AER 525 Fall 2015

Assignment 1 - Solutions

→ Question 1

In general, for a matrix to be a rotation matrix it must:

- be square
- have columns/rows which are unit vectors
- have columns/rows which are mutually orthogonal
- have a determinant of magnitude one
- be invertible where its inverse is equal to its transpose

→ R_1 is valid

→ R_2 is not valid - columns/rows are not unit vectors
- the $|\det(R_2)| \neq 1$
- $R_2^{-1} \neq R_2^T$

→ R_3 is valid

→ R_4 is not valid - columns/rows are not unit vectors
- the $|\det(R_4)| \neq 1$
- $R_4^{-1} \neq R_4^T$

→ R_5 is not valid - it is not square (cannot invert)

→ R_6 is valid

Question 2

- Find the transformation between each frame

$$ {}^{A}_{B}T = R_X(15°) = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c(15) & -s(15) & 0 \\ 0 & s(15) & c(15) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{B}_{C}T = R_Z(10°)D_Y(6) = \left[\begin{array}{c|c} {}^{B}_{C}R & {}^{B}P_{CORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \begin{bmatrix} c(10) & -s(10) & 0 & 0 \\ s(10) & c(10) & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{C}_{D}T = D_Z(3) = \left[\begin{array}{c|c} {}^{C}_{D}R & {}^{C}P_{DORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Next, either multiple the transformations together or compound the nontrivial components as in Chapter 2:

$$ {}^{A}_{D}T = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T = \left[\begin{array}{c|c} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}_{B}R\,{}^{B}P_{CORG} + {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}P_{DORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right] $$

$$ = \begin{bmatrix} 0.9848 & -0.1736 & 0 & 0 \\ 0.1677 & 0.9513 & -0.2588 & 5.0191 \\ 0.0449 & 0.2549 & 0.9659 & 4.4507 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Use this transformation to find ${}^{A}P$:

$$ {}^{A}P = {}^{A}_{D}T\,{}^{D}P = \begin{bmatrix} 1.9696 \\ 4.8369 \\ 6.4724 \end{bmatrix} $$

→ Question 3

An orthogonal matrix has the interesting property that its inverse is equal to its transpose (it must be square to be orthogonal!).

Start by setting the product of our given matrices equal to a matrix K:

$$K = R_1 R_2 \dots R_{N-1} R_N \qquad (3.1)$$

Recall that

$$R_1 R_1^T = I, \quad R_2 R_2^T = I, \dots \quad R_N R_N^T = I$$

Take (3.1) and post-multiply both sides by the transpose of the last orthogonal matrix on the right-hand side. Repeat.

$$\begin{aligned} K &= R_1 R_2 \dots R_{N-1} R_N \\ K R_N^T &= R_1 R_2 \dots R_{N-1} R_N R_N^T \\ K R_N^T &= R_1 R_2 \dots R_{N-1} \\ K R_N^T R_{N-1}^T &= R_1 R_2 \dots R_{N-1} R_{N-1}^T \\ &\vdots \\ K R_N^T R_{N-1}^T \dots R_2^T R_1^T &= I \end{aligned}$$

We can see that $K^T = R_N^T R_{N-1}^T \dots R_2^T R_1^T$ is also an orthogonal matrix ($K^T = K^{-1}$), therefore the product of orthogonal matrices is also an orthogonal matrix

→ Question 4

We seek to find the equivalent angle-axis representation of $\{B\}$ rotated relative to $\{A\}$

→ Start by finding ${}^{A}_{B}R$

We know ${}^{A}G$ lies along ${}^{A}\hat{X}_B$ and ${}^{A}H$ lies along ${}^{A}\hat{Z}_B$, finding ${}^{A}\hat{X}_B$ and ${}^{A}\hat{Z}_B$ is done by finding the unit vectors in those directions.

$$ {}^{A}\hat{X}_B = \frac{1}{|{}^{A}G|}\,{}^{A}G = \frac{1}{\sqrt{2^2+1^2+2^2}}\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} $$

$$ {}^{A}\hat{Z}_B = \frac{1}{|{}^{A}H|}\,{}^{A}H = \frac{1}{\sqrt{(-1)^2+(-2)^2+2^2}}\begin{bmatrix}-1\\-2\\2\end{bmatrix} = \begin{bmatrix}-1/3\\-2/3\\2/3\end{bmatrix} $$

