Chapter 7 Indefinite Integration and its Applications

EXERCISE 7A

Level 1

1. Find $\int (-\frac{1}{5})dx$.

SOLUTION

$$\int (-\frac{1}{5})dx = \underline{\underline{-\frac{1}{5}x + C}}$$

2. Find $\int 4x^3 dx$.

SOLUTION

$$\begin{aligned}\int 4x^3 dx &= 4 \cdot \frac{1}{4}x^4 + C \\ &= \underline{\underline{x^4 + C}}\end{aligned}$$

3. Find $\int (-2x^{-3})dx$.

SOLUTION

$$\begin{aligned}\int (-2x^{-3})dx &= -2 \cdot \frac{1}{-2}x^{-2} + C \\ &= \underline{\underline{\frac{1}{x^2} + C}}\end{aligned}$$

4. Find $\int \frac{3}{x^2}dx$.

SOLUTION

$$\int \frac{3}{x^2}dx = 3(-1)x^{-1} + C$$
$$= \underline{\underline{-\frac{3}{x} + C}}$$

5. Find $\int \frac{dx}{5x^4}$.

SOLUTION

$$\int \frac{dx}{5x^4} = \frac{1}{5}\cdot\frac{1}{-3}x^{-3} + C$$
$$= \underline{\underline{-\frac{1}{15x^3} + C}}$$

6. Find $\int 8\sqrt{x}dx$.

SOLUTION

$$\int 8\sqrt{x}dx = 8\cdot\frac{2}{3}x^{\frac{3}{2}} + C$$
$$= \underline{\underline{\frac{16}{3}x^{\frac{3}{2}} + C}}$$

7. Find $\int 4x^{-\frac{1}{3}}dx$.

SOLUTION

$$\int 4x^{-\frac{1}{3}}dx = 4\cdot\frac{3}{2}x^{\frac{2}{3}} + C$$
$$= \underline{\underline{6x^{\frac{2}{3}} + C}}$$

8. Find $\int \frac{6dx}{\sqrt[5]{x^3}}$.

SOLUTION

$$\int \frac{6dx}{\sqrt[5]{x^3}} = 6\int x^{-\frac{3}{5}}dx$$
$$= 6 \cdot \frac{5}{2} x^{\frac{2}{5}} + C$$
$$= \underline{\underline{15x^{\frac{2}{5}} + C}}$$

9. Find $\int 1.2x^{1.2}dx$.

SOLUTION

$$\int 1.2x^{1.2}dx = 1.2 \cdot \frac{1}{2.2} x^{2.2} + C$$
$$= \underline{\underline{\frac{6}{11} x^{2.2} + C}}$$

10. Find $\int (5x^4 + 2x^2 - 3)dx$.

SOLUTION

$$\int (5x^4 + 2x^2 - 3)dx = 5 \cdot \frac{1}{5} x^5 + 2 \cdot \frac{1}{3} x^3 - 3x + C$$
$$= \underline{\underline{x^5 + \frac{2}{3} x^3 - 3x + C}}$$

11. Find $\int (3x^4 + 6x^2)dx$.

SOLUTION

$$\int (3x^4 + 6x^2)dx = 3 \cdot \frac{1}{5} x^5 + 6 \cdot \frac{1}{3} x^3 + C$$
$$= \underline{\underline{\frac{3}{5} x^5 + 2x^3 + C}}$$

12. Find $\int (\frac{3}{x^3} + 4 - 2x^2)dx$.

SOLUTION

$$\int (\frac{3}{x^3} + 4 - 2x^2)dx = 3 \cdot \frac{1}{-2} x^{-2} + 4x - 2 \cdot \frac{1}{3} x^3 + C$$
$$= \underline{\underline{-\frac{3}{2x^2} + 4x - \frac{2}{3} x^3 + C}}$$

13. Find $\int 2x^2(x^3 - 4x)dx$.

SOLUTION

$$\int 2x^2(x^3 - 4x)dx = \int (2x^5 - 8x^3)dx$$
$$= 2 \cdot \frac{1}{6} x^6 - 8 \cdot \frac{1}{4} x^4 + C$$
$$= \underline{\underline{\frac{1}{3} x^6 - 2x^4 + C}}$$

14. Find $\int (x+1)(3x-2)dx$.

SOLUTION

$$\int (x+1)(3x-2)dx = \int (3x^2 + x - 2)dx$$
$$= 3 \cdot \frac{1}{3} x^3 + \frac{x^2}{2} - 2x + C$$
$$= \underline{\underline{x^3 + \frac{x^2}{2} - 2x + C}}$$

15. Find $\int (2x+3)(4-7x)dx$.

SOLUTION

$$\int (2x+3)(4-7x)dx = \int (-14x^2 - 13x + 12)dx$$
$$= -14 \cdot \frac{1}{3} x^3 - 13 \cdot \frac{1}{2} x^2 + 12x + C$$
$$= \underline{\underline{-\frac{14}{3} x^3 - \frac{13}{2} x^2 + 12x + C}}$$

16. Find $\int (x-\frac{1}{x})^2 dx$.

SOLUTION

$$\int (x-\frac{1}{x})^2 dx = \int (x^2 - 2 + \frac{1}{x^2})dx$$
$$= \frac{x^3}{3} - 2x + (-1)x^{-1} + C$$
$$= \underline{\underline{\frac{x^3}{3} - 2x - \frac{1}{x} + C}}$$

17. Find $\int \sqrt{x}(\sqrt{x}-3)^2 dx$.

SOLUTION

$$\int \sqrt{x}(\sqrt{x}-3)^2 dx = \int \sqrt{x}(x - 6\sqrt{x} + 9)dx$$
$$= \int (x^{\frac{3}{2}} - 6x + 9\sqrt{x})dx$$
$$= \frac{2}{5}x^{\frac{5}{2}} - 6 \cdot \frac{1}{2}x^2 + 9 \cdot \frac{2}{3}x^{\frac{3}{2}} + C$$
$$= \underline{\underline{\frac{2}{5}x^{\frac{5}{2}} - 3x^2 + 6x^{\frac{3}{2}} + C}}$$

18. Find $\int \frac{x^5 - 2x + 3}{6x^3} dx$.

SOLUTION

$$\int \frac{x^5 - 2x + 3}{6x^3} dx = \int (\frac{1}{6}x^2 - \frac{1}{3}x^{-2} + \frac{1}{2}x^{-3})dx$$
$$= \frac{1}{6} \cdot \frac{1}{3}x^3 - \frac{1}{3}(-1)x^{-1} + \frac{1}{2} \cdot \frac{1}{-2}x^{-2} + C$$
$$= \underline{\underline{\frac{x^3}{18} + \frac{1}{3x} - \frac{1}{4x^2} + C}}$$

19. Find $\int \frac{(\sqrt{x^3}-\frac{2}{\sqrt{x}})^3}{x^2}dx$.

SOLUTION

$$\int \frac{(\sqrt{x^3}-\frac{2}{\sqrt{x}})^3}{x^2}dx = \int \frac{(x^{\frac{3}{2}})^3+3(x^{\frac{3}{2}})^2(-\frac{2}{\sqrt{x}})+3(x^{\frac{3}{2}})(-\frac{2}{\sqrt{x}})^2+(-\frac{2}{\sqrt{x}})^3}{x^2}dx$$

$$= \int \frac{x^{\frac{9}{2}}-6x^{\frac{5}{2}}+12x^{\frac{1}{2}}-8x^{-\frac{3}{2}}}{x^2}dx$$

$$= \int (x^{\frac{5}{2}}-6x^{\frac{1}{2}}+12x^{-\frac{3}{2}}-8x^{-\frac{7}{2}})dx$$

$$= \frac{2}{7}x^{\frac{7}{2}}-6\cdot\frac{2}{3}x^{\frac{3}{2}}+12(-2)x^{-\frac{1}{2}}-8(-\frac{2}{5})x^{-\frac{5}{2}}+C$$

$$= \underline{\underline{\frac{2}{7}x^{\frac{7}{2}}-4x^{\frac{3}{2}}-24x^{-\frac{1}{2}}+\frac{16}{5}x^{-\frac{5}{2}}+C}}$$

20. Find $\int \frac{4x^2-9}{2x+3}dx$.

SOLUTION

$$\int \frac{4x^2-9}{2x+3}dx = \int \frac{(2x+3)(2x-3)}{2x+3}dx$$

$$= \int (2x-3)dx$$

$$= \underline{\underline{x^2-3x+C}}$$

21. Find $\int \frac{x^{\frac{3}{2}}-8}{\sqrt{x}-2}dx$.

SOLUTION

$$\int \frac{x^{\frac{3}{2}}-8}{\sqrt{x}-2}dx = \int \frac{(x^{\frac{1}{2}}-2)(x+2x^{\frac{1}{2}}+4)}{x^{\frac{1}{2}}-2}dx$$

$$= \int (x+2x^{\frac{1}{2}}+4)dx$$

$$= \frac{1}{2}x^2+2\cdot\frac{2}{3}x^{\frac{3}{2}}+4x+C$$

$$= \underline{\underline{\frac{1}{2}x^2+\frac{4}{3}x^{\frac{3}{2}}+4x+C}}$$

22. Find $\int \frac{e^x dx}{3}$.

SOLUTION

$$\int \frac{e^x dx}{3} = \underline{\underline{\frac{1}{3}e^x + C}}$$

23. Find $\int (x+e)(x-e)dx$.

SOLUTION

$$\begin{aligned}\int (x+e)(x-e)dx &= \int (x^2 - e^2)dx \\ &= \underline{\underline{\frac{x^3}{3} - e^2 x + C}}\end{aligned}$$

24. Find $\int \frac{e^{2x}-4}{e^x-2}dx$.

SOLUTION

$$\begin{aligned}\int \frac{e^{2x}-4}{e^x-2}dx &= \int \frac{(e^x+2)(e^x-2)}{e^x-2}dx \\ &= \int (e^x+2)dx \\ &= \underline{\underline{e^x + 2x + C}}\end{aligned}$$

25. Find $\int \ln e^{x^2} dx$.

SOLUTION

$$\begin{aligned}\int \ln e^{x^2} dx &= \int x^2 dx \\ &= \underline{\underline{\frac{x^3}{3} + C}}\end{aligned}$$

26. Find $\int e^{\ln\sqrt[3]{x}}dx$.

SOLUTION

$$\int e^{\ln\sqrt[3]{x}}dx = \int x^{\frac{1}{3}}dx$$
$$= \frac{3}{4}x^{\frac{4}{3}} + C$$

27. Find $\int e^{-\ln 5x}dx$.

SOLUTION

$$\int e^{-\ln 5x}dx = \int e^{\ln\frac{1}{5x}}dx$$
$$= \int \frac{1}{5x}dx$$
$$= \frac{1}{5}\ln|x| + C$$

28. Find $\int \frac{1}{x\ln 7}dx$.

SOLUTION

$$\int \frac{1}{x\ln 7}dx = \frac{1}{\ln 7}\int \frac{1}{x}dx = \frac{\ln|x|}{\ln 7} + C$$

29. Find $\int (2x+7)(6+\frac{5}{2x})dx$.

SOLUTION

$$\int (2x+7)(6+\frac{5}{2x})dx = \int (12x+47+\frac{35}{2x})dx$$
$$= 12\cdot\frac{1}{2}x^2 + 47x + \frac{35}{2}\ln|x| + C$$
$$= 6x^2 + 47x + \frac{35}{2}\ln|x| + C$$

30. Find $\int (2x-\frac{1}{x})(\frac{3}{x^2}-4)dx$.

SOLUTION

$$\int (2x-\frac{1}{x})(\frac{3}{x^2}-4)dx = \int (-8x+\frac{10}{x}-\frac{3}{x^3})dx$$
$$= -8\cdot\frac{1}{2}x^2+10\ln|x|-3\cdot\frac{1}{-2}x^{-2}+C$$
$$= \underline{\underline{-4x^2+10\ln|x|+\frac{3}{2x^2}+C}}$$

31. Find $\int \frac{x^3-5x+6}{2x^2}dx$.

SOLUTION

$$\int \frac{x^3-5x+6}{2x^2}dx = \int (\frac{1}{2}x-\frac{5}{2}x^{-1}+3x^{-2})dx$$
$$= \frac{1}{2}\cdot\frac{1}{2}x^2-\frac{5}{2}\ln|x|+3(-1)x^{-1}+C$$
$$= \underline{\underline{\frac{1}{4}x^2-\frac{5}{2}\ln|x|-\frac{3}{x}+C}}$$

32. Find $\int \frac{(3x+5)^2}{15x^2}dx$.

SOLUTION

$$\int \frac{(3x+5)^2}{15x^2}dx = \int \frac{9x^2+30x+25}{15x^2}dx$$
$$= \int (\frac{3}{5}+\frac{2}{x}+\frac{5}{3x^2})dx$$
$$= \frac{3}{5}x+2\ln|x|+\frac{5}{3}(-1)x^{-1}+C$$
$$= \underline{\underline{\frac{3}{5}x+2\ln|x|-\frac{5}{3x}+C}}$$

33. Find $\int (\frac{x-2}{2x})^2 dx$.

SOLUTION

$$\begin{aligned}\int (\frac{x-2}{2x})^2 dx &= \int (\frac{1}{2}-\frac{1}{x})^2 dx \\ &= \int (\frac{1}{4}-\frac{1}{x}+\frac{1}{x^2})dx \\ &= \frac{1}{4}x-\ln|x|+(-1)x^{-1}+C \\ &= \underline{\underline{\frac{1}{4}x-\ln|x|-\frac{1}{x}+C}}\end{aligned}$$

34. Find $\int \frac{(2\sqrt{x}-7)^2}{x^{\frac{3}{2}}} dx$.

SOLUTION

$$\begin{aligned}\int \frac{(2\sqrt{x}-7)^2}{x^{\frac{3}{2}}} dx &= \int \frac{4x-28\sqrt{x}+49}{x^{\frac{3}{2}}} dx \\ &= \int (4x^{-\frac{1}{2}}-28x^{-1}+49x^{-\frac{3}{2}})dx \\ &= 4\cdot 2x^{\frac{1}{2}}-28\ln|x|+49(-2)x^{-\frac{1}{2}}+C \\ &= \underline{\underline{8x^{\frac{1}{2}}-28\ln|x|-98x^{-\frac{1}{2}}+C}}\end{aligned}$$

35. Find $\int e^{2x} dx$.

SOLUTION

$$\begin{aligned}\int e^{2x} dx &= \int (e^2)^x dx \\ &= \frac{(e^2)^x}{\ln(e^2)}+C \\ &= \underline{\underline{\frac{1}{2}e^{2x}+C}}\end{aligned}$$

36. Find $\int 6e^{3x-2}dx$.

SOLUTION

$$\int 6e^{3x-2}dx = 6e^{-2}\int (e^3)^x\,dx$$
$$= 6e^{-2}\left[\frac{(e^3)^x}{\ln(e^3)}\right] + C$$
$$= 6e^{-2}\left(\frac{1}{3}e^{3x}\right) + C$$
$$= \underline{\underline{2e^{3x-2} + C}}$$

37. Find $\int 7^x\,dx$.

SOLUTION

$$\int 7^x\,dx = \underline{\underline{\frac{7^x}{\ln 7} + C}}$$

38. Find $\int 2^{3-5x}\,dx$.

SOLUTION

$$\int 2^{3-5x}\,dx = 2^3\int (2^{-5})^x\,dx$$
$$= 2^3\left[\frac{(2^{-5})^x}{\ln(2^{-5})}\right] + C$$
$$= \underline{\underline{-\frac{2^{3-5x}}{5\ln 2} + C}}$$

39. Find $\int \frac{dx}{7e^{4x-5}}$.

SOLUTION

$$\int \frac{dx}{7e^{4x-5}} = \frac{1}{7}\int e^{5-4x}dx$$
$$= \frac{e^5}{7}\int (e^{-4})^x\,dx$$
$$= \frac{e^5}{7}\left[\frac{(e^{-4})^x}{\ln(e^{-4})}\right] + C$$
$$= \frac{e^5}{7}\left(\frac{e^{-4x}}{-4}\right) + C$$
$$= \underline{\underline{-\frac{e^{5-4x}}{28} + C}}$$

40. Find $\int (e^x + e^{-x})^2 dx$.

SOLUTION

$$\begin{aligned}\int (e^x + e^{-x})^2 dx &= \int (e^{2x} + 2 + e^{-2x})dx \\ &= \int [(e^2)^x + 2 + (e^{-2})^x]dx \\ &= \frac{(e^2)^x}{\ln(e^2)} + 2x + \frac{(e^{-2})^x}{\ln(e^{-2})} + C \\ &= \underline{\underline{\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C}}\end{aligned}$$

41. Find $\int (e^x + 1)^3 dx$.

SOLUTION

$$\begin{aligned}\int (e^x + 1)^3 dx &= \int (e^{3x} + 3e^{2x} + 3e^x + 1)dx \\ &= \int [(e^3)^x + 3(e^2)^x + 3e^x + 1]dx \\ &= \frac{(e^3)^x}{\ln(e^3)} + 3\cdot\frac{(e^2)^x}{\ln(e^2)} + 3e^x + x + C \\ &= \underline{\underline{\frac{1}{3}e^{3x} + \frac{3}{2}e^{2x} + 3e^x + x + C}}\end{aligned}$$

42. Find $\int (6^{x+1} - 5x^6)dx$.

SOLUTION

$$\begin{aligned}\int (6^{x+1} - 5x^6)dx &= 6\int 6^x dx - 5\int x^6 dx \\ &= 6\cdot\frac{6^x}{\ln 6} - 5\cdot\frac{1}{7}x^7 + C \\ &= \underline{\underline{\frac{6^{x+1}}{\ln 6} - \frac{5}{7}x^7 + C}}\end{aligned}$$

43. Find $\int (5e^{5x} - 5x^{-5})dx$.

SOLUTION

$$\int (5e^{5x} - 5x^{-5})dx = 5\int (e^5)^x dx - 5\int x^{-5} dx$$
$$= 5 \cdot \frac{(e^5)^x}{\ln(e^5)} - 5 \cdot \frac{1}{-4} x^{-4} + C$$
$$= \underline{\underline{e^{5x} + \frac{5}{4x^4} + C}}$$

44. Find $\int (\frac{e^{3x}}{3} - \frac{4^x}{5})dx$.

SOLUTION

$$\int (\frac{e^{3x}}{3} - \frac{4^x}{5})dx = \frac{1}{3}\int (e^3)^x dx - \frac{1}{5}\int 4^x dx$$
$$= \frac{1}{3} \cdot \frac{(e^3)^x}{\ln(e^3)} - \frac{1}{5} \cdot \frac{4^x}{\ln 4} + C$$
$$= \underline{\underline{\frac{1}{9}e^{3x} - \frac{4^x}{5\ln 4} + C}}$$

Level 2

45. **(a)** Find $\frac{d}{dx}\sqrt{x^2 + 4x + 3}$.

(b) Hence find $\int \frac{x+2}{\sqrt{x^2 + 4x + 3}} dx$.

SOLUTION

(a) $$\frac{d}{dx}\sqrt{x^2 + 4x + 3} = (\frac{1}{2})(\frac{2x+4}{\sqrt{x^2 + 4x + 3}})$$
$$= \underline{\underline{\frac{x+2}{\sqrt{x^2 + 4x + 3}}}}$$

(b) $\because \quad \frac{d}{dx}\sqrt{x^2 + 4x + 3} = \frac{x+2}{\sqrt{x^2 + 4x + 3}}$

$\therefore \quad \int \frac{x+2}{\sqrt{x^2 + 4x + 3}} dx = \underline{\underline{\sqrt{x^2 + 4x + 3} + C}}$

46. **(a)** Find $\frac{d}{dx}(xe^x)$.

(b) Hence find $\int (x+1)e^x dx$.

SOLUTION

(a) $\frac{d}{dx}(xe^x) = \underline{\underline{xe^x + e^x}}$

(b) $\because \quad \frac{d}{dx}(xe^x) = xe^x + e^x$

$\therefore \quad \int (xe^x + e^x)dx = xe^x + C$

$\int (x+1)e^x dx = \underline{\underline{xe^x + C}}$

47. It is given that $f(x) = x\sqrt{x-3}$.

(a) Find $f'(x)$.

