1. a). Lagrangian Method:

$$K_1 = \frac{1}{2}(m_1 l_{g1}^2 \dot{\theta}_1^2 + \tilde{I}_1 \dot{\theta}_1^2)$$

$$P_1 = m_1 g l_{g1} \sin\theta_1$$

$$K_2 = \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2}\tilde{I}_2 \dot{\theta}_1^2$$

$$P_2 = m_2 g d_2 \sin\theta_1$$

$$\mathcal{L} = K_1 + K_2 - P_1 - P_2$$

$$= \frac{1}{2}\left[(m_1 l_{g1}^2 + \tilde{I}_1 + m_2 d_2^2 + \tilde{I}_2)\dot{\theta}_1^2 + m_2 \dot{d}_2^2\right]$$

$$- (m_1 l_{g1} + m_2 d_2) g \sin\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = (m_1 l_{g1}^2 + \tilde{I}_1 + m_2 d_2^2 + \tilde{I}_2)\dot{\theta}_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{d}_2} = m_2 \dot{d}_2$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -(m_1 l_{g1} + m_2 d_2) g \cos\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial d_2} = -m_2 g \sin\theta_1 + m_2 \dot{\theta}_1^2 d_2$$

Using Lagrangian equations, we have

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1}$$

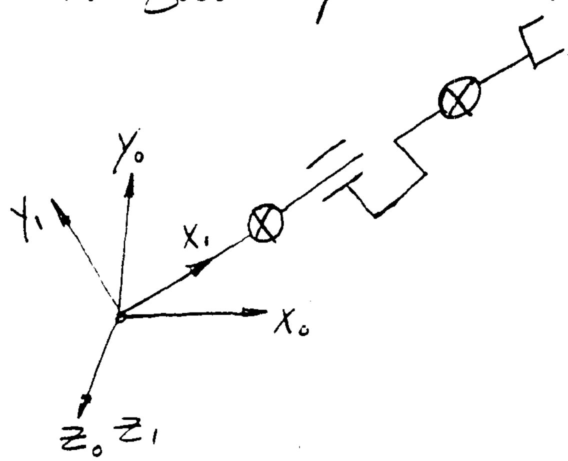
$$= (m_1 l g_1^2 + \tilde{I}_1 + m_2 d_2^2 + \tilde{I}_2)\ddot{\theta}_1 + 2 m_2 d_2 \dot{d}_2 \dot{\theta}_1$$

$$+ (m_1 l g_1 + m_2 d_2) g \cos\theta_1$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{d}_2}\right) - \frac{\partial \mathcal{L}}{\partial d_2}$$

$$= m_2 \ddot{d}_2 + m_2 g \sin\theta_1 - m_2 \dot{\theta}_1^2 d_2$$

b). Newton-Euler Formulation



$$R_{01} = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_{10} = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\omega}_1^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\theta}_1$$

$$\underline{\dot{\omega}}_1^0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ddot{\theta}_1$$

$$\underset{\rightarrow}{V}_1 = \underset{\rightarrow}{\omega}_1 \times \underset{\rightarrow}{r}_1$$

$$= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} lg_1 \cos\theta_1 \\ lg_1 \sin\theta_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -lg_1 \dot{\theta}_1 \sin\theta_1 \\ lg_1 \dot{\theta}_1 \cos\theta_1 \\ 0 \end{pmatrix}$$

$$\dot{\underset{\rightarrow}{V}}_1 = \dot{\underset{\rightarrow}{\omega}}_1 \times \underset{\rightarrow}{r}_1 + \underset{\rightarrow}{\omega}_1 \times (\underset{\rightarrow}{\omega}_1 \times \underset{\rightarrow}{r}_1)$$

$$= \begin{pmatrix} -lg_1 \ddot{\theta}_1 \sin\theta_1 \\ lg_1 \ddot{\theta}_1 \cos\theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -lg_1 \dot{\theta}_1^2 \cos\theta_1 \\ -lg_1 \dot{\theta}_1^2 \sin\theta_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -lg_1 \ddot{\theta}_1 \sin\theta_1 - lg_1 \dot{\theta}_1^2 \cos\theta_1 \\ lg_1 \ddot{\theta}_1 \cos\theta_1 - lg_1 \dot{\theta}_1^2 \sin\theta_1 \\ 0 \end{pmatrix}$$

$$\dot{\underset{\rightarrow}{\omega}}_2 = \dot{\underset{\rightarrow}{\omega}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ddot{\theta}_1$$

