Chapter 4

Inverse Kinematics

Goal

To find the values of joint variables (manipulator configuration) that will place the end-effector at a desired position and orientation relative to the base, given the manipulator geometry (link lengths, offsets, twist angles, and the location of the base).

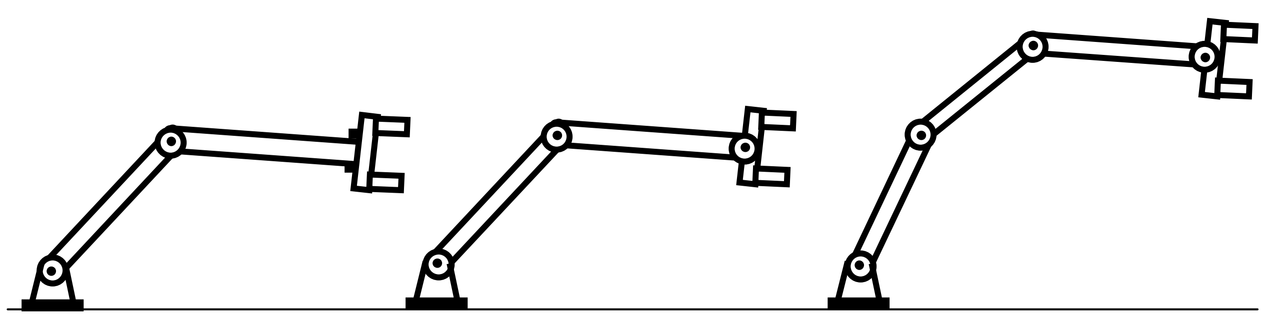
4.1 Solvability

Given the numerical value of the homogeneous transformation matrix ${}^{0}T_{n}$, we attempt to find values of joint variables (θ or d, depending on the type of the joint) q_1, q_2, …, q_n. If the dimension of the task space is m, then there are m independent equations with n unknown joint variables:

a) $m > n$: that is, the number of robot degrees of freedom is not sufficient to provide all possibilities of end-effector position and orientation. Hence, the inverse kinematics problem may not have a solution.

b) $m = n$: that is, there are enough equations to solve for the unknowns. However, these equations are nonlinear and transcendental. Hence, there may be one or more solutions (but finite) to the inverse kinematics problem.

c) $m < n$: that is, there are more degrees of freedom than required to provide the desired end-effector position and orientation. Hence, there are infinite solutions to the inverse kinematics problem. This situation is called ***Redundancy***.

