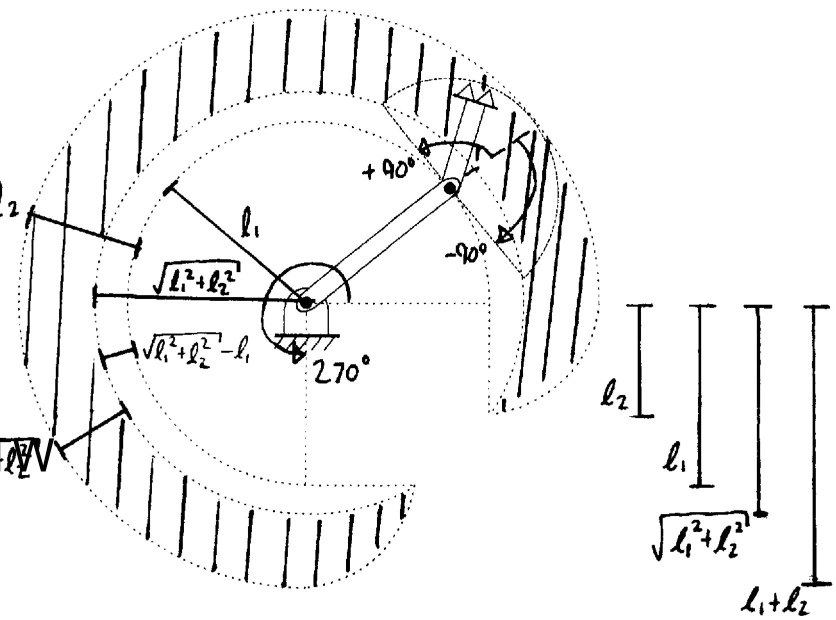
AER 525 Fall 2015

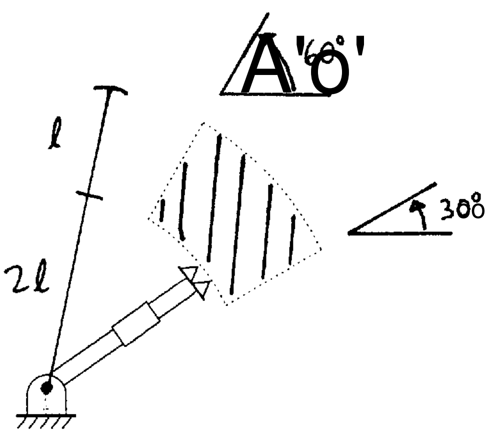
Assignment # 2 - Solutions

Question 1

a)



b)

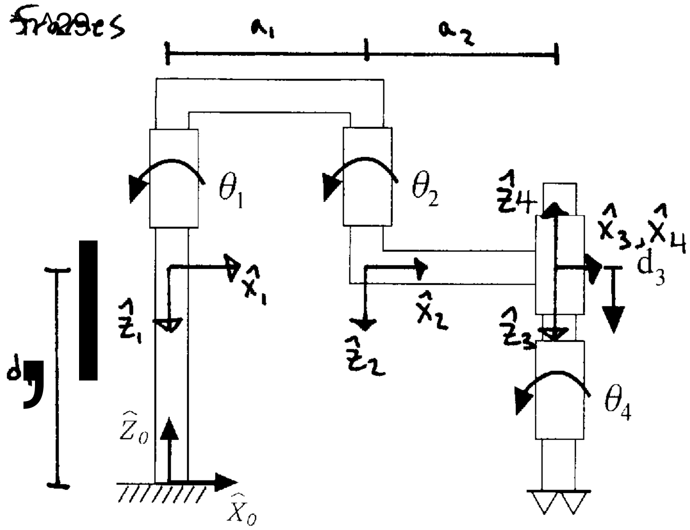


Question 2

- given

$$ {}^B_T T = \begin{bmatrix} c\phi & -s\phi & 0 & x \\ s\phi & c\phi & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- assign frames



