

Chapter 8

Definite Integration

EXERCISE 8A*Level 1*

1. Evaluate
- $\int_2^4 6x^3 dx$
- .

SOLUTION

$$\begin{aligned}
 \int_2^4 6x^3 dx &= 6\left[\frac{x^4}{4}\right]_2^4 \\
 &= \frac{6}{4}(4^4 - 2^4) \\
 &= \underline{\underline{360}}
 \end{aligned}$$

2. Evaluate
- $\int_0^2 (5 - 3x^2) dx$
- .

SOLUTION

$$\begin{aligned}
 \int_0^2 (5 - 3x^2) dx &= [5x - x^3]_0^2 \\
 &= [5(2) - 2^3] - [5(0) - 0^3] \\
 &= \underline{\underline{2}}
 \end{aligned}$$

3. Evaluate
- $\int_{-1}^3 (x^3 - 3x^2 + 1) dx$
- .

SOLUTION

$$\begin{aligned}
 \int_{-1}^3 (x^3 - 3x^2 + 1) dx &= \left[\frac{x^4}{4} - x^3 + x\right]_{-1}^3 \\
 &= \left(\frac{3^4}{4} - 3^3 + 3\right) - \left[\frac{(-1)^4}{4} - (-1)^3 + (-1)\right] \\
 &= \underline{\underline{-4}}
 \end{aligned}$$

4. Evaluate $\int_1^2 (3x^2 - 6x + 7) dx$.

SOLUTION

$$\begin{aligned}\int_1^2 (3x^2 - 6x + 7) dx &= [x^3 - 3x^2 + 7x]_1^2 \\ &= [2^3 - 3(2)^2 + 7(2)] - [1^3 - 3(1)^2 + 7(1)] \\ &= \underline{\underline{5}}\end{aligned}$$

5. Evaluate $\int_1^2 (x^{-2} + x^2) dx$.

SOLUTION

$$\begin{aligned}\int_1^2 (x^{-2} + x^2) dx &= [-x^{-1} + \frac{x^3}{3}]_1^2 \\ &= (-2^{-1} + \frac{2^3}{3}) - (-1^{-1} + \frac{1^3}{3}) \\ &= \frac{17}{\underline{\underline{6}}}\end{aligned}$$

6. Evaluate $\int_0^9 3\sqrt{x} dx$.

SOLUTION

$$\begin{aligned}\int_0^9 3\sqrt{x} dx &= [2x^{\frac{3}{2}}]_0^9 \\ &= 2(9)^{\frac{3}{2}} - 2(0)^{\frac{3}{2}} \\ &= \underline{\underline{54}}\end{aligned}$$

7. Evaluate $\int_1^4 \sqrt{\frac{2}{x}} dx$.

SOLUTION

$$\begin{aligned}\int_1^4 \sqrt{\frac{2}{x}} dx &= \sqrt{2} \int_1^4 x^{-\frac{1}{2}} dx \\ &= \sqrt{2} [2x^{\frac{1}{2}}]_1^4 \\ &= \sqrt{2} [2(4)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}}] \\ &= \underline{\underline{2\sqrt{2}}}\end{aligned}$$

8. Evaluate $\int_1^9 (9x^2 - \frac{6}{\sqrt{x}}) dx$.

 **SOLUTION**

$$\begin{aligned}\int_1^9 (9x^2 - \frac{6}{\sqrt{x}}) dx &= [3x^3 - 12x^{\frac{1}{2}}]_1^9 \\ &= [3(9)^3 - 12(9)^{\frac{1}{2}}] - [3(1)^3 - 12(1)^{\frac{1}{2}}] \\ &= \underline{\underline{2160}}\end{aligned}$$

9. Evaluate $\int_1^8 (x^{\frac{1}{3}} + 4x^{-\frac{2}{3}}) dx$.

 **SOLUTION**

$$\begin{aligned}\int_1^8 (x^{\frac{1}{3}} + 4x^{-\frac{2}{3}}) dx &= [\frac{3}{4}x^{\frac{4}{3}} + 12x^{\frac{1}{3}}]_1^8 \\ &= [\frac{3}{4}(8)^{\frac{4}{3}} + 12(8)^{\frac{1}{3}}] - [\frac{3}{4}(1)^{\frac{4}{3}} + 12(1)^{\frac{1}{3}}] \\ &= \underline{\underline{\frac{93}{4}}}\end{aligned}$$

10. Evaluate $\int_1^2 x(x+1) dx$.

 **SOLUTION**

$$\begin{aligned}\int_1^2 x(x+1) dx &= \int_1^2 (x^2 + x) dx \\ &= [\frac{x^3}{3} + \frac{x^2}{2}]_1^2 \\ &= (\frac{2^3}{3} + \frac{2^2}{2}) - (\frac{1^3}{3} + \frac{1^2}{2}) \\ &= \underline{\underline{\frac{23}{6}}}\end{aligned}$$

11. Evaluate $\int_{-2}^2 (3x+2)(7x+5) dx$.

SOLUTION

$$\begin{aligned}\int_{-2}^2 (3x+2)(7x+5) dx &= \int_{-2}^2 (21x^2 + 29x + 10) dx \\&= \left[7x^3 + \frac{29}{2}x^2 + 10x \right]_{-2}^2 \\&= \left[7(2)^3 + \frac{29}{2}(2)^2 + 10(2) \right] - \left[7(-2)^3 + \frac{29}{2}(-2)^2 + 10(-2) \right] \\&= \underline{\underline{152}}\end{aligned}$$