To find ${}^{A}\hat{Y}_B$, we find a unit vector mutually orthogonal to ${}^{A}\hat{X}_B$ and ${}^{A}\hat{Z}_B$. From linear algebra, the cross product achieves this.

$$ {}^{A}\hat{Y}_B = {}^{A}\hat{Z}_B \times {}^{A}\hat{X}_B = \begin{bmatrix}Z_x\\Z_y\\Z_z\end{bmatrix} \times \begin{bmatrix}X_x\\X_y\\X_z\end{bmatrix} = \begin{bmatrix}0 & -Z_z & Z_y\\ Z_z & 0 & -Z_x\\ -Z_y & Z_x & 0\end{bmatrix}\begin{bmatrix}X_x\\X_y\\X_z\end{bmatrix} $$

$$ = \begin{bmatrix}0 & -2/3 & -2/3\\ 2/3 & 0 & 1/3\\ 2/3 & -1/3 & 0\end{bmatrix}\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix} = \begin{bmatrix}-2/9-4/9\\ 4/9+2/9\\ 4/9-1/9\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix} $$

You can verify that they are mutually orthogonal using the dot product $({}^{A}\hat{X}_B \cdot {}^{A}\hat{Y}_B = 0,\ {}^{A}\hat{X}_B \cdot {}^{A}\hat{Z}_B = 0, \dots)$

From chapter 2, we know

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} $$

Also from Chapter 2, we know the representation of angle-axis from a rotation matrix is

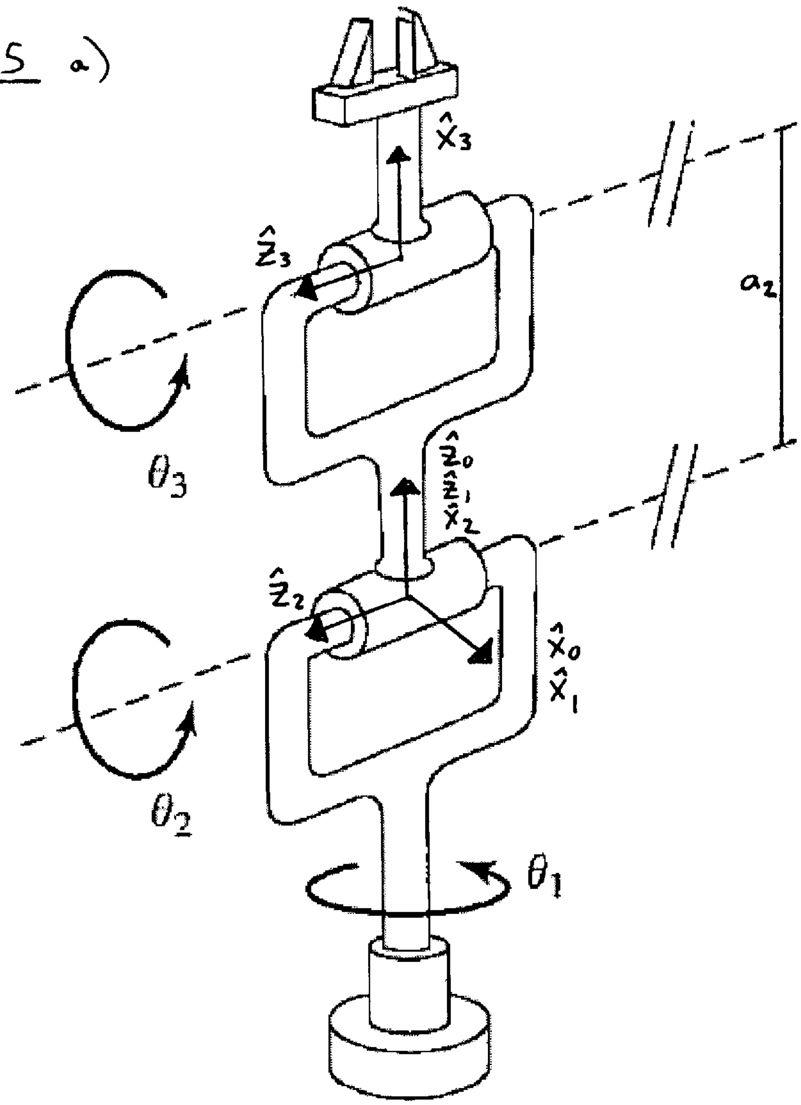
$$ {}^{A}_{B}R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

