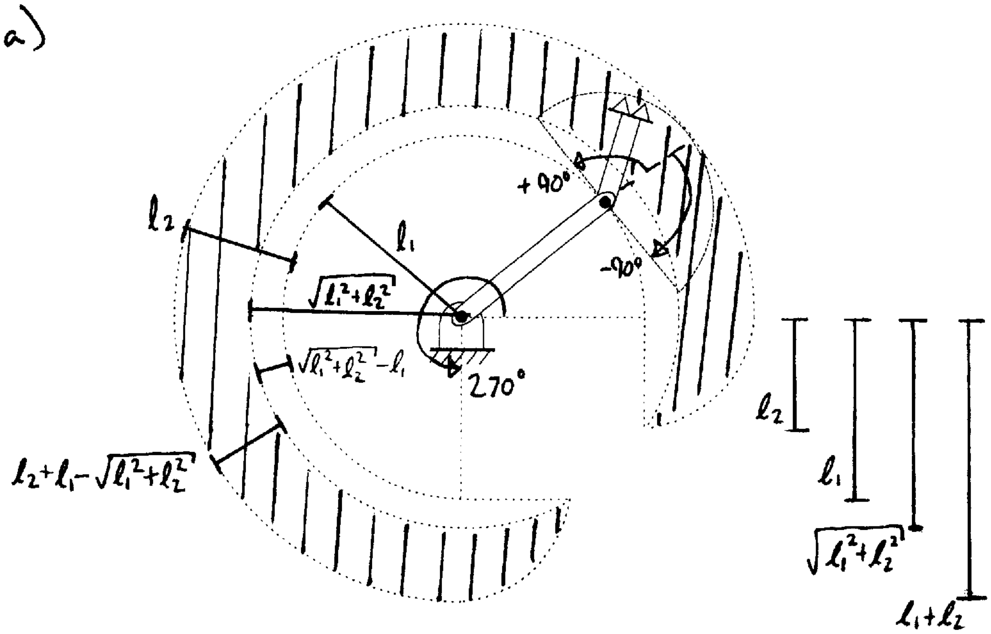
AER 525 Fall 2015

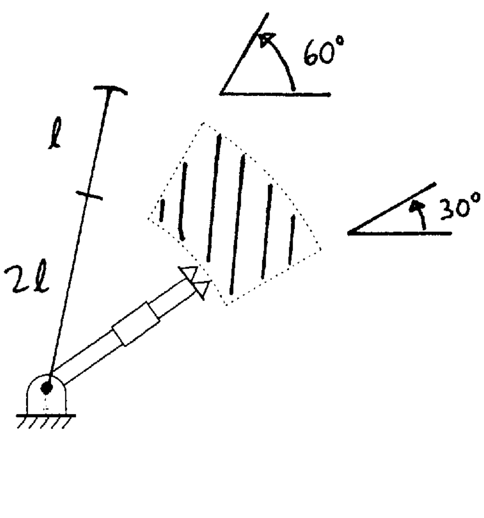
Assignment 2 - Solutions

Question 1

a)



b)

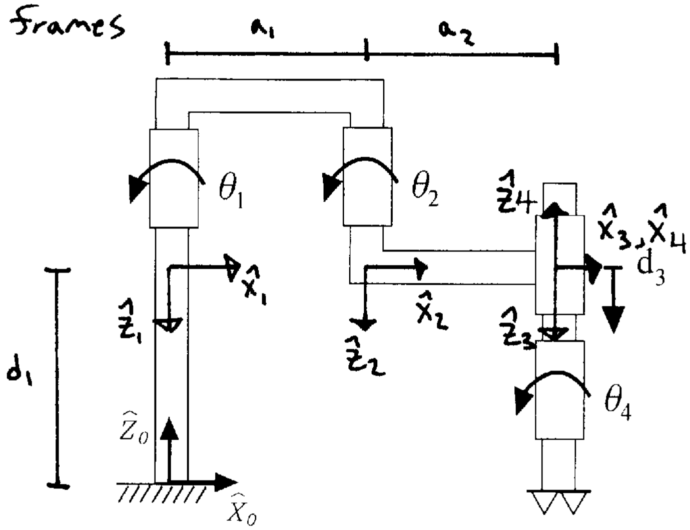


Question 2

- given

$$
{}^{B}_{W}T = \begin{bmatrix} c\phi & -s\phi & 0 & X \\ s\phi & c\phi & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- assign frames