12. Evaluate $\int_{-1}^2 (6-4x)(3x-1) dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^2 (6-4x)(3x-1) dx &= \int_{-1}^2 (-12x^2 + 22x - 6) dx \\&= \left[-4x^3 + 11x^2 - 6x \right]_{-1}^2 \\&= \left[-4(2)^3 + 11(2)^2 - 6(2) \right] - \left[-4(-1)^3 + 11(-1)^2 - 6(-1) \right] \\&= \underline{\underline{-21}}\end{aligned}$$

13. Evaluate $\int_{-1}^2 (2x^2-1)^2 dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^2 (2x^2-1)^2 dx &= \int_{-1}^2 (4x^4 - 4x^2 + 1) dx \\&= \left[\frac{4}{5}x^5 - \frac{4}{3}x^3 + x \right]_{-1}^2 \\&= \left[\frac{4}{5}(2)^5 - \frac{4}{3}(2)^3 + 2 \right] - \left[\frac{4}{5}(-1)^5 - \frac{4}{3}(-1)^3 + (-1) \right] \\&= \underline{\underline{\frac{87}{5}}}\end{aligned}$$

14. Evaluate $\int_4^9 (2\sqrt{x} - 3)^2 dx$.

SOLUTION

$$\begin{aligned}\int_4^9 (2\sqrt{x} - 3)^2 dx &= \int_4^9 (4x - 12\sqrt{x} + 9) dx \\ &= [2x^2 - 8x^{\frac{3}{2}} + 9x]_4^9 \\ &= [2(9)^2 - 8(9)^{\frac{3}{2}} + 9(9)] - [2(4)^2 - 8(4)^{\frac{3}{2}} + 9(4)] \\ &= \underline{\underline{23}}\end{aligned}$$

15. Evaluate $\int_{-3}^0 3e^x dx$.

SOLUTION

$$\begin{aligned}\int_{-3}^0 3e^x dx &= [3e^x]_{-3}^0 \\ &= 3e^0 - 3e^{-3} \\ &= \underline{\underline{3 - 3e^{-3}}}\end{aligned}$$

16. Evaluate $\int_1^4 (3x^2 - 12x^{-2} + 5e^x) dx$.

SOLUTION

$$\begin{aligned}\int_1^4 (3x^2 - 12x^{-2} + 5e^x) dx &= [x^3 + \frac{12}{x} + 5e^x]_1^4 \\ &= (4^3 + \frac{12}{4} + 5e^4) - (1^3 + \frac{12}{1} + 5e^1) \\ &= (67 + 5e^4) - (13 + 5e) \\ &= \underline{\underline{54 + 5e^4 - 5e}}\end{aligned}$$

17. Evaluate $\int_{-1}^1 e^{-2x} dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^1 e^{-2x} dx &= \int_{-1}^1 (e^{-2})^x dx \\&= \left[\frac{(e^{-2})^x}{\ln e^{-2}} \right]_{-1}^1 \\&= \left[-\frac{1}{2} e^{-2x} \right]_{-1}^1 \\&= -\frac{1}{2} e^{-2(1)} - \left[-\frac{1}{2} e^{-2(-1)} \right] \\&= \frac{1}{2} e^2 - \frac{1}{2} e^{-2} \\&= \underline{\underline{\frac{1}{2} e^2 - \frac{1}{2} e^{-2}}}}\end{aligned}$$

18. Evaluate $\int_1^3 6e^{4x+5} dx$.

SOLUTION

$$\begin{aligned}\int_1^3 6e^{4x+5} dx &= \int_1^3 6e^5 \cdot e^{4x} dx \\&= 6e^5 \int_1^3 (e^4)^x dx \\&= 6e^5 \left[\frac{(e^4)^x}{\ln e^4} \right]_1^3 \\&= 6e^5 \left[\frac{e^{4x}}{4} \right]_1^3 \\&= \frac{3}{2} e^5 [e^{4(3)} - e^{4(1)}] \\&= \underline{\underline{\frac{3}{2} (e^{17} - e^9)}}\end{aligned}$$

19. Evaluate $\int_1^2 5^x dx$.

SOLUTION

$$\begin{aligned}\int_1^2 5^x dx &= \left[\frac{5^x}{\ln 5} \right]_1^2 \\&= \frac{5^2}{\ln 5} - \frac{5^1}{\ln 5} \\&= \underline{\underline{\frac{20}{\ln 5}}}\end{aligned}$$

20. Evaluate $\int_0^2 (2x^3 - 2^{3x}) dx$.

SOLUTION

$$\begin{aligned}
 \int_0^2 (2x^3 - 2^{3x}) dx &= \left[\frac{x^4}{2} - \frac{(2^3)^x}{\ln 2^3} \right]_0^2 \\
 &= \left[\frac{x^4}{2} - \frac{2^{3x}}{3 \ln 2} \right]_0^2 \\
 &= \left[\frac{2^4}{2} - \frac{2^{3(2)}}{3 \ln 2} \right] - \left[\frac{0^4}{2} - \frac{2^{3(0)}}{3 \ln 2} \right] \\
 &= \left(8 - \frac{64}{3 \ln 2} \right) - \left(-\frac{1}{3 \ln 2} \right) \\
 &= 8 - \frac{21}{\ln 2}
 \end{aligned}$$

