Chapter 6

Statics

Goal

To study the relationship between the force/moment (generalized force) applied on the manipulator end-effector and the torque (force) produced at the joints to hold the manipulator in a static or quasi-static equilibrium.

6.1 Transformation of Forces and Moments

Consider that the force and moment vectors applied at point A of the rigid body and expressed in frame $\{A\}$ is known. It is desired to derive the equivalent force and moment that are applied at point B and expressed in frame $\{B\}$:

$$\begin{cases} {}_B\hat{f} = {}_A\hat{f} \\ {}_B\hat{g} = {}_A\hat{g} - \hat{p}_{AB} \times {}_A\hat{f} \end{cases}$$

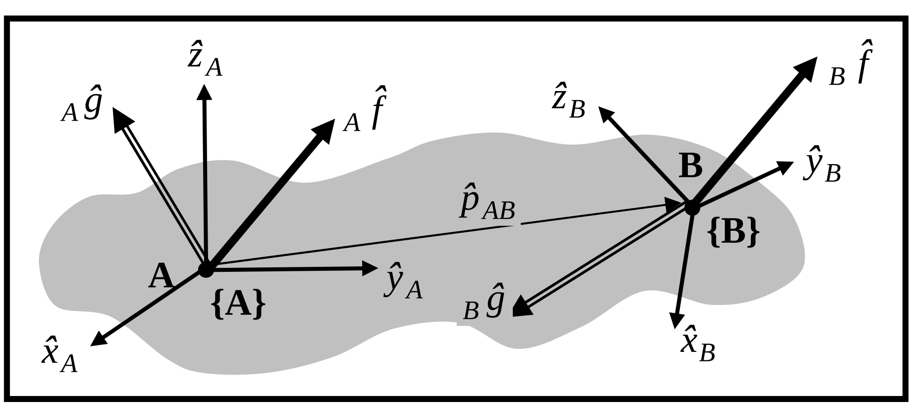
$$\begin{cases} {}^B_B f = {}^B_A f = {}^A R_B^T \, {}^A_A f \\ {}^B_B g = {}^B_A g - {}^B\tilde{p}_{AB} \, {}^B_A f = {}^A R_B^T \, {}^A_A g - {}^A R_B^T \, {}^A\tilde{p}_{AB} \, {}^A_A f = {}^A R_B^T \, {}^A_A g + \left({}^A\tilde{p}_{AB} \, {}^A R_B\right)^T \, {}^A_A f \end{cases}.$$

Therefore, in a matrix form we have:

$$ {}^B_B F = \begin{bmatrix} {}^B_B f \\ {}^B_B g \end{bmatrix} = \begin{bmatrix} {}^A R_B^T & [0] \\ \left({}^A\tilde{p}_{AB} \, {}^A R_B\right)^T & {}^A R_B^T \end{bmatrix} \begin{bmatrix} {}^A_A f \\ {}^A_A g \end{bmatrix} = {}^A S_B^T \begin{bmatrix} {}^A_A f \\ {}^A_A g \end{bmatrix} = {}^A S_B^T \, {}^A_A F $$

where

$$ {}^A S_B = \begin{bmatrix} {}^A R_B & {}^A\tilde{p}_{AB} \, {}^A R_B \\ [0] & {}^A R_B \end{bmatrix} $$



NOTE: By using Coriolis theorem, it can be shown that:

$$
{}^{A}d_{B}\boldsymbol{p} = \begin{bmatrix} \left({}^{A}d_{B}p\right)_x \\ \left({}^{A}d_{B}p\right)_y \\ \left({}^{A}d_{B}p\right)_z \\ \left({}^{A}d\gamma\right)_x \\ \left({}^{A}d\gamma\right)_y \\ \left({}^{A}d\gamma\right)_z \end{bmatrix} = {}^{A}S_{B}\left({}^{B}d_{A}\boldsymbol{p}\right) = \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}\tilde{p}_{AB}\,{}^{A}R_{B} \\ \hline 0 & {}^{A}R_{B} \end{array}\right] \begin{bmatrix} \left({}^{B}d_{A}p\right)_x \\ \left({}^{B}d_{A}p\right)_y \\ \left({}^{B}d_{A}p\right)_z \\ \left({}^{B}d\gamma\right)_x \\ \left({}^{B}d\gamma\right)_y \\ \left({}^{B}d\gamma\right)_z \end{bmatrix}
$$

