Chapter 3

Forward Kinematics

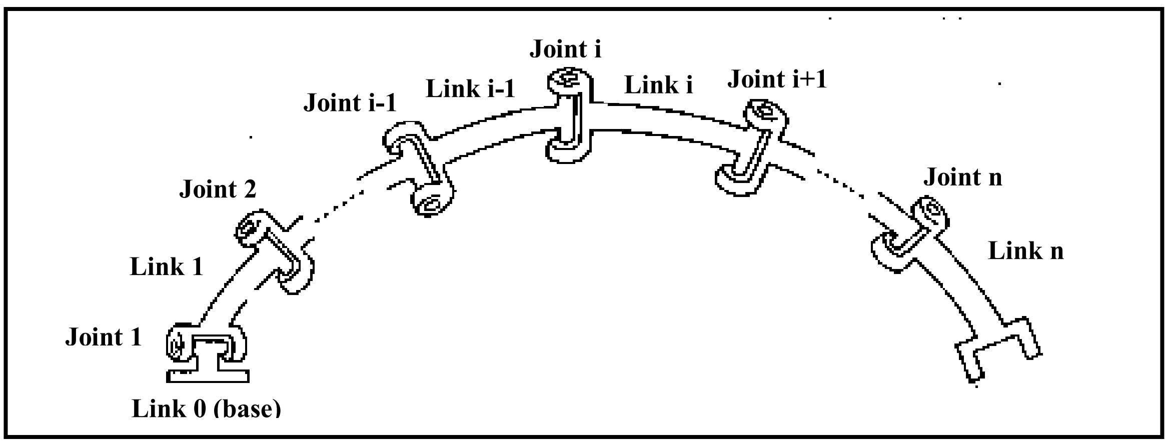
Goal

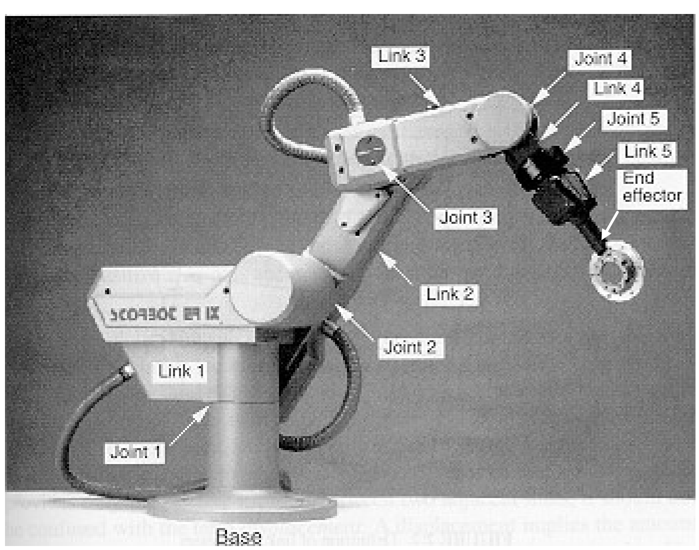
To establish basic robot parameters (Denavit-Hartenberg notation), link transformation matrices, and to determine the location of the robot end-effector with respect to a reference coordinate frame as a result of the relative motion of each pair of adjacent links, i.e., to formulate the Forward Kinematics problem.

3.1 Links and Joints: Numbers and Parameters

Consider an n d.o.f. serial manipulator. The links are numbered starting from the immobile base of the arm, called link 0. The first moving body is link 1, and so on, out to the free end of the arm, which is link n.

For the purpose of obtaining the kinematic equations of the manipulator, *a link is considered only as a rigid body which defines the relationship between two neighboring joint axes*.





Joint axes are defined by lines in the space; joint axis i is defined by a line, about which link i rotates (if revolute) or translates (if prismatic) relative to link i-1.

Two parameters are required to specify each link, which define the relative location of the two successive joint axes in the space. The first one is the ***length*** of link i, a_i, which is the length of the common normal between the axes i and $i+1$. The second is the ***twist*** of link i, α_i, which is the angle measured from axis i to axis $i+1$ in the right-hand sense about a_i, in a plane whose normal is the mutually perpendicular line to axes i and $i+1$.

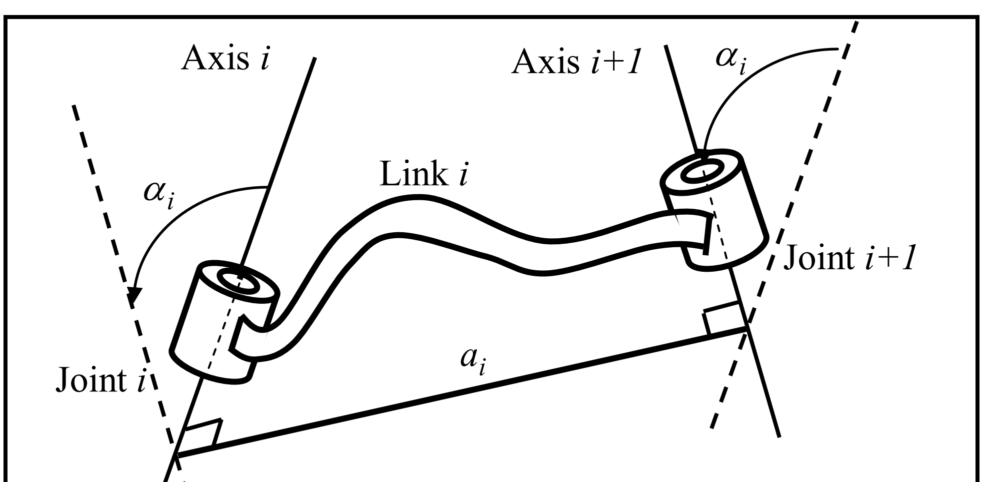
NOTE: In the case where axes i and $i+1$ intersect, the length of link i is zero, and twist is measured in the plane containing both axes.

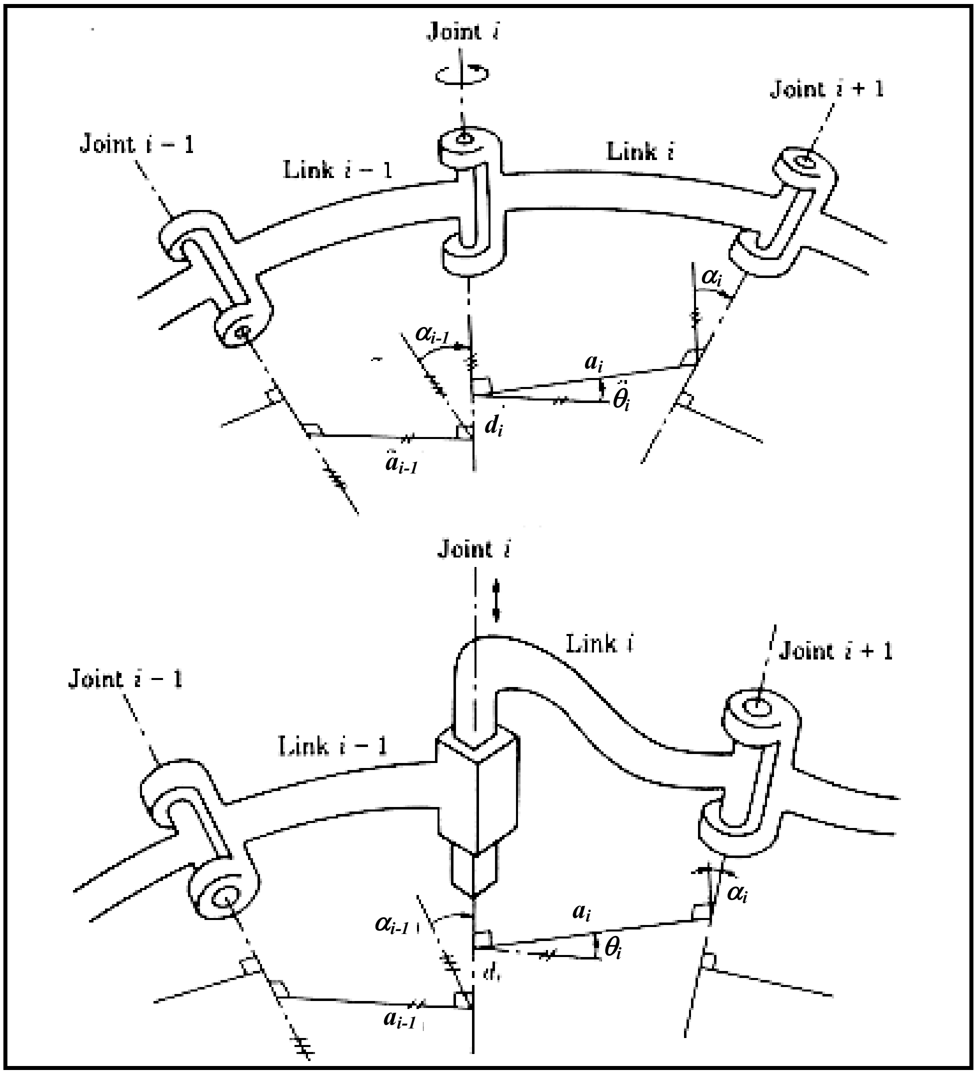
NOTE: In the case where axes i and $i+1$ are parallel, there are many common normal lines to the axes, but the length of link i still has a unique magnitude.

