates to the new one.

PROOF: Assuming that this axis exists, for any vector $\hat{p}$ along this axis we have:

$$^{A}p = {}^{A}R_{B}\,{}^{B}p = {}^{B}p$$

The above equation is a special case of the following general eigenvalue problem:

$$^{A}R_{B}\,{}^{B}p = \lambda\,{}^{B}p \quad \Rightarrow \quad \left({}^{A}R_{B} - \lambda I\right){}^{B}p = 0,$$

where **{A}** is the original and **{B}** is the rotated frame, and I denotes a 3×3 identity matrix. Hence, the axis of rotation can be found by solving the eigenvalue problem for rotation matrix $^{A}R_{B}$, as follows:

$$\left|{}^{A}R_{B} - \lambda I\right| = \begin{vmatrix} r_{11}-\lambda & r_{12} & r_{13} \\ r_{21} & r_{22}-\lambda & r_{23} \\ r_{31} & r_{32} & r_{33}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad \lambda^{3} - tr\left({}^{A}R_{B}\right)\lambda^{2} + tr\left({}^{A}R_{B}\right)\lambda - 1 = 0 \quad (\textit{Characteristic Equation})$$

where $\quad tr\left({}^{A}R_{B}\right) = r_{11} + r_{22} + r_{33}$.

The characteristic equation has three distinct roots with the corresponding eigenvectors:

$$\begin{cases} \lambda_{1} = \boldsymbol{cos}\,\phi + i\,\boldsymbol{sin}\,\phi = e^{i\phi} & \Rightarrow \quad \hat{b} = \dfrac{\hat{m}}{\sqrt{2}} + i\dfrac{\hat{n}}{\sqrt{2}} \quad \text{(Complex Eigenvector)} \\ \lambda_{2} = \boldsymbol{cos}\,\phi - i\,\boldsymbol{sin}\,\phi = e^{-i\phi} & \Rightarrow \quad \hat{b}^{*} = \dfrac{\hat{m}}{\sqrt{2}} - i\dfrac{\hat{n}}{\sqrt{2}} \quad \text{(Complex Eigenvector)} \\ \lambda_{3} = 1 & \Rightarrow \quad \hat{e} \quad \text{(Real Eigenvector)} \end{cases}$$

where

$$\phi = \boldsymbol{cos}^{-1}\left(\frac{tr\left({}^{A}R_{B}\right) - 1}{2}\right).$$

Hence,

$$ {}^{A}R_{B}\,{}^{B}b = \lambda_1\,{}^{B}b \ , \ {}^{A}R_{B}\,{}^{B}b^{*} = \lambda_2\,{}^{B}b^{*} \ , \ {}^{A}R_{B}\,{}^{B}e = {}^{B}e $$

The complex eigenvectors $\hat{b}$ and $\hat{b}^{*}$ can be replaced by the following real orthogonal axis:

$$ \begin{cases} \hat{m} = \dfrac{\sqrt{2}}{2}\left(\hat{b} + \hat{b}^{*}\right) \\ \hat{n} = \dfrac{-i\sqrt{2}}{2}\left(\hat{b} - \hat{b}^{*}\right) \end{cases} $$

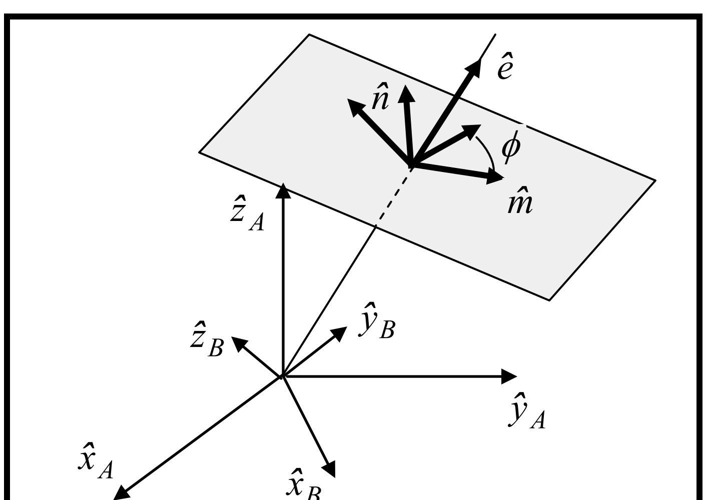
Then,

$$ \begin{cases} {}^{A}m = {}^{A}R_{B}\,{}^{B}m = \dfrac{\sqrt{2}}{2}\,{}^{A}R_{B}\left({}^{B}b + {}^{B}b^{*}\right) = \dfrac{\sqrt{2}}{2}\left(\lambda_1\,{}^{B}b + \lambda_2\,{}^{B}b^{*}\right) = (\boldsymbol{cos}\,\phi)\,{}^{B}m - (\boldsymbol{sin}\,\phi)\,{}^{B}n \\ {}^{A}n = {}^{A}R_{B}\,{}^{B}n = \dfrac{-i\sqrt{2}}{2}\,{}^{A}R_{B}\left({}^{B}b - {}^{B}b^{*}\right) = \dfrac{-i\sqrt{2}}{2}\left(\lambda_1\,{}^{B}b - \lambda_2\,{}^{B}b^{*}\right) = (\boldsymbol{sin}\,\phi)\,{}^{B}m + (\boldsymbol{cos}\,\phi)\,{}^{B}n \\ {}^{A}e = {}^{A}R_{B}\,{}^{B}e = {}^{B}e \end{cases} $$

$$ {}^{A}R_{B}\underbrace{\begin{bmatrix} {}^{B}m & {}^{B}n & {}^{B}e \end{bmatrix}}_{{}^{B}P} = \underbrace{\begin{bmatrix} {}^{B}m & {}^{B}n & {}^{B}e \end{bmatrix}}_{{}^{B}P} \begin{bmatrix} \boldsymbol{cos}\,\phi & -\boldsymbol{sin}\,\phi & 0 \\ \boldsymbol{sin}\,\phi & \boldsymbol{cos}\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} = {}^{B}PR_{z}^{T}(\phi) $$

$$ {}^{B}P^{T}\,{}^{B}R_{A}\,{}^{B}P = {}^{A}P^{T}\,{}^{B}R_{A}\,{}^{A}P = R_{z}(\phi) $$ **(Similarity Transformation)**