- D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	180°	0	d_1	θ_1
2	0	a_1	0	θ_2
3	0	a_2	d_3	0
4	180°	0	0	$-\theta_4$

- link transformation matrices

$$ {}^0_1 T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ -s_1 & -c_1 & 0 & 0 \\ 0 & 0 & -1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1_2 T = \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{3}_{4}T = \begin{bmatrix} c(-\theta_4) & -s(-\theta_4) & 0 & 0 \\ s(-\theta_4) & c(-\theta_4) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

= 2D to 3 transformation matrix

$$
{}^{0}_{4}T = \begin{bmatrix} c(\theta_1+\theta_2-(-\theta_4)) & s(\theta_1+\theta_2-(-\theta_4)) & 0 & a_2 c_{12} + a_1 c_1 \\ -s(\theta_1+\theta_2-(-\theta_4)) & c(\theta_1+\theta_2-(-\theta_4)) & 0 & -a_2 s_{12} - a_1 s_1 \\ 0 & 0 & 1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$\because$ $c(\theta) = c(-\theta)$ and $s(\theta) = -s(-\theta)$

$$
{}^{0}_{4}T = \begin{bmatrix} c(\theta_1+\theta_2-\theta_4) & -s(\theta_1+\theta_2-\theta_4) & 0 & a_2 c_{12} + a_1 c_1 \\ s(\theta_1+\theta_2-\theta_4) & c(\theta_1+\theta_2-\theta_4) & 0 & -a_2 s_{12} - a_1 s_1 \\ 0 & 0 & 1 & d_1 - d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Compare ${}^{B}_{3}T$ to ${}^{0}_{4}T$, we get

from r_{11} → $c\phi = c(\theta_1+\theta_2-\theta_4)$ (2.1)

from p_x → $x = a_2 c_{12} + a_1 c_1$ (2.2)

from p_y → $y = -a_2 s_{12} - a_1 s_1$ (2.3)

from p_z → $z = d_1 - d_3$ (2.4)

- take $(4z),t2)^2 + (z1,3)^2 = X_t^2 + y^2$

$$y^2 + y^2 = a_1^2 + a_2^2 + 2a_1 a_2 c_2$$

$$c_2 := \frac{x_2^2 + y_2^2 - a_1^2 - a_2^2}{2a_1 a_2} \quad \text{where } -1 \le c_2 \le 1$$

$$s_2 := \pm\sqrt{1 - c_2^2}$$

$\therefore \quad \theta_2 := \mathrm{Atan2}(s_2, c_2)$ with two possible solutions

- next, Compare (2,z)) and (2,3))

$$x := a_1 c_1 + a_2(c_1 c_2 - s_1 s_2) = k_1 c_1 - k_2 s_1$$

$$y := a_1 s_1 + a_2(s_1 c_2 + c_1 s_2) = k_1 s_1 + k_2 c_1$$

where $k_1 := a_1 + a_2 c_2$ and $k_2 := a_2 s_2$

- let $r := \sqrt{k_1^2 + k_2^2}$ and $\gamma := \mathrm{Atan2}(k_2, k_1)$

then $k_1 = r\cos\gamma$, $k_2 = r\sin\gamma$ $\Rightarrow$ $x/r = \cos\gamma c_1 - \sin\gamma s_1 = c\gamma_1$, $y/r = \cos\gamma s_1 + \sin\gamma c_1 = s\gamma_1$