Two more parameters are required to specify location of the links relative to each other. Two adjacent links, i-1 and i, have a common joint axis i between them. The first parameter, called ***offset*** of link i, d_i, is along this common joint axis, from the point of intersection of the joint axis i and a_{i-1} to the point of intersection of the joint axis i and a_i. The second parameter describes the amount of rotation about the common axis i from link i-1 (a_{i-1}) to link i (a_i), and is called the ***angle*** of joint i, θ_i.

First and Last Links in the Chain:

At the ends of the manipulator chain, we usually assign:





$$a_0 = a_n = 0.0$$
$$\alpha_0 = \alpha_n = 0.0$$

$$\begin{cases} \text{if joint } 1 \text{ is revolute}: & \begin{cases} d_1 = 0.0 \\ \theta_1 \text{ is chosen arbitrarily (zero)} \end{cases} \\ \text{if joint } 1 \text{ is prismatic}: & \begin{cases} \theta_1 = 0.0 \\ d_1 \text{ is chosen arbitrarily (zero)} \end{cases} \end{cases}$$

$$\begin{cases} \text{if joint } n \text{ is revolute}: & \begin{cases} d_n = 0.0 \\ \theta_n \text{ is chosen arbitrarily (zero)} \end{cases} \\ \text{if joint } n \text{ is prismatic}: & \begin{cases} \theta_n = 0.0 \\ d_n \text{ is chosen arbitrarily (zero)} \end{cases} \end{cases}$$

Robot Parameters

Any robot can be described kinematically by giving the values of four quantities for each link. Two parameters describe the link itself, and two parameters describe the link's connection to the adjacent link. If link i starts with a revolute joint, θ_i is called the ***joint variable***, and the other three quantities would be the "fixed" ***link parameters***. If link i starts with a prismatic joint, d_i is the joint variable and the other three quantities are "fixed" link parameters. The definition of mechanisms by means of these quantities is called the ***Denavit-Hartenberg (DH) notation***.

Joint Type	***Joint Variable***	***Fixed Link Parameters***
Revolute	**Angle** θ_i	**Link Length** a_i **Twist Angle** α_i **Link Offset** d_i
Prismatic	**Offset** d_i	**Link Length** a_i **Twist Angle** α_i **Joint Angle** θ_i

3.2 Link Frames

3.2.1 Standard Denavit-Hartenberg Convention

For each link, we define a Cartesian coordinate frame attached to the link. The link frames are named by numbers according to the link to which they are attached.

⇨ The origin of frame $\{i\}$ attached to link i is located at the point of intersection of joint $i+1$ axis and the common normal between joint axes i and $i+1$.

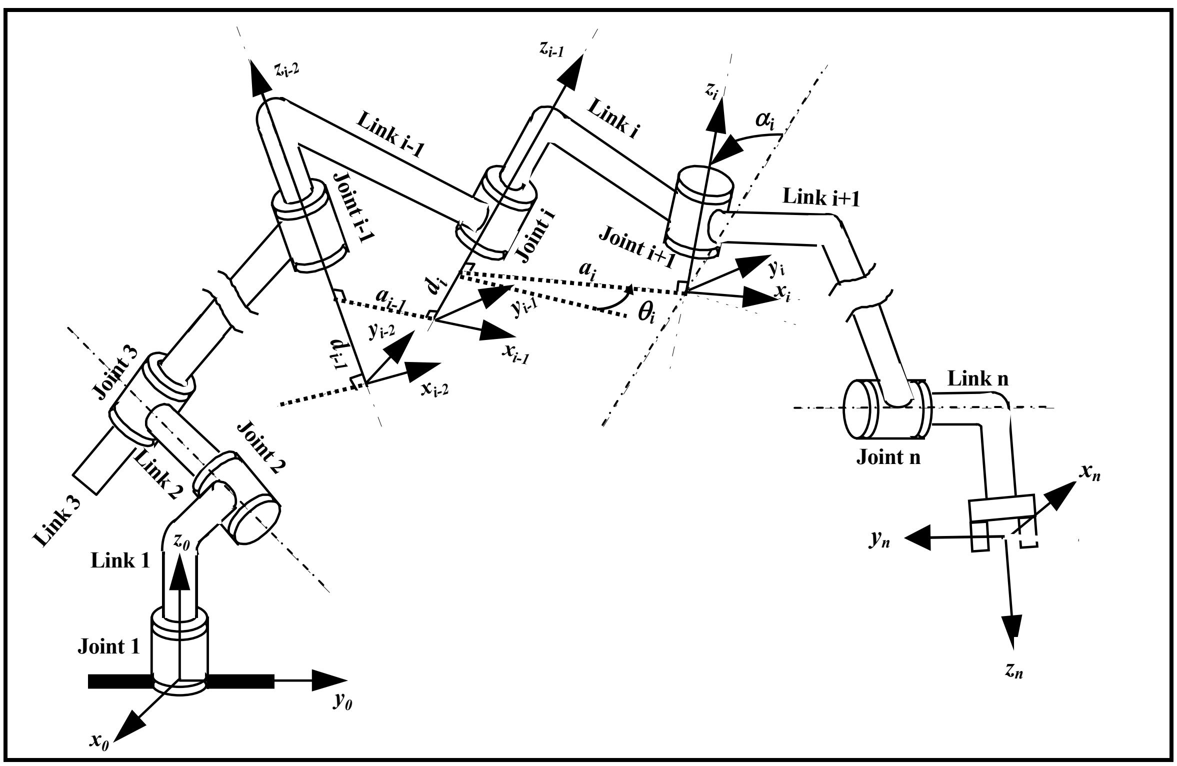
⇨ The axis $\hat{z}_i$ is aligned with the joint axis $i+1$. The positive direction of $\hat{z}_i$ can be chosen arbitrarily.

⇨ The axis $\hat{x}_i$ points along a_i in the direction from joint i to joint $i+1$ axes.

⇨ The axis $\hat{y}_i$ is determined by the right-hand rule to complete the frame $\{i\}$.

NOTE: In the case of two intersecting joint axes i and $i+1$, $a_i = 0$ and $\hat{x}_i$ is normal to the plane of $\hat{z}_{i-1}$ and $\hat{z}_i$ with an arbitrary direction. The origin of the frame $\{i\}$ is at the point of intersection.

NOTE: In the case where joint axes i and $i+1$ are parallel, one of the common normal lines is chosen as $\hat{x}_i$ arbitrarily, but the one that makes $d_i = 0$ is preferable.



NOTE: In the case where joint axes i and $i+1$ coincide, The origin of frame $\{i\}$ is chosen arbitrarily, and $\hat{x}_i$ is also chosen normal to the joint axes arbitrarily. However, the choice that makes more link parameters equal to zero is preferable.

First and Last Links in the Chain:

Frame $\{0\}$ is attached to the base that is usually stationary, and the position and orientation of other link frames are described in terms of this frame. The origin of frame $\{0\}$ can be defined at any convenient location. One choice would be to locate frame $\{0\}$ so that it coincides with frame $\{1\}$ when joint variable 1 is zero. Axis $\hat{z}_0$ of frame $\{0\}$ is aligned with the axis of joint 1.

The last frame attached to the end-effector is called *end-effector or hand coordinate frame*. The hand frame can be located anywhere at the end-effector as long as the axis $\hat{x}_n$ is normal to the last joint axis. For joint n revolute, the axis $\hat{x}_n$ is usually chosen along $\hat{x}_{n-1}$ when $\theta_n = 0.0$, and the origin of frame $\{n\}$ is chosen so that $d_n = 0.0$, due to simplicity. For joint n prismatic, the axis $\hat{x}_n$ is chosen so that $\theta_n = 0.0$, and the origin of frame $\{n\}$ is chosen at the intersection of $\hat{x}_{n-1}$ and joint axis n when $d_n = 0.0$, due to simplicity. In some occasions, the axis $\hat{z}_n$ is defined along the direction of approach of a gripper.