- D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	180°	0	$-d_1$	θ_1
2	0	a_1	0	θ_2
3	0	a_2	d_3	0
4	180°	0	0	$-\theta_4$

- link transformation matrices

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ -s_1 & -c_1 & 0 & 0 \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}T = \begin{bmatrix} c(-\theta_4) & -s(-\theta_4) & 0 & 0 \\ -s(-\theta_4) & -c(-\theta_4) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- $\{0\}$ to $\{4\}$ transformation matrix

$$
{}^{0}_{4}T = \begin{bmatrix} c(\theta_1+\theta_2-(-\theta_4)) & s(\theta_1+\theta_2-(-\theta_4)) & 0 & a_2c_{12}+a_1c_1 \\ -s(\theta_1+\theta_2-(-\theta_4)) & c(\theta_1+\theta_2-(-\theta_4)) & 0 & -a_2s_{12}-a_1s_1 \\ 0 & 0 & 1 & d_1-d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$\because \quad c(\gamma) = c(-\gamma) \quad$ and $\quad s(\gamma) = -s(-\gamma)$

$$
{}^{0}_{4}T = \begin{bmatrix} c(-\theta_1-\theta_2-\theta_4) & -s(-\theta_1-\theta_2-\theta_4) & 0 & a_2c_{12}+a_1c_1 \\ s(-\theta_1-\theta_2-\theta_4) & c(-\theta_1-\theta_2-\theta_4) & 0 & -a_2s_{12}-a_1s_1 \\ 0 & 0 & 1 & d_1-d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- compare ${}^{B}_{W}T$ to ${}^{0}_{4}T$, we get

from $r_{11} \rightarrow \quad c_\phi = c(-\theta_1-\theta_2-\theta_4) \qquad (2.1)$

from $p_x \rightarrow \quad x = a_2c_{12} + a_1c_1 \qquad (2.2)$

from $p_y \rightarrow \quad y = -a_2s_{12} - a_1s_1 \qquad (2.3)$

from $p_z \rightarrow \quad z = d_1 - d_3 \qquad (2.4)$

- take $(2,2)^2 + (2,3)^2 = x^2 + y^2$

$$x^2 + y^2 = a_1^2 + a_2^2 + 2a_1a_2c_2$$

$$c_2 = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2} \quad \text{where} \quad -1 \le c_2 \le 1$$

$$s_2 = \pm\sqrt{1-c_2^2}$$

$\therefore \quad \theta_2 = \mathrm{Atan2}(s_2, c_2)$ with two possible solutions

- next, compare (2,2) and (2,3)

$$x = a_1c_1 + a_2(c_1c_2 - s_1s_2) = k_1c_1 - k_2s_1$$

$$y = -a_1s_1 - a_2(c_1s_2 + s_1c_2) = -k_1s_1 - k_2c_1$$

where $k_1 = a_1 + a_2c_2$ and $k_2 = a_2s_2$

- let $r = \sqrt{k_1^2 + k_2^2}$ and $\gamma = \mathrm{Atan2}(k_2, k_1)$

then $\begin{aligned} k_1 &= rc\gamma \\ k_2 &= rs\gamma \end{aligned} \rightarrow \begin{aligned} x/r &= c\gamma c_1 - s\gamma s_1 = c\gamma 1 \\ y/r &= c\gamma s_1 + s\gamma c_1 = s\gamma 1 \end{aligned}$

$$\therefore \quad \gamma + \theta_1 = \mathrm{Atan2}(y/r, x/r) = \mathrm{Atan2}(y, x)$$

$$\theta_1 = \mathrm{Atan2}(y, x) - \mathrm{Atan2}(k_2, k_1)$$

- finally, from (2,1)