21. Evaluate $\int_{-2}^0 [8e^{-4x} + 18(3^{2x})] dx$.

SOLUTION

$$\begin{aligned}
 \int_{-2}^0 [8e^{-4x} + 18(3^{2x})] dx &= \int_{-2}^0 \{8(e^{-4})^x + 18[(3^2)^x]\} dx \\
 &= \left[8 \times \frac{(e^{-4})^x}{\ln e^{-4}} + 18 \times \frac{(3^2)^x}{\ln 3^2} \right]_{-2}^0 \\
 &= \left[\frac{8e^{-4x}}{-4} + \frac{18(3^{2x})}{2 \ln 3} \right]_{-2}^0 \\
 &= \left[-2e^{-4x} + \frac{9(3^{2x})}{\ln 3} \right]_{-2}^0 \\
 &= \left\{ -2e^{-4(0)} + \frac{9[3^{2(0)}]}{\ln 3} \right\} - \left\{ -2e^{-4(-2)} + \frac{9[3^{2(-2)}]}{\ln 3} \right\} \\
 &= \left(-2 + \frac{9}{\ln 3} \right) - \left(-2e^8 + \frac{1}{9 \ln 3} \right) \\
 &= 2e^8 + \frac{80}{9 \ln 3} - 2
 \end{aligned}$$

22. Evaluate $\int_0^2 3^x e^x dx$.

 **SOLUTION**

$$\begin{aligned}\int_0^2 3^x e^x dx &= \int_0^2 (3e)^x dx \\&= \left[\frac{(3e)^x}{\ln 3e} \right]_0^2 \\&= \left[\frac{(3e)^x}{\ln 3 + \ln e} \right]_0^2 \\&= \left[\frac{(3e)^x}{\ln 3 + 1} \right]_0^2 \\&= \frac{(3e)^2}{\ln 3 + 1} - \frac{(3e)^0}{\ln 3 + 1} \\&= \frac{9e^2 - 1}{\ln 3 + 1}\end{aligned}$$

23. Evaluate $\int_{-2}^0 5^x e^{-2x} dx$.

 **SOLUTION**

$$\begin{aligned}\int_{-2}^0 5^x e^{-2x} dx &= \int_{-2}^0 (5e^{-2})^x dx \\&= \left[\frac{(5e^{-2})^x}{\ln(5e^{-2})} \right]_{-2}^0 \\&= \left[\frac{(5e^{-2})^x}{\ln 5 + \ln e^{-2}} \right]_{-2}^0 \\&= \left[\frac{(5e^{-2})^x}{\ln 5 - 2} \right]_{-2}^0 \\&= \frac{(5e^{-2})^0}{\ln 5 - 2} - \frac{(5e^{-2})^{-2}}{\ln 5 - 2} \\&= \frac{1}{\ln 5 - 2} - \frac{e^4}{25(\ln 5 - 2)} \\&= \frac{25 - e^4}{25(\ln 5 - 2)}\end{aligned}$$

24. Evaluate $\int_2^6 (\frac{3}{x} + \frac{1}{6x}) dx$.

SOLUTION

$$\begin{aligned}\int_2^6 (\frac{3}{x} + \frac{1}{6x}) dx &= \int_2^6 \frac{19}{6x} dx \\ &= \frac{19}{6} \int_2^6 \frac{1}{x} dx \\ &= \frac{19}{6} [\ln|x|]_2^6 \\ &= \frac{19}{6} (\ln 6 - \ln 2) \\ &= \frac{19}{6} \ln 3\end{aligned}$$

25. Evaluate $\int_1^4 \frac{3x + 4x^2 - 2\sqrt{x}}{x^2} dx$.

SOLUTION

$$\begin{aligned}\int_1^4 \frac{3x + 4x^2 - 2\sqrt{x}}{x^2} dx &= \int_1^4 (3x^{-1} + 4 - 2x^{-\frac{3}{2}}) dx \\ &= [3\ln|x| + 4x + 4x^{-\frac{1}{2}}]_1^4 \\ &= [3\ln 4 + 4(4) + 4(4)^{-\frac{1}{2}}] - [3\ln 1 + 4(1) + 4(1)^{-\frac{1}{2}}] \\ &= (3\ln 4 + 18) - 8 \\ &= \underline{\underline{6\ln 2 + 10}}\end{aligned}$$

26. Evaluate $\int_1^3 (x-2)(4+\frac{3}{x^2}) dx$.

SOLUTION

$$\begin{aligned}\int_1^3 (x-2)(4+\frac{3}{x^2}) dx &= \int_1^3 (4x - 8 + \frac{3}{x} - \frac{6}{x^2}) dx \\ &= [2x^2 - 8x + 3\ln|x| + \frac{6}{x}]_1^3 \\ &= [2(3)^2 - 8(3) + 3\ln 3 + \frac{6}{3}] - [2(1)^2 - 8(1) + 3\ln 1 + \frac{6}{1}] \\ &= \underline{\underline{3\ln 3 - 4}}\end{aligned}$$

27. Suppose $f(x)$ is a continuous function and that $\int_0^5 f(x) dx = 7$. Evaluate the following definite integrals.

(a) $\int_0^5 2f(x) dx$

(b) $\int_0^5 [2 - f(x)] dx$

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_0^5 2f(x) dx &= 2 \int_0^5 f(x) dx \\ &= 2(7) \\ &= \underline{\underline{14}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_0^5 [2 - f(x)] dx &= \int_0^5 2 dx - \int_0^5 f(x) dx \\ &= 2[x]_0^5 - 7 \\ &= 2(5 - 0) - 7 \\ &= \underline{\underline{3}} \end{aligned}$$