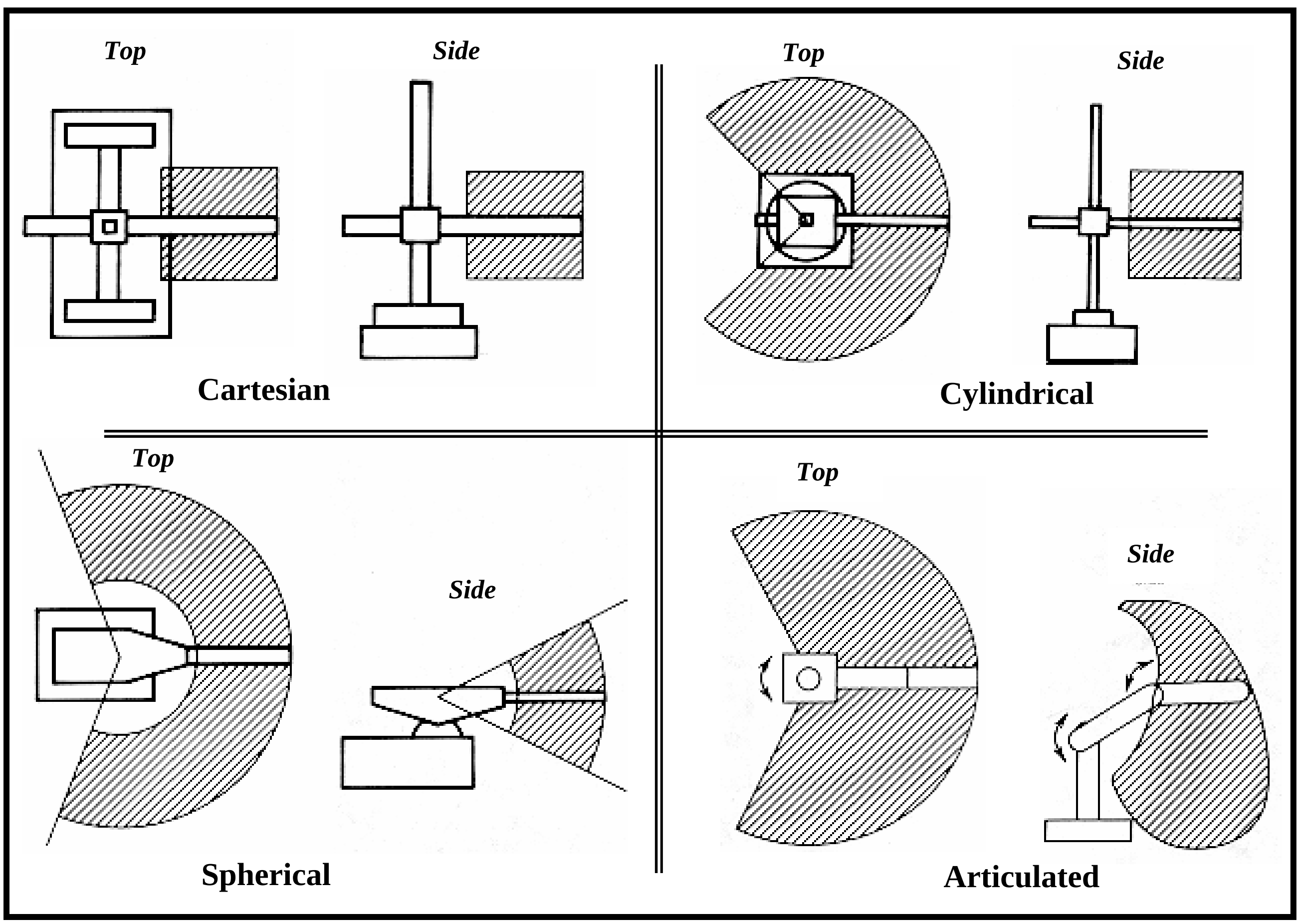


(a) ***m>n*** ***(m=3 , n=2)*** **(b)** ***m=n*** ***(m=3 , n=3)*** **(c)** ***m<n*** ***(m=3 , n=4)***

<u>NOTE</u>: Generally, the solutions of the inverse kinematics problem are not necessarily in the analytical closed form, but must be computed through numerical methods. However, for a 6 d.o.f. manipulator, closed-form solutions are guaranteed if three neighboring joint axes intersect at a point (Pieper's Theorem). This includes manipulators with three consecutive parallel axes, since they meet at a point at infinity. The majority of industrial robots have a spherical wrist attached to the end-effector providing its orientation, and the first three joints assign the wrist point location in the space. Hence, the *end-effector orientation* and the *wrist-point position* problems are de-coupled and can be solved separately.

Manipulator Workspace:

The workspace of a manipulator is defined as the set of all end-effector locations (positions and orientations of the end-effector frame) that can be reached by arbitrary choices of joint variables within the corresponding ranges, regarding all the physical limitations such as link lengths and joint limits. If both end-effector position and



orientation are considered, the workspace is the *complete* or *dexterous workspace*; disregarding the orientation of the end-effector gives the *reachable workspace*, i.e., the volume of space which the manipulator can reach in at least one orientation. Obviously, the dexterous workspace is a subset of reachable workspace.

NOTE: The necessary condition for a solution of inverse kinematics to exist is that the specified goal point must lie within the reachable workspace and $m \le n$.

4.2 The Algebraic Solution

For an n d.o.f. manipulator, the process of solving the inverse kinematics problem is as follows:

STEP 1) Assign the Denavit-Hartenberg parameters and link coordinate frames, and derive the homogenous transformation matrices ${}^{0}T_{1},\ {}^{1}T_{2},\ \ldots,{}^{n-1}T_{n}$, and obtain ${}^{0}T_{n}$ as a function of joint variables. (Forward Kinematics Problem)

STEP 2) Start from the following matrix equation:

$$\left({}^{0}T_{1}\right)^{-1}{}^{0}T_{n}={}^{1}T_{2}\,{}^{2}T_{3}\cdots{}^{n-1}T_{n}\,,$$

and equalize suitable elements of the matrices on both sides of the above equation to reach to "***simple***" trigonometric equations for solving joint variables.