$$ \theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) = \cos^{-1}\left(\frac{2/3 + 2/3 + 2/3 - 1}{2}\right) = 60^{\circ} $$

$$ \hat{K} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \frac{1}{2\left(\frac{\sqrt{3}}{2}\right)} \begin{bmatrix} 1/3 + 2/3 \\ -1/3 - 2/3 \\ 1/3 + 2/3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} $$

Therefore the angle-axis representation of $\{B\}$ relative to $\{A\}$ is a 60° rotation about $\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}^T$.

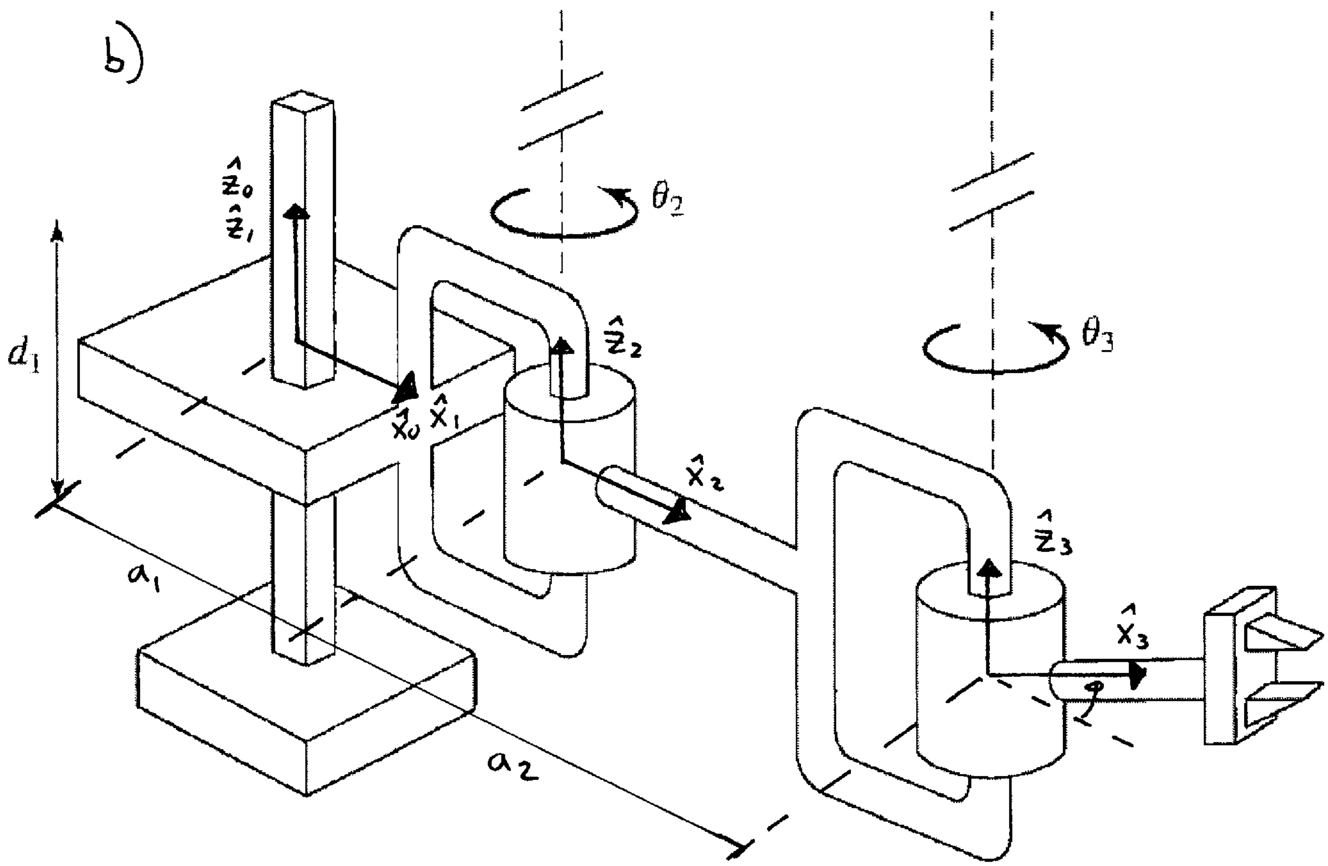
→ Question 5 a)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$90°$	0	0	$90° + \theta_2$
3	0	a_2	0	θ_3