$\therefore \quad \gamma + \theta_1 := \mathrm{Atan2}(y/r, x/r) = \mathrm{Atan2}(y, x)$

$$\theta_1 := \mathrm{Atan2}(y, x) - \mathrm{Atan2}(k_2, k_1)$$

- finally, from (2,t))

$$\phi := \theta_1 + \theta_2 + \theta_4$$

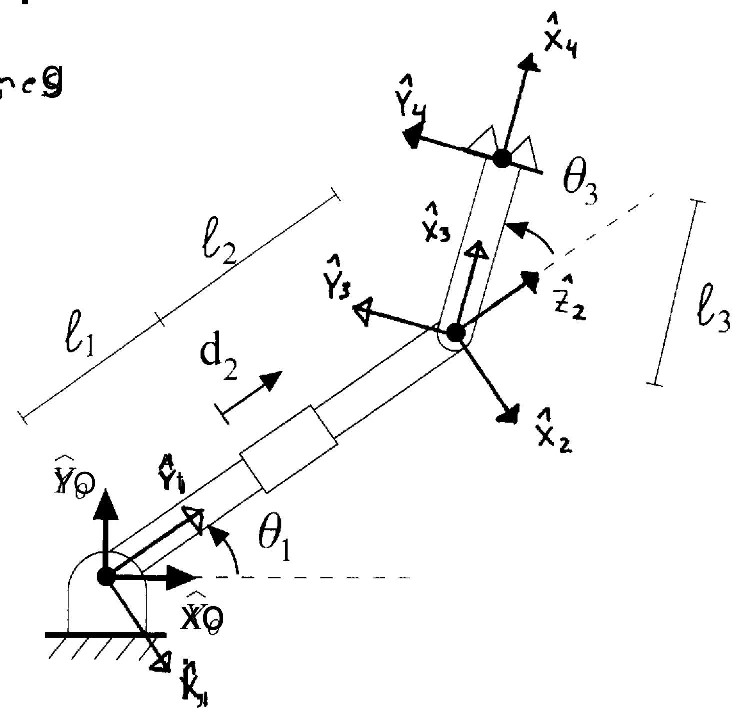
$$\therefore \quad \theta_4 = \phi - \theta_1 - \theta_2$$

ADL = 41

*→ Question 3

a) Find ${}^{B}_{T}T$

- assign frames



↓ D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	$-90° + \theta_1$
2	$-90°$	0	$\ell_1+\ell_2+d_2$	0
3	$90°$	0	0	$90° + \theta_3$
4	0	ℓ_3	0	0

- link transformation matrices

$$ {}^{0}_{1}T = \begin{bmatrix} c(-90+\theta_1) & -s(-90+\theta_1) & 0 & 0 \\ s(-90+\theta_1) & c(-90+\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_1 & c_1 & 0 & 0 \\ -c_1 & s_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$
{}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell_1+\ell_2+d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c(90+\theta_3) & -s(90+\theta_3) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s(90+\theta_3) & c(90+\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ c_3 & -s_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & \ell_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Total {0} to {4} (or {B} to {T}) transformation matrix

$$
{}^{0}_{4}T = {}^{B}_{T}T = \begin{bmatrix} c_{13} & -s_{13} & 0 & (\ell_1+\ell_2+d_2)c_1 + \ell_3 c_{13} \\ s_{13} & c_{13} & 0 & (\ell_1+\ell_2+d_2)s_1 + \ell_3 s_{13} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

b) Find the Jacobian

- We will find the complete Jacobian in the tool frame
- use the recursive link velocity method

- start with

$$ {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- joint 1 revolute

$$ {}^1\omega_1 = {}^1_0R \, \cancel{{}^0\omega_0} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} $$

$$ {}^1v_1 = {}^1_0R \left(\cancel{{}^0v_0} + \cancel{{}^0\omega_0} \times {}^0P_1 \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- joint 2 prismatic

$$ {}^2\omega_2 = {}^2_1R \, {}^1\omega_1 = \begin{bmatrix} [illegible] & 0 & 0 \\ [illegible] & [illegible] & 1 \\ 0 & [illegible] & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot\theta_1 \\ 0 \end{bmatrix} $$

$$ {}^2v_2 = {}^2_1R \left(\cancel{{}^1v_1} + {}^1\omega_1 \times {}^1P_2 \right) + \begin{bmatrix} 0 \\ 0 \\ \dot d_2 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & [illegible] \\ 0 & 0 & [illegible] \end{bmatrix} \left(\begin{bmatrix} 0 & -\dot\theta_1 & [illegible] \\ \dot\theta_1 & 0 & 0 \\ [illegible] & [illegible] & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 + l_2 + d_2 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ \dot d_2 \end{bmatrix} $$

$$ = \begin{bmatrix} -\dot\theta_1 \left(l_1 + (l_2 + d_2) \right) \\ 0 \\ \dot d_2 \end{bmatrix} $$