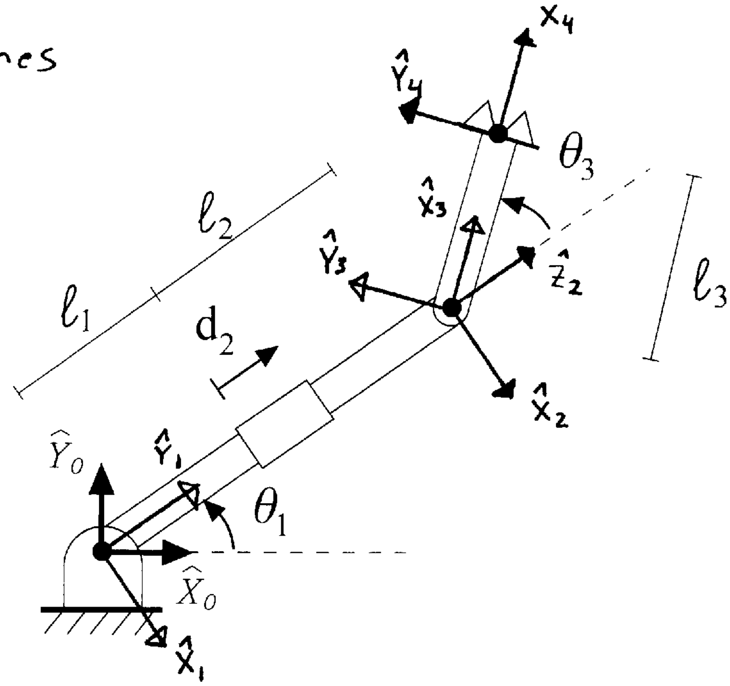
$$\phi = -\theta_1 - \theta_2 - \theta_4$$

$$\therefore \quad \theta_4 = -\theta_1 - \theta_2 - \phi$$

Question 3

a) Find ${}^{B}_{T}T$

- assign frames



- D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	$-90° + \theta_1$
2	$-90°$	0	$\ell_1 + \ell_2 + d_2$	0
3	$90°$	0	0	$90° + \theta_3$
4	0	ℓ_3	0	0

- link transformation matrices

$$
{}^{0}_{1}T = \begin{bmatrix} c(-90+\theta_1) & -s(-90+\theta_1) & 0 & 0 \\ s(-90+\theta_1) & c(-90+\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_1 & c_1 & 0 & 0 \\ -c_1 & s_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & l_1+l_2+d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c(90+\theta_3) & -s(90+\theta_3) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s(90+\theta_3) & c(90+\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- $\{0\}$ to $\{4\}$ (or $\{B\}$ to $\{T\}$) transformation matrix

$$
{}^{0}_{4}T = {}^{B}_{T}T = \begin{bmatrix} c_{13} & -s_{13} & 0 & (l_1+l_2+d_2)c_1 + l_3c_{13} \\ s_{13} & c_{13} & 0 & (l_1+l_2+d_2)s_1 + l_3s_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

b) Find the Jacobian

- we will find the complete Jacobian in the tool frame
- use the recursive link velocity method

- start with

$$ {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- joint 1 revolute

$$ {}^1\omega_1 = {}^1_0R\, \cancel{{}^0\omega_0}^{\,0} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^1v_1 = {}^1_0R\left(\cancel{{}^0v_0}^{\,0} + \cancel{{}^0\omega_0}^{\,0} \times {}^0P_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- joint 2 prismatic

$$ {}^2\omega_2 = {}^2_1R\, {}^1\omega_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^2v_2 = {}^2_1R\left(\cancel{{}^1v_1}^{\,0} + {}^1\omega_1 \times {}^1P_2\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 + l_2 + d_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} $$

$$ = \begin{bmatrix} -\dot{\theta}_1 (l_1 + l_2 + d_2) \\ 0 \\ \dot{d}_2 \end{bmatrix} $$

- joint 3 revolute

$$
{}^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} -s_3 & 0 & c_3 \\ -c_3 & 0 & -s_3 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_3 \end{bmatrix}
$$

$$
{}^3V_3 = {}^3_2R\left({}^2V_2 + \cancelto{0}{{}^2\omega_2 \times {}^2P_3}\right)
$$

$$
= \begin{bmatrix} -s_3 & 0 & c_3 \\ -c_3 & 0 & -s_3 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} -\dot\theta_1(l_1+l_2+d_2) \\ 0 \\ \dot d_2 \end{bmatrix} \right) = \begin{bmatrix} s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2 \\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2 \\ 0 \end{bmatrix}
$$