This means that for any rotation matrix ${}^{A}R_{B}$ there is a plane (containing $\hat{m}$ and $\hat{n}$) in which the rotation happens around an axis ($\hat{e}$) normal to the plane by an angle ϕ. ■



Based on the properties of similarity transformation:

$$tr(R) = tr(R_z) \;\Rightarrow\; r_{11} + r_{22} + r_{33} = 1 + 2\cos\phi \;\Rightarrow\; \phi = \cos^{-1}\frac{r_{11} + r_{22} + r_{33} - 1}{2}$$

$$^{A}e = {}^{B}e = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \frac{1}{2\sin\phi}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

For obtaining the rotation matrix from the rotation axis, first the following corollary is proved:

COROLLARY: For every (right-handed) coordinate frame **{A}** (with unit axes $\hat{m}$, $\hat{n}$ and $\hat{e}$), the following relationships hold, where frame **{B}** is the result of the rotation of **{A}** under rotation matrix $^{A}R_B$.

$$^{B}m\left(^{B}m\right)^T + {}^{B}n\left(^{B}n\right)^T + {}^{B}e\left(^{B}e\right)^T = I$$

$$^{B}n\left(^{B}m\right)^T - {}^{B}m\left(^{B}n\right)^T = {}^{B}\tilde{e}$$

PROOF:

$$^{A}m = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; ^{A}n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\; ^{A}e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\Rightarrow\; ^{A}m\left(^{A}m\right)^T + {}^{A}n\left(^{A}n\right)^T + {}^{A}e\left(^{A}e\right)^T = I$$

$$\begin{cases} ^{A}m = {}^{A}R_B\, {}^{B}m \\ ^{A}n = {}^{A}R_B\, {}^{B}n \\ ^{A}e = {}^{A}R_B\, {}^{B}e \end{cases} \Rightarrow \begin{cases} ^{A}m\left(^{A}m\right)^T = {}^{A}R_B\, {}^{B}m\left(^{B}m\right)^T\left(^{A}R_B\right)^T \\ ^{A}n\left(^{A}n\right)^T = {}^{A}R_B\, {}^{B}n\left(^{B}n\right)^T\left(^{A}R_B\right)^T \\ ^{A}e\left(^{A}e\right)^T = {}^{A}R_B\, {}^{B}e\left(^{B}e\right)^T\left(^{A}R_B\right)^T \end{cases}$$

By adding the above three equations:

$$\Rightarrow\; I = {}^{A}R_B\left[{}^{B}m\left(^{B}m\right)^T + {}^{B}n\left(^{B}n\right)^T + {}^{B}e\left(^{B}e\right)^T\right]\left(^{A}R_B\right)^T$$

$$\Rightarrow\; \left(^{A}R_B\right)^T I\, {}^{A}R_B = \left[{}^{B}m\left(^{B}m\right)^T + {}^{B}n\left(^{B}n\right)^T + {}^{B}e\left(^{B}e\right)^T\right] = I\,. \;\blacksquare$$

Also

$$\begin{cases} \hat{m} = -\hat{e} \times \hat{n} \\ \hat{n} = \hat{e} \times \hat{m} \end{cases} \Rightarrow \begin{cases} {}^B m = -{}^B\tilde{e}\,{}^B n \\ {}^B n = {}^B\tilde{e}\,{}^B m \end{cases}$$

$$\Rightarrow \quad {}^B n \left({}^B m\right)^T - {}^B m \left({}^B n\right)^T = {}^B\tilde{e}\,{}^B m \left({}^B m\right)^T + {}^B\tilde{e}\,{}^B n \left({}^B n\right)^T = {}^B\tilde{e}\left[I - {}^B e \left({}^B e\right)^T\right] = {}^B\tilde{e} \quad \blacksquare$$

From the similarity transformation, we have:

$${}^B R_A = {}^B P R_z \left({}^B P\right)^T = \begin{bmatrix} {}^B m & {}^B n & {}^B e \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \left({}^B m\right)^T \\ \left({}^B n\right)^T \\ \left({}^B e\right)^T \end{bmatrix}$$

$$\Rightarrow \quad {}^B R_A = ee^T + \left(I - ee^T\right)\cos\phi + \tilde{e}\sin\phi \qquad \textbf{(Rodrigues' Formula)}$$

EXAMPLE:

Given

$$R = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$$

a) Verify that it represents a rotation.

b) Find the axis and angle of rotation.

SOLUTION:

a) $RR^T = R^T R = I$ and $\det(R) = 1$ imply that the matrix is a proper orthogonal matrix representing a rotation.

b) $tr(R) = 2 = 1 + 2\cos\phi \quad \Rightarrow \quad \cos\phi = 0.5 \quad \Rightarrow \quad \phi = \pm 60^\circ$ (to be identified)

$$R\,e = e \quad \Rightarrow \quad e = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

And

$$\begin{cases} e_x \, \boldsymbol{sin}\,\phi = -1/2 \\ e_y \, \boldsymbol{sin}\,\phi = 1/2 \\ e_z \, \boldsymbol{sin}\,\phi = -1/2 \end{cases} \quad \Rightarrow \quad \phi = -60^\circ$$

EXAMPLE:

Determine the rotation matrix R representing a rotation of 90 degrees about an axis having three equal direction cosines with respect to a fixed frame **{O}**.

