Chapter 5

Differential Kinematics

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Goal

To establish a relationship between joint velocities and end-effector linear and angular velocities.

$$\begin{pmatrix}\text{End-Effector}\\ \text{Linear \& Angular}\\ \text{Velocities}\end{pmatrix} = \begin{pmatrix} \; ? \; \end{pmatrix}\begin{pmatrix}\text{Joint}\\ \text{Velocities}\end{pmatrix}$$

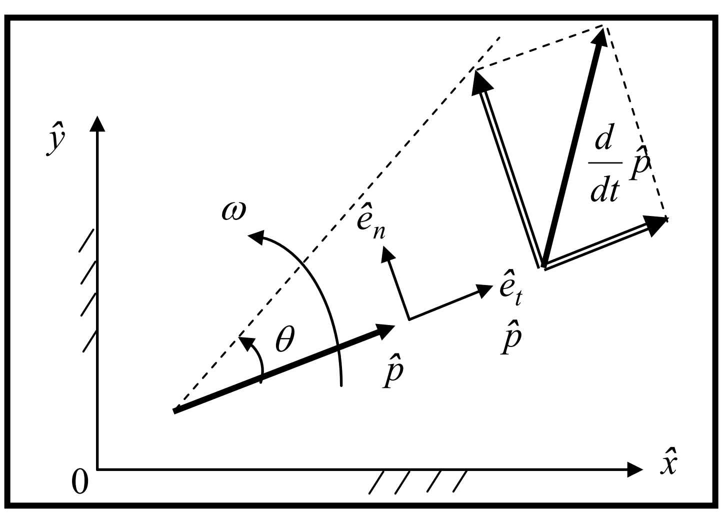
5.1 Differentiation of Vectors

There is a major difference between differentiation of scalars and vectors (non-zero rank tensors) that can be expressed in two ways:

a) Unlike scalars, which have only one property to change (magnitude), vectors possess two properties to change, i.e., their magnitude and their direction. Hence, the change of a vector $\hat{p}$ with respect to a fixed frame $\{0\}$ consists of two components:

$$\frac{d}{dt}\hat{p} = \underbrace{\left(\frac{d}{dt}\|\hat{p}\|\right)}_{\text{due to magnitude}} \hat{e}_t + \underbrace{\left(\frac{d\theta}{dt}\|\hat{p}\|\right)}_{\text{due to direction}} \hat{e}_n$$

$$= \left(\frac{\frac{d}{dt}\|\hat{p}\|}{\|\hat{p}\|}\right)\hat{p} + \hat{\omega} \times \hat{p}$$



b) Unlike scalars, vectors are coordinate-dependent quantities, i.e., their rate of change varies from frame to frame. For instance, if vector $\hat{p}$ is attached to frame $\{\mathrm{B}\}$ and frame $\{\mathrm{B}\}$ (with its attached vector) is moving with respect to frame $\{\mathrm{C}\}$, then

change of $\hat{p}$ with respect to frame $\{\mathrm{B}\}$ $= \dfrac{d_B}{dt}\hat{p} = {}_B\hat{v}_p = \hat{0}$;

change of $\hat{p}$ with respect to frame $\{\mathrm{C}\}$ $= \dfrac{d_C}{dt}\hat{p} = {}_C\hat{v}_p \neq \hat{0}$.

Therefore, in general, two reference frames are needed to interpret the vector differentiation: one *with respect* (*relative*) *to* which the change of the vector is measured (observed), and another *in* which the changed vector is expressed:

${}^A(d_B\hat{p}) = {}^A d_B p \equiv$ Differential change of vector $\hat{p}$ with respect (relative) to frame $\{\mathrm{B}\}$, expressed in frame $\{\mathrm{A}\}$.

NOTE: For convenience, when both frames $\{\mathrm{A}\}$ and $\{\mathrm{B}\}$ are the same in the above notation, the *subscript* is omitted.

NOTE: $d^A p$ or ${}^A\dot{p}$ by itself is *scalar* differentiation of the elements of a 3×1 matrix. However, vector-wise, it means that vector $\hat{p}$ is differentiated with respect to frame $\{\mathrm{A}\}$, i.e., ${}^A(d_A\hat{p})$ or ${}^A dp$.

Relation Between Vector Time Derivative <u>Relative to</u> Different Frames:

$$\frac{d_A\hat{p}}{dt} = \frac{d_B\hat{p}}{dt} + \hat{\omega}_{AB} \times \hat{p} \qquad \text{(Coriolis Theorem)}$$

where $\hat{\omega}_{AB}$ is the vector of the angular velocity of frame $\{\mathrm{B}\}$ with respect to $\{\mathrm{A}\}$.

Relation Between Vector Time Derivative <u>Expressed in</u> Different Frames:

$${}^A\left(\frac{d_C}{dt}\hat{p}\right) = {}^A R_B \left({}^B\left(\frac{d_C}{dt}\hat{p}\right)\right)$$

Hence, the Coriolis Theorem can be stated in the corresponding coordinate frames as:

$$\frac{{}^{A}dp}{dt} = {}^{A}R_{B}\left(\frac{{}^{B}dp}{dt}\right) + {}^{A}\tilde{\omega}_{AB}\,{}^{A}R_{B}\left({}^{B}p\right)$$

where
$$ {}^{A}\tilde{\omega}_{AB} = \begin{bmatrix} 0 & -\omega_{AB}^{z} & \omega_{AB}^{y} \\ \omega_{AB}^{z} & 0 & -\omega_{AB}^{x} \\ -\omega_{AB}^{y} & \omega_{AB}^{x} & 0 \end{bmatrix}.$$

Vector $\hat{\omega}_{AB}$ is the rate of change of the rotation of frame $\{\mathrm{B}\}$ with respect to frame $\{\mathrm{A}\}$, and can be computed from the rotation matrix:

Consider a vector $\hat{q}$ attached to frame $\{\mathrm{B}\}$ while frame $\{\mathrm{B}\}$ is rotating (only rotation) relative to frame $\{\mathrm{A}\}$. The following set of three *algebraic* equations holds:

$${}^{A}q(t) = {}^{A}R_{B}(t)\left({}^{B}q\right).$$

Note that matrices ${}^{A}q$ and ${}^{A}R_{B}$ are functions of time t but ${}^{B}q$ is a constant matrix. *Scalar* differentiation of the above set of equations with respect to time yields:

$$\frac{d}{dt}\left({}^{A}q(t)\right) = {}^{A}\dot{R}_{B}(t)\,{}^{B}q + {}^{A}R_{B}(t)\left(\frac{d}{dt}\left({}^{B}q\right)\right) = {}^{A}\dot{R}_{B}\,{}^{B}q = \left({}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1}\right){}^{A}q\ .$$

On the other hand, vector-wise and according to the Coriolis theorem, since $\frac{{}^{B}dq}{dt} = 0$

we have:
$$\frac{{}^{A}dq}{dt} = {}^{A}R_{B}\left(\frac{{}^{B}dq}{dt}\right) + {}^{A}\tilde{\omega}_{AB}\,{}^{A}q = {}^{A}\tilde{\omega}_{AB}\,{}^{A}q\ .$$

From the above two equations we conclude:

$${}^{A}\tilde{\omega}_{AB} = {}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1} \quad \text{or} \quad {}^{A}\dot{R}_{B} = {}^{A}\tilde{\omega}_{AB}\,{}^{A}R_{B} \qquad \textbf{(*)}$$

or in terms of its components:

$$\begin{cases} \omega_{AB}^{x} = \dot{r}_{31}r_{21} + \dot{r}_{32}r_{22} + \dot{r}_{33}r_{23} \\ \omega_{AB}^{y} = \dot{r}_{11}r_{31} + \dot{r}_{12}r_{32} + \dot{r}_{13}r_{33} \\ \omega_{AB}^{z} = \dot{r}_{21}r_{11} + \dot{r}_{22}r_{12} + \dot{r}_{23}r_{13} \end{cases}$$

NOTE: The physical meaning of the angular velocity vector $\hat{\omega}_{AB}$ is that at any instant there is an *instantaneous screw axis* abut which the rotation occurs and hence the rate of the change of the rotation can be characterized by this axis and a scalar speed of rotation. An infinitesimal rotation of $\{B\}$ with respect to $\{A\}$ results in an infinitesimal change of ${}^{A}R_{B}$ as $d_{A}{}^{A}R_{B}$. This can be viewed as an infinitesimal rotation $d\phi$ about an instantaneous screw axis $\hat{e}$. A differential form of equation (*) can be written as:

$$d_{A}{}^{A}R_{B} = {}^{A}\tilde{\Phi}_{AB}\,{}^{A}R_{B}$$

where

$${}^{A}\tilde{\Phi}_{AB} = \begin{bmatrix} 0 & -\left({}^{A}d\gamma\right)_{z} & \left({}^{A}d\gamma\right)_{y} \\ \left({}^{A}d\gamma\right)_{z} & 0 & -\left({}^{A}d\gamma\right)_{x} \\ -\left({}^{A}d\gamma\right)_{y} & \left({}^{A}d\gamma\right)_{x} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\left({}^{A}e\right)_{z} d\phi & \left({}^{A}e\right)_{y} d\phi \\ \left({}^{A}e\right)_{z} d\phi & 0 & -\left({}^{A}e\right)_{x} d\phi \\ -\left({}^{A}e\right)_{y} d\phi & \left({}^{A}e\right)_{x} d\phi & 0 \end{bmatrix}$$

Note that since ${}^{A}R_{B}$ is the same in both frames then:

$$d_{A}{}^{A}R_{B} = d_{B}{}^{A}R_{B}$$

and also we have

$${}^{A}\tilde{\Phi}_{AB} = {}^{A}R_{B}\,{}^{B}\tilde{\Phi}_{AB}\,{}^{B}R_{A}\ .$$

Therefore,

$$d\,{}^{A}R_{B} = {}^{A}\tilde{\Phi}_{AB}\,{}^{A}R_{B} = {}^{A}R_{B}\,{}^{B}\tilde{\Phi}_{AB}$$

5.2 Differentiation of Homogeneous Transformation Matrices

Since

$$
{}^{A}T_{B} = \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}p_{AB} \\ \hline 0 & 1 \end{array}\right]
$$

then

$$
d_{A}\,{}^{A}T_{B} = \left[\begin{array}{c|c} d_{A}\,{}^{A}R_{B} & {}^{A}\left(d_{A}\hat{p}_{AB}\right) \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} {}^{A}\tilde{\Phi}_{AB}\,{}^{A}R_{B} & {}^{A}R_{B}\,{}^{B}\left(d_{B}\hat{p}_{AB}\right) + {}^{A}\tilde{\Phi}_{AB}\,{}^{A}R_{B}\,{}^{B}p_{AB} \\ \hline 0 & 0 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} {}^{A}\tilde{\Phi}_{AB} & {}^{A}R_{B}\,{}^{B}\left(d_{B}\hat{p}_{AB}\right) \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}R_{B}\,{}^{B}p_{AB} \\ \hline 0 & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} {}^{A}\tilde{\Phi}_{AB} & {}^{A}\left(d_{B}\hat{p}_{AB}\right) \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} {}^{A}R_{B} & {}^{A}p_{AB} \\ \hline 0 & 1 \end{array}\right]
$$

$$
= {}^{A}\Delta_{B}\,{}^{A}T_{B}
$$

Based on the above formulation, the time derivative of the homogeneous transformation matrix is obtained as:

$$
{}^{A}\dot{T}_{B} = {}^{A}\Omega_{B}\,{}^{A}T_{B}
$$

where

$$
{}^{A}\Omega_{B} = {}^{A}\dot{T}_{B}\,{}^{A}T_{B}^{-1} = \left[\begin{array}{c|c} {}^{A}\tilde{\omega}_{AB} & {}^{A}_{B}v_{AB} \\ \hline 0 & 0 \end{array}\right]
$$

where ${}^{A}_{B}v_{AB}$ is the *velocity* of the origin of frame $\{B\}$ from the origin of frame $\{A\}$ (or the time rate of change of $\hat{p}_{AB}$) as observed from (with respect to) frame $\{B\}$ and expressed in frame $\{A\}$.