(b) Hence find $\int \frac{x-2}{\sqrt{x-3}} dx$.

SOLUTION

(a) $f'(x) = x(\frac{1}{2})(\frac{1}{\sqrt{x-3}}) + \sqrt{x-3}$

$= \frac{x + 2(x-3)}{2\sqrt{x-3}}$

$= \frac{3x-6}{2\sqrt{x-3}}$

$= \underline{\underline{\frac{3(x-2)}{2\sqrt{x-3}}}}$

(b) $\because \quad f'(x) = \frac{3(x-2)}{2\sqrt{x-3}}$

$\therefore \quad \int \frac{3(x-2)}{2\sqrt{x-3}} dx = x\sqrt{x-3} + C_1$

$\int \frac{x-2}{\sqrt{x-3}} dx = \underline{\underline{\frac{2}{3}x\sqrt{x-3} + C}}$ where $C = \frac{2}{3}C_1$

48. (a) Find $\dfrac{d}{dx}(x^2e^{-4x})$.

(b) Hence find $\int (2x-1)xe^{-4x}dx$.

SOLUTION

(a) $\dfrac{d}{dx}(x^2e^{-4x}) = 2xe^{-4x} + x^2(-4e^{-4x})$

$$= \underline{\underline{2(1-2x)xe^{-4x}}}$$

(b) $\because \quad \dfrac{d}{dx}(x^2e^{-4x}) = 2(1-2x)xe^{-4x}$

$\therefore \quad \int 2(1-2x)xe^{-4x}dx = x^2e^{-4x} + C_1$

$\int (2x-1)xe^{-4x}dx = \underline{\underline{-\dfrac{1}{2}x^2e^{-4x} + C}}$ where $C = -\dfrac{1}{2}C_1$

49. (a) Show that $\int \dfrac{xdx}{px^2+q} = \dfrac{1}{2p}\ln\left|px^2+q\right| + C$, where p and q are constants, $p \neq 0$ and $x^2 \neq -\dfrac{q}{p}$.

(b) Hence find the following indefinite integrals.

(i) $\int \dfrac{xdx}{3-5x^2}$

(ii) $\int \dfrac{xdx}{(3+2x)(2x-3)}$

SOLUTION

(a) $\dfrac{d}{dx}[\dfrac{1}{2p}\ln\left|px^2+q\right|] = \dfrac{1}{2p}\cdot\dfrac{2px}{px^2+q}$

$$= \frac{x}{px^2+q}$$

$\therefore \quad \int \dfrac{xdx}{px^2+q} = \dfrac{1}{2p}\ln\left|px^2+q\right| + C$

(b) (i) Take $p = -5$ and $q = 3$,

$\int \dfrac{xdx}{3-5x^2} = \dfrac{1}{2(-5)}\ln\left|3-5x^2\right| + C$

$$= \underline{\underline{-\frac{1}{10}\ln\left|3-5x^2\right| + C}}$$

(ii) $\displaystyle\int \frac{xdx}{(3+2x)(2x-3)} = \int \frac{xdx}{4x^2-9}$

Take $p = 4$ and $q = -9$,

$$\int \frac{xdx}{(3+2x)(2x-3)} = \frac{1}{2(4)}\ln\left|4x^2-9\right| + C$$

$$= \underline{\underline{\frac{1}{8}\ln\left|4x^2-9\right| + C}}$$

Level 3

50. (a) Show that $\displaystyle\int \frac{e^{px}dx}{e^{px}+1} = \frac{1}{p}\ln(e^{px}+1) + C$, where p is a non-zero constant.

(b) Hence find the following indefinite integrals.

(i) $\displaystyle\int \frac{e^{2x}dx}{e^{2x}+1}$

(ii) $\displaystyle\int \frac{dx}{e^{2x}+1}$

SOLUTION

(a) $\displaystyle\frac{d}{dx}[\frac{1}{p}\ln(e^{px}+1)] = \frac{1}{p}\cdot\frac{pe^{px}}{e^{px}+1}$

$$= \frac{e^{px}}{e^{px}+1}$$

$$\therefore \quad \int \frac{e^{px}dx}{e^{px}+1} = \frac{1}{p}\ln(e^{px}+1) + C$$

(b) (i) Take $p = 2$,

$$\int \frac{e^{2x}dx}{e^{2x}+1} = \underline{\underline{\frac{1}{2}\ln(e^{2x}+1) + C}}$$

(ii) $\displaystyle\int \frac{dx}{e^{2x}+1} = \int \frac{e^{-2x}dx}{e^{-2x}(e^{2x}+1)}$

$$= \int \frac{e^{-2x}dx}{1+e^{-2x}}$$

Take $p = -2$,

$$\int \frac{dx}{e^{2x}+1} = \frac{1}{-2}\ln(e^{-2x}+1) + C$$

$$= \underline{\underline{-\frac{1}{2}\ln(e^{-2x}+1) + C}}$$

51. (a) Show that $\int (1+\ln x)dx = x\ln x + C$.

(b) Hence find $\int \ln x\, dx$.

SOLUTION

(a) $\because$ $\frac{d}{dx}(x\ln x) = x(\frac{1}{x}) + \ln x$

$= 1 + \ln x$

$\therefore$ $\int (1+\ln x)dx = x\ln x + C$

(b)

$$\int (1+\ln x)dx = x\ln x + C$$

$$\int dx + \int \ln x\, dx = x\ln x + C$$

$$x + C_1 + \int \ln x\, dx = x\ln x + C$$

$$\int \ln x\, dx = \underline{\underline{x\ln x - x + C_2}} \text{ where } C_2 = C - C_1$$

52. (a) Find $\frac{d}{dx}(x^2 \ln x)$.

(b) Hence find $\int x\ln x\, dx$.

SOLUTION

(a) $\frac{d}{dx}(x^2 \ln x) = x^2(\frac{1}{x}) + 2x\ln x$

$= \underline{\underline{x + 2x\ln x}}$

(b) $\because$ $\frac{d}{dx}(x^2 \ln x) = x + 2x\ln x$

$\therefore$

$$\int (x + 2x\ln x)dx = x^2 \ln x + C$$

$$\int x dx + 2\int x\ln x\, dx = x^2 \ln x + C$$

$$\frac{1}{2}x^2 + C_1 + 2\int x\ln x\, dx = x^2 \ln x + C$$

$$2\int x\ln x\, dx = x^2 \ln x - \frac{1}{2}x^2 + C - C_1$$

$$\int x\ln x\, dx = \underline{\underline{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C_2}} \text{ where } C_2 = \frac{1}{2}(C - C_1)$$

53. **(a)** Find $\frac{d}{dx}(x^x)$, where $x > 0$.

(b) Find $\int x^x(1+\ln x)dx$, where $x > 0$.

SOLUTION

(a) Let $y = x^x$.

$$\ln y = \ln x^x$$
$$= x\ln x$$
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln x)$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = x(\frac{1}{x}) + \ln x$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \ln x$$
$$\frac{dy}{dx} = y(1+\ln x)$$
$$= x^x(1+\ln x)$$
$$\therefore \quad \frac{d}{dx}(x^x) = \underline{\underline{x^x(1+\ln x)}}$$

(b) $\because \quad \frac{d}{dx}(x^x) = x^x(1+\ln x)$

$\therefore \quad \int x^x(1+\ln x)dx = \underline{\underline{x^x + C}}$

EXERCISE 7B

Level 1

54. Find $\int (x+2)^2 dx$.

SOLUTION

$$\int (x+2)^2 dx = \int (x+2)^2 d(x+2)$$
$$= \underline{\underline{\frac{1}{3}(x+2)^3 + C}}$$

55. Find $\int (8-x)^6 dx$.

SOLUTION

$$\int (8-x)^6 dx = -\int (8-x)^6 d(8-x)$$
$$= \underline{\underline{-\frac{1}{7}(8-x)^7 + C}}$$

56. Find $\int \frac{dx}{x+4}$.

SOLUTION

$$\int \frac{dx}{x+4} = \int \frac{d(x+4)}{x+4}$$
$$= \underline{\underline{\ln|x+4| + C}}$$

57. Find $\int \frac{dx}{1-3x}$.

SOLUTION

$$\int \frac{dx}{1-3x} = -\frac{1}{3}\int \frac{d(1-3x)}{1-3x}$$
$$= \underline{\underline{-\frac{1}{3}\ln|1-3x| + C}}$$

58. Find $\int \sqrt{2x+6}\, dx$.

SOLUTION

$$\int \sqrt{2x+6}\, dx = \int \frac{1}{2}\sqrt{2x+6}\, d(2x+6)$$
$$= \frac{1}{2}\cdot\frac{2}{3}(2x+6)^{\frac{3}{2}} + C$$
$$= \underline{\underline{\frac{1}{3}(2x+6)^{\frac{3}{2}} + C}}$$

59. Find $\int \frac{dx}{\sqrt{8-7x}}$.

SOLUTION

$$\begin{aligned}\int \frac{dx}{\sqrt{8-7x}} &= -\frac{1}{7}\int \frac{d(8-7x)}{\sqrt{8-7x}} \\ &= -\frac{1}{7}\cdot\frac{1}{2}\sqrt{8-7x}+C \\ &= \underline{\underline{-\frac{1}{14}\sqrt{8-7x}+C}}\end{aligned}$$

60. Find $\int (1-2x)^{\frac{1}{3}}dx$.

SOLUTION

$$\begin{aligned}\int (1-2x)^{\frac{1}{3}}dx &= -\frac{1}{2}\int (1-2x)^{\frac{1}{3}}d(1-2x) \\ &= -\frac{1}{2}\cdot\frac{3}{4}(1-2x)^{\frac{4}{3}}+C \\ &= \underline{\underline{-\frac{3}{8}(1-2x)^{\frac{4}{3}}+C}}\end{aligned}$$

61. Find $\int \frac{dx}{(5-\frac{x}{3})^4}$.

SOLUTION

$$\begin{aligned}\int \frac{dx}{(5-\frac{x}{3})^4} &= -3\int \frac{d(5-\frac{x}{3})}{(5-\frac{x}{3})^4} \\ &= -3\cdot\frac{1}{-3}(5-\frac{x}{3})^{-3}+C \\ &= \underline{\underline{(5-\frac{x}{3})^{-3}+C}}\end{aligned}$$

62. Find $\int \frac{x^2+6x+9}{x^4-18x^2+81}dx$.

SOLUTION

$$\begin{aligned}\int \frac{x^2+6x+9}{x^4-18x^2+81}dx &= \int \frac{(x+3)^2}{(x^2-9)^2}dx \\ &= \int \frac{(x+3)^2}{(x+3)^2(x-3)^2}dx \\ &= \int \frac{1}{(x-3)^2}dx \\ &= \int \frac{1}{(x-3)^2}d(x-3) \\ &= \underline{\underline{-\frac{1}{x-3}+C}}\end{aligned}$$

63. Find $\int x(x^2+3)dx$.

SOLUTION

$$\begin{aligned}\int x(x^2+3)dx &= \int \frac{1}{2}(x^2+3)d(x^2+3) \\ &= \frac{1}{2}\cdot\frac{1}{2}(x^2+3)^2+C \\ &= \underline{\underline{\frac{1}{4}(x^2+3)^2+C}}\end{aligned}$$

64. Find $\int x^4\sqrt{3-2x^5}dx$.

SOLUTION

$$\begin{aligned}\int x^4\sqrt{3-2x^5}dx &= -\frac{1}{10}\int \sqrt{3-2x^5}\,d(3-2x^5) \\ &= -\frac{1}{10}\cdot\frac{2}{3}(3-2x^5)^{\frac{3}{2}}+C \\ &= \underline{\underline{-\frac{1}{15}(3-2x^5)^{\frac{3}{2}}+C}}\end{aligned}$$

65. Find $\int x^n(px^{n+1}+q)^r\,dx$, where n, p, q and r are non-zero constants, $n \neq -1$ and $r \neq -1$.

SOLUTION

$$\begin{aligned}\int x^n(px^{n+1}+q)^r\,dx &= \int \frac{1}{p(n+1)}\cdot(px^{n+1}+q)^r\,d(px^{n+1}+q)\\ &= \frac{1}{p(n+1)}\cdot\frac{(px^{n+1}+q)^{r+1}}{r+1}+C\\ &= \underline{\underline{\frac{(px^{n+1}+q)^{r+1}}{p(n+1)(r+1)}+C}}\end{aligned}$$

66. Find $\int (x^2+3x-5)^3(2x+3)dx$.

SOLUTION

$$\begin{aligned}\int (x^2+3x-5)^3(2x+3)dx &= \int (x^2+3x-5)^3\,d(x^2+3x-5)\\ &= \underline{\underline{\frac{1}{4}(x^2+3x-5)^4+C}}\end{aligned}$$

67. Find $\int (3x^2-2)\sqrt{x^3-2x}\,dx$.

SOLUTION

$$\begin{aligned}\int (3x^2-2)\sqrt{x^3-2x}\,dx &= \int \sqrt{x^3-2x}\,d(x^3-2x)\\ &= \underline{\underline{\frac{2}{3}(x^3-2x)^{\frac{3}{2}}+C}}\end{aligned}$$

68. Find $\int \frac{x-3}{(x^2-6x+5)^3}dx$.

SOLUTION

$$\begin{aligned}\int \frac{x-3}{(x^2-6x+5)^3}dx &= \int \frac{1}{2}\cdot\frac{2x-6}{(x^2-6x+5)^3}dx\\ &= \int \frac{1}{2}\cdot\frac{1}{(x^2-6x+5)^3}d(x^2-6x+5)\\ &= \frac{1}{2}\cdot\frac{1}{-2}(x^2-6x+5)^{-2}+C\\ &= \underline{\underline{-\frac{1}{4(x^2-6x+5)^2}+C}}\end{aligned}$$

69. Find $\int \frac{x^2-1}{x^3-3x-5}\,dx$.

SOLUTION

$$\int \frac{x^2-1}{x^3-3x-5}\,dx = \int \frac{1}{3}\cdot\frac{3x^2-3}{x^3-3x-5}\,dx$$
$$= \int \frac{1}{3}\cdot\frac{1}{x^3-3x-5}\,d(x^3-3x-5)$$
$$= \underline{\underline{\frac{1}{3}\ln\left|x^3-3x-5\right|+C}}$$

70. Find $\int \frac{x(x+1)(x-1)}{(7+2x^2-x^4)^{\frac{3}{4}}}\,dx$.

SOLUTION

$$\int \frac{x(x+1)(x-1)}{(7+2x^2-x^4)^{\frac{3}{4}}}\,dx = \int \frac{x^3-x}{(7+2x^2-x^4)^{\frac{3}{4}}}\,dx$$
$$= -\frac{1}{4}\int \frac{-4x^3+4x}{(7+2x^2-x^4)^{\frac{3}{4}}}\,dx$$
$$= -\frac{1}{4}\int \frac{1}{(7+2x^2-x^4)^{\frac{3}{4}}}\,d(7+2x^2-x^4)$$
$$= -\frac{1}{4}\cdot 4(7+2x^2-x^4)^{\frac{1}{4}}+C$$
$$= \underline{\underline{-(7+2x^2-x^4)^{\frac{1}{4}}+C}}$$

71. Find $\int \frac{1}{x^2}(1+\frac{1}{x})^{\frac{5}{2}}\,dx$.

SOLUTION

$$\int \frac{1}{x^2}(1+\frac{1}{x})^{\frac{5}{2}}\,dx = -\int (1+\frac{1}{x})^{\frac{5}{2}}\,d(1+\frac{1}{x})$$
$$= \underline{\underline{-\frac{2}{7}(1+\frac{1}{x})^{\frac{7}{2}}+C}}$$

72. Find $\int \frac{dx}{x^2\sqrt{\frac{3}{2}-\frac{2}{x}}}$.

SOLUTION

$$\int \frac{dx}{x^2\sqrt{\frac{3}{2}-\frac{2}{x}}} = \int \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}-\frac{2}{x}}}\,d(\frac{3}{2}-\frac{2}{x})$$

$$=\frac{1}{2}\cdot 2\sqrt{\frac{3}{2}-\frac{2}{x}}+C$$

$$=\underline{\underline{\sqrt{\frac{3}{2}-\frac{2}{x}}+C}}$$

73. Find $\int e^{4x+7}dx$.

SOLUTION

$$\int e^{4x+7}dx = \int \frac{1}{4}e^{4x+7}d(4x+7)$$

$$=\underline{\underline{\frac{1}{4}e^{4x+7}+C}}$$

74. Find $\int e^{5-3x}dx$.

SOLUTION

$$\int e^{5-3x}dx = -\frac{1}{3}\int e^{5-3x}d(5-3x)$$

$$=\underline{\underline{-\frac{1}{3}e^{5-3x}+C}}$$

75. Find $\int (e^{5x}+3e^{-3x})dx$.

SOLUTION

$$\int (e^{5x}+3e^{-3x})dx = \int e^{5x}dx + 3\int e^{-3x}dx$$

$$=\int \frac{1}{5}e^{5x}d(5x) + 3\int (-\frac{1}{3})e^{-3x}d(-3x)$$

$$=\underline{\underline{\frac{1}{5}e^{5x}-e^{-3x}+C}}$$

76. Find $\int 2xe^{x^2}dx$.

SOLUTION

$$\int 2xe^{x^2}dx = \int e^{x^2}d(x^2)$$
$$= \underline{\underline{e^{x^2} + C}}$$

77. Find $\int x^2e^{-x^3}dx$.

SOLUTION

$$\int x^2e^{-x^3}dx = -\frac{1}{3}\int e^{-x^3}d(-x^3)$$
$$= \underline{\underline{-\frac{1}{3}e^{-x^3} + C}}$$

78. Find $\int \frac{\ln 6x}{x}dx$.

SOLUTION

Let $u = \ln 6x$,

then $du = \frac{1}{6x}\cdot 6dx$

$= \frac{dx}{x}$

$$\therefore \quad \int \frac{\ln 6x}{x}dx = \int udu$$
$$= \frac{u^2}{2} + C$$
$$= \underline{\underline{\frac{1}{2}(\ln 6x)^2 + C}}$$

79. Find $\int \frac{dx}{2x\ln\frac{x}{2}}$.

SOLUTION

Let $u = \ln\frac{x}{2}$,

then $du = \frac{2}{x}\cdot\frac{1}{2}dx$

$= \frac{dx}{x}$

$$\therefore \quad \int \frac{dx}{2x\ln\frac{x}{2}} = \int \frac{du}{2u}$$
$$= \frac{1}{2}\ln|u| + C$$
$$= \underline{\underline{\frac{1}{2}\ln\left|\ln\frac{x}{2}\right| + C}}$$

80. Find $\int \frac{dx}{\sqrt{x}(4\sqrt{x}-1)}$.

SOLUTION

$$\int \frac{dx}{\sqrt{x}(4\sqrt{x}-1)} = \int \frac{1}{2}\cdot\frac{1}{4\sqrt{x}-1}d(4\sqrt{x}-1)$$
$$= \underline{\underline{\frac{1}{2}\ln\left|4\sqrt{x}-1\right| + C}}$$

Level 2

81. Find $\int \frac{e^x}{e^{2x}+2e^x+1}dx$.

SOLUTION

$$\int \frac{e^x}{e^{2x}+2e^x+1}dx = \int \frac{e^x}{(e^x+1)^2}dx$$
$$= \int \frac{1}{(e^x+1)^2}d(e^x+1)$$
$$= \underline{\underline{-\frac{1}{e^x+1} + C}}$$

82. Find $\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx$.

SOLUTION

$$\int \frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}dx = \int \frac{1}{2}\cdot\frac{2e^{2x}+2e^{-2x}}{e^{2x}-e^{-2x}}dx$$
$$= \int \frac{1}{2}\cdot\frac{1}{e^{2x}-e^{-2x}}d(e^{2x}-e^{-2x})$$
$$= \underline{\underline{\frac{1}{2}\ln\left|e^{2x}-e^{-2x}\right| + C}}$$

83. Find $\int x(2^{x^2+2})dx$.

SOLUTION

$$\int x(2^{x^2+2})dx = \int \frac{1}{2}\cdot 2^{x^2+2}d(x^2+2)$$
$$= \frac{2^{x^2+2}}{2\ln 2}+C$$
$$= \underline{\underline{\frac{2^{x^2+1}}{\ln 2}+C}}$$

84. Find $\int \frac{5^{\sqrt{x}-2}}{\sqrt{x}}dx$.

SOLUTION

$$\int \frac{5^{\sqrt{x}-2}}{\sqrt{x}}dx = \int 2\cdot 5^{\sqrt{x}-2}d(\sqrt{x}-2)$$
$$= \underline{\underline{\frac{2(5^{\sqrt{x}-2})}{\ln 5}+C}}$$

85. Find $\int \frac{\sqrt{\ln\sqrt{x}}}{x}dx$.

SOLUTION

$$\int \frac{\sqrt{\ln\sqrt{x}}}{x}dx = \int \frac{\sqrt{\frac{1}{2}\ln x}}{x}dx$$
$$= \int \sqrt{\frac{1}{2}}\cdot\sqrt{\ln x}d(\ln x)$$
$$= \frac{1}{\sqrt{2}}\cdot\frac{2}{3}(\ln x)^{\frac{3}{2}}+C$$
$$= \underline{\underline{\frac{\sqrt{2}}{3}(\ln x)^{\frac{3}{2}}+C}}$$

86. Find $\int \frac{dx}{6x\ln x^6}$.

SOLUTION

$$\begin{aligned}\int \frac{dx}{6x\ln x^6} &= \int \frac{dx}{36x\ln x} \\ &= \int \frac{1}{36\ln x} d(\ln x) \\ &= \underline{\underline{\frac{1}{36}\ln|\ln x| + C}}\end{aligned}$$

87. **(a)** Express $\log_{10} x$ in terms of $\ln x$.