$$\underset{\rightarrow}{V}_2 = \underset{\rightarrow}{\omega}_2 \times \underset{\rightarrow}{r}_2 + \dot{d}_2 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -d_2 \dot{\theta}_1 \sin\theta_1 + \dot{d}_2 \cos\theta_1 \\ d_2 \dot{\theta}_1 \cos\theta_1 + \dot{d}_2 \sin\theta_1 \\ 0 \end{pmatrix}$$

$$\dot{\underline{V}}_2 = \begin{pmatrix} -d_2\ddot{\theta}_1\sin\theta_1 - d_2\dot{\theta}_1^2\cos\theta_1 + \ddot{d}_2\cos\theta_1 - 2\dot{d}_2\dot{\theta}_1\sin\theta_1 \\ d_2\ddot{\theta}_1\cos\theta_1 - d_2\dot{\theta}_1^2\sin\theta_1 + \ddot{d}_2\sin\theta_1 + 2\dot{d}_2\dot{\theta}_1\cos\theta_1 \\ 0 \end{pmatrix}$$

$$\tau_2 = R_{10}\, m_2\, \dot{\underline{V}}_2 \big|_x + m_2 g \sin\theta_1$$

$$= \cos\theta_1 \left(-d_2\ddot{\theta}_1\sin\theta_1 - d_2\dot{\theta}_1^2\cos\theta_1 + \ddot{d}_2\cos\theta_1 - 2\dot{d}_2\dot{\theta}_1\sin\theta_1\right) m_2$$

$$+ \sin\theta_1 \left(d_2\ddot{\theta}_1\cos\theta_1 - d_2\dot{\theta}_1^2\sin\theta_1 + \ddot{d}_2\sin\theta_1 + 2\dot{d}_2\dot{\theta}_1\cos\theta_1\right) m_2 + m_2 g\sin\theta_1$$

$$= m_2\ddot{d}_2 + m_2 g\sin\theta_1 - m_2\dot{\theta}_1^2 d_2$$

The same as in (a).

$$\tau_1 = \left[R_{10}\, m_2\, \dot{\underline{V}}_2 \big|_y + m_2 g\cos\theta_1\right] d_2$$

$$+ \tilde{I}_1\ddot{\theta}_1 + \tilde{I}_2\ddot{\theta}_1 + R_{10}\, m_1\, \dot{\underline{V}}_1 \big|_y\, l_{g1}$$

$$+ m_1 g\, l_{g1} \cdot \cos\theta_1$$

$$= -d_2 m_2\sin\theta_1\left(-d_2\ddot{\theta}_1\sin\theta_1 - d_2\dot{\theta}_1^2\cos\theta_1 + \ddot{d}_2\cos\theta_1 - 2\dot{d}_2\dot{\theta}_1\sin\theta_1\right)$$

$$+ d_2 m_2\cos\theta_1\left(d_2\ddot{\theta}_1\cos\theta_1 - d_2\dot{\theta}_1^2\sin\theta_1 + \ddot{d}_2\sin\theta_1 + 2\dot{d}_2\dot{\theta}_1\cos\theta_1\right)$$

$$+ m_2 g d_2\cos\theta_1 + (\tilde{I}_1 + \tilde{I}_2)\ddot{\theta}_1 + m_1 g\, l_{g1}\cos\theta_1$$

$$- \sin\theta_1\left(-l_{g1}\ddot{\theta}_1\sin\theta_1 - l_{g1}\dot{\theta}_1^2\cos\theta_1\right) \cancel{l_{g1}} m_1 +$$

$$\cos\theta_1\left(l_{g1}\ddot{\theta}_1\cos\theta_1 - l_{g1}\dot{\theta}_1^2\sin\theta_1\right) l_{g1} m_1$$

$$= m_2 d_2^2 \ddot{\theta}_1 + 2 m_2 \dot{d}_2 d_2 \dot{\theta}_1 + m_2 g d_2 \cos\theta_1$$

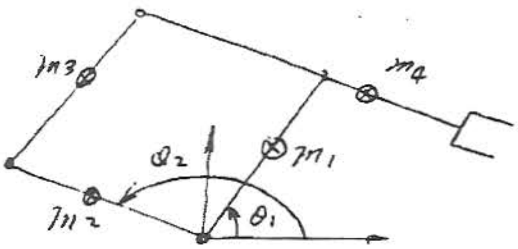
$$+ (\tilde{I}_1 + \tilde{I}_2)\ddot{\theta}_1 + m_1 g l_{g1} \cos\theta_1 + m_1 l_{g1}^2 \ddot{\theta}_1$$

$$= (m_1 l_{g1}^2 + \tilde{I}_1 + m_2 d_2^2 + \tilde{I}_2)\ddot{\theta}_1 + 2 m_2 \dot{d}_2 d_2 \dot{\theta}_1$$

$$+ (m_2 d_2 + m_1 l_{g1}) g \cos\theta_1$$

The same as obtained in (1)

2.