STEP 3) If required, continue to the next equations as follows and repeat step 2, until all joint variables are solved:

$$\left({}^{1}T_{2}\right)^{-1}\left({}^{0}T_{1}\right)^{-1}{}^{0}T_{n}={}^{2}T_{3}\,{}^{3}T_{4}\cdots{}^{n-1}T_{n}$$

$$\vdots$$

$$\left({}^{2}T_{3}\right)^{-1}\left({}^{1}T_{2}\right)^{-1}\left({}^{0}T_{1}\right)^{-1}{}^{0}T_{n}={}^{3}T_{4}\cdots{}^{n-1}T_{n}$$

Simple Trigonemtric Equations

a) $\cos q = A$; (A is constant)

2 solutions: $q = \pm \mathbf{Atan2}\left(\dfrac{\sqrt{1-A^2}}{A}\right)$.

b) $-A\sin q + B\cos q = 0$; (A and B are constant)

2 solutions: $\begin{cases} q_1 = \mathbf{Atan2}\left(\dfrac{B}{A}\right) \\ q_2 = q_1 + 180^o \end{cases}$

NOTE: If both A and B are zero, the joint variable is undefined and one d.o.f. is lost (singular configuration). This is known as a *degeneracy*, and the joint variable is arbitrarily set to zero.

c) $-A\sin q + B\cos q = C$; (A and B and C are constant)

Solution: Define: $\begin{cases} A = r\cos\phi \\ B = r\sin\phi \end{cases}$; where $\begin{cases} r = \sqrt{A^2 + B^2} \\ \phi = \mathbf{Atan2}\left(\dfrac{B}{A}\right) \end{cases}$.

By substitution, obtain: $\begin{cases} \sin(\phi - q) = C/r \\ \cos(\phi - q) = \pm\sqrt{1 - \left(C/r\right)^2} \end{cases}$.

2 solutions: $q = \text{Atan}\,2\left(\dfrac{B}{A}\right) - \text{Atan}\,2\left(\dfrac{C}{\pm\sqrt{r^2 - C^2}}\right)$

d) $\begin{cases} A\cos q_1 + B\cos(q_1 + q_2) + C\sin(q_1 + q_2) = D \\ A\sin q_1 + B\sin(q_1 + q_2) - C\cos(q_1 + q_2) = H \end{cases}$ (A, B, D, H are constant)

Solution: take the square of both equations and add them up:

$$C\,\boldsymbol{sin}\,q_2 + B\,\boldsymbol{cos}\,q_2 = \frac{D^2 + H^2 - A^2 - B^2 - C^2}{2A} = F$$

Define: $\begin{cases} \boldsymbol{cos}\,q_2 = \dfrac{1-t^2}{1+t^2} \\ \boldsymbol{sin}\,q_2 = \dfrac{2t}{1+t^2} \end{cases}$; where $t = \boldsymbol{tan}\dfrac{q_2}{2}$

By substitution, obtain: $(F+B)t^2 - 2Ct + (F-B) = 0$.

Hence,

2 solutions: $$q_2 = 2\,\boldsymbol{Atan2}\left(\frac{C \pm \sqrt{C^2 + B^2 - F^2}}{B+F}\right);$$

Solving for q_1 in the original equations: $\begin{cases} M\,\boldsymbol{cos}\,q_1 - N\,\boldsymbol{sin}\,q_1 = D \\ N\,\boldsymbol{cos}\,q_1 + M\,\boldsymbol{sin}\,q_1 = H \end{cases}$;

where $\begin{cases} M = A + B\,\boldsymbol{cos}\,q_2 + C\,\boldsymbol{sin}\,q_2 \\ N = B\,\boldsymbol{sin}\,q_2 - C\,\boldsymbol{cos}\,q_2 \end{cases}$.

Hence, $\boldsymbol{cos}\,q_1 = \dfrac{DM + HN}{M^2 + N^2}$, and $\boldsymbol{sin}\,q_1 = \dfrac{H - N\,\boldsymbol{cos}\,q_1}{M}$;

And therefore,

$$q_1 = \boldsymbol{Atan2}\left(\frac{\boldsymbol{sin}\,q_1}{\boldsymbol{cos}\,q_1}\right)$$

<u>NOTE</u>: Determining both the *sine* and *cosine* functions of the desired joint variable, and then applying the two-argument *arctangent* function ensures that we have found all solutions, and that the solved variable is in the proper quadrant.

<u>NOTE</u>: Before using the algebraic approach, it is always proper to check the geometrical relations through which the joint variables may be calculated more easily.

