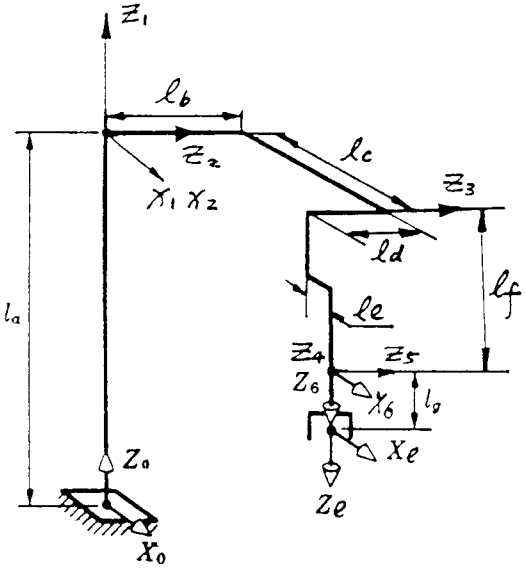
1. Set the frames as shown in the figure.



Directly, from the figure, we can find

$$\underset{\rightarrow}{Z}_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underset{\rightarrow}{Z}_2^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underset{\rightarrow}{Z}_3^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{\rightarrow}{Z}_4^0 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \qquad \underset{\rightarrow}{Z}_5^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underset{\rightarrow}{Z}_6^0 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

and also since $\theta_4 = \theta_5 = \theta_6 = 0$

$$p_x = l_c \cos\theta_2 \cos\theta_1 + l_e \cos(\theta_2+\theta_3)\cos\theta_1 - (l_b - l_d)\sin\theta_1 - l_f \sin(\theta_2+\theta_3)\cos\theta_1$$

$$p_y = l_c \cos\theta_2 \sin\theta_1 + l_e \cos(\theta_2+\theta_3)\sin\theta_1 + (l_b - l_d)\cos\theta_1 - l_f \sin(\theta_2+\theta_3)\sin\theta_1$$

$$p_z = l_f \cos(\theta_2+\theta_3) - l_c \sin\theta_2 - l_e \sin(\theta_2+\theta_3)$$

Hence,

$$\left.\frac{dp_x}{d\theta_1}\right|_{\theta_1=0} = -(l_b - l_d)\cos\theta_1 \Big|_{\theta_1=0} = -l_b + l_d$$

$$\left.\frac{dp_y}{d\theta_1}\right|_{\theta_1=0} = -(l_c + l_e)$$

$$\frac{dp_z}{d\theta_1} = 0 \qquad \left.\frac{dp_x}{d\theta_2}\right|_{\substack{\theta_1=0\\ \theta_2=0}} = 0 \qquad \left.\frac{dp_y}{d\theta_2}\right|_{\substack{\theta_1=0\\ \theta_2=0}} = 0$$

$$\left.\frac{dp_z}{d\theta_2}\right|_{\theta_2=0} = -(l_c + l_e)$$

and

$$\left.\frac{dp_x}{d\theta_3}\right|_{\substack{\theta_1=0\\ \theta_2=0\\ \theta_3=0}} = 0 \qquad \left.\frac{dp_y}{d\theta_3}\right|_{\substack{\theta_1=0\\ \theta_2=0\\ \theta_3=0}} = 0 \qquad \left.\frac{dp_z}{d\theta_3}\right|_{\substack{\theta_1=0\\ \theta_2=0\\ \theta_3=0}} = -l_e$$

$$\left.\frac{dp_x}{d\theta_4}\right|_{\theta_5=0} = 0 \qquad \left.\frac{dp_y}{d\theta_4}\right|_{\theta_5=0} = 0 \qquad \left.\frac{dp_z}{d\theta_4}\right|_{\theta_5=0} = 0$$

$$\left.\frac{dp_x}{d\theta_5}\right|_{\theta_i=0 \quad i=1,2,3,4} = l_g \qquad \left.\frac{dp_y}{d\theta_5}\right|_{\theta_i=0,\ i=1,2,3,4} = 0$$

$$\left.\frac{dp_z}{d\theta_5}\right|_{\theta_i=0 \quad i=1,2,3,4} = 0$$

So we can write the Jacobian at this special configuration as

$$J = \begin{bmatrix} -l_b-l_d & 0 & 0 & 0 & l_g & 0 \\ l_c+l_e & 0 & 0 & 0 & 0 & 0 \\ 0 & -l_c-l_e & -l_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}$$

$$J^T = \begin{bmatrix} -l_b-l_d & l_c+l_e & 0 & 0 & 0 & 1 \\ 0 & 0 & -l_c-l_e & 0 & 1 & 0 \\ 0 & 0 & -l_e & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ l_g & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Now $F = \begin{bmatrix} f_e \\ n_e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 10 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore \quad \underline{\tau} = J^T F$$

$$= \begin{bmatrix} 0 \\ -100(l_c + l_e) \\ -100\, l_e \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.