$l_a = l_c = l_1$
$l_b = l_2$

$K_1 = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} m_1 l_{ga}^2 \dot{\theta}_1^2$

$P_1 = m_1 g l_{ga} \sin\theta_1$

$K_2 = \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_{gb}^2 \dot{\theta}_2^2$

$P_2 = m_2 g \, l_{gb} \sin\theta_2$

$x_{c3} = l_2 C_2 + l_{gc} C_1$
$y_{c3} = l_2 S_2 + l_{gc} S_1$

$\dot{x}_{c3} = -l_2 S_2 \cdot \dot{\theta}_2 - l_{gc} S_1 \dot{\theta}_1$

$\dot{y}_{c3} = l_2 C_2 \dot{\theta}_2 + l_{gc} C_1 \dot{\theta}_1$

$v_{c3}^2 = \dot{x}_{c3}^2 + \dot{y}_{c3}^2$

$= l_2^2 \dot{\theta}_2^2 + l_{gc}^2 \dot{\theta}_1^2 - 2 l_2 l_{gc} \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1$

$K_3 = \frac{1}{2} I_3 \dot{\theta}_1^2 + \frac{1}{2} m_3 (l_2^2 \dot{\theta}_2^2 + l_{gc}^2 \dot{\theta}_1^2 - 2 l_2 l_{gc} \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$

$P_3 = m_3 g y_{c3} = m_3 g (l_2 S_2 + l_{gc} S_1)$

Similarly we obtain

$$K_4 = \frac{1}{2} I_4 \dot{\theta}_2^2 + \frac{1}{2} m_4 [l_1^2 \dot{\theta}_1^2 + l_{gd}^2 \dot{\theta}_2^2 - 2 l_1 l_{gd} \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$P_4 = m_4 [l_1 S_1 - l_{gd} S_2] g$$

$$K = K_1 + K_2 + K_3 + K_4$$

$$= \frac{1}{2} [I_1 + m_1 l_{ga}^2 + I_3 + m_3 l_{gc}^2 + m_4 l_1^2] \dot{\theta}_1^2$$

$$+ \frac{1}{2} [I_2 + I_4 + m_3 l_2^2 + m_4 l_{gd}^2 + m_2 l_{gb}^2] \dot{\theta}_2^2$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd}) \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$P = P_1 + P_2 + P_3 + P_4$$

$$= m_1 g l_{ga} \sin\theta_1 + m_2 g l_{gb} \sin\theta_2 + m_3 g (l_2 \sin\theta_2 + l_{gc} \sin\theta_1)$$

$$+ m_4 g (l_1 \sin\theta_1 - l_{gd} \sin\theta_2)$$

$$\mathcal{L} = K - P$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = [I_1 + m_1 l_{ga}^2 + I_3 + m_3 l_{gc}^2 + m_4 l_1^2] \dot{\theta}_1$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd}) \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = -(m_3 l_2 l_{gc} + m_4 l_1 l_{gd}) \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$
$$+ (m_1 l_{ga} + m_3 l_{gc} + m_4 l_1) g \cos\theta_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = [I_2 + I_4 + m_3 l_2^2 + m_4 l_{gd}^2 + m_2 l_{gb}^2] \dot{\theta}_2$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd}) \dot{\theta}_1 \cos(\theta_2 - \theta_1)$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = (m_3 l_2 l_{gc} + m_4 l_1 l_{gd}) \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$
$$+ (m_2 l_{gb} + m_3 l_2 - m_4 l_{gd}) g \cos\theta_2$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1}$$

$$= \left[I_1 + m_1 l_{ga}^2 + I_3 + m_3 l_{gc}^2 + m_4 l_1^2\right]\ddot{\theta}_1$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\ddot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$+ (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\dot{\theta}_2(\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)$$

$$+ (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$

$$- (m_1 l_{ga} + m_3 l_{gc} + m_4 l_1) g \cos\theta_1$$

$$= \left[I_1 + m_1 l_{ga}^2 + I_3 + m_3 l_{gc}^2 + m_4 l_1^2\right]\ddot{\theta}_1$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\ddot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$+ (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\dot{\theta}_2^2 \sin(\theta_2 - \theta_1)$$

$$- (m_1 l_{ga} + m_3 l_{gc} + m_4 l_1) g \cos\theta_1$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2}$$