4.3 Examples

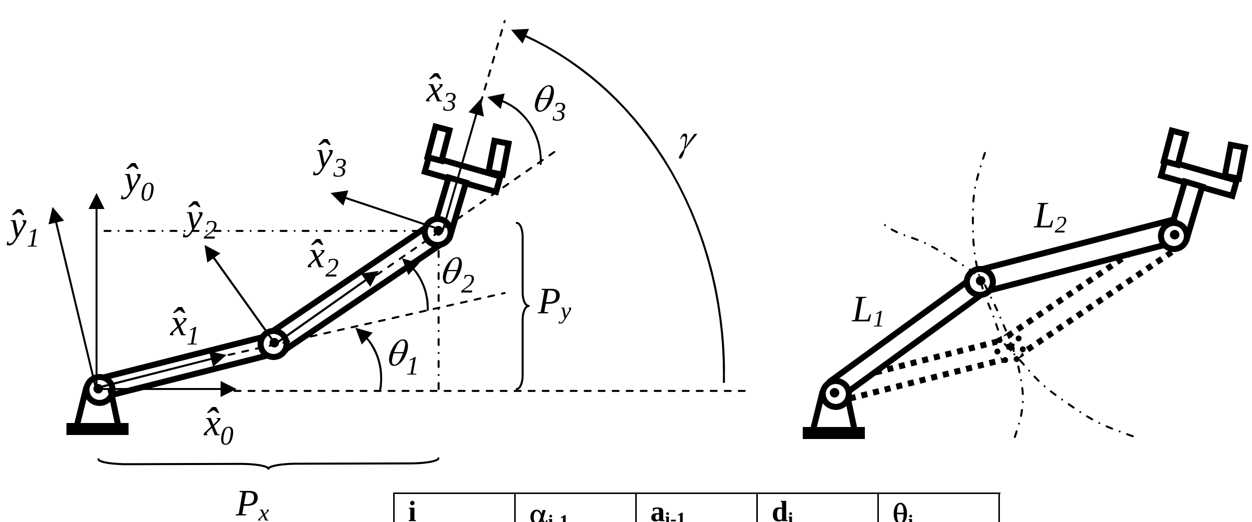
Example 1:

For the 3 d.o.f. planar manipulator shown in the figure, the x-y location (p_x and p_y) and the orientation (γ) of the end-effector is given. Find the corresponding joint angles.

Solution: By inspection of the geometry, the kinematics equations are immediately obtained as follows:

$$\begin{cases} p_x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ p_y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \\ \gamma = \theta_1 + \theta_2 + \theta_3 \end{cases}$$

The first two equations are of type (d) (with $C=0$), and can be solved for θ_1 and θ_2, and θ_3 is obtained next from the third equation. As shown in the figure, two solutions exist for the inverse kinematics problem



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

Example 2: FANUC S-900W Robot

The forward kinematics problem of this robot was solved in Chapter 3, and the position and orientation of the end-effector were formulated as functions of joint variables. Having the end-effector position and orientation, the same equations can be used to solve the inverse kinematics. However, they are highly nonlinear and difficult to solve; hence, numerical methods must be applied. Nevertheless, having a wrist in the system (three consecutive coinciding joint axes) necessitates the existence of a closed-form solution for the inverse kinematics as follows:

a) ***Wrist-Point Position***

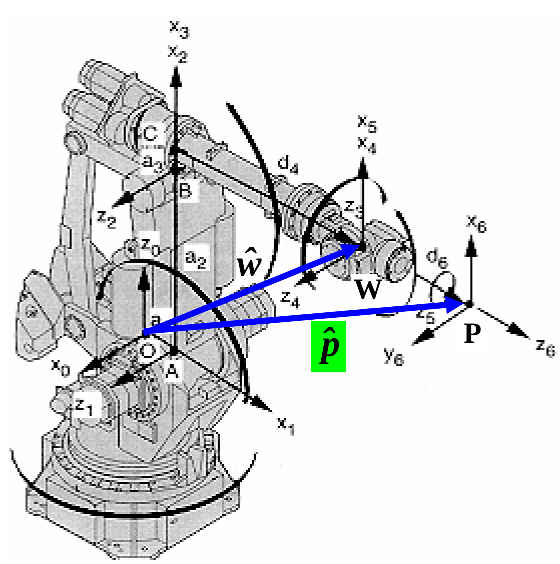
The position of the wrist point does not depend on the rotations of the last three joints. Hence,

$$ {}^{0}w = {}^{0}T_{3}\,{}^{3}\left(\overline{CW}\right) = {}^{0}T_{3}\begin{bmatrix}0 & 0 & d_4 & 1\end{bmatrix}^{T}. $$