(b) Given that $x > 0$, find $\int \log_{10} x^{\frac{1}{x}} dx$.

SOLUTION

(a) $\log_{10} x = \underline{\underline{\frac{\ln x}{\ln 10}}}$

(b)
$$\begin{aligned}\int \log_{10} x^{\frac{1}{x}} dx &= \int \frac{\log_{10} x}{x} dx \\ &= \int \frac{\ln x}{x\ln 10} dx \\ &= \frac{1}{\ln 10}\int \ln x d(\ln x) \\ &= \frac{1}{\ln 10}\cdot\frac{1}{2}(\ln x)^2 + C \\ &= \underline{\underline{\frac{(\ln x)^2}{2\ln 10} + C}}\end{aligned}$$

88. **(a)** Express $\log_7 x$ in terms of $\ln x$.

(b) Given that $x > 0$, find $\int \frac{dx}{x\log_7 x^2}$.

SOLUTION

(a) $\log_7 = \underline{\underline{\frac{\ln x}{\ln 7}}}$

(b) $$\int \frac{dx}{x\log_7 x^2} = \int \frac{dx}{2x\log_7 x}$$
$$= \frac{1}{2}\int \frac{\ln 7}{x\ln x}dx$$
$$= \frac{\ln 7}{2}\int \frac{1}{\ln x}d(\ln x)$$
$$= \underline{\underline{\frac{\ln 7\ln|\ln x|}{2} + C}}$$

89. Find $\int x(x-2)^7 dx$.

SOLUTION

Let $u = x-2$, then $du = dx$.

When $u = x-2$, $x = u+2$.

$$\therefore \quad \int x(x-2)^7 dx = \int (u+2)u^7 du$$
$$= \int (u^8 + 2u^7)du$$
$$= \frac{1}{9}u^9 + 2\cdot\frac{1}{8}u^8 + C$$
$$= \underline{\underline{\frac{1}{9}(x-2)^9 + \frac{1}{4}(x-2)^8 + C}}$$

90. Find $\int (x+1)(5-x)^{2\,009} dx$.

SOLUTION

Let $u = 5-x$, then $du = -dx$.

When $u = 5-x$, $x = 5-u$.

$$\therefore \quad \int (x+1)(5-x)^{2\,009} dx = -\int [(5-u)+1]u^{2\,009} du$$
$$= \int (u-6)u^{2\,009} du$$
$$= \int (u^{2\,010} - 6u^{2\,009})du$$
$$= \frac{1}{2\,011}u^{2\,011} - 6\cdot\frac{1}{2\,010}u^{2\,010} + C$$
$$= \underline{\underline{\frac{1}{2\,011}(5-x)^{2\,011} - \frac{1}{335}(5-x)^{2\,010} + C}}$$

91. Find $\int \frac{x+3}{x-1}dx$.

SOLUTION

Let $u = x-1$, then $du = dx$.

When $u = x-1$, $x = u+1$.

$$\begin{aligned}\therefore \quad \int \frac{x+3}{x-1}dx &= \int \frac{(u+1)+3}{u}du \\ &= \int \frac{u+4}{u}du \\ &= \int (1+\frac{4}{u})du \\ &= u + 4\ln|u| + C_1 \\ &= x-1+4\ln|x-1|+C_1 \\ &= \underline{\underline{x+4\ln|x-1|+C}} \text{ where } C = C_1 - 1\end{aligned}$$

92. Find $\int \frac{2x+3}{x+2}dx$.

SOLUTION

Let $u = x+2$, then $du = dx$.

When $u = x+2$, $x = u-2$.

$$\begin{aligned}\therefore \quad \int \frac{2x+3}{x+2}dx &= \int \frac{2(u-2)+3}{u}du \\ &= \int \frac{2u-1}{u}du \\ &= \int (2-\frac{1}{u})du \\ &= 2u - \ln|u| + C_1 \\ &= 2(x+2) - \ln|x+2| + C_1 \\ &= \underline{\underline{2x - \ln|x+2| + C}} \text{ where } C = C_1 + 4\end{aligned}$$

93. Find $\int \frac{3xdx}{\sqrt{4x-1}}$.

SOLUTION

Let $u = 4x-1$, then $du = 4dx$.

When $u = 4x-1$, $x = \frac{u+1}{4}$.

$$\therefore \quad \int \frac{3xdx}{\sqrt{4x-1}} = \int \frac{3x}{4\sqrt{4x-1}} \cdot 4dx$$

$$= \frac{3}{4}\int \frac{u+1}{4} \cdot \frac{1}{\sqrt{u}} du$$

$$= \frac{3}{16}\int (u^{\frac{1}{2}} + u^{-\frac{1}{2}})du$$

$$= \frac{3}{16}(\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}) + C$$

$$= \frac{1}{8}u^{\frac{3}{2}} + \frac{3}{8}u^{\frac{1}{2}} + C$$

$$= \underline{\underline{\frac{1}{8}(4x-1)^{\frac{3}{2}} + \frac{3}{8}(4x-1)^{\frac{1}{2}} + C}}$$

94. Find $\int \frac{xdx}{(2x+3)^{\frac{5}{3}}}$.

SOLUTION

Let $u = 2x+3$, then $du = 2dx$.

When $u = 2x+3$, $x = \frac{u-3}{2}$.

$$\therefore \quad \int \frac{xdx}{(2x+3)^{\frac{5}{3}}} = \int \frac{x}{2(2x+3)^{\frac{5}{3}}} \cdot 2dx$$

$$= \frac{1}{2}\int \frac{u-3}{2} \cdot \frac{1}{u^{\frac{5}{3}}} du$$

$$= \frac{1}{4}\int (u^{-\frac{2}{3}} - 3u^{-\frac{5}{3}})du$$

$$= \frac{1}{4}[3u^{\frac{1}{3}} - 3(-\frac{3}{2}u^{-\frac{2}{3}})] + C$$

$$= \frac{3}{4}u^{\frac{1}{3}} + \frac{9}{8}u^{-\frac{2}{3}} + C$$

$$= \underline{\underline{\frac{3}{4}(2x+3)^{\frac{1}{3}} + \frac{9}{8}(2x+3)^{-\frac{2}{3}} + C}}$$

95. Find $\int x^2(3x-1)^{\frac{1}{3}}dx$.

SOLUTION

Let $u=3x-1$, then $du=3dx$.

When $u=3x-1$, $x=\frac{u+1}{3}$.

$$\begin{aligned}
\therefore \quad \int x^2(3x-1)^{\frac{1}{3}}dx &= \int \frac{1}{3}x^2(3x-1)^{\frac{1}{3}}\cdot 3dx \\
&= \int \frac{1}{3}\left(\frac{u+1}{3}\right)^2 u^{\frac{1}{3}}du \\
&= \frac{1}{27}\int (u^2+2u+1)u^{\frac{1}{3}}du \\
&= \frac{1}{27}\int (u^{\frac{7}{3}}+2u^{\frac{4}{3}}+u^{\frac{1}{3}})du \\
&= \frac{1}{27}\left(\frac{3}{10}u^{\frac{10}{3}}+2\cdot\frac{3}{7}u^{\frac{7}{3}}+\frac{3}{4}u^{\frac{4}{3}}\right)+C \\
&= \frac{1}{90}u^{\frac{10}{3}}+\frac{2}{63}u^{\frac{7}{3}}+\frac{1}{36}u^{\frac{4}{3}}+C \\
&= \underline{\underline{\frac{1}{90}(3x-1)^{\frac{10}{3}}+\frac{2}{63}(3x-1)^{\frac{7}{3}}+\frac{1}{36}(3x-1)^{\frac{4}{3}}+C}}
\end{aligned}$$

96. Find $\int \frac{x^2dx}{\sqrt{5-2x}}$.

SOLUTION

Let $u=5-2x$, then $du=-2dx$.

When $u=5-2x$, $x=\frac{5-u}{2}$.

$$\therefore \quad \int \frac{x^2dx}{\sqrt{5-2x}} = -\frac{1}{2}\int \frac{x^2}{\sqrt{5-2x}} \cdot (-2)dx$$
$$= -\frac{1}{2}\int (\frac{5-u}{2})^2 \cdot \frac{1}{\sqrt{u}}du$$
$$= -\frac{1}{8}\int \frac{u^2-10u+25}{\sqrt{u}}du$$
$$= -\frac{1}{8}\int (u^{\frac{3}{2}}-10u^{\frac{1}{2}}+25u^{-\frac{1}{2}})du$$
$$= -\frac{1}{8}(\frac{2}{5}u^{\frac{5}{2}}-10\cdot\frac{2}{3}u^{\frac{3}{2}}+25\cdot 2u^{\frac{1}{2}})+C$$
$$= -\frac{1}{20}u^{\frac{5}{2}}+\frac{5}{6}u^{\frac{3}{2}}-\frac{25}{4}u^{\frac{1}{2}}+C$$
$$= \underline{\underline{-\frac{1}{20}(5-2x)^{\frac{5}{2}}+\frac{5}{6}(5-2x)^{\frac{3}{2}}-\frac{25}{4}(5-2x)^{\frac{1}{2}}+C}}$$

97. Let $u = x^2+1$, find $\int \frac{4x^3}{x^2+1}dx$.

SOLUTION

Let $u = x^2+1$, then $du = 2xdx$.

When $u = x^2+1$, $x^2 = u-1$.

$$\therefore \quad \int \frac{4x^3}{x^2+1}dx = \int \frac{2x^2}{x^2+1}\cdot 2xdx$$
$$= \int \frac{2(u-1)}{u}du$$
$$= 2\int (1-\frac{1}{u})du$$
$$= 2u - 2\ln|u| + C_1$$
$$= 2(x^2+1) - 2\ln(x^2+1) + C_1$$
$$= \underline{\underline{2x^2 - 2\ln(x^2+1) + C}} \text{ where } C = C_1 + 2$$

98. Let $u = x^3-2$, find $\int \frac{x^5}{x^3-2}dx$.

SOLUTION

Let $u = x^3-2$, then $du = 3x^2dx$.

When $u = x^3-2$, $x^3 = u+2$.

$$\begin{aligned}\therefore \quad \int \frac{x^5}{x^3-2}dx &= \int \frac{x^3}{3(x^3-2)}\cdot 3x^2dx \\ &= \int \frac{u+2}{3u}du \\ &= \frac{1}{3}\int (1+\frac{2}{u})du \\ &= \frac{1}{3}(u+2\ln|u|)+C_1 \\ &= \frac{1}{3}u+\frac{2}{3}\ln|u|+C_1 \\ &= \frac{1}{3}(x^3-2)+\frac{2}{3}\ln|x^3-2|+C_1 \\ &= \underline{\underline{\frac{1}{3}x^3+\frac{2}{3}\ln|x^3-2|+C}} \text{ where } C=C_1-\frac{2}{3}\end{aligned}$$

99. Let $u=x^3+5$, find $\int x^5(x^3+5)^{\frac{3}{4}}dx$.

SOLUTION

Let $u=x^3+5$, then $du=3x^2dx$.

When $u=x^3+5$, $x^3=u-5$.

$$\begin{aligned}\therefore \quad \int x^5(x^3+5)^{\frac{3}{4}}dx &= \int \frac{1}{3}\cdot x^3(x^3+5)^{\frac{3}{4}}\cdot 3x^2dx \\ &= \frac{1}{3}\int (u-5)u^{\frac{3}{4}}du \\ &= \frac{1}{3}\int (u^{\frac{7}{4}}-5u^{\frac{3}{4}})du \\ &= \frac{1}{3}(\frac{4}{11}u^{\frac{11}{4}}-5\cdot\frac{4}{7}u^{\frac{7}{4}})+C \\ &= \frac{4}{33}u^{\frac{11}{4}}-\frac{20}{21}u^{\frac{7}{4}}+C \\ &= \underline{\underline{\frac{4}{33}(x^3+5)^{\frac{11}{4}}-\frac{20}{21}(x^3+5)^{\frac{7}{4}}+C}}\end{aligned}$$

100. Let $u=1-2x^2$, find $\int \frac{2x^3dx}{(1-2x^2)^{\frac{1}{3}}}$.

SOLUTION

Let $u=1-2x^2$, then $du=-4xdx$. When $u=1-2x^2$, $x^2=\frac{1-u}{2}$.

$$\begin{aligned}
\therefore \quad \int \frac{2x^3dx}{(1-2x^2)^{\frac{1}{3}}} &= -\int \frac{x^2}{2(1-2x^2)^{\frac{1}{3}}}\cdot(-4x)dx \\
&= -\frac{1}{2}\int \frac{1-u}{2}\cdot\frac{1}{u^{\frac{1}{3}}}du \\
&= -\frac{1}{4}\int (u^{-\frac{1}{3}}-u^{\frac{2}{3}})du \\
&= -\frac{1}{4}(\frac{3}{2}u^{\frac{2}{3}}-\frac{3}{5}u^{\frac{5}{3}})+C \\
&= -\frac{3}{8}u^{\frac{2}{3}}+\frac{3}{20}u^{\frac{5}{3}}+C \\
&= \underline{\underline{-\frac{3}{8}(1-2x^2)^{\frac{2}{3}}+\frac{3}{20}(1-2x^2)^{\frac{5}{3}}+C}}
\end{aligned}$$

101. Let $u=x^2-2$, find $\int \frac{x(x^2-3)(x^2+1)}{x^2-2}dx$.

SOLUTION

Let $u=x^2-2$, then $du=2xdx$. When $u=x^2-2$, $x^2=u+2$.

$$\begin{aligned}
\therefore \quad \int \frac{x(x^2-3)(x^2+1)}{x^2-2}dx &= \int \frac{(x^2-3)(x^2+1)}{2(x^2-2)}\cdot 2xdx \\
&= \frac{1}{2}\int \frac{[(u+2)-3][(u+2)+1]}{u}du \\
&= \frac{1}{2}\int \frac{(u-1)(u+3)}{u}du \\
&= \frac{1}{2}\int \frac{u^2+2u-3}{u}du \\
&= \frac{1}{2}\int (u+2-\frac{3}{u})du \\
&= \frac{1}{2}(\frac{1}{2}u^2+2u-3\ln|u|)+C_1 \\
&= \frac{1}{4}u^2+u-\frac{3}{2}\ln|u|+C_1 \\
&= \frac{1}{4}(x^2-2)^2+x^2-2-\frac{3}{2}\ln|x^2-2|+C_1 \\
&= \underline{\underline{\frac{1}{4}(x^2-2)^2+x^2-\frac{3}{2}\ln|x^2-2|+C}} \text{ where } C=C_1-2
\end{aligned}$$

102. Find $\int \frac{x^2-4x-12}{x^2-4x+4}dx$.

SOLUTION

$$\int \frac{x^2-4x-12}{x^2-4x+4}dx = \int \frac{(x-6)(x+2)}{(x-2)^2}dx$$

Let $u = x-2$, then $du = dx$. When $u = x-2$, $x = u+2$.

$$\begin{aligned}
\therefore \quad \int \frac{x^2-4x-12}{x^2-4x+4}dx &= \int \frac{[(u+2)-6][(u+2)+2]}{u^2}du \\
&= \int \frac{(u-4)(u+4)}{u^2}du \\
&= \int \frac{u^2-16}{u^2}du \\
&= \int (1-\frac{16}{u^2})du \\
&= u+\frac{16}{u}+C_1 \\
&= x-2+\frac{16}{x-2}+C_1 \\
&= \underline{\underline{x+\frac{16}{x-2}+C}} \quad \text{where } C = C_1-2
\end{aligned}$$

103. Find $\int \frac{x^2-2x+5}{x^2+2x+1}dx$.

SOLUTION

$$\int \frac{x^2-2x+5}{x^2+2x+1}dx = \int \frac{x^2-2x+5}{(x+1)^2}dx$$

Let $u = x+1$, then $du = dx$. When $u = x+1$, $x = u-1$.

$$\begin{aligned}
\therefore \quad \int \frac{x^2-2x+5}{x^2+2x+1}dx &= \int \frac{(u-1)^2-2(u-1)+5}{u^2}du \\
&= \int \frac{u^2-4u+8}{u^2}du \\
&= \int (1-\frac{4}{u}+\frac{8}{u^2})du \\
&= u-4\ln|u|-\frac{8}{u}+C_1 \\
&= x+1-4\ln|x+1|-\frac{8}{x+1}+C_1 \\
&= \underline{\underline{x-4\ln|x+1|-\frac{8}{x+1}+C}} \quad \text{where } C = C_1+1
\end{aligned}$$

104. It is given that $x > 0$. Find $\int \sqrt{x^4 + 2x^2}\, dx$.

SOLUTION

$$\begin{aligned}\int \sqrt{x^4 + 2x^2}\, dx &= \int x\sqrt{x^2+2}\, dx \\ &= \int \frac{1}{2}\sqrt{x^2+2}\, d(x^2+2) \\ &= \frac{1}{2} \cdot \frac{2}{3}(x^2+2)^{\frac{3}{2}} + C \\ &= \underline{\underline{\frac{1}{3}(x^2+2)^{\frac{3}{2}} + C}}\end{aligned}$$

105. Given that $x > 0$, use the substitution $u = \frac{1}{x^2}$ to find $\int \frac{\sqrt{x^2+3}}{x^4}\, dx$.

SOLUTION

Let $u = \frac{1}{x^2}$, then $du = -\frac{2}{x^3}\, dx$.

$$\begin{aligned}\therefore \quad \int \frac{\sqrt{x^2+3}}{x^4}\, dx &= -\frac{1}{2}\int \frac{\sqrt{x^2+3}}{x} \cdot (-\frac{2}{x^3}) dx \\ &= -\frac{1}{2}\int \sqrt{1+\frac{3}{x^2}} \cdot (-\frac{2}{x^3}) dx \\ &= -\frac{1}{2}\int \sqrt{1+3u}\, du \\ &= -\frac{1}{2}\int \frac{1}{3}\sqrt{1+3u}\, d(1+3u) \\ &= -\frac{1}{6} \cdot \frac{2}{3}(1+3u)^{\frac{3}{2}} + C \\ &= \underline{\underline{-\frac{1}{9}(1+\frac{3}{x^2})^{\frac{3}{2}} + C}}\end{aligned}$$

106. Given that $x > 0$, use the substitution $u = \frac{1}{x}$ to find $\int \frac{\sqrt[3]{4x^3 - 2}}{x^5}\, dx$.

SOLUTION

Let $u = \frac{1}{x}$, then $du = -\frac{1}{x^2} dx$.

$$\therefore \quad \int \frac{\sqrt[3]{4x^3-2}}{x^5}dx = -\int \frac{(4x^3-2)^{\frac{1}{3}}}{x^3}\cdot(-\frac{1}{x^2})dx$$

$$= -\int \frac{1}{x^2}(4-\frac{2}{x^3})^{\frac{1}{3}}\cdot(-\frac{1}{x^2})dx$$

$$= -\int u^2(4-2u^3)^{\frac{1}{3}}du$$

$$= -\int (-\frac{1}{6})(4-2u^3)^{\frac{1}{3}}d(4-2u^3)$$

$$= \frac{1}{6}\cdot\frac{3}{4}(4-2u^3)^{\frac{4}{3}}+C$$

$$= \underline{\underline{\frac{1}{8}(4-\frac{2}{x^3})^{\frac{4}{3}}+C}}$$

107. (a) Prove that $\frac{4}{x^2-4} = \frac{1}{x-2} - \frac{1}{x+2}$.