- tool

$$
{}^4\omega_4 = {}^4_3R\,{}^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_3 \end{bmatrix}
$$

$$
{}^4V_4 = {}^4_3R\left({}^3V_3 + {}^3\omega_3 \times {}^3P_4\right)
$$

$$
= \begin{bmatrix} s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2 \\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -(\dot\theta_1+\dot\theta_3) & 0 \\ \dot\theta_1+\dot\theta_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2 \\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2 + l_3(\dot\theta_1+\dot\theta_3) \\ 0 \end{bmatrix}
$$

- therefore the Jacobian is

$$
{}^4J(\theta_1, d_2, \theta_3) = \begin{bmatrix} {}^4V_4 \\ {}^4\omega_4 \end{bmatrix} = \begin{bmatrix} s_3(l_1+l_2+d_2) & c_3 & 0 \\ c_3(l_1+l_2+d_2)+l_3 & -s_3 & l_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}
$$

c) Find ${}^0v_4$ and ${}^0\omega_4$

- first, rewrite the Jacobian in $\{0\}$

$$
{}^0J(\theta_1, d_2, \theta_3) = \begin{bmatrix} {}^0_4R & 0_3 \\ 0_3 & {}^0_4R \end{bmatrix} {}^4J(\theta_1, d_2, \theta_3)
$$

$$
= \begin{bmatrix} (c_{13}s_3 - s_{13}c_3)(l_1+l_2+d_2) - s_{13}l_3 & c_{13}c_3 + s_{13}s_3 & -l_3 s_{13} \\ (s_{13}s_3 + c_{13}c_3)(l_1+l_2+d_2) + c_{13}l_3 & s_{13}c_3 - c_{13}s_3 & l_3 c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -s_1(l_1+l_2+d_2) - s_{13}l_3 & c_1 & -l_3 s_{13} \\ c_1(l_1+l_2+d_2) + c_{13}l_3 & s_1 & l_3 c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}
$$

- since $\begin{bmatrix} {}^0v_4 \\ {}^0\omega_4 \end{bmatrix} = {}^0J(\theta_1, d_2, \theta_3) \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{\theta}_3 \end{bmatrix}$

- therefore the translational and angular velocities of the tool frame relative to the base frame are

$$
{}^0v_x = [-s_1(l_1+l_2+d_2) - s_{13}l_3]\dot{\theta}_1 + c_1\dot{d}_2 - l_3 s_{13}\dot{\theta}_3
$$

$$
{}^0v_y = [c_1(l_1+l_2+d_2) + c_{13}l_3]\dot{\theta}_1 + s_1\dot{d}_2 + l_3 c_{13}\dot{\theta}_3
$$

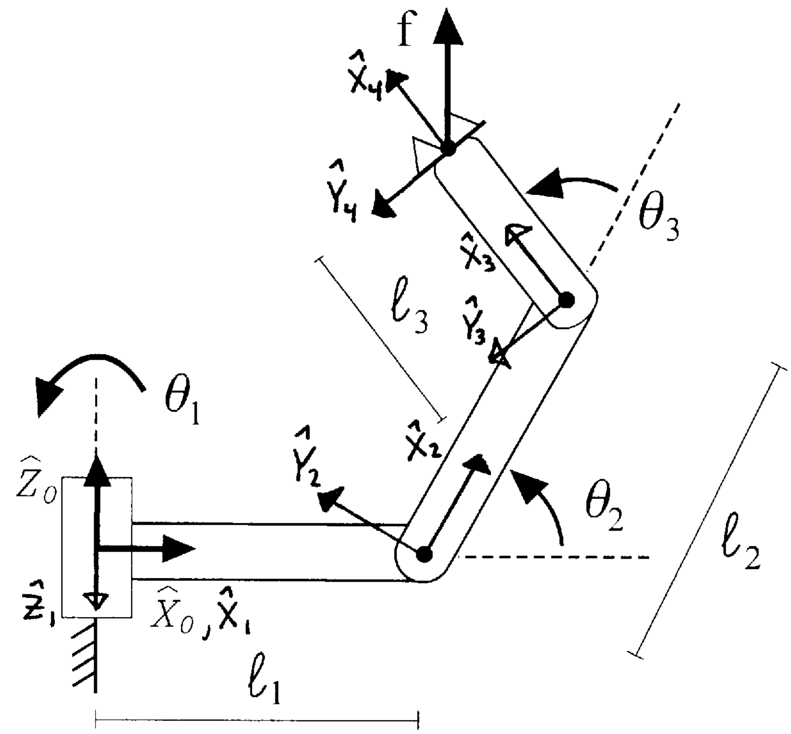
$$
{}^0v_z = {}^0\omega_x = {}^0\omega_y = 0
$$