- joint 3 revolute

$$^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{bmatrix}0\\0\\\dot\theta_3\end{bmatrix} = \begin{bmatrix}-s_3 & 0 & c_3\\ -c_3 & 0 & -s_3\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}0\\-\dot\theta_1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_3\end{bmatrix}$$

$$= \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_3\end{bmatrix}$$

$$^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right)$$

$$= \begin{bmatrix}-s_3 & 0 & c_3\\ -c_3 & 0 & -s_3\\ 0 & -1 & 0\end{bmatrix}\left(\begin{bmatrix}-\dot\theta_1(l_1+l_2+d_2)\\ 0\\ \dot d_2\end{bmatrix}\right) = \begin{bmatrix}s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2\\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2\\ 0\end{bmatrix}$$

- fixed

$$^4\omega_4 = {}^4_3R\,{}^3\omega_3 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_3\end{bmatrix} = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_3\end{bmatrix}$$

$$^4v_4 = {}^4_3R\left({}^3v_3 + {}^3\omega_3 \times {}^3P_4\right)$$

$$= \begin{bmatrix}s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2\\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2\\ 0\end{bmatrix} + \begin{bmatrix}0 & -(\dot\theta_1+\dot\theta_3) & 0\\ \dot\theta_1+\dot\theta_3 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}l_3\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}s_3\dot\theta_1(l_1+l_2+d_2)+c_3\dot d_2\\ c_3\dot\theta_1(l_1+l_2+d_2)-s_3\dot d_2+l_3(\dot\theta_1+\dot\theta_3)\\ 0\end{bmatrix}$$

- therefore the Jacobian is

$$^4J(\theta_1,d_2,\theta_3) = \begin{bmatrix}^4v_4\\ ^4\omega_4\end{bmatrix} = \begin{bmatrix}s_3(l_1+l_2+d_2) & c_3 & 0\\ c_3(l_1+l_2+d_2)+l_3 & -s_3 & l_3\\ 0&0&0\\ 0&0&0\\ 0&0&0\\ 1&0&1\end{bmatrix}$$

c) Find ${}^0v_4$ and ${}^0\omega_4$

First, rewrite the Jacobian in {0}

$$ {}^0J(\theta_1, d_2, \theta_3) = \begin{bmatrix} {}^0_4R & 0_3 \\ 0 & {}^0_4R \end{bmatrix} {}^4J(\theta_1, d_2, \theta_3) $$

$$ = \begin{bmatrix} (c_{13}s_3 - s_{13}c_3)(l_1+l_2+d_2) - s_{13}l_3 & c_{13}c_3 + s_{13}s_3 & +l_3 s_{13} \\ (s_{13}s_3 + c_{13}c_3)(l_1+l_2+d_2) + c_{13}l_3 & s_{13}c_3 - c_{13}s_3 & l_3 c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} -s_1(l_1+l_2+d_2) - s_{13}l_3 & c_1 & -l_3 s_{13} \\ c_1(l_1+l_2+d_2) + c_{13}l_3 & s_1 & l_3 c_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} $$

Since $\begin{bmatrix} {}^0v_4 \\ {}^0\omega_4 \end{bmatrix} = {}^0J(\theta_1, d_2, \theta_3) \begin{bmatrix} \dot\theta_1 \\ \dot d_2 \\ \dot\theta_3 \end{bmatrix}$