(b) Hence find $\int \frac{dx}{x^2-4}$.

SOLUTION

(a) R.H.S. $= \frac{1}{x-2} - \frac{1}{x+2}$

$$= \frac{(x+2)-(x-2)}{(x-2)(x+2)}$$

$$= \frac{4}{x^2-4}$$

$=$ L.H.S.

$$\therefore \quad \frac{4}{x^2-4} = \frac{1}{x-2} - \frac{1}{x+2}$$

(b) $\int \frac{dx}{x^2-4} = \int \frac{1}{4}\cdot\frac{4}{x^2-4}dx$

$$= \frac{1}{4}\int (\frac{1}{x-2}-\frac{1}{x+2})dx \quad \text{[From the result of \textbf{(a)}]}$$

$$= \frac{1}{4}(\int \frac{1}{x-2}dx - \int \frac{1}{x+2}dx)$$

$$= \frac{1}{4}[\int \frac{1}{x-2}d(x-2) - \int \frac{1}{x+2}d(x+2)]$$

$$= \frac{1}{4}(\ln|x-2| - \ln|x+2|)+C$$

$$= \underline{\underline{\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C}}$$

108. (a) If $\dfrac{2x+1}{[x(x+1)]^2} \equiv \dfrac{A}{x^2} + \dfrac{B}{(x+1)^2}$, find the values of constants A and B.

(b) Hence find $\displaystyle\int \frac{2x+1}{[x(x+1)]^2}\,dx$.

SOLUTION

(a) $$\frac{2x+1}{[x(x+1)]^2} \equiv \frac{A}{x^2} + \frac{B}{(x+1)^2} \equiv \frac{A(x+1)^2 + Bx^2}{x^2(x+1)^2}$$

i.e.
$$A(x+1)^2 + Bx^2 \equiv 2x+1$$
$$A(x^2+2x+1) + Bx^2 \equiv 2x+1$$
$$(A+B)x^2 + 2Ax + A \equiv 2x+1$$

$$\therefore \begin{cases} A+B=0 \\ 2A=2 \\ A=1 \end{cases}$$

$$\therefore \quad \underline{\underline{A=1,\ B=-1}}$$

(b) $$\int \frac{2x+1}{[x(x+1)]^2}\,dx = \int \left[\frac{1}{x^2} + \frac{-1}{(x+1)^2}\right]dx \quad \text{[From the result of (a)]}$$
$$= \int \frac{1}{x^2}\,dx - \int \frac{1}{(x+1)^2}\,dx$$
$$= \int \frac{1}{x^2}\,dx - \int \frac{1}{(x+1)^2}\,d(x+1)$$
$$= \underline{\underline{-\frac{1}{x} + \frac{1}{x+1} + C}}$$

109. (a) If $\dfrac{x-2}{(x+1)^3} \equiv \dfrac{P}{x+1} + \dfrac{Q}{(x+1)^2} + \dfrac{R}{(x+1)^3}$, find the values of constants P, Q and R.

(b) Hence find $\displaystyle\int \frac{x-2}{(x+1)^3}\,dx$.

SOLUTION

(a) $$\frac{x-2}{(x+1)^3} \equiv \frac{P}{x+1} + \frac{Q}{(x+1)^2} + \frac{R}{(x+1)^3}$$
$$\equiv \frac{P(x+1)^2 + Q(x+1) + R}{(x+1)^3}$$

i.e. $P(x+1)^2+Q(x+1)+R\equiv x-2$

$P(x^2+2x+1)+Q(x+1)+R\equiv x-2$

$Px^2+(2P+Q)x+P+Q+R\equiv x-2$

$\therefore \begin{cases} P=0 \\ 2P+Q=1 \\ P+Q+R=-2 \end{cases}$

$\therefore \underline{\underline{P=0,\ Q=1,\ R=-3}}$

(b) $\int \frac{x-2}{(x+1)^3}dx=\int [\frac{0}{x+1}+\frac{1}{(x+1)^2}+\frac{-3}{(x+1)^3}]dx$ [From the result of **(a)**]

$$=\int [\frac{1}{(x+1)^2}-\frac{3}{(x+1)^3}]d(x+1)$$

$$=-\frac{1}{x+1}-3\cdot\frac{1}{-2}(x+1)^{-2}+C$$

$$=\underline{\underline{-\frac{1}{x+1}+\frac{3}{2(x+1)^2}+C}}$$

Level 3

110. Let $u=e^{2x}-1$, find $\int \frac{e^x}{e^x-e^{-x}}dx$.

SOLUTION

Let $u=e^{2x}-1$, then $du=2e^{2x}dx$.

$$\therefore \int \frac{e^x}{e^x-e^{-x}}dx=\int \frac{e^x}{e^x-\frac{1}{e^x}}dx$$

$$=\int \frac{e^x}{\frac{e^{2x}-1}{e^x}}dx$$

$$=\int \frac{e^{2x}}{e^{2x}-1}dx$$

$$=\int \frac{1}{2}\cdot\frac{1}{e^{2x}-1}\cdot 2e^{2x}dx$$

$$=\frac{1}{2}\int \frac{1}{u}du$$

$$=\frac{1}{2}\ln|u|+C$$

$$=\underline{\underline{\frac{1}{2}\ln\left|e^{2x}-1\right|+C}}$$

111. Let $u = 1 - e^{-3x}$, find $\int \frac{dx}{e^{3x} - 1}$.

SOLUTION

Let $u = 1 - e^{-3x}$, then $du = 3e^{-3x}dx$.

$$\begin{aligned}\therefore \quad \int \frac{dx}{e^{3x}-1} &= \int \frac{1}{3e^{-3x}(e^{3x}-1)} \cdot 3e^{-3x}dx \\ &= \frac{1}{3}\int \frac{1}{1-e^{-3x}} \cdot 3e^{-3x}dx \\ &= \frac{1}{3}\int \frac{1}{u}du \\ &= \frac{1}{3}\ln|u| + C \\ &= \underline{\underline{\frac{1}{3}\ln\left|1-e^{-3x}\right| + C}}\end{aligned}$$

112. Let $u = \frac{x+1}{x-2}$, find $\int \frac{dx}{x^2 - x - 2}$.

SOLUTION

Let $u = \frac{x+1}{x-2}$,

$$\begin{aligned}\text{then } du &= \frac{(x-2)(1)-(x+1)(1)}{(x-2)^2}dx \\ &= -\frac{3}{(x-2)^2}dx\end{aligned}$$

$$\begin{aligned}\therefore \quad \int \frac{dx}{x^2-x-2} &= \int \frac{dx}{(x+1)(x-2)} \\ &= -\frac{1}{3}\int \frac{x-2}{x+1} \cdot [\frac{-3}{(x-2)^2}]dx \\ &= -\frac{1}{3}\int \frac{1}{u}du \\ &= -\frac{1}{3}\ln|u| + C \\ &= \underline{\underline{-\frac{1}{3}\ln\left|\frac{x+1}{x-2}\right| + C}}\end{aligned}$$

113. Let $u=\dfrac{2x-5}{2x-3}$, find $\displaystyle\int \frac{dx}{(2x-5)(2x-3)}$.

SOLUTION

Let $u=\dfrac{2x-5}{2x-3}$,

then $du=\dfrac{(2x-3)(2)-(2x-5)(2)}{(2x-3)^2}dx$

$$=\frac{4}{(2x-3)^2}dx$$

$$\therefore \quad \int \frac{dx}{(2x-5)(2x-3)}=\int \frac{1}{4}\cdot\frac{2x-3}{2x-5}\cdot[\frac{4}{(2x-3)^2}]dx$$

$$=\frac{1}{4}\int \frac{1}{u}du$$

$$=\frac{1}{4}\ln|u|+C$$

$$=\underline{\underline{\frac{1}{4}\ln\left|\frac{2x-5}{2x-3}\right|+C}}$$

114. Find $\displaystyle\int (x+1)^2(x+2)^{\frac{2}{3}}dx$.

SOLUTION

Let $u=x+2$, then $du=dx$.

When $u=x+2$, $x=u-2$.

$$\therefore \quad \int (x+1)^2(x+2)^{\frac{2}{3}}dx=\int [(u-2)+1]^2u^{\frac{2}{3}}du$$

$$=\int (u-1)^2u^{\frac{2}{3}}du$$

$$=\int (u^2-2u+1)u^{\frac{2}{3}}du$$

$$=\int (u^{\frac{8}{3}}-2u^{\frac{5}{3}}+u^{\frac{2}{3}})du$$

$$=\frac{3}{11}u^{\frac{11}{3}}-2\cdot\frac{3}{8}u^{\frac{8}{3}}+\frac{3}{5}u^{\frac{5}{3}}+C$$

$$=\underline{\underline{\frac{3}{11}(x+2)^{\frac{11}{3}}-\frac{3}{4}(x+2)^{\frac{8}{3}}+\frac{3}{5}(x+2)^{\frac{5}{3}}+C}}$$

115. Find $\int \frac{(x-1)^2}{(2-3x)^{\frac{4}{3}}}dx$.

SOLUTION

Let $u = 2-3x$, then $du = -3dx$.

When $u = 2-3x$, $x = \frac{2-u}{3}$.

$$\begin{aligned}
\therefore \quad \int \frac{(x-1)^2}{(2-3x)^{\frac{4}{3}}}dx &= -\frac{1}{3}\int \frac{(x-1)^2}{(2-3x)^{\frac{4}{3}}}\cdot(-3)dx \\
&= -\frac{1}{3}\int \frac{(\frac{2-u}{3}-1)^2}{u^{\frac{4}{3}}}du \\
&= -\frac{1}{3}\int \frac{(\frac{-1-u}{3})^2}{u^{\frac{4}{3}}}du \\
&= -\frac{1}{27}\int (u+1)^2u^{-\frac{4}{3}}du \\
&= -\frac{1}{27}\int (u^2+2u+1)u^{-\frac{4}{3}}du \\
&= -\frac{1}{27}\int (u^{\frac{2}{3}}+2u^{-\frac{1}{3}}+u^{-\frac{4}{3}})du \\
&= -\frac{1}{27}(\frac{3}{5}u^{\frac{5}{3}}+2\cdot\frac{3}{2}u^{\frac{2}{3}}-3u^{-\frac{1}{3}})+C \\
&= -\frac{1}{45}u^{\frac{5}{3}}-\frac{1}{9}u^{\frac{2}{3}}+\frac{1}{9}u^{-\frac{1}{3}}+C \\
&= \underline{\underline{-\frac{1}{45}(2-3x)^{\frac{5}{3}}-\frac{1}{9}(2-3x)^{\frac{2}{3}}+\frac{1}{9}(2-3x)^{-\frac{1}{3}}+C}}
\end{aligned}$$

116. **(a)** Find $\int \frac{2x+2}{x^2+2x+2}dx$.

(b) Hence find $\int \frac{x^2dx}{x^2+2x+2}$.

SOLUTION

(a)
$$\begin{aligned}
\int \frac{2x+2}{x^2+2x+2} &= \int \frac{1}{x^2+2x+2}d(x^2+2x+2) \\
&= \underline{\underline{\ln(x^2+2x+2)+C}}
\end{aligned}$$

(b) $\displaystyle\int \frac{x^2dx}{x^2+2x+2} = \int \frac{(x^2+2x+2)-(2x+2)}{x^2+2x+2}dx$

$\displaystyle = \int (1-\frac{2x+2}{x^2+2x+2})dx$

$\displaystyle = \int dx - \int \frac{2x+2}{x^2+2x+2}dx$

$= x-[\ln(x^2+2x+2)+C]$

$= \underline{\underline{x-\ln(x^2+2x+2)+C_1}}$ where $C_1 = -C$

117. **(a)** Find $\displaystyle\int \frac{x-4}{x^2-8x+32}dx$.

(b) Hence find $\displaystyle\int \frac{x^2dx}{x^2-8x+32}$.

SOLUTION

(a) $\displaystyle\int \frac{x-4}{x^2-8x+32}dx = \int \frac{2x-8}{2(x^2-8x+32)}dx$

$\displaystyle = \frac{1}{2}\int \frac{1}{x^2-8x+32}d(x^2-8x+32)$

$\displaystyle = \underline{\underline{\frac{1}{2}\ln(x^2-8x+32)+C}}$

(b) $\displaystyle\int \frac{x^2dx}{x^2-8x+32} = \int \frac{(x^2-8x+32)+(8x-32)}{x^2-8x+32}dx$

$\displaystyle = \int (1+\frac{8x-32}{x^2-8x+32})dx$

$\displaystyle = \int dx + 8\int \frac{x-4}{x^2-8x+32}dx$

$\displaystyle = x+8[\frac{1}{2}\ln(x^2-8x+32)+C]$

$= \underline{\underline{x+4\ln(x^2-8x+32)+C_1}}$ where $C_1 = 8C$

118. **(a)** If $\displaystyle\frac{4x^2+11x-59}{(x+4)(2x-5)} \equiv P + \frac{Q}{x+4} + \frac{R}{2x-5}$, find the values of constants P, Q and R.

(b) Find $\displaystyle\int \frac{4x^2+11x-59}{2x^2+3x-20}dx$.

SOLUTION

(a) $$\frac{4x^2+11x-59}{(x+4)(2x-5)} \equiv P+\frac{Q}{x+4}+\frac{R}{2x-5}$$
$$\equiv \frac{P(x+4)(2x-5)+Q(2x-5)+R(x+4)}{(x+4)(2x-5)}$$

i.e. $$P(x+4)(2x-5)+Q(2x-5)+R(x+4) \equiv 4x^2+11x-59$$
$$P(2x^2+3x-20)+Q(2x-5)+R(x+4) \equiv 4x^2+11x-59$$
$$2Px^2+(3P+2Q+R)x+(-20P-5Q+4R) \equiv 4x^2+11x-59$$

$$\therefore \begin{cases} 2P=4 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \\ 3P+2Q+R=11 \ldots\ldots\ldots\ldots\ldots (2) \\ -20P-5Q+4R=-59 \ldots\ldots\ldots (3) \end{cases}$$

From (1), $P=2$

Substitute $P=2$ into (2),

$$3(2)+2Q+R=11$$
$$2Q+R=5$$
$$R=5-2Q \ \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Substitute $P=2$ and (4) into (3),

$$-20(2)-5Q+4(5-2Q)=-59$$
$$-40-5Q+20-8Q=-59$$
$$-13Q=-39$$
$$Q=3$$

Substitute $Q=3$ into (4),

$$R=5-2(3)$$
$$=-1$$

$\therefore$ $\underline{\underline{P=2,\ Q=3,\ R=-1}}$

(b) $$\int \frac{4x^2+11x-59}{2x^2+3x-20}dx = \int \frac{4x^2+11x-59}{(x+4)(2x-5)}dx$$
$$= \int \left(2+\frac{3}{x+4}+\frac{-1}{2x-5}\right)dx \quad \text{[From the result of (a)]}$$
$$= \int 2dx + \int \frac{3}{x+4}dx - \int \frac{1}{2x-5}dx$$
$$= \int 2dx + \int \frac{3}{x+4}d(x+4) - \int \frac{1}{2}\cdot\frac{1}{2x-5}d(2x-5)$$
$$= \underline{\underline{2x+3\ln|x+4|-\frac{1}{2}\ln|2x-5|+C}}$$

119. (a) If $\dfrac{6x^2+3x-10}{(x^2-2)(x+3)} \equiv \dfrac{Px+Q}{x^2-2}+\dfrac{R}{x+3}$, find the values of constants P, Q and R.

(b) Hence find $\displaystyle\int \frac{6x^2+3x-10}{x^3+3x^2-2x-6}\,dx$.

SOLUTION

(a) $$\frac{6x^2+3x-10}{(x^2-2)(x+3)} \equiv \frac{Px+Q}{x^2-2}+\frac{R}{x+3}$$
$$\equiv \frac{(Px+Q)(x+3)+R(x^2-2)}{(x^2-2)(x+3)}$$

i.e. $$(Px+Q)(x+3)+R(x^2-2) \equiv 6x^2+3x-10$$
$$Px^2+(3P+Q)x+3Q+Rx^2-2R \equiv 6x^2+3x-10$$
$$(P+R)x^2+(3P+Q)x+(3Q-2R) \equiv 6x^2+3x-10$$

$$\therefore \quad \begin{cases} P+R=6 \text{ ....(1)} \\ 3P+Q=3 \text{ ....(2)} \\ 3Q-2R=-10 \text{ .... (3)} \end{cases}$$

From (1),
$P=6-R$ ....(4)

Substitute (4) into (2),
$$3(6-R)+Q=3$$
$$18-3R+Q=3$$
$$Q=3R-15 \text{ ....(5)}$$

Substitute (5) into (3),
$$3(3R-15)-2R=-10$$
$$9R-45-2R=-10$$
$$7R=35$$
$$R=5$$

Substitute $R=5$ into (4),
$$P=6-5$$
$$=1$$

Substitute $R=5$ into (5),
$$Q=3(5)-15$$
$$=0$$

$\therefore \quad \underline{\underline{P=1,\ Q=0,\ R=5}}$

(b) $\int \frac{6x^2+3x-10}{x^3+3x^2-2x-6}dx = \int \frac{6x^2+3x-10}{(x^2-2)(x+3)}dx$

$= \int (\frac{x}{x^2-2}+\frac{5}{x+3})dx$ [From the result of **(a)**]

$= \int \frac{x}{x^2-2}dx + \int \frac{5}{x+3}dx$

$= \int \frac{1}{2}\cdot\frac{1}{x^2-2}d(x^2-2) + \int \frac{5}{x+3}d(x+3)$

$= \underline{\underline{\frac{1}{2}\ln\left|x^2-2\right| + 5\ln\left|x+3\right| + C}}$

EXERCISE 7C

Level 1

120. The slope at any point (x, y) of a curve is $2x+5$, and the curve passes through (2, 8). Find the equation of the curve.

SOLUTION

$\because \quad \frac{dy}{dx} = 2x+5$

$\therefore \quad y = \int (2x+5)dx$

$= x^2+5x+C$

Substitute (2, 8) into $y = x^2+5x+C$,

$8 = 2^2+5(2)+C$

$C = -6$

$\therefore$ $\underline{\underline{\text{The equation of the curve is } y = x^2+5x-6.}}$

121. The slope at any point (x, y) of a curve is $(3x-1)(9x-1)$, and the curve cuts the x-axis at (1, 0). Find the equation of the curve.

SOLUTION

$\because \quad \frac{dy}{dx} = (3x-1)(9x-1)$

$\therefore \quad y = \int (3x-1)(9x-1)dx$

$= \int (27x^2-12x+1)dx$

$= 9x^3-6x^2+x+C$

Substitute (1, 0) into $y = 9x^3 - 6x^2 + x + C$,

$0 = 9(1)^3 - 6(1)^2 + 1 + C$

$C = -4$

$\therefore$ The equation of the curve is $y = 9x^3 - 6x^2 + x - 4$.