SOLUTION:

The rotation axis is defined as:

$$e = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Axis $\hat{m}$ can be assigned as follows:

$${}^{O}m = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}, \text{ so that } \begin{cases} \hat{m}\cdot\hat{e} = 0 \;\Rightarrow\; m_x + m_y + m_z = 0 \\ \|\hat{m}\| = 1 \;\Rightarrow\; m_x^2 + m_y^2 + m_z^2 = 1 \end{cases}$$

Choose $m_x = 0$, then $m_y = \pm\sqrt{2}/2$ and $m_z = \mp\sqrt{2}/2$. Then choose

$${}^{O}m = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Axis $\hat{n}$ is obtained from $\hat{n} = \hat{e} \times \hat{m}$. Hence,

$$ {}^{O}n = \tilde{e}\ {}^{O}m = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} $$

Now, matrix P can be formed as:

$$ {}^{O}P = \begin{bmatrix} {}^{O}m & {}^{O}n & e \end{bmatrix} = \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, $$

and

$$ R_z\left(90^\circ\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

Therefore,

$$ {}^{A}R_O = {}^{O}P\, R_z \left({}^{O}P\right)^T = \frac{1}{3} \begin{bmatrix} 1 & 1-\sqrt{3} & 1+\sqrt{3} \\ 1+\sqrt{3} & 1 & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} & 1 \end{bmatrix} $$

NOTE: The rotation matrix R can also be directly obtained from Rodrigues' formula.

2.2 Coordinate Transformation

Goal

To introduce the translation, rotation, and homogeneous transformation operators.

2.2.1 Translational Operators

A translation moves a point in the space a finite distance along the direction of a given vector.

EXAMPLE:

Point P_1 is addressed by vector $\hat{p}_1$ that is described in frame $\{A\}$ as ${}^{A}p_1$. We move this point along vector $\hat{q}$, described in $\{A\}$ by ${}^{A}q$, to the point P_2 that is addressed by vector $\hat{p}_2$ described in $\{A\}$ as ${}^{A}p_2$.

The new vector can be calculated as:

$$\hat{p}_2 = \hat{p}_1 + \hat{q} \qquad (\textit{Vector Sum})$$

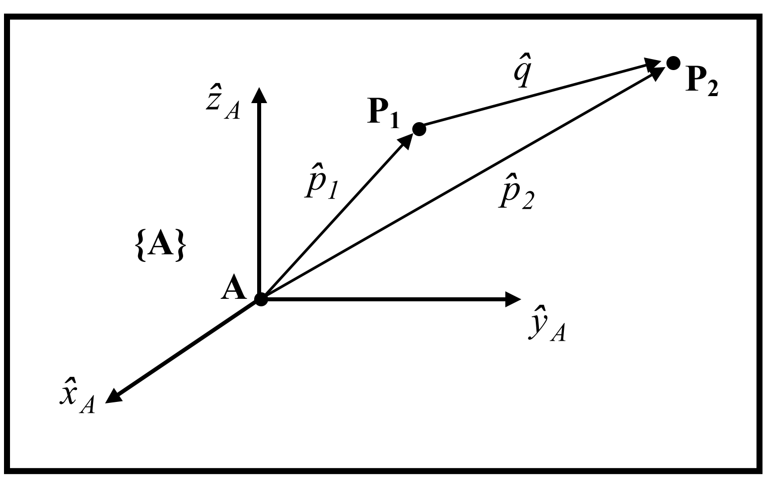
or:

$${}^{A}p_2 = {}^{A}p_1 + {}^{A}q \qquad (\textit{Matrix Sum}).$$

An operator $D_{\hat{q}}(q)$ can be defined for the above operation such that:

$$\begin{bmatrix} {}^{A}p_2 \\ \hline 1 \end{bmatrix}_{4\times 1} = D_{\hat{q}}(q) \begin{bmatrix} {}^{A}p_1 \\ \hline 1 \end{bmatrix}_{4\times 1}$$

where q is the "***signed magnitude***" of the translation along vector $\hat{q}$. $D_{\hat{q}}(q)$ is a 4×4 matrix represented as:



$$D_{\hat{q}}(q) = \begin{bmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & {}^{A}q \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right]$$

where

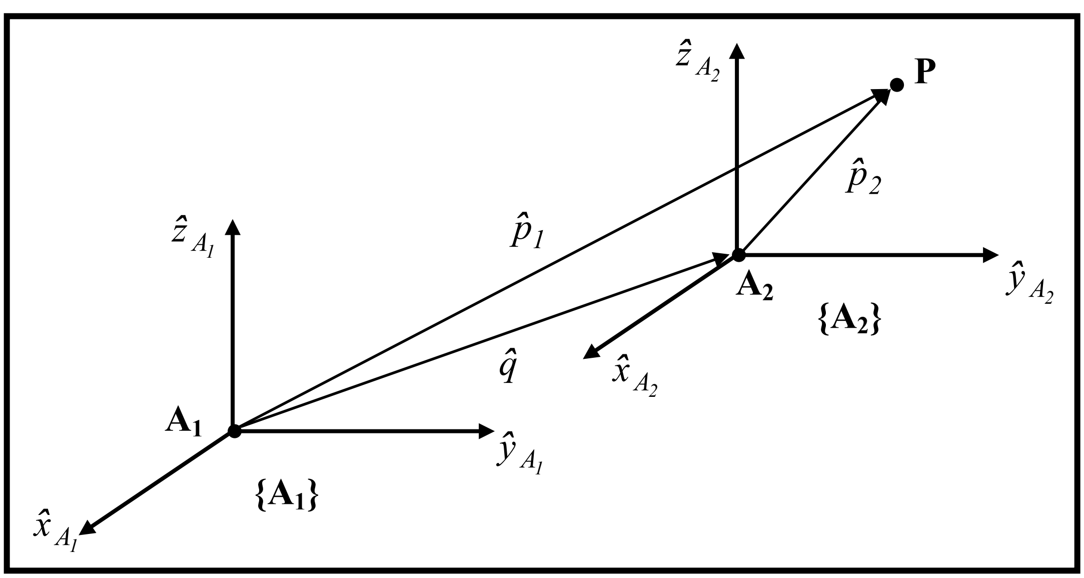
$${}^{A}q = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \text{ and } \quad |q| = \sqrt{q_x^2 + q_y^2 + q_z^2}$$