Therefore the translational and angular velocities of the tool frame relative to the base frame are

$$ {}^0v_x = [-s_1(l_1+l_2+d_2) - s_{13}l_3]\dot\theta_1 + c_1\dot d_2 - l_3 s_{13}\dot\theta_3 $$

$$ {}^0v_y = [c_1(l_1+l_2+d_2) + c_{13}l_3]\dot\theta_1 + s_1\dot d_2 + l_3 c_{13}\dot\theta_3 $$

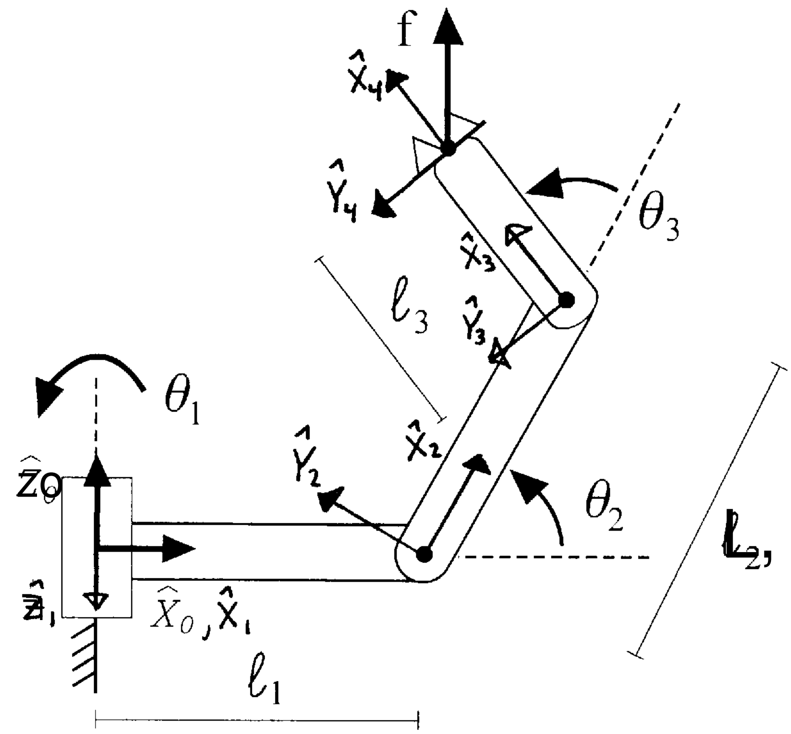
$$ {}^0v_z = {}^0\omega_x = {}^0\omega_y = 0 $$

$$ {}^0\omega_z = \dot\theta_1 + \dot\theta_3 $$

⇒ Question 4

a) Find ${}^{B}_{T}T$

- assign frames



- D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	180°	0	0	θ_1
2	-90°	l_1	0	θ_2
3	0	l_2	0	θ_3
4	0	l_3	0	0

- link transformation matrices

$${}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ -s_1 & -c_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A2-to

$${}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

= {0} to {B} (or {B} to {T}) transformation matrix

$${}^{0}_{4}T = {}^{B}_{T}T = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 & l_3c_1c_{23} + l_2c_1c_2 \\ s_1c_{23} & -s_1s_{23} & -c_1 & l_3s_1c_{23} + l_2s_1c_2 \\ s_{23} & c_{23} & 0 & l_3s_{23} + l_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Find the Jacobian

- we will find the complete Jacobian in the tool frame
- use the recursive link velocity method

= start with

$${}^{0}\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad {}^{0}v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