$$
{}^0\omega_z = \dot{\theta}_1 + \dot{\theta}_3
$$

→ Question 4

a) Find ${}^{B}_{T}T$

- assign frames



- D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$180°$	0	0	θ_1
2	$-90°$	ℓ_1	0	θ_2
3	0	ℓ_2	0	θ_3
4	0	ℓ_3	0	0

- link transformation matrices

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ -s_1 & -c_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & \ell_1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- $\{0\}$ to $\{4\}$ (or $\{B\}$ to $\{T\}$) transformation matrix

$$
{}^0_4T = {}^B_TT = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & -s_1 & l_3c_1c_{23} + l_2c_1c_2 + l_1c_1 \\ -s_1c_{23} & s_1s_{23} & -c_1 & -l_3s_1c_{23} - l_2s_1c_2 - l_1s_1 \\ s_{23} & c_{23} & 0 & l_3s_{23} + l_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

b) Find the Jacobian

- we will find the complete Jacobian in the tool frame
- use the recursive link velocity method

- start with

$$
{}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

- joint 1 revolute

$$
{}^1\omega_1 = {}^1_0R\,\cancelto{0}{{}^0\omega_0} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}
$$

$$
{}^1v_1 = {}^1_0R\left(\cancelto{0}{{}^0v_0} + \cancelto{0}{{}^0\omega_0} \times {}^0P_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

- joint 2 revolute

$$ {}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} $$

$$ = \begin{bmatrix} -s_2\dot\theta_1 \\ -c_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} $$

$$ {}^2V_2 = {}^2_1R\left(\cancelto{0}{{}^1V_1} + {}^1\omega_1 \times {}^1P_2\right) $$

$$ = \begin{bmatrix} c_2 & 0 & -s_2 \\ -s_2 & 0 & -c_2 \\ 0 & 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & -\dot\theta_1 & 0 \\ \dot\theta_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ l_1\dot\theta_1 \end{bmatrix} $$

- joint 3 revolute

$$ {}^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_2\dot\theta_1 \\ -c_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} $$

$$ = \begin{bmatrix} (-s_2c_3 - c_2s_3)\dot\theta_1 \\ (s_2s_3 - c_2c_3)\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} = \begin{bmatrix} -s_{23}\dot\theta_1 \\ -c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} $$

$$ {}^3V_3 = {}^3_2R\left({}^2V_2 + {}^2\omega_2 \times {}^2P_3\right) $$

$$ = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ l_1\dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 & -\dot\theta_2 & -c_2\dot\theta_1 \\ \dot\theta_2 & 0 & s_2\dot\theta_1 \\ c_2\dot\theta_1 & -s_2\dot\theta_1 & 0 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} l_2s_3\dot\theta_2 \\ l_2c_3\dot\theta_2 \\ (l_2c_2 + l_1)\dot\theta_1 \end{bmatrix} $$

- tool

$$^4\omega_4 = {}^4_3R\,{}^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -s_{23}\dot\theta_1 \\ -c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix} = \begin{bmatrix} -s_{23}\dot\theta_1 \\ -c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix}$$

$$\begin{aligned} ^4V_4 &= {}^4_3R\left({}^3V_3 + {}^3\omega_3 \times {}^3P_4\right) \\ &= \begin{bmatrix} l_2 s_3 \dot\theta_2 \\ l_2 c_3 \dot\theta_2 \\ (l_2 c_2 + l_1)\dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 & -(\dot\theta_2+\dot\theta_3) & -c_{23}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3 & 0 & s_{23}\dot\theta_1 \\ c_{23}\dot\theta_1 & -s_{23}\dot\theta_1 & 0 \end{bmatrix}\begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} l_2 s_3 \dot\theta_2 \\ (l_2 c_3 + l_3)\dot\theta_2 + l_3\dot\theta_3 \\ (l_2 c_2 + l_3 c_{23} + l_1)\dot\theta_1 \end{bmatrix} \end{aligned}$$