Link Parameters in terms of the Link Frames:

$a_i \equiv$ the length of the common normal between $\hat{z}_{i-1}$ and $\hat{z}_i$ along $\hat{x}_i$ (link length);

$\alpha_i \equiv$ the angle between $\hat{z}_{i-1}$ and $\hat{z}_i$ measured about $\hat{x}_i$ (twist angle);

$d_i \equiv$ the distance from $\hat{x}_{i-1}$ to $\hat{x}_i$ measured along $\hat{z}_{i-1}$ (link offset);

$\theta_i \equiv$ the angle between $\hat{x}_{i-1}$ and $\hat{x}_i$ measured about $\hat{z}_{i-1}$ (joint angle);

NOTE: We usually choose $a_i \geq 0$; however, α_i, d_i and θ_i are signed quantities.

NOTE: Several different coordinate systems can be defined for a manipulator, due to various possible choices of the positive $\hat{z}$ and $\hat{x}$ axes.

Summary of link frame attachment procedure:

1) Draw a schematic diagram of the robot, and starting from the base link, number the links and joints sequentially. The base is numbered as link 0 and the last link is the end-effector. Joint i connects link i-1 to link i.

2) Identify the axis $\hat{z}_i$ pointing along the $(i+1)^{\text{th}}$ joint axis. Choose axis $\hat{z}_n$ arbitrarily.

3) Identify the common normal lines between the joint axes, and assign the points of intersection as the origins of link frames; the origin of frame $\{\mathrm{i}\}$ is at the intersection of $\hat{z}_i$ and the common normal between $\hat{z}_i$ and $\hat{z}_{i-1}$. Choose the origin of frame $\{0\}$ arbitrarily (at a convenient location.)

4) Assign $\hat{x}_i$ pointing along the common normal lines pointing from the origin of frame $\{\mathrm{i\text{-}1}\}$ toward $\hat{z}_i$. Choose axis $\hat{x}_0$ at the origin of frame $\{0\}$ arbitrarily normal to $\hat{z}_0$.

5) Assign $\hat{y}_i$ perpendicular to $\hat{x}_i$ and $\hat{z}_i$ according to the right-hand rule.

3.2.2 Modified Denavit-Hartenberg Convention

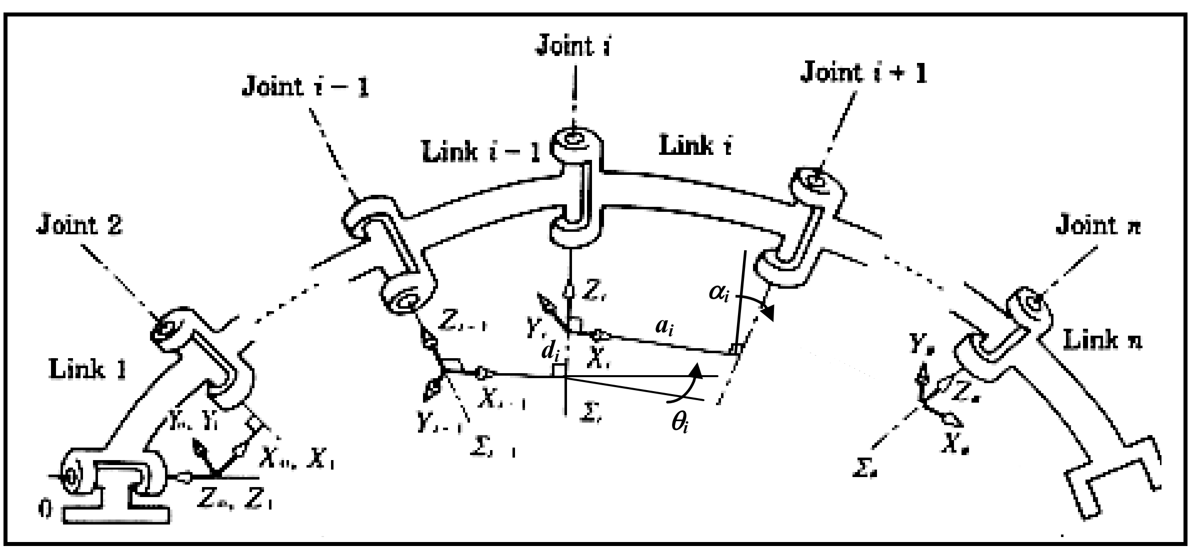
Some researchers have suggested a modified version of DH convention in which the $\hat{z}_i$ axis of frame $\{i\}$ locates along the (i)th joint axis instead of the $(i+1)$th joint axis. Since both methods are used extensively, the modified DH is also explained in the following:

For each link, we define a frame attached to the link. The link frames are named by number according to the link to which they are attached. The origin of frame $\{i\}$ attached to link i is located at the point of intersection of joint i axis and the common normal between joint axes i and $i+1$. The axis $\hat{z}_i$ is coincident with the joint axis i, in the direction from the point of intersection of the joint axis i and a_{i-1} to the point of intersection of the joint axis i and a_i. The axis $\hat{x}_i$ points along a_i in the direction from joint i to joint $i+1$. And, the axis $\hat{y}_i$ is formed by the right-hand rule to complete the frame $\{i\}$.

NOTE: In the case of $a_i = 0$, $\hat{x}_i$ is normal to the plane of $\hat{z}_i$ and $\hat{z}_{i+1}$ with an arbitrary direction.

NOTE: In the case where joint axes i and $i+1$ are parallel, one of the common normal lines is chosen as $\hat{x}_i$ arbitrarily, but the one that makes $d_i = 0$ is preferable.

NOTE: In the case where joint axes i and $i+1$ coincide, The origin of frame $\{i\}$ is chosen arbitrarily, and $\hat{x}_i$ is also chosen normal to the joint axes arbitrarily. However, the choice that makes more link parameters equal zero is preferable.



First and Last Links in the Chain:

Frame $\{0\}$ is attached to the base that is usually stationary, and the position and orientation of other link frames are described in terms of this frame. Frame $\{0\}$ can be defined arbitrarily if not assigned. For simplicity, we choose $\hat{z}_0$ along $\hat{z}_1$, and locate frame $\{0\}$ so that it coincides with frame $\{1\}$ when joint variable 1 is zero.

For joint n revolute, the axis $\hat{x}_n$ is chosen along $\hat{x}_{n-1}$ when $\theta_n = 0.0$, and the origin of frame $\{n\}$ is chosen so that $d_n = 0.0$, due to simplicity. For joint n prismatic, the axis $\hat{x}_n$ is chosen so that $\theta_n = 0.0$, and the origin of frame $\{n\}$ is chosen at the intersection of $\hat{x}_{n-1}$ and joint axis n when $d_n = 0.0$, due to simplicity.

Link Parameters in terms of the Link Frames:

$a_i \equiv$ the distance from $\hat{z}_i$ to $\hat{z}_{i+1}$ measured along $\hat{x}_i$;

$\alpha_i \equiv$ the angle between $\hat{z}_i$ and $\hat{z}_{i+1}$ measured about $\hat{x}_i$;

$d_i \equiv$ the distance from $\hat{x}_{i-1}$ to $\hat{x}_i$ measured along $\hat{z}_i$;

$\theta_i \equiv$ the angle between $\hat{x}_{i-1}$ and $\hat{x}_i$ measured about $\hat{z}_i$;

NOTE: We usually choose $a_i \geq 0$; however, α_i, d_i and θ_i are signed quantities.

Summary of link frame attachment procedure:

1) Draw a schematic diagram of the robot, and assign appropriate numbers to the links and joints.

2) Identify the axes $\hat{z}_i$ pointing along the joint axes.

3) Identify the common normal lines between the joint axes, and assign the points of intersection as the origins of link frames.

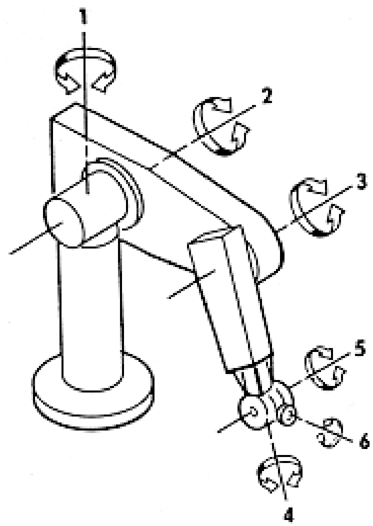
4) Assign $\hat{x}_i$ pointing along the common normal lines.