On the other hand,

$$ \hat{w} = \hat{p} - \overline{WP} \quad \Rightarrow \quad {}^{0}w = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} - \begin{bmatrix} {}^{0}R_{6}\begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix} \\ 1 \end{bmatrix} = \begin{bmatrix} p_x - d_6 a_x \\ p_y - d_6 a_y \\ p_z - d_6 a_z \\ 1 \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \\ 1 \end{bmatrix} \quad \text{(Known)} $$

By equating the above two equations, three scalar equations will be obtained that can be solved for three unknowns θ_1, θ_2, and θ_3. However, by pre-multiplying by the inverse



of ${}^{0}T_1$ simpler equations can be obtained:

$$\left({}^{0}T_1\right)^{-1}\,{}^{0}w = {}^{1}T_3\,{}^{3}w\,;$$

or

$$\begin{cases} w_x c1 + w_y s1 - a_1 = a_2 c2 + a_3 c23 + d_4 s23 \\ w_z = a_2 s2 + a_3 s23 - d_4 c23 \end{cases} \quad \textbf{\textit{Type (d)}}$$

$$w_x s1 - w_y c1 = 0 \qquad \textbf{\textit{Type (b)}}$$

First, by solving the third equation θ_1 will be obtained:

$$\theta_1^1 = \boldsymbol{Atan2}\left(\frac{w_y}{w_x}\right), \quad \text{and} \quad \theta_1^2 = \theta_1^1 + 180^\circ$$

θ_1^1 is the *front-reach* solution and θ_1^2 is the *back-reach* solution, and due to mechanical constraints the second solution is not feasible for this manipulator.

Next, by solving the first two equations, θ_2 and θ_3 will be obtained:

$$\theta_3 = 2\,\boldsymbol{Atan2}\left(\frac{d_4 \pm \sqrt{d_4^2 + a_3^2 - \left(\dfrac{\left(w_x c1 + w_y s1 - a_1\right)^2 + w_z^2 - a_2^2 - a_3^2 - d_4^2}{2a_2}\right)^2}}{a_3 + \dfrac{\left(w_x c1 + w_y s1 - a_1\right)^2 + w_z^2 - a_2^2 - a_3^2 - d_4^2}{2a_2}}\right)$$

The two solutions of θ_3 are due to the *fully-stretched* and *folded-back* configurations. In case of no real root, the assigned wrist-point position is not reachable. And,

$$\theta_2 = \boldsymbol{Atan2}\left(\frac{\boldsymbol{sin}\,\theta_2}{\boldsymbol{cos}\,\theta_2}\right)$$

where

$$\begin{cases} \boldsymbol{cos}\,\theta_2 = \dfrac{\left(w_x c1 + w_y s1 - a_1\right)\left(a_2 + a_3 c3 + d_4 s3\right) + w_z\left(a_3 s3 - d_4 c3\right)}{\left(a_2 + a_3 c3 + d_4 s3\right)^2 + \left(a_3 s3 - d_4 c3\right)^2} \\ \boldsymbol{sin}\,\theta_2 = \dfrac{w_z - \left(a_3 s3 - d_4 c3\right)\boldsymbol{cos}\,\theta_2}{\left(a_2 + a_3 c3 + d_4 s3\right)} \end{cases}$$

In conclusion, given the wrist-point location, mathematically there are at most 4 possible arm configurations, but due to the mechanical constraints, only two of them are physically possible.

b) ***End-effector Orientation***

By solving the first three joints, ${}^{0}T_{3}$ is known and the forward kinematics equation can be transformed to:

$$ {}^{3}T_{6} = \left({}^{0}T_{3}\right)^{-1}\, {}^{0}T_{6} \ . $$

Equating the $(3,3)$ elements of the above equation yields:

$$ \boldsymbol{cos}\,\theta_5 = \left(a_x c1s23 + a_y s1s23 - a_z c23\right), $$

and hence,

$$ \boldsymbol{sin}\,\theta_5 = \pm\sqrt{1-\left(a_x c1s23 + a_y s1s23 - a_z c23\right)^2} \ . $$