NOTE:

$$
{}^{A}\dot{T}_{B} = \frac{d_{A}}{dt}\,{}^{A}T_{B}
$$

Example:

Given

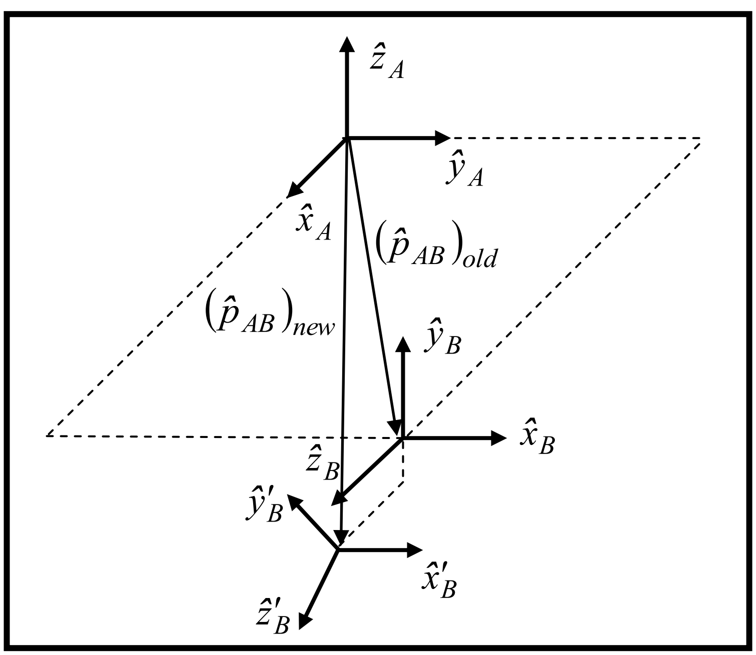
$$ {}^{A}T_{B} = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

Derive the new homogeneous transformation matrix if frame $\{B\}$ has a small rotation 0.1 about $\hat{y}_A$ axis with a small translation -0.5 and 1 along $\hat{z}_A$ and $\hat{x}_A$ axes, respectively.

Solution: The motion is observed from frame $\{A\}$. Therefore, we have

$$ {}^{A}\widetilde{\Phi}_{AB} = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0 \\ -0.1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad {}^{A}d_{A}p_{AB} = \begin{bmatrix} 1 \\ 0 \\ -0.5 \end{bmatrix} $$

By using the Coriolis theorem:



$$ {}^{A}d_{B}p_{AB} = {}^{A}d_{A}p_{AB} + {}^{A}\tilde{\Phi}_{BA}\,{}^{A}p_{AB} = {}^{A}d_{A}p_{AB} - {}^{A}\tilde{\Phi}_{AB}\,{}^{A}p_{AB} = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} $$

Hence,

$$ d_{A}\,{}^{A}T_{B} = {}^{A}\Delta_{B}\,{}^{A}T_{B} = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

$$ \left({}^{A}T_{B}\right)_{new} \cong \left({}^{A}T_{B}\right)_{old} + d_{A}\,{}^{A}T_{B} = \begin{bmatrix} 0 & 0.1 & 1 & 11 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & -0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

5.3 Jacobian

5.3.1 Definition

Consider function $\boldsymbol{f} : \Re^n \rightarrow \Re^m$ defined as:

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}) \;\Leftrightarrow\; \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} f_1(q_1, q_2, \cdots, q_n) \\ f_2(q_1, q_2, \cdots, q_n) \\ \vdots \\ f_m(q_1, q_2, \cdots, q_n) \end{bmatrix}.$$

According to the differential theorem:

$$d\boldsymbol{x} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{f}}{\partial q_i} dq_i \;\Leftrightarrow\; \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \cdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ \vdots \\ dq_n \end{bmatrix} = [J]_{m \times n} \begin{bmatrix} dq_1 \\ dq_2 \\ \vdots \\ dq_n \end{bmatrix}$$

Matrix J is called the *Jacobian matrix* or simply *Jacobian*.

5.3.2 Manipulator Jacobian

For a general spatial n d.o.f. manipulator, the Jacobian matrix represents a linear transformation that maps the velocity components of the joint space into velocity components of the task space:

$$\dot{\boldsymbol{p}} = \begin{bmatrix} \begin{bmatrix} \text{end - effector point linear velocity along } \hat{x}_0 \; (v_x) \\ \text{end - effector point linear velocity along } \hat{y}_0 \; (v_y) \\ \text{end - effector point linear velocity along } \hat{z}_0 \; (v_z) \end{bmatrix} \\ \begin{bmatrix} \text{end - effector angular velocity about } \hat{x}_0 \; (\omega_x) \\ \text{end - effector angular velocity about } \hat{y}_0 \; (\omega_y) \\ \text{end - effector angular velocity about } \hat{z}_0 \; (\omega_z) \end{bmatrix} \end{bmatrix}_{6 \times 1} = [J]_{6 \times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n \times 1} = J(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Hence, Jacobian for the robot manipulators can be generally defined as:

$$J = \begin{bmatrix} \frac{\partial p_x}{\partial q_1} & \frac{\partial p_x}{\partial q_2} & \cdots & \frac{\partial p_x}{\partial q_n} \\ \frac{\partial p_y}{\partial q_1} & \frac{\partial p_y}{\partial q_2} & \cdots & \frac{\partial p_y}{\partial q_n} \\ \frac{\partial p_z}{\partial q_1} & \frac{\partial p_z}{\partial q_2} & \cdots & \frac{\partial p_z}{\partial q_n} \\ \frac{\partial \gamma_x}{\partial q_1} & \frac{\partial \gamma_x}{\partial q_2} & \cdots & \frac{\partial \gamma_x}{\partial q_n} \\ \frac{\partial \gamma_y}{\partial q_1} & \frac{\partial \gamma_y}{\partial q_2} & \cdots & \frac{\partial \gamma_y}{\partial q_n} \\ \frac{\partial \gamma_z}{\partial q_1} & \frac{\partial \gamma_z}{\partial q_2} & \cdots & \frac{\partial \gamma_z}{\partial q_n} \end{bmatrix}$$