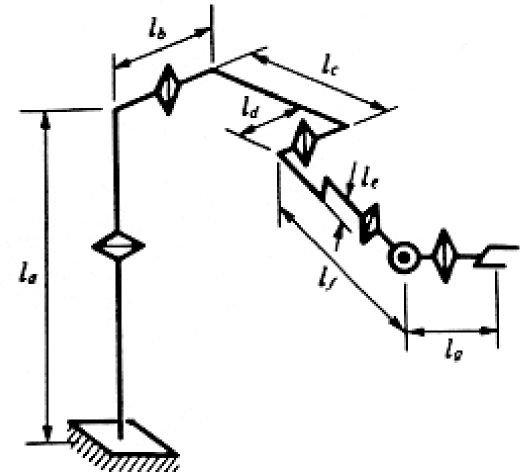
5) Assign $\hat{y}_i$ perpendicular to $\hat{x}_i$ and $\hat{z}_i$ according to the right-hand rule.

6) Assign frame $\{0\}$ to match frame $\{1\}$ when the first joint variable is zero. And, for frame $\{n\}$, choose the origin location and assign $\hat{x}_n$ arbitrarily, but generally so as to cause as many link parameters as possible equal zero.

3.3 Examples

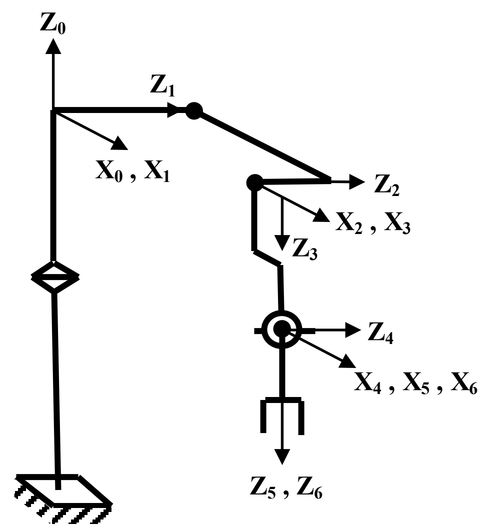
3.3.1 PUMA Robot





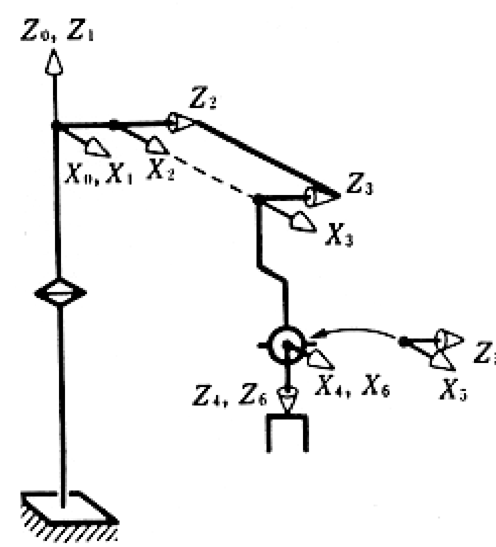
Standard Denavit Hartenberg

i	a_i	α_i	d_i	θ_i
1	0	$-90°$	0	θ_1 (0°)
2	l_c	$0°$	$(l_b - l_d)$	θ_2 (0°)
3	l_e	$-90°$	0	θ_3 (0°)
4	0	$90°$	l_f	θ_4 (0°)
5	0	$-90°$	0	θ_5 (0°)
6	0	$0°$	0	θ_6 (0°)

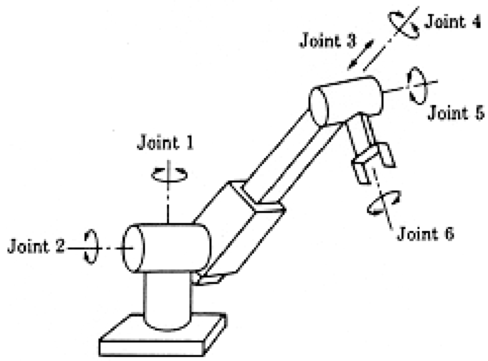


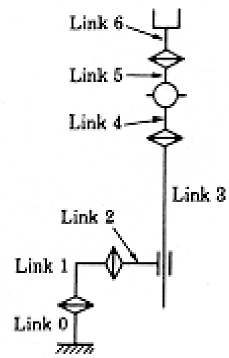
Modified Denavit Hartenberg

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$0°$	0	θ_1 (0°)
2	0	$-90°$	$(l_b - l_d)$	θ_2 (0°)
3	l_c	$0°$	0	θ_3 (0°)
4	l_e	$-90°$	l_f	θ_4 (0°)
5	0	$90°$	0	θ_5 (0°)
6	0	$-90°$	0	θ_6 (0°)

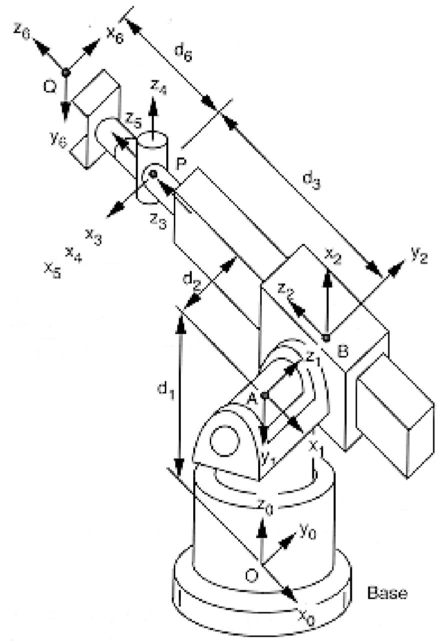


3.3.2 Stanford Manipulator





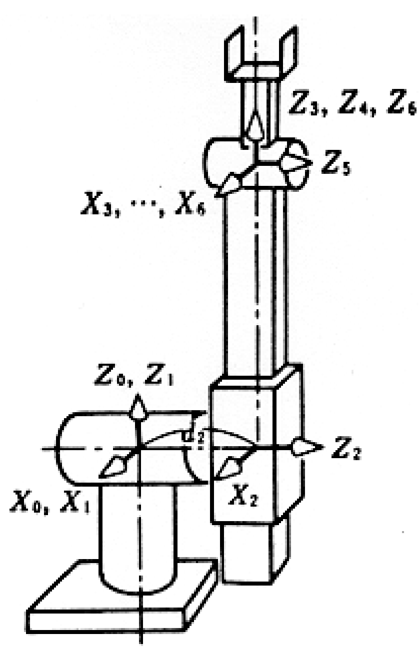
Standard Denavit Hartenberg



i	a_i	α_i	d_i	θ_i
1	0	-90°	d_1	θ_1 (0)
2	0	90°	d_2	θ_2 (-90°)
3	0	0°	d_3 (d_3)	–90°
4	0	-90°	0	θ_4 (0)
5	0	90°	0	θ_5 (0)
6	0	0°	d_6	θ_6 (180°)

Modified Denavit Hartenberg

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0°	0	θ_1 (0°)
2	0	-90°	d_2	θ_2 (0°)
3	0	90°	d_3 (d_3)	0°
4	0	0°	0	θ_4 (0°)
5	0	-90°	0	θ_5 (0°)
6	0	90°	0	θ_6 (0°)



3.4 Forward Kinematics Problem

3.4.1 Different Configuration spaces for Robot Manipulators

The configuration of a robot manipulator can be specified using either of the following algebraic spaces:

i) The *joint space* $\boldsymbol{Q}$ is the set of all possible vectors of joint variables. The dimension of the joint vector is equal to the number of joints (or degrees-of-freedom), i.e., $\boldsymbol{Q} \subset \Re^n$. Each joint variable is defined as an angle $\theta \in [0,2\pi)$ for a revolute joint, or a linear translation $d \in \Re$ for a prismatic joint. Let $q \in \boldsymbol{Q}$ denote the vector of generalized coordinates.

ii) The *task space* is the set of pairs (p,R), where $p \in \Re^3$ is the position vector of the origin of link coordinate frame, and $R \in SO(3)$ represents the orientation of the link frame, both with respect to a general reference frame. Here, $SO(3)$ denotes the group of 3×3 proper rotation matrices. Thus, the task space is a *Special Euclidean* group $SE(3)$ defined as follows:

$$SE(3) = \left\{ (p,R):\ p \in \Re^3, R \in SO(3) \right\} = \Re^3 \times SO(3).$$

The dimension of the task space m may be less than 6 in particular cases such as planar manipulators.

Using the above notation, the forward kinematics, in general, is a mapping T defined as:

$$T : \boldsymbol{Q} \to SE(3).$$

This mapping can be represented by a 4×4 *homogeneous transformation* matrix defined as $\left[\begin{array}{c:c} R(q) & p(q) \\ \hdashline [0] & 1 \end{array}\right]$.