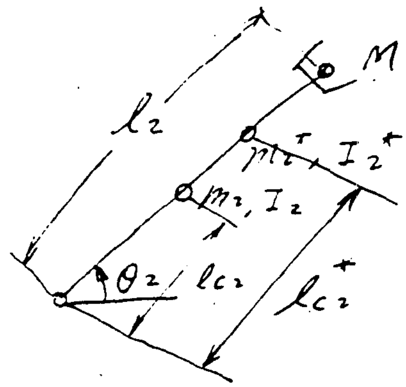
$$= \left[I_2 + I_4 + m_3 l_2^2 + m_4 l_{gd}^2 + m_2 l_{gb}^2\right]\ddot{\theta}_2$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\ddot{\theta}_1 \cos(\theta_2 - \theta_1)$$

$$- (m_3 l_2 l_{gc} + m_4 l_1 l_{gd})\dot{\theta}_2^2 \sin(\theta_2 - \theta_1)$$

$$- (m_2 l_{gb} + m_3 l_2 - m_4 l_{gd}) g \cos\theta_2$$

3.



i) $m_2^* = m_2 + M$

$$m_2 g c_2 (\ell_{c2}^* - \ell_{c2}) C_2 + M g C_2 (\ell_2 - \ell_{c2}^*) C_2 = 0$$

or

$$m_2 (\ell_{c2}^* - \ell_{c2}) + M (\ell_2 - \ell_{c2}^*) = 0$$

or

$$\ell_{c2}^* = \frac{M\ell_2 + m_2 \ell_{c2}}{m + M}$$

$$I_2^* = I_2 + m_2 (\ell_{c2}^* - \ell_{c2})^2 + M (\ell_2 - \ell_{c2}^*)^2$$

ii) Given : $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2, \tau_1, \tau_2$

Find : M

$$\underline{\omega}_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ; \quad \underline{\dot{\omega}}_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ;$$

$$\underline{v}_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} ; \quad \underline{\dot{v}}_0^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{01} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\omega}_1^0 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} ; \quad \underline{\omega}_1^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \leftarrow (R_{10}\, \underline{\omega}_1^0)$$

$$\underline{\omega}_2^1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} ; \quad \underline{\omega}_2^2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \leftarrow (R_{21}\, \underline{\omega}_2^1)$$

$$\underline{\dot{\omega}}_1^0 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} ; \quad \underline{\dot{\omega}}_1^1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$\underline{\dot{\omega}}_2^1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} ; \quad \underline{\dot{\omega}}_2^2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$\underline{\dot{v}}_1^0 = \underline{\dot{v}}_0^0 + \underline{\dot{\tilde{\omega}}}_1^0 \underline{r}_1^0 + (\underline{\tilde{\omega}}_1^0)^2 \underline{r}_1^0$$

But

$$\underline{r}_1^0 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}$$

$$\therefore \quad \underline{\dot{v}}_1^0 = \begin{bmatrix} -\ddot{\theta}_1 l_1 s_1 - \dot{\theta}_1^2 l_1 c_1 \\ g + \ddot{\theta}_1 l_1 c_1 - \dot{\theta}_1^2 l_1 s_1 \\ 0 \end{bmatrix}$$

$$\dot{v}_1^1 = R_{10} \dot{v}_1^0 = \begin{bmatrix} -\dot{\theta}_1^2 l_1 + s_1 g \\ \ddot{\theta}_1 l_1 + c_1 g \\ 0 \end{bmatrix}$$

$$\underline{\dot{v}}_2^1 = \dot{v}_1^1 + \dot{\tilde{\omega}}_2^1 \underline{r}_2^1 + (\underline{\tilde{\omega}}_2^1)^2 \underline{r}_2^1$$

