Velocity and acceleration

$v = \frac{ds}{dt}$

$a = \frac{dv}{dt} = v\frac{dv}{ds}$

Special cases

(a = 0) $s = s_0 + v_0 t$

$v^2 = v_0^2 + 2\int_{s_0}^{s} a(s)\,ds$

(const a) $v = v_0 + a_0 t$

$s = s_0 + v_0 t + \frac{1}{2}a_0 t^2$

$v^2 = v_0^2 + 2a_0(s - s_0)$

$v = v_0 + \int_0^t a(t)\,dt$

$s = s_0 + v_0 t + \int_0^t \int_0^t a(t)\,dt\,dt$

Rectangular coordinates

$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$

$\vec{a} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$

$|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$

SUVAT equation (a const)

$v = v_0 + a_0 t$

$s = s_0 + v_0 t + \frac{1}{2}a_0 t^2$

$v^2 = v_0^2 + 2a_0(s - s_0)$

$s = \frac{(v + v_0)t}{2}$

Projectile motion

$v_{0x} = v_0 \cos\theta$

$v_{0y} = v_0 \sin\theta$

$a_x = 0$

$x = x_0 + v_0 t$

$a_y = a_{0y}$

$y = y_0 + v_{0y} t + \frac{1}{2}a_{0y} t^2$

Normal-tangential system

$\overrightarrow{v_{n-t}} = v\overrightarrow{u_t}$

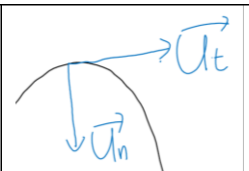
$\overrightarrow{u_b} = \overrightarrow{u_t} \times \overrightarrow{u_n}$

$v = \frac{ds}{dt}$

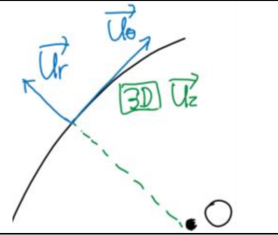
$\vec{a} = \dot{v}\overrightarrow{u_t} + v\dot{\theta}\overrightarrow{u_n}$

$= \dot{v}\overrightarrow{u_t} + \frac{v^2}{\rho}\overrightarrow{u_n}$

$\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{1.5}}{\left|\frac{d^2y}{dx^2}\right|}$



Cylindrical r − θ system



$\vec{r} = r\overrightarrow{u_r}$

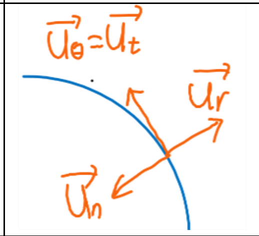
$\vec{v} = \dot{r}\overrightarrow{u_r} + r\dot{\theta}\overrightarrow{u_\theta}$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\overrightarrow{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\overrightarrow{u_\theta}$

$\dot{\overrightarrow{u_r}} = \dot{\theta}\overrightarrow{u_\theta}$

$\dot{\overrightarrow{u_\theta}} = -\dot{\theta}\overrightarrow{u_r}$

Circular motion



Dependent motion

- Rope has constant length
- Define good datum lines (fixed position)
- Find fixed length if possible
- Divide the rope into sections if needed

$L_T = s_A + s_B$

Then $v_A + v_B = 0$

$a_A + a_B = 0$

Relative motion

$\overrightarrow{r_B} = \overrightarrow{r_A} + \overrightarrow{r_{B/A}}$

$\overrightarrow{v_B} = \overrightarrow{v_A} + \overrightarrow{v_{B/A}}$

$\overrightarrow{a_B} = \overrightarrow{a_A} + \overrightarrow{a_{B/A}}$

Gravitational force

$\vec{g} = -9.81\vec{\jmath}\,\mathrm{ms}^{-2}$

$F = m\vec{g}$

Frictional force (oppose motion)