- therefore the Jacobian is

$$^4J(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} ^4V_4 \\ ^4\omega_4 \end{bmatrix} = \begin{bmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ l_1 + l_2 c_2 + l_3 c_{23} & 0 & 0 \\ -s_{23} & 0 & 0 \\ -c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- alternatively, in frame {0} the Jacobian is

$$^0J = \begin{bmatrix} ^0_4R & 0_3 \\ 0_3 & ^0_4R \end{bmatrix}{}^4J = \begin{bmatrix} -s_1(l_1 + c_2 l_2 + c_{23} l_3) & -c_1(s_2 l_2 + s_{23} l_3) & -c_1 s_{23} l_3 \\ -c_1(l_1 + c_2 l_2 + c_{23} l_3) & s_1(s_2 l_2 + s_{23} l_3) & s_1 s_{23} l_3 \\ 0 & c_2 l_2 + c_{23} l_3 & c_{23} l_3 \\ 0 & -s_1 & -s_1 \\ 0 & -c_1 & -c_1 \\ -1 & 0 & 0 \end{bmatrix}$$

c) & d) Are there any singularities?

- start by reducing the Jacobian to a square matrix by taking the top three rows

$$ {}^{4}J_{3x3} = \begin{bmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ l_1 + l_2 c_2 + l_3 c_{23} & 0 & 0 \end{bmatrix} $$

- taking the determinant will give the singularity conditions once it is set to zero

$$ \det({}^{4}J_{3x3}) = \begin{vmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ l_1 + l_2 c_2 + l_3 c_{23} & 0 & 0 \end{vmatrix} $$

$$ = -l_2 s_3 \left[-l_3 (l_1 + l_2 c_2 + l_3 c_{23}) \right] $$

- therefore singularities occur when $\det({}^{4}J_{3x3}) = 0$
- so when

$$ s_3 = 0 \qquad (4.1) $$

or

$$ l_1 + l_2 c_2 + l_3 c_{23} = 0 \qquad (4.2) $$

- in order to distinguish between boundary and interior singularities, more information is needed about the link dimensions.

→ Case 1: $l_1 > l_2 + l_3$

(4.1): $s_3 = 0 \rightarrow \theta_3 = 0°$ or $\theta_3 = 180°$

(4.2): $|l_1 + l_2 c_2 + l_3 c_{23}| \neq 0$ $\because$ $\min(c_2) = -1$ & $\min(c_{23}) = -1$

- If $l_2 > l_3$, then $\theta_3 = 0°$ and $\theta_3 = 180°$ are both workspace-boundary singularities.

- If $l_2 \leq l_3$, then $\theta_3 = 0°$ is a workspace-boundary singularity and $\theta_3 = 180°$ is a workspace interior singularity.

→ Case 2: $l_1 = l_2 + l_3$

(4.1): $s_3 = 0 \rightarrow \theta_3 = 0°$ or $\theta_3 = 180°$

(4.2): $l_1 + l_2 c_2 + l_3 c_{23} = 0$ only when $\theta_2 = 180°$ and $\theta_3 = 0°$

- If $l_2 > l_3$, then $\theta_3 = 0°$ when $\theta_2 \neq 180°$ and $\theta_3 = 180°$ are workspace-boundary singularities. Also, $\theta_3 = 0°$ when $\theta_2 = 180°$ is a workspace-interior singularity.

- If $l_2 \leq l_3$, then $\theta_3 = 0°$ when $\theta_2 \neq 180°$ are workspace-boundary singularities. Also, $\theta_3 = 180°$ and $\theta_3 = 0°$ when $\theta_2 = 180°$ are workspace-interior singularities.