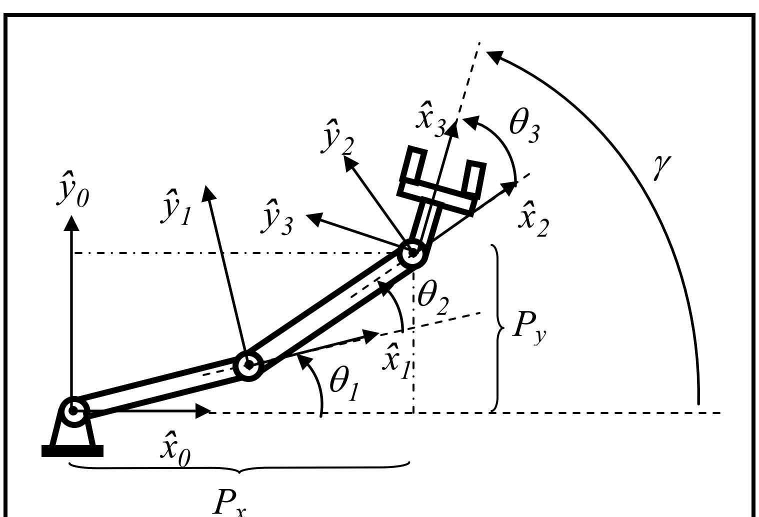
NOTE: The end-effector linear and angular velocities are relative to and expressed in the stationary base frame.

NOTE: The Jacobian matrix is a function of manipulator configuration (joint variables), and varies when the joints move.

NOTE: If the dimension of task space m is less than 6 then the manipulator Jacobian has m rows, as it relates the joint variables to the available components of the end-effector motion.

Example:

The kinematics equations of the 3 d.o.f. manipulator shown in figure below are:



$$\begin{cases} p_x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ p_y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \\ \gamma = \theta_1 + \theta_2 + \theta_3 \end{cases}$$

By differentiating the above equations with respect to the stationary base frame the velocity relationship is obtained:

$$\begin{cases} v_x = \dfrac{dp_x}{dt} = -L_1\left(\dfrac{d\theta_1}{dt}\right)\sin\theta_1 - L_2\left(\dfrac{d\theta_1}{dt} + \dfrac{d\theta_2}{dt}\right)\sin(\theta_1 + \theta_2) \\ v_y = \dfrac{dp_y}{dt} = L_1\left(\dfrac{d\theta_1}{dt}\right)\cos\theta_1 + L_2\left(\dfrac{d\theta_1}{dt} + \dfrac{d\theta_2}{dt}\right)\cos(\theta_1 + \theta_2) \\ \omega_z = \dfrac{d\gamma}{dt} = \dfrac{d\theta_1}{dt} + \dfrac{d\theta_2}{dt} + \dfrac{d\theta_3}{dt} \end{cases}$$

The velocity relationship of the 3 d.o.f. manipulator can be written in the following matrix form:

$$\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} = \overbrace{\begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1+\theta_2) & -L_2 \sin(\theta_1+\theta_2) & 0 \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) & L_2 \cos(\theta_1+\theta_2) & 0 \\ 1 & 1 & 1 \end{bmatrix}}^{\boldsymbol{J}} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}$$

5.3.3 Formulation of the Manipulator Jacobian (Standard DH)

The forward kinematics of a n d.o.f. manipulator is formulated as:

$$ {}^0T_n = {}^0T_1 \, {}^1T_2 \cdots\cdots {}^{i-1}T_i \cdots\cdots {}^{n-2}T_{n-1} \, {}^{n-1}T_n \, .$$

Taking the derivative of the above equation yields:

$$ {}^0\dot T_n = \left({}^0\dot T_1 \, {}^1T_2 \cdots {}^{n-1}T_n\right) + \left({}^0T_1 \, {}^1\dot T_2 \cdots {}^{n-1}T_n\right) + \cdots + \left({}^0T_1 \, {}^1T_2 \cdots {}^{n-1}\dot T_n\right)$$

Post-multiplying the above equation by ${}^0T_n^{-1}$ results in:

$$ {}^0\dot T_n \, {}^0T_n^{-1} = \left({}^0\dot T_1 \, {}^0T_1^{-1}\right) + {}^0T_1\left({}^1\dot T_2 \, {}^1T_2^{-1}\right){}^0T_1^{-1} + \left({}^0T_1 \, {}^1T_2\right)\left({}^2\dot T_3 \, {}^2T_3^{-1}\right)\left({}^0T_1 \, {}^1T_2\right)^{-1} + \cdots$$

or

$$ {}^{0}\Omega_n = {}^{0}\Omega_1 + {}^{0}T_1\,{}^{1}\Omega_2\,{}^{0}T_1^{-1} + {}^{0}T_2\,{}^{2}\Omega_3\,{}^{0}T_2^{-1} + \cdots + {}^{0}T_{n-1}\,{}^{n-1}\Omega_n\,{}^{0}T_{n-1}^{-1} \qquad (**) $$

Based on the DH convention (standard) of defining the link coordinate frames, matrices ${}^{i-1}\Omega_i$ are obtained as follows:

Joint *i* Revolute :

$$ {}^{i-1}\Omega_i^R = \left[\begin{array}{c|c} {}^{i-1}\tilde{\omega}_{(i-1)i} & {}^{i-1}_{\ \ i}v_{(i-1)i} \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{i-1}\tilde{z}_{i-1} & 0 \\ \hline 0 & 0 \end{array}\right] $$

Joint *i* Prismatic :