A Different Translation Operation

Suppose that point P is fixed in the space, but we translate (no rotation) frame $\{A_1\}$ to frame $\{A_2\}$ along vector $\hat{q}$. We can use the same operator to interpret the relation between ${}^{A_1}p_1$ and ${}^{A_2}p_2$:

$$\begin{bmatrix} {}^{A_1}p_1 \\ \hline 1 \end{bmatrix}_{4\times 1} = D_{\hat{q}}(q) \begin{bmatrix} {}^{A_2}p_2 \\ \hline 1 \end{bmatrix}_{4\times 1}$$

NOTE: In this case, ${}^{A_1}p_2 = {}^{A_2}p_2$, since there is no rotation between the frames.



2.2.2 Rotational Operators

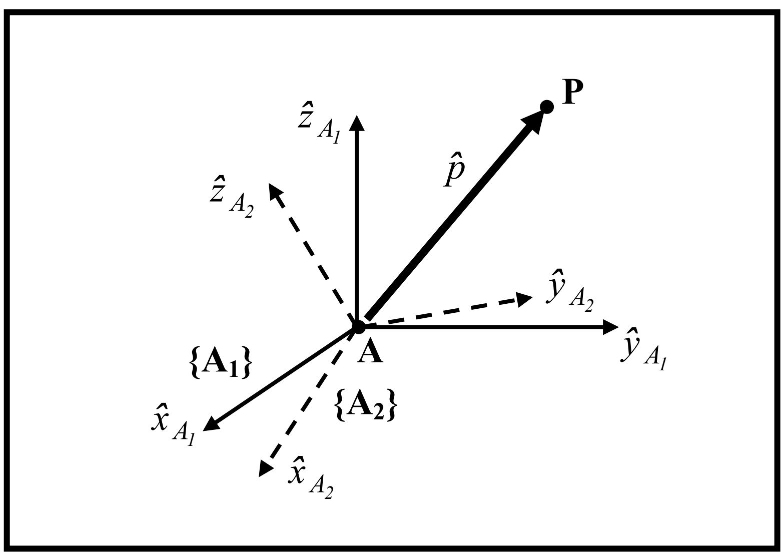
A rotational operator rotates a body (frame) in the space about a fixed point (the origin of the coordinate frame). A fixed vector $\hat{p}$ in the space (representing point P) will then have different descriptions in the original ({A_1}) and the rotated ({A_2}) frames:

$$^{A_2}p = {}^{A_2}R_{A_1} \; {}^{A_1}p \, .$$

This operation can also be interpreted as follows:

$$\begin{bmatrix} {}^{A_2}p \\ \hline 1 \end{bmatrix}_{4\times 1} = \left[\begin{array}{c|c} \begin{bmatrix} {}^{A_2}R_{A_1} \end{bmatrix}_{3\times 3} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] \begin{bmatrix} {}^{A_1}p \\ \hline 1 \end{bmatrix}_{4\times 1}$$

NOTE: The first 3 rows of the last column are zero, as no translation happens in the mapping.



A Different Rotation Operation

The rotation matrix can also be viewed as an operator that rotates a vector $\hat{p}_1$ to a new vector $\hat{p}_2$ in a fixed coordinate frame $\{A\}$. The descriptions of the two vectors $\hat{p}_1$ and $\hat{p}_2$ in frame $\{A\}$ are related as:

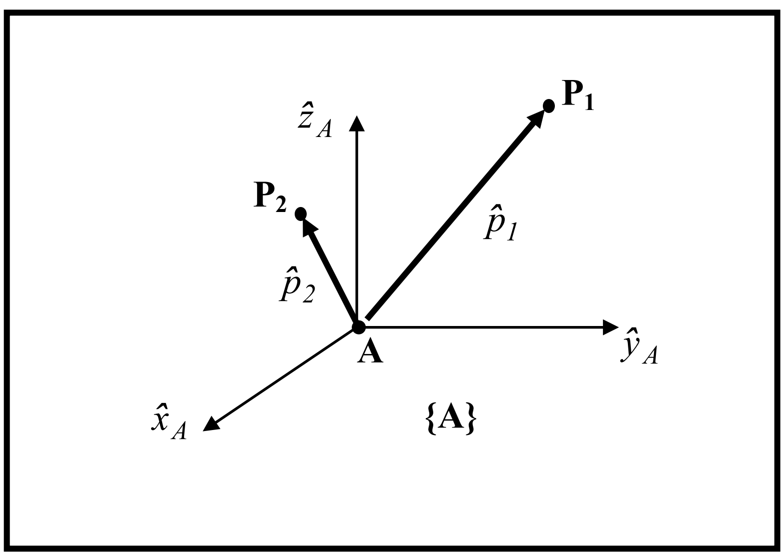
$$^{A}p_1 = {}^{2}R_1 \; {}^{A}p_2$$

or

$$\begin{bmatrix} {}^{A}p_1 \\ \hline 1 \end{bmatrix}_{4\times 1} = \left[\begin{array}{c|c} \left[{}^{2}R_1\right]_{3\times 3} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] \begin{bmatrix} {}^{A}p_2 \\ \hline 1 \end{bmatrix}_{4\times 1}$$

NOTE: ${}^{2}R_1$ is the rotation matrix of frame $\{A\}$ with respect to a (hypothetical) frame $\{B\}$ that can be obtained by a "reverse" rotation of $\hat{p}_1$ to $\hat{p}_2$.