* This is one of many possible solutions

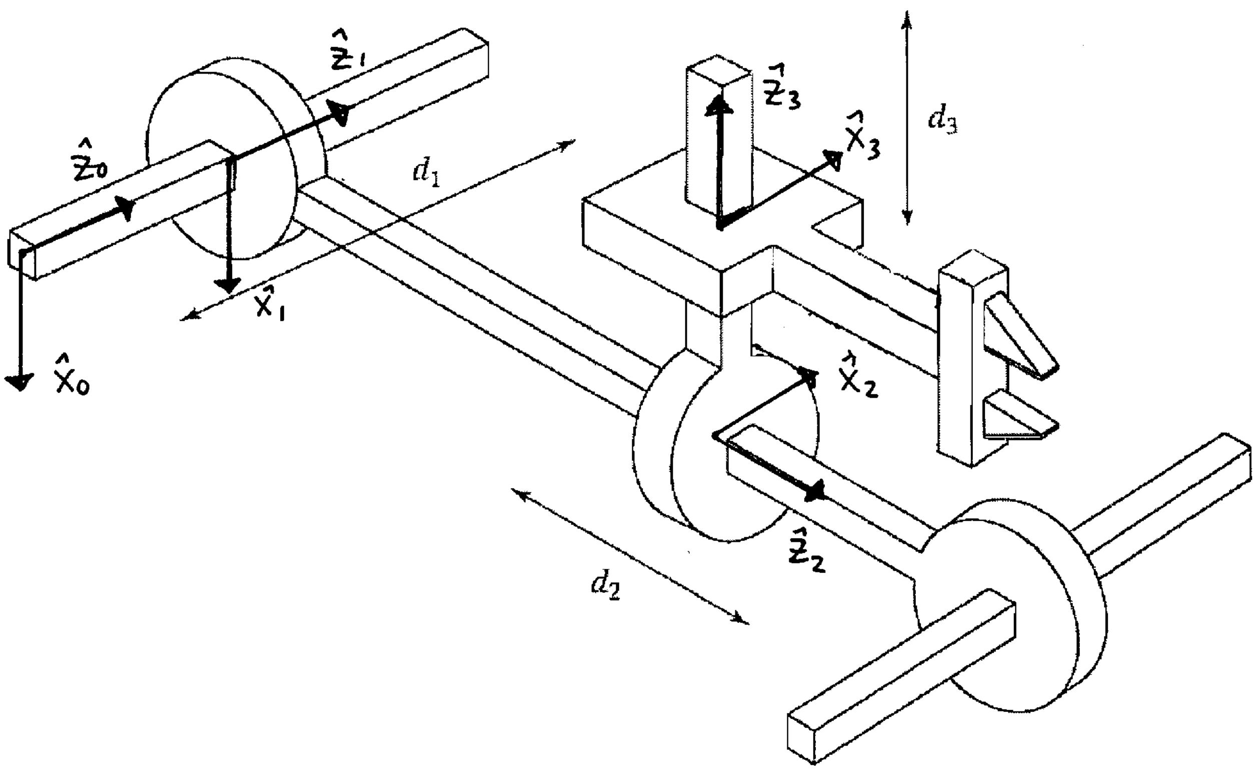
b)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	0	a_1	0	θ_2
3	0	a_2	0	θ_3