Static $|F_{fsmax}| = \mu_s F_N$

Kinetic $|F_{fk}| = \mu_k F_N$

$F_{fk} > \mu_k F_N$ velocity decrease

$F_{fk} = \mu_k F_N$ velocity same

$F_{fk} < \mu_k F_N$ velocity increase

Spring force

$F_s = -kx$

k is spring constant

x is deviation from rest

Equilibrium (x-y-z)

$\sum F_x = ma_x = m\ddot{x}$

$\sum F_y = ma_y = m\ddot{y}$

$\sum F_z = ma_z = m\ddot{z}$

Equilibrium (n-t)

$\sum F_n = ma_n = mv\dot{\theta} = \frac{mv^2}{\rho}$

$\sum F_t = ma_t = m\dot{v}$

Equilibrium (r − θ)

$\sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$

$\sum F_\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Work and motion

$dU = \vec{F}\cdot d\vec{r} = F\cos\theta\,dr$

$U_{P>P'} = \int_P^{P'} dU = \int_P^{P'} \vec{F}\,d\vec{r} = \int_P^{P'} F\cos\theta\,ds$

Work by gravitational F

$U_g = -W\Delta y$

$= -mg(y_2 - y_1)$

*Always negative

Work by kinetic friction

*Against motion> negative

$U_f = -F_f\Delta x$

Work by spring

$U_s = \int_{x_1}^{x_2} -kx\,dx$

$= -\frac{1}{2}k(x_2^2 - x_1^2)$

Kinetic energy

$T = \frac{1}{2}mv^2$

$T_1 + U_{1>2} = T_2$

$\frac{1}{2}m_i v_{i1}^2 + \int_{s_{i1}}^{s_{i2}} \overrightarrow{f_{it}}\,ds + \int_{s_{i1}}^{s_{i2}} \overrightarrow{f_{it}}\,ds = \frac{1}{2}m_i v_{i2}^2$

Internal force is zero

If particles connected by inextensible cable

$\int_{s_{i1}}^{s_{i2}} \overrightarrow{f_{it}}\,ds = 0$

Work done by force

$U_g = -mg\Delta y$

$U_s = -\frac{1}{2}k(s_2^2 - s_1^2)$

$U_f = -F_{fk}\Delta s$

Potential energy

$V_g = mgh$

$V_s = \frac{1}{2}kx^2$

Conservation of energy

$T_1 + V_1 + U_{1>2} = T_2 + V_2$

If $(\mathbf{U_{1>2} = 0})$

$T_1 + V_1 = T_2 + V_2$

Linear momentum

$\vec{L} = m\vec{v}$

Elastic collision

$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2}$

Inelastic collision

$m_1 v_{i1} + m_2 v_{i2} = (m_1 + m_2)v_f$

Conservation of momentum: Constant force

$\int_{t_1}^{t_2} \vec{F}\,dt = \vec{F}\Delta t$

Conservation of momentum: Avg force

$\int_{t_1}^{t_2} \vec{F}\,dt = \overrightarrow{F_{avg}}\Delta t$

Conservation of momentum: $\sum \boldsymbol{F} = \mathbf{0}$ || $\Delta t = 0$

$m\overrightarrow{v_1} = m\overrightarrow{v_2}$

$\overrightarrow{L_1} = \overrightarrow{L_2}$

Multiple particles

$\sum m_i(\overrightarrow{v_{i1}}) + \sum \int_{t_1}^{t_2} \overrightarrow{F_i}\,dt = \sum m_i(\overrightarrow{v_{i2}})$

Moment

$\overrightarrow{M_0} = \overrightarrow{r_0} \times \vec{F}$

Angular momentum

$\overrightarrow{H_0} = \overrightarrow{r_0} \times m\vec{v}$

$|\overrightarrow{H_0}| = r_0 mv\sin\theta = r_0 m v_\theta$

Principle of angular momentum and impulse

$\overrightarrow{H_{01}} + \int_{t_1}^{t_2} \sum \overrightarrow{\mu_0}\,dt = \overrightarrow{H_{02}}$