NOTE: In the rotation operation, the length of a vector preserves.



2.2.3 Transformation Operator

The general movement of a body (frame) in space that consists of both translation and rotation is interpreted by the transformation operator.

Consider a body located at point A with an orientation identified by frame $\{A\}$. It moves to point B and rotates to a new frame $\{B\}$. In order to specify the new status of the body, we must identify:

i) ***The location of point B:*** It is assigned by a known point (such as A) and a vector (such as $\hat{p}_{AB}$).

ii) ***The orientation of the body:*** It is assigned by the rotation matrix ${}^{A}R_{B}$.

A fixed point P will be identified differently in the original ({A}) and the moved ({B}) frames:

$$\hat{p}_A = \hat{p}_B + \hat{p}_{AB} \qquad (\textit{Vector Sum})$$

or

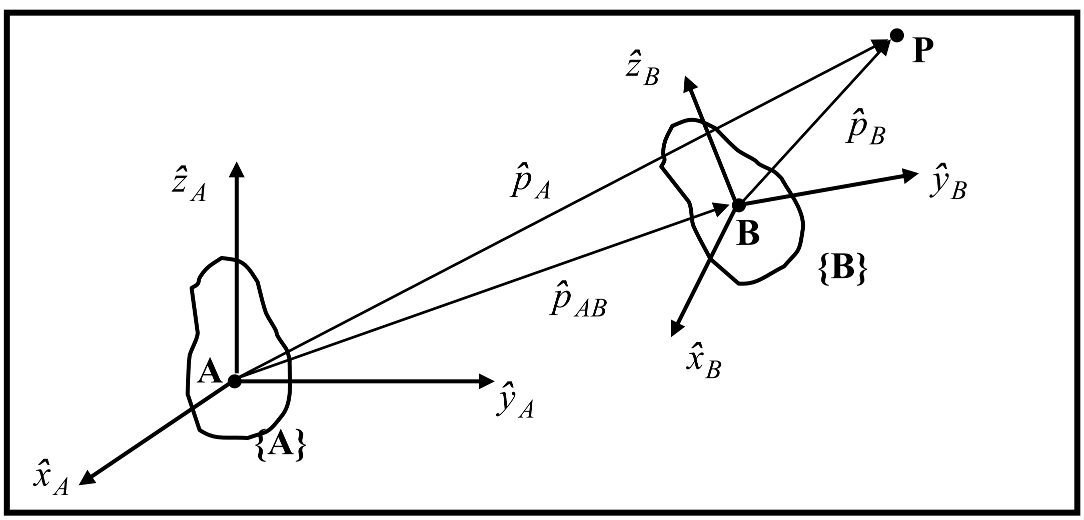
$${}^{A}p_A = {}^{A}p_B + {}^{A}p_{AB} \qquad (\textit{Matrix Sum})$$

or

$${}^{A}p_A = {}^{A}R_B\,{}^{B}p_B + {}^{A}p_{AB} \qquad (\textit{Matrix Sum})$$

The transformation operation can be represented as:

$$\begin{bmatrix} {}^{A}p_A \\ \hline 1 \end{bmatrix}_{4\times 1} = \left[\begin{array}{ccc|c} & \left({}^{A}R_B\right)_{3\times 3} & & \left({}^{A}p_{AB}\right)_{3\times 1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4} \begin{bmatrix} {}^{B}p_B \\ \hline 1 \end{bmatrix}_{4\times 1} \quad \textbf{OR} \quad {}^{A}p_A = {}^{A}T_B\ {}^{B}p_B$$



The operator T is called ***Homogeneous Transform***, and it has linear properties, i.e.:

$$\begin{cases} T(x_1 + x_2) = T(x_1) + T(x_2) \\ T(\alpha\, x) = \alpha\, T(x) \ ; \qquad \alpha \text{ is a scalar} \end{cases}$$

A Different Transformation Operation

The transformation matrix can also be viewed as an operator that moves, i.e., rotates and translates, a vector $\hat{p}_1$ to a new vector $\hat{p}_2$ in a fixed coordinate frame $\{\mathrm{A}\}$. The descriptions of the two vectors $\hat{p}_1$ and $\hat{p}_2$ in frame $\{\mathrm{A}\}$ are related as:

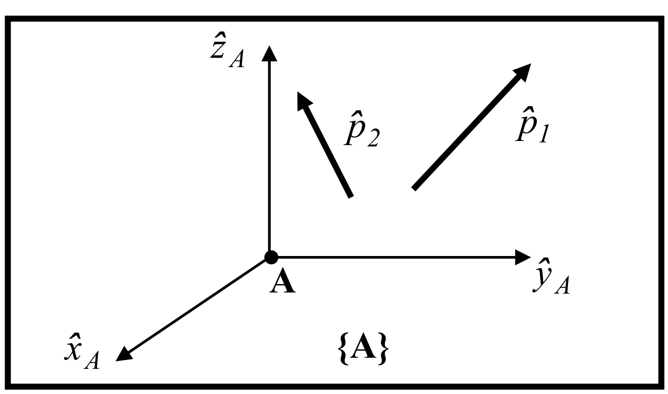
$$^{A}p_1 = {}^{2}T_1\ {}^{A}p_2 \quad ; \quad \text{or} \qquad {}^{A}p_2 = \left({}^{2}T_1\right)^{-1}\ {}^{A}p_1$$

NOTE: $^{2}T_1$ is the transformation matrix of frame $\{\mathrm{A}\}$ with respect to a (hypothetical) frame $\{\mathrm{B}\}$ that can be obtained by a "reverse" transformation of $\hat{p}_1$ to $\hat{p}_2$.