(a) $H_{0e} = H_{01} H_{12} H_{23} H_{3e}$

$$H_{01} = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{23} = \begin{bmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{3e} = \begin{bmatrix} 1 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{02} = \begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 & 0 \\ S_1 C_2 & C_1 & S_1 S_2 & 0 \\ -S_2 & 0 & C_2 & 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{03} = \begin{bmatrix} C_1 C_2 C_3 - S_1 S_3 & -C_1 C_2 S_3 - S_1 C_3 & C_1 S_2 & 0.1 C_1 S_2 \\ S_1 C_2 C_3 + C_1 S_3 & -S_1 C_2 S_3 + C_1 C_3 & S_1 S_2 & 0.1 S_1 S_2 \\ -S_2 C_3 & S_2 S_3 & C_2 & 0.1 C_2 + 0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{0e} = \left[\begin{array}{ccc|c} \multicolumn{3}{c|}{As\ in\ H_{03}} & \begin{matrix} 0.1 C_1 C_2 C_3 - 0.1 S_1 S_3 + 0.15 C_1 S_2 \\ 0.1 S_1 C_2 C_3 + 0.1 C_1 S_3 + 0.15 S_1 S_2 \\ -0.1 S_2 C_3 + 0.15 C_2 + 0.4 \end{matrix} \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Now,

$$Z_0^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Z_0^1 = \begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix}$$

$$Z_0^2 = \begin{bmatrix} C_1 S_2 \\ S_1 S_2 \\ C_2 \end{bmatrix}$$

$$\therefore \quad J_0 = \begin{bmatrix} 0.1\, S_1 C_2 C_3 - 0.1\, C_1 S_3 - 0.15\, S_1 S_2 & -0.1\, C_1 S_2 C_3 + 0.15\, C_1 C_2 & -0.1\, S_1 C_3 - 0.1\, C_1 C_2 S_3 - 0.1\, S_1 C_3 \\ 0.1\, C_1 C_2 C_3 - 0.1\, S_1 S_3 + 0.15\, C_1 S_2 & -0.1\, S_1 S_2 C_3 + 0.15\, S_1 C_2 & -0.1\, S_1 C_2 S_3 + 0.1\, C_1 C_3 \\ 0 & -0.1\, C_2 C_3 - 0.15\, S_2 & 0.1\, S_2 S_3 \\ 0 & -S_1 & C_1 S_2 \\ 0 & C_1 & S_1 S_2 \\ 1 & 0 & C_2 \end{bmatrix}$$

(b) $Z = J^T F_0$

When, $\theta_1 = 90, \quad \theta_2 = 45 \quad \theta_3 = 0,$

$$J_0^T = \begin{bmatrix} -0.177 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.0354 & -0.177 & -1 & 0 & 0 \\ -0.1 & 0 & 0 & 0 & 0.707 & 0.707 \end{bmatrix}$$

$$F_o^T = \left[f_x^o \quad f_y^o \quad f_z^o, \quad m_x^o, \quad m_y^o, \quad m_z^o \right]^T$$

From problem, $f_N = f_z^o = -10\ N$

$f_t = f_x^o = -8\ N$

and

$$\begin{bmatrix} m_x^o \\ m_y^o \\ m_z^o \end{bmatrix} = \begin{bmatrix} R_{oe} \end{bmatrix} \begin{bmatrix} m_x^e \\ m_y^e \\ m_z^e \end{bmatrix}$$

$$= R_{oe} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since $m_y^e = m_z^e = 0$.

$$\therefore \begin{bmatrix} m_x^o \\ m_y^o \\ m_z^o \end{bmatrix} = \begin{bmatrix} C_1 C_2 C_3 - S_1 S_3 \\ S_1 C_2 C_3 + C_1 S_3 \\ -S_2 C_3 \end{bmatrix}$$

Substituting values for θ_1, θ_2 & θ_3 we get,

$$\begin{bmatrix} m_x^0 & m_y^0 & m_z^0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{0.04}{\sqrt{2}} & \frac{0.04}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \quad F_0 = \begin{bmatrix} -8 \\ 0 \\ -10 \\ 0 \\ 0.04/\sqrt{2} \\ 0.04/\sqrt{2} \end{bmatrix}$$

Since we already know J_0^T, we can calculate τ from

$$\tau = J_0^T F_0$$

$$= \begin{bmatrix} 1.39 \\ 1.77 \\ 0.8 \end{bmatrix}$$

(c) Given : $\left[f_x^3 \quad f_y^3 \quad f_z^3 \quad m_x^3 \quad m_y^3 \quad m_z^3 \right]^T$

find : $\left[f_{tx}^0 \quad f_{ty}^0 \quad f_{tz}^0 \quad m_{tx}^0 \quad m_{ty}^0 \quad m_{tz}^0 \right]^T$

where f_{tx}, f_{ty}, f_{tz} are the forces at the tool-tip and m_{tx}, m_{ty}, m_{tz} are the corresponding moments.