→ Case 3: $l_1 < l_2 + l_3$

(4.1): $s_3 = 0$ → $\theta_3 = 0°$ or $\theta_3 = 180°$

(4.2): $l_1 + l_2 c_2 + l_3 c_{23} = 0$ for many angles...

- a special case occurs when $\theta_3 = 0°$

$$l_1 + (l_2 + l_3) c_{2'} = 0 \rightarrow c_{2'} = \frac{-l_1}{l_2 + l_3}$$

$$\therefore s_{2'} = \pm\sqrt{1 - \frac{l_1^2}{(l_2+l_3)^2}}$$

then $\theta_2' = \mathrm{Atan2}(s_{2'}, c_{2'})$

- If $l_2 > l_3$, then $\theta_3 = 0°$ when $\theta_2 < |\theta_2'|$ and $\theta_3 = 180°$ are workspace-boundary singularities. Also, $\theta_3 = 0°$ when $\theta_2 \geq |\theta_2'|$ and any other combination of θ_2 and θ_3 which satisfies $l_1 + l_2 c_2 + l_3 c_{23} = 0$ are workspace-interior singularities

- If $l_2 \leq l_3$, then $\theta_3 = 0°$ when $\theta_2 < |\theta_2'|$ are workspace-boundary singularities. Also, $\theta_3 = 180°$ and $\theta_3 = 0°$ when $\theta_2 \geq |\theta_2'|$ and any other combination of θ_2 and θ_3 which satisfies $l_1 + l_2 c_2 + l_3 c_{23} = 0$ are workspace-interior singularities.

e) Find the joint torques necessary to keep the manipulator in place as shown in the figure when a force f is applied

- first, describe ${}^0F$ in $\{4\}$

$$ {}^0F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} \quad \text{then} \quad {}^4F = {}^4_0R\,{}^0F = \begin{bmatrix} s_{23}\, f \\ c_{23}\, f \\ 0 \end{bmatrix} \equiv \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} \quad \text{and} \quad {}^4n_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- using the force/moment recursive equations

- link 3

$$ {}^3f_3 = {}^3_4R\,{}^4f_4 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^3n_3 = {}^3_4R\,\cancel{{}^4n_4}^{\,0} + {}^3P_4 \times {}^3f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_3 \\ 0 & l_3 & 0 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix} $$

- link 2

$$ {}^2f_2 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ 0 \end{bmatrix} $$

$$ {}^2n_2 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_2 \\ 0 & l_2 & 0 \end{bmatrix} \begin{bmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ (l_3 + c_3 l_2) f_y + (s_3 l_2) f_x \end{bmatrix} $$

- link 3

$$ {}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \begin{bmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} (c_2c_3 - s_2s_3)f_x + (-c_2s_3 - s_2c_3)f_y \\ 0 \\ (-s_2c_3 - c_2s_3)f_x + (s_2s_3 - c_2c_3)f_y \end{bmatrix} = \begin{bmatrix} c_{23}f_x - s_{23}f_y \\ 0 \\ -s_{23}f_x - c_{23}f_y \end{bmatrix} $$

$$ {}^1n_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ -s_2 & -c_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (s_3 l_2)f_x + (l_3 + c_3 l_2)f_y \end{bmatrix} $$

$$ + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_1 \\ 0 & l_1 & 0 \end{bmatrix} \begin{bmatrix} c_{23}f_x - s_{23}f_y \\ 0 \\ -s_{23}f_x - c_{23}f_y \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ l_1(s_{23}f_x + c_{23}f_y) \\ 0 \end{bmatrix} $$

- finding the scalar torques

$$ \tau_1 = {}^1\hat{Z}_1^T \, {}^1n_1 = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ l_1(s_{23}f_x + c_{23}f_y) \\ 0 \end{bmatrix} = 0 $$

$$ \tau_2 = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 0 \\ (l_3 + c_3 l_2)f_y + (s_3 l_2)f_x \end{bmatrix} $$

$$ = (l_3 + c_3 l_2)f_y + (s_3 l_2)f_x = \left[s_{23} s_3 l_2 + c_{23}(l_3 + c_3 l_2) \right] f $$

$$ \tau_3 = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix} = l_3 f_y = c_{23} l_3 f $$