Since

$$\underline{r}_2^1 = \begin{bmatrix} l_2 c_2 \\ l_2 s_2 \\ 0 \end{bmatrix}$$

we have

$$\underline{\dot{v}}_2^1 = \begin{bmatrix} -\dot{\theta}_1^2 l_1 + g s_1 - (\ddot{\theta}_1 + \ddot{\theta}_2) l_2 s_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 l_2 c_2 \\ \ddot{\theta}_1 l_1 + g c_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) l_2 c_2 - (\dot{\theta}_1 + \dot{\theta}_2)^2 l_2 s_2 \\ 0 \end{bmatrix}$$

$$\underline{\dot{v}}_2^{\,2} = R_{21}\ \underline{\dot{v}}_2^{\,1}$$

$$= \begin{bmatrix} S_2 l_1 \ddot{\theta}_1 - l_2(\dot{\theta}_1+\dot{\theta}_2)^2 - c_2 l_1 \dot{\theta}_1^2 + s_{12} g \\ l_2(\ddot{\theta}_1+\ddot{\theta}_2) + l_1 c_2 \ddot{\theta}_1 + \dot{\theta}_1^2 l_1 S_2 + c_{12} g \\ 0 \end{bmatrix}$$

$$\bullet\ \underline{\dot{v}}_1^{\,1} = \begin{bmatrix} -\dot{\theta}_1^2 l_{c_1} + S_1 g \\ \ddot{\theta}_1 l_{c_1} + c_1 g \\ 0 \end{bmatrix}$$

$(\underline{\dot{v}}_{1c}^{1})$

$$\bullet\ \underline{\dot{v}}_2^{\,2} = \begin{bmatrix} s_2 l_1 \ddot{\theta}_1 - l_{c_2}(\dot{\theta}_1+\dot{\theta}_2)^2 - c_2 l_1 \dot{\theta}_1^2 + S_{12} g \\ l_{c_2}(\ddot{\theta}_1+\ddot{\theta}_2) + l_1 c_2 \ddot{\theta}_1 + \dot{\theta}_1^2 l_1 S_2 + c_{12} g \\ 0 \end{bmatrix}$$

$\bullet\ \underline{\dot{v}}_3^{\,3} = \underline{\dot{v}}_2^{\,2}$ since accel. of pt. mass

$\bullet\ F_3^{\,3} = M\ {}^{\bullet}\underline{\dot{v}}_3^{\,3} = M\,\dot{v}_2^{\,2}$ (Here, ${}^{\bullet}F_3^{\,3} = f_{3c}^{\,3}$ and ${}^{\bullet}\underline{\dot{v}}_3^{\,3} = \underline{\dot{v}}_{3c}^{\,3}$)

~~Since~~ Also ${}^{2}\underline{f}_3^{\,3} = {}^{\bullet}\underline{F}_3^{\,3}$

$${}^{\bullet}G_3^{\,3} = {}^{\bullet}I_3^{\,3}\ \dot{\underline{\omega}}_3^{\,3} + \underline{\omega}_3^{\,3}\ {}^{\bullet}I_3^{\,3}\ \underline{\omega}_3^{\,3}$$

Since ${}^{\bullet}I_3^{\,3} = 0$, ${}^{\bullet}G_3^{\,3} = 0$ (Pt. mass is treated here as an addit...

$$ {}^{0}\underline{F}_2^{\,2} = m_2 \, {}^{0}\underline{\dot{v}}_2^{\,2} $$

$$ {}^{0}\underline{G}_2^{\,2} = {}^{0}\underline{I}_2^{\,2} \, \underline{\dot{\omega}}_2^{\,2} + \underline{\tilde{\omega}}_2^{\,2} \, {}^{0}\underline{I}_2^{\,2} \, \underline{\omega}_2^{\,2} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ I_2 \end{bmatrix} (\ddot{\theta}_1 + \ddot{\theta}_2) $$

$$ {}^{1}\underline{g}_2^{\,2} = {}^{0}\underline{G}_2^{\,2} + \underbrace{R_{23} \left({}^{2}\underline{g}_3^{\,3} + \underline{\tilde{r}}_2^{\,3} \, {}^{2}\underline{f}_3^{\,3} \right)} + $$

$$ ({}^{0}\underline{r}_2^{\,2} + \underline{r}_2^{\,2}) \; {}^{0}\underline{F}_2^{\,2} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ (\ddot{\theta}_1 + \ddot{\theta}_2) I_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[0 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_2 \\ 0 & l_2 & 0 \end{bmatrix} \right. $$

