Chapter 8 Definite Integration

Exercise 8A

Level 1

1. Evaluate $\int_2^4 6x^3 dx$.

SOLUTION

$$\int_2^4 6x^3 dx = 6[\frac{x^4}{4}]_2^4$$
$$= \frac{6}{4}(4^4 - 2^4)$$
$$= \underline{\underline{360}}$$

2. Evaluate $\int_0^2 (5-3x^2)\, dx$.

SOLUTION

$$\int_0^2 (5-3x^2)\, dx = [5x - x^3]_0^2$$
$$= [5(2) - 2^3] - [5(0) - 0^3]$$
$$= \underline{\underline{2}}$$

3. Evaluate $\int_{-1}^3 (x^3 - 3x^2 + 1)\, dx$.

SOLUTION

$$\int_{-1}^3 (x^3 - 3x^2 + 1)\, dx = [\frac{x^4}{4} - x^3 + x]_{-1}^3$$
$$= (\frac{3^4}{4} - 3^3 + 3) - [\frac{(-1)^4}{4} - (-1)^3 + (-1)]$$
$$= \underline{\underline{-4}}$$

4. Evaluate $\int_1^2 (3x^2 - 6x + 7)\,dx$.

SOLUTION

$$\begin{aligned}\int_1^2 (3x^2 - 6x + 7)\,dx &= [x^3 - 3x^2 + 7x]_1^2 \\ &= [2^3 - 3(2)^2 + 7(2)] - [1^3 - 3(1)^2 + 7(1)] \\ &= \underline{\underline{5}}\end{aligned}$$

5. Evaluate $\int_1^2 (x^{-2} + x^2)\,dx$.

SOLUTION

$$\begin{aligned}\int_1^2 (x^{-2} + x^2)\,dx &= [-x^{-1} + \frac{x^3}{3}]_1^2 \\ &= (-2^{-1} + \frac{2^3}{3}) - (-1^{-1} + \frac{1^3}{3}) \\ &= \underline{\underline{\frac{17}{6}}}\end{aligned}$$

6. Evaluate $\int_0^9 3\sqrt{x}\,dx$.

SOLUTION

$$\begin{aligned}\int_0^9 3\sqrt{x}\,dx &= [2x^{\frac{3}{2}}]_0^9 \\ &= 2(9)^{\frac{3}{2}} - 2(0)^{\frac{3}{2}} \\ &= \underline{\underline{54}}\end{aligned}$$

7. Evaluate $\int_1^4 \sqrt{\frac{2}{x}}\,dx$.

SOLUTION

$$\begin{aligned}\int_1^4 \sqrt{\frac{2}{x}}\,dx &= \sqrt{2}\int_1^4 x^{-\frac{1}{2}}\,dx \\ &= \sqrt{2}[2x^{\frac{1}{2}}]_1^4 \\ &= \sqrt{2}[2(4)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}] \\ &= \underline{\underline{2\sqrt{2}}}\end{aligned}$$

8. Evaluate $\int_1^9 (9x^2 - \frac{6}{\sqrt{x}})\,dx$.

SOLUTION

$$\int_1^9 (9x^2 - \frac{6}{\sqrt{x}})\,dx = [3x^3 - 12x^{\frac{1}{2}}]_1^9$$
$$= [3(9)^3 - 12(9)^{\frac{1}{2}}] - [3(1)^3 - 12(1)^{\frac{1}{2}}]$$
$$= \underline{\underline{2\,160}}$$

9. Evaluate $\int_1^8 (x^{\frac{1}{3}} + 4x^{-\frac{2}{3}})\,dx$.

SOLUTION

$$\int_1^8 (x^{\frac{1}{3}} + 4x^{-\frac{2}{3}})\,dx = [\frac{3}{4}x^{\frac{4}{3}} + 12x^{\frac{1}{3}}]_1^8$$
$$= [\frac{3}{4}(8)^{\frac{4}{3}} + 12(8)^{\frac{1}{3}}] - [\frac{3}{4}(1)^{\frac{4}{3}} + 12(1)^{\frac{1}{3}}]$$
$$= \underline{\underline{\frac{93}{4}}}$$

10. Evaluate $\int_1^2 x(x+1)\,dx$.

SOLUTION

$$\int_1^2 x(x+1)\,dx = \int_1^2 (x^2 + x)\,dx$$
$$= [\frac{x^3}{3} + \frac{x^2}{2}]_1^2$$
$$= (\frac{2^3}{3} + \frac{2^2}{2}) - (\frac{1^3}{3} + \frac{1^2}{2})$$
$$= \underline{\underline{\frac{23}{6}}}$$

11. Evaluate $\int_{-2}^{2} (3x+2)(7x+5)\,dx$.

SOLUTION

$$\begin{aligned}\int_{-2}^{2} (3x+2)(7x+5)\,dx &= \int_{-2}^{2} (21x^2+29x+10)\,dx \\ &= [7x^3+\frac{29}{2}x^2+10x]_{-2}^{2} \\ &= [7(2)^3+\frac{29}{2}(2)^2+10(2)]-[7(-2)^3+\frac{29}{2}(-2)^2+10(-2)] \\ &= \underline{\underline{152}}\end{aligned}$$

12. Evaluate $\int_{-1}^{2} (6-4x)(3x-1)\,dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^{2} (6-4x)(3x-1)\,dx &= \int_{-1}^{2} (-12x^2+22x-6)\,dx \\ &= [-4x^3+11x^2-6x]_{-1}^{2} \\ &= [-4(2)^3+11(2)^2-6(2)]-[-4(-1)^3+11(-1)^2-6(-1)] \\ &= \underline{\underline{-21}}\end{aligned}$$

13. Evaluate $\int_{-1}^{2} (2x^2-1)^2\,dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^{2} (2x^2-1)^2\,dx &= \int_{-1}^{2} (4x^4-4x^2+1)\,dx \\ &= [\frac{4}{5}x^5-\frac{4}{3}x^3+x]_{-1}^{2} \\ &= [\frac{4}{5}(2)^5-\frac{4}{3}(2)^3+2]-[\frac{4}{5}(-1)^5-\frac{4}{3}(-1)^3+(-1)] \\ &= \underline{\underline{\frac{87}{5}}}\end{aligned}$$

14. Evaluate $\int_4^9 (2\sqrt{x}-3)^2\,dx$.

SOLUTION

$$\begin{aligned}\int_4^9 (2\sqrt{x}-3)^2\,dx &= \int_4^9 (4x-12\sqrt{x}+9)\,dx\\ &= [2x^2-8x^{\frac{3}{2}}+9x]_4^9\\ &= [2(9)^2-8(9)^{\frac{3}{2}}+9(9)]-[2(4)^2-8(4)^{\frac{3}{2}}+9(4)]\\ &= \underline{\underline{23}}\end{aligned}$$

15. Evaluate $\int_{-3}^0 3e^x\,dx$.

SOLUTION

$$\begin{aligned}\int_{-3}^0 3e^x dx &= [3e^x]_{-3}^0\\ &= 3e^0-3e^{-3}\\ &= \underline{\underline{3-3e^{-3}}}\end{aligned}$$

16. Evaluate $\int_1^4 (3x^2-12x^{-2}+5e^x)\,dx$.

SOLUTION

$$\begin{aligned}\int_1^4 (3x^2-12x^{-2}+5e^x)\,dx &= [x^3+\frac{12}{x}+5e^x]_1^4\\ &= (4^3+\frac{12}{4}+5e^4)-(1^3+\frac{12}{1}+5e^1)\\ &= (67+5e^4)-(13+5e)\\ &= \underline{\underline{54+5e^4-5e}}\end{aligned}$$

17. Evaluate $\int_{-1}^{1} e^{-2x}\,dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^{1} e^{-2x}\,dx &= \int_{-1}^{1} (e^{-2})^x\,dx\\ &= [\frac{(e^{-2})^x}{\ln e^{-2}}]_{-1}^{1}\\ &= [-\frac{1}{2}e^{-2x}]_{-1}^{1}\\ &= -\frac{1}{2}e^{-2(1)} - [-\frac{1}{2}e^{-2(-1)}]\\ &= \underline{\underline{\frac{1}{2}e^{2} - \frac{1}{2}e^{-2}}}\end{aligned}$$

18. Evaluate $\int_{1}^{3} 6e^{4x+5}\,dx$.

SOLUTION

$$\begin{aligned}\int_{1}^{3} 6e^{4x+5}\,dx &= \int_{1}^{3} 6e^{5}\cdot e^{4x}\,dx\\ &= 6e^5\int_{1}^{3} (e^{4})^x\,dx\\ &= 6e^5[\frac{(e^4)^x}{\ln e^4}]_1^3\\ &= 6e^5[\frac{e^{4x}}{4}]_1^3\\ &= \frac{3}{2}e^5[e^{4(3)} - e^{4(1)}]\\ &= \underline{\underline{\frac{3}{2}(e^{17} - e^{9})}}\end{aligned}$$

19. Evaluate $\int_{1}^{2} 5^x\,dx$.

SOLUTION

$$\begin{aligned}\int_{1}^{2} 5^x\,dx &= [\frac{5^x}{\ln 5}]_1^2\\ &= \frac{5^2}{\ln 5} - \frac{5^1}{\ln 5}\\ &= \underline{\underline{\frac{20}{\ln 5}}}\end{aligned}$$

20. Evaluate $\int_0^2 (2x^3 - 2^{3x})\,dx$.

SOLUTION

$$\int_0^2 (2x^3 - 2^{3x})\,dx = [\frac{x^4}{2} - \frac{(2^3)^x}{\ln 2^3}]_0^2$$

$$= [\frac{x^4}{2} - \frac{2^{3x}}{3\ln 2}]_0^2$$

$$= [\frac{2^4}{2} - \frac{2^{3(2)}}{3\ln 2}] - [\frac{0^4}{2} - \frac{2^{3(0)}}{3\ln 2}]$$

$$= (8 - \frac{64}{3\ln 2}) - (-\frac{1}{3\ln 2})$$

$$= \underline{\underline{8 - \frac{21}{\ln 2}}}$$

21. Evaluate $\int_{-2}^0 [8e^{-4x} + 18(3^{2x})]dx$.

SOLUTION

$$\int_{-2}^0 [8e^{-4x} + 18(3^{2x})]dx = \int_{-2}^0 \{8(e^{-4})^x + 18[(3^2)^x]\}dx$$

$$= [8 \times \frac{(e^{-4})^x}{\ln e^{-4}} + 18 \times \frac{(3^2)^x}{\ln 3^2}]_{-2}^0$$

$$= [\frac{8e^{-4x}}{-4} + \frac{18(3^{2x})}{2\ln 3}]_{-2}^0$$

$$= [-2e^{-4x} + \frac{9(3^{2x})}{\ln 3}]_{-2}^0$$

$$= \{-2e^{-4(0)} + \frac{9[3^{2(0)}]}{\ln 3}\} - \{-2e^{-4(-2)} + \frac{9[3^{2(-2)}]}{\ln 3}\}$$

$$= (-2 + \frac{9}{\ln 3}) - (-2e^8 + \frac{1}{9\ln 3})$$

$$= \underline{\underline{2e^8 + \frac{80}{9\ln 3} - 2}}$$

22. Evaluate $\int_0^2 3^x e^x \, dx$.

SOLUTION

$$\begin{aligned}\int_0^2 3^x e^x \, dx &= \int_0^2 (3e)^x \, dx \\ &= \left[\frac{(3e)^x}{\ln 3e}\right]_0^2 \\ &= \left[\frac{(3e)^x}{\ln 3 + \ln e}\right]_0^2 \\ &= \left[\frac{(3e)^x}{\ln 3 + 1}\right]_0^2 \\ &= \frac{(3e)^2}{\ln 3 + 1} - \frac{(3e)^0}{\ln 3 + 1} \\ &= \underline{\underline{\frac{9e^2 - 1}{\ln 3 + 1}}}\end{aligned}$$

23. Evaluate $\int_{-2}^0 5^x e^{-2x} \, dx$.

SOLUTION

$$\begin{aligned}\int_{-2}^0 5^x e^{-2x} \, dx &= \int_{-2}^0 (5e^{-2})^x \, dx \\ &= \left[\frac{(5e^{-2})^x}{\ln(5e^{-2})}\right]_{-2}^0 \\ &= \left[\frac{(5e^{-2})^x}{\ln 5 + \ln e^{-2}}\right]_{-2}^0 \\ &= \left[\frac{(5e^{-2})^x}{\ln 5 - 2}\right]_{-2}^0 \\ &= \frac{(5e^{-2})^0}{\ln 5 - 2} - \frac{(5e^{-2})^{-2}}{\ln 5 - 2} \\ &= \frac{1}{\ln 5 - 2} - \frac{e^4}{25(\ln 5 - 2)} \\ &= \underline{\underline{\frac{25 - e^4}{25(\ln 5 - 2)}}}\end{aligned}$$

24. Evaluate $\int_2^6 (\frac{3}{x}+\frac{1}{6x})\,dx$.

SOLUTION

$$\begin{aligned}\int_2^6 (\frac{3}{x}+\frac{1}{6x})\,dx &= \int_2^6 \frac{19}{6x}\,dx \\ &= \frac{19}{6}\int_2^6 \frac{1}{x}\,dx \\ &= \frac{19}{6}[\ln|x|]_2^6 \\ &= \frac{19}{6}(\ln 6-\ln 2) \\ &= \underline{\underline{\frac{19}{6}\ln 3}}\end{aligned}$$