= joint 1 revolute

$${}^{1}\omega_1 = {}^{1}_{0}R\,{}^{0}\omega_0 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}$$

$${}^{1}v_1 = {}^{1}_{0}R\left({}^{0}v_0 + {}^{0}\omega_0 \times {}^{0}P_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- joint 2 revolute

$$
{}^2\omega_2 = {}^2_1R\,{}^1\omega_1 + \begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix} = \begin{bmatrix}c_2 & 0 & -s_2\\ -s_2 & 0 & -c_2\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix}
$$

$$
= \begin{bmatrix}-s_2\dot\theta_1\\ -c_2\dot\theta_1\\ \dot\theta_2\end{bmatrix}
$$

$$
{}^2v_2 = {}^2_1R\left({}^1v_1 + {}^1\omega_1 \times {}^1P_2\right)
$$

$$
= \begin{bmatrix}c_2 & 0 & -s_2\\ -s_2 & 0 & -c_2\\ 0 & 1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & -\dot\theta_1 & 0\\ \dot\theta_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}l_1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\l_1\dot\theta_1\end{bmatrix}
$$

- joint 3 revolute

$$
{}^3\omega_3 = {}^3_2R\,{}^2\omega_2 + \begin{bmatrix}0\\0\\\dot\theta_3\end{bmatrix} = \begin{bmatrix}c_3 & s_3 & 0\\ -s_3 & c_3 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-s_2\dot\theta_1\\ -c_2\dot\theta_1\\ \dot\theta_2\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_3\end{bmatrix}
$$

$$
= \begin{bmatrix}(-s_2c_3 - c_2s_3)\dot\theta_1\\ (s_2s_3 - c_2c_3)\dot\theta_1\\ \dot\theta_2 + \dot\theta_3\end{bmatrix} = \begin{bmatrix}-s_{23}\dot\theta_1\\ -c_{23}\dot\theta_1\\ \dot\theta_2 + \dot\theta_3\end{bmatrix}
$$

$$
{}^3v_3 = {}^3_2R\left({}^2v_2 + {}^2\omega_2 \times {}^2P_3\right)
$$

$$
= \begin{bmatrix}c_3 & s_3 & 0\\ -s_3 & c_3 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\0\\l_1\dot\theta_1\end{bmatrix} + \begin{bmatrix}0 & -\dot\theta_2 & -c_2\dot\theta_1\\ \dot\theta_2 & 0 & s_2\dot\theta_1\\ c_2\dot\theta_1 & -s_2\dot\theta_1 & 0\end{bmatrix}\begin{bmatrix}l_2\\0\\0\end{bmatrix}\right) = \begin{bmatrix}l_2 s_3\dot\theta_2\\ l_2 c_3\dot\theta_2\\ (l_2c_2 + l_1)\dot\theta_1\end{bmatrix}
$$

$={}^{q}0{}_{0}|$

$$^{4}\omega_4 = {}^{4}_{3}R\,{}^{3}\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_{23}\dot\theta_1 \\ c_{23}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3 \end{bmatrix} = \begin{bmatrix} s_{23}\dot\theta_1 \\ c_{23}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3 \end{bmatrix}$$

$$^{4}v_4 = {}^{4}_{3}R\left({}^{3}v_3 + {}^{3}\omega_3 \times {}^{3}P_4\right)$$

$$= \begin{bmatrix} l_2 s_3\,\dot\theta_2 \\ l_2 c_3\,\dot\theta_2 \\ (l_2 c_2 + l_1)\dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 & -(\dot\theta_2+\dot\theta_3) & c_{23}\dot\theta_1 \\ \dot\theta_2+\dot\theta_3 & 0 & s_{23}\dot\theta_1 \\ -c_{23}\dot\theta_1 & -s_{23}\dot\theta_1 & 0 \end{bmatrix}\begin{bmatrix} l_3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} l_2 s_3\,\dot\theta_2 \\ (l_2 c_3 + l_3)\,\dot\theta_2 + l_3\,\dot\theta_3 \\ (l_2 c_2 + l_3 c_{23} + l_1)\dot\theta_1 \end{bmatrix}$$