Let $$\left[F_m^m \right]^T = \left[f_x^3 \quad f_y^3 \quad f_z^3 \quad m_x^3 \quad m_y^3 \quad m_z^3 \right]^T$$

and $$\left[F_t^t \right]^T = \left[f_{tx}^t \quad f_{ty}^t \quad f_{tz}^t \quad m_{tx}^t \quad m_{ty}^t \quad m_{tz}^t \right]^T$$

Then $$F_m^m = \begin{bmatrix} R_{mt} & 0 \\ \tilde{P}_{mt}^m R_{mt} & R_{mt} \end{bmatrix} F_t^t$$

Or $$F_t^t = \begin{bmatrix} R_{mt}^T & 0 \\ -R_{mt}^T \tilde{P}_{mt}^m & R_{mt}^T \end{bmatrix} F_m^m \rightarrow (1)$$

where $\tilde{p}^{m}_{mt} = \Omega^{m}_{mt}$, the skew-symmetric matrix of the position vector $\vec{p}^{m}_{mt}$, which gives the position of the tool-tip with respect to the 'force-measurement' coordinate frame $O_3 - x_3 y_3 z_3$.

In order to get the forces at the tool-tip w.r.t base frame, ~~from O we have~~ we use (1) as follows.

$$F_t^{o} = \begin{bmatrix} R_{ot} & 0 \\ 0 & R_{ot} \end{bmatrix} \begin{bmatrix} F_t^{t} \end{bmatrix}$$

Or

$$F_t^{o} = \begin{bmatrix} R_{ot} & 0 \\ 0 & R_{ot} \end{bmatrix} \begin{bmatrix} R_{mt}^{T} & 0 \\ -R_{mt}^{T}\tilde{p}^{m}_{mt} & R_{mt}^{T} \end{bmatrix} F_m^{m}$$

$$= \begin{bmatrix} R_{om} & 0 \\ R_{om}\tilde{p}^{m}_{mt} & R_{om} \end{bmatrix} F_m^{m}$$

Since $R_{om} = R_{o3}$, we have and $\tilde{p}^{m}_{mt} = \tilde{p}^{3}_{3e}$

$$F_t^{\,0} = \begin{bmatrix} R_{03} & 0 \\ R_{03}\,\tilde{P}_{3t}^{\,3} & R_{03} \end{bmatrix} F_m^{\,m}$$

Since $F_m^{\,m}$ is the measured force inputs, (force-sense) we can find the forces & moments at the tool-tip with respect to the base frame using this expression.

3.

$\theta_1 = \pi/2 = 90^\circ$

$\theta_2 = 3\pi/4 = 135^\circ$

$\theta_3 = -\pi/2 = -90^\circ$

	Stiffness	Link	Link length	
Joint 1 :	4×10^5 Nm/rad	Link 1	1 m	$= l_0$
" 2 :	2×10^5 Nm/rad	" 2	1 m	$= l_1$
" 3 :	1×10^5 Nm/rad	" 3	1.5 m	$= l_2$

$$^0T_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^1T_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^0T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}+1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0.5\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 2.5\frac{\sqrt{2}}{2}+1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad ^0Z_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad ^0Z_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$^0P_{03} = \begin{bmatrix} 0 \\ 0.5\frac{\sqrt{2}}{2} \\ 2.5\frac{\sqrt{2}}{2}+1 \end{bmatrix}, \quad ^0P_{13} = \begin{bmatrix} 0 \\ 0.5\frac{\sqrt{2}}{2} \\ 2.5\frac{\sqrt{2}}{2} \end{bmatrix}, \quad ^0P_{23} = \begin{bmatrix} 0 \\ -0.5\frac{\sqrt{2}}{2} \\ 1.5\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$^0\tilde{Z}_0 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad ^0\tilde{Z}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = {}^0\tilde{Z}_2$$

$$^0J = \begin{bmatrix} -0.5\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & -2.5\frac{\sqrt{2}}{2} & -1.5\frac{\sqrt{2}}{2} \\ 0 & 0.5\frac{\sqrt{2}}{2} & -0.5\frac{\sqrt{2}}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\Delta p = JK^{-1}J^T F = CF$$

$$\boxed{C = JK^{-1}J^T}$$

$$K = \begin{bmatrix} 4\times10^5 & 0 & 0 \\ 0 & 2\times10^5 & 0 \\ 0 & 0 & 1\times10^5 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} \frac{1}{4\times10^5} & 0 & 0 \\ 0 & \frac{1}{2\times10^5} & 0 \\ 0 & 0 & \frac{1}{1\times10^5} \end{bmatrix}$$