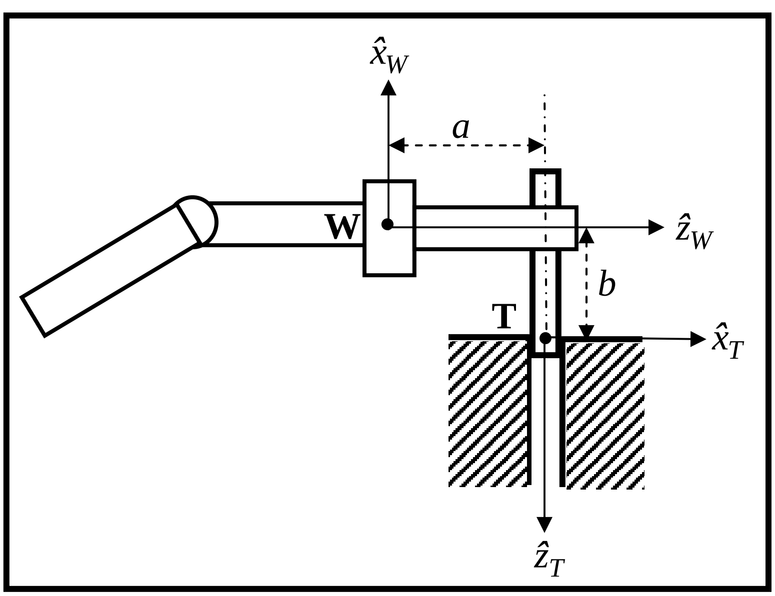
NOTE:

$$
{}^{B}S_{A} = \left({}^{A}S_{B}\right)^{-1} = \left[\begin{array}{c|c} {}^{B}R_{A} & {}^{B}\tilde{p}_{BA}\,{}^{B}R_{A} \\ \hline 0 & {}^{B}R_{A} \end{array}\right]
$$

Example:

The hand shown in figure below is inserting a peg in a hole. A force sensor is attached to the wrist at point W, and measures the forces and moment as:

$$
{}^{W}_{W}f = \begin{bmatrix} 30 \\ 0 \\ -50 \end{bmatrix} \text{(N)} \quad ; \quad {}^{W}_{W}g = \begin{bmatrix} 0 \\ -0.5 \\ 0.2 \end{bmatrix} \text{(Nm)}
$$



Find the forces and moment applied at point T of the peg at the moment shown in the figure.

Solution:

$$^{W}R_{T} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} ; \quad ^{W}p_{WT} = \begin{bmatrix} -b \\ 0 \\ a \end{bmatrix} ,$$

$$^{W}S_{T} = \left[\begin{array}{c:c} ^{W}R_{T} & ^{W}\tilde{p}_{WT}\, ^{W}R_{T} \\ \hdashline 0 & ^{W}R_{T} \end{array}\right] = \left[\begin{array}{ccc:ccc} 0 & 0 & -1 & 0 & -a & 0 \\ 0 & 1 & 0 & b & 0 & -a \\ 1 & 0 & 0 & 0 & -b & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$^{T}_{T}F = \begin{bmatrix} \left(^{T}f\right)_{x} \\ \left(^{T}f\right)_{y} \\ \left(^{T}f\right)_{z} \\ \left(^{T}g\right)_{x} \\ \left(^{T}g\right)_{y} \\ \left(^{T}g\right)_{z} \end{bmatrix} = {}^{W}S_{T}^{T} \left[\begin{array}{c} ^{W}_{W}f \\ \hdashline ^{W}_{W}g \end{array}\right] = \left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & b & 0 & 0 & 0 & 1 \\ -a & 0 & -b & 0 & 1 & 0 \\ 0 & -a & 0 & -1 & 0 & 0 \end{array}\right] \begin{bmatrix} 30 \\ 0 \\ -50 \\ 0 \\ -0.5 \\ 0.2 \end{bmatrix} = \begin{bmatrix} -50 \\ 0 \\ -30 \\ 0.2 \\ -30a + 50b - 0.5 \\ 0 \end{bmatrix}$$