28. Suppose $g(x)$ is a continuous function and that $\int_1^2 g(x) dx = 3$. Evaluate the following definite integrals.

(a) $\int_1^2 [6x^2 - g(x)] dx$

(b) $\int_1^2 \frac{5x + x^2 g(x)}{x^2} dx$

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_1^2 [6x^2 - g(x)] dx &= \int_1^2 6x^2 dx - \int_1^2 g(x) dx \\ &= [2x^3]_1^2 - 3 \\ &= [2(2)^3 - 2(1)^3] - 3 \\ &= \underline{\underline{11}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^2 \frac{5x + x^2 g(x)}{x^2} dx &= \int_1^2 [5x^{-1} + g(x)] dx \\ &= \int_1^2 5x^{-1} dx + \int_1^2 g(x) dx \\ &= 5[\ln|x|]_1^2 + 3 \\ &= 5(\ln 2 - \ln 1) + 3 \\ &= \underline{\underline{5\ln 2 + 3}} \end{aligned}$$

29. Suppose $f(x)$ is a continuous function, and that $\int_1^3 f(x) dx = -1$ and $\int_1^5 f(x) dx = 2$. Evaluate the following definite integrals.

(a) $\int_3^5 f(z) dz$

(b) $\int_5^3 f(u) du$

SOLUTION

(a) $\int_1^3 f(x) dx + \int_3^5 f(x) dx = \int_1^5 f(x) dx$

$$-1 + \int_3^5 f(z) dz = 2$$

$$\int_3^5 f(z) dz = \underline{\underline{3}}$$

(b) $\int_5^3 f(u) du = \int_5^3 f(z) dz$

$$= -\int_3^5 f(z) dz$$

$$= \underline{\underline{-3}} \quad [\text{From the result of (a)}]$$

30. Suppose $h(x)$ is a continuous function, and that $\int_{-2}^2 h(x) dx = 8$ and $\int_{-5}^2 h(x) dx = -2$. Evaluate the following definite integrals.

(a) $\int_{-5}^{-2} h(r) dr$

(b) $\int_{-2}^{-5} h(u) du$

SOLUTION

(a) $\int_{-5}^{-2} h(x) dx + \int_{-2}^2 h(x) dx = \int_{-5}^2 h(x) dx$

$$\int_{-5}^{-2} h(r) dr + 8 = -2$$

$$\int_{-5}^{-2} h(r) dr = \underline{\underline{-10}}$$

(b) $\int_{-2}^{-5} h(u) du = \int_{-2}^{-5} h(r) dr$

$$= -\int_{-5}^{-2} h(r) dr$$

$$= -(-10) \quad [\text{From the result of (a)}]$$

$$= \underline{\underline{10}}$$

31. Suppose $g(x)$ and $h(x)$ are continuous functions, and that $\int_{-1}^3 g(x)dx = 5$, $\int_3^6 g(x)dx = 8$ and $\int_{-1}^6 h(x)dx = -4$. Evaluate the following definite integrals.

(a) $\int_{-1}^6 g(x)dx$

(b) $\int_{-1}^6 [3h(x) + 2g(x)]dx$

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_{-1}^6 g(x)dx &= \int_{-1}^3 g(x)dx + \int_3^6 g(x)dx \\ &= 5 + 8 \\ &= \underline{\underline{13}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_{-1}^6 [3h(x) + 2g(x)]dx &= 3\int_{-1}^6 h(x)dx + 2\int_{-1}^6 g(x)dx \\ &= 3(-4) + 2(13) \quad [\text{From the result of (a)}] \\ &= \underline{\underline{14}} \end{aligned}$$

32. Suppose $p(x)$ and $q(x)$ are continuous functions, and that $\int_{-1}^4 p(x)dx = 3$, $\int_1^4 p(x)dx = 1$ and $\int_{-1}^1 q(x)dx = -5$. Evaluate the following definite integrals.

(a) $\int_{-1}^1 p(x)dx$

(b) $\int_{-1}^1 5[p(x) - 3q(x)]dx$

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_{-1}^1 p(x)dx + \int_1^4 p(x)dx &= \int_{-1}^4 p(x)dx \\ \int_{-1}^1 p(x)dx + 1 &= 3 \\ \int_{-1}^1 p(x)dx &= \underline{\underline{2}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_{-1}^1 5[p(x) - 3q(x)]dx &= \int_{-1}^1 [5p(x) - 15q(x)]dx \\ &= 5\int_{-1}^1 p(x)dx - 15\int_{-1}^1 q(x)dx \\ &= 5(2) - 15(-5) \quad [\text{From the result of (a)}] \\ &= \underline{\underline{85}} \end{aligned}$$

33. Suppose $f(x)$ is a continuous function, and that $\int_{-3}^1 f(x)dx = p$ and $\int_1^4 f(x)dx = q$, where p and q are constants. Express the following definite integrals in terms of p and q .