$$C = 1\times10^{-4}\begin{bmatrix} 0.0031 & 0 & 0 & 0 & 0 & -0.0088 \\ 0 & 0.2688 & 0.0063 & -0.1945 & 0 & 0 \\ 0 & 0.0063 & 0.0188 & -0.0177 & 0 & 0 \\ 0 & -0.1945 & -0.0177 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0088 & 0 & 0 & 0 & 0 & 0.025 \end{bmatrix}$$

$$C = J K^{-1} J^T$$

$$C = \begin{bmatrix} -0.3536 & 0 & 0 \\ 0 & -1.7678 & -1.0607 \\ 0 & 0.3536 & -0.3536 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.25\times10^{-5} & 0 & 0 \\ 0 & 0.5\times10^{-5} & 0 \\ 0 & 0 & 1\times10^{-5} \end{bmatrix} \begin{bmatrix} -0.3536 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1.7678 & 0.3536 & 1 & 0 & 0 \\ 0 & -1.0607 & -0.3536 & 1 & 0 & 0 \end{bmatrix}$$

$$C = 1\times10^{-4} \begin{bmatrix} -0.0088 & 0 & 0 \\ 0 & -0.0884 & -0.1061 \\ 0 & 0.0177 & -0.0354 \\ 0 & 0.05 & 0.1 \\ 0 & 0 & 0 \\ 0.025 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.3536 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1.7678 & 0.3536 & 1 & 0 & 0 \\ 0 & -1.0607 & -0.3536 & 1 & 0 & 0 \end{bmatrix}$$

$$C = 1\times10^{-4} \begin{bmatrix} 0.0031 & 0 & 0 & 0 & 0 & -0.0088 \\ 0 & 0.2688 & 0.0063 & -0.1945 & 0 & 0 \\ 0 & 0.0063 & 0.0188 & -0.0177 & 0 & 0 \\ 0 & -0.1945 & -0.0177 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0088 & 0 & 0 & 0 & 0 & 0.025 \end{bmatrix}$$

C : end-point compliance for the given configuration

Assume $\|F\| = F^T F = 1$

$\|\Delta p\| = \Delta p^T \Delta p = F^T C^T C F = F^T C^2 F$

use Lagrange multiplier λ

$$L = F^T C^2 F - \lambda (F^T F - 1)$$

$$\frac{\partial L}{\partial \lambda} = 0 \rightarrow -F^T F + 1 = 0$$

$$\frac{\partial L}{\partial F} = 0 \rightarrow C^2 F - \lambda F = 0 \rightarrow C^2 F = \lambda F$$

$$C^2 = C^T C$$

$$= 1\times10^{-8}\begin{bmatrix} 0.00008789 & 0 & 0 & 0 & 0 & -0.0002486 \\ 0 & 0.1101 & 0.0052 & -0.0815 & 0 & 0 \\ 0 & 0.0052 & 0.0007031 & -0.0042 & 0 & 0 \\ 0 & -0.0815 & -0.0042 & 0.0606 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0002486 & 0 & 0 & 0 & 0 & 0.0007031 \end{bmatrix}$$

(eigenvalues of C^2
$\det(C^2 - \lambda I) = 0$
eigenvectors
$C^2 v = \lambda v$)

$\longrightarrow$ use MATLAB to calculate eigenvalues & eigenvectors

Eigenvalue	Eigenvector
0	$[0\ 0\ 0\ 0\ 1\ 0]^T$
0	$[-0.9428\ 0\ 0\ 0\ 0\ -0.3333]^T$
3.549×10^{-25}	$[0\ 0.5\ 0.5\ 0.7071\ 0\ 0]^T$
5.854×10^{-12}	$[0\ 0.3255\ -0.8651\ 0.3816\ 0\ 0]^T$
7.910×10^{-12}	$[-0.3333\ 0\ 0\ 0\ 0\ 0.9428]^T$
1.708×10^{-9}	$[0\ -0.8025\ -0.0394\ 0.5953\ 0\ 0]^T$

$\therefore$ when ${}^0_3F = \begin{bmatrix} 0 \\ -0.8025 \\ -0.0394 \\ 0.5953 \\ 0 \\ 0 \end{bmatrix}$, the max. compliance occurs

$$\max \Delta p = CF = 10^{-4}\cdot\begin{bmatrix} 0 \\ -0.3317 \\ -0.0163 \\ 0.2460 \\ 0 \\ 0 \end{bmatrix}, \quad \|\Delta p\| = 1.708\times10^{-9}$$

when ${}^0_3F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ \sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix}$ the min compliance occurs

$$\min \Delta p = CF = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ for both directions of min. compliance

For Assumption made on page 6
$\|{}^0_3F\| = 1$