6.2 The Manipulator Statics Relationship

We apply the *principle of virtual work* to derive a transformation between the joint torque and end-effector forces. Consider the manipulator shown in the figure below, which is initially in static equilibrium, i.e., its weight is compensated by some initial joint torque. We apply an external generalized force ${}_e\hat{F} = \left({}_e\hat{f}\ ,\ {}_e\hat{g}\right)$ at the end-effector and joint torque $\boldsymbol{\tau} = \begin{bmatrix}\tau_1 & \tau_2 & \cdots & \tau_n\end{bmatrix}^T$ at the joints, while virtual joint displacements $\delta\,\boldsymbol{q} = \begin{bmatrix}\delta\, q_1 & \delta\, q_2 & \cdots & \delta\, q_n\end{bmatrix}^T$ and virtual end-effector displacements ${}^0\delta_0\ \boldsymbol{p} = \begin{bmatrix}\delta\, x & \delta\, y & \delta\, z & \delta\,\phi_x & \delta\,\phi_y & \delta\,\phi_z\end{bmatrix}^T$ occur. Assuming that frictional forces at the joints are negligible, the virtual work produced by the end-effector generalized force and joint torque is zero. Hence,

$$\delta W = \tau^T \delta\,\boldsymbol{q} + {}^0_eF^{T\ 0}\delta_0\ \boldsymbol{p} = 0\ .$$

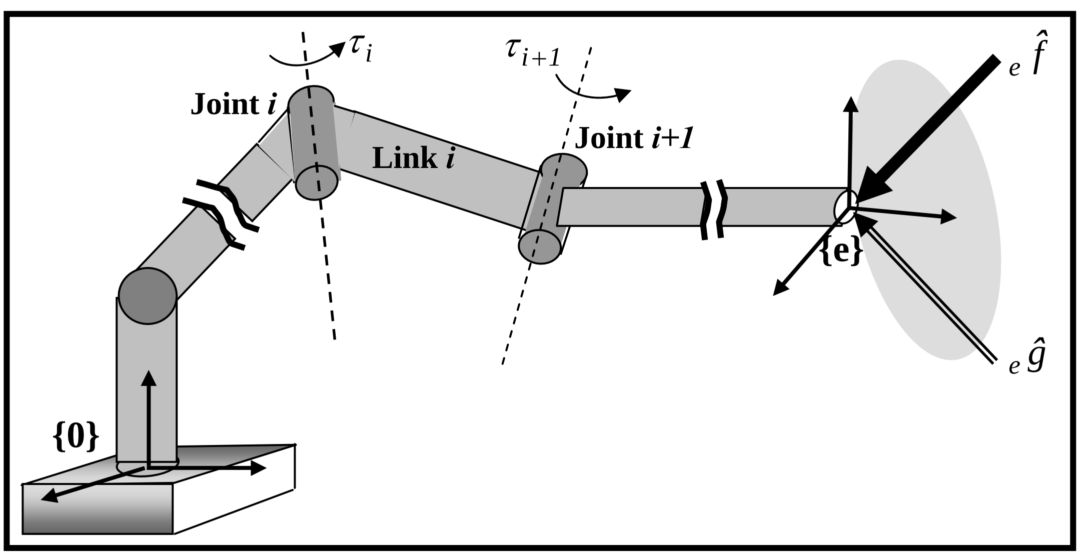
Knowing that ${}^0\delta_0\ \boldsymbol{p} = {}^0J\ \delta\,\boldsymbol{q}$ leads the above equation to:

$$\left(\tau^T + {}^0_eF^{T\ 0}J\right)\delta\,\boldsymbol{q} = 0\,.$$

Since the above equation holds for any arbitrary virtual displacement $\delta\,\boldsymbol{q}$, we conclude that:

$$\tau^T + {}^0_eF^{T\ 0}J = 0\ ,$$

or by taking the transpose:



$$\tau = -{}^{0}J^{T}\,{}^{0}_{e}F \;.$$

The above equation is a mapping from the m-dimensional task (force) space into the n-dimensional joint (torque) space. Since the Jacobian is configuration dependent, the mapping is also configuration dependent.

NOTE: In the above equation, ${}^{0}_{e}F$ is the set of *external* generalized force applied to the end-effector. In some references, the end-effector generalized force ${}^{0}_{e}F'$ applied to the environment is used in the above equation transforming it into the following:

$$\tau = {}^{0}J^{T}\,{}^{0}_{e}F' \;.$$

NOTE: If the generalized force is expressed in the end-effector frame {e}, then the *end-effector* Jacobian matrix is used for the statics relationship, i.e.,

$$\tau = -{}^{e}J^{T}\,{}^{e}_{e}F \;.$$

Example:

By using the Jacobian formulation of the Stanford manipulator, which was derived in Chapter 5, the equivalent joints torque for the external generalized force on the end-effector is formulated as:

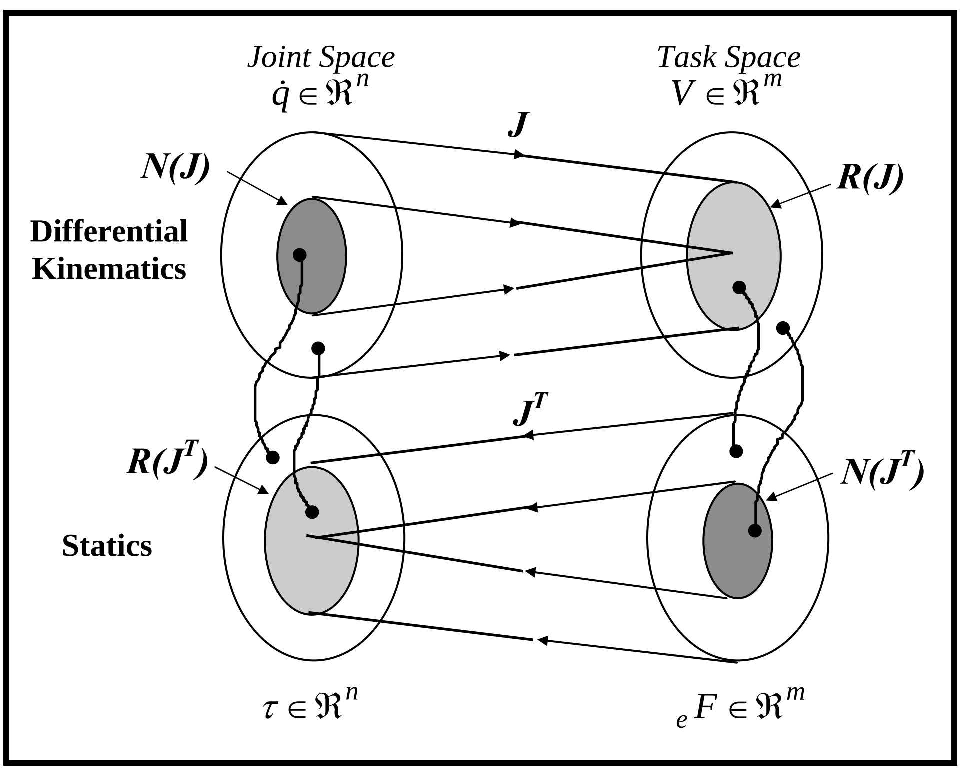
$$\begin{cases}
\tau_1 = \left(-d_3 s1s2 - d_2 c1\right) f_x + \left(-d_3 c1s2 - d_2 s1\right) f_y + g_z \\
\tau_2 = d_3 c1c2\, f_x + d_3 s1c2\, f_y - d_3 s2\, f_z - s1 g_x + c1 g_y \\
f_3 = c1s2\, f_x + s1s2\, f_y + c2\, f_z \\
\tau_4 = c1s2 g_x + s1s2 g_y + c2 g_z \\
\tau_5 = \left(-s1s4 + c1c2c4\right) g_x + \left(c1s4 + s1c2c4\right) g_y - s2c4 g_z \\
\tau_6 = \left(s1c4s5 + c1c2s4s5 + c1s2c5\right) g_x + \left(-c1c4s5 + s1c2s4s5 + s1s2c5\right) g_y + \left(-s2s4s5 + c2c5\right) g_z
\end{cases}$$