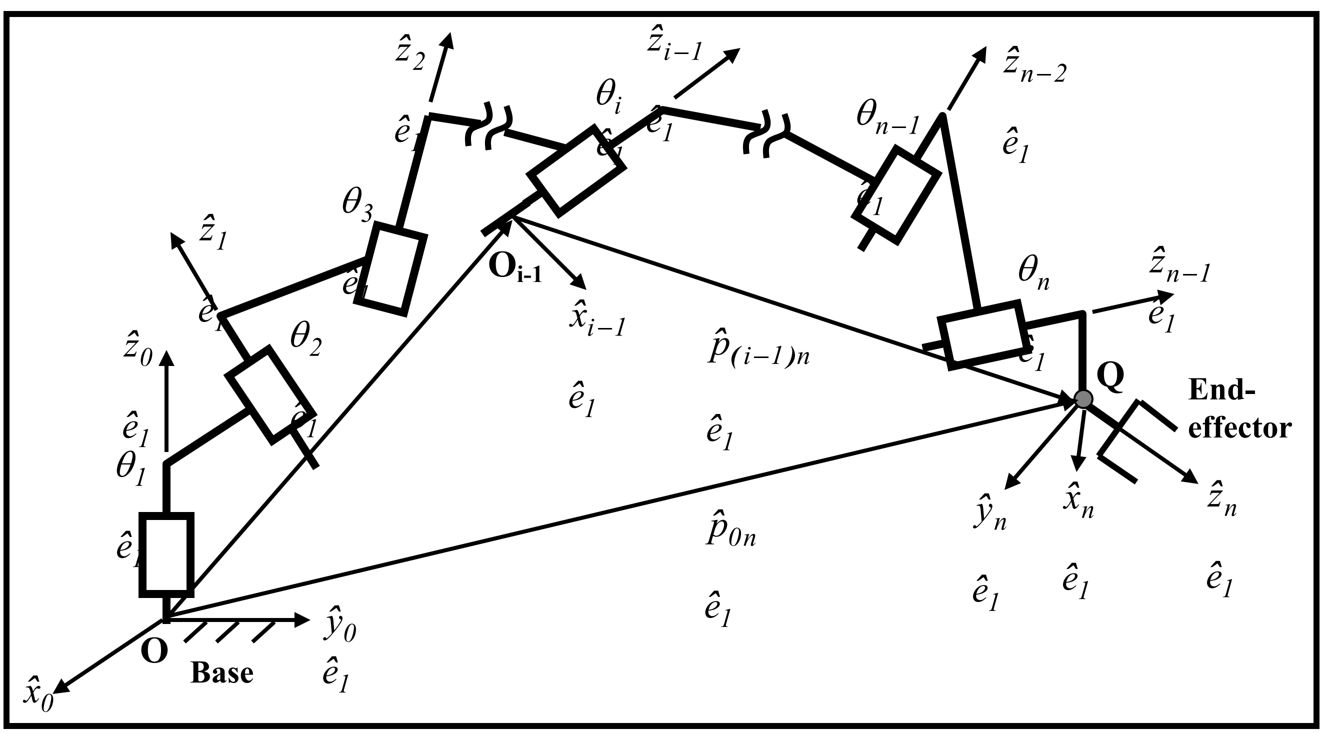
$$ {}^{i-1}\Omega_i^P = \left[\begin{array}{c|c} {}^{i-1}\tilde{\omega}_{(i-1)i} & {}^{i-1}_{\ \ i}v_{(i-1)i} \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & \dot{d}_i\,{}^{i-1}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

A general formulation can be introduced as:

$$ {}^{i-1}\Omega_i = \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{i-1}\tilde{z}_{i-1} & \dot{d}_i\,{}^{i-1}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

where $\dot{d} = 0$ for a revolute joint and $\dot{\theta} = 0$ for a prismatic joint.

Substituting Ω into equation (**) and knowing that



$$ {}^{i-1}T_i = \left[\begin{array}{c|c} {}^{i-1}R_i & {}^{i-1}p_{(i-1)i} \\ \hline 0 & 1 \end{array}\right] $$

will result in:

$$ \left[\begin{array}{c|c} {}^{0}\tilde{\omega}_{0n} & {}^{0}_{n}v_{0n} \\ \hline 0 & 0 \end{array}\right] = \sum_{i=1}^{n} \left[\begin{array}{c|c} \dot{\theta}_i\left({}^{0}R_{i-1}\,{}^{i-1}\tilde{z}_{i-1}\,{}^{0}R_{i-1}^{T}\right) & -\dot{\theta}_i\left({}^{0}R_{i-1}\,{}^{i-1}\tilde{z}_{i-1}\,{}^{0}R_{i-1}^{T}\right){}^{0}p_{0(i-1)} + \dot{d}_i\,{}^{0}R_{i-1}\,{}^{i-1}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

or

$$ \left[\begin{array}{c|c} {}^{0}\tilde{\omega}_{0n} & {}^{0}_{0}v_{0n} - {}^{0}\tilde{\omega}_{0n}\,{}^{0}p_{0n} \\ \hline 0 & 0 \end{array}\right] = \sum_{i=1}^{n} \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1} & -\dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{0(i-1)} + \dot{d}_i\,{}^{0}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

$$ \left[\begin{array}{c|c} {}^{0}\tilde{\omega}_{0n} & {}^{0}_{0}v_{0n} \\ \hline 0 & 0 \end{array}\right] - \left[\begin{array}{c|c} 0 & {}^{0}\tilde{\omega}_{0n}\,{}^{0}p_{0n} \\ \hline 0 & 0 \end{array}\right] = \sum_{i=1}^{n} \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1} & -\dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\left({}^{0}p_{0n} - {}^{0}p_{(i-1)n}\right) + \dot{d}_i\,{}^{0}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

$$ \left[\begin{array}{c|c} {}^{0}\tilde{\omega}_{0n} & {}^{0}_{0}v_{0n} \\ \hline 0 & 0 \end{array}\right] - \sum_{i=1}^{n} \left[\begin{array}{c|c} 0 & {}^{0}\tilde{\omega}_{(i-1)i}\,{}^{0}p_{0n} \\ \hline 0 & 0 \end{array}\right] = \sum_{i=1}^{n} \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1} & \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{(i-1)n} + \dot{d}_i\,{}^{0}z_{i-1} \\ \hline 0 & 0 \end{array}\right] - \sum_{i=1}^{n} \left[\begin{array}{c|c} 0 & \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{0n} \\ \hline 0 & 0 \end{array}\right] $$