Inverse of the Transformation Operator

$$^{A}T_B = \left[\begin{array}{c|c} {}^{A}R_B & {}^{A}p_{AB} \\ \hline [0] & 1 \end{array}\right]$$

$$^{B}T_A = \left[\begin{array}{c|c} {}^{B}R_A & {}^{B}p_{BA} \\ \hline [0] & 1 \end{array}\right] = \left[\begin{array}{c|c} \left({}^{A}R_B\right)^T & {}^{B}R_A\,{}^{A}p_{BA} \\ \hline [0] & 1 \end{array}\right] = \left[\begin{array}{c|c} \left({}^{A}R_B\right)^T & -\left({}^{A}R_B\right)^T\,{}^{A}p_{AB} \\ \hline [0] & 1 \end{array}\right] = \left({}^{A}T_B\right)^{-1}$$



Cascade (Compound) Transformations

Consider two consequent transformations: first, frame $\{A\}$ is moved to frame $\{B\}$, and then frame $\{B\}$ is moved to frame $\{C\}$. A point P is identified in frame $\{C\}$, i.e., ${}^{C}p_C$ is known, and we want to describe its location with respect to frame $\{A\}$.

$${}^{B}p_B = {}^{B}T_C \; {}^{C}p_C$$

and

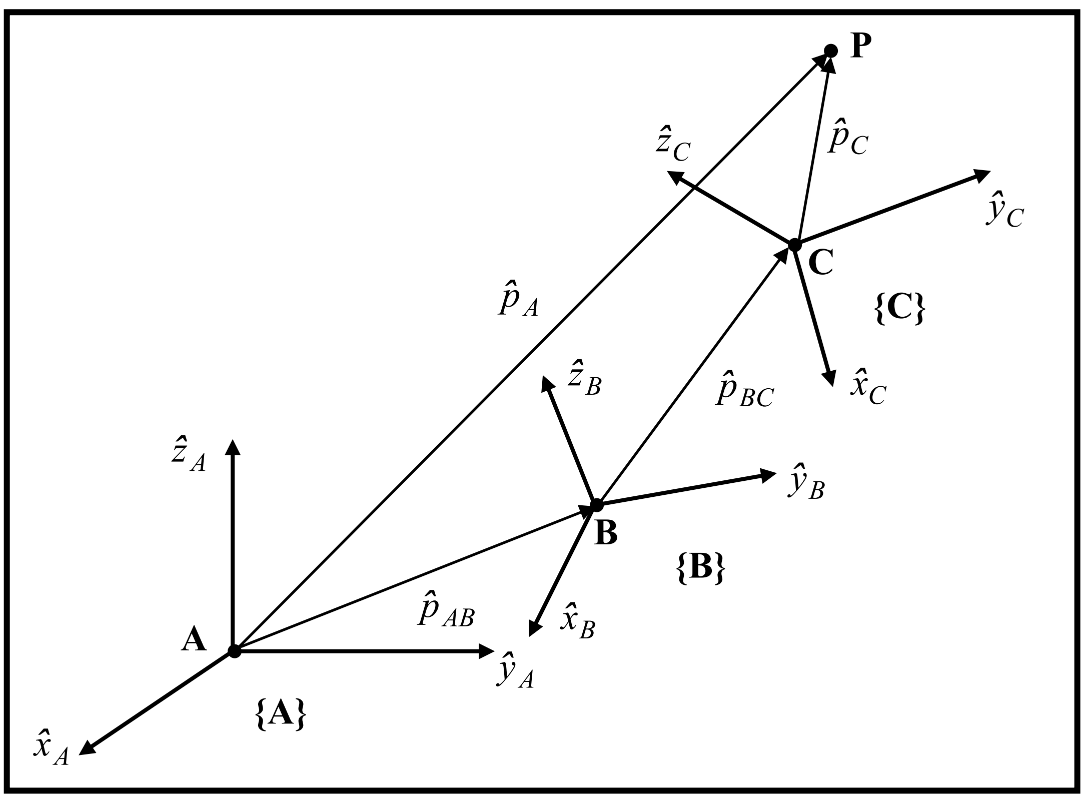
$${}^{A}p_A = {}^{A}T_B \; {}^{B}p_B .$$

Hence,

$${}^{A}p_A = {}^{A}T_B \; {}^{B}T_C \; {}^{C}p_C .$$

So,

$${}^{A}T_C = {}^{A}T_B \; {}^{B}T_C = \left[\begin{array}{c|c} {}^{A}R_B \, {}^{B}R_C & {}^{A}R_B \; {}^{B}p_{BC} + {}^{A}p_{AB} \\ \hline [0] & 1 \end{array}\right]$$



EXAMPLE:

For the control of manipulator motion, we require position and orientation of the links as well as position of the target point in the space. For this purpose, one coordinate frame is attached to each link at a *specific* point (known), and transformations of the frames with respect to each other are considered. For example, in figure below, The target point P is identified by the camera located at and attached to the frame $\{B\}$, i.e., ${}^{B}p_{B}$ is known. The location of P with respect to frame $\{A\}$ is needed.