122. The slope at any point (x, y) of a curve is $4e^{2x} - e^x$, and the y-intercept of the curve is 5. Find the equation of the curve.

SOLUTION

$\because \dfrac{dy}{dx} = 4e^{2x} - e^x$

$$\begin{aligned} \therefore \quad y &= \int (4e^{2x} - e^x)dx \\ &= 4\int e^{2x}dx - \int e^x dx \\ &= 4\int \frac{1}{2}e^{2x}d(2x) - \int e^x dx \\ &= 2e^{2x} - e^x + C \end{aligned}$$

Substitute (0, 5) into $y = 2e^{2x} - e^x + C$,

$5 = 2e^{2(0)} - e^0 + C$

$C = 4$

$\therefore$ The equation of the curve is $y = 2e^{2x} - e^x + 4$.

123. The slope at any point (x, y) of a curve is $2x - \dfrac{30}{x} + \dfrac{50}{x^2}$, and the curve passes through (5, 10). Find the equation of the curve.

SOLUTION

$\because \dfrac{dy}{dx} = 2x - \dfrac{30}{x} + \dfrac{50}{x^2}$

$$\begin{aligned} \therefore \quad y &= \int \left(2x - \frac{30}{x} + \frac{50}{x^2}\right)dx \\ &= x^2 - 30\ln|x| - \frac{50}{x} + C \end{aligned}$$

Substitute (5, 10) into $y = x^2 - 30\ln|x| - \dfrac{50}{x} + C$,

$10 = 5^2 - 30\ln|5| - \dfrac{50}{5} + C$

$C = 30\ln 5 - 5$

$\therefore$ The equation of the curve is

$$y = x^2 - 30\ln|x| - \frac{50}{x} + 30\ln 5 - 5$$

$$\underline{\underline{y = x^2 - 30\ln\left|\frac{x}{5}\right| - \frac{50}{x} - 5}}$$

124. It is given that $\frac{dy}{dx} = 5e^x + 2(5^x)$. When $x = 0$, $y = 3$. Express y in terms of x.

SOLUTION

$\because$ $\frac{dy}{dx} = 5e^x + 2(5^x)$

$\therefore$ $y = \int [5e^x + 2(5^x)]dx$

$$= 5e^x + \frac{2(5^x)}{\ln 5} + C$$

Substitute (0, 3) into $y = 5e^x + \frac{2(5^x)}{\ln 5} + C$,

$$3 = 5e^0 + \frac{2(5^0)}{\ln 5} + C$$

$$C = -\frac{2}{\ln 5} - 2$$

$\therefore$ The equation of the curve is

$$y = 5e^x + \frac{2(5^x)}{\ln 5} - \frac{2}{\ln 5} - 2$$

$$\underline{\underline{y = 5e^x + \frac{2(5^x - 1)}{\ln 5} - 2}}$$

125. The slope at any point (x, y) of a curve C is $\frac{dy}{dx} = -3x^2 + 2x - 1$, and the y-intercept of C is 1.

(a) Find the equation of C.

(b) Find the equation of the tangent to C at the point where C cuts the x-axis.

SOLUTION

(a) $\because$ $\frac{dy}{dx} = -3x^2 + 2x - 1$

$\therefore$ $y = \int (-3x^2 + 2x - 1)dx$

$$= -x^3 + x^2 - x + C_1$$

When $x = 0$, $y = 1$.

$\therefore \quad 1 = -0^3 + 0^2 - 0 + C_1$

$C_1 = 1$

$\therefore$ The equation of C is $y = -x^3 + x^2 - x + 1$.

(b) When $y = 0$,

$-x^3 + x^2 - x + 1 = 0$

$-(x^2 + 1)(x - 1) = 0$

$x - 1 = 0$

$x = 1$

$\therefore$ C cuts the x-axis at (1, 0).

Slope of the tangent at $(1, 0) = \left.\dfrac{dy}{dx}\right|_{x=1}$

$= -3(1)^2 + 2(1) - 1$

$= -2$

The equation of the tangent to C at (1, 0) is

$y - 0 = -2(x - 1)$

$2x + y - 2 = 0$

126. It is given that $\dfrac{dy}{dx} = 6x^2 + kx + 8$, where k is a constant. When $x = 1$, $\dfrac{d^2y}{dx^2} = 6$ and $y = 24$.

(a) Find the value of k.

(b) Express y in terms of x.

SOLUTION

(a) $\dfrac{dy}{dx} = 6x^2 + kx + 8$

$\dfrac{d^2y}{dx^2} = 12x + k$

$\because \quad \left.\dfrac{d^2y}{dx^2}\right|_{x=1} = 6$

$\therefore \quad 12(1) + k = 6$

$k = -6$

(b) $\because \quad \frac{dy}{dx} = 6x^2 - 6x + 8$

$\therefore \quad y = \int (6x^2 - 6x + 8)dx$

$= 2x^3 - 3x^2 + 8x + C$

Substitute (1, 24) into $y = 2x^3 - 3x^2 + 8x + C$,

$24 = 2(1)^3 - 3(1)^2 + 8(1) + C$

$C = 17$

$\therefore \quad \underline{\underline{y = 2x^3 - 3x^2 + 8x + 17}}$

127. It is given that $f'(x) = ax^2 - 3x - 6$, where a is a constant. If $f(-6) = 120$ and $f'(\frac{1}{a}) = -5$, find the value of a and $f(x)$.

SOLUTION

$\because \quad f'(\frac{1}{a}) = -5$

$\therefore \quad a(\frac{1}{a^2}) - 3(\frac{1}{a}) - 6 = -5$

$-\frac{2}{a} = 1$

$a = \underline{\underline{-2}}$

$\because \quad f'(x) = -2x^2 - 3x - 6$

$\therefore \quad f(x) = \int (-2x^2 - 3x - 6)dx$

$= -\frac{2}{3}x^3 - \frac{3}{2}x^2 - 6x + C$

$\because \quad f(-6) = 120$

$\therefore \quad -\frac{2}{3}(-6)^3 - \frac{3}{2}(-6)^2 - 6(-6) + C = 120$

$C = -6$

$\therefore \quad \underline{\underline{f(x) = -\frac{2}{3}x^3 - \frac{3}{2}x^2 - 6x - 6}}$

128. It is given that $f'(x)=x^2-ax-5$, where $a>0$. If $f(-3)=4$ and $f'(\frac{1}{a})=-2$, find the value of a and $f(x)$.

SOLUTION

$\because \quad f'(\frac{1}{a})=-2$

$\therefore \quad (\frac{1}{a})^2-a(\frac{1}{a})-5=-2$

$$\frac{1}{a^2}=4$$

$$a^2=\frac{1}{4}$$

$$a=\underline{\underline{\frac{1}{2}}} \text{ or } -\frac{1}{2} \text{ (rejected)}$$

$\because \quad f'(x)=x^2-\frac{1}{2}x-5$

$\therefore \quad f(x)=\int(x^2-\frac{1}{2}x-5)dx$

$$=\frac{x^3}{3}-\frac{x^2}{4}-5x+C$$

$\because \quad f(-3)=4$

$\therefore \quad \frac{(-3)^3}{3}-\frac{(-3)^2}{4}-5(-3)+C=4$

$$C=\frac{1}{4}$$

$\therefore \quad \underline{\underline{f(x)=\frac{x^3}{3}-\frac{x^2}{4}-5x+\frac{1}{4}}}$

129. An agent estimates that the rate of change of the unit value $V(t)$ (in dollars) of a share with respect to time after t days can be modelled by $V'(t)=3t^2-\frac{25t}{2}+12$ $(0\le t\le 5)$. It is given that the current unit value of the share is \$47.4.

(a) Find $V(t)$.

(b) Find the unit value of the share after 3 days.

SOLUTION

(a) $\because \quad V'(t)=3t^2-\frac{25t}{2}+12$

$\therefore \quad V(t)=\int(3t^2-\frac{25t}{2}+12)dt$

$$=t^3-\frac{25t^2}{4}+12t+C$$

$\because \quad V(0) = 47.4$

$\therefore \quad 47.4 = 0^3 - \dfrac{25(0)^2}{4} + 12(0) + C$

$C = 47.4$

$\therefore \quad \underline{\underline{V(t) = t^3 - \dfrac{25t^2}{4} + 12t + 47.4}}$

(b) $V(3) = 3^3 - \dfrac{25(3)^2}{4} + 12(3) + 47.4$

$= 54.15$

$\therefore$ The unit value of the share will be \$54.15 after 3 days.

130. The rate of change of the volume V cm^3 of a balloon with respect to time t seconds $(0 \le t \le 5)$ can be modelled as follows.

$$\frac{dV}{dt} = 4t^3 - 2t$$

It is given that the volume of the balloon is 16 cm^3 after 2 seconds.

(a) Express V in terms of t.

(b) Find the volume of the balloon after 4 seconds.

(c) When the volume of the balloon is 76 cm^3, find the value of t.

SOLUTION

(a) $\because \quad \dfrac{dV}{dt} = 4t^3 - 2t$

$\therefore \quad V = \int (4t^3 - 2t)dt$

$= t^4 - t^2 + C$

When $t = 2$, $V = 16$.

$\therefore \quad 16 = 2^4 - 2^2 + C$

$C = 4$

$\therefore \quad \underline{\underline{V = t^4 - t^2 + 4}}$

(b) When $t = 4$,

$V = 4^4 - 4^2 + 4$

$= 244$

$\therefore$ The volume of the balloon is 244 cm^3 after 4 seconds.

(c) When $V = 76$,

$$76 = t^4 - t^2 + 4$$
$$t^4 - t^2 - 72 = 0$$
$$(t^2 - 9)(t^2 + 8) = 0$$
$$t^2 = 9 \quad \text{or} \quad t^2 = -8 \text{ (rejected)}$$
$$t = \underline{\underline{3}} \quad \text{or} \quad -3 \text{ (rejected)}$$

131. A particle starts moving along the x-axis from $x = 5$. Its velocity at any time t (in seconds) is given by $v = 8t - 3$ where $t \geq 0$.

(a) Find the velocity and acceleration, in magnitude and direction, when the particle starts to move.

(b) Express x in terms of t.

(c) Prove that the particle will not pass through the origin at any time.

SOLUTION

(a) Let a units/s^2 be the acceleration of the particle at time t s.

$$a = \frac{dv}{dt}$$
$$= 8$$

When $t = 0$,

$$v = 8(0) - 3$$
$$= -3$$
$$a = 8$$

$\therefore$ $\underline{\underline{\text{The required velocity is 3 units/s in the negative direction.}}}$

$\underline{\underline{\text{The required acceleration is 8 units/s}^2 \text{ in the positive direction.}}}$

(b) $\because$ $v = 8t - 3$

$$\frac{dx}{dt} = 8t - 3$$

$\therefore$
$$x = \int (8t - 3)dt$$
$$= 4t^2 - 3t + C$$

When $t = 0$, $x = 5$.

$$5 = 4(0)^2 - 3(0) + C$$
$$C = 5$$

$\therefore$ $\underline{\underline{x = 4t^2 - 3t + 5}}$

(c) When the particle passes through the origin, $x = 0$.

Consider $4t^2 - 3t + 5 = 0$,

$$\begin{aligned}\Delta &= (-3)^2 - 4(4)(5)\\ &= -71\\ &< 0\end{aligned}$$

$\therefore$ The equation $4t^2 - 3t + 5 = 0$ has no real solutions.

$\therefore$ The particle will not pass through the origin.

132. It is known that $C(x)$ (in dollars) is the total cost of printing x commemorative albums, where the rate of change of $C(x)$ with respect to x can be modelled by $C'(x) = 10 + 0.36\sqrt{x}$. Due to the fixed costs, a cost of $2 000 will be incurred even if no commemorative albums are printed.

(a) Find $C(x)$.

(b) Find the total cost of printing 900 commemorative albums.

SOLUTION

(a) $\because$ $C'(x) = 10 + 0.36\sqrt{x}$

$\therefore$ $$\begin{aligned}C(x) &= \int (10 + 0.36\sqrt{x})dx\\ &= 10x + 0.24x^{\frac{3}{2}} + C_1\end{aligned}$$

$\because$ $C(0) = 2\ 000$

$\therefore$ $$\begin{aligned}10(0) + 0.24(0)^{\frac{3}{2}} + C_1 &= 2\ 000\\ C_1 &= 2\ 000\end{aligned}$$

$\therefore$ $\underline{\underline{C(x) = 10x + 0.24x^{\frac{3}{2}} + 2\ 000}}$

(b) $$\begin{aligned}C(900) &= 10(900) + 0.24(900)^{\frac{3}{2}} + 2\ 000\\ &= 17\ 480\end{aligned}$$

$\therefore$ The total cost of printing 900 commemorative albums is $17 480.

133. Based on the past experience, Vicki predicts that the rate of change of her Mathematics test score S with respect to the number of hours t spent in studying the night before the test can be modelled by $\frac{dS}{dt} = 3t - 2 + \frac{4}{t} + \frac{10}{t^2}$ $(1 \le t \le 4)$. If Vicki has spent 1 hour in studying the night before the test, she predicts that she will obtain a score of 65 in the Mathematics test.

(a) Express S in terms of t.

(b) It is given that Vicki has spent 3 hours in studying the night before the test. What will be her predicted score? (Give your answer correct to the nearest integer.)

SOLUTION

(a) $\because \quad \dfrac{dS}{dt} = 3t - 2 + \dfrac{4}{t} + \dfrac{10}{t^2}$

$\therefore \quad S = \int (3t - 2 + \dfrac{4}{t} + \dfrac{10}{t^2})dt$

$= \dfrac{3t^2}{2} - 2t + 4\ln t - \dfrac{10}{t} + C$

When $t = 1,\ S = 65.$

$\therefore \quad 65 = \dfrac{3(1)^2}{2} - 2(1) + 4\ln 1 - \dfrac{10}{1} + C$

$C = 75.5$

$\therefore \quad \underline{\underline{S = \dfrac{3t^2}{2} - 2t + 4\ln t - \dfrac{10}{t} + 75.5}}$

(b) When $t = 3$,

$S = \dfrac{3(3)^2}{2} - 2(3) + 4\ln 3 - \dfrac{10}{3} + 75.5$

$= 84$ (corr. to the nearest integer)

$\therefore$ The predicted score of Vicki will be 84.

134. Some residents in a city are infected with a particular type of influenza. The rate of spread of the virus can be modelled by $\dfrac{dN}{dt} = 50e^{0.2t}$ $(0 \le t \le 20)$, where N is the total number of people infected and t is the number of days elapsed since the outbreak of the disease. It is given that the total number of people infected is 450 after 2 days since the outbreak of the disease.

(a) Express N in terms of t.

(b) Find the total number of people infected after 10 days since the outbreak of the disease. (Give your answer correct to the nearest integer.)

SOLUTION

(a) $\because \quad \dfrac{dN}{dt} = 50e^{0.2t}$

$\therefore \quad N = \int 50e^{0.2t}dt$

$= 50\int \dfrac{1}{0.2} e^{0.2t} d(0.2t)$

$= 250e^{0.2t} + C$

When $t = 2,\ N = 450.$

$\therefore \quad 450 = 250e^{0.2(2)} + C$

$C = 450 - 250e^{0.4}$

$\therefore \quad N = 250e^{0.2t} + 450 - 250e^{0.4}$

$\underline{\underline{N = 250(e^{0.2t} - e^{0.4}) + 450}}$

(b) When $t=10$,

$N=250[e^{0.2(10)}-e^{0.4}]+450$

$=1\,924$ (corr. to the nearest integer)

$\therefore$ The total number of people infected is 1 924 after 10 days since the outbreak of the disease.

135. The cooling process of a cup of hot coffee can be modelled by $\dfrac{dT}{dt}=-4.8e^{-0.08t}$, where T (in °C) is the temperature of the cup of coffee after t minutes. It is given that the initial temperature is 85°C.

(a) Express T in terms of t.

(b) When will the temperature of the cup of coffee decrease to 45°C? (Give your answer correct to 3 significant figures.)

(c) Find the temperature of the cup of coffee after a very long time.

SOLUTION

(a) $\because \quad \dfrac{dT}{dt}=-4.8e^{-0.08t}$

$$\therefore \quad T=\int(-4.8e^{-0.08t})dt$$
$$=-4.8\int\frac{1}{-0.08}e^{-0.08t}d(-0.08t)$$
$$=60e^{-0.08t}+C$$

When $t=0$, $T=85$.

$$\therefore \quad 85=60e^{-0.08(0)}+C$$
$$C=25$$
$$\therefore \quad \underline{\underline{T=60e^{-0.08t}+25}}$$

(b) When $T=45$,

$$45=60e^{-0.08t}+25$$
$$20=60e^{-0.08t}$$
$$e^{-0.08t}=\frac{1}{3}$$
$$t=13.7 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The temperature of the cup of coffee will decrease to 45°C after 13.7 minutes.

(c) $$\lim_{t\to\infty}T=\lim_{t\to\infty}(60e^{-0.08t}+25)$$
$$=60\cdot\lim_{t\to\infty}e^{-0.08t}+25$$
$$=25$$

$\therefore$ The temperature of the cup of coffee is 25°C after a very long time.

136. It is given that $P(x)$ (in dollars) is the total profit of producing x handbags, where the rate of change of $P(x)$ with respect to x can be modelled by $P'(x) = 100 + 200e^{-0.05x}$. Due to the fixed costs incurred, a loss of \$3 000 will be made even if no handbags are produced.

(a) Find $P(x)$.

(b) Find the total profit of producing 20 handbags. (Give your answer correct to the nearest dollar.)

(c) When the number of handbags produced increases from 20 to 30, find the increase in profit. (Give your answer correct to the nearest dollar.)

SOLUTION

(a) $\because \quad P'(x) = 100 + 200e^{-0.05x}$

$$\therefore \quad P(x) = \int (100 + 200e^{-0.05x})dx$$
$$= \int 100dx + 200\int \frac{1}{-0.05} e^{-0.05x} d(-0.05x)$$
$$= 100x - 4\,000e^{-0.05x} + C$$

$\because \quad P(0) = -3\,000$

$\therefore \quad 100(0) - 4\,000e^{-0.05(0)} + C = -3\,000$

$C = 1\,000$

$\therefore \quad \underline{\underline{P(x) = 100x - 4\,000e^{-0.05x} + 1\,000}}$

(b) $P(20) = 100(20) - 4\,000e^{-0.05(20)} + 1\,000$

$= 1\,528$ (corr. to the nearest integer)

$\therefore$ The total profit of producing 20 handbags is \$1 528.

(c) $P(30) - P(20) = 100(30) - 4\,000e^{-0.05(30)} + 1\,000 - 1\,528.5$

$= 1\,579$ (corr. to the nearest integer)

$\therefore$ The increase in profit is \$1 579.

Level 2

137. The slope at any point (x, y) of a curve is $\dfrac{e^x}{e^x + 1}$, and the y-intercept of the curve is $\ln 4$. Find the equation of the curve.

SOLUTION

$$\because \quad \frac{dy}{dx} = \frac{e^x}{e^x + 1}$$

$$\therefore \quad y = \int \frac{e^x}{e^x + 1} dx$$
$$= \int \frac{1}{e^x + 1} d(e^x + 1)$$
$$= \ln(e^x + 1) + C$$

Substitute $(0, \ln 4)$ into $y = \ln(e^x + 1) + C$,

$$\ln 4 = \ln(e^0 + 1) + C$$
$$C = \ln 4 - \ln 2$$
$$= \ln 2$$

$\therefore$ The equation of the curve is $y = \ln(e^x + 1) + \ln 2$.

138. The slope at any point (x, y) of a curve is $\dfrac{x^2+1}{\sqrt{2x^3+6x+8}}$, and the x-intercept of the curve is 2. Find the equation of the curve.