= therefore the Jacobian is

$$^{4}J(\theta_1,\theta_2,\theta_3) = \begin{bmatrix} ^{4}v_4 \\ ^{4}\omega_4 \end{bmatrix} = \begin{bmatrix} 0 & l_2 s_3 & 0 \\ 0 & l_2 c_3 + l_3 & l_3 \\ l_1 + l_2 c_2 + l_3 c_{23} & 0 & 0 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

= alternatively, in frame {0} the Jacobian is

$$^{0}J = \begin{bmatrix} ^{0}_{4}R & 0_3 \\ 0_3 & ^{0}_{4}R \end{bmatrix} {}^{4}J = \begin{bmatrix} -s_1(l_1 + c_2 l_2 + c_{23} l_3) & -c_1(s_2 l_2 + s_{23} l_3) & -c_1 s_{23} l_3 \\ c_1(l_1 + c_2 l_2 + c_{23} l_3) & -s_1(s_2 l_2 + s_{23} l_3) & -s_1 s_{23} l_3 \\ 0 & c_2 l_2 + c_{23} l_3 & c_{23} l_3 \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$A2 = \setminus 3$

6) 4 d) Are there any singularities?

- Start by reducing the Jacobian to a square matrix by taking the top three rows

$$J_{3\times3} = \begin{bmatrix} 0 & \ell_2 s_3 & 0 \\ 0 & \ell_2 c_3 + \ell_3 & \ell_3 \\ \ell_1 + \ell_2 c_2 + \ell_3 c_{23} & 0 & 0 \end{bmatrix}$$

- taking the determinant will give the singularity conditions once it is set to zero

$$\det(J_{3\times3}) = \begin{vmatrix} 0 & \ell_2 s_3 & 0 \\ 0 & \ell_2 c_3 + \ell_3 & \ell_3 \\ \ell_1 + \ell_2 c_2 + \ell_3 c_{23} & 0 & 0 \end{vmatrix}$$

$$= -\ell_2 s_3 \left[-\ell_3 \left(\ell_1 + \ell_2 c_2 + \ell_3 c_{23} \right) \right]$$

- therefore singularities occur when $\det(J_{3\times3}) = 0$
- so when

$$s_3 = 0 \qquad (\theta_3)$$

or

$$\ell_1 + \ell_2 c_2 + \ell_3 c_{23} = 0 \qquad (4,2)$$

- in order to distinguish between boundary and interior singularities, more information is needed about the link dimensions.

* Case 1: $\ell_1 > \ell_2 + \ell_3$

(q1): $s_3 = 0 \Rightarrow \theta_3 = 0°$ or $\theta_3 = 180°$

(q2): $\|\ell_1 + \ell_2 c_2 + \ell_3 c_{23}\| \neq 0$ ∴ arm$(q_2) = -1$ & arm$(q_3) = 1$

- If $\ell_2 > \ell_3$, then $\theta_3 = 0°$ and $\theta_3 = 180°$ are both workspace-boundary singularities.

- If $\ell_2 \le \ell_3$, then $\theta_3 = 0°$ is a workspace-boundary singularity and $\theta_3 = 180°$ is a workspace-interior singularity.

* Case 2: $\ell_1 = \ell_2 + \ell_3$

(q1): $s_3 = 0 \Rightarrow \theta_3 = 0°$ or $\theta_3 = 180°$

(q2): $\ell_1 + \ell_2 c_2 + \ell_3 c_{23} = 0$ only when $\theta_2 = 180°$ and $\theta_3 = 0°$

- If $\ell_2 > \ell_3$, then $\theta_3 = 0°$ when $\theta_2 \neq 180°$ are workspace-boundary singularities. Also, $\theta_3 = 0°$ when $\theta_2 = 180°$ is a workspace-interior singularity.

- If $\ell_2 \le \ell_3$, then $\theta_3 = 0°$ when $\theta_2 \neq 180°$ are workspace-boundary singularities. Also, $\theta_3 = 180°$ and $\theta_3 = 0°$ when $\theta_2 = 180°$ are workspace-interior singularities.