* This is one of many possible solutions

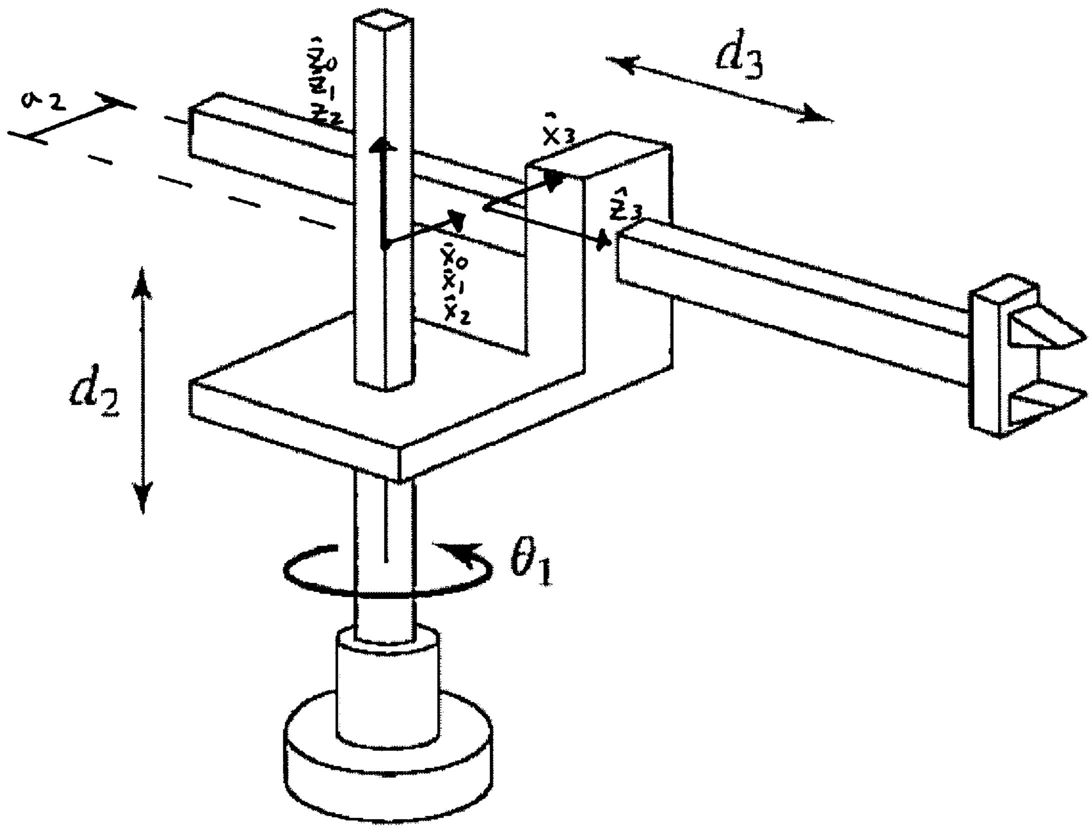
c)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	d_1	0
2	90°	0	d_2	90°
3	-90°	0	d_3	0

* This is one of many possible solutions

d)