SOLUTION

$$\because \quad \frac{dy}{dx} = \frac{x^2+1}{\sqrt{2x^3+6x+8}}$$

$$\therefore \quad y = \int \frac{x^2+1}{\sqrt{2x^3+6x+8}}\,dx$$
$$= \int \frac{1}{6} \cdot \frac{1}{\sqrt{2x^3+6x+8}}\,d(2x^3+6x+8)$$
$$= \frac{1}{3}\sqrt{2x^3+6x+8} + C$$

Substitute $(2, 0)$ into $y = \frac{1}{3}\sqrt{2x^3+6x+8} + C$,

$$0 = \frac{1}{3}\sqrt{2(2)^3+6(2)+8} + C$$
$$C = -2$$

$\therefore$ The equation of the curve is $y = \frac{1}{3}\sqrt{2x^3+6x+8} - 2$.

139. At any point on a certain curve, $\dfrac{d^2y}{dx^2} = 6x - 3$. Find the equation of the curve if it passes through the point $(1, 1)$ and the slope is 1 at that point.

SOLUTION

$$\because \quad \frac{d^2y}{dx^2} = 6x - 3$$

$$\therefore \quad \frac{dy}{dx} = \int (6x-3)\,dx$$
$$= 3x^2 - 3x + C_1$$

At the point (1, 1),

$$\frac{dy}{dx}=1$$

$$3(1)^2-3(1)+C_1=1$$

$$\therefore \quad C_1=1$$

i.e. $\frac{dy}{dx}=3x^2-3x+1$

$$\therefore \quad y=\int(3x^2-3x+1)dx$$

$$=x^3-\frac{3}{2}x^2+x+C_2$$

$\because$ The curve passes through (1, 1).

$$\therefore \quad 1=1^3-\frac{3}{2}(1)^2+1+C_2$$

$$C_2=\frac{1}{2}$$

$\therefore$ The equation of the curve is $y=x^3-\frac{3}{2}x^2+x+\frac{1}{2}$.

140. At any point on a certain curve, $\frac{d^2y}{dx^2}=2x^2-3$. If the equation of the tangent to the curve at the point (3, 1) is $x+2y-5=0$, find the equation of the curve.

SOLUTION

$$\because \quad \frac{d^2y}{dx^2}=2x^2-3$$

$$\therefore \quad \frac{dy}{dx}=\int(2x^2-3)dx$$

$$=\frac{2}{3}x^3-3x+C_1$$

$\because$ Slope of the tangent at (3, 1) $=-\frac{1}{2}$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=3}=-\frac{1}{2}$$

$$\frac{2}{3}(3)^3-3(3)+C_1=-\frac{1}{2}$$

$$C_1=-\frac{19}{2}$$

i.e. $\dfrac{dy}{dx}=\dfrac{2}{3}x^3-3x-\dfrac{19}{2}$

$\therefore$ $y=\int(\dfrac{2}{3}x^3-3x-\dfrac{19}{2})dx$

$=\dfrac{1}{6}x^4-\dfrac{3}{2}x^2-\dfrac{19}{2}x+C_2$

$\because$ The curve passes through the point (3, 1).

$\therefore$ $1=\dfrac{1}{6}(3)^4-\dfrac{3}{2}(3)^2-\dfrac{19}{2}(3)+C_2$

$C_2=\dfrac{59}{2}$

$\therefore$ The equation of the curve is $y=\dfrac{1}{6}x^4-\dfrac{3}{2}x^2-\dfrac{19}{2}x+\dfrac{59}{2}$.

141. At any point on a certain curve, $\dfrac{d^2y}{dx^2}=e^{2x}+2e^{-x}+1$. If the equation of the tangent to the curve at the origin O is $x=3y$, find the equation of the curve.

SOLUTION

$\because$ $\dfrac{d^2y}{dx^2}=e^{2x}+2e^{-x}+1$

$\therefore$ $\dfrac{dy}{dx}=\int(e^{2x}+2e^{-x}+1)dx$

$=\int e^{2x}dx+2\int e^{-x}dx+\int dx$

$=\int \dfrac{1}{2}e^{2x}d(2x)-2\int e^{-x}d(-x)+\int dx$

$=\dfrac{1}{2}e^{2x}-2e^{-x}+x+C_1$

$\because$ Slope of the tangent at the origin $(0,\ 0)=\dfrac{1}{3}$

$\therefore$ $\left.\dfrac{dy}{dx}\right|_{x=0}=\dfrac{1}{3}$

$\dfrac{1}{2}e^{2(0)}-2e^{-0}+0+C_1=\dfrac{1}{3}$

$C_1=\dfrac{11}{6}$

i.e. $\dfrac{dy}{dx}=\dfrac{1}{2}e^{2x}-2e^{-x}+x+\dfrac{11}{6}$

$\therefore\quad y=\int(\dfrac{1}{2}e^{2x}-2e^{-x}+x+\dfrac{11}{6})dx$

$=\dfrac{1}{2}\int e^{2x}dx-2\int e^{-x}dx+\int(x+\dfrac{11}{6})dx$

$=\dfrac{1}{2}\int\dfrac{1}{2}e^{2x}d(2x)+2\int e^{-x}d(-x)+\int(x+\dfrac{11}{6})dx$

$=\dfrac{1}{4}e^{2x}+2e^{-x}+\dfrac{x^2}{2}+\dfrac{11x}{6}+C_2$

$\because$ The curve passes through the point (0, 0).

$\therefore\quad 0=\dfrac{1}{4}e^{2(0)}+2e^{-0}+\dfrac{0^2}{2}+\dfrac{11(0)}{6}+C_2$

$C_2=-\dfrac{9}{4}$

$\therefore$ The equation of the curve is $y=\dfrac{1}{4}e^{2x}+2e^{-x}+\dfrac{x^2}{2}+\dfrac{11x}{6}-\dfrac{9}{4}$.

142. At any point on a certain curve, $\dfrac{d^2y}{dx^2}=(3x-2)^{-\frac{4}{3}}$. Find the equation of the curve if it passes through the points $(1,-\dfrac{3}{2})$ and (22, 5).

SOLUTION

$\because\quad \dfrac{d^2y}{dx^2}=(3x-2)^{-\frac{4}{3}}$

$\therefore\quad \dfrac{dy}{dx}=\int(3x-2)^{-\frac{4}{3}}dx$

$=\int\dfrac{1}{3}(3x-2)^{-\frac{4}{3}}d(3x-2)$

$=-(3x-2)^{-\frac{1}{3}}+C_1$

$\therefore\quad y=\int[-(3x-2)^{-\frac{1}{3}}+C_1]dx$

$=-\int(3x-2)^{-\frac{1}{3}}dx+\int C_1dx$

$=-\int\dfrac{1}{3}(3x-2)^{-\frac{1}{3}}d(3x-2)+\int C_1dx$

$=-\dfrac{1}{2}(3x-2)^{\frac{2}{3}}+C_1x+C_2$

$\because$ The curve passes through the point $(1, -\frac{3}{2})$.

$\therefore$ $$-\frac{3}{2} = -\frac{1}{2}[3(1)-2]^{\frac{2}{3}} + C_1(1) + C_2$$

$$C_1 + C_2 = -1 \text{..........................(1)}$$

$\because$ The curve passes through the point (22, 5).

$\therefore$ $$5 = -\frac{1}{2}[3(22)-2]^{\frac{2}{3}} + C_1(22) + C_2$$

$$22C_1 + C_2 = 13 \text{......................(2)}$$

(2) – (1): $(22C_1 + C_2) - (C_1 + C_2) = 13 - (-1)$

$$21C_1 = 14$$

$$C_1 = \frac{2}{3}$$

Substitute $C_1 = \frac{2}{3}$ into (1),

$$\frac{2}{3} + C_2 = -1$$

$$C_2 = -\frac{5}{3}$$

$\therefore$ The equation of the curve is $y = -\frac{1}{2}(3x-2)^{\frac{2}{3}} + \frac{2x}{3} - \frac{5}{3}$.

143. Given that $f''(x) = x^2 + 2$ and $f(0) = f(2) = 1$, find $f(x)$.

SOLUTION

$\because$ $f''(x) = x^2 + 2$

$\therefore$ $$f'(x) = \int (x^2 + 2)dx$$

$$= \frac{1}{3}x^3 + 2x + C_1$$

$$f(x) = \int (\frac{1}{3}x^3 + 2x + C_1)dx$$

$$= \frac{1}{12}x^4 + x^2 + C_1 x + C_2$$

$\because \quad f(0) = 1$

$\therefore \quad \frac{1}{12}(0)^4 + 0^2 + C_1(0) + C_2 = 1$

$C_2 = 1$

$\because \quad f(2) = 1$

$\therefore \quad \frac{1}{12}(2)^4 + 2^2 + C_1(2) + 1 = 1$

$C_1 = -\frac{8}{3}$

$\therefore \quad \underline{\underline{f(x) = \frac{1}{12}x^4 + x^2 - \frac{8}{3}x + 1}}$

144. At any point on a certain curve, $\frac{d^2y}{dx^2} = 3 + 2x - x^2$. If (5, 35) is the maximum point of the curve, find the equation of the curve.

SOLUTION

$\because \quad \frac{d^2y}{dx^2} = 3 + 2x - x^2$

$= -x^2 + 2x + 3$

$\therefore \quad \frac{dy}{dx} = \int (-x^2 + 2x + 3)dx$

$= -\frac{1}{3}x^3 + x^2 + 3x + C_1$

$\because$ (5, 35) is the maximum point of the curve.

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=5} = 0$

$-\frac{1}{3}(5)^3 + 5^2 + 3(5) + C_1 = 0$

$C_1 = \frac{5}{3}$

i.e. $\frac{dy}{dx} = -\frac{1}{3}x^3 + x^2 + 3x + \frac{5}{3}$

$\therefore \quad y = \int (-\frac{1}{3}x^3 + x^2 + 3x + \frac{5}{3})dx$

$= -\frac{1}{12}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{5}{3}x + C_2$

$\because$ The curve passes through the point (5, 35).

$\therefore$ $35 = -\frac{1}{12}(5)^4 + \frac{1}{3}(5)^3 + \frac{3}{2}(5)^2 + \frac{5}{3}(5) + C_2$

$C_2 = -\frac{5}{12}$

$\therefore$ The equation of the curve is $\underline{\underline{y = -\frac{1}{12}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x^2 + \frac{5}{3}x - \frac{5}{12}}}$.

145. At any point on a certain curve, $\frac{d^2y}{dx^2} = 6e^{-2x} + 4$. If (0, 1) is the minimum point of the curve, find the equation of the curve.

SOLUTION

$\because$ $\frac{d^2y}{dx^2} = 6e^{-2x} + 4$

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \int (6e^{-2x} + 4)dx \\ &= 6\int e^{-2x}dx + \int 4dx \\ &= 6\int \left(-\frac{1}{2}\right)e^{-2x}d(-2x) + \int 4dx \\ &= -3e^{-2x} + 4x + C_1\end{aligned}$$

$\because$ (0, 1) is the minimum point of the curve.

$\therefore$ $\left.\frac{dy}{dx}\right|_{x=0} = 0$

$-3e^{-2(0)} + 4(0) + C_1 = 0$

$C_1 = 3$

i.e. $\frac{dy}{dx} = -3e^{-2x} + 4x + 3$

$$\begin{aligned}\therefore \quad y &= \int (-3e^{-2x} + 4x + 3)dx \\ &= -3\int e^{-2x}dx + \int (4x+3)dx \\ &= -3\int \left(-\frac{1}{2}\right)e^{-2x}d(-2x) + \int (4x+3)dx \\ &= \frac{3}{2}e^{-2x} + 2x^2 + 3x + C_2\end{aligned}$$

$\because$ The curve passes through the point (0, 1).

$\therefore$ $1 = \frac{3}{2}e^{-2(0)} + 2(0)^2 + 3(0) + C_2$

$C_2 = -\frac{1}{2}$

$\therefore$ The equation of the curve is $y = \frac{3}{2}e^{-2x} + 2x^2 + 3x - \frac{1}{2}$.

146. The slope at any point (x, y) of a curve C is given by $\frac{dy}{dx} = 4x + 5$, and the straight line $y = 7 - 3x$ is the tangent to the curve at point A.

(a) Find the coordinates of A.

(b) Find the equation of C.

SOLUTION

(a) Let (x_1, y_1) be the coordinates of A.

Slope of the tangent $= -3$

Slope of the tangent at $(x_1, y_1) = \left.\frac{dy}{dx}\right|_{x=x_1}$

$= 4x_1 + 5$

$\therefore$ $4x_1 + 5 = -3$

$x_1 = -2$

When $x_1 = -2$,

$y_1 = 7 - 3(-2)$

$= 13$

$\therefore$ The coordinates of A are $(-2, 13)$.

(b) $\because$ $\frac{dy}{dx} = 4x + 5$

$\therefore$ $y = \int (4x + 5)dx$

$= 2x^2 + 5x + C_1$

$\because$ The curve passes through the point $(-2, 13)$.

$\therefore$ $13 = 2(-2)^2 + 5(-2) + C_1$

$C_1 = 15$

$\therefore$ The equation of C is $y = 2x^2 + 5x + 15$.

147. At any point on a certain curve, $\frac{d^2y}{dx^2} = kx^2 + 3x$, where k is a constant and $\left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{dy}{dx}\right|_{x=-4}$. It is given that the curve passes through the points (–2, 5) and (4, 20).

(a) Find the value of k.

(b) Find the equation of the curve.

SOLUTION

(a) $\because \quad \frac{d^2y}{dx^2} = kx^2 + 3x$

$$\therefore \quad \frac{dy}{dx} = \int (kx^2 + 3x)dx$$
$$= \frac{k}{3}x^3 + \frac{3}{2}x^2 + C_1$$

$\because \quad \left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{dy}{dx}\right|_{x=-4}$

$$\therefore \quad \frac{k}{3}(2)^3 + \frac{3}{2}(2)^2 + C_1 = \frac{k}{3}(-4)^3 + \frac{3}{2}(-4)^2 + C_1$$
$$\frac{8k}{3} + 6 = -\frac{64k}{3} + 24$$
$$24k = 18$$
$$k = \underline{\underline{\frac{3}{4}}}$$

(b) $\because \quad \frac{dy}{dx} = \frac{1}{4}x^3 + \frac{3}{2}x^2 + C_1$

$$\therefore \quad y = \int (\frac{1}{4}x^3 + \frac{3}{2}x^2 + C_1)dx$$
$$= \frac{1}{16}x^4 + \frac{1}{2}x^3 + C_1x + C_2$$

$\because$ The curve passes through the point (–2, 5).

$$\therefore \quad 5 = \frac{1}{16}(-2)^4 + \frac{1}{2}(-2)^3 + C_1(-2) + C_2$$
$$2C_1 - C_2 = -8 \text{........................(1)}$$

$\because$ The curve passes through the point (4, 20).

$$\therefore \quad 20 = \frac{1}{16}(4)^4 + \frac{1}{2}(4)^3 + C_1(4) + C_2$$
$$4C_1 + C_2 = -28 \text{......................(2)}$$

$$(1) + (2): \ (2C_1 - C_2) + (4C_1 + C_2) = -8 - 28$$
$$6C_1 = -36$$
$$C_1 = -6$$

Substitute $C_1 = -6$ into (1),

$$2(-6) - C_2 = -8$$
$$C_2 = -4$$

$\therefore$ The equation of the curve is $y = \frac{1}{16}x^4 + \frac{1}{2}x^3 - 6x - 4$.

148. At any point on a certain curve, $\frac{d^2y}{dx^2} = kx - 6$, where k is a constant and $\left.\frac{dy}{dx}\right|_{x=-2} = \left.\frac{dy}{dx}\right|_{x=1}$. If the curve touches the x-axis at $x = -4$, find the equation of the curve.

SOLUTION

$\because$ $\frac{d^2y}{dx^2} = kx - 6$

$\therefore$ $\frac{dy}{dx} = \int (kx - 6)dx$

$$= \frac{k}{2}x^2 - 6x + C_1$$

$\because$ $\left.\frac{dy}{dx}\right|_{x=-2} = \left.\frac{dy}{dx}\right|_{x=1}$

$\therefore$ $\frac{k}{2}(-2)^2 - 6(-2) + C_1 = \frac{k}{2}(1)^2 - 6(1) + C_1$

$$2k + 12 = \frac{k}{2} - 6$$
$$\frac{3k}{2} = -18$$
$$k = -12$$

$\therefore$ $\frac{dy}{dx} = -6x^2 - 6x + C_1$

$\because$ The curve touches the x-axis at $x = -4$.

$\therefore$ $\left.\frac{dy}{dx}\right|_{x=-4} = 0$

$$-6(-4)^2 - 6(-4) + C_1 = 0$$
$$C_1 = 72$$

$\because$ $\frac{dy}{dx} = -6x^2 - 6x + 72$

$\therefore$ $y = \int (-6x^2 - 6x + 72)dx$

$$= -2x^3 - 3x^2 + 72x + C_2$$

$\because$ The curve passes through the point (−4, 0).

$\therefore$ $0 = -2(-4)^3 - 3(-4)^2 + 72(-4) + C_2$

$C_2 = 208$

$\therefore$ The equation of the curve is $y = -2x^3 - 3x^2 + 72x + 208.$

149. The initial number of germs in a sample is 5 000. When the sample is placed under ultraviolet, the rate of change of the number of germs N with respect to time can be modelled by $\frac{dN}{dt} = -\frac{3\,000t}{t^2+1}$ $(0 \le t \le 4)$, where t is the number of hours elapsed since the sample is placed under ultraviolet.

(a) Express N in terms of t.

(b) After how long will the number of germs reduce to 1 000? (Give your answer correct to the nearest 0.1 hour.)

SOLUTION

(a) $\because$ $\frac{dN}{dt} = -\frac{3\,000t}{t^2+1}$

$$\begin{aligned}\therefore\quad N &= \int \left(-\frac{3\,000t}{t^2+1}\right)dt \\ &= -\int \frac{1}{2}\cdot\frac{3\,000}{t^2+1}\,d(t^2+1) \\ &= -1\,500\ln(t^2+1) + C\end{aligned}$$

When $t = 0$, $N = 5\,000$.

$\therefore$ $5\,000 = -1\,500\ln(0^2+1) + C$

$C = 5\,000$

$\therefore$ $N = -1\,500\ln(t^2+1) + 5\,000$

(b) When $N = 1\,000$,

$$\begin{aligned}1\,000 &= -1\,500\ln(t^2+1) + 5\,000 \\ \ln(t^2+1) &= \frac{4\,000}{1\,500} \\ t^2+1 &= e^{\frac{8}{3}} \\ t^2 &= e^{\frac{8}{3}} - 1 \\ t &= 3.7 \text{ (corr. to 1 d.p.) or } -3.7 \text{ (corr. to 1 d.p.) (rejected)}\end{aligned}$$

$\therefore$ The number of germs will reduce to 1 000 after 3.7 hours.

150. In a study, a student has been arranged to read a lot of vocabularies in t hours. It is found that the rate of change of the number of vocabularies N that the student can memorize is given by $\frac{dN}{dt}=\frac{78\,400t}{(2t^2+7)^3}$ $(0\le t\le 3)$. Given that when $t=0,\ N=0$.

(a) Let $u=2t^2+7$, express N in terms of t.

(b) If the student has read the vocabularies for 2 hours, how many vocabularies can he memorize? (Give your answer correct to the nearest integer.)

SOLUTION

(a) Let $u=2t^2+7$, then $du=4t\,dt$.

$$\because \quad \frac{dN}{dt}=\frac{78\,400t}{(2t^2+7)^3}$$

$$\therefore \quad N=\int\frac{19\,600}{(2t^2+7)^3}\cdot 4t\,dt$$

$$=\int\frac{19\,600}{u^3}du$$

$$=-\frac{9\,800}{u^2}+C$$

$$=-\frac{9\,800}{(2t^2+7)^2}+C$$

When $t=0,\ N=0$.

$$\therefore \quad 0=-\frac{9\,800}{[2(0)^2+7]^2}+C$$

$$C=200$$

$$\therefore \quad \underline{\underline{N=-\frac{9\,800}{(2t^2+7)^2}+200}}$$

(b) When $t=2$, $N=-\frac{9\,800}{[2(2)^2+7]^2}+200$

$=156$ (corr. to the nearest integer)

$\therefore$ He can memorize 156 vocabularies.