(a) $\int_{-3}^4 f(x)dx$

(b) $\int_2^4 f(x)dx - \int_2^1 f(x)dx$

(c) $\int_{-3}^6 f(x)dx + \int_6^4 f(t)dt$

SOLUTION

(a)
$$\begin{aligned}\int_{-3}^4 f(x)dx &= \int_{-3}^1 f(x)dx + \int_1^4 f(x)dx \\ &= \underline{\underline{p+q}}\end{aligned}$$

(b)
$$\begin{aligned}\int_2^4 f(x)dx - \int_2^1 f(x)dx &= \int_2^4 f(x)dx + \int_1^2 f(x)dx \\ &= \int_1^4 f(x)dx \\ &= \underline{\underline{q}}\end{aligned}$$

(c)
$$\begin{aligned}\int_{-3}^6 f(x)dx + \int_6^4 f(t)dt &= \int_{-3}^6 f(x)dx + \int_6^4 f(x)dx \\ &= \int_{-3}^4 f(x)dx + \int_4^6 f(x)dx - \int_4^6 f(x)dx \\ &= \int_{-3}^4 f(x)dx \\ &= \underline{\underline{p+q}} \quad [\text{From the result of (a)}]\end{aligned}$$

34. Suppose $f(x)$ is a continuous function, and that $\int_{-3}^1 f(x)dx = a$, $\int_1^3 f(x)dx = b$ and $\int_1^7 f(x)dx = c$, where a , b and c are constants. Express the following definite integrals in terms of a , b and c .

(a) $\int_{-3}^7 [f(x) - 2]dx$

(b) $\int_3^7 2f(x)dx$

(c) $\int_3^{-3} [f(x) - x^2]dx$

(d) $\int_{-3}^{-1} [f(x) + 1]dx - \int_7^{-1} f(t)dt$

SOLUTION

$$\begin{aligned}
 \text{(a)} \quad \int_{-3}^7 [f(x) - 2] dx &= \int_{-3}^7 f(x) dx - \int_{-3}^7 2 dx \\
 &= \int_{-3}^1 f(x) dx + \int_1^7 f(x) dx - [2x]_{-3}^7 \\
 &= a + c - [2(7) - 2(-3)] \\
 &= \underline{\underline{a + c - 20}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_1^3 f(x) dx + \int_3^7 f(x) dx &= \int_1^7 f(x) dx \\
 b + \int_3^7 f(x) dx &= c \\
 \int_3^7 f(x) dx &= c - b \\
 \therefore \int_3^7 2f(x) dx &= 2 \int_3^7 f(x) dx \\
 &= \underline{\underline{2(c - b)}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int_3^{-3} [f(x) - x^2] dx &= - \int_{-3}^3 [f(x) - x^2] dx \\
 &= - \int_{-3}^3 f(x) dx + \int_{-3}^3 x^2 dx \\
 &= - \left[\int_{-3}^1 f(x) dx + \int_1^3 f(x) dx \right] + \left[\frac{x^3}{3} \right]_{-3}^3 \\
 &= -(a + b) + \left[\frac{3^3}{3} - \frac{(-3)^3}{3} \right] \\
 &= \underline{\underline{-a - b + 18}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int_{-3}^{-1} [f(x) + 1] dx - \int_7^{-1} f(t) dt &= \int_{-3}^{-1} f(x) dx + \int_{-3}^{-1} dx - \int_7^{-1} f(x) dx \\
 &= \int_{-3}^{-1} f(x) dx + [x]_{-3}^{-1} + \int_{-1}^7 f(x) dx \\
 &= \int_{-3}^7 f(x) dx + [(-1) - (-3)] \\
 &= \int_{-3}^1 f(x) dx + \int_1^7 f(x) dx + 2 \\
 &= \underline{\underline{a + c + 2}}
 \end{aligned}$$

Level 2

35. (a) Factorize $x^3 - 3x^2 - x + 3$.

(b) Hence evaluate $\int_1^2 \frac{x^3 - 3x^2 - x + 3}{x+1} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad x^3 - 3x^2 - x + 3 &= x^2(x-3) - (x-3) \\ &= (x^2-1)(x-3) \\ &= \underline{\underline{(x-1)(x+1)(x-3)}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^2 \frac{x^3 - 3x^2 - x + 3}{x+1} dx &= \int_1^2 \frac{(x-1)(x+1)(x-3)}{x+1} dx \quad [\text{From the result of (a)}] \\ &= \int_1^2 (x-1)(x-3) dx \\ &= \int_1^2 (x^2 - 4x + 3) dx \\ &= \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^2 \\ &= \left[\frac{2^3}{3} - 2(2)^2 + 3(2) \right] - \left[\frac{1^3}{3} - 2(1)^2 + 3(1) \right] \\ &= \underline{\underline{-\frac{2}{3}}} \end{aligned}$$

36. Evaluate $\int_{-1}^1 \frac{x^3 + 8}{x+2} dx$.

SOLUTION

$$\begin{aligned} \int_{-1}^1 \frac{x^3 + 8}{x+2} dx &= \int_{-1}^1 \frac{(x+2)(x^2 - 2x + 4)}{x+2} dx \\ &= \int_{-1}^1 (x^2 - 2x + 4) dx \\ &= \left[\frac{x^3}{3} - x^2 + 4x \right]_{-1}^1 \\ &= \left[\frac{1^3}{3} - 1^2 + 4(1) \right] - \left[\frac{(-1)^3}{3} - (-1)^2 + 4(-1) \right] \\ &= \underline{\underline{\frac{26}{3}}} \end{aligned}$$