$\overrightarrow{\mu_0} = \frac{d\vec{H_0}}{dt}$

Conservation of linear momentum

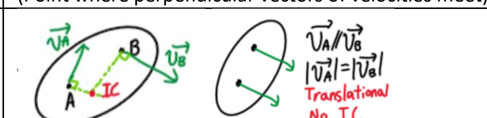
$\sum H_{01i} = \sum H_{02i}$

Rigid body motions

- Translation
- Fixed rotation
- General motion

Instantaneous centre of zero velocity

(Point where perpendicular vectors of velocities meet)



Fixed rotation

Angular displacement $\vec{\theta}$

Angular velocity $\vec{\omega}$

Angular acceleration $\vec{\alpha}$

$\overrightarrow{v_P} = \vec{\omega} \times \vec{r}$

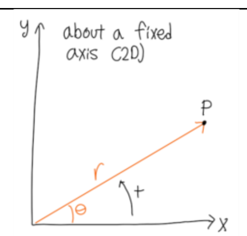
$\overrightarrow{a_P} = \vec{\omega} \times \vec{v} + \vec{\alpha} \times \vec{r}$

If α constant,

$\omega = \omega_0 + \alpha_c t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$

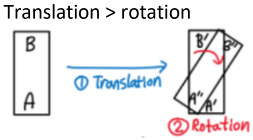
$\omega^2 = \omega_0^2 + 2\alpha_c\theta$



General motion

Decompose the motion

Translation > rotation



$\omega = \frac{v_B - v_A}{r_{B/A}}$

$\overrightarrow{v_B} = \overrightarrow{v_A} + \vec{\omega} \times \overrightarrow{r_{B/A}}$

$\overrightarrow{a_B}$

$= \overrightarrow{a_A} + \vec{\omega} \times \overrightarrow{v_{B/A}} + \vec{\alpha} \times \overrightarrow{r_{B/A}}$

$= \overrightarrow{a_A} - \omega^2\overrightarrow{r_{B/A}} + \alpha\overrightarrow{r_{B/A}}$

$|a_{B/At}| = r\alpha$

$|a_{B/An}| = \omega^2 r$

Translation

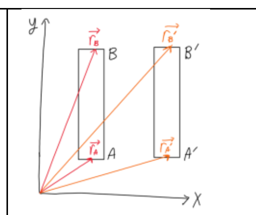
$\overrightarrow{r_B} = \overrightarrow{r_A} + \overrightarrow{r_{B/A}}$

$\overrightarrow{v_B} = \overrightarrow{v_A}$

$\overrightarrow{a_B} = \overrightarrow{a_A}$

Magnitude don't change

Direction don't change



Force, Moment, Angular Momentum

$\vec{F} = m\overrightarrow{a_G} = \sum_i m_i\overrightarrow{a_i}$

$\overrightarrow{H_A} = \overrightarrow{r_{p/A}} \times m\vec{v}$

$= \overrightarrow{r_{p/A}} \times \vec{L}$

$\dot{\overrightarrow{H_G}} = \sum \overrightarrow{M_G}$

($\dot{\overrightarrow{H_G}} = \sum \overrightarrow{M_G}$ X)

$\sum H = wI$

$H_0 = wI_0$

$H_G = wI_G$

$\sum M = I\alpha$

$M_G = I_G\alpha$

$M_0 = I_0\alpha$ (pinned at O)

$M_A = I_A\alpha$ (rolling, no slip)