Therefore,

$$ \left[\begin{array}{c|c} {}^{0}\tilde{\omega}_{0n} & {}^{0}_{0}v_{0n} \\ \hline 0 & 0 \end{array}\right] = \sum_{i=1}^{n} \left[\begin{array}{c|c} \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1} & \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{(i-1)n} + \dot{d}_i\,{}^{0}z_{i-1} \\ \hline 0 & 0 \end{array}\right] $$

$$ \Rightarrow \begin{cases} {}^{0}_{0}v_{0n} = \displaystyle\sum_{i=1}^{n} \dot{\theta}_i\,{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{(i-1)n} + \dot{d}_i\,{}^{0}z_{i-1} \\ {}^{0}_{0}\omega_{0n} = \displaystyle\sum_{i=1}^{n} \dot{\theta}_i\,{}^{0}z_{i-1} \end{cases} . $$

Since

$$ \left[\begin{array}{c} {}^{0}_{0}v_{0n} \\ \hline {}^{0}_{0}\omega_{0n} \end{array}\right] = {}^{0}J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, $$

then

$$
{}^{0}J = \left[\begin{array}{c:c:c:c:c}
\cdots & \overbrace{{}^{0}\tilde{z}_{i-1}\,{}^{0}p_{(i-1)n}}^{\text{joint } i \text{ revolute}} & \cdots & \overbrace{{}^{0}z_{j-1}}^{\text{joint } j \text{ prismatic}} & \cdots \\
\hdashline
\cdots & {}^{0}z_{i-1} & \cdots & 0 & \cdots
\end{array}\right]
$$

NOTE: ${}^{0}J$ means that the *absolute* (related to the stationary base frame) linear and angular velocities of the end-effector are expressed in the base (inertia) coordinate frame $\{0\}$.

NOTE: The Jacobian matrix can also be expressed in the end-effector coordinate frame $\{n\}$ or $\{e\}$, i.e., the *absolute* linear and angular velocities are expressed in the end-effector frame. Therefore,

$$
\left[\begin{array}{c} {}_{0}^{e}v_{0n} \\ \hdashline {}_{0}^{e}\omega_{0n} \end{array}\right] = {}^{e}J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} .
$$

We can write

$$
\left[\begin{array}{c} {}_{0}^{0}v_{on} \\ \hdashline {}_{0}^{0}\omega_{on} \end{array}\right] = \left[\begin{array}{c} {}^{0}R_e\,{}_{0}^{e}v_{on} \\ \hdashline {}^{0}R_e\,{}_{0}^{e}\omega_{on} \end{array}\right] = \left[\begin{array}{c:c} {}^{0}R_e & 0 \\ \hdashline 0 & {}^{0}R_e \end{array}\right] \left[\begin{array}{c} {}_{0}^{e}v_{on} \\ \hdashline {}_{0}^{e}\omega_{on} \end{array}\right] = \left[\begin{array}{c:c} {}^{0}R_e & 0 \\ \hdashline 0 & {}^{0}R_e \end{array}\right] {}^{e}J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = {}^{0}J \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}
$$

for all $\dot{\boldsymbol{q}}$. Hence,

$$
{}^{0}J = \left[\begin{array}{c:c} {}^{0}R_e & 0 \\ \hdashline 0 & {}^{0}R_e \end{array}\right] {}^{e}J \quad \text{or} \quad {}^{e}J = \left[\begin{array}{c:c} {}^{0}R_e^T & 0 \\ \hdashline 0 & {}^{0}R_e^T \end{array}\right] {}^{0}J ,
$$

and expansion of the above equation yields

$$
{}^{e}J = \left[\begin{array}{c:c:c:c:c}
\cdots & \overbrace{{}^{e}\tilde{z}_{i-1}\,{}^{e}p_{(i-1)n}}^{\text{joint } i \text{ revolute}} & \cdots & \overbrace{{}^{e}z_{j-1}}^{\text{joint } j \text{ prismatic}} & \cdots \\
\hdashline
\cdots & {}^{e}z_{i-1} & \cdots & 0 & \cdots
\end{array}\right]
$$

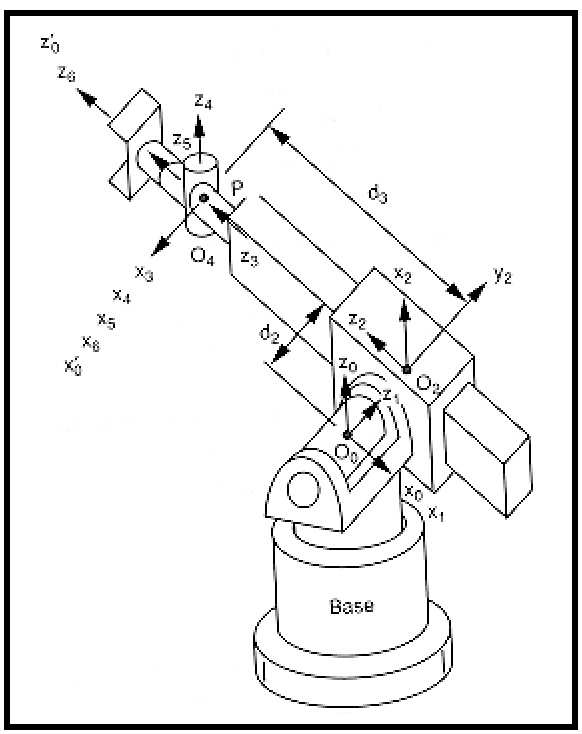
Example: Jacobian of the Stanford Manipulator

In order to simplify the analysis, the base frame is chosen to coincide with frame $\{1\}$ and the last frame (end-effector) is located at the wrist point, eliminating parameters d_1 and d_6. The directions of the joint axes expressed in the base frame are derived as:

$$^0z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$^0z_1 = {}^0R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s1 \\ c1 \\ 0 \end{bmatrix},$$

$$^0z_2 = {}^0z_3 = {}^0R_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c1s2 \\ s1s2 \\ c2 \end{bmatrix},$$



$$ {}^{0}z_4 = {}^{0}R_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s1s4 + c1c2c4 \\ c1s4 + s1c2c4 \\ -s2c4 \end{bmatrix}, $$

$$ {}^{0}z_5 = {}^{0}R_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s1c4s5 + c1c2s4s5 + c1s2c5 \\ -c1c4s5 + s1c2s4s5 + s1s2c5 \\ -s2s4s5 + c2c5 \end{bmatrix}. $$