25. Evaluate $\int_1^4 \frac{3x+4x^2-2\sqrt{x}}{x^2}\,dx$.

SOLUTION

$$\begin{aligned}\int_1^4 \frac{3x+4x^2-2\sqrt{x}}{x^2}\,dx &= \int_1^4 (3x^{-1}+4-2x^{-\frac{3}{2}})\,dx \\ &= [3\ln|x|+4x+4x^{-\frac{1}{2}}]_1^4 \\ &= [3\ln 4+4(4)+4(4)^{-\frac{1}{2}}]-[3\ln 1+4(1)+4(1)^{-\frac{1}{2}}] \\ &= (3\ln 4+18)-8 \\ &= \underline{\underline{6\ln 2+10}}\end{aligned}$$

26. Evaluate $\int_1^3 (x-2)(4+\frac{3}{x^2})\,dx$.

SOLUTION

$$\begin{aligned}\int_1^3 (x-2)(4+\frac{3}{x^2})\,dx &= \int_1^3 (4x-8+\frac{3}{x}-\frac{6}{x^2})\,dx \\ &= [2x^2-8x+3\ln|x|+\frac{6}{x}]_1^3 \\ &= [2(3)^2-8(3)+3\ln 3+\frac{6}{3}]-[2(1)^2-8(1)+3\ln 1+\frac{6}{1}] \\ &= \underline{\underline{3\ln 3-4}}\end{aligned}$$

27. Suppose $f(x)$ is a continuous function and that $\int_0^5 f(x)\,dx = 7$. Evaluate the following definite integrals.

(a) $\int_0^5 2f(x)\,dx$ **(b)** $\int_0^5 [2 - f(x)]\,dx$

SOLUTION

(a) $\int_0^5 2f(x)\,dx = 2\int_0^5 f(x)\,dx$

$= 2(7)$

$= \underline{\underline{14}}$

(b) $\int_0^5 [2 - f(x)]\,dx = \int_0^5 2\,dx - \int_0^5 f(x)\,dx$

$= 2[x]_0^5 - 7$

$= 2(5-0) - 7$

$= \underline{\underline{3}}$

28. Suppose $g(x)$ is a continuous function and that $\int_1^2 g(x)\,dx = 3$. Evaluate the following definite integrals.

(a) $\int_1^2 [6x^2 - g(x)]\,dx$ **(b)** $\int_1^2 \frac{5x + x^2 g(x)}{x^2}\,dx$

SOLUTION

(a) $\int_1^2 [6x^2 - g(x)]\,dx = \int_1^2 6x^2\,dx - \int_1^2 g(x)\,dx$

$= [2x^3]_1^2 - 3$

$= [2(2)^3 - 2(1)^3] - 3$

$= \underline{\underline{11}}$

(b) $\int_1^2 \frac{5x + x^2 g(x)}{x^2}\,dx = \int_1^2 [5x^{-1} + g(x)]\,dx$

$= \int_1^2 5x^{-1}\,dx + \int_1^2 g(x)\,dx$

$= 5[\ln|x|]_1^2 + 3$

$= 5(\ln 2 - \ln 1) + 3$

$= \underline{\underline{5\ln 2 + 3}}$

29. Suppose $f(x)$ is a continuous function, and that $\int_1^3 f(x)\,dx = -1$ and $\int_1^5 f(x)\,dx = 2$. Evaluate the following definite integrals.

(a) $\int_3^5 f(z)\,dz$ **(b)** $\int_5^3 f(u)\,du$

SOLUTION

(a)
$$\int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_1^5 f(x)\,dx$$
$$-1 + \int_3^5 f(z)\,dz = 2$$
$$\int_3^5 f(z)\,dz = \underline{\underline{3}}$$

(b)
$$\int_5^3 f(u)\,du = \int_5^3 f(z)\,dz$$
$$= -\int_3^5 f(z)\,dz$$
$$= \underline{\underline{-3}} \quad \text{[From the result of } \textbf{(a)}\text{]}$$

30. Suppose $h(x)$ is a continuous function, and that $\int_{-2}^{2} h(x)\,dx = 8$ and $\int_{-5}^{2} h(x)\,dx = -2$. Evaluate the following definite integrals.

(a) $\int_{-5}^{-2} h(r)\,dr$ **(b)** $\int_{-2}^{-5} h(u)\,du$

SOLUTION

(a)
$$\int_{-5}^{-2} h(x)dx + \int_{-2}^{2} h(x)\,dx = \int_{-5}^{2} h(x)\,dx$$
$$\int_{-5}^{-2} h(r)\,dr + 8 = -2$$
$$\int_{-5}^{-2} h(r)\,dr = \underline{\underline{-10}}$$

(b)
$$\int_{-2}^{-5} h(u)\,du = \int_{-2}^{-5} h(r)\,dr$$
$$= -\int_{-5}^{-2} h(r)\,dr$$
$$= -(-10) \quad \text{[From the result of } \textbf{(a)}\text{]}$$
$$= \underline{\underline{10}}$$

31. Suppose $g(x)$ and $h(x)$ are continuous functions, and that $\int_{-1}^{3} g(x)\,dx = 5$, $\int_{3}^{6} g(x)\,dx = 8$ and $\int_{-1}^{6} h(x)\,dx = -4$. Evaluate the following definite integrals.

(a) $\int_{-1}^{6} g(x)\,dx$ **(b)** $\int_{-1}^{6} [3h(x)+2g(x)]dx$

SOLUTION

(a)
$$\begin{aligned}\int_{-1}^{6} g(x)\,dx &= \int_{-1}^{3} g(x)\,dx + \int_{3}^{6} g(x)\,dx \\ &= 5+8 \\ &= \underline{\underline{13}}\end{aligned}$$

(b)
$$\begin{aligned}\int_{-1}^{6} [3h(x)+2g(x)]dx &= 3\int_{-1}^{6} h(x)\,dx + 2\int_{-1}^{6} g(x)\,dx \\ &= 3(-4)+2(13) \quad \text{[From the result of } \textbf{(a)} \text{]} \\ &= \underline{\underline{14}}\end{aligned}$$

32. Suppose $p(x)$ and $q(x)$ are continuous functions, and that $\int_{-1}^{4} p(x)\,dx = 3$, $\int_{1}^{4} p(x)\,dx = 1$ and $\int_{-1}^{1} q(x)\,dx = -5$. Evaluate the following definite integrals.

(a) $\int_{-1}^{1} p(x)\,dx$ **(b)** $\int_{-1}^{1} 5[p(x)-3q(x)]dx$

SOLUTION

(a)
$$\begin{aligned}\int_{-1}^{1} p(x)\,dx + \int_{1}^{4} p(x)\,dx &= \int_{-1}^{4} p(x)\,dx \\ \int_{-1}^{1} p(x)\,dx + 1 &= 3 \\ \int_{-1}^{1} p(x)\,dx &= \underline{\underline{2}}\end{aligned}$$

(b)
$$\begin{aligned}\int_{-1}^{1} 5[p(x)-3q(x)]dx &= \int_{-1}^{1} [5p(x)-15q(x)]dx \\ &= 5\int_{-1}^{1} p(x)\,dx - 15\int_{-1}^{1} q(x)\,dx \\ &= 5(2)-15(-5) \quad \text{[From the result of } \textbf{(a)} \text{]} \\ &= \underline{\underline{85}}\end{aligned}$$

33. Suppose $f(x)$ is a continuous function, and that $\int_{-3}^{1} f(x)\,dx = p$ and $\int_{1}^{4} f(x)\,dx = q$, where p and q are constants. Express the following definite integrals in terms of p and q.

(a) $\int_{-3}^{4} f(x)\,dx$ **(b)** $\int_{2}^{4} f(x)\,dx - \int_{2}^{1} f(x)\,dx$

(c) $\int_{-3}^{6} f(x)\,dx + \int_{6}^{4} f(t)\,dt$

SOLUTION

(a) $$\int_{-3}^{4} f(x)\,dx = \int_{-3}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$$
$$= \underline{\underline{p+q}}$$

(b) $$\int_{2}^{4} f(x)\,dx - \int_{2}^{1} f(x)\,dx = \int_{2}^{4} f(x)\,dx + \int_{1}^{2} f(x)\,dx$$
$$= \int_{1}^{4} f(x)\,dx$$
$$= \underline{\underline{q}}$$

(c) $$\int_{-3}^{6} f(x)\,dx + \int_{6}^{4} f(t)\,dt = \int_{-3}^{6} f(x)\,dx + \int_{6}^{4} f(x)\,dx$$
$$= \int_{-3}^{4} f(x)\,dx + \int_{4}^{6} f(x)\,dx - \int_{4}^{6} f(x)\,dx$$
$$= \int_{-3}^{4} f(x)\,dx$$
$$= \underline{\underline{p+q}}$$ [From the result of **(a)**]

34. Suppose $f(x)$ is a continuous function, and that $\int_{-3}^{1} f(x)\,dx = a$, $\int_{1}^{3} f(x)\,dx = b$ and $\int_{1}^{7} f(x)\,dx = c$, where a, b and c are constants. Express the following definite integrals in terms of a, b and c.

(a) $\int_{-3}^{7} [f(x)-2]\,dx$ **(b)** $\int_{3}^{7} 2f(x)\,dx$

(c) $\int_{3}^{-3} [f(x)-x^2]\,dx$ **(d)** $\int_{-3}^{-1} [f(x)+1]\,dx - \int_{7}^{-1} f(t)\,dt$

SOLUTION

(a) $\int_{-3}^{7}[f(x)-2]\,dx=\int_{-3}^{7}f(x)\,dx-\int_{-3}^{7}2\,dx$

$=\int_{-3}^{1}f(x)\,dx+\int_{1}^{7}f(x)\,dx-[2x]_{-3}^{7}$

$=a+c-[2(7)-2(-3)]$

$=\underline{\underline{a+c-20}}$

(b) $\int_{1}^{3}f(x)\,dx+\int_{3}^{7}f(x)\,dx=\int_{1}^{7}f(x)\,dx$

$b+\int_{3}^{7}f(x)\,dx=c$

$\int_{3}^{7}f(x)\,dx=c-b$

$\therefore\quad \int_{3}^{7}2f(x)\,dx=2\int_{3}^{7}f(x)\,dx$

$=\underline{\underline{2(c-b)}}$

(c) $\int_{3}^{-3}[f(x)-x^2]\,dx=-\int_{-3}^{3}[f(x)-x^2]\,dx$

$=-\int_{-3}^{3}f(x)\,dx+\int_{-3}^{3}x^2\,dx$

$=-[\int_{-3}^{1}f(x)\,dx+\int_{1}^{3}f(x)\,dx]+[\frac{x^3}{3}]_{-3}^{3}$

$=-(a+b)+[\frac{3^3}{3}-\frac{(-3)^3}{3}]$

$=\underline{\underline{-a-b+18}}$

(d) $\int_{-3}^{-1}[f(x)+1]\,dx-\int_{7}^{-1}f(t)\,dt=\int_{-3}^{-1}f(x)\,dx+\int_{-3}^{-1}dx-\int_{7}^{-1}f(x)\,dx$

$=\int_{-3}^{-1}f(x)\,dx+[x]_{-3}^{-1}+\int_{-1}^{7}f(x)\,dx$

$=\int_{-3}^{7}f(x)\,dx+[(-1)-(-3)]$

$=\int_{-3}^{1}f(x)\,dx+\int_{1}^{7}f(x)\,dx+2$

$=\underline{\underline{a+c+2}}$

Level 2

35. **(a)** Factorize $x^3 - 3x^2 - x + 3$.

(b) Hence evaluate $\int_1^2 \frac{x^3 - 3x^2 - x + 3}{x+1}\,dx$.

SOLUTION

(a) $$\begin{aligned} x^3 - 3x^2 - x + 3 &= x^2(x-3) - (x-3) \\ &= (x^2 - 1)(x-3) \\ &= \underline{\underline{(x-1)(x+1)(x-3)}} \end{aligned}$$

(b) $$\begin{aligned} \int_1^2 \frac{x^3 - 3x^2 - x + 3}{x+1}\,dx &= \int_1^2 \frac{(x-1)(x+1)(x-3)}{x+1}\,dx \quad [\text{From the result of } \textbf{(a)}] \\ &= \int_1^2 (x-1)(x-3)\,dx \\ &= \int_1^2 (x^2 - 4x + 3)\,dx \\ &= [\frac{x^3}{3} - 2x^2 + 3x]_1^2 \\ &= [\frac{2^3}{3} - 2(2)^2 + 3(2)] - [\frac{1^3}{3} - 2(1)^2 + 3(1)] \\ &= \underline{\underline{-\frac{2}{3}}} \end{aligned}$$

36. Evaluate $\int_{-1}^1 \frac{x^3 + 8}{x+2}\,dx$.

SOLUTION

$$\begin{aligned} \int_{-1}^1 \frac{x^3 + 8}{x+2}\,dx &= \int_{-1}^1 \frac{(x+2)(x^2 - 2x + 4)}{x+2}\,dx \\ &= \int_{-1}^1 (x^2 - 2x + 4)\,dx \\ &= [\frac{x^3}{3} - x^2 + 4x]_{-1}^1 \\ &= [\frac{1^3}{3} - 1^2 + 4(1)] - [\frac{(-1)^3}{3} - (-1)^2 + 4(-1)] \\ &= \underline{\underline{\frac{26}{3}}} \end{aligned}$$