151. The rate of change of the density $D(T)$ (in g/cm^3) of a substance with respect to the temperature T (in °C) can be modelled by $D'(T)=-\frac{0.1(T-4)}{T^2-8T+25}$ $(0\le T\le 20)$. When the temperature is 0°C, the density of the substance is $0.95\ \text{g/cm}^3$.

(a) **(i)** Let $u=T^2-8T+25$, find $\frac{du}{dT}$.

(ii) Find $D(T)$.

(b) It is given that $D(T_1)=D(0)$, where $T_1\neq 0$. Find the value of T_1.

(c) Find the highest density of the substance. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) **(i)** $u = T^2 - 8T + 25$

$$\frac{du}{dT} = \underline{\underline{2T - 8}}$$

(ii) $\because \quad D'(T) = -\dfrac{0.1(T-4)}{T^2 - 8T + 25}$

$$\therefore \quad D(T) = \int [-\frac{0.1(T-4)}{T^2 - 8T + 25}]dT$$

$$= -0.1\int \frac{1}{2} \cdot \frac{1}{T^2 - 8T + 25} \cdot (2T-8)dT$$

$$= -0.05\int \frac{1}{u}du$$

$$= -0.05\ln|u|$$

$$= -0.05\ln(T^2 - 8T + 25) + C$$

$\because \quad D(0) = 0.95$

$$\therefore \quad -0.05\ln[0^2 - 8(0) + 25] + C = 0.95$$

$$C = 0.95 + 0.05\ln 25$$

$$\therefore \quad D(T) = -0.05\ln(T^2 - 8T + 25) + 0.95 + 0.05\ln 25$$

$$\underline{\underline{D(T) = -0.05\ln\frac{T^2 - 8T + 25}{25} + 0.95}}$$

(b) $\because \quad D(T_1) = D(0)$

$$\therefore \quad -0.05\ln\frac{T_1^2 - 8T_1 + 25}{25} + 0.95 = 0.95$$

$$\ln\frac{T_1^2 - 8T_1 + 25}{25} = 0$$

$$\frac{T_1^2 - 8T_1 + 25}{25} = 1$$

$$T_1^2 - 8T_1 + 25 = 25$$

$$T_1^2 - 8T_1 = 0$$

$$T_1(T_1 - 8) = 0$$

$$T_1 = 0 \text{ (rejected) or } T_1 = \underline{\underline{8}}$$

(c) When $D'(T) = 0$, $T = 4$.

$D(0) = 0.95$

$$D(4) = -0.05\ln\frac{4^2 - 8(4) + 25}{25} + 0.95$$

$$= 1.00 \text{ (corr. to 3 sig. fig.)}$$

$$D(20) = -0.05\ln\frac{20^2 - 8(20) + 25}{25} + 0.95$$
$$= 0.832 \text{ (corr. to 3 sig. fig.)}$$

T	$T = 0$	$0 < T < 4$	$T = 4$	$4 < T < 20$	$T = 20$
$D(T)$	0.95	↗	1.00	↘	0.832
$D'(T)$		+	0	–	

$\therefore$ When $T = 4$, $D(T)$ attains its greatest value.

$\therefore$ The highest density of the substance is 1.00 g/cm^3.

152. The rate of change of the power P (in kW) of an engine with respect to its temperature T (in °C) can be modelled by $\dfrac{dP}{dT} = \dfrac{14\,658e^{-0.015T}}{(2+698e^{-0.015T})^2}$. Given that when $T = 0$, $P = 50$.

(a) Let $u = 2 + 698e^{-0.015T}$, express P in terms of T.

(b) When the temperature of the engine is 500°C, find the power of the engine. (Give your answer correct to 3 significant figures.)

(c) If the smallest power required to operate a machine is 500 kW, what is the lowest temperature of the engine when the machine is operating? (Give your answer correct to 3 significant figures.)

SOLUTION

(a) Let $u = 2 + 698e^{-0.015T}$, then $du = -10.47e^{-0.015T}\,dT$.

$$\because \quad \frac{dP}{dT} = \frac{14\,658e^{-0.015T}}{(2+698e^{-0.015T})^2}$$

$$\therefore \quad P = \int \frac{14\,658e^{-0.015T}}{(2+698e^{-0.015T})^2}\,dT$$
$$= \int \frac{-1\,400}{(2+698e^{-0.015T})^2}\cdot(-10.47e^{-0.015T})\,dT$$
$$= -1\,400\int \frac{du}{u^2}$$
$$= \frac{1\,400}{u} + C$$
$$= \frac{1\,400}{2+698e^{-0.015T}} + C$$

When $T=0$, $P=50$.

$$\therefore \quad 50=\frac{1\,400}{2+698e^{-0.015(0)}}+C$$

$$C=48$$

$$\therefore \quad \underline{\underline{P=\frac{1\,400}{2+698e^{-0.015T}}+48}}$$

(b) When $T=500$,

$$P=\frac{1\,400}{2+698e^{-0.015(500)}}+48$$

$$=635 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The required power is 635 kW.

(c) When $P\geq 500$,

$$\frac{1\,400}{2+698e^{-0.015T}}+48\geq 500$$

$$\frac{1\,400}{2+698e^{-0.015T}}\geq 452$$

$$\frac{1\,400}{452}\geq 2+698e^{-0.015T}$$

$$698e^{-0.015T}\leq\frac{124}{113}$$

$$e^{-0.015T}\leq\frac{124}{(698)(113)}$$

$$T\geq 430 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ When the machine is operating, the lowest temperature of the engine is 430°C.

153. A researcher finds that the rate of change of the number of butterflies in a forest can be modelled by $\dfrac{dN}{dt}=\dfrac{30}{(e^{\frac{t}{6}}+e^{-\frac{t}{12}})^3}$, where N (in thousands) is the number of butterflies in the forest and $t(\geq 0)$ is the number of years elapsed since the start of the research. Given that when $t=0$, $N=30$.

(a) **(i)** Prove that $\dfrac{dN}{dt}=\dfrac{30e^{\frac{t}{4}}}{(e^{\frac{t}{4}}+1)^3}$.

(ii) Express N in terms of t.

(b) Estimate the number of butterflies in the forest after a very long time.

SOLUTION

(a) **(i)** $$\frac{dN}{dt}=\frac{30}{(e^{\frac{t}{6}}+e^{-\frac{t}{12}})^3}$$
$$=\frac{30}{(e^{\frac{t}{6}}+\frac{1}{e^{\frac{t}{12}}})^3}$$
$$=\frac{30}{(\frac{e^{\frac{t}{4}}+1}{e^{\frac{t}{12}}})^3}$$
$$=\frac{30e^{\frac{t}{4}}}{(e^{\frac{t}{4}}+1)^3}$$

(ii) Let $u=e^{\frac{t}{4}}+1$, then $du=\frac{1}{4}e^{\frac{t}{4}}dt$.

$$\because \quad \frac{dN}{dt}=\frac{30e^{\frac{t}{4}}}{(e^{\frac{t}{4}}+1)^3}$$
$$\therefore \quad N=\int\frac{30e^{\frac{t}{4}}}{(e^{\frac{t}{4}}+1)^3}dt$$
$$=\int\frac{120}{(e^{\frac{t}{4}}+1)^3}\cdot\frac{1}{4}e^{\frac{t}{4}}dt$$
$$=\int\frac{120}{u^3}du$$
$$=-\frac{60}{u^2}+C$$
$$=-\frac{60}{(e^{\frac{t}{4}}+1)^2}+C$$

When $t=0$, $N=30$.

$$\therefore \quad 30=-\frac{60}{(e^{\frac{0}{4}}+1)^2}+C$$
$$C=45$$
$$\therefore \quad N=45-\frac{60}{(e^{\frac{t}{4}}+1)^2}$$

(b) $\lim_{t\to\infty} N = \lim_{t\to\infty}[45 - \frac{60}{(e^{\frac{t}{4}}+1)^2}]$

$$= 45 - \lim_{t\to\infty}\frac{60}{(e^{\frac{t}{4}}+1)^2}$$

$$= 45$$

$\therefore$ The number of butterflies in the forest is 45 thousand after a very long time.

154. The rate of change of the population $P(t)$ (in thousands) of a city after t years can be modelled by $P'(t) = 6 \cdot 5^{0.02t}$ $(0 \le t \le 20)$. It is given that the current population of the city is 150 000.

(a) Find $P(t)$.

(b) Find the population of the city after 5 years. (Give your answer correct to 3 significant figures.)

(c) **(i)** Find $P''(t)$.

(ii) Describe the behaviour of $P(t)$ and $P'(t)$ in the 5th year.

SOLUTION

(a) $\because$ $P'(t) = 6 \cdot 5^{0.02t}$

$\therefore$ $P(t) = \int 6 \cdot 5^{0.02t}\, dt$

$$= 6\int \frac{1}{0.02} \cdot 5^{0.02t}\, d(0.02t)$$

$$= \frac{300 \cdot 5^{0.02t}}{\ln 5} + C$$

$\because$ $P(0) = 150$

$\therefore$ $\frac{300 \cdot 5^{0.02(0)}}{\ln 5} + C = 150$

$$C = 150 - \frac{300}{\ln 5}$$

$\therefore$ $P(t) = \frac{300 \cdot 5^{0.02t}}{\ln 5} + 150 - \frac{300}{\ln 5}$

$$P(t) = \frac{300(5^{0.02t} - 1)}{\ln 5} + 150$$

(b) $P(5) = \frac{300[5^{0.02(5)} - 1]}{\ln 5} + 150$

$= 183$ (corr. to 3 sig. fig.)

$\therefore$ The population of the city is 183 thousand after 5 years.

(c) **(i)** $P'(t) = 6 \cdot 5^{0.02t}$

$P''(t) = 6(5^{0.02t})(\ln 5)(0.02)$

$= \underline{\underline{0.12(5^{0.02t})\ln 5}}$

(ii) When $4 \le t \le 5$, $P'(t) = 6 \cdot 5^{0.02t} > 0$

$\therefore$ $P(t)$ will increase in the 5th year.

When $4 \le t \le 5$, $P''(t) = 0.12(5^{0.02t})\ln 5 > 0$

$\therefore$ $P'(t)$ will increase in the 5th year.

155. After heating a substance for t minutes, the rate of change of its temperature with respect to time can be modelled by $\frac{dT}{dt} = \lambda e^{-0.25t}$, where T (in °C) is the temperature of the substance after heating it for t minutes, λ is a constant. It is given that the initial temperature of the substance and the temperature after heating it for 8 minutes are 20°C and 250°C respectively.

(a) Find the value of λ. (Give your answer correct to 4 decimal places.)

(b) Take $\lambda = 66.5$, express T in terms of t.

(c) Find the temperature of the substance after a very long time.

SOLUTION

(a) $\because$ $\frac{dT}{dt} = \lambda e^{-0.25t}$

$\therefore$ $T = \int \lambda e^{-0.25t} dt$

$= \lambda \int (-\frac{1}{0.25}) e^{-0.25t} d(-0.25t)$

$= -4\lambda e^{-0.25t} + C$

When $t = 0$, $T = 20$.

$\therefore$ $20 = -4\lambda e^{-0.25(0)} + C$

$20 = -4\lambda + C$ ....(1)

When $t = 8$, $T = 250$.

$\therefore$ $250 = -4\lambda e^{-0.25(8)} + C$

$250 = -4\lambda e^{-2} + C$ ....(2)

(2) – (1): $250 - 20 = (-4\lambda e^{-2} + C) - (-4\lambda + C)$

$4\lambda(1 - e^{-2}) = 230$

$\lambda = \underline{\underline{66.4998}}$ (corr. to 4 d.p.)

(b) Substitute $\lambda = 66.5$ into (1),

$$20 = -4(66.5) + C$$
$$C = 286$$
$$\therefore \quad T = -4(66.5)e^{-0.25t} + 286$$
$$\underline{\underline{T = -266e^{-0.25t} + 286}}$$

(c)
$$\begin{aligned}\lim_{t\to\infty} T &= \lim_{t\to\infty}(-266e^{-0.25t} + 286)\\ &= -266\lim_{t\to\infty} e^{-0.25t} + 286\\ &= -266(0) + 286\\ &= 286\end{aligned}$$

$\therefore$ <u>The temperature of the substance is 286°C after a very long time.</u>

156. The acceleration a m/s^2 of a car after t seconds since the start of the race can be modelled by $a = \dfrac{k(25-t)}{8e^{0.04t} + 3t}$ $(0 \le t \le 60)$, where k is a constant. Let v m/s be the velocity of the car after t seconds since the start of the race. Given that when $t = 0$, $v = 0$ and $a = 15$.

(a) **(i)** Find the value of k.

(ii) Let $u = 8 + 3te^{-0.04t}$, find $\dfrac{du}{dt}$.

(iii) Express v in terms of t.

(b) Has the velocity of the car ever exceeded 60 m/s in the first minute after the start of the race? Explain briefly.

SOLUTION

(a) **(i)** When $t = 0$, $a = 15$.

$$\therefore \quad 15 = \frac{k(25-0)}{8e^{0.04(0)} + 3(0)}$$
$$15 = \frac{25k}{8}$$
$$k = \underline{\underline{\frac{24}{5}}}$$

(ii)
$$\begin{aligned}u &= 8 + 3te^{-0.04t}\\ \frac{du}{dt} &= 3[(1)e^{-0.04t} + (t)(-0.04e^{-0.04t})]\\ &= 3e^{-0.04t}(1 - 0.04t)\\ &= \underline{\underline{0.12e^{-0.04t}(25-t)}}\end{aligned}$$

(iii) $\because \dfrac{dv}{dt}=\dfrac{24(25-t)}{5(8e^{0.04t}+3t)}$

$\therefore \quad v=\dfrac{24}{5}\int\dfrac{25-t}{8e^{0.04t}+3t}dt$

$=\dfrac{24}{5}\int\dfrac{1}{0.12}\cdot\dfrac{0.12e^{-0.04t}(25-t)}{8+3te^{-0.04t}}dt$

$=40\int\dfrac{1}{u}du$

$=40\ln|u|+C$

$=40\ln(8+3te^{-0.04t})+C$

When $t=0,\ v=0$.

$\therefore \quad 0=40\ln[8+3(0)e^{-0.04(0)}]$

$C=-40\ln 8$

$\therefore \quad v=40\ln(8+3te^{-0.04t})-40\ln 8$

$v=40\ln(1+\dfrac{3te^{-0.04t}}{8})$

(b) $\dfrac{dv}{dt}=\dfrac{24(25-t)}{5(8e^{0.04t}+3t)}$

When $\dfrac{dv}{dt}=0,\ t=25$.

When $t=0,\ v=0$.

When $t=25$,

$v=40\ln[1+\dfrac{3(25)e^{-0.04(25)}}{8}]$

$=59.7$ (corr. to 3 sig. fig.)

When $t=60$,

$v=40\ln[1+\dfrac{3(60)e^{-0.04(60)}}{8}]$

$=44.5$ (corr. to 3 sig. fig.)

t	$t=0$	$0<t<25$	$t=25$	$25<t<60$	$t=60$
v	0		59.7		44.5
$\dfrac{dv}{dt}$		+	0	−	

$\therefore$ When $t=25$, v attains its greatest value.

$\therefore$ The highest velocity of the car is 59.7 m/s.

$\therefore$ The velocity of the car has not exceeded 60 m/s in the first minute after the start of the race.

157. After starting the business for t years, the rate of change of the total operating expense $E(t)$ (in thousand dollars) of a company with respect to time t can be modelled by $E'(t) = 1\,000k \cdot 2^{kt}$ $(1 \le t \le 15)$, where k is a positive constant. It is given that the initial operating expense of the company is \$0, and the total operating expense of the company after starting the business for 4 years is 3 times of the total operating expense after starting the business for 2 years.

(a) Find the value of k. Hence find $E(t)$.

(b) Find the operating expense of the company in the third year after the start of the business. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) $\because \quad E'(t) = 1\,000k \cdot 2^{kt}$

$$\therefore \quad E(t) = \int 1\,000k \cdot 2^{kt}\,dt$$
$$= 1\,000\int 2^{kt}\,d(kt)$$
$$= \frac{1\,000 \cdot 2^{kt}}{\ln 2} + C$$

$\because \quad E(0) = 0$

$$\therefore \quad 0 = \frac{1\,000 \cdot 2^{k(0)}}{\ln 2} + C$$
$$C = -\frac{1\,000}{\ln 2}$$
$$\therefore \quad E(t) = \frac{1\,000 \cdot 2^{kt}}{\ln 2} - \frac{1\,000}{\ln 2}$$
$$= \frac{1\,000(2^{kt} - 1)}{\ln 2}$$

$\because \quad E(4) = 3E(2)$

$$\therefore \quad \frac{1\,000[2^{k(4)} - 1]}{\ln 2} = 3 \times \frac{1\,000[2^{k(2)} - 1]}{\ln 2}$$
$$2^{4k} - 1 = 3(2^{2k} - 1)$$
$$(2^{2k})^2 - 3(2^{2k}) + 2 = 0$$
$$(2^{2k} - 2)(2^{2k} - 1) = 0$$
$$2^{2k} = 2 \quad \text{or} \quad 2^{2k} = 1$$
$$2^{2k} = 2^1 \quad \text{or} \quad 2^{2k} = 2^0$$
$$2k = 1 \quad \text{or} \quad 2k = 0$$
$$k = \underline{\underline{0.5}} \quad \text{or} \quad k = 0 \text{ (rejected)}$$

$$\therefore \quad \underline{\underline{E(t) = \frac{1\,000(2^{0.5t} - 1)}{\ln 2}}}$$

(b) $E(3)-E(2)=\dfrac{1\,000[2^{0.5(3)}-1]}{\ln 2}-\dfrac{1\,000[2^{0.5(2)}-1]}{\ln 2}$

$=1\,200$ (corr. to 3 sig. fig.)

$\therefore$ The operating expense of the company in the third year after the start of the business is 1 200 thousand dollars.

158. A principal of \$50 000 is deposited in a bank at an interest rate of r% p.a. compounded continuously, where r is a constant. It is given that the rate of change of the amount $A(t)$ (in dollars) after t years can be modelled by $A'(t)=\dfrac{rA(t)}{100}$ $(t\geq 0)$, and the amount after 9 years is twice the principal.

(a) Rewrite $A'(t)=\dfrac{rA(t)}{100}$ as $\dfrac{A'(t)}{A(t)}=\dfrac{r}{100}$, express $A(t)$ in terms of r and t.

(b) Find the value of r. (Give your answer correct to 2 significant figures.)

(c) Using the value of r obtained in **(b)**, find the interest obtained after 15 years. (Give your answer correct to the nearest dollar.)

SOLUTION

(a)
$$A'(t)=\frac{rA(t)}{100}$$
$$\frac{A'(t)}{A(t)}=\frac{r}{100}$$
$$\int\frac{A'(t)}{A(t)}dt=\int\frac{r}{100}dt$$
$$\int\frac{1}{A(t)}d[A(t)]=\int\frac{r}{100}dt$$
$$\ln A(t)=\frac{rt}{100}+C$$
$$A(t)=e^{\frac{rt}{100}+C}$$
$$=e^{C}e^{\frac{rt}{100}}$$

$\because$ $A(0)=50\,000$

$\therefore$ $e^{C}e^{\frac{r(0)}{100}}=50\,000$

$e^{C}=50\,000$

$\therefore$ $A(t)=50\,000e^{\frac{rt}{100}}$

(b) $\because$ $\quad A(9) = 50\,000 \times 2$

$$= 100\,000$$

$\therefore$ $$50\,000e^{\frac{r(9)}{100}} = 100\,000$$

$$e^{\frac{9r}{100}} = 2$$

$$\frac{9r}{100} = \ln 2$$

$$r = \underline{\underline{7.7}} \text{ (corr. to 2 sig. fig.)}$$

(c) $$A(15) - A(0) = 50\,000e^{\frac{7.7(15)}{100}} - 50\,000$$

$$= 108\,701 \text{ (corr. to the nearest integer)}$$

$\therefore$ $\underline{\underline{\text{The interest obtained after 15 years is \$108 701.}}}$

159. The rate of increase of the volume V (in million square metres) of rubbish delivering from a city to a landfill after t years can be modelled by $\frac{dV}{dt} = k(V - 30)$, where k is a constant. It is given that the current volume of rubbish in the landfill is 45 million square metres, and it will increase by 1.5 million square metres after 1 year.