37. Evaluate $\int_{-2}^2 \frac{16}{4+x} dx - \int_{-2}^2 \frac{x^2}{4+x} dx$.

SOLUTION

$$\begin{aligned} \int_{-2}^2 \frac{16}{4+x} dx - \int_{-2}^2 \frac{x^2}{4+x} dx &= \int_{-2}^2 \left(\frac{16}{4+x} - \frac{x^2}{4+x} \right) dx \\ &= \int_{-2}^2 \frac{16-x^2}{4+x} dx \\ &= \int_{-2}^2 \frac{(4-x)(4+x)}{4+x} dx \\ &= \int_{-2}^2 (4-x) dx \\ &= \left[4x - \frac{x^2}{2} \right]_{-2}^2 \\ &= \left[4(2) - \frac{2^2}{2} \right] - \left[4(-2) - \frac{(-2)^2}{2} \right] \\ &= \underline{\underline{16}} \end{aligned}$$

38. Evaluate $\int_0^2 (e^{x^2} - e^{2x}) dx - \int_0^2 (e^{x^2} + e^x) dx$.

SOLUTION

$$\begin{aligned} \int_0^2 (e^{x^2} - e^{2x}) dx - \int_0^2 (e^{x^2} + e^x) dx &= \int_0^2 (e^{x^2} - e^{2x} - e^{x^2} - e^x) dx \\ &= \int_0^2 [-(e^2)^x - e^x] dx \\ &= \left[-\frac{(e^2)^x}{\ln e^2} - e^x \right]_0^2 \\ &= \left[-\frac{e^{2x}}{2} - e^x \right]_0^2 \\ &= \left[-\frac{e^{2(2)}}{2} - e^2 \right] - \left[-\frac{e^{2(0)}}{2} - e^0 \right] \\ &= -\frac{e^4}{2} - e^2 - \left(-\frac{3}{2} \right) \\ &= \underline{\underline{-\frac{e^4}{2} - e^2 + \frac{3}{2}}} \end{aligned}$$

39. Evaluate $\int_2^5 (3x^x + 8x) dx + 3 \int_5^2 t^t dt + 8 \int_{-1}^2 t dt$.

SOLUTION

$$\begin{aligned} \int_2^5 (3x^x + 8x) dx + 3 \int_5^2 t^t dt + 8 \int_{-1}^2 t dt &= \int_2^5 3x^x dx + \int_2^5 8x dx - \int_2^5 3x^x dx + \int_{-1}^2 8x dx \\ &= \int_{-1}^5 8x dx \\ &= [4x^2]_{-1}^5 \\ &= 4(5)^2 - 4(-1)^2 \\ &= \underline{\underline{96}} \end{aligned}$$

40. Evaluate $\int_1^5 \frac{2e^{2x} dx}{e^{2x} - 2e^{-x}} + 4 \int_5^1 \frac{e^{-x} dx}{e^{2x} - 2e^{-x}}$.

SOLUTION

$$\begin{aligned} \int_1^5 \frac{2e^{2x} dx}{e^{2x} - 2e^{-x}} + 4 \int_5^1 \frac{e^{-x} dx}{e^{2x} - 2e^{-x}} &= \int_1^5 \frac{2e^{2x} dx}{e^{2x} - 2e^{-x}} - \int_1^5 \frac{4e^{-x} dx}{e^{2x} - 2e^{-x}} \\ &= \int_1^5 \frac{2e^{2x} - 4e^{-x}}{e^{2x} - 2e^{-x}} dx \\ &= 2 \int_1^5 \frac{e^{2x} - 2e^{-x}}{e^{2x} - 2e^{-x}} dx \\ &= 2 \int_1^5 dx \\ &= 2[x]_1^5 \\ &= 2(5 - 1) \\ &= \underline{\underline{8}} \end{aligned}$$

41. If $\int_{-1}^2 \frac{1}{f(x)-3} dx = 5$, evaluate $\int_{-1}^2 \frac{f(x)}{f(x)-3} dx$.

SOLUTION

$$\begin{aligned}\int_{-1}^2 \frac{f(x)}{f(x)-3} dx &= \int_{-1}^2 \frac{[f(x)-3]+3}{f(x)-3} dx \\&= \int_{-1}^2 \left[1 + \frac{3}{f(x)-3}\right] dx \\&= \int_{-1}^2 dx + 3 \int_{-1}^2 \frac{1}{f(x)-3} dx \\&= [x]_{-1}^2 + 3(5) \\&= [2 - (-1)] + 15 \\&= \underline{\underline{18}}\end{aligned}$$

42. Suppose $f(x)$ and $g(x)$ are continuous functions and that $\int_2^4 \frac{f(x)}{f(x)-2g(x)} dx = 10$. Evaluate the following definite integrals.

(a) $\int_4^2 \frac{2f(x)}{f(x)-2g(x)} dx$

(b) $\int_2^4 \frac{g(t)}{f(t)-2g(t)} dt$

SOLUTION

$$\begin{aligned}\text{(a)} \quad \int_4^2 \frac{2f(x)}{f(x)-2g(x)} dx &= -2 \int_2^4 \frac{f(x)}{f(x)-2g(x)} dx \\&= -2(10) \\&= \underline{\underline{-20}}\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad \int_2^4 \frac{g(t)}{f(t)-2g(t)} dt &= \int_2^4 \frac{g(x)}{f(x)-2g(x)} dx \\
&= -\frac{1}{2} \int_2^4 \frac{-2g(x)}{f(x)-2g(x)} dx \\
&= -\frac{1}{2} \int_2^4 \frac{[f(x)-2g(x)]-f(x)}{f(x)-2g(x)} dx \\
&= -\frac{1}{2} \int_2^4 \left[1 - \frac{f(x)}{f(x)-2g(x)}\right] dx \\
&= -\frac{1}{2} \int_2^4 dx + \frac{1}{2} \int_2^4 \frac{f(x)}{f(x)-2g(x)} dx \\
&= -\frac{1}{2} [x]_2^4 + \frac{1}{2} (10) \\
&= -\frac{1}{2} (4-2) + 5 \\
&= \underline{\underline{4}}
\end{aligned}$$

43. It is given that $y = (4x-3)^6$.

(a) Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_{\frac{1}{2}}^1 (4x-3)^5 dx$.