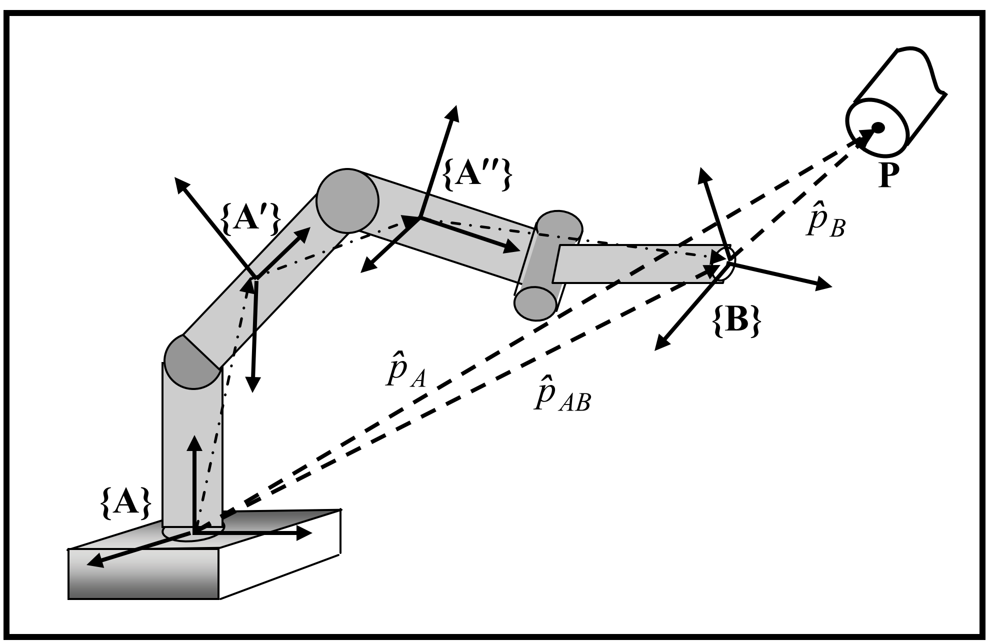
First, we obtain the position and orientation of frame $\{B\}$ with respect to frame $\{A\}$ by cascade transformations:

$$ {}^{A}T_{B} = \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}p_{AB} \\ \hline [0] & 1 \end{array}\right] = {}^{A}T_{A'} \ {}^{A'}T_{A''} \ {}^{A''}T_{B} $$

Each of the transformation matrices between the adjucent links is uniqely defined. Next:

$$ {}^{A}p_{A} = {}^{A}T_{B} \ {}^{B}p_{B} $$

NOTE: The *mutual* transformation matrices are functions of the angles (or displacements) of the joints that connect the adjacent links.



2.2.4 The Screw Motion

The general motion of a coordinate frame (rigid body) can be considered as a pure rotation about an axis of rotation plus a pure translation along directions parallel to the rotation axis, featuring a screw motion.

THEOREM:

Under the most general motion of a rigid body, there exists a line such that the motion is obtained as a rotation about the line and a translation along this line, where the translation is of minimum magnitude in the Euclidean sense.

PROOF:

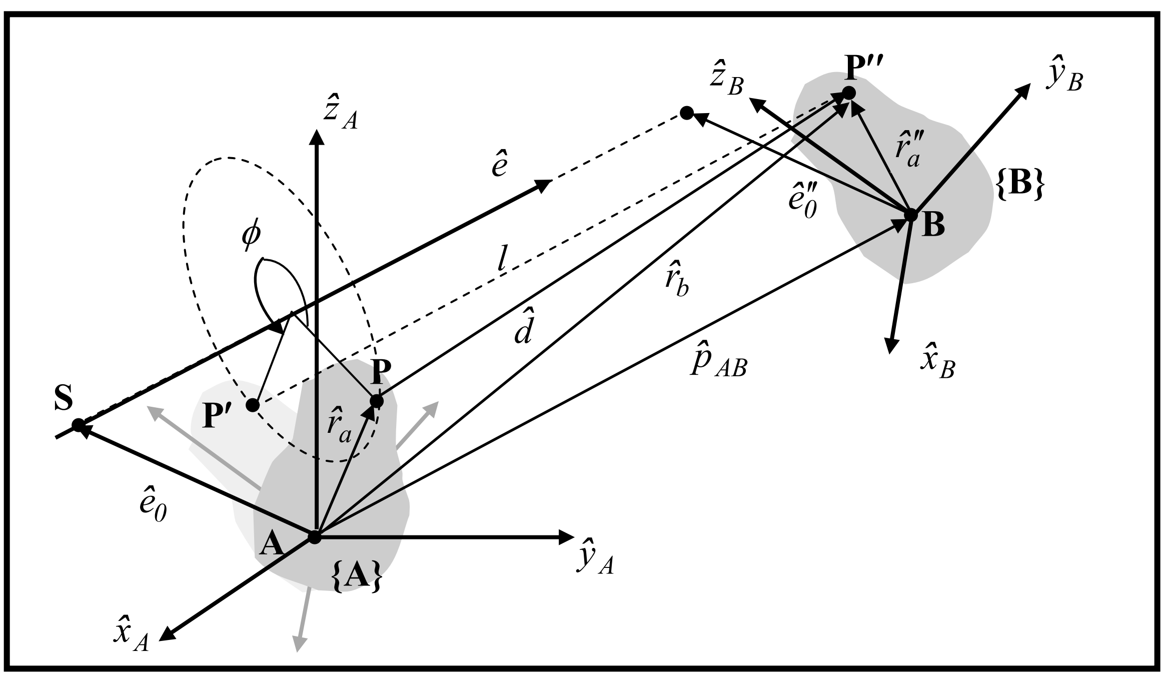
The translation of point P of the body to point P'' is represented by vector $\hat{d}$. In order to identify those points of the body whose translation is minimum, the Euclidean norm of $\hat{d}$ must be minimized with respect to the body vector $\hat{r}''_a$.

$$\hat{d} = \hat{r}_b - \hat{r}_a = \left(\hat{p}_{AB} + \hat{r}''_a\right) - \hat{r}_a$$

$${}^A d = {}^A p_{AB} + {}^A R_B\, {}^B r''_a - {}^A r_a = {}^A p_{AB} + {}^A R_B\, {}^B r''_a - {}^B r''_a = {}^A p_{AB} + \left({}^A R_B - I\right) {}^B r''_a$$

$$f\left({}^B r''_a\right) = \|d\|^2 = \left({}^A d\right)^T {}^A d = \left({}^B r''_a\right)^T \left({}^A R_B - I\right)^T \left({}^A R_B - I\right) {}^B r''_a$$

$$+ {}^A p^T_{AB}\, {}^A p_{AB} + {}^A p^T_{AB} \left({}^A R_B - I\right) {}^B r''_a + {}^B r''^T_a \left({}^A R_B - I\right)^T {}^A p_{AB}$$



$$\frac{\partial f}{\partial\, {}^{B}r''_a} = \left(\frac{\partial\, {}^{A}d}{\partial\, {}^{B}r''_a}\right)^T \frac{\partial f}{\partial\, {}^{A}d} = 0$$

$$\left({}^{A}R_B - I\right)^T \left(2\,{}^{A}d\right) = 0, \qquad \text{for} \quad \boldsymbol{min}\left({}^{A}d\right) = {}^{A}d^*$$

$${}^{B}R_A\, {}^{A}d^* = {}^{A}d^* \quad \text{or} \quad {}^{B}d^* = {}^{A}d^*$$