3.4.2 The Manipulator Homogeneous Transformation Matrices

The objective is to find the transformation matrix ${}^0T_n$ of the last link frame $\{\mathrm{n}\}$ (end-effector) relative to the base frame $\{0\}$. This will be obtained by compound transformations between frames of the adjacent links:

$$^{0}T_{n} = {}^{0}T_{1}\ {}^{1}T_{2} \cdots\cdots {}^{i-1}T_{i} \cdots\cdots {}^{n-2}T_{n-1}\ {}^{n-1}T_{n}$$

Standard Denavit Hartenberg

The transformation matrix $^{i-1}T_i$ is obtained by four consequent transformations from frame {i-1} to frame {i}:

a) from frame {i-1} to frame {R} by a translation d_i along $\hat{z}_{i-1}$;

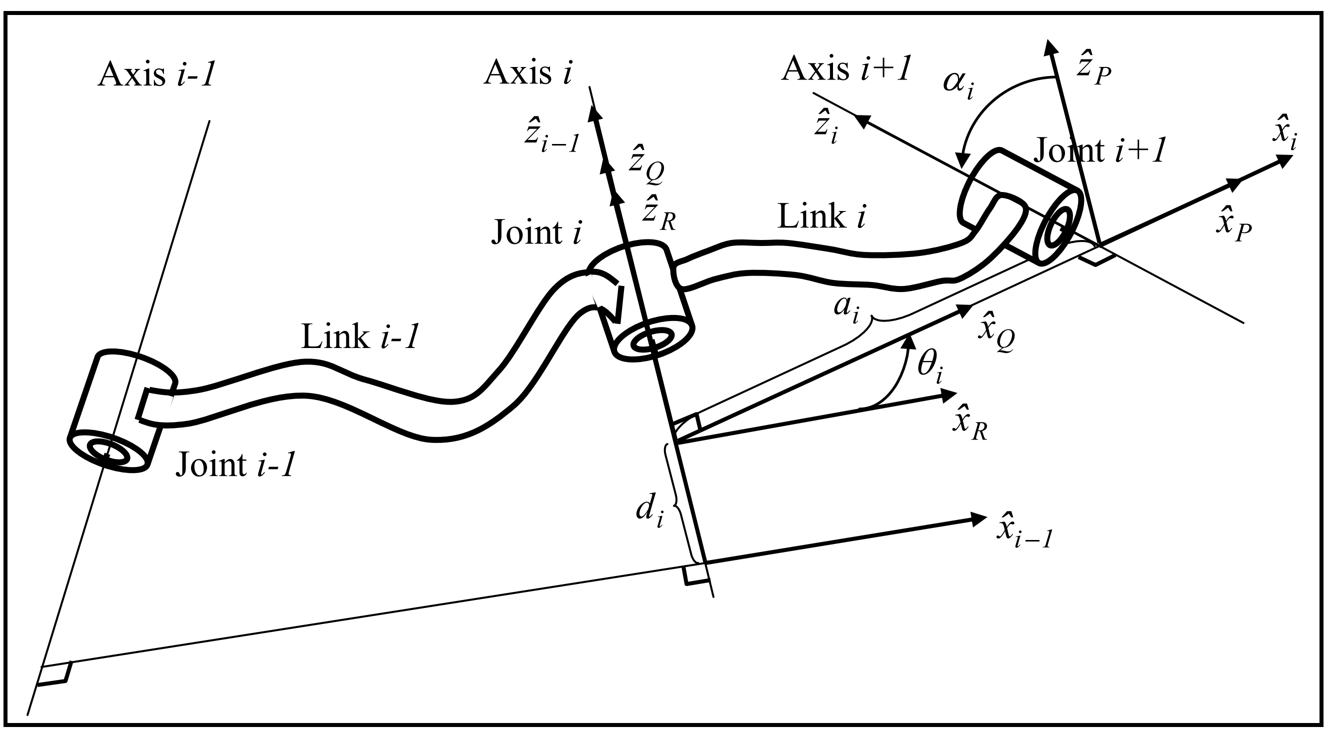
b) from frame {R} to frame {Q} by a rotation θ_i about $\hat{z}_R$;

c) from frame {Q} to frame {P} by a translation a_i along $\hat{x}_Q$;

d) from frame {P} to frame {i} by a rotation α_i about $\hat{x}_P$.

$$^{i-1}T_i = {}^{i-1}T_R\ {}^{R}T_Q\ {}^{Q}T_P\ {}^{P}T_i$$

$$^{i-1}T_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$
{}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Modified Denavit Hartenberg

The transformation matrix ${}^{i-1}T_i$ is obtained by four consequent transformations from frame {i-1} to frame {i}:

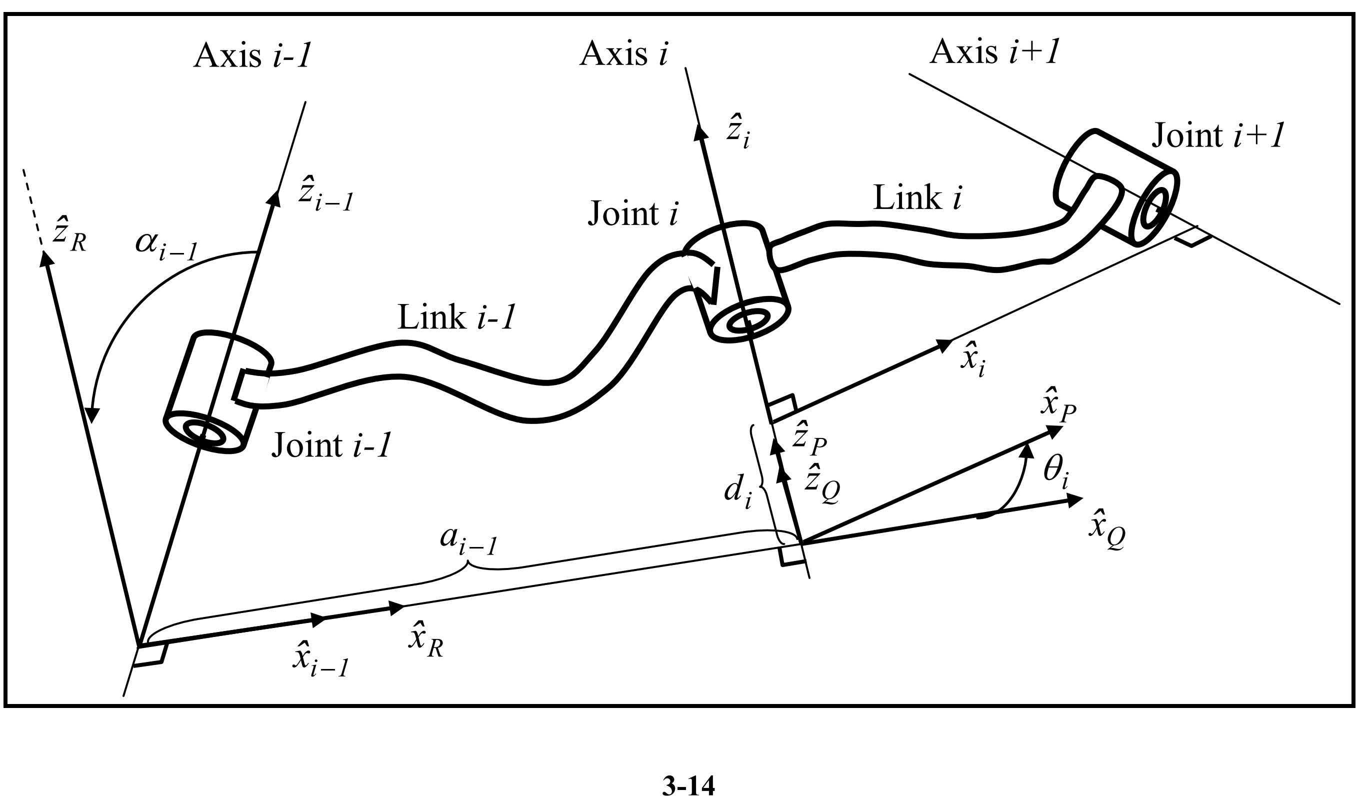
a) from frame {i-1} to frame {R} by a rotation α_{i-1} about $\hat{x}_{i-1}$;

b) from frame {R} to frame {Q} by a translation a_{i-1} along $\hat{x}_R$;

c) from frame {Q} to frame {P} by a rotation θ_i about $\hat{z}_Q$;

d) from frame {P} to frame {i} by a translation d_i along $\hat{z}_P$.

$$
{}^{i-1}T_i = {}^{i-1}T_R \, {}^{R}T_Q \, {}^{Q}T_P \, {}^{P}T_i
$$



$$
{}^{i-1}T_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -d_i s\alpha_{i-1} \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & d_i c\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

NOTE: In the transformation matrix ${}^{i-1}T_i$, θ_i is variable if joint i is revolute, or d_i is variable if joint i is prismatic. The overall transformation ${}^{0}T_n$ will be a function of all n joint variables.