The position vectors ${}^{0}p_{(i-1)n}$ can be obtained from the following backward recursive formulation:

$$ {}^{0}p_{(i-1)n} = {}^{0}R_{i-1} \, {}^{i-1}p_{(i-1)i} + {}^{0}p_{in} $$

where

$$ {}^{i-1}p_{(i-1)i} = \begin{bmatrix} a_i c\theta_i \\ a_i s\theta_i \\ d_i \end{bmatrix}. $$

Hence,

$$ {}^{0}p_{36} = {}^{0}p_{46} = {}^{0}p_{56} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, $$

$$ {}^{0}p_{26} = \begin{bmatrix} d_3c1s2 \\ d_3s1s2 \\ d_3c2 \end{bmatrix}, $$

$$ {}^{0}p_{16} = {}^{0}p_{06} = \begin{bmatrix} d_3c1s2 - d_2s1 \\ d_3s1s2 + d_2c1 \\ d_3c2 \end{bmatrix}. $$

The Jacobian matrix defined as:

$$\begin{bmatrix} {}^0v_{06}^x & {}^0v_{06}^y & {}^0v_{06}^z & {}^0\omega_{06}^x & {}^0\omega_{06}^y & {}^0\omega_{06}^z \end{bmatrix}^T = {}^0J \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{d}_3 & \dot{\theta}_4 & \dot{\theta}_5 & \dot{\theta}_6 \end{bmatrix}^T$$

is derived as:

$${}^0J = \begin{bmatrix} -d_3s1s2 - d_2c1 & d_3c1c2 & c1s2 & 0 & 0 & 0 \\ -d_3c1s2 - d_2s1 & d_3s1c2 & s1s2 & 0 & 0 & 0 \\ 0 & -d_3s2 & c2 & 0 & 0 & 0 \\ 0 & -s1 & 0 & c1s2 & -s1s4 + c1c2c4 & s1c4s5 + c1c2s4s5 + c1s2c5 \\ 0 & c1 & 0 & s1s2 & c1s4 + s1c2c4 & -c1c4s5 + s1c2s4s5 + s1s2c5 \\ 1 & 0 & 0 & c2 & -s2c4 & -s2s4s5 + c2c5 \end{bmatrix}$$

5.4 Inverse Differential Kinematics

Given the end-effector velocities in the task space (with dimension m), the question is whether there exists a corresponding set of n joint velocities. Similar to the inverse kinematics problem, 3 different cases can be distinguished:

a) $m > n$: that is, the number of robot degrees of freedom is not sufficient to provide all possible end-effector movements. Hence, the inverse differential kinematics problem may not have a solution.

b) $m < n$: that is, there are more degrees of freedom than required to provide the desired end-effector motion. Hence, there are infinite solutions to the inverse differential kinematics problem. This case is called ***Redundancy***.

c) $m = n$: that is, there are enough equations to solve for the unknowns:

$$\dot{\boldsymbol{q}} = J^{-1}(\boldsymbol{q})\,\dot{\boldsymbol{p}}$$

However, at some configurations, the determinant of the Jacobian matrix may become zero, hence the inverse of Jacobian does not exist for that specific configuration, and consequently the inverse differential kinematics does not have any solution.

c.1) $\boldsymbol{det}[J(\boldsymbol{q})] \neq 0$: Unique solution.

c.2) $\boldsymbol{det}[J(\boldsymbol{q} = \boldsymbol{q}_S)] = 0$: No solution. ***Singular Configuration***.

5.4.1 Singularity

At a singular configuration, there is at least one direction in the task space along (about) which it is impossible to move (translate or rotate) the end-effector, no matter which joint velocities are selected. Mathematically, the determinant of the Jacobian matrix becomes zero at a singular configuration, as the matrix is no longer full rank and its column vectors are linearly dependent. Singularity can be categorized into two groups:

a) ***Workspace Boundary Singularities:*** those that occur when the manipulator is fully stretched out or folded back on itself such that the end-effector is at the boundary of its workspace.

b) ***Workspace Interior Singularities:*** those that occur away from the workspace boundary, and generally are caused by two or more joints lining up.

Singular configurations should usually be avoided since most of the manipulators are designed for tasks in which all degrees of freedom are required. Furthermore, near singular configurations, the joint velocities required to maintain the desired end-effector velocity in certain directions may become extremely large. The most common singular configurations for 6 d.o.f. manipulators are listed below:

i) ***Two collinear revolute joint axes*****:** this type occurs in spherical wrist assemblies that have three mutually perpendicular axes intersecting at one point. Rotating the second joint may align the first and third joints, and then the Jacobian will have two linearly dependent columns. Mechanical restrictions are usually imposed on the wrist design to prevent the wrist axes from generating a singularity of the wrist;

ii) ***Three parallel coplanar revolute joint axes*****:** this type occurs, for instance, in an elbow manipulator that consists of a 3 d.o.f. manipulator with a spherical wrist when it is fully extended or fully retracted;

iii) ***Four revolute joint axes intersecting in one point*****;**

iv) ***Four coplanar revolute joints*****;**

v) ***Six revolute joints intersecting along a line*****;**

vi) ***A prismatic joint axis perpendicular to two parallel coplanar revolute joints*****.**

In addition to the Jacobian singularities, the motion of the manipulator is restricted if the joint variables are constrained to a certain interval. In this case, a reduction in the number of degrees of freedom may occur when one or more joints reach the limit of their allowed motion.

Example: The 3 d.o.f. Manipulator

The determinant of the Jacobian matrix for the 3 d.o.f. manipulator equals:

$$\boldsymbol{det}(J) = L_1 L_2 \, \boldsymbol{sin}(\theta_2)$$

So, the singular configuration is when:

$$\theta_2 = 0.0^o \quad \textbf{or} \quad \theta_2 = 180.0^o,$$

which locates the end-effector point at the workspace boundary.