Parallel Axis Theorem

$I_p = I_G + md^2$

d – distance between P and G

Moment of inertia can be added/subtracted

Square moment of inertia

$I_G = \frac{1}{12}ml^2$

$I_A = \frac{1}{3}ml^2$

G – centre of gravity

A – end part of square

Circle moment of inertia

$I_G = \frac{1}{2}mR^2$

$I_G = mk_G^2$

$I = \frac{1}{2}\rho t\pi R^4$

$k_G = \sqrt{\frac{I_G}{m}} = \frac{R}{\sqrt{2}}$

k_G – radius of gyration

Donut moment of inertia

$I_G = \frac{1}{2}m(R_0 + R_i)^2$

R_0 – outer radius

R_i – inner radius

$I_G = \frac{1}{2}\rho t\pi(R_0^4 - R_i^4)$

$m = \rho t\pi(R_0^2 - R_i^2)$

Kinetic Energy ($\mathbf{T_1 + U_{1\to2} = T_2}$)

$T_{Body} = \frac{1}{2}I_{IC}w^2$

$= \frac{1}{2}w^2(I_G + md^2)$

$T = T_{rotate} + T_{translate}$

$= \frac{1}{2}I_G w^2 + \frac{1}{2}mv_G^2$

Work by forces

$U_g = -mg\Delta h$

$U_e = -\frac{1}{2}k(s_2^2 - s_1^2)$

$\vec{M} = \vec{r} \times \vec{F}$

$U_M = M(\theta_2 - \theta_1)$

Conservation of energy

$T_1 + V_1 + U_{1\to2} = T_2 + V_2$

$U_{1\to2}$ – work of nonconservative F

$T_1 + V_1 = T_2 + V_2$

If no nonconservative F

$V = V_e + V_g$

$V_e = \frac{1}{2}ks^2$

$V_g = mgh_G$

Momentum, impulse and angular momentum

$\vec{L} = m\vec{v}$

$\overrightarrow{L_1} + \sum \int_{t_1}^{t_2} \overrightarrow{F_i}\,dt = \overrightarrow{L_2}$

$\vec{H} = \vec{r} \times m\vec{v} = Iw$

$\vec{M} = \vec{r} \times \vec{F} = I\alpha$

$\overrightarrow{H_{G1}} + \sum \int_{t_1}^{t_2} \overrightarrow{M_G}\,dt = \overrightarrow{H_{G2}}$

$\overrightarrow{H_{O1}} + \sum \int_{t_1}^{t_2} \overrightarrow{M_O}\,dt = \overrightarrow{H_{O2}}$

$\int \vec{F}\,dt = 0 \to \overrightarrow{L_1} = \overrightarrow{L_2}$

$\int \Sigma\overrightarrow{M_G}\,dt = 0 \to \overrightarrow{H_{G1}} = \overrightarrow{H_{G2}}$

$\int \Sigma\overrightarrow{M_O}\,dt = 0 \to \overrightarrow{H_{O1}} = \overrightarrow{H_{O2}}$

Second order differential equations

$m\ddot{x} + c\dot{x} + kx = 0$

$x = e^{\lambda t}$

$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0$

$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

$c^2 - 4km > 0$

$|c| > \sqrt{4km}$

$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

2 real λs

$c^2 - 4km = 0$

$|c| = \sqrt{4km}$

$x = (A + Bt)e^{\lambda t}$

one real λ

$c^2 - 4km < 0$

2 complex λs

$\lambda_1 = a + bi$ $\lambda_2 = a - bi$

$x = Ae^{at}e^{bti} + Be^{at}e^{-bti}$

$= e^{at}(\alpha\cos bt + \beta\sin bt)$

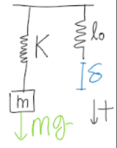
Undamped free vibration – spring motion (horizontal)

$$\ddot{x} + \frac{k}{m}x = 0$$
$$\ddot{x} + w_n^2 x = 0$$
$$w_n = \sqrt{\frac{k}{m}}$$

$$x = A\sin w_n t + B\cos w_n t = C\sin(w_n t + \theta)$$
$$C = \sqrt{A^2 + B^2} \quad \tau = \frac{2\pi}{w_n} = \frac{1}{f}$$
$$\theta = \tan^{-1}\frac{B}{A} \quad w_n = 2\pi f$$