NOTE: *Physical* expression of the transformation matrix ${}^{0}T_n$ is as follows:

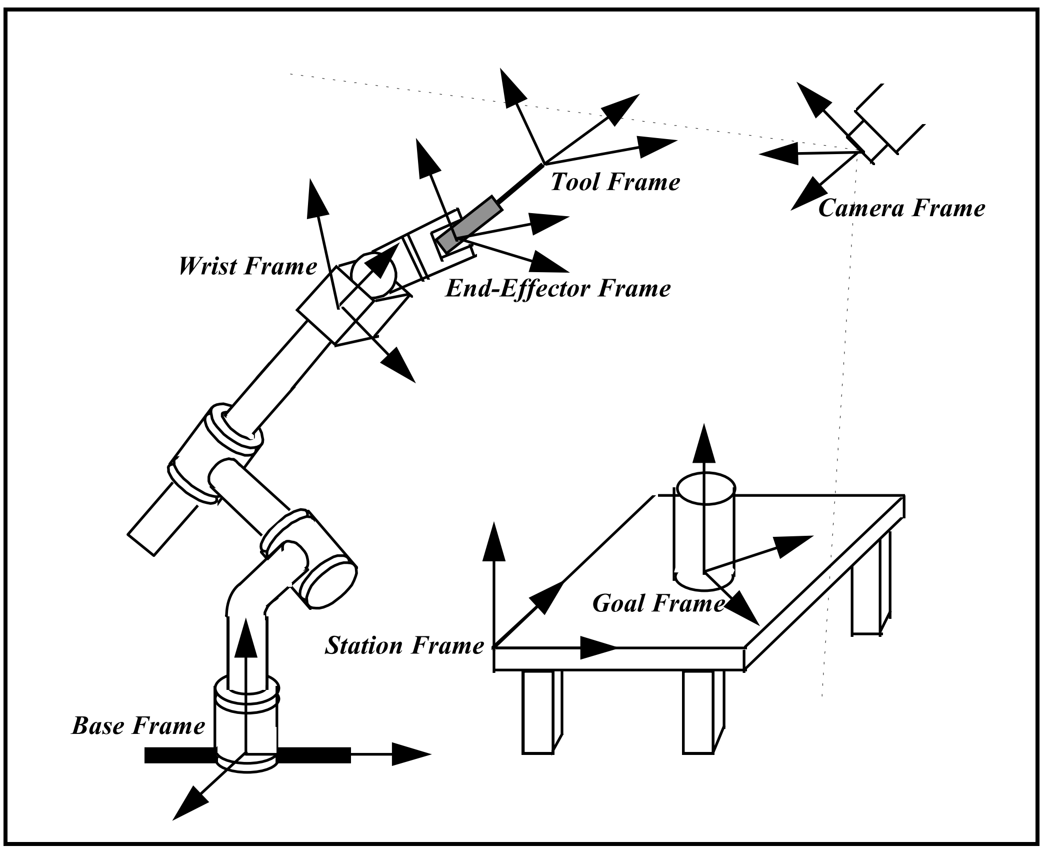
$$
{}^{0}T_n = \left[\begin{array}{c|c} {}^{0}R_n & {}^{0}p_{0n} \\ \hline [0] & 1 \end{array}\right] = \left[\begin{array}{c:c:c:c} n_{0n}^{x} & o_{0n}^{x} & a_{0n}^{x} & p_{0n}^{x} \\ n_{0n}^{y} & o_{0n}^{y} & a_{0n}^{y} & p_{0n}^{y} \\ n_{0n}^{z} & o_{0n}^{z} & a_{0n}^{z} & p_{0n}^{z} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] = \left(\begin{array}{c:c:c|c} \text{Expression of the end-effector } x \text{ axis in the base frame} & \text{Expression of the end-effector } y \text{ axis in the base frame} & \text{Expression of the end-effector } z \text{ axis in the base frame} & \text{End-effector point defined in the base frame} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)
$$

3.4.3 Some Popular Coordinate Frames

Some of the standard frames commonly used in industrial applications are shown in the figure. The position of the origin and the orientation of each frame with respect to the base frame is obtained by successive multiplications of the intermediate homogeneous transformation matrices. For example, the representation of the tool frame with respect to the base frame is determined by

$$^{0}T_{t} = {}^{0}T_{n}\,{}^{n}T_{t},$$

where $^{n}T_{t}$ and $^{0}T_{n}$ are the homogeneous transformation matrices between the end-effector and the tool frames and between the end-effector and the base frames, respectively.



3.4.4 Examples

FANUC S-900W (Standard DH Convention)

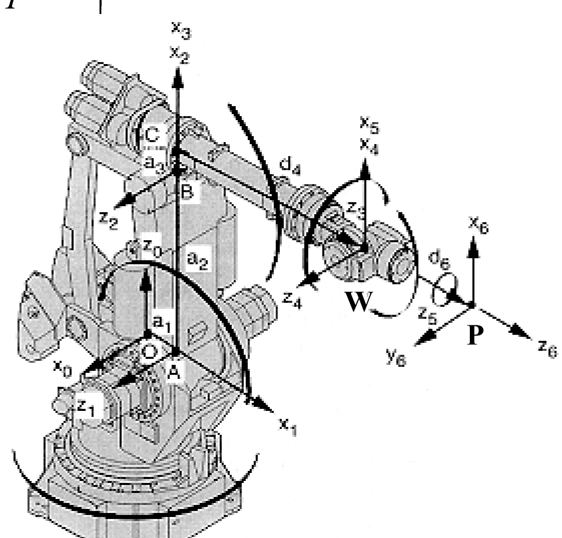
$$ {}^{0}T_{1}=\begin{bmatrix} c1 & 0 & s1 & a_1c1 \\ s1 & 0 & -c1 & a_1s1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ {}^{1}T_{2}=\begin{bmatrix} c2 & -s2 & 0 & a_2c2 \\ s2 & c2 & 0 & a_2s2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ {}^{2}T_{3}=\begin{bmatrix} c3 & 0 & s3 & a_3c3 \\ s3 & 0 & -c3 & a_3s3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}T_{4}=\begin{bmatrix} c4 & 0 & -s4 & 0 \\ s4 & 0 & c4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ {}^{4}T_{5}=\begin{bmatrix} c5 & 0 & s5 & 0 \\ s5 & 0 & -c5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};\ {}^{5}T_{6}=\begin{bmatrix} c6 & -s6 & 0 & 0 \\ s6 & c6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}T_{3}=\begin{bmatrix} c1c23 & s1 & c1s23 & c1(a_1+a_2c2+a_3c23) \\ s1c23 & -c1 & s1s23 & s1(a_1+a_2c2+a_3c23) \\ s23 & 0 & -c23 & a_2s2+a_3s23 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}T_{6}=\begin{bmatrix} c4c5c6-s4s6 & -c4c5s6-s4c6 & c4s5 & d_6c4s5 \\ s4c5c6+c4s6 & -s4c5s6+c4c6 & s4s5 & d_6s4s5 \\ -s5c6 & s5s6 & c5 & d_4+d_6c5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

i	a_i	α_i	d_i	θ_i
1	a_1	90°	0	θ_1 (90°)
2	a_2	0°	0	θ_2 (90°)
3	a_3	90°	0	θ_3 (0°)
4	0	-90°	d_4	θ_4 (0)
5	0	90°	0	θ_5 (0)
6	0	0°	d_6	θ_6 (0°)



$$
{}^{0}T_{6} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where,

$$
\begin{cases}
n_x = c1[c23(c4c5c6 - s4s6) - s23s5c6] + s1(s4c5c6 + c4s6); \\
n_y = s1[c23(c4c5c6 - s4s6) - s23s5c6] - c1(s4c5c6 + c4s6); \\
n_z = s23(c4c5c6 - s4s6) + c23s5c6 ; \\
o_x = c1[- c23(c4c5s6 + s4c6) + s23s5s6] + s1(- s4c5s6 - c4c6); \\
o_y = s1[- c23(c4c5s6 + s4c6) + s23s5s6] - c1(- s4c5s6 + c4c6); \\
o_z = -s23(c4c5s6 + s4c6) - c23s5s6 ; \\
a_x = c1(c23c4s5 + s23c5) + s1s4s5 ; \\
a_y = s1(c23c4s5 + s23c5) - c1s4s5 ; \\
a_z = s23c4s5 - c23c5 ; \\
p_x = c1[a_1 + a_2c2 + a_3c23 + d_4s23 + d_6(c23c4s5 + s23c5)] + d_6s1s4s5 ; \\
p_y = s1[a_1 + a_2c2 + a_3c23 + d_4s23 + d_6(c23c4s5 + s23c5)] - d_6c1s4s5 ; \\
p_z = a_2s2 + a_3s23 - d_4c23 + d_6(s23c4s5 - c23c5).
\end{cases}
$$