37. Evaluate $\int_{-2}^{2}\frac{16}{4+x}dx-\int_{-2}^{2}\frac{x^2}{4+x}dx$.

SOLUTION

$$\begin{aligned}\int_{-2}^{2}\frac{16}{4+x}dx-\int_{-2}^{2}\frac{x^2}{4+x}dx&=\int_{-2}^{2}(\frac{16}{4+x}-\frac{x^2}{4+x})dx\\&=\int_{-2}^{2}\frac{16-x^2}{4+x}dx\\&=\int_{-2}^{2}\frac{(4-x)(4+x)}{4+x}dx\\&=\int_{-2}^{2}(4-x)dx\\&=[4x-\frac{x^2}{2}]_{-2}^{2}\\&=[4(2)-\frac{2^2}{2}]-[4(-2)-\frac{(-2)^2}{2}]\\&=\underline{\underline{16}}\end{aligned}$$

38. Evaluate $\int_{0}^{2}(e^{x^2}-e^{2x})dx-\int_{0}^{2}(e^{x^2}+e^{x})dx$.

SOLUTION

$$\begin{aligned}\int_{0}^{2}(e^{x^2}-e^{2x})dx-\int_{0}^{2}(e^{x^2}+e^{x})dx&=\int_{0}^{2}(e^{x^2}-e^{2x}-e^{x^2}-e^{x})dx\\&=\int_{0}^{2}[-(e^2)^x-e^x]dx\\&=[-\frac{(e^2)^x}{\ln e^2}-e^x]_0^2\\&=[-\frac{e^{2x}}{2}-e^x]_0^2\\&=[-\frac{e^{2(2)}}{2}-e^2]-[-\frac{e^{2(0)}}{2}-e^0]\\&=-\frac{e^4}{2}-e^2-(-\frac{3}{2})\\&=\underline{\underline{-\frac{e^4}{2}-e^2+\frac{3}{2}}}\end{aligned}$$

39. Evaluate $\int_2^5 (3x^x + 8x)\,dx + 3\int_5^2 t^t\,dt + 8\int_{-1}^2 t\,dt$.

SOLUTION

$$\int_2^5 (3x^x + 8x)\,dx + 3\int_5^2 t^t\,dt + 8\int_{-1}^2 t\,dt = \int_2^5 3x^x\,dx + \int_2^5 8x\,dx - \int_2^5 3x^x\,dx + \int_{-1}^2 8x\,dx$$

$$= \int_{-1}^5 8x\,dx$$

$$= [4x^2]_{-1}^5$$

$$= 4(5)^2 - 4(-1)^2$$

$$= \underline{\underline{96}}$$

40. Evaluate $\int_1^5 \frac{2e^{2x}\,dx}{e^{2x} - 2e^{-x}} + 4\int_5^1 \frac{e^{-x}\,dx}{e^{2x} - 2e^{-x}}$.

SOLUTION

$$\int_1^5 \frac{2e^{2x}\,dx}{e^{2x} - 2e^{-x}} + 4\int_5^1 \frac{e^{-x}\,dx}{e^{2x} - 2e^{-x}} = \int_1^5 \frac{2e^{2x}\,dx}{e^{2x} - 2e^{-x}} - \int_1^5 \frac{4e^{-x}\,dx}{e^{2x} - 2e^{-x}}$$

$$= \int_1^5 \frac{2e^{2x} - 4e^{-x}}{e^{2x} - 2e^{-x}}\,dx$$

$$= 2\int_1^5 \frac{e^{2x} - 2e^{-x}}{e^{2x} - 2e^{-x}}\,dx$$

$$= 2\int_1^5 dx$$

$$= 2[x]_1^5$$

$$= 2(5-1)$$

$$= \underline{\underline{8}}$$

41. If $\int_{-1}^{2} \frac{1}{f(x)-3} dx = 5$, evaluate $\int_{-1}^{2} \frac{f(x)}{f(x)-3} dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^{2} \frac{f(x)}{f(x)-3} dx &= \int_{-1}^{2} \frac{[f(x)-3]+3}{f(x)-3} dx \\ &= \int_{-1}^{2} [1+\frac{3}{f(x)-3}] dx \\ &= \int_{-1}^{2} dx + 3\int_{-1}^{2} \frac{1}{f(x)-3} dx \\ &= [x]_{-1}^{2} + 3(5) \\ &= [2-(-1)]+15 \\ &= \underline{\underline{18}}\end{aligned}$$

42. Suppose $f(x)$ and $g(x)$ are continuous functions and that $\int_{2}^{4} \frac{f(x)}{f(x)-2g(x)} dx = 10$. Evaluate the following definite integrals.

(a) $\int_{4}^{2} \frac{2f(x)}{f(x)-2g(x)} dx$

(b) $\int_{2}^{4} \frac{g(t)}{f(t)-2g(t)} dt$

SOLUTION

(a) $$\begin{aligned}\int_{4}^{2} \frac{2f(x)}{f(x)-2g(x)} dx &= -2\int_{2}^{4} \frac{f(x)}{f(x)-2g(x)} dx \\ &= -2(10) \\ &= \underline{\underline{-20}}\end{aligned}$$

(b) $$\int_2^4 \frac{g(t)}{f(t)-2g(t)}dt = \int_2^4 \frac{g(x)}{f(x)-2g(x)}dx$$
$$= -\frac{1}{2}\int_2^4 \frac{-2g(x)}{f(x)-2g(x)}dx$$
$$= -\frac{1}{2}\int_2^4 \frac{[f(x)-2g(x)]-f(x)}{f(x)-2g(x)}dx$$
$$= -\frac{1}{2}\int_2^4 [1-\frac{f(x)}{f(x)-2g(x)}]dx$$
$$= -\frac{1}{2}\int_2^4 dx + \frac{1}{2}\int_2^4 \frac{f(x)}{f(x)-2g(x)}dx$$
$$= -\frac{1}{2}[x]_2^4 + \frac{1}{2}(10)$$
$$= -\frac{1}{2}(4-2)+5$$
$$= \underline{\underline{4}}$$

43. It is given that $y=(4x-3)^6$.

(a) Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_{\frac{1}{2}}^{1}(4x-3)^5 dx$.

SOLUTION

(a) $$y=(4x-3)^6$$
$$\frac{dy}{dx}=6(4x-3)^5(4)$$
$$=\underline{\underline{24(4x-3)^5}}$$

(b) $\because$ $$\frac{d}{dx}(4x-3)^6 = 24(4x-3)^5$$
$\therefore$ $$\int_{\frac{1}{2}}^{1} 24(4x-3)^5 dx = [(4x-3)^6]_{\frac{1}{2}}^{1}$$
$$24\int_{\frac{1}{2}}^{1}(4x-3)^5 dx = [4(1)-3]^6 - [4(\frac{1}{2})-3]^6$$
$$24\int_{\frac{1}{2}}^{1}(4x-3)^5 dx = 0$$
$$\int_{\frac{1}{2}}^{1}(4x-3)^5 dx = \underline{\underline{0}}$$

44. It is given that $y = x\ln x$.

(a) Find $\dfrac{dy}{dx}$.

(b) Hence evaluate $\int_1^e \ln x\,dx$.

SOLUTION

(a)
$$y = x\ln x$$
$$\frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x$$
$$= \underline{\underline{1+\ln x}}$$

(b)
$$\because \quad \frac{d}{dx}(x\ln x) = 1 + \ln x$$
$$\therefore \quad \int_1^e (1+\ln x)\,dx = [x\ln x]_1^e$$
$$\int_1^e dx + \int_1^e \ln x\,dx = e\ln e - 1\ln 1$$
$$[x]_1^e + \int_1^e \ln x\,dx = e$$
$$e - 1 + \int_1^e \ln x\,dx = e$$
$$\int_1^e \ln x\,dx = \underline{\underline{1}}$$

45. It is given that $y = \dfrac{e^x}{e^x+1}$.

(a) Find $\dfrac{dy}{dx}$.

(b) Hence evaluate $\int_0^1 \dfrac{e^x}{(e^x+1)^2}\,dx$.

SOLUTION

(a)
$$y = \frac{e^x}{e^x+1}$$
$$\frac{dy}{dx} = \frac{(e^x+1)(e^x) - (e^x)(e^x)}{(e^x+1)^2}$$
$$= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2}$$
$$= \underline{\underline{\frac{e^x}{(e^x+1)^2}}}$$

(b) $\because \quad \dfrac{d}{dx}\left(\dfrac{e^x}{e^x+1}\right)=\dfrac{e^x}{(e^x+1)^2}$

$$\therefore \quad \int_0^1 \frac{e^x}{(e^x+1)^2}\,dx = \left[\frac{e^x}{e^x+1}\right]_0^1$$

$$= \frac{e^1}{e^1+1}-\frac{e^0}{e^0+1}$$

$$= \frac{e}{e+1}-\frac{1}{2}$$

$$= \frac{2e-(e+1)}{2(e+1)}$$

$$= \underline{\underline{\frac{e-1}{2(e+1)}}}$$

46. It is given that $y=\ln(10^x+8)$.

(a) Find $\dfrac{dy}{dx}$.

(b) Hence evaluate $\displaystyle\int_0^2 \frac{10^x}{10^x+8}\,dx$.

SOLUTION

(a) $y=\ln(10^x+8)$

$$\frac{dy}{dx}=\underline{\underline{\frac{10^x\ln 10}{10^x+8}}}$$

(b) $\because \quad \dfrac{d}{dx}\ln(10^x+8)=\dfrac{10^x\ln 10}{10^x+8}$

$$\therefore \quad \int_0^2 \frac{10^x \ln 10}{10^x+8}\,dx = [\ln(10^x+8)]_0^2$$

$$\ln 10\int_0^2 \frac{10^x}{10^x+8}\,dx = \ln(10^2+8)-\ln(10^0+8)$$

$$\int_0^2 \frac{10^x}{10^x+8}\,dx = \frac{\ln 108-\ln 9}{\ln 10}$$

$$= \underline{\underline{\frac{\ln 12}{\ln 10}}}$$

47. It is given that $k > 1$.

(a) Express the value of $\int_1^k (6x+2)\,dx$ in terms of k.

(b) If $\int_1^k (6x+2)\,dx = 51$, find the value of k.

SOLUTION

(a)
$$\begin{aligned}\int_1^k (6x+2)\,dx &= [3x^2+2x]_1^k \\ &= (3k^2+2k)-[3(1)^2+2(1)] \\ &= \underline{\underline{3k^2+2k-5}}\end{aligned}$$

(b)
$$\begin{aligned}\int_1^k (6x+2)\,dx &= 51 \\ 3k^2+2k-5 &= 51 \\ 3k^2+2k-56 &= 0 \\ (k-4)(3k+14) &= 0\end{aligned}$$
$$k = \underline{\underline{4}} \quad \text{or} \quad k = -\frac{14}{3} \text{ (rejected)}$$

48. It is given that $k < 3$.

(a) Express the value of $\int_k^3 (5-4x)\,dx$ in terms of k.

(b) If $\int_k^3 (5-4x)\,dx = 15$, find the value of k.

SOLUTION

(a)
$$\begin{aligned}\int_k^3 (5-4x)\,dx &= [5x-2x^2]_k^3 \\ &= [5(3)-2(3)^2]-(5k-2k^2) \\ &= \underline{\underline{2k^2-5k-3}}\end{aligned}$$

(b)
$$\begin{aligned}\int_k^3 (5-4x)\,dx &= 15 \\ 2k^2-5k-3 &= 15 \\ 2k^2-5k-18 &= 0 \\ (k+2)(2k-9) &= 0\end{aligned}$$
$$k = \underline{\underline{-2}} \quad \text{or} \quad k = \frac{9}{2} \text{ (rejected)}$$

Level 3

49. It is given that $y = x\sqrt{4-x^2}$.

(a) Prove that $\dfrac{dy}{dx} = \dfrac{4-2x^2}{\sqrt{4-x^2}}$.

(b) Hence prove that $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx - \frac{\sqrt{3}}{2}$.

PROOF

(a) $y = x\sqrt{4-x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{4-x^2} + x\cdot\frac{-2x}{2\sqrt{4-x^2}}\\ &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}\\ &= \frac{4-x^2-x^2}{\sqrt{4-x^2}}\\ &= \frac{4-2x^2}{\sqrt{4-x^2}}\end{aligned}$$

(b) $\because$ $\dfrac{d}{dx}(x\sqrt{4-x^2}) = \dfrac{4-2x^2}{\sqrt{4-x^2}}$

$\therefore$

$$\begin{aligned}\int_0^1 \frac{4-2x^2}{\sqrt{4-x^2}}\,dx &= [x\sqrt{4-x^2}]_0^1\\ 4\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx - 2\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= (1)\sqrt{4-1^2} - (0)\sqrt{4-0^2}\\ 2\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= 4\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx - \sqrt{3}\\ \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx &= 2\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx - \frac{\sqrt{3}}{2}\end{aligned}$$

50. It is given that $f(x)=\begin{cases}2x & (\text{where } 0\le x\le 2)\\ x+2 & (\text{where } x>2)\end{cases}$. Evaluate $\int_0^5 f(x)\,dx$.

SOLUTION

$$\begin{aligned}\int_0^5 f(x)\,dx &= \int_0^2 f(x)\,dx+\int_2^5 f(x)\,dx\\ &= \int_0^2 2x\,dx+\int_2^5 (x+2)\,dx\\ &= [x^2]_0^2+[\frac{x^2}{2}+2x]_2^5\\ &= (2^2-0^2)+\{[\frac{5^2}{2}+2(5)]-[\frac{2^2}{2}+2(2)]\}\\ &= \underline{\underline{\frac{41}{2}}}\end{aligned}$$

51. It is given that $f(x)=\begin{cases}3x^2+2e^x & (\text{where } x\le 0)\\ 4x+2 & (\text{where } x>0)\end{cases}$. Evaluate $\int_{-2}^4 f(x)\,dx$.