 **SOLUTION**

(a) $y = (4x-3)^6$

$$\begin{aligned}
\frac{dy}{dx} &= 6(4x-3)^5 (4) \\
&= \underline{\underline{24(4x-3)^5}}
\end{aligned}$$

(b) $\therefore \frac{d}{dx} (4x-3)^6 = 24(4x-3)^5$

$$\therefore \int_{\frac{1}{2}}^1 24(4x-3)^5 dx = [(4x-3)^6]_{\frac{1}{2}}^1$$

$$24 \int_{\frac{1}{2}}^1 (4x-3)^5 dx = [4(1)-3]^6 - [4(\frac{1}{2})-3]^6$$

$$24 \int_{\frac{1}{2}}^1 (4x-3)^5 dx = 0$$

$$\int_{\frac{1}{2}}^1 (4x-3)^5 dx = \underline{\underline{0}}$$

44. It is given that $y = x \ln x$.

(a) Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_1^e \ln x \, dx$.

SOLUTION

(a) $y = x \ln x$

$$\begin{aligned}\frac{dy}{dx} &= x \cdot \frac{1}{x} + \ln x \\ &= \underline{\underline{1 + \ln x}}\end{aligned}$$

(b) $\therefore \frac{d}{dx}(x \ln x) = 1 + \ln x$

$$\therefore \int_1^e (1 + \ln x) \, dx = [x \ln x]_1^e$$

$$\int_1^e dx + \int_1^e \ln x \, dx = e \ln e - 1 \ln 1$$

$$[x]_1^e + \int_1^e \ln x \, dx = e$$

$$e - 1 + \int_1^e \ln x \, dx = e$$

$$\int_1^e \ln x \, dx = \underline{\underline{1}}$$

45. It is given that $y = \frac{e^x}{e^x + 1}$.

(a) Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_0^1 \frac{e^x}{(e^x + 1)^2} \, dx$.

SOLUTION

(a) $y = \frac{e^x}{e^x + 1}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^x + 1)(e^x) - (e^x)(e^x)}{(e^x + 1)^2} \\ &= \frac{e^{2x} + e^x - e^{2x}}{(e^x + 1)^2} \\ &= \underline{\underline{\frac{e^x}{(e^x + 1)^2}}}\end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \therefore \frac{d}{dx} \left(\frac{e^x}{e^x + 1} \right) &= \frac{e^x}{(e^x + 1)^2} \\
 \therefore \int_0^1 \frac{e^x}{(e^x + 1)^2} dx &= \left[\frac{e^x}{e^x + 1} \right]_0^1 \\
 &= \frac{e^1}{e^1 + 1} - \frac{e^0}{e^0 + 1} \\
 &= \frac{e}{e + 1} - \frac{1}{2} \\
 &= \frac{2e - (e + 1)}{2(e + 1)} \\
 &= \underline{\underline{\frac{e - 1}{2(e + 1)}}}
 \end{aligned}$$

46. It is given that $y = \ln(10^x + 8)$.

(a) Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_0^2 \frac{10^x}{10^x + 8} dx$.

SOLUTION

(a) $y = \ln(10^x + 8)$

$$\frac{dy}{dx} = \frac{10^x \ln 10}{\underline{\underline{10^x + 8}}}$$

(b) $\therefore \frac{d}{dx} \ln(10^x + 8) = \frac{10^x \ln 10}{10^x + 8}$

$$\therefore \int_0^2 \frac{10^x \ln 10}{10^x + 8} dx = [\ln(10^x + 8)]_0^2$$

$$\ln 10 \int_0^2 \frac{10^x}{10^x + 8} dx = \ln(10^2 + 8) - \ln(10^0 + 8)$$

$$\int_0^2 \frac{10^x}{10^x + 8} dx = \frac{\ln 108 - \ln 9}{\ln 10}$$

$$= \underline{\underline{\frac{\ln 12}{\ln 10}}}$$

47. It is given that $k > 1$.

(a) Express the value of $\int_1^k (6x+2)dx$ in terms of k .

(b) If $\int_1^k (6x+2)dx = 51$, find the value of k .

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_1^k (6x+2)dx &= [3x^2 + 2x]_1^k \\ &= (3k^2 + 2k) - [3(1)^2 + 2(1)] \\ &= \underline{\underline{3k^2 + 2k - 5}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^k (6x+2)dx &= 51 \\ 3k^2 + 2k - 5 &= 51 \\ 3k^2 + 2k - 56 &= 0 \\ (k-4)(3k+14) &= 0 \\ k &= \underline{\underline{4}} \quad \text{or} \quad k = -\frac{14}{3} \text{ (rejected)} \end{aligned}$$

48. It is given that $k < 3$.

(a) Express the value of $\int_k^3 (5-4x)dx$ in terms of k .

(b) If $\int_k^3 (5-4x)dx = 15$, find the value of k .