$$ \left. M (\underline{\dot{v}}_2^{\,2}) \right] + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_{c_2} \\ 0 & l_{c_2} & 0 \end{bmatrix} m_2 \, {}^{0}\underline{\dot{v}}_2^{\,2} $$

$$ {}^{1}\underline{g}_2^{\,2} = \begin{bmatrix} 0 \\ 0 \\ (\ddot{\theta}_1 + \ddot{\theta}_2) I_2 + m l_2 (\dot{\theta}_1^2 l_1 s_2 + c_{12} g + c_2 \ddot{\theta}_1 l_1 + (\ddot{\theta}_1 + \ddot{\theta} \\ + m_2 l_{c2} (\dot{\theta}_1^2 l_1 s_2 + c_{12} g + c_2 \ddot{\theta}_1 l_1 + l_{c_2} (\ddot{\theta}_1 + \ddot{\theta} \end{bmatrix} $$

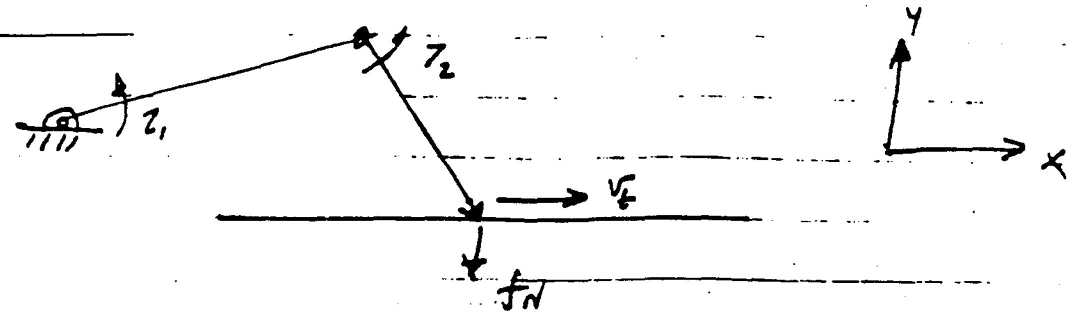
$$ {}^{1}\underline{g}_2^{\,1} = R_{12} \, {}^{1}\underline{g}_2^{\,2} = {}^{1}\underline{g}_2^{\,2} $$

$$\tau_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \, {}^{1}g_2'$$

$\therefore$

$$\therefore \quad \boxed{M = \frac{\tau_2 - (\ddot{\theta}_1 + \ddot{\theta}_2) I_2 - m_2 l_{c2} \left[\dot{\theta}_1 l_1 s_2 + c_{12} g + c_2 \ddot{\theta}_1 l_1 + l_{c2} (\ddot{\theta}_1 + \ddot{\theta}_2) \right]}{l_2 \left(\dot{\theta}_1^2 l_1 s_2 + c_{12} g + c_2 \ddot{\theta}_1 l_1 + l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \right)}}$$

4.



$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

$$\dot{x} = -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) = 0.5 \quad (a)$$
$$\dot{y} = l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) = 0 \quad (b)$$

$$\ddot{x} = l_1 c_1 \dot{\theta}_1^2 - l_1 s_1 \ddot{\theta}_1 - l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 - l_2 s_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) = 0 \quad (c)$$

$$\ddot{y} = -l_1 s_1 \dot{\theta}_1^2 + l_1 c_1 \ddot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2)^2 + l_2 c_{12}(\ddot{\theta}_1 + \ddot{\theta}_2) = 0 \quad (d)$$

From (a) + (b), we can get $\dot{\theta}_1$ + $\dot{\theta}_2$

$$\dot{\theta}_1 = -0.25 \text{ rad/s}, \quad \dot{\theta}_2 = 0.683 \text{ rad/s}$$

Substituting the above in (c) + (d) we can get $\ddot{\theta}_1$, $\ddot{\theta}_2$

$$\ddot{\theta}_1 = -0.1875 \text{ rad/s}^2, \quad \ddot{\theta}_2 = 0.25 \text{ rad/s}^2$$