SOLUTION

$$\begin{aligned}\int_{-2}^4 f(x)\,dx &= \int_{-2}^0 f(x)\,dx+\int_0^4 f(x)\,dx\\ &= \int_{-2}^0 (3x^2+2e^x)\,dx+\int_0^4 (4x+2)\,dx\\ &= [x^3+2e^x]_{-2}^0+[2x^2+2x]_0^4\\ &= \{(0^3+2e^0)-[(-2)^3+2e^{-2}]\}+\{[2(4)^2+2(4)]-[2(0)^2+2(0)]\}\\ &= (10-2e^{-2})+40\\ &= \underline{\underline{50-2e^{-2}}}\end{aligned}$$

52. Let $f(x)$ be a function defined on $x\ge 0$. It is given that $f'(x)=e^{4bx}+ae^{2bx}+3$, where a and b are constants, $f'(0)=8$ and $f'(1)=9.7$.

(a) Find the values of a and b.

(b) Using the values of a and b obtained in **(a)**, evaluate $\int_0^1 f'(x)\,dx$.

(c) If $f(0)=5$, find $f(1)$.

(Give your answers correct to 2 significant figures if necessary.)

SOLUTION

(a) $\because \quad f'(0)=8$

$\therefore \quad e^{4b(0)}+ae^{2b(0)}+3=8$

$$1+a(1)+3=8$$

$$a=\underline{\underline{4}}$$

$\because \quad f'(1)=9.7$

$\therefore \quad e^{4b(1)}+4e^{2b(1)}+3=9.7$

$$(e^{2b})^2+4(e^{2b})-6.7=0$$

$\therefore \quad e^{2b}=\dfrac{-4\pm\sqrt{4^2-4(1)(-6.7)}}{2(1)}$

$$2b=\ln\frac{-4+\sqrt{42.8}}{2} \quad \text{or} \quad \ln\frac{-4-\sqrt{42.8}}{2} \text{ (rejected)}$$

$$b=\frac{1}{2}\ln\frac{-4+\sqrt{42.8}}{2}$$

$$=\underline{\underline{0.12}} \text{ (corr. to 2 sig. fig.)}$$

(b) From **(a)**, $f'(x)=e^{4(0.12)x}+4e^{2(0.12)x}+3$

$$=e^{0.48x}+4e^{0.24x}+3$$

$$\int_0^1 f'(x)\,dx=\int_0^1 (e^{0.48x}+4e^{0.24x}+3)\,dx$$

$$=[\frac{(e^{0.48})^x}{\ln e^{0.48}}+\frac{4(e^{0.24})^x}{\ln e^{0.24}}+3x]_0^1$$

$$=[\frac{e^{0.48x}}{0.48}+\frac{4e^{0.24x}}{0.24}+3x]_0^1$$

$$=[\frac{e^{0.48(1)}}{0.48}+\frac{4e^{0.24(1)}}{0.24}+3(1)]-[\frac{e^{0.48(0)}}{0.48}+\frac{4e^{0.24(0)}}{0.24}+3(0)]$$

$$=\underline{\underline{8.8}} \text{ (corr. to 2 sig. fig.)}$$

(c) $f(1)-f(0)=\int_0^1 f'(x)dx$

$$f(1)=\int_0^1 f'(x)dx+f(0)$$

$$=8.80+5$$

$$=\underline{\underline{14}} \text{ (corr. to 2 sig. fig.)}$$

EXERCISE 8B

Level 1

53. Evaluate $\int_0^1 (2-3x)^4\,dx$.

SOLUTION

$$\begin{aligned}\int_0^1 (2-3x)^4\,dx &= -\frac{1}{3}\int_0^1 (2-3x)^4\cdot(-3)\,dx\\ &= -\frac{1}{3}\int_0^1 (2-3x)^4\,d(2-3x)\\ &= -\frac{1}{3}\left[\frac{(2-3x)^5}{5}\right]_0^1\\ &= -\frac{1}{3}\left\{\frac{[2-3(1)]^5}{5}-\frac{[2-3(0)]^5}{5}\right\}\\ &= \underline{\underline{\frac{11}{5}}}\end{aligned}$$

54. Evaluate $\int_1^2 \frac{dx}{(4x-1)^3}$.

SOLUTION

$$\begin{aligned}\int_1^2 \frac{dx}{(4x-1)^3} &= \frac{1}{4}\int_1^2 \frac{4dx}{(4x-1)^3}\\ &= \frac{1}{4}\int_1^2 (4x-1)^{-3}\,d(4x-1)\\ &= \frac{1}{4}\left[\frac{(4x-1)^{-2}}{-2}\right]_1^2\\ &= \frac{1}{4}\left\{\frac{[4(2)-1]^{-2}}{-2}-\frac{[4(1)-1]^{-2}}{-2}\right\}\\ &= \underline{\underline{\frac{5}{441}}}\end{aligned}$$

55. Evaluate $\int_3^{11} \sqrt{2x+3}\,dx$.

SOLUTION

$$\begin{aligned}\int_3^{11} \sqrt{2x+3}\,dx &= \frac{1}{2}\int_3^{11} \sqrt{2x+3}\cdot 2\,dx \\ &= \frac{1}{2}\int_3^{11} (2x+3)^{\frac{1}{2}}\,d(2x+3) \\ &= \frac{1}{2}[\frac{2(2x+3)^{\frac{3}{2}}}{3}]_3^{11} \\ &= \frac{1}{2}\{\frac{2[2(11)+3]^{\frac{3}{2}}}{3} - \frac{2[2(3)+3]^{\frac{3}{2}}}{3}\} \\ &= \underline{\underline{\frac{98}{3}}}\end{aligned}$$

56. Evaluate $\int_2^{14} \frac{dx}{2x-1}$.

SOLUTION

$$\begin{aligned}\int_2^{14} \frac{dx}{2x-1} &= \frac{1}{2}\int_2^{14} \frac{2dx}{2x-1} \\ &= \frac{1}{2}\int_2^{14} \frac{d(2x-1)}{2x-1} \\ &= \frac{1}{2}[\ln|2x-1|]_2^{14} \\ &= \frac{1}{2}\{\ln[2(14)-1] - \ln[2(2)-1]\} \\ &= \frac{1}{2}(\ln 27 - \ln 3) \\ &= \frac{1}{2}\ln 9 \\ &= \underline{\underline{\ln 3}}\end{aligned}$$

57. Evaluate $\int_0^2 e^{2x+1}\,dx$.

SOLUTION

$$\begin{aligned}\int_0^2 e^{2x+1}\,dx &= \frac{1}{2}\int_0^2 e^{2x+1}\cdot 2\,dx\\ &= \frac{1}{2}\int_0^2 e^{2x+1}\,d(2x+1)\\ &= \frac{1}{2}[e^{2x+1}]_0^2\\ &= \frac{1}{2}[e^{2(2)+1}-e^{2(0)+1}]\\ &= \underline{\underline{\frac{e^5-e}{2}}}\end{aligned}$$

58. Evaluate $\int_2^3 2x(x^2-4)^4\,dx$.

SOLUTION

$$\begin{aligned}\int_2^3 2x(x^2-4)^4\,dx &= \int_2^3 (x^2-4)^4\cdot 2x\,dx\\ &= \int_2^3 (x^2-4)^4\,d(x^2-4)\\ &= [\frac{(x^2-4)^5}{5}]_2^3\\ &= \frac{(3^2-4)^5}{5}-\frac{(2^2-4)^5}{5}\\ &= \underline{\underline{625}}\end{aligned}$$

59. Evaluate $\int_1^3 \frac{x\,dx}{3x^2+1}$.

SOLUTION

$$\int_1^3 \frac{x\,dx}{3x^2+1} = \frac{1}{6}\int_1^3 \frac{6x\,dx}{3x^2+1}$$

$$= \frac{1}{6}\int_1^3 \frac{d(3x^2+1)}{3x^2+1}$$

$$= \frac{1}{6}[\ln(3x^2+1)]_1^3$$

$$= \frac{1}{6}\{\ln[3(3)^2+1] - \ln[3(1)^2+1]\}$$

$$= \frac{1}{6}(\ln 28 - \ln 4)$$

$$= \underline{\underline{\frac{1}{6}\ln 7}}$$

60. Evaluate $\int_3^{10} \frac{2x+3}{\sqrt{x^2+3x-2}}\,dx$.

SOLUTION

$$\int_3^{10} \frac{2x+3}{\sqrt{x^2+3x-2}}\,dx = \int_3^{10} (x^2+3x-2)^{-\frac{1}{2}}\,d(x^2+3x-2)$$

$$= [2(x^2+3x-2)^{\frac{1}{2}}]_3^{10}$$

$$= 2[10^2+3(10)-2]^{\frac{1}{2}} - 2[3^2+3(3)-2]^{\frac{1}{2}}$$

$$= 2\sqrt{128} - 8$$

$$= \underline{\underline{16\sqrt{2} - 8}}$$

61. Evaluate $\int_4^8 \frac{x+3}{16-6x-x^2}\,dx$.

SOLUTION

$$\begin{aligned}\int_4^8 \frac{x+3}{16-6x-x^2}\,dx &= -\frac{1}{2}\int_4^8 \frac{-2x-6}{16-6x-x^2}\,dx \\ &= -\frac{1}{2}\int_4^8 \frac{1}{16-6x-x^2}\,d(16-6x-x^2) \\ &= -\frac{1}{2}[\ln\left|16-6x-x^2\right|]_4^8 \\ &= -\frac{1}{2}[\ln\left|16-6(8)-8^2\right|-\ln\left|16-6(4)-4^2\right|] \\ &= -\frac{1}{2}(\ln 96-\ln 24) \\ &= -\frac{1}{2}\ln 4 \\ &= \underline{\underline{-\ln 2}}\end{aligned}$$

62. Evaluate $\int_0^1 (x+1)e^{x^2+2x-3}dx$.

SOLUTION

$$\begin{aligned}\int_0^1 (x+1)e^{x^2+2x-3}dx &= \frac{1}{2}\int_0^1 e^{x^2+2x-3}\cdot(2x+2)\,dx \\ &= \frac{1}{2}\int_0^1 e^{x^2+2x-3}d(x^2+2x-3) \\ &= \frac{1}{2}[e^{x^2+2x-3}]_0^1 \\ &= \frac{1}{2}[e^{1^2+2(1)-3}-e^{0^2+2(0)-3}] \\ &= \frac{1}{2}(e^0-e^{-3}) \\ &= \underline{\underline{\frac{1}{2}(1-e^{-3})}}\end{aligned}$$

63. Evaluate $\int_0^2 \frac{x^2}{x^3+4}\,dx$.

SOLUTION

$$\begin{aligned}\int_0^2 \frac{x^2}{x^3+4}\,dx &= \frac{1}{3}\int_0^2 \frac{1}{x^3+4}\cdot 3x^2\,dx \\ &= \frac{1}{3}\int_0^2 \frac{1}{x^3+4}\,d(x^3+4) \\ &= \frac{1}{3}[\ln\left|x^3+4\right|]_0^2 \\ &= \frac{1}{3}[\ln(2^3+4)-\ln(0^3+4)] \\ &= \frac{1}{3}(\ln 12-\ln 4) \\ &= \underline{\underline{\frac{1}{3}\ln 3}}\end{aligned}$$

64. Evaluate $\int_4^9 \frac{(2\sqrt{x}-3)^4}{\sqrt{x}}\,dx$.

SOLUTION

$$\begin{aligned}\int_4^9 \frac{(2\sqrt{x}-3)^4}{\sqrt{x}}\,dx &= \int_4^9 (2\sqrt{x}-3)^4\cdot x^{-\frac{1}{2}}\,dx \\ &= \int_4^9 (2\sqrt{x}-3)^4\,d(2\sqrt{x}-3) \\ &= [\frac{1}{5}(2\sqrt{x}-3)^5]_4^9 \\ &= \frac{1}{5}(2\sqrt{9}-3)^5-\frac{1}{5}(2\sqrt{4}-3)^5 \\ &= \underline{\underline{\frac{242}{5}}}\end{aligned}$$

65. Evaluate $\int_1^9 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx$.

SOLUTION

$$\int_1^9 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx = \int_1^9 4e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx$$
$$= \int_1^9 4e^{\sqrt{x}} d(\sqrt{x})$$
$$= [4e^{\sqrt{x}}]_1^9$$
$$= 4e^{\sqrt{9}} - 4e^{\sqrt{1}}$$
$$= \underline{\underline{4e^3 - 4e}}$$

66. Evaluate $\int_0^{\ln 2} e^x (e^x - 1)^{\frac{3}{2}} dx$.

SOLUTION

$$\int_0^{\ln 2} e^x (e^x - 1)^{\frac{3}{2}} dx = \int_0^{\ln 2} (e^x - 1)^{\frac{3}{2}} \cdot e^x dx$$
$$= \int_0^{\ln 2} (e^x - 1)^{\frac{3}{2}} d(e^x - 1)$$
$$= [\frac{2(e^x - 1)^{\frac{5}{2}}}{5}]_0^{\ln 2}$$
$$= \frac{2(e^{\ln 2} - 1)^{\frac{5}{2}}}{5} - \frac{2(e^0 - 1)^{\frac{5}{2}}}{5}$$
$$= \underline{\underline{\frac{2}{5}}}$$