This implies that a point S located on the axis of rotation of ${}^{A}R_B$ has the minimum translation in the general rigid body motion. Now, we can determine that all points with the same property (minimal norm displacement) lie on the axis of rotation. For another point S located on the axis of rotation $\hat{e}$ we have:

$${}^{B}s = {}^{B}r''^*_a + \alpha\, {}^{B}e$$

$${}^{A}d_s = {}^{A}p_{AB} + \left({}^{A}R_B - I\right)\left({}^{B}r''^*_a + \alpha\, {}^{B}e\right)$$

$$\left({}^{A}R_B - I\right)\left(\alpha\, {}^{B}e\right) = 0 \;\Rightarrow\; {}^{A}d_s = {}^{A}p_{AB} + \left({}^{A}R_B - I\right){}^{B}r''^*_a = {}^{A}d^*$$

Therefore, all points S of minimum magnitude displacement $\hat{d}^*$ lie on the axis of rotation of ${}^{A}R_B$.■

A general displacement of a rigid body in the space can therefore be represented by an axis of rotation $\hat{e}$ (${}^{A}e = \begin{bmatrix} e_x & e_y & e_z \end{bmatrix}^T$), its location in the space by specifying an arbitrary point S on it by a vector $\hat{e}_0$ (${}^{A}e_0 = \begin{bmatrix} e_{0x} & e_{0y} & e_{0z} \end{bmatrix}^T$), the rotation angle ϕ, and the translation l. Not all the eight parameters are independent, as only two of the three parameters associated with the direction of screw axis are independent since they must satisfy:

$$\left({}^{A}e\right)^T {}^{A}e = 1;$$

and similarly, only two of the three parameters associated with the location of the screw axis are independent since S can be any point on the screw axis. For convenience, one can choose $\hat{e}_0$ to be normal to the screw axis:

$$\left({}^{A}e_0\right)^T {}^{A}e = 0 \;.$$

The homogeneous transformation of the rigid body motion with respect to the reference frame $\{A\}$ is the mapping between $\hat{p}_a$ and $\hat{p}_b$ so that:

$$^{A}p_a = {}^{A}T_B \, {}^{B}p_b = \left[\begin{array}{c|c} {}^{A}R_B & {}^{A}p_{AB} \\ \hline [0] & 1 \end{array}\right] {}^{B}p_b \ .$$

The rotation matrix $^{A}R_B$ can be obtained from the screw parameters by Rodrigues' formula:

$$^{B}R_A = ee^T + \left(I - ee^T\right)\boldsymbol{cos}\,\phi + \tilde{e}\,\boldsymbol{sin}\,\phi \ ;$$

and
$$\hat{p}_{AB} = \hat{e}_0 + l\hat{e} - \hat{e}_0''$$

$$\Rightarrow \quad {}^{A}p_{AB} = {}^{A}e_0 + l\,{}^{A}e - {}^{A}R_B\,{}^{B}e_0'' = {}^{A}e_0 + l\,{}^{A}e - {}^{A}R_B\,{}^{A}e_0 = l\,{}^{A}e - \left({}^{A}R_B - I\right){}^{A}e_0 \ .$$

By expanding the above formulations, elements of the homogeneous transformation matrix are obtained based on the screw parameters:

$$\begin{cases} t_{11} = \left(e_x^2 - 1\right)\left(1 - \boldsymbol{cos}\,\phi\right) + 1; \\ t_{12} = e_x e_y\left(1 - \boldsymbol{cos}\,\phi\right) - e_z\,\boldsymbol{sin}\,\phi; \\ t_{13} = e_x e_z\left(1 - \boldsymbol{cos}\,\phi\right) + e_y\,\boldsymbol{sin}\,\phi; \\ t_{21} = e_y e_x\left(1 - \boldsymbol{cos}\,\phi\right) + e_z\,\boldsymbol{sin}\,\phi; \\ t_{22} = \left(e_y^2 - 1\right)\left(1 - \boldsymbol{cos}\,\phi\right) + 1; \\ t_{23} = e_y e_z\left(1 - \boldsymbol{cos}\,\phi\right) - e_x\,\boldsymbol{sin}\,\phi; \\ t_{31} = e_z e_x\left(1 - \boldsymbol{cos}\,\phi\right) - e_y\,\boldsymbol{sin}\,\phi; \\ t_{32} = e_z e_y\left(1 - \boldsymbol{cos}\,\phi\right) + e_x\,\boldsymbol{sin}\,\phi; \\ t_{33} = \left(e_z^2 - 1\right)\left(1 - \boldsymbol{cos}\,\phi\right) + 1; \\ t_{14} = le_x - e_{0x}\left(t_{11} - 1\right) - e_{0y}t_{12} - e_{0z}t_{13}; \\ t_{24} = le_y - e_{0x}t_{21} - e_{0y}\left(t_{22} - 1\right) - e_{0z}t_{23}; \\ t_{34} = le_z - e_{0x}t_{31} - e_{0y}t_{32} - e_{0z}\left(t_{33} - 1\right); \\ t_{41} = 0; \quad t_{42} = 0; \quad t_{43} = 0; \quad t_{44} = 1. \end{cases}$$