Inverse Dynamics

To solve for the reqd. joint torques, we ~~can~~ use

$$\underline{\tau} = \cancel{f}\,\tau(q, \dot{q}, \ddot{q})_{\tau} + J^T \underline{f}_N$$

$$J_0 = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 0.366 & 0.866 \\ 1.366 & 0.5 \end{bmatrix}$$

$$\underline{f}_N = \begin{bmatrix} 0 \\ 50 \end{bmatrix}$$

$$J^T \underline{f}_N = \begin{bmatrix} -68.3 \\ -25.0 \end{bmatrix}$$

$$\tau_2 \Big|_{q,\dot{q},\ddot{q}} = (\ddot{\theta}_1 + \ddot{\theta}_2) I_2 + m_2 l_{c2} \left[(\ddot{\theta}_1 c_2 l_1 + (\ddot{\theta}_1 + \ddot{\theta}_2) l_{c2} + \dot{\theta}_1^2 l_1 s_2 + c_{12} g \right]$$

(see the example in the course notes)

Substituting the values we get

$$\tau_2 \Big|_{q,\dot{q},\ddot{q}} = 23.57 \ N.m.$$

Similarly,

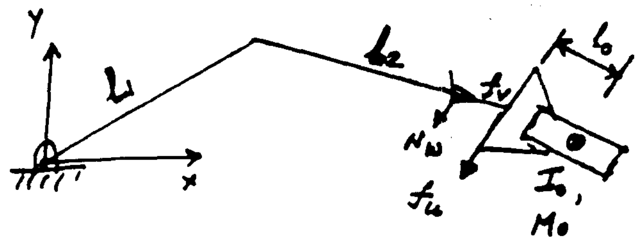
$$\tau_1 = \ddot{\theta}_1 I_1 + \tau_2 + l_1 m_2 \left[l_1 \ddot{\theta}_1 + c_2 l_{c_2} (\ddot{\theta}_1 + \ddot{\theta}_2) - l_{c_2} s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + c_1 g \right] + l_{c_1} m_1 \left(l_{c_1} \ddot{\theta}_1 + c_1 g \right)$$

Or substituting the values in the R.H.S,

$$\tau_1 = 191.46$$

$$\underline{\tau} = \begin{bmatrix} 191.46 \\ 23.46 \end{bmatrix} + \begin{bmatrix} -68.3 \\ -25.0 \end{bmatrix} = \begin{bmatrix} 123.2 \\ -1.4 \end{bmatrix} \ Nm$$

5.



Let f_v, f_u and N_w be the forces & moments sensed by the wrist sensor.

$$X_0 = L_1 c_1 + (L_2 + l_0) c_{12}$$

$$Y_0 = L_1 s_1 + (L_2 + l_0) s_{12}$$

$$\dot{X}_0 = -L_1 s_1 \dot{\theta}_1 - (L_2 + l_0) s_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{Y}_0 = + L_1 c_1 \dot{\theta}_1 + (L_2 + l_0) c_{12} (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\ddot{X}_0 = -L_1 s_1 \ddot{\theta}_1 - L_1 c_1 \dot{\theta}_1^2 - (L_2 + l_0) s_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) - (L_2 + l_0) c_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\ddot{Y}_0 = L_1 c_1 \ddot{\theta}_1 - L_1 s_1 \dot{\theta}_1^2 + (L_2 + l_0) c_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) - (L_2 + l_0) s_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2$$

Now,

$$-f_v c_{12} - f_u s_{12} = M_0 \ddot{X}_0 \qquad (a)$$

$$-f_v s_{12} + f_u c_{12} = M_0 \ddot{Y}_0 \qquad (b)$$

$$-N_w - f_u l_0 = I_0 \ddot{\theta}_0$$

Substituting for $\ddot{X}_0$, $\ddot{Y}_0$, and $\ddot{\theta}_0 = \ddot{\theta}_1 + \ddot{\theta}_2$

we get

$$-f_v C_{12} + f_u S_{12} = M_0 \big(L_1 S_1 \ddot{\theta}_1 - L_1 C_1 \dot{\theta}_1^2 - (L_2 + l_0) S_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) - (L_2 + l_0) C_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 \qquad (a')$$

$$-f_v S_{12} + f_u C_{12} = M_0 \big(L_1 C_1 \ddot{\theta}_1 - L_1 S_1 \dot{\theta}_1^2 + (L_2 + l_0) C_{12} (\ddot{\theta}_1 + \ddot{\theta}_2) - (L_2 + l_0) S_{12} (\dot{\theta}_1 + \dot{\theta}_2)^2 \qquad (b')$$

$$-N_w - f_u l_0 = I_0 (\ddot{\theta}_1 + \ddot{\theta}_2) \qquad (c')$$

From (a') (b') and (c') we can determine l_0, M_0 and I_0 which are the unknowns. (if Joint accelerations are known).