67. Evaluate $\int_1^{e^4} \frac{\sqrt{\ln x}}{x} dx$.

SOLUTION

$$\begin{aligned}\int_1^{e^4} \frac{\sqrt{\ln x}}{x} dx &= \int_1^{e^4} (\ln x)^{\frac{1}{2}} \cdot \frac{1}{x} dx \\ &= \int_1^{e^4} (\ln x)^{\frac{1}{2}} d(\ln x) \\ &= [\frac{2(\ln x)^{\frac{3}{2}}}{3}]_1^{e^4} \\ &= \frac{2(\ln e^4)^{\frac{3}{2}}}{3} - \frac{2(\ln 1)^{\frac{3}{2}}}{3} \\ &= \underline{\underline{\frac{16}{3}}}\end{aligned}$$

68. Evaluate $\int_1^{e} \frac{3+\ln x^2}{x} dx$.

SOLUTION

$$\begin{aligned}\int_1^{e} \frac{3+\ln x^2}{x} dx &= \int_1^{e} \frac{3+2\ln x}{x} dx \\ &= \int_1^{e} (\frac{3}{x} + \frac{2\ln x}{x}) dx \\ &= \int_1^{e} \frac{3dx}{x} + \int_1^{e} 2\ln x \cdot \frac{1}{x} dx \\ &= \int_1^{e} \frac{3dx}{x} + \int_1^{e} 2\ln x \, d(\ln x) \\ &= [3\ln|x|]_1^e + [(\ln x)^2]_1^e \\ &= 3(\ln e - \ln 1) + [(\ln e)^2 - (\ln 1)^2] \\ &= \underline{\underline{4}}\end{aligned}$$

69. Prove that $\int_5^{20} \frac{\ln\sqrt{2x}}{x}\,dx = 2(\ln 2)^2 + (\ln 2)(\ln 5)$.

PROOF

$$\begin{aligned}
\int_5^{20} \frac{\ln\sqrt{2x}}{x}\,dx &= \int_5^{20} \frac{\frac{1}{2}\ln 2x}{x}\,dx \\
&= \frac{1}{2}\int_5^{20} \ln 2x \cdot \frac{1}{x}\,dx \\
&= \frac{1}{2}\int_5^{20} (\ln 2x)\,d(\ln 2x) \\
&= \frac{1}{2}[\frac{1}{2}(\ln 2x)^2]_5^{20} \\
&= \frac{1}{4}\{[\ln 2(20)]^2 - [\ln 2(5)]^2\} \\
&= \frac{1}{4}[(\ln 40)^2 - (\ln 10)^2] \\
&= \frac{1}{4}(\ln 40 - \ln 10)(\ln 40 + \ln 10) \\
&= \frac{1}{4}\ln 4 \cdot \ln 400 \\
&= \frac{1}{4}(2\ln 2)(2\ln 20) \\
&= \ln 2 \cdot \ln 20 \\
&= (\ln 2)(2\ln 2 + \ln 5) \\
&= 2(\ln 2)^2 + (\ln 2)(\ln 5)
\end{aligned}$$

Level 2

70. Evaluate $\int_{10}^{100} \frac{\log x^2}{x}\,dx$.

SOLUTION

$$\begin{aligned}
\int_{10}^{100} \frac{\log x^2}{x}\,dx &= \ln 10 \int_{10}^{100} \frac{2\log x}{x\ln 10}\,dx \\
&= 2\ln 10 \int_{10}^{100} \log x\,d(\log x) \\
&= 2(\ln 10)[\frac{1}{2}(\log x)^2]_{10}^{100} \\
&= 2(\ln 10)[\frac{1}{2}(\log 100)^2 - \frac{1}{2}(\log 10)^2] \\
&= 2(\ln 10)(2 - \frac{1}{2}) \\
&= \underline{\underline{3\ln 10}}
\end{aligned}$$

71. Evaluate $\int_0^1 x^3 3^{x^4}\,dx$.

SOLUTION

$$\begin{aligned}\int_0^1 x^3 3^{x^4}\,dx &= \frac{1}{4}\int_0^1 3^{x^4}\cdot 4x^3dx\\ &= \frac{1}{4}\int_0^1 3^{x^4}\,d(x^4)\\ &= \frac{1}{4}\left[\frac{3^{x^4}}{\ln 3}\right]_0^1\\ &= \frac{1}{4}\left(\frac{3^{1^4}}{\ln 3}-\frac{3^{0^4}}{\ln 3}\right)\\ &= \underline{\underline{\frac{1}{2\ln 3}}}\end{aligned}$$

72. Evaluate $\int_1^2 \frac{10^x}{10^x+5}\,dx$.

SOLUTION

$$\begin{aligned}\int_1^2 \frac{10^x}{10^x+5}\,dx &= \frac{1}{\ln 10}\int_1^2 \frac{10^x\ln 10}{10^x+5}\,dx\\ &= \frac{1}{\ln 10}\int_1^2 \frac{d(10^x+5)}{10^x+5}\\ &= \frac{1}{\ln 10}[\ln(10^x+5)]_1^2\\ &= \frac{1}{\ln 10}[\ln(10^2+5)-\ln(10^1+5)]\\ &= \frac{1}{\ln 10}(\ln 105-\ln 15)\\ &= \underline{\underline{\frac{\ln 7}{\ln 10}}}\end{aligned}$$

73. Evaluate $\int_{\frac{1}{2}}^1 x(2x-1)^{99}\,dx$.

SOLUTION

Let $u=2x-1$. Then $du=2dx$.

When $x=\frac{1}{2}$, $u=0$;

when $x=1$, $u=1$.

$$\therefore \quad \int_{\frac{1}{2}}^{1} x(2x-1)^{99}\,dx = \int_{\frac{1}{2}}^{1} \frac{1}{2}x(2x-1)^{99}\cdot 2\,dx$$
$$= \int_{0}^{1} \frac{1}{2}\left(\frac{u+1}{2}\right)\cdot u^{99}\,du$$
$$= \int_{0}^{1} \left(\frac{u^{100}}{4}+\frac{u^{99}}{4}\right)du$$
$$= \left[\frac{u^{101}}{404}+\frac{u^{100}}{400}\right]_0^1$$
$$= \left(\frac{1^{101}}{404}+\frac{1^{100}}{400}\right)-\left(\frac{0^{101}}{404}+\frac{0^{100}}{400}\right)$$
$$= \underline{\underline{\frac{201}{40\,400}}}$$

74. Evaluate $\int_0^3 (x+2)\sqrt{4-x}\,dx$.

SOLUTION

Let $u = 4-x$. Then $du = -dx$.
When $x=0$, $u=4$;
when $x=3$, $u=1$.

$$\therefore \quad \int_0^3 (x+2)\sqrt{4-x}\,dx = -\int_0^3 (x+2)\sqrt{4-x}\cdot(-1)\,dx$$
$$= -\int_4^1 (6-u)\sqrt{u}\,du$$
$$= \int_1^4 \left(6u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$
$$= \left[4u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^4$$
$$= \left[4(4)^{\frac{3}{2}} - \frac{2}{5}(4)^{\frac{5}{2}}\right] - \left[4(1)^{\frac{3}{2}} - \frac{2}{5}(1)^{\frac{5}{2}}\right]$$
$$= \underline{\underline{\frac{78}{5}}}$$

75. Evaluate $\int_2^7 \frac{x+3}{\sqrt{x+2}}\,dx$.

SOLUTION

Let $u = x+2$. Then $du = dx$.
When $x=2$, $u=4$;
when $x=7$, $u=9$.

$$\therefore \quad \int_2^7 \frac{x+3}{\sqrt{x+2}}\,dx = \int_4^9 \frac{u+1}{\sqrt{u}}\,du$$

$$= \int_4^9 (\sqrt{u} + \frac{1}{\sqrt{u}})\,du$$

$$= [\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}]_4^9$$

$$= [\frac{2}{3}(9)^{\frac{3}{2}} + 2(9)^{\frac{1}{2}}] - [\frac{2}{3}(4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}}]$$

$$= \underline{\underline{\frac{44}{3}}}$$

76. Evaluate $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

SOLUTION

Let $u = 1 - x^2$. Then $du = -2xdx$.

When $u = 1 - x^2$, $x^2 = 1 - u$.

When $x = 0$, $u = 1$;

when $x = 1$, $u = 0$.

$$\therefore \quad \int_0^1 x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_0^1 x^2\sqrt{1-x^2}\cdot(-2x)\,dx$$

$$= -\frac{1}{2}\int_1^0 (1-u)\sqrt{u}\,du$$

$$= \frac{1}{2}\int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}})\,du$$

$$= \frac{1}{2}[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}]_0^1$$

$$= \frac{1}{2}\{[\frac{2}{3}(1)^{\frac{3}{2}} - \frac{2}{5}(1)^{\frac{5}{2}}] - [\frac{2}{3}(0)^{\frac{3}{2}} - \frac{2}{5}(0)^{\frac{5}{2}}]\}$$

$$= \underline{\underline{\frac{2}{15}}}$$

77. Evaluate $\int_0^4 \frac{x^3}{\sqrt{5x^2+1}}\,dx$.

SOLUTION

Let $u=5x^2+1$. Then $du=10x\,dx$.

When $u=5x^2+1$, $x^2=\frac{u-1}{5}$.

When $x=0$, $u=1$;

when $x=4$, $u=81$.

$$\begin{aligned}
\therefore\quad \int_0^4 \frac{x^3}{\sqrt{5x^2+1}}\,dx &= \int_0^4 \frac{x^2}{10\sqrt{5x^2+1}}\cdot 10x\,dx \\
&= \int_1^{81} \frac{u-1}{5}\cdot\frac{1}{10\sqrt{u}}\,du \\
&= \frac{1}{50}\int_1^{81}\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du \\
&= \frac{1}{50}\left[\frac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}\right]_1^{81} \\
&= \frac{1}{50}\left\{\left[\frac{2}{3}(81)^{\frac{3}{2}}-2(81)^{\frac{1}{2}}\right]-\left[\frac{2}{3}(1)^{\frac{3}{2}}-2(1)^{\frac{1}{2}}\right]\right\} \\
&= \underline{\underline{\frac{704}{75}}}
\end{aligned}$$

78. Evaluate $\int_9^{16} \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$.

SOLUTION

Let $u=1+\sqrt{x}$. Then $du=\frac{1}{2\sqrt{x}}\,dx$.

When $u=1+\sqrt{x}$, $x=(u-1)^2$.

When $x=9$, $u=4$;

when $x=16$, $u=5$.

$$\therefore \quad \int_9^{16} \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \int_9^{16} \frac{2x}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$
$$= \int_4^5 \frac{2(u-1)^2}{u}\,du$$
$$= \int_4^5 \frac{2u^2-4u+2}{u}\,du$$
$$= \int_4^5 (2u-4+\frac{2}{u})\,du$$
$$= [u^2-4u+2\ln|u|]_4^5$$
$$= [5^2-4(5)+2\ln 5]-[4^2-4(4)+2\ln 4]$$
$$= (5+2\ln 5)-2\ln 4$$
$$= \underline{\underline{5+2\ln 5-4\ln 2}}$$

79. Evaluate $\int_0^4 \frac{\sqrt{x}-4}{\sqrt{x}+4}\,dx$.

SOLUTION

Let $u=\sqrt{x}+4$. Then $du=\frac{1}{2\sqrt{x}}\,dx$.

When $u=\sqrt{x}+4$, $\sqrt{x}=u-4$.

When $x=0$, $u=4$;
when $x=4$, $u=6$.

$$\therefore \quad \int_0^4 \frac{\sqrt{x}-4}{\sqrt{x}+4}\,dx = \int_0^4 \frac{2\sqrt{x}(\sqrt{x}-4)}{\sqrt{x}+4} \cdot \frac{1}{2\sqrt{x}}\,dx$$
$$= \int_4^6 \frac{2(u-4)(u-4-4)}{u}\,du$$
$$= \int_4^6 \frac{2u^2-24u+64}{u}\,du$$
$$= \int_4^6 (2u-24+\frac{64}{u})\,du$$
$$= [u^2-24u+64\ln|u|]_4^6$$
$$= [6^2-24(6)+64\ln 6]-[4^2-24(4)+64\ln 4]$$
$$= (-108+64\ln 6)-(-80+64\ln 4)$$
$$= \underline{\underline{-28+64\ln\frac{3}{2}}}$$

80. Evaluate $\int_0^1 \frac{x[\ln(1+5x^2)]^2}{1+5x^2}\,dx$.

[Hint: Let $u = \ln(1+5x^2)$.]

SOLUTION

Let $u = \ln(1+5x^2)$. Then $du = \frac{10x}{1+5x^2}\,dx$.

When $x = 0$, $u = 0$;

when $x = 1$, $u = \ln 6$.

$$\begin{aligned}\therefore \quad \int_0^1 \frac{x[\ln(1+5x^2)]^2}{1+5x^2}\,dx &= \int_0^1 \frac{[\ln(1+5x^2)]^2}{10}\cdot\frac{10x}{1+5x^2}\,dx \\ &= \int_0^{\ln 6} \frac{u^2}{10}\,du \\ &= [\frac{u^3}{30}]_0^{\ln 6} \\ &= \frac{(\ln 6)^3}{30} - \frac{0^3}{30} \\ &= \underline{\underline{\frac{(\ln 6)^3}{30}}}\end{aligned}$$

81. Evaluate $\int_0^1 e^x e^{e^x}\,dx$.

SOLUTION

Let $u = e^{e^x}$. Then $du = e^{e^x} e^x\,dx$.

When $x = 0$, $u = e$;

when $x = 1$, $u = e^e$.

$$\begin{aligned}\therefore \quad \int_0^1 e^x e^{e^x}\,dx &= \int_e^{e^e} du \\ &= [u]_e^{e^e} \\ &= \underline{\underline{e^e - e}}\end{aligned}$$

82. Evaluate $\int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,dx$.

SOLUTION

Let $u = e^x - e^{-x}$. Then $du = (e^x + e^{-x})\,dx$.