Example: Stanford Manipulator

The necessary and sufficient condition for being the determinant of the Jacobian equal to zero is that

$$d_3^2 s2s5 = 0 \ .$$

The workspace boundary of the wrist point P is determined by the upper and lower limits of the sliding distance along the prismatic joint, i.e., the extreme values of d_3.

a) $s2 = 0$ (*Workspace Boundary*): The arm points either vertically up or down, and the position of the wrist-point is confined on a cylindrical surface of radius d_2.

b) $s5 = 0$ (*Workspace Interior*): The sixth joint axis is in line the fourth one, and the wrist can perform a self-motion with no effect on the orientation of the end-effector.

c) $d_3 = 0$ (*Workspace Boundary*): Any rotation about the second joint axis does not have any effect on the linear velocity of the wrist-point.

Assuming $d_3 \neq 0$:

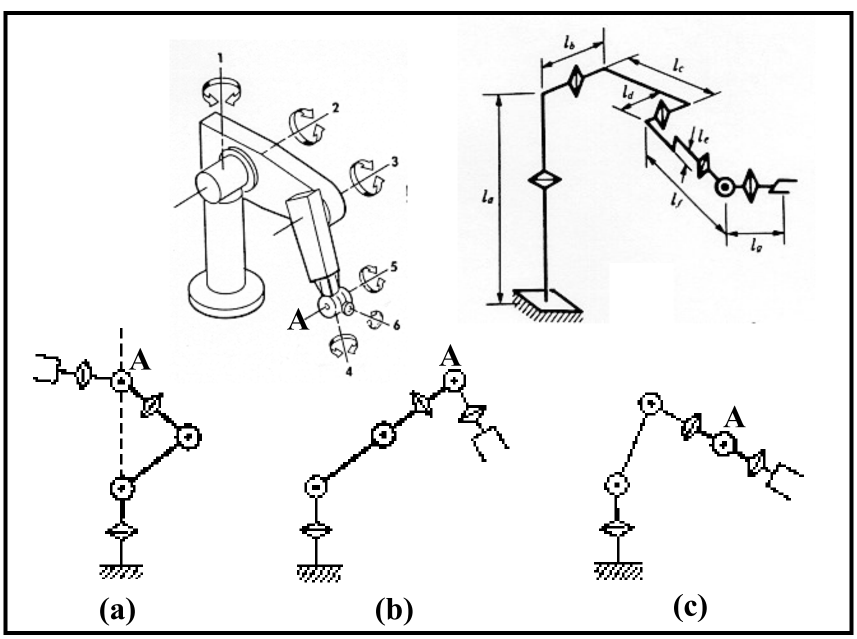
i) $s2$ or $s5 = 0$: *one d.o.f. is lost*;

ii) $s2$ and $s5 = 0$: *two d.o.f. are lost*;

iii) $s2$ and $s5 = 0$, and d_3 is at its extremes : *three d.o.f. are lost*.

Example: PUMA-Type Manipulator, with no offsets $(l_b = l_d = l_e = 0)$

Point A is the wrist point.



a) ***Shoulder Singularity*** (*Workspace Interior*): Joints #1, #2, #5 are lined up. The wrist point A can not move out of the plane.

b) ***Elbow Singularity*** (*Workspace Boundary*): The wrist point A is stretched out.

c) ***Wrist Singularity*** (*Workspace Interior*): Two axes of the wrist are aligned. No end-effector *pitch* is possible.

5.4.2 Redundancy

A *kinematically redundant* manipulator is one that has more than the minimum number of degrees of freedom required to attain a desired motion. In this case, an infinite number of configurations can be obtained for a desired end-effector motion. For a general redundant manipulator with n degrees of freedom $(n > 6)$, the Jacobian is not square, and there are only $(n-6)$ arbitrary variables in the general solution of the inverse differential kinematics mapping, assuming that the Jacobian is full rank. Additional constraints are needed to limit the solution to a unique one. Therefore, redundancy provides the opportunity for choice or decision. It is typically used to optimize some secondary criteria while achieving the primary goal of following a specified end-effector trajectory. Some of the secondary criteria are robot singularity and obstacle avoidance, minimization of joint velocity and torque, increasing the system precision by an optimal distribution of arm compliance, and improving the load carrying capacity by optimizing the transmission ratio between the input torque and output forces. As an example, one common approach is to choose the minimum (in the least squares sense) joint velocities that provide the desired end-effector motion through the following problem definition:

<u>Problem</u>: For a redundant manipulator ($m < n$), Find $\dot{\boldsymbol{q}}$ so that $\dot{\boldsymbol{p}} = J\dot{\boldsymbol{q}}$, while minimizing the following cost function:

$$G(\dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}}^T W \dot{\boldsymbol{q}}$$

where

$$W = W^T > 0 \text{ (positive-definite)},$$

and it can be chosen as the identity matrix.

<u>Solution</u>: Using the Lagrange multiplier λ results in:

$$G'(\dot{\boldsymbol{q}}, \lambda) = \dot{\boldsymbol{q}}^T W \dot{\boldsymbol{q}} - \lambda^T (J\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}}) \; ; \qquad \lambda \in \Re^m$$

Minimum of $G(\dot{\boldsymbol{q}}, \lambda)$ occurs at

$$\frac{\partial G'}{\partial \dot{\boldsymbol{q}}} = 0 \quad \Rightarrow \quad 2W\dot{\boldsymbol{q}} - J^T \lambda = 0 \quad \Rightarrow \quad \dot{\boldsymbol{q}} = \frac{1}{2} W^{-1} J^T \lambda$$

$$\frac{\partial G'}{\partial \lambda} = 0 \quad \Rightarrow \quad J\dot{\boldsymbol{q}} - \dot{\boldsymbol{p}} = 0 \qquad \Rightarrow \quad \dot{\boldsymbol{p}} = \frac{1}{2} J W^{-1} J^T \lambda$$

$JW^{-1}J^T$ is square and full rank. Therefore,

$$\lambda = 2\left(JW^{-1}J^T\right)^{-1} \dot{\boldsymbol{p}} \; ,$$

$$\dot{\boldsymbol{q}} = W^{-1} J^T \left(JW^{-1}J^T\right)^{-1} \dot{\boldsymbol{p}} \; .$$

If $W = I$, then

$$\dot{\boldsymbol{q}} = J^T \left(JJ^T\right)^{-1} \dot{\boldsymbol{p}} \; ,$$

and $J^T \left(JJ^T\right)^{-1}$ is known as *non-weighted pseudo-inverse* of the Jacobian matrix.