Vertical spring

$$\sum F_y = 0$$
$$mg - k(l - l_0) = 0$$
$$\delta_{eq} = l - l_0 = \frac{mg}{k}$$



Parallel / Series spring

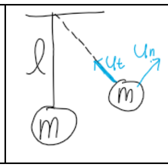
Parallel	Series
$k_{eq} = \sum_i k_i$	$\frac{1}{k_{eq}} = \sum_i \frac{1}{k_i}$

$$\ddot{x} + \frac{k_{eq}}{m}x = 0$$

Undamped free vibration – pendulum motion

$$-mg\sin\theta = ma_t$$
$$s = l\theta$$
$$ml\ddot{\theta} = -mg\sin\theta = -mg\theta$$
$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

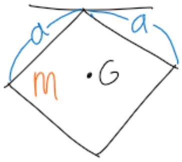
$$w_n = \sqrt{\frac{g}{l}}$$
$$\ddot{\theta} + w_n^2\theta = 0$$



Bar pendulum

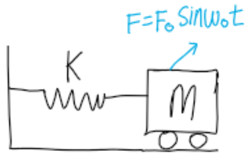
$$w_n = \sqrt{\frac{3g}{2l}}$$

Square pendulum



$$w_n = \sqrt{\frac{3g}{2\sqrt{2}a}}$$

Undamped forced vibration



w_0 forcing frequency

$\delta_0 = \frac{F_0}{k}$ static deflection

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin w_0 t$$
$$x = x_c + x_p$$

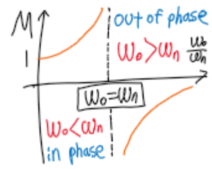
$x_c = A\sin w_n t + B\cos w_n t$ (Transient)

$x_p = C\sin w_0 t$ (steady)

$$x_{pmax} = C = \frac{F_0}{(w_n^2 - w_0^2)m} = \frac{F_0/k}{\left(1-\left(\frac{w_0}{w_n}\right)^2\right)} = \frac{\delta_0}{\left(1-\left(\frac{w_0}{w_n}\right)^2\right)}$$

$$M = \frac{x_{pmax}}{F_0/k} = \frac{1}{1-\left(\frac{w_0}{w_k}\right)^2}$$

Magnification factor



Damping coefficient

$$F_d = -c\dot{x}$$

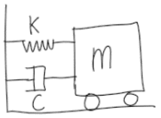
(c is damping coefficient)

$$c_c = \sqrt{\frac{k}{m}}2m = 2mw_n$$

(critical damping coeff)

Damping equation

$$m\ddot{x} + c\dot{x} + kx = 0$$
$$x = e^{\lambda t}$$
$$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0$$
$$\lambda = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$
$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$



Overdamped ($c > c_c$)

2 real, negative λs

No vibration

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} > 0$$

Critically damped ($c = c_c$)

one real λ

No vibration

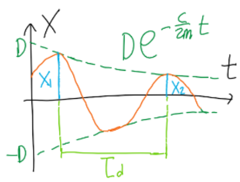
c_c is smallest c which system won't vibrate

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0$$
$$x = (A + Bt)e^{\lambda t}$$

Underdamped ($c < c_c$)

2 complex λs

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} < 0$$
$$x = D\left[e^{-\frac{c}{2m}t}\sin(w_d t + \theta)\right]$$
$$w_d = w_n\sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$
$$\ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\left(\frac{c}{c_c}\right)}{\sqrt{1-\left(\frac{c}{c_c}\right)^2}}$$
$$\tau_d = \frac{2\pi}{w_d} > \tau$$



$$x = x_0 e^{-\frac{c}{2m}t}$$

Electrical circuit analogy

Mass <-> inductance

Damp coeff. <-> resistance

Spring con. <-> 1/capacitance

Force <-> voltage

Displacement <-> charge

Velocity <-> current