Therefore, in general for each set of θ_1, θ_2, θ_3 two solutions exist for θ_5:

$$ \theta_5 = \pm \boldsymbol{Atan2}\left(\frac{s5}{c5}\right) $$

Next, equating the elements $(1,3)$ and elements $(2,3)$ results:

$$ \begin{cases} \boldsymbol{cos}\,\theta_4 = \dfrac{a_x c1c23 + a_y s1c23 + a_z s23}{s5} \\ \boldsymbol{sin}\,\theta_4 = \dfrac{a_x s1 - a_y c1}{s5} \end{cases} . $$

Hence, corresponding to each solution set of θ_1, θ_2, θ_3, and θ_5 a unique solution of θ_4 can be obtained:

$$ \theta_4 = \boldsymbol{Atan2}\left(\frac{s4}{c4}\right). $$

Similarly, equating the elements $(3,1)$ and elements $(3,2)$ yields:

$$\begin{cases} \boldsymbol{cos}\,\theta_6 = -\dfrac{n_x c1s23 + n_y s1s23 - n_z c23}{s5} \\ \boldsymbol{sin}\,\theta_6 = \dfrac{o_x c1s23 + o_y s1s23 - o_z c23}{s5} \end{cases};$$

and hence, for each set of θ_1, θ_2, θ_3, and θ_5, a unique solution if θ_6 is obtained as:

$$\theta_6 = \boldsymbol{Atan2}\left(\frac{s6}{c6}\right).$$

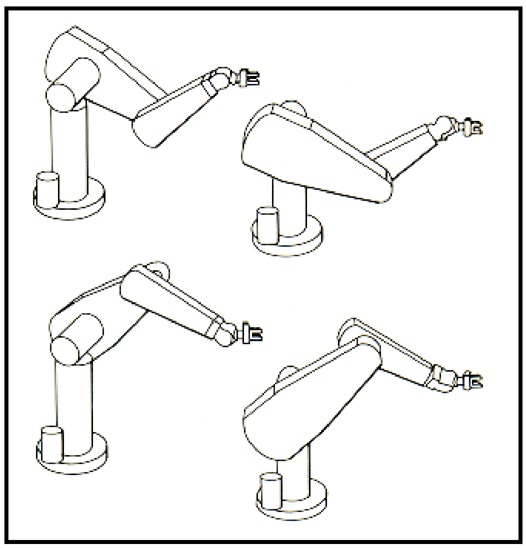
In case $\theta_5 = 0$ or π the sixth joint axis ($\hat{z}_5$) is in the line with the fourth joint axis ($\hat{z}_3$) and hence θ_4 or θ_6 are not independent (degeneracy). In this case, one of them can be arbitrarily set to zero. For example $\theta_4 = 0$ and by θ_6 can be uniquely obtained from elements (1,1) and (2,1) as:

$$\begin{cases} (c1c23)\boldsymbol{cos}\,\theta_6 + s1\,\boldsymbol{sin}\,\theta_6 = n_x \\ (s1c23)\boldsymbol{cos}\,\theta_6 - c1\,\boldsymbol{sin}\,\theta_6 = n_y \end{cases}.$$

In conclusion, corresponding to each solution set of the first three joints, there are two possible wrist configurations. Hence, mathematically a total of eight configurations are possible. However, due to physical limitations, four of them are feasible. When $\theta_5 = 0$ or π, wrist is in singular configuration and only sum or difference of θ_4 and θ_6 can be computed.

Example 3: PUMA 560 Robot

Closed-form solutions exist for the inverse kinematics formulation of the PUMA 560 robot. The algebraic solving process is similar to Example 2. As a result, for each desired end-effector position and orientation inside the workspace, 4 solutions exist as shown in the figure below.



4.4 Repeatability and Accuracy

Industrial robots are usually moved to target points that have been already taught. A *taught point* is the one that the manipulator is moved to physically, and then the joint position sensors are read and recorded. When the robot is commanded to return to that point, each joint is moved to the stored value. In this *teach and playback* process, the inverse kinematics problem does not arise, as the target points are not specified in the task space. The specification of how precisely the robot can return to the taught points, is the ***repeatability*** characteristic of the robot. Manipulators with low joint backlash, friction, and flexibility usually have high repeatability.

For more complicated tasks, the target points are not taught but defined in the task space (*computed points*). For example, the robot is equipped with a vision system for locating a part that the robot must grasp. In these cases, the inverse kinematics problem must be solved to obtain the corresponding joint angles. The precision with which the computed points can be attained illustrates the ***accuracy*** characteristic of the robot. The accuracy of the robot is lower bounded by the repeatability, and depends on how accurately the robot parameters are estimated and how precisely the inverse kinematics problem is solved.