When $x = \ln 2$, $u = \frac{3}{2}$;

when $x = \ln 4$, $u = \frac{15}{4}$.

$$\begin{aligned}\therefore \quad \int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,dx &= \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{du}{\sqrt{u}} \\ &= [2\sqrt{u}]_{\frac{3}{2}}^{\frac{15}{4}} \\ &= 2\sqrt{\frac{15}{4}} - 2\sqrt{\frac{3}{2}} \\ &= \underline{\underline{\sqrt{15} - \sqrt{6}}}\end{aligned}$$

83. Evaluate $\int_0^{\ln 2} \frac{e^x dx}{(e^x + 1)\ln(e^x + 1)}$.

[Hint: Let $u = \ln(e^x + 1)$.]

SOLUTION

Let $u = \ln(e^x + 1)$. Then $du = \frac{e^x dx}{e^x + 1}$.

When $x = 0$, $u = \ln 2$;

when $x = \ln 2$, $u = \ln 3$.

$$\begin{aligned}\therefore \quad \int_0^{\ln 2} \frac{e^x dx}{(e^x + 1)\ln(e^x + 1)} &= \int_{\ln 2}^{\ln 3} \frac{du}{u} \\ &= [\ln|u|]_{\ln 2}^{\ln 3} \\ &= \ln(\ln 3) - \ln(\ln 2) \\ &= \underline{\underline{\ln\left(\frac{\ln 3}{\ln 2}\right)}}\end{aligned}$$

84. Evaluate $\int_{\ln 2}^{\ln 5} \frac{(e^x+5)e^x}{e^x+1}\,dx$.

SOLUTION

Let $u = e^x + 1$. Then $du = e^x dx$.

When $x = \ln 2$, $u = 3$;

when $x = \ln 5$, $u = 6$.

$$\begin{aligned}
\therefore \quad \int_{\ln 2}^{\ln 5} \frac{(e^x+5)e^x}{e^x+1}\,dx &= \int_3^6 \frac{u+4}{u}\,du \\
&= \int_3^6 (1+\frac{4}{u})\,du \\
&= [u + 4\ln|u|]_3^6 \\
&= (6+4\ln 6) - (3+4\ln 3) \\
&= \underline{\underline{3+4\ln 2}}
\end{aligned}$$

85. **(a)** Evaluate $\int_0^3 \frac{e^x}{1+3e^x}\,dx$.

(b) Hence, or otherwise, evaluate $\int_0^3 \frac{1}{1+3e^x}\,dx$.

SOLUTION

(a)
$$\begin{aligned}
\int_0^3 \frac{e^x}{1+3e^x}\,dx &= \frac{1}{3}\int_0^3 \frac{3e^x}{1+3e^x}\,dx \\
&= \frac{1}{3}\int_0^3 \frac{1}{1+3e^x}\,d(1+3e^x) \\
&= \frac{1}{3}[\ln|1+3e^x|]_0^3 \\
&= \frac{1}{3}[\ln(1+3e^3) - \ln(1+3e^0)] \\
&= \underline{\underline{\frac{1}{3}\ln\frac{1+3e^3}{4}}}
\end{aligned}$$

(b) $$\int_0^3 \frac{1}{1+3e^x}dx = \int_0^3 \frac{(1+3e^x)-3e^x}{1+3e^x}dx$$
$$= \int_0^3 (1-\frac{3e^x}{1+3e^x})dx$$
$$= \int_0^3 dx - 3\int_0^3 \frac{e^x}{1+3e^x}dx$$
$$= [x]_0^3 - 3(\frac{1}{3}\ln\frac{1+3e^3}{4}) \quad \text{[From the result of (a)]}$$
$$= \underline{\underline{3-\ln\frac{1+3e^3}{4}}}$$

86. **(a)** Evaluate $\int_4^8 \frac{x-4}{x^2-8x+32}dx$.

(b) Hence, or otherwise, evaluate $\int_4^8 \frac{x^2dx}{x^2-8x+32}$.

SOLUTION

(a) $$\int_4^8 \frac{x-4}{x^2-8x+32}dx = \frac{1}{2}\int_4^8 \frac{2x-8}{x^2-8x+32}dx$$
$$= \frac{1}{2}\int_4^8 \frac{1}{x^2-8x+32}d(x^2-8x+32)$$
$$= \frac{1}{2}[\ln(x^2-8x+32)]_4^8$$
$$= \frac{1}{2}\{\ln[8^2-8(8)+32]-\ln[4^2-8(4)+32]\}$$
$$= \frac{1}{2}(\ln 32 - \ln 16)$$
$$= \underline{\underline{\frac{1}{2}\ln 2}}$$

(b) $$\int_4^8 \frac{x^2\,dx}{x^2-8x+32} = \int_4^8 \frac{(x^2-8x+32)+(8x-32)}{x^2-8x+32}dx$$
$$= \int_4^8 (1+\frac{8x-32}{x^2-8x+32})dx$$
$$= \int_4^8 dx + 8\int_4^8 \frac{x-4}{x^2-8x+32}dx$$
$$= [x]_4^8 + 8(\frac{1}{2}\ln 2) \quad \text{[From the result of (a)]}$$
$$= (8-4)+4\ln 2$$
$$= \underline{\underline{4+4\ln 2}}$$

87. **(a)** Evaluate $\int_1^4 \frac{e^{3x}+1}{e^{3x}+3x+1}dx$.

(b) Hence, or otherwise, evaluate $\int_1^4 \frac{x\,dx}{e^{3x}+3x+1}$.

SOLUTION

(a)
$$\begin{aligned}\int_1^4 \frac{e^{3x}+1}{e^{3x}+3x+1}dx &= \int_1^4 \frac{3e^{3x}+3}{3(e^{3x}+3x+1)}dx \\ &= \frac{1}{3}\int_1^4 \frac{1}{e^{3x}+3x+1}d(e^{3x}+3x+1) \\ &= \frac{1}{3}[\ln\left|e^{3x}+3x+1\right|]_1^4 \\ &= \frac{1}{3}\{\ln[e^{3(4)}+3(4)+1]-\ln[e^{3(1)}+3(1)+1]\} \\ &= \underline{\underline{\frac{1}{3}\ln\frac{e^{12}+13}{e^3+4}}}\end{aligned}$$

(b)
$$\begin{aligned}\int_1^4 \frac{x\,dx}{e^{3x}+3x+1} &= \int_1^4 \frac{3x\,dx}{3(e^{3x}+3x+1)} \\ &= \frac{1}{3}\int_1^4 \frac{(e^{3x}+3x+1)-(e^{3x}+1)}{e^{3x}+3x+1}dx \\ &= \frac{1}{3}\int_1^4 (1-\frac{e^{3x}+1}{e^{3x}+3x+1})dx \\ &= \frac{1}{3}\int_1^4 dx-\frac{1}{3}\int_1^4 \frac{e^{3x}+1}{e^{3x}+3x+1}dx \\ &= \frac{1}{3}[x]_1^4-\frac{1}{3}(\frac{1}{3}\ln\frac{e^{12}+13}{e^3+4}) \quad \text{[From the result of (a)]} \\ &= \frac{1}{3}(4-1)-\frac{1}{9}\ln\frac{e^{12}+13}{e^3+4} \\ &= \underline{\underline{1-\frac{1}{9}\ln\frac{e^{12}+13}{e^3+4}}}\end{aligned}$$

88. **(a)** If $\dfrac{11}{(3x-4)(2x+1)} \equiv \dfrac{A}{3x-4} + \dfrac{B}{2x+1}$, find the values of constants A and B.

(b) Hence evaluate $\displaystyle\int_2^5 \frac{11}{(3x-4)(2x+1)}\,dx$.

SOLUTION

(a)
$$\frac{11}{(3x-4)(2x+1)} \equiv \frac{A}{3x-4} + \frac{B}{2x+1}$$
$$\equiv \frac{A(2x+1)+B(3x-4)}{(3x-4)(2x+1)}$$

i.e. $A(2x+1)+B(3x-4) \equiv 11$

$(2A+3B)x+(A-4B) \equiv 11$

$$\therefore \begin{cases} 2A+3B=0 \text{ ....(1)} \\ A-4B=11 \text{ .... (2)} \end{cases}$$

(1) − 2 × (2):

$(2A+3B)-2(A-4B)=0-2(11)$

$11B=-22$

$B=-2$

Substitute $B=-2$ into (1),

$2A+3(-2)=0$

$A=3$

$\therefore$ $\underline{\underline{A=3,\ B=-2}}$

(b)
$$\int_2^5 \frac{11}{(3x-4)(2x+1)}\,dx = \int_2^5 \left(\frac{3}{3x-4}+\frac{-2}{2x+1}\right)dx \quad \text{[From the result of (a)]}$$
$$= \int_2^5 \frac{1}{3x-4}\cdot 3\,dx - \int_2^5 \frac{1}{2x+1}\cdot 2\,dx$$
$$= \int_2^5 \frac{1}{3x-4}\,d(3x-4) - \int_2^5 \frac{1}{2x+1}\,d(2x+1)$$
$$= [\ln|3x-4|]_2^5 - [\ln|2x+1|]_2^5$$
$$= (\ln 11 - \ln 2) - (\ln 11 - \ln 5)$$
$$= \ln 5 - \ln 2$$
$$= \underline{\underline{\ln\frac{5}{2}}}$$

89. (a) If $\dfrac{17-2x}{(2x-3)(5-x)} \equiv \dfrac{A}{2x-3} + \dfrac{B}{5-x}$, find the values of constants A and B.

(b) Hence evaluate $\displaystyle\int_2^4 \frac{17-2x}{(2x-3)(5-x)}\,dx$.

SOLUTION

(a)
$$\frac{17-2x}{(2x-3)(5-x)} \equiv \frac{A}{2x-3} + \frac{B}{5-x}$$
$$\equiv \frac{A(5-x)+B(2x-3)}{(2x-3)(5-x)}$$

i.e. $A(5-x)+B(2x-3) \equiv 17-2x$

$(2B-A)x+(5A-3B) \equiv 17-2x$

$\therefore \begin{cases} 2B-A=-2 \text{ .... (1)} \\ 5A-3B=17 \text{ .... (2)} \end{cases}$

From (1), $A = 2B+2$ ....(3)

Substitute (3) into (2),

$5(2B+2)-3B = 17$

$7B = 7$

$B = 1$

Substitute $B=1$ into (3),

$A = 2(1)+2$

$= 4$

$\therefore$ $\underline{\underline{A=4,\ B=1}}$

(b)
$$\int_2^4 \frac{17-2x}{(2x-3)(5-x)}\,dx = \int_2^4 \left(\frac{4}{2x-3}+\frac{1}{5-x}\right)dx$$
$$= \int_2^4 \frac{2}{2x-3}\cdot 2\,dx - \int_2^4 \frac{1}{5-x}\cdot(-1)\,dx$$
$$= 2\int_2^4 \frac{1}{2x-3}\,d(2x-3) - \int_2^4 \frac{1}{5-x}\,d(5-x)$$
$$= 2[\ln|2x-3|]_2^4 - [\ln|5-x|]_2^4$$
$$= 2(\ln 5 - \ln 1) - (\ln 1 - \ln 3)$$
$$= \underline{\underline{2\ln 5 + \ln 3}}$$

90. It is given that $k > 0$.

(a) Express the value of $\int_0^k \frac{dx}{\sqrt{x+1}}$ in terms of k.

(b) If $\int_0^k \frac{dx}{\sqrt{x+1}} = 4$, find the value of k.

SOLUTION

(a)
$$\begin{aligned}\int_0^k \frac{dx}{\sqrt{x+1}} &= \int_0^k (x+1)^{-\frac{1}{2}} d(x+1) \\ &= [2(x+1)^{\frac{1}{2}}]_0^k \\ &= 2(k+1)^{\frac{1}{2}} - 2(0+1)^{\frac{1}{2}} \\ &= \underline{\underline{2\sqrt{k+1}-2}}\end{aligned}$$

(b)
$$\begin{aligned}\int_0^k \frac{dx}{\sqrt{x+1}} &= 4 \\ 2\sqrt{k+1}-2 &= 4 \\ 2\sqrt{k+1} &= 6 \\ \sqrt{k+1} &= 3 \\ k+1 &= 9 \\ k &= \underline{\underline{8}}\end{aligned}$$

91. It is given that $k > -1$.

(a) Express the value of $\int_{-1}^k \frac{dx}{4x+5}$ in terms of k.

(b) If $\int_{-1}^k \frac{dx}{4x+5} = \frac{1}{2}$, find the value of k.

SOLUTION

(a)
$$\begin{aligned}\int_{-1}^k \frac{dx}{4x+5} &= \frac{1}{4}\int_{-1}^k \frac{4dx}{4x+5} \\ &= \frac{1}{4}\int_{-1}^k \frac{d(4x+5)}{4x+5} \\ &= \frac{1}{4}[\ln|4x+5|]_{-1}^k \\ &= \frac{1}{4}\{\ln(4k+5) - \ln[4(-1)+5]\} \\ &= \underline{\underline{\frac{1}{4}\ln(4k+5)}}\end{aligned}$$

(b) $\int_{-1}^{k} \frac{dx}{4x+5} = \frac{1}{2}$

$$\frac{1}{4}\ln(4k+5) = \frac{1}{2}$$

$$\ln(4k+5) = 2$$

$$4k+5 = e^2$$

$$k = \underline{\underline{\frac{e^2-5}{4}}}$$

92. It is given that $k < -\frac{1}{2}$.

(a) Express the value of $\int_{k}^{-\frac{1}{2}} \frac{dx}{e^{2x+1}}$ in terms of k.