CHECK: From the figure,

$$
\theta_1 = \theta_2 = 90^\circ \;\; ; \;\; \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0^\circ
$$

$$
{}^{0}T_{6} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a_1 + d_4 + d_6 \\ 1 & 0 & 0 & a_2 + a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

PUMA 560 (Modified DH Convention)

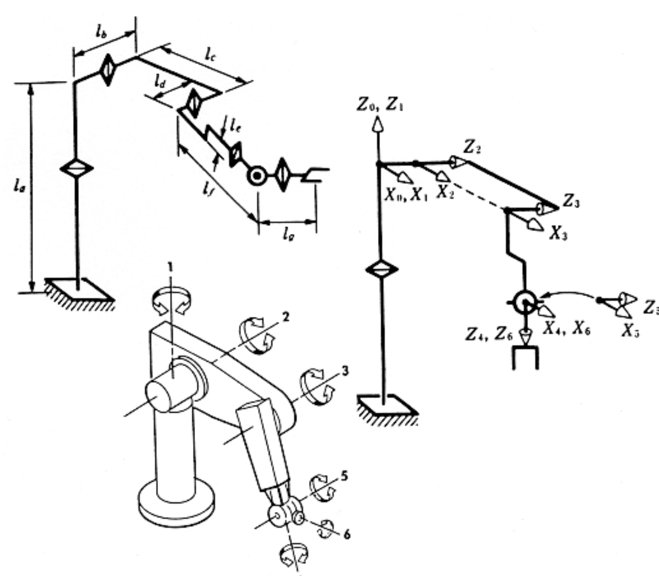
$$
{}^{0}T_{1}=\begin{bmatrix} c1 & -s1 & 0 & 0\\ s1 & c1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ {}^{1}T_{2}=\begin{bmatrix} c2 & -s2 & 0 & 0\\ 0 & 0 & 1 & (l_b-l_d)\\ -s2 & -c2 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ {}^{2}T_{3}=\begin{bmatrix} c3 & -s3 & 0 & l_c\\ s3 & c3 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};
$$

$$
{}^{3}T_{4}=\begin{bmatrix} c4 & -s4 & 0 & l_e\\ 0 & 0 & 1 & l_f\\ -s4 & -c4 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ {}^{4}T_{5}=\begin{bmatrix} c5 & -s5 & 0 & 0\\ 0 & 0 & -1 & 0\\ s5 & c5 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix};\ {}^{5}T_{6}=\begin{bmatrix} c6 & -s6 & 0 & 0\\ 0 & 0 & 1 & 0\\ -s6 & -c6 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{0}T_{3}=\begin{bmatrix} c1c23 & -c1s23 & -s1 & l_c c1c2-(l_b-l_d)s1\\ s1c23 & -s1s23 & c1 & l_c s1c2+(l_b-l_d)c1\\ -s23 & -c23 & 0 & -l_c s2\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

$$
{}^{3}T_{6}=\begin{bmatrix} c4c5c6-s4s6 & -c4c5s6-s4c6 & -c4s5 & l_e\\ s5c6 & -s5s6 & c5 & l_f\\ -s4c5c6-c4s6 & s4c5s6-c4c6 & s4s5 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0°	0	θ_1 (0°)
2	0	-90°	$(l_b$-$l_d)$	θ_2 (0°)
3	l_c	0°	0	θ_3 (0°)
4	l_e	-90°	l_f	θ_4 (0°)
5	0	90°	0	θ_5 (0°)
6	0	-90°	0	θ_6 (0°)



$$ {}^0T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

where,

$$ \begin{cases} n_x = c1[c23(c4c5c6 - s4s6) - s23s5c6] + s1(s4c5c6 + c4s6); \\ n_y = s1[c23(c4c5c6 - s4s6) - s23s5c6] - c1(s4c5c6 + c4s6); \\ n_z = -s23(c4c5c6 - s4s6) - c23s5c6\,; \\ o_x = c1[-c23(c4c5s6 + s4c6) + s23s5s6] - s1(s4c5s6 - c4c6); \\ o_y = s1[-c23(c4c5s6 + s4c6) + s23s5s6] + c1(s4c5s6 - c4c6); \\ o_z = s23(c4c5s6 + s4c6) + c23s5s6\,; \\ a_x = -c1(c23c4s5 + s23c5) - s1s4s5\,; \\ a_y = -s1(c23c4s5 + s23c5) + c1s4s5\,; \\ a_z = s23c4s5 - c23c5\,; \\ p_x = c1(l_c c2 + l_e c23 - l_f s23) - (l_b - l_d)s1\,; \\ p_y = s1(l_c c2 + l_e c23 - l_f s23) + (l_b - l_d)c1\,; \\ p_z = -l_c s2 - l_e s23 - l_f c23\,. \end{cases} $$

CHECK: From the figure:

$$ \theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = \theta_6 = 0^\circ $$

$$ {}^0T_6 = \begin{bmatrix} 1 & 0 & 0 & l_c + l_e \\ 0 & -1 & 0 & l_b - l_d \\ 0 & 0 & -1 & -l_f \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

3.4.5 Computational Considerations

Forward kinematics calculations can be performed recursively to save the computation time, since the open-chain manipulator can be seen as being constructed by adding a link to the previous links. The following algorithm illustrates a fast backward recursive formulation for calculating the forward kinematics, based on the standard DH convention. Transcendental functions are a major computational expense in forward kinematics calculations when standard software is used. Instead, lookup table implementations of these functions may reduce the required calculation time by a factor of two to three, or more. Moreover, using fixed-point representation instead of floating point can speed up the operations. A 24-bit representation of joint variables is adequate due to the typically small dynamic range of these variables.

LOOP : FOR i=n-1 to 1

1) SET : $$ {}^{i}R_n = \begin{bmatrix} n_{i,n}^x & o_{i,n}^x & a_{i,n}^x \\ n_{i,n}^y & o_{i,n}^y & a_{i,n}^y \\ n_{i,n}^z & o_{i,n}^z & a_{i,n}^z \end{bmatrix} = \begin{bmatrix} \left[{}^{i}R_n^x\right] \\ \left[{}^{i}R_n^y\right] \\ \left[{}^{i}R_n^z\right] \end{bmatrix} $$

2) CALCULATE : $$ \begin{cases} \left[{}^{i}M_n\right] = \boldsymbol{cos}\,\alpha_i \left[{}^{i}R_n^y\right] - \boldsymbol{sin}\,\alpha_i \left[{}^{i}R_n^z\right] \\ r_{i,n} = \boldsymbol{cos}\,\alpha_i p_{i,n}^y - \boldsymbol{sin}\,\alpha_i p_{i,n}^z \\ s_{i,n} = \begin{cases} p_{i,n}^x + a_i & : \text{ if joint } i \text{ is revolute} \\ p_{i,n}^x & : \text{ if joint } i \text{ is prismatic} \end{cases} \end{cases} $$

3) CALCULATE : $$ \begin{cases} \left[{}^{i-1}R_n^x\right] = \boldsymbol{cos}\,\theta_i \left[{}^{i}R_n^x\right] - \boldsymbol{sin}\,\theta_i \left[{}^{i}M_n\right] \\ \left[{}^{i-1}R_n^y\right] = \boldsymbol{sin}\,\theta_i \left[{}^{i}R_n^x\right] + \boldsymbol{cos}\,\theta_i \left[{}^{i}M_n\right] \\ \left[{}^{i-1}R_n^z\right] = \boldsymbol{sin}\,\alpha_i \left[{}^{i}R_n^y\right] + \boldsymbol{cos}\,\alpha_i \left[{}^{i}R_n^z\right] \\ p_{i-1,n}^x = \boldsymbol{cos}\,\theta_i s_{i,n} - \boldsymbol{sin}\,\theta_i r_{i,n} \\ p_{i-1,n}^y = \boldsymbol{sin}\,\theta_i s_{i,n} + \boldsymbol{cos}\,\theta_i r_{i,n} \\ p_{i-1,n}^z = \boldsymbol{sin}\,\alpha_i p_{i,n}^y + \boldsymbol{cos}\,\alpha_i p_{i,n}^z + d_i \end{cases} $$