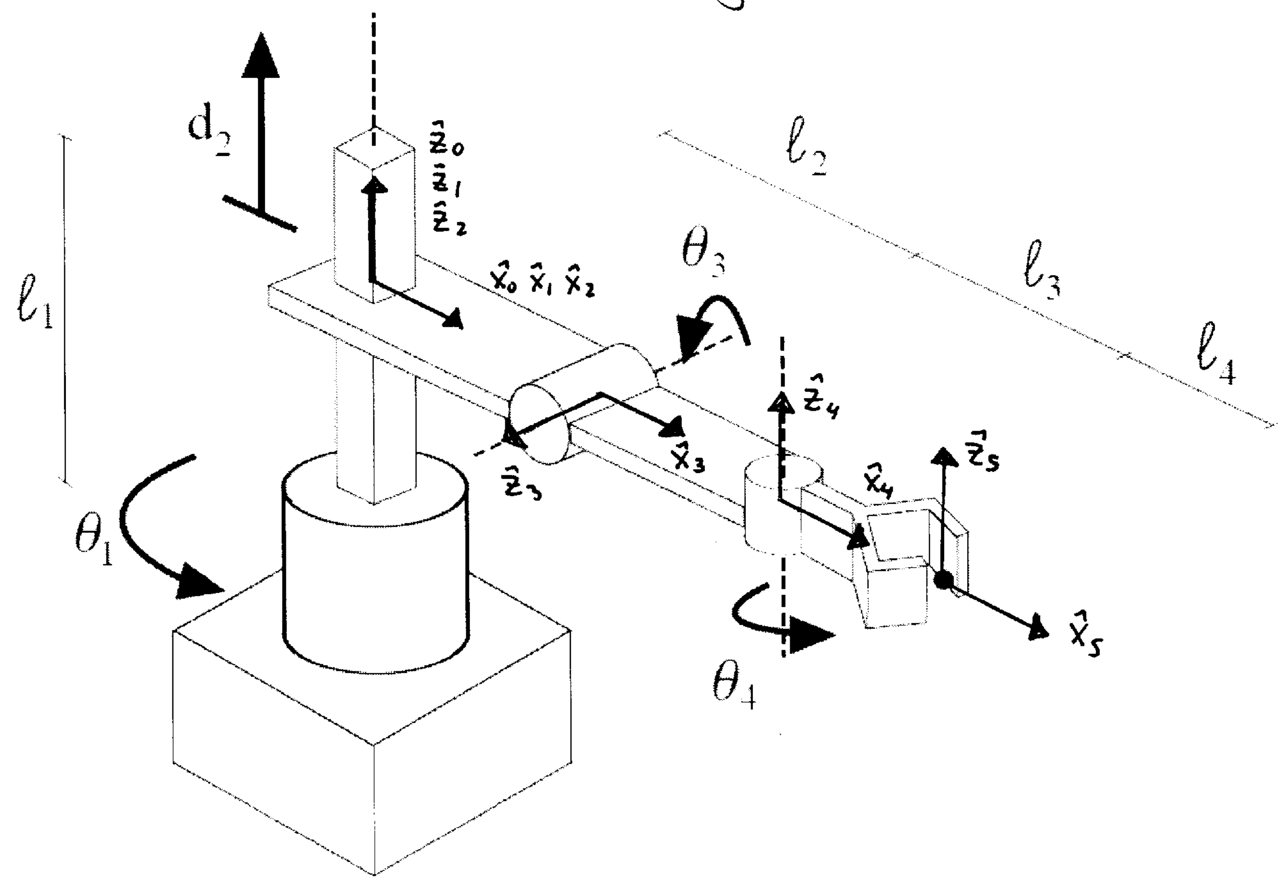


i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	0	d_2	0
3	90°	a_2	d_3	0

* This is one of many possible solutions

→ Question 6

→ Assigning frames



→ D-H Table

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	0	d_2	0
3	$90°$	ℓ_2	0	θ_3
4	$-90°$	ℓ_3	0	θ_4
5	0	ℓ_4	0	0

*This is one of many possible frame assignments

- Link Transformations

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ 0 & 0 & -1 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ -s_4 & -c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} 1 & 0 & 0 & l_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- The kinematic model from {0} to {5}

$$ {}^{0}_{5}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{23} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

* note that this transformation will be different for other frame assignments

where

$$ r_{11} = c_1 c_3 c_4 - s_1 s_4 \qquad r_{13} = -c_1 s_3 $$
$$ r_{21} = s_1 c_3 c_4 + c_1 s_4 \qquad r_{23} = -s_1 s_3 $$
$$ r_{31} = s_3 s_4 \qquad r_{33} = c_3 $$

$$ r_{12} = -c_1 c_3 s_4 - s_1 c_4 \qquad p_x = (c_1 c_3 c_4 - s_1 s_4) l_4 + c_1 c_3 l_3 + c_1 l_2 $$
$$ r_{22} = -s_1 c_3 s_4 + c_1 c_4 \qquad p_y = (s_1 c_3 c_4 + c_1 s_4) l_4 + s_1 c_3 l_3 + s_1 l_2 $$
$$ r_{32} = -s_3 s_4 \qquad p_z = (s_3 c_4) l_4 + s_3 l_3 + d_2 $$