(b) If $\int_{k}^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} = 1$, find the value of k.

SOLUTION

(a) $\int_{k}^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} = \int_{k}^{-\frac{1}{2}} e^{-2x-1}\,dx$

$$= -\frac{1}{2}\int_{k}^{-\frac{1}{2}} e^{-2x-1}\cdot(-2)\,dx$$

$$= -\frac{1}{2}\int_{k}^{-\frac{1}{2}} e^{-2x-1}\,d(-2x-1)$$

$$= -\frac{1}{2}[e^{-2x-1}]_{k}^{-\frac{1}{2}}$$

$$= -\frac{1}{2}[e^{-2(-\frac{1}{2})-1} - e^{-2k-1}]$$

$$= \underline{\underline{\frac{1}{2}(e^{-2k-1}-1)}}$$

(b) $\int_{k}^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} = 1$

$$\frac{1}{2}(e^{-2k-1} - 1) = 1$$

$$e^{-2k-1} - 1 = 2$$

$$e^{-2k-1} = 3$$

$$-2k - 1 = \ln 3$$

$$k = \underline{\underline{\frac{-1-\ln 3}{2}}}$$

93. Suppose $f(x)$ is a continuous function and that $\int_0^4 f(x)\,dx = 10$. Evaluate $\int_0^2 f(2u)\,du$.

[Hint: Let $u = \frac{x}{2}$.]

SOLUTION

Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}dx$.

When $u = 0$, $x = 0$;

when $u = 2$, $x = 4$.

$$\therefore \quad \int_0^2 f(2u)\,du = \int_0^4 f(x) \cdot \frac{1}{2}dx$$

$$= \frac{1}{2}\int_0^4 f(x)\,dx$$

$$= \frac{1}{2}(10)$$

$$= \underline{\underline{5}}$$

94. Suppose $g(x)$ is a continuous function and that $\int_4^7 g(x)\,dx = 12$. Evaluate $\int_2^3 g(3u-2)\,du$.

[Hint: Let $u = \frac{x+2}{3}$.]

SOLUTION

Let $u = \frac{x+2}{3}$. Then $du = \frac{1}{3}dx$.

When $u = 2$, $x = 4$;

when $u = 3$, $x = 7$.

$$\therefore \quad \int_2^3 g(3u-2)\,du = \int_4^7 g(x)\cdot\frac{1}{3}\,dx$$

$$= \frac{1}{3}\int_4^7 g(x)\,dx$$

$$= \frac{1}{3}(12)$$

$$= \underline{\underline{4}}$$

95. It is given that $f(-x) = -f(x)$ and $\int_3^5 f(x)dx = 2$.

(a) Prove that $\int_0^a f(x)\,dx = -\int_{-a}^0 f(x)\,dx$.

(b) Hence evaluate $\int_{-3}^5 f(x)\,dx$.

SOLUTION

(a) Let $u = -x$. Then $du = -dx$.
When $x = 0$, $u = 0$;
when $x = a$, $u = -a$.

$$\therefore \quad \int_0^a f(x)\,dx = \int_0^{-a} f(-u)\cdot(-1)\,du$$

$$= \int_{-a}^0 f(-u)\,du$$

$$= \int_{-a}^0 [-f(u)]du$$

$$= -\int_{-a}^0 f(x)\,dx$$

(b) $$\int_{-3}^5 f(x)\,dx = \int_{-3}^0 f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$

$$= \int_{-3}^0 f(x)\,dx + \left[-\int_{-3}^0 f(x)\,dx\right] + 2$$ [From the result of **(a)**]

$$= \underline{\underline{2}}$$

96. It is given that $f(x)=f(-x)$ and $\int_0^6 f(x)\,dx=a$.

(a) Express the value of $\int_{-6}^{0} f(x)\,dx$ in terms of a.

(b) Hence express the value of $\int_{-6}^{6} [f(x)+3x^2]\,dx$ in terms of a.

SOLUTION

(a) Let $u=-x$. Then $du=-dx$.
When $x=-6$, $u=6$;
when $x=0$, $u=0$.

$$\begin{aligned}\therefore\quad \int_{-6}^{0} f(x)\,dx &= \int_{6}^{0} f(-u)\cdot(-1)\,du\\ &= \int_0^6 f(-u)\,du\\ &= \int_0^6 f(u)\,du\\ &= \int_0^6 f(x)\,dx\\ &= \underline{\underline{a}}\end{aligned}$$

(b) $$\begin{aligned}\int_{-6}^{6}[f(x)+3x^2]\,dx &= \int_{-6}^{6} f(x)\,dx+\int_{-6}^{6} 3x^2\,dx\\ &= \int_{-6}^{0} f(x)\,dx+\int_{0}^{6} f(x)\,dx+[x^3]_{-6}^{6}\\ &= a+a+[6^3-(-6)^3] \quad [\text{From the result of } \textbf{(a)}]\\ &= \underline{\underline{2a+432}}\end{aligned}$$

97. **(a)** Prove that $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

(b) Hence, or otherwise, evaluate $\int_0^5 \sqrt{5-x}\,dx$.

SOLUTION

(a) Let $u=a-x$. Then $du=-dx$.
When $x=0$, $u=a$;
when $x=a$, $u=0$.

$$\therefore \quad \int_0^a f(x)\,dx = \int_a^0 f(a-u)\cdot(-1)\,du$$
$$= \int_0^a f(a-u)\,du$$
$$= \int_0^a f(a-x)\,dx$$

(b) $\int_0^5 \sqrt{5-x}\,dx = \int_0^5 \sqrt{5-(5-x)}\,dx$ [From the result of **(a)**]

$$= \int_0^5 x^{\frac{1}{2}}\,dx$$
$$= [\frac{2}{3}x^{\frac{3}{2}}]_0^5$$
$$= \frac{2}{3}(5)^{\frac{3}{2}} - \frac{2}{3}(0)^{\frac{3}{2}}$$
$$= \underline{\underline{\frac{10\sqrt{5}}{3}}}$$

98. (a) Prove that $\int_a^b f(x)\,dx = \int_{a-b}^0 f(a-x)\,dx$.

(b) Hence, or otherwise, evaluate $\int_6^7 (x-3)(6-x)^{99}\,dx$.

SOLUTION

(a) Let $u = a - x$. Then $du = -dx$.
When $x = a$, $u = 0$;
when $x = b$, $u = a - b$.

$$\therefore \quad \int_a^b f(x)\,dx = \int_0^{a-b} f(a-u)\cdot(-1)\,du$$
$$= \int_{a-b}^0 f(a-u)\,du$$
$$= \int_{a-b}^0 f(a-x)\,dx$$

(b) $\int_6^7 (x-3)(6-x)^{99}\,dx = \int_{6-7}^{0} [(6-x)-3][6-(6-x)]^{99}\,dx$ [From the result of **(a)**]

$$= \int_{-1}^{0} (3-x)x^{99}\,dx$$

$$= \int_{-1}^{0} (3x^{99}-x^{100})\,dx$$

$$= [\frac{3}{100}x^{100} - \frac{1}{101}x^{101}]_{-1}^{0}$$

$$= [\frac{3}{100}(0)^{100} - \frac{1}{101}(0)^{101}] - [\frac{3}{100}(-1)^{100} - \frac{1}{101}(-1)^{101}]$$

$$= \underline{\underline{-\frac{403}{10100}}}$$

99. **(a)** Prove that $\int_a^b f(x)\,dx = \int_0^{b-a} f(x+a)\,dx$.

(b) Hence, or otherwise, evaluate $\int_4^8 (8-2x)^{\frac{1}{3}}\,dx$.

SOLUTION

(a) Let $u = x - a$. Then $du = dx$.
When $x = a$, $u = 0$;
when $x = b$, $u = b - a$.

$$\therefore \quad \int_a^b f(x)\,dx = \int_0^{b-a} f(u+a)\,du$$

$$= \int_0^{b-a} f(x+a)\,dx$$

(b) $\int_4^8 (8-2x)^{\frac{1}{3}}\,dx = \int_0^{8-4} [8-2(x+4)]^{\frac{1}{3}}\,dx$ [From the result of **(a)**]

$$= \int_0^4 (-2x)^{\frac{1}{3}}\,dx$$

$$= -\frac{1}{2}\int_0^4 (-2x)^{\frac{1}{3}} \cdot (-2)\,dx$$

$$= -\frac{1}{2}\int_0^4 (-2x)^{\frac{1}{3}}\,d(-2x)$$

$$= -\frac{1}{2}[\frac{3}{4}(-2x)^{\frac{4}{3}}]_0^4$$

$$= -\frac{1}{2}\{\frac{3}{4}[-2(4)]^{\frac{4}{3}} - \frac{3}{4}[-2(0)]^{\frac{4}{3}}\}$$

$$= \underline{\underline{-6}}$$

Level 3

100. (a) Evaluate $\int_{-2}^{2} \frac{2x+2}{x^2+2x+4}\,dx$.

(b) Hence, or otherwise, evaluate $\int_{-2}^{2} \frac{3x^2+2x+8}{x^2+2x+4}\,dx$.

SOLUTION

(a)
$$\begin{aligned}\int_{-2}^{2} \frac{2x+2}{x^2+2x+4}\,dx &= \int_{-2}^{2} \frac{d(x^2+2x+4)}{x^2+2x+4}\\ &= [\ln(x^2+2x+4)]_{-2}^{2}\\ &= \ln[2^2+2(2)+4]-\ln[(-2)^2+2(-2)+4]\\ &= \ln 12 - \ln 4\\ &= \underline{\underline{\ln 3}}\end{aligned}$$

(b)
$$\begin{aligned}\int_{-2}^{2} \frac{3x^2+2x+8}{x^2+2x+4}\,dx &= \int_{-2}^{2} \frac{(3x^2+6x+12)-(4x+4)}{x^2+2x+4}\,dx\\ &= \int_{-2}^{2} \frac{3(x^2+2x+4)}{x^2+2x+4}\,dx - \int_{-2}^{2} \frac{2(2x+2)}{x^2+2x+4}\,dx\\ &= 3\int_{-2}^{2} dx - 2\int_{-2}^{2} \frac{2x+2}{x^2+2x+4}\,dx\\ &= 3[x]_{-2}^{2} - 2\ln 3\\ &= 3[2-(-2)] - 2\ln 3\\ &= \underline{\underline{12 - 2\ln 3}}\end{aligned}$$

101. (a) Evaluate $\int_{0}^{6} \frac{x-3}{x^2-6x+18}\,dx$.

(b) Hence evaluate $\int_{0}^{\sqrt{6}} \frac{x^5\,dx}{x^4-6x^2+18}$.

SOLUTION

(a) $\displaystyle\int_0^6 \frac{x-3}{x^2-6x+18}dx = \frac{1}{2}\int_0^6 \frac{2x-6}{x^2-6x+18}dx$

$$= \frac{1}{2}\int_0^6 \frac{1}{x^2-6x+18}d(x^2-6x+18)$$

$$= \frac{1}{2}[\ln\left|x^2-6x+18\right|]_0^6$$

$$= \frac{1}{2}\{\ln[6^2-6(6)+18]-\ln[0^2-6(0)+18]\}$$

$$= \frac{1}{2}(\ln 18-\ln 18)$$

$$= \underline{\underline{0}}$$

(b) Let $u = x^2$. Then $du = 2xdx$.

When $x = 0$, $u = 0$;

when $x = \sqrt{6}$, $u = 6$.

$\displaystyle\int_0^{\sqrt{6}} \frac{x^5\,dx}{x^4-6x^2+18} = \frac{1}{2}\int_0^{\sqrt{6}} \frac{x^4}{x^4-6x^2+18}\cdot 2x\,dx$

$$= \frac{1}{2}\int_0^6 \frac{u^2}{u^2-6u+18}du$$

$$= \frac{1}{2}\int_0^6 \frac{(u^2-6u+18)+(6u-18)}{u^2-6u+18}du$$

$$= \frac{1}{2}\int_0^6 (1+\frac{6u-18}{u^2-6u+18})du$$

$$= \frac{1}{2}\int_0^6 du + 3\int_0^6 \frac{u-3}{u^2-6u+18}du$$

$$= \frac{1}{2}[u]_0^6 + 3(0) \quad \text{[From the result of } \textbf{(a)}\text{]}$$

$$= \frac{1}{2}(6-0)$$

$$= \underline{\underline{3}}$$

102. (a) If $\dfrac{x+2}{(x^2-2)(x+1)} \equiv \dfrac{Ax+B}{x^2-2} + \dfrac{C}{x+1}$, find the values of constants A, B and C.

(b) Hence evaluate $\displaystyle\int_3^4 \frac{x+2}{x^3+x^2-2x-2}\,dx$.