NEXT i

3.4.6 Screw Formulation of Forward Kinematics

In Chapter 2, the homogeneous transformation matrix was formulated based on the screw parameters. This method can be extended to an n d.o.f. serial manipulator through the following procedure:

a) Assign a *Reference Configuration* for the manipulator: The reference configuration is specified in terms of the end-effector location:

$$^0n_{e0}\,,\ ^0o_{e0}\,,\ ^0a_{e0}\,,\ ^0q_{e0} \quad \Rightarrow T_{ref} = \begin{bmatrix} ^0n_{e0} & ^0o_{e0} & ^0a_{e0} & ^0q_{e0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

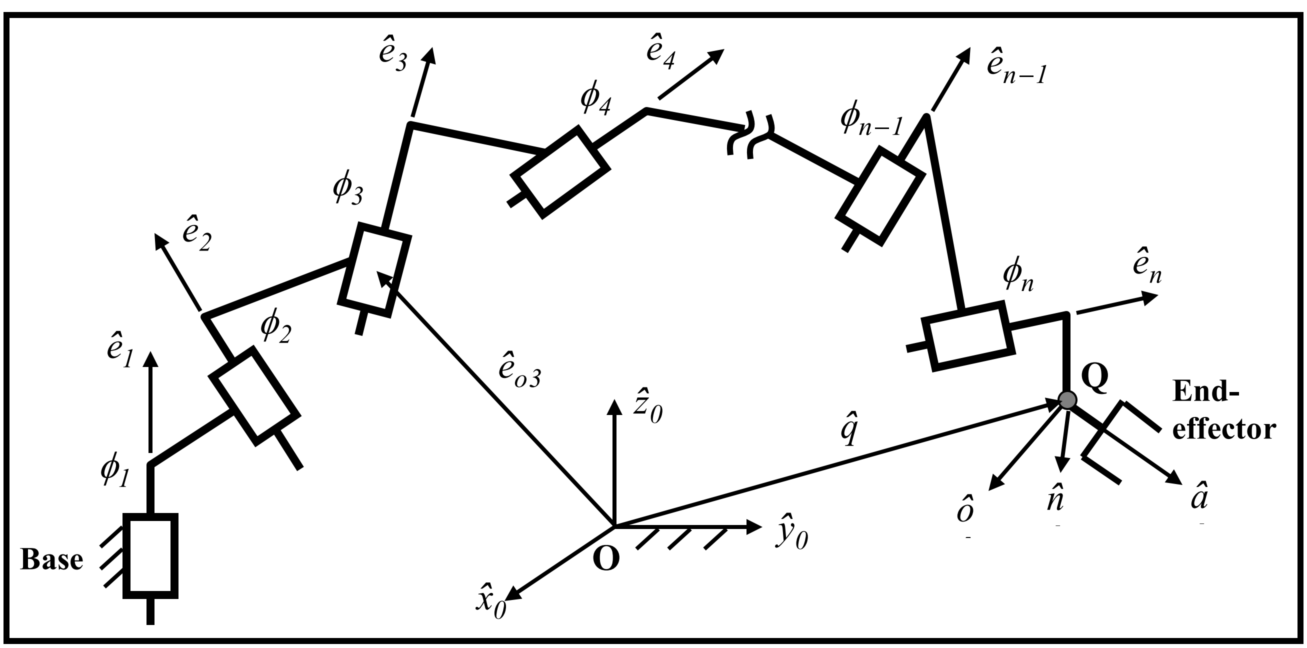
and the location and orientation of the joint axes:

$$^0e_i \text{ and } ^0e_{0i} \quad \text{for } i=1,2,\ldots,n\,.$$

b) Specify the *Target Configuration* of the manipulator: The target configuration is specified in terms of the desired end-effector location and orientation:

$$^0n_e\,,\ ^0o_e\,,\ ^0a_e\,,\ ^0q_e\,.$$

c) Arrange the successive screw displacements: The end-effector displacement from a reference configuration to a target configuration can be considered as the resultant of n successive screw displacements, i.e., a rotation about the nth joint axis followed by another rotation about the joint axis $(n\text{-}1)$ and so on. Since all screw displacements occur about the joint axes at the reference configuration, the resulting screw displacement is obtained by pre-multiplying the joint screw homogeneous transformation matrices:



$$ {}^{0}T_{e} = T_{1}T_{2}\cdots T_{n-1}T_{n}T_{ref} $$

NOTE: In the method of successive screw displacements, only one fixed coordinate frame and one end-effector coordinate frame are required.

NOTE: The screw parameters used in the above transformations are not the same as DH parameters. The joint variables of a screw displacement represent the absolute (with respect to the reference frame) angles of rotation (for revolute joints) or distances of translation (for prismatic joints). For a revolute joint the screw angle ϕ is variable and the screw distance l is zero, while for a prismatic joint ϕ is zero and l is a variable.

NOTE: To obtain the actual joint displacements, one must subtract the joint variables associated with the reference configuration from that of the target configuration. One of the advantages of this method is that the reference configuration can be selected arbitrarily, such as the *home position* of a manipulator, where all the information about the location of the end-effector and the joint axes is known.

EXAMPLE: SCARA Robot

The Reference Configuration can be assigned as:

$$ {}^0n_{e0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; {}^0o_{e0} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; {}^0a_{e0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^0q_{e0} = \begin{bmatrix} 0 \\ L_1 + L_2 \\ L_0 \end{bmatrix} $$

$$ T_{ref} = \begin{bmatrix} {}^0n_{e0} & {}^0o_{e0} & {}^0a_{e0} & {}^0q_{e0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & L_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

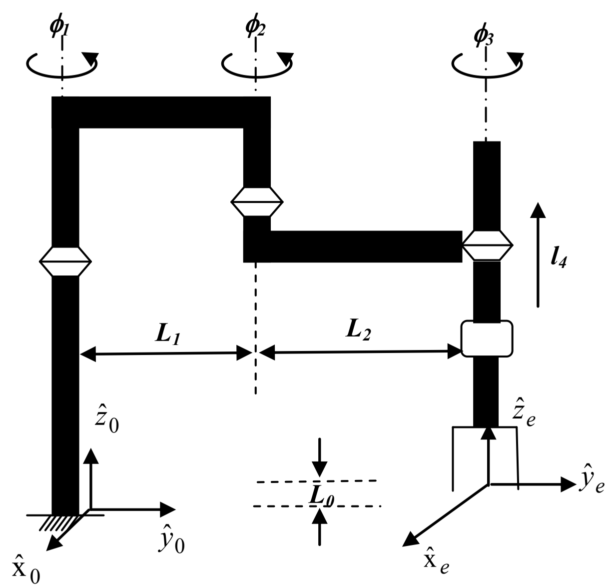
Then, the target configuration can be obtained by successive joint screw displacements:

$$ {}^0e_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^0e_{04} = \begin{bmatrix} 0 \\ L_1 + L_2 \\ 0 \end{bmatrix}, \; T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^0e_{03} = \begin{bmatrix} 0 \\ L_1 + L_2 \\ 0 \end{bmatrix}, \; T_3 = \begin{bmatrix} \cos\phi_3 & -\sin\phi_3 & 0 & (L_1 + L_2)\sin\phi_3 \\ \sin\phi_3 & \cos\phi_3 & 0 & (L_1 + L_2)(1-\cos\phi_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^0e_{02} = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix}, $$

$$ T_2 = \begin{bmatrix} \cos\phi_2 & -\sin\phi_2 & 0 & L_1\sin\phi_2 \\ \sin\phi_2 & \cos\phi_2 & 0 & L_1(1-\cos\phi_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$ {}^0e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; {}^0e_{01} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; T_1 = \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & 0 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0T_e = T_1 T_2 T_3 T_4 T_{ref} $$

$$ {}^0T_e = \begin{bmatrix} \cos(\phi_1+\phi_2+\phi_3) & -\sin(\phi_1+\phi_2+\phi_3) & 0 & -L_1\sin\phi_1 - L_2\sin(\phi_1+\phi_2) \\ \sin(\phi_1+\phi_2+\phi_3) & \cos(\phi_1+\phi_2+\phi_3) & 0 & L_1\cos\phi_1 + L_2\cos(\phi_1+\phi_2) \\ 0 & 0 & 1 & L_0 + l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$