6.3 Duality Between Differential Kinematics and Statics

The statics is closely related to the differential kinematics, as the manipulator Jacobian is being used for both mappings. As represented in figure below:

a) In Differential kinematics, the Jacobian is a linear mapping from the *n*-dimensional vector space $\Re^n$ to the m-dimensional space $\Re^m$. Note that n (number of joints) can be more than m in the case of redundant manipulators. The range subspace $R(J)$ represents all possible end-effector velocities that can be generated by the n joint velocities in each configuration. If $n > m$, there exists a null space $N(J)$ of the Jacobian mapping that corresponds to all joint velocity vectors $\dot{q}$ that produce no net velocity at the end-effector. In a singular configuration, J is not full rank, and the subspace $R(J)$ does not cover the entire vector space $\Re^m$, i.e., there exists at least one direction in which the end-effector can not be moved.

b) The statics relationship is also a linear mapping from $\Re^m$ to $\Re^n$ provided by the transpose of the Jacobian. The range subspace $R(J^T)$ and null subspace $N(J^T)$



can be identified from the Jacobian mapping. The null subspace $N(J^T)$ corresponds to the end-effector generalized forces that can be balanced without input torque at the joints, as the load is borne entirely by the structure of the arm linkages. For a redundant manipulator, the range subspace $R(J^T)$ does not cover the entire space $\Re^n$, and there are some sets of input joint torque that can not be balanced by any end-effector generalize force. These configurations correspond to the null space of the kinematics mapping $N(J)$ which contains the joint velocity vectors that produce no end-effector motion.

The duality concept can be summarized as follows:

i) In a singular configuration, there exists at least one direction in which the end-effector can not be moved. In this direction, the end-effector generalized force is entirely balanced by the manipulator structure and does not require any input joint torque or force.

ii) In each configuration of a redundant manipulator, there is at least one direction in which joint velocities produce no end-effector velocity. In this direction, the joint torque and forces can not be balanced by any end-effector generalized force. Therefore, in order to maintain a stationary arm configuration, no input joint torque or force that generates end-effector force should be applied.

iii) For a general manipulator, at each configuration, the directions of possible motion of the end-effector also define the directions in which generalized forces that are applied to the end-effector can be entirely balanced by the joint torque and forces.

6.4 Manipulator Stiffness

The external generalized force applied by the contact surface causes the end-effector to deflect by an amount, which depends on the manipulator stiffness and the applied force. The stiffness of the arm's endpoint determines the strength of manipulator and positioning accuracy in the presence of force and load disturbance. Further, some control algorithms adjust the stiffness to accommodate endpoint forces with acceptable displacements.

For industrial robots, the major source of end-effector deflection occurs in the mechanical transmission mechanism and servo drive system. In some applications such as aerospace, long-link robots are used, which adds the arm flexibility to the sources of compliance.