SOLUTION

(a) $$\frac{x+2}{(x^2-2)(x+1)} \equiv \frac{Ax+B}{x^2-2} + \frac{C}{x+1}$$
$$\equiv \frac{(Ax+B)(x+1)+C(x^2-2)}{(x^2-2)(x+1)}$$

i.e. $$(Ax+B)(x+1)+C(x^2-2) \equiv x+2$$
$$Ax^2+(A+B)x+B+Cx^2-2C \equiv x+2$$
$$(A+C)x^2+(A+B)x+(B-2C) \equiv x+2$$

$$\therefore \quad \begin{cases} A+C=0 \text{ ....}(1) \\ A+B=1 \text{ ....}(2) \\ B-2C=2 \text{ ....}(3) \end{cases}$$

From (1), $A=-C$ ....(4)

Substitute (4) into (2),

$-C+B=1$ ....(5)

(3) − (5): $(B-2C)-(-C+B)=2-1$
$$-C=1$$
$$C=-1$$

Substitute $C=-1$ into (3),

$B-2(-1)=2$
$$B=0$$

Substitute $C=-1$ into (4),

$A=-(-1)$
$\quad =1$

$\therefore \quad \underline{\underline{A=1,\ B=0,\ C=-1}}$

(b) $\displaystyle\int_3^4 \frac{x+2}{x^3+x^2-2x-2}dx = \int_3^4 \frac{x+2}{(x^2-2)(x+1)}dx$

$\displaystyle = \int_3^4 (\frac{x}{x^2-2}+\frac{-1}{x+1})dx$ [From the result of **(a)**]

$\displaystyle = \int_3^4 \frac{x}{x^2-2}dx - \int_3^4 \frac{1}{x+1}dx$

$\displaystyle = \frac{1}{2}\int_3^4 \frac{2x}{x^2-2}dx - \int_3^4 \frac{1}{x+1}dx$

$\displaystyle = \frac{1}{2}\int_3^4 \frac{1}{x^2-2}d(x^2-2) - \int_3^4 \frac{1}{x+1}d(x+1)$

$\displaystyle = \frac{1}{2}[\ln\left|x^2-2\right|]_3^4 - [\ln\left|x+1\right|]_3^4$

$\displaystyle = \frac{1}{2}(\ln 14 - \ln 7) - (\ln 5 - \ln 4)$

$\displaystyle = \frac{1}{2}\ln 2 - \ln 5 + 2\ln 2$

$\displaystyle = \underline{\underline{\frac{5}{2}\ln 2 - \ln 5}}$

103. (a) Evaluate $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}}dx$.

(b) If $\displaystyle\frac{1}{u(u-1)} \equiv \frac{A}{u}+\frac{B}{u-1}$, find the values of constants A and B.

(c) Hence evaluate $\displaystyle\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x-1)}dx$.

SOLUTION

(a) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}}dx = \int_{\ln 2}^{\ln 3} \frac{1}{1-e^{-x}}d(1-e^{-x})$

$\displaystyle = [\ln\left|1-e^{-x}\right|]_{\ln 2}^{\ln 3}$

$\displaystyle = \ln(1-e^{-\ln 3}) - \ln(1-e^{-\ln 2})$

$\displaystyle = \ln(1-e^{\ln\frac{1}{3}}) - \ln(1-e^{\ln\frac{1}{2}})$

$\displaystyle = \ln(1-\frac{1}{3}) - \ln(1-\frac{1}{2})$

$\displaystyle = \ln\frac{2}{3} - \ln\frac{1}{2}$

$\displaystyle = \underline{\underline{\ln\frac{4}{3}}}$

(b) $\dfrac{1}{u(u-1)} \equiv \dfrac{A}{u} + \dfrac{B}{u-1}$

$$\equiv \frac{A(u-1)+Bu}{u(u-1)}$$

i.e. $A(u-1)+Bu \equiv 1$

$(A+B)u - A \equiv 1$

$\therefore \begin{cases} A+B=0 \\ -A=1 \end{cases}$

$\therefore$ $\underline{\underline{A=-1,\ B=1}}$

(c) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x-1)}\,dx = \int_{\ln 2}^{\ln 3} [\frac{-1}{e^x} + \frac{1}{e^x-1}]\,dx$ [From the result of **(b)**]

$$= -\int_{\ln 2}^{\ln 3} e^{-x}dx + \int_{\ln 2}^{\ln 3} \frac{1}{e^x-1}\,dx$$

$$= \int_{\ln 2}^{\ln 3} e^{-x}d(-x) + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{e^{-x}(e^x-1)}\,dx$$

$$= [e^{-x}]_{\ln 2}^{\ln 3} + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}}\,dx$$

$= [e^{-\ln 3} - e^{-\ln 2}] + \ln\dfrac{4}{3}$ [From the result of **(a)**]

$$= (\frac{1}{3} - \frac{1}{2}) + \ln\frac{4}{3}$$

$$= \underline{\underline{-\frac{1}{6} + \ln\frac{4}{3}}}$$

104. (a) Prove that $\displaystyle\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(-x)\,dx$.

(b) Hence, or otherwise, evaluate $\displaystyle\int_{-2}^{2} \frac{x(e^{x^2} - e^{-x^2})}{1+x^2}\,dx$.

SOLUTION

(a) Let $u=-x$. Then $du=-dx$.

When $x=-a$, $u=a$;

when $x=0$, $u=0$.

$$\therefore \quad \int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-u)\cdot(-1)du$$

$$= \int_{0}^{a} f(-u)du$$

$$= \int_{0}^{a} f(-x)dx$$

(b) $$\int_{-2}^{2} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx = \int_{-2}^{0} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx + \int_{0}^{2} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx$$

$$= \int_{0}^{2} \frac{-x[e^{(-x)^2}-e^{-(-x)^2}]}{1+(-x)^2}dx + \int_{0}^{2} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx$$ [From the result of **(a)**]

$$= -\int_{0}^{2} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx + \int_{0}^{2} \frac{x(e^{x^2}-e^{-x^2})}{1+x^2}dx$$

$$= \underline{\underline{0}}$$

105. (a) (i) Prove that $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$.

(ii) Hence prove that $\int_{0}^{a} f(x)\,dx = \frac{1}{2}\int_{0}^{a} [f(x)+f(a-x)]dx$.

(b) From the result of **(a)**, evaluate $\int_{0}^{3} \frac{dx}{e^{3-2x}+1}$.

SOLUTION

(a) (i) Let $u = a - x$. Then $du = -dx$.

When $x = 0$, $u = a$;

when $x = a$, $u = 0$.

$$\therefore \quad \int_{0}^{a} f(x)dx = \int_{a}^{0} f(a-u)\cdot(-1)du$$

$$= \int_{0}^{a} f(a-u)du$$

$$= \int_{0}^{a} f(a-x)dx$$

(ii) $\frac{1}{2}\int_0^a [f(x)+f(a-x)]dx = \frac{1}{2}\int_0^a f(x)\,dx + \frac{1}{2}\int_0^a f(a-x)\,dx$

$= \frac{1}{2}\int_0^a f(x)\,dx + \frac{1}{2}\int_0^a f(x)\,dx$ [From the result of **(a)(i)**]

$= \int_0^a f(x)dx$

$\therefore \quad \int_0^a f(x)\,dx = \frac{1}{2}\int_0^a [f(x)+f(a-x)]dx$

(b) $\int_0^3 \frac{dx}{e^{3-2x}+1} = \frac{1}{2}\int_0^3 [\frac{1}{e^{3-2x}+1} + \frac{1}{e^{3-2(3-x)}+1}]dx$ [From the result of **(a)(ii)**]

$= \frac{1}{2}\int_0^3 (\frac{1}{e^{3-2x}+1} + \frac{1}{e^{2x-3}+1})\,dx$

$= \frac{1}{2}\int_0^3 \frac{(e^{2x-3}+1)+(e^{3-2x}+1)}{(e^{3-2x}+1)(e^{2x-3}+1)}dx$

$= \frac{1}{2}\int_0^3 \frac{e^{2x-3}+e^{3-2x}+2}{1+e^{3-2x}+e^{2x-3}+1}dx$

$= \frac{1}{2}\int_0^3 \frac{e^{2x-3}+e^{3-2x}+2}{e^{2x-3}+e^{3-2x}+2}dx$

$= \frac{1}{2}\int_0^3 dx$

$= \frac{1}{2}[x]_0^3$

$= \underline{\underline{\frac{3}{2}}}$

106. (a) Prove that $\int_0^1 \frac{x^4}{x^4+(1-x)^4}dx = \int_0^1 \frac{(1-x)^4}{x^4+(1-x)^4}dx.$

[Hint: Let $u=1-x$.]

(b) Hence, or otherwise, evaluate $\int_0^1 \frac{x^4}{x^4+(1-x)^4}dx.$

SOLUTION

(a) Let $u=1-x$. Then $du=-dx$.

When $x=0$, $u=1$;

when $x=1$, $u=0$.

$$\therefore \quad \int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx = \int_1^0 \frac{(1-u)^4}{(1-u)^4+u^4}\cdot(-1)\,du$$

$$= \int_0^1 \frac{(1-u)^4}{u^4+(1-u)^4}\,du$$

$$= \int_0^1 \frac{(1-x)^4}{x^4+(1-x)^4}\,dx$$

(b) $$\int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx = \int_0^1 \frac{[x^4+(1-x)^4]-(1-x)^4}{x^4+(1-x)^4}\,dx$$

$$= \int_0^1 [1-\frac{(1-x)^4}{x^4+(1-x)^4}]\,dx$$

$$= \int_0^1 dx - \int_0^1 \frac{(1-x)^4}{x^4+(1-x)^4}\,dx$$

$$= [x]_0^1 - \int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx \quad \text{[From the result of (a)]}$$

$$2\int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx = 1-0$$

$$\int_0^1 \frac{x^4}{x^4+(1-x)^4}\,dx = \underline{\underline{\frac{1}{2}}}$$

107. (a) Prove that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.

(b) Hence, or otherwise, evaluate $\int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}\,dx$.

SOLUTION

(a) Let $u=a+b-x$. Then $du=-dx$.
When $x=a$, $u=b$;
when $x=b$, $u=a$.

$$\therefore \quad \int_a^b f(x)\,dx = \int_b^a f(a+b-u)\cdot(-1)\,du$$

$$= \int_a^b f(a+b-u)\,du$$

$$= \int_a^b f(a+b-x)\,dx$$

(b) $\displaystyle\int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}dx = \int_5^{10} \frac{(5+10-x)^{10}}{(5+10-x)^{10}+[15-(5+10-x)]^{10}}dx$ [From the result of **(a)**]

$$= \int_5^{10} \frac{(15-x)^{10}}{(15-x)^{10}+x^{10}}dx$$

$$\therefore \quad \int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}dx = \int_5^{10} \frac{[x^{10}+(15-x)^{10}]-(15-x)^{10}}{x^{10}+(15-x)^{10}}dx$$

$$= \int_5^{10} [1-\frac{(15-x)^{10}}{x^{10}+(15-x)^{10}}]dx$$

$$= \int_5^{10} dx - \int_5^{10} \frac{(15-x)^{10}}{x^{10}+(15-x)^{10}}dx$$

$$= [x]_5^{10} - \int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}dx$$

$$2\int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}dx = 10-5$$

$$\therefore \quad \int_5^{10} \frac{x^{10}}{x^{10}+(15-x)^{10}}dx = \underline{\underline{\frac{5}{2}}}$$

108. Let $f(x)$ be a function defined on $x \geq 0$. It is given that $f'(x) = \dfrac{90}{2+ae^{-kx}}$, where a and k are positive constants.

(a) **(i)** Express $\ln[\dfrac{90}{f'(x)}-2]$ as a linear function of x.

(ii) It is given that the graph of the linear function obtained in **(a)(i)** passes through (1, 2.44) and the intercept on the vertical axis is 2.89. Find the values of a and k. (Give your answers correct to 2 significant figures if necessary.)

(b) **(i)** Prove that $f'(x) = \dfrac{90e^{kx}}{2e^{kx}+a}$.

(ii) It is given that $f(0)=100$. Using the values of a and k obtained in **(a)(ii)**, find the value of $f(5)$. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) (i)

$$f'(x) = \frac{90}{2 + ae^{-kx}}$$

$$2 + ae^{-kx} = \frac{90}{f'(x)}$$

$$\frac{90}{f'(x)} - 2 = ae^{-kx}$$

$$\ln[\frac{90}{f'(x)} - 2] = \ln(ae^{-kx})$$

$$\underline{\underline{\ln[\frac{90}{f'(x)} - 2] = \ln a - kx}}$$

(ii) $\ln a =$ Intercept on the vertical axis

$= 2.89$

$a = e^{2.89}$

$= \underline{\underline{18}}$ (corr. to 2 sig. fig.)

$\because$ The graph passes through (1, 2.44).

$\therefore$ $2.44 = 2.89 - k(1)$

$k = \underline{\underline{0.45}}$

(b) (i)

$$f'(x) = \frac{90}{2 + ae^{-kx}}$$

$$= \frac{90}{e^{-kx}(2e^{kx} + a)}$$

$$= \frac{90e^{kx}}{2e^{kx} + a}$$

(ii) From **(a)(ii)**, $f'(x) = \dfrac{90e^{0.45x}}{2e^{0.45x} + 18}$.

Let $u = 2e^{0.45x} + 18$. Then $du = 0.9e^{0.45x}dx$.

When $x = 0$, $u = 20$;

when $x = 5$, $u = 2e^{2.25} + 18$.

$$\int_0^5 f'(x)\,dx = \int_0^5 \frac{90e^{0.45x}}{2e^{0.45x}+18}\,dx$$

$$= \int_0^5 \frac{100}{2e^{0.45x}+18}\cdot(0.9e^{0.45x})\,dx$$

$$= \int_{20}^{2e^{2.25}+18} \frac{100}{u}\,du$$

$$= 100[\ln|u|]_{20}^{2e^{2.25}+18}$$

$$= 100[\ln(2e^{2.25}+18) - \ln 20]$$

$$= 100\ln\frac{2e^{2.25}+18}{20}$$

$$f(5) - f(0) = \int_0^5 f'(x)\,dx$$

$$f(5) = \int_0^5 f'(x)\,dx + f(0)$$

$$= 100\ln\frac{2e^{2.25}+18}{20} + 100$$

$$= \underline{\underline{161}}\ \text{(corr. to 3 sig. fig.)}$$