(a) Rewrite $\frac{dV}{dt} = k(V - 30)$ as $\frac{1}{V-30} \cdot \frac{dV}{dt} = k$. Hence express V in terms of k and t.

(b) Find the value of k. (Give your answer correct to 2 significant figures.)

(c) The government estimates that the landfill can only hold rubbish of 100 million square metres. Using the value of k obtained in **(b)**, when will the landfill be saturated? (Give your answer correct to 3 significant figures.)

SOLUTION

(a)

$$\frac{dV}{dt} = k(V - 30)$$

$$\frac{1}{V-30} \cdot \frac{dV}{dt} = k$$

$$\int \frac{1}{V-30} \cdot \frac{dV}{dt}\, dt = \int k\, dt$$

$$\int \frac{1}{V-30}\, dV = \int k\, dt$$

$$\int \frac{1}{V-30}\, d(V - 30) = \int k\, dt$$

$$\ln(V - 30) = kt + C \quad [\because\ V > 30]$$

$$V - 30 = e^{kt+C}$$

$$V - 30 = e^C e^{kt}$$

$$V = 30 + e^C e^{kt}$$

When $t = 0$, $V = 45$.

$45 = 30 + e^C e^{k(0)}$

$e^C = 15$

$\therefore \quad \underline{\underline{V = 30 + 15e^{kt}}}$

(b) When $t = 1$,

$V = 45 + 1.5$

$= 46.5$

$\therefore \quad 46.5 = 30 + 15e^{k(1)}$

$16.5 = 15e^k$

$e^k = 1.1$

$k = \ln 1.1$

$= \underline{\underline{0.095}}$ (corr. to 2 sig. fig.)

(c) When $V = 100$,

$100 = 30 + 15e^{0.095t}$

$70 = 15e^{0.095t}$

$e^{0.095t} = \dfrac{70}{15}$

$0.095t = \ln \dfrac{70}{15}$

$t = 16.2$ (corr. to 3 sig. fig.)

$\therefore$ The landfill will be saturated after 16.2 years.

Level 3

160. (a) Show that $\dfrac{d}{dx}[(x-3)^n(x+1)] = (x-3)^{n-1}[(n+1)x + n - 3]$, where n is a rational number.

(b) The slope at any point (x, y) of a curve C is given by $\dfrac{dy}{dx} = (x-3)^{2\,008}(1\,005x + 1\,003)$. If C passes through the point (3, 3), find the equation of C.

SOLUTION

(a) L.H.S. $= \dfrac{d}{dx}[(x-3)^n(x+1)]$

$= (x-3)^n(1) + n(x-3)^{n-1}(x+1)$

$= (x-3)^{n-1}[(x-3) + n(x+1)]$

$= (x-3)^{n-1}[(n+1)x + n - 3]$

$=$ R.H.S.

$\therefore \quad \dfrac{d}{dx}[(x-3)^n(x+1)] = (x-3)^{n-1}[(n+1)x + n - 3]$

(b) From **(a)**, take $n = 2\,009$,

$$\frac{d}{dx}[(x-3)^{2\,009}(x+1)] = (x-3)^{2\,008}(2\,010x+2\,006)$$

$\therefore \quad \int (x-3)^{2\,008}(2\,010x+2\,006)dx = (x-3)^{2\,009}(x+1)+C_1$

$\because \quad \frac{dy}{dx} = (x-3)^{2\,008}(1\,005x+1\,003)$

$$\therefore \quad y = \int (x-3)^{2\,008}(1\,005x+1\,003)dx$$
$$= \frac{1}{2}\int (x-3)^{2\,008}(2\,010x+2\,006)dx$$
$$= \frac{1}{2}(x-3)^{2\,009}(x+1)+C_1$$

$\because$ C passes through the point (3, 3).

$$\therefore \quad 3 = \frac{1}{2}(3-3)^{2\,009}(3+1)+C_1$$
$$C_1 = 3$$

$\therefore$ The equation of C is $y = \frac{1}{2}(x-3)^{2\,009}(x+1)+3$.

161. (a) Show that $\frac{d}{dx}[(2x-1)(x-1)^n] = (x-1)^{n-1}[2(n+1)x-n-2]$, where n is a rational number.

(b) The slope at any point (x, y) of a curve C is given by $\frac{dy}{dx} = (x-1)^{998}(2\,000x-1\,001)$. If the y-intercept of C is 2, find the equation of C.

SOLUTION

(a)
$$\text{L.H.S.} = \frac{d}{dx}[(2x-1)(x-1)^n]$$
$$= 2(x-1)^n + (2x-1)(n)(x-1)^{n-1}$$
$$= (x-1)^{n-1}[2(x-1)+n(2x-1)]$$
$$= (x-1)^{n-1}[(2n+2)x-n-2]$$
$$= (x-1)^{n-1}[2(n+1)x-n-2]$$
$$= \text{R.H.S.}$$

$\therefore \quad \frac{d}{dx}[(2x-1)(x-1)^n] = (x-1)^{n-1}[2(n+1)x-n-2]$

(b) From **(a)**, take $n = 999$,

$$\frac{d}{dx}[(2x-1)(x-1)^{999}] = (x-1)^{998}(2\,000x-1\,001)$$

$\therefore \quad \int (x-1)^{998}(2\,000x-1\,001)dx = (2x-1)(x-1)^{999} + C_1$

$\because \quad \frac{dy}{dx} = (x-1)^{998}(2\,000x-1\,001)$

$\therefore \quad y = \int (x-1)^{998}(2\,000x-1\,001)dx$

$\quad = (2x-1)(x-1)^{999} + C_1$

$\because$ C passes through the point (0, 2).

$\therefore \quad 2 = [2(0)-1](0-1)^{999} + C_1$

$2 = (-1)(-1)^{999} + C_1$

$C_1 = 1$

$\therefore$ The equation of C is $y = (2x-1)(x-1)^{999} + 1.$

162. The slope at any point (x, y) of a curve C is given by $\frac{dy}{dx} = 3x^2 - 4x - 5$, and the straight line $y = -x-1$ is the tangent to the curve at point A.

(a) Find the possible coordinates of A.

(b) Given that the y-intercept of C is greater than 0, find the equation of C.

(c) Find the other point of intersection of the tangent and C.

SOLUTION

(a) Let (x_1, y_1) be the coordinates of A.

Slope of the tangent $= -1$

Slope of the tangent at $(x_1, y_1) = \left.\frac{dy}{dx}\right|_{x=x_1}$

$= 3x_1^2 - 4x_1 - 5$

$\therefore \quad 3x_1^2 - 4x_1 - 5 = -1$

$3x_1^2 - 4x_1 - 4 = 0$

$(3x_1 + 2)(x_1 - 2) = 0$

$x_1 = -\frac{2}{3}$ or $x_1 = 2$

When $x_1 = -\frac{2}{3}$,

$y_1 = -(-\frac{2}{3}) - 1$

$= -\frac{1}{3}$

When $x_1 = 2$,

$y_1 = -2 - 1$

$= -3$

$\therefore$ The possible coordinates of A are $(-\frac{2}{3}, -\frac{1}{3})$ and $(2, -3)$.

(b) $\because$ $\frac{dy}{dx} = 3x^2 - 4x - 5$

$\therefore$ $y = \int (3x^2 - 4x - 5)dx$

$= x^3 - 2x^2 - 5x + C_1$

If the coordinates of A are $(-\frac{2}{3}, -\frac{1}{3})$,

$\because$ C passes through the point $(-\frac{2}{3}, -\frac{1}{3})$.

$\therefore$ $-\frac{1}{3} = (-\frac{2}{3})^3 - 2(-\frac{2}{3})^2 - 5(-\frac{2}{3}) + C_1$

$C_1 = -\frac{67}{27}$

When $x = 0$,

$y = 0^3 - 2(0)^2 - 5(0) - \frac{67}{27}$

$= -\frac{67}{27}$

< 0

$\because$ The y-intercept < 0

$\therefore$ The coordinates of A are not $(-\frac{2}{3}, -\frac{1}{3})$.

If the coordinates of A are $(2, -3)$,

$\because$ C passes through the point $(2, -3)$.

$\therefore$ $-3 = 2^3 - 2(2)^2 - 5(2) + C_1$

$C_1 = 7$

When $x = 0$,

$y = 0^3 - 2(0)^2 - 5(0) + 7$

$= 7$

> 0

$\because$ The y-intercept > 0

$\therefore$ The coordinates of A are $(2, -3)$.

$\therefore$ The equation of C is $y = x^3 - 2x^2 - 5x + 7$.

(c) $\begin{cases} y = -x - 1 \text{ ....(1)} \\ y = x^3 - 2x^2 - 5x + 7 \text{ ....(2)} \end{cases}$

$(2) - (1)$: $y - y = (x^3 - 2x^2 - 5x + 7) - (-x - 1)$

$$x^3 - 2x^2 - 4x + 8 = 0$$
$$(x-2)(x^2-4) = 0$$
$$(x+2)(x-2)^2 = 0$$
$$x = -2 \quad \text{or} \quad x = 2 \text{ (rejected)}$$

When $x = -2$,

$$y = (-2)^3 - 2(-2)^2 - 5(-2) + 7$$
$$= 1$$

$\therefore$ The other point of intersection is $(-2, 1)$.

163. An ant walks along a straight line. Let s cm, v cm/s and a cm/s^2 be the displacement, velocity and acceleration of the ant at time t s respectively.

(a) Prove that $a = \frac{1}{2}\frac{d}{ds}(v^2)$.

(b) Given that when $v = 0$, $s = 3$. If $a = \frac{1}{2s^2}$, express s in terms of v.

SOLUTION

(a) L.H.S. $= a$

$$= \frac{dv}{dt}$$
$$= \frac{dv}{ds} \cdot \frac{ds}{dt}$$

R.H.S. $= \frac{1}{2} \cdot \frac{d}{ds}(v^2)$

$$= \frac{1}{2} \cdot 2v \cdot \frac{dv}{ds}$$
$$= v \cdot \frac{dv}{ds}$$
$$= \frac{ds}{dt} \cdot \frac{dv}{ds}$$
$$= \text{L.H.S.}$$

$\therefore \quad a = \frac{1}{2}\frac{d}{ds}(v^2)$

(b) $\because \quad a = \dfrac{1}{2s^2}$

$\therefore \quad \dfrac{1}{2}\dfrac{d}{ds}(v^2) = \dfrac{1}{2s^2}$

$\dfrac{d}{ds}(v^2) = \dfrac{1}{s^2}$

$\therefore \quad v^2 = \displaystyle\int \frac{1}{s^2}\,ds$

$v^2 = -\dfrac{1}{s} + C$

When $v = 0$, $s = 3$.

$\therefore \quad 0^2 = -\dfrac{1}{3} + C$

$C = \dfrac{1}{3}$

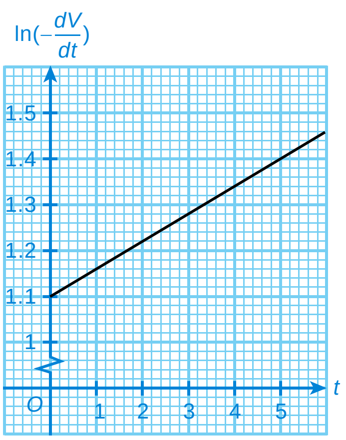
$\therefore \quad v^2 = -\dfrac{1}{s} + \dfrac{1}{3}$

$\dfrac{1}{s} = \dfrac{1}{3} - v^2$

$\dfrac{1}{s} = \dfrac{1-3v^2}{3}$

$\therefore \quad \underline{\underline{s = \dfrac{3}{1-3v^2}}}$

164. An ice cube of volume $250\,\text{cm}^3$ starts melting after taking out from the refrigerator. The rate of change of the volume V (in cm^3) of the ice cube can be modelled by $\dfrac{dV}{dt} = -ake^{kt}$ $(0 \le t \le 25)$, where a and k are positive constants, t (in minutes) is the time elapsed since the ice cube starts melting. The figure below shows the graph of $\ln(-\dfrac{dV}{dt})$ against t.



(a) **(i)** Express $\ln(-\frac{dV}{dt})$ as a linear function of t.

(ii) Use the given graph to find the values of a and k. (Give your answers correct to 4 decimal places if necessary.)

(iii) Take $a = 50$ and $k = 0.06$, express V in terms of t.

(b) After how long will the ice cube melt to half of its original volume? (Give your answer correct to 3 significant figures.)

SOLUTION

(a) **(i)**

$$\frac{dV}{dt} = -ake^{kt}$$

$$-\frac{dV}{dt} = ake^{kt}$$

$$\ln(-\frac{dV}{dt}) = \ln(ake^{kt})$$

$$\underline{\underline{\ln(-\frac{dV}{dt}) = \ln ak + kt}}$$

(ii) From the graph,

$$k = \text{Slope}$$

$$= \frac{1.4 - 1.1}{5 - 0}$$

$$= \underline{\underline{0.06}}$$

$\ln ak =$ Intercept on the vertical axis

$$\ln ak = 1.1$$

$$0.06a = e^{1.1}$$

$$a = \underline{\underline{50.0694}} \text{ (corr. to 4 d.p.)}$$

(iii) $\because \quad \frac{dV}{dt} = -50(0.06)e^{0.06t}$

$$\therefore \quad V = \int (-50)(0.06)e^{0.06t}dt$$

$$= -50\int e^{0.06t}d(0.06t)$$

$$= -50e^{0.06t} + C$$

When $t = 0$, $V = 250$.

$$\therefore \quad 250 = -50e^{0.06(0)} + C$$

$$C = 300$$

$$\therefore \quad \underline{\underline{V = 300 - 50e^{0.06t}}}$$

(b) When $V=\frac{1}{2}\times 250=125$,

$$125=300-50e^{0.06t}$$

$$50e^{0.06t}=175$$

$$e^{0.06t}=3.5$$

$$0.06t=\ln 3.5$$

$$t=20.9 \text{ (corr. to 3 sig. fig.)}$$

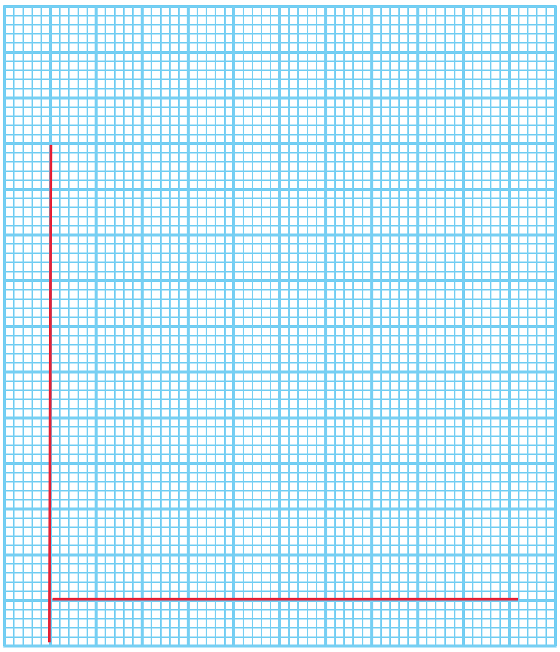
$\therefore$ The ice cube will melt to half of its original volume after 20.9 minutes.

165. The manager of a company finds that if the company invests N million dollars in promoting a new product, $P(N)$% of the residents will recognize the new product after three months, where $P'(N)=\frac{35ake^{-kN}}{2(1-a)}$ $(N\geq 0)$, a and k are positive constants. The following table shows the corresponding values of N and $P'(N)$.

N	3	6	9	12	15
$P'(N)$	6.695	4.269	2.722	1.736	1.107

(a) **(i)** Express $\ln P'(N)$ as a linear function of N.

(ii) Plot the graph of $\ln P'(N)$ against N.



(iii) Use the graph to find the values of a and k. (Give your answers correct to 2 decimal places.)

(b) Suppose 25% of the residents will recognize the new product after three months even if no promotion has been launched for the new product. Using the values of a and k obtained in **(a)(iii)**, find $P(N)$.

(c) The manager of the company claims that they can let 98% of the residents recognize the new product after three months. Do you agree? Explain briefly.

SOLUTION

(a) (i)

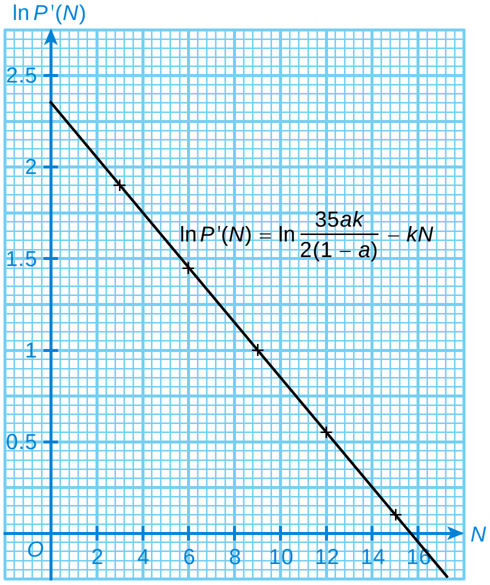
$$P'(N) = \frac{35ake^{-kN}}{2(1-a)}$$

$$\ln P'(N) = \ln \frac{35ake^{-kN}}{2(1-a)}$$

$$\ln P'(N) = \ln \frac{35ak}{2(1-a)} - kN$$

(ii) We convert the given data into the following.

N	3	6	9	12	15
$\ln P'(N)$	1.901	1.451	1.001	0.552	0.102



(iii) From the graph above,

$$-k = \text{Slope}$$

$$= \frac{0.102 - 1.901}{15 - 3}$$

$$= -0.15 \text{ (corr. to 2 d.p.)}$$

$$k = \underline{\underline{0.15}}$$

$$\ln\frac{35ak}{2(1-a)} = \text{Intercept on the vertical axis}$$
$$= 2.35$$
$$\frac{35a(0.150)}{2(1-a)} = e^{2.35}$$
$$5.25a = 2(1-a)e^{2.35}$$
$$(5.25+2e^{2.35})a = 2e^{2.35}$$
$$a = \underline{\underline{0.80}} \text{ (corr. to 2 d.p.)}$$

(b) $\because$ $P'(N) = \dfrac{35(0.80)(0.15)e^{-0.15N}}{2(1-0.80)}$

$$= 10.5e^{-0.15N}$$

$\therefore$ $P(N) = \int 10.5e^{-0.15N}\,dN$

$$= 10.5\int (-\frac{1}{0.15})e^{-0.15N}\,d(-0.15N)$$
$$= -70e^{-0.15N} + C$$

$\because$ $P(0) = 25$

$\therefore$ $-70e^{-0.15(0)} + C = 25$

$$C = 95$$

$\therefore$ $\underline{\underline{P(N) = -70e^{-0.15N} + 95}}$

(c) $P'(N) = 10.5e^{-0.15N}$

$$> 0$$

$\therefore$ $P(N)$ is an increasing function.

$$\lim_{N\to\infty} P(N) = \lim_{N\to\infty}(-70e^{-0.15N} + 95)$$
$$= -70\cdot\lim_{N\to\infty} e^{0.15N} + 95$$
$$= -70(0) + 95$$
$$= 95$$

$\therefore$ At most 95% of the residents will recognize the new product after three months.

$\therefore$ The company cannot let 98% of the residents recognize the new product after three months.

$\therefore$ I do not agree with the manager of the company.