At each joint, we model the stiffness of the drive system and transmission mechanism by a spring constant k_i that relates the deflection Δq_i at joint i to the applied torque or force τ_i :

$$\tau_i = k_i \Delta q_i \;; \qquad i = 1, 2, ..., n$$

Therefore, the joint stiffness equations can form the following matrix equation:

$$\boldsymbol{\tau} = K\, \Delta\boldsymbol{q}$$

where K is an $n \times n$ diagonal matrix of k_i's. Replacing $\boldsymbol{\tau}$ and $\Delta\boldsymbol{q}$ from the statics and differential kinematics, respectively, yield:

$$J^T \boldsymbol{F} = K\, J^{-1} \Delta\boldsymbol{p}$$

or

$$\Delta\boldsymbol{p} = JK^{-1}\, J^T \boldsymbol{F} = C\, \boldsymbol{F}$$

Matrix C is an $m \times m$ matrix called *compliance matrix*. The compliance matrix is symmetric, and it depends not only on the stiffness of each joint but also on the Jacobian matrix, and hence, the compliance matrix is configuration dependent.

If $m = n$ and the Jacobian is non-singular (square full-rank matrix), the compliance matrix in invertible, and we can have:

$$\boldsymbol{F} = K_e\, \Delta\boldsymbol{p}$$

where

$$K_e = C^{-1} = \left(J^T\right)^{-1} K\ J^{-1}\ .$$

Matrix K_e is called *stiffness matrix*, which is also configuration dependent. At the configuration where the Jacobian is singular the stiffness becomes infinite in at least one direction. The reason is that in this case there exists a null space $N(J^T)$ in which the end-effector generalized force is mapped into zero joint torque. Therefore, any finite end-effector force applied in the direction involved in the null space $N(J^T)$ will cause no joint torque, and hence no joint deflection occurs, and consequently no end-effector deflection happens, so that stiffness is infinite (if links are assumed to be rigid.).

At each configuration, the direction of the end-effector force along which the manipulator deflection is minimum/maximum can be obtained. Let's assume that the applied force has a unit magnitude:

$$\|\boldsymbol{F}\| = \boldsymbol{F}^T \boldsymbol{F} = 1\ .$$

The amount of deflection is calculated as:

$$\|\Delta\boldsymbol{p}\| = \Delta\boldsymbol{p}^T \Delta\boldsymbol{p} = \boldsymbol{F}^T C^T C\boldsymbol{F} = \boldsymbol{F}^T C^2 \boldsymbol{F}\ .$$

To solve the problem, we construct the following Lagrangian with the Lagrange multiplier λ:

$$L = \boldsymbol{F}^T C^2 \boldsymbol{F} - \lambda\left(\boldsymbol{F}^T \boldsymbol{F} - 1\right).$$

The necessary condition to take the extreme values of the end-effector deflection is

$$\begin{cases} \dfrac{\partial L}{\partial \lambda} = 0 \ \Rightarrow\ -\boldsymbol{F}^T \boldsymbol{F} + 1 = 0 \\ \dfrac{\partial L}{\partial \boldsymbol{F}} = 0 \ \Rightarrow\ C^2 \boldsymbol{F} - \lambda\,\boldsymbol{F} = 0 \ \Rightarrow\ C^2 \boldsymbol{F} = \lambda\,\boldsymbol{F} \end{cases}$$

The second equation is an eigenvalue problem in which λ is the eigenvalue of matrix C^2, and we can write:

$$\|\Delta\boldsymbol{p}^*\| = \boldsymbol{F}^T C^2 \boldsymbol{F} = \boldsymbol{F}^T \lambda\,\boldsymbol{F} = \lambda$$

Since $C^2 > 0$, all eigenvalues are positive, and the largest and smallest ones are corresponding to the maximum and minimum arm deflection, respectively. The directions in which the maximum and minimum deflections occur are obtained by the eigenvectors corresponding to the maximum and minimum eigenvalues, respectively. These directions are referred to as *principal directions*. If transformed compliance matrix in *the principal coordinate frame* becomes diagonal. This implies that when the end-effector force is applied in a principal direction, the deflection also occurs in the same principal direction.