Case 3: $l_1 < l_2 + l_3$

(4.1): $s_3 = 0 \rightarrow \theta_3 = 0°$ or $\theta_3 = 180°$

(4.2): $l_1 + l_2 c_2 + l_3 c_{23} = 0$ for many angles

→ special case occurs when $\theta_3 = 0°$

$$l_1 + (l_2 + l_3)\, c_2' = 0 \rightarrow c_2' = \frac{-l_1}{l_2 + l_3}$$

$$\therefore\ s_2' = \pm\sqrt{1 - \frac{l_1^2}{(l_2 + l_3)^2}}$$

then $\theta_2' = \mathrm{Atan2}(s_2', c_2')$

- If $l_2 > l_3$, then $\theta_3 = 0°$ when $\theta_2 \in \{\theta_2'\}$ and $\theta_3 = 180°$ are workspace-boundary singularities. Also, $\theta_3 = 0°$ when $\theta_2 \notin \{\theta_2'\}$ and any other combination of θ_2 and θ_3 which satisfies $l_1 + l_2 c_2 + l_3 c_{23} = 0$ are workspace-interior singularities.

- If $l_2 \le l_3$, then $\theta_3 = 0°$ with $\theta_2 \in \{\theta_2'\}$ are workspace-boundary singularities. Also, $\theta_3 = 180°$ and $\theta_3 = 0°$ when $\theta_2 \notin \{\theta_2'\}$ and any other combination of θ_2 and θ_3 which satisfies $l_1 + l_2 c_2 + l_3 c_{23} = 0$ are workspace-interior singularities.

e) Find the joint torques necessary to keep the manipulator in place as shown in the figure when a force F is applied

- First, describe ^0F in {4}

$^0F = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$, then $^4F = {}^4_0R\,{}^0F = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$ and $^4n_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- using the force/moment recursive equations

- link 3

$$^3f_3 = {}^3_4R\,{}^4f_4 = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix}$$

$$^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_3 \\ 0 & l_3 & 0 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix}$$

- link 2

$$^2f_2 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ 0 \end{bmatrix}$$

$$^2n_2 = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_2 \\ 0 & l_2 & 0 \end{bmatrix}\begin{bmatrix} f_x c_3 - f_y s_3 \\ f_x s_3 + f_y c_3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ (l_3 + c_3 l_2) f_y + (s_3 l_2) f_x \end{bmatrix}$$

- [link 3]

$$^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} f_x c_3 - f_y s_3 \\ 0 \\ f_x s_3 + f_y c_3 \end{bmatrix}$$

$$= \begin{bmatrix} (c_2c_3 - s_2s_3)f_x + (-c_2s_3 - s_2c_3)f_y \\ 0 \\ (s_2c_3 + c_2s_3)f_x + (-s_2s_3 + c_2c_3)f_y \end{bmatrix} = \begin{bmatrix} c_{23} f_x - s_{23} f_y \\ 0 \\ s_{23} f_x + c_{23} f_y \end{bmatrix}$$

$$^1n_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (s_3 l_2) f_x + (l_3 + c_3 l_2) f_y \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -l_1 \\ 0 & l_1 & 0 \end{bmatrix} \begin{bmatrix} c_{23} f_x - s_{23} f_y \\ 0 \\ s_{23} f_x + c_{23} f_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ l_1 (s_{23} f_x + c_{23} f_y) \\ 0 \end{bmatrix}$$

* finding the joint torques

$$\tau_1 = {}^1n_1^T \, {}^1\hat{Z}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l_1 (s_{23} f_x + c_{23} f_y) \\ 0 \end{bmatrix} = 0$$

$$\tau_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ (l_3 + c_3 l_2) f_y + (s_3 l_2) f_x \end{bmatrix}$$

$$= (l_3 + c_3 l_2) f_y + (s_3 l_2) f_x$$

$$\tau_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l_3 f_y \end{bmatrix} = l_3 f_y$$