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_k^3 (5-4x)dx &= [5x - 2x^2]_k^3 \\ &= [5(3) - 2(3)^2] - (5k - 2k^2) \\ &= \underline{\underline{2k^2 - 5k - 3}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_k^3 (5-4x)dx &= 15 \\ 2k^2 - 5k - 3 &= 15 \\ 2k^2 - 5k - 18 &= 0 \\ (k+2)(2k-9) &= 0 \\ k &= \underline{\underline{-2}} \quad \text{or} \quad k = \frac{9}{2} \text{ (rejected)} \end{aligned}$$

Level 3

49. It is given that $y = x\sqrt{4-x^2}$.

(a) Prove that $\frac{dy}{dx} = \frac{4-2x^2}{\sqrt{4-x^2}}$.

(b) Hence prove that $\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = 2 \int_0^1 \frac{1}{\sqrt{4-x^2}} dx - \frac{\sqrt{3}}{2}$.

 PROOF

(a) $y = x\sqrt{4-x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{4-x^2} + x \cdot \frac{-2x}{2\sqrt{4-x^2}} \\ &= \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} \\ &= \frac{4-x^2-x^2}{\sqrt{4-x^2}} \\ &= \frac{4-2x^2}{\sqrt{4-x^2}}\end{aligned}$$

(b) $\therefore \frac{d}{dx}(x\sqrt{4-x^2}) = \frac{4-2x^2}{\sqrt{4-x^2}}$

$$\therefore \int_0^1 \frac{4-2x^2}{\sqrt{4-x^2}} dx = [x\sqrt{4-x^2}]_0^1$$

$$4 \int_0^1 \frac{1}{\sqrt{4-x^2}} dx - 2 \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = (1)\sqrt{4-1^2} - (0)\sqrt{4-0^2}$$

$$2 \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = 4 \int_0^1 \frac{1}{\sqrt{4-x^2}} dx - \sqrt{3}$$

$$\int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = 2 \int_0^1 \frac{1}{\sqrt{4-x^2}} dx - \frac{\sqrt{3}}{2}$$

50. It is given that $f(x) = \begin{cases} 2x & (\text{where } 0 \leq x \leq 2) \\ x+2 & (\text{where } x > 2) \end{cases}$. Evaluate $\int_0^5 f(x) dx$.

SOLUTION

$$\begin{aligned} \int_0^5 f(x) dx &= \int_0^2 f(x) dx + \int_2^5 f(x) dx \\ &= \int_0^2 2x dx + \int_2^5 (x+2) dx \\ &= [x^2]_0^2 + \left[\frac{x^2}{2} + 2x \right]_2^5 \\ &= (2^2 - 0^2) + \left\{ \left[\frac{5^2}{2} + 2(5) \right] - \left[\frac{2^2}{2} + 2(2) \right] \right\} \\ &= \underline{\underline{\frac{41}{2}}} \end{aligned}$$

51. It is given that $f(x) = \begin{cases} 3x^2 + 2e^x & (\text{where } x \leq 0) \\ 4x + 2 & (\text{where } x > 0) \end{cases}$. Evaluate $\int_{-2}^4 f(x) dx$.

SOLUTION

$$\begin{aligned} \int_{-2}^4 f(x) dx &= \int_{-2}^0 f(x) dx + \int_0^4 f(x) dx \\ &= \int_{-2}^0 (3x^2 + 2e^x) dx + \int_0^4 (4x + 2) dx \\ &= [x^3 + 2e^x]_{-2}^0 + [2x^2 + 2x]_0^4 \\ &= \{(0^3 + 2e^0) - [(-2)^3 + 2e^{-2}]\} + \{[2(4)^2 + 2(4)] - [2(0)^2 + 2(0)]\} \\ &= (10 - 2e^{-2}) + 40 \\ &= \underline{\underline{50 - 2e^{-2}}} \end{aligned}$$

52. Let $f(x)$ be a function defined on $x \geq 0$. It is given that $f'(x) = e^{4bx} + ae^{2bx} + 3$, where a and b are constants, $f'(0) = 8$ and $f'(1) = 9.7$.

(a) Find the values of a and b .

(b) Using the values of a and b obtained in (a), evaluate $\int_0^1 f'(x) dx$.

(c) If $f(0) = 5$, find $f(1)$.

(Give your answers correct to 2 significant figures if necessary.)

SOLUTION

$$(a) \quad \because f'(0) = 8$$

$$\therefore e^{4b(0)} + ae^{2b(0)} + 3 = 8$$

$$1 + a(1) + 3 = 8$$

$$a = \underline{\underline{4}}$$

$$\therefore f'(1) = 9.7$$

$$\therefore e^{4b(1)} + 4e^{2b(1)} + 3 = 9.7$$

$$(e^{2b})^2 + 4(e^{2b}) - 6.7 = 0$$

$$\therefore e^{2b} = \frac{-4 \pm \sqrt{4^2 - 4(1)(-6.7)}}{2(1)}$$

$$2b = \ln \frac{-4 + \sqrt{42.8}}{2} \quad \text{or} \quad \ln \frac{-4 - \sqrt{42.8}}{2} \quad (\text{rejected})$$

$$b = \frac{1}{2} \ln \frac{-4 + \sqrt{42.8}}{2}$$

$$= \underline{\underline{0.12}} \quad (\text{corr. to 2 sig. fig.})$$

$$(b) \quad \text{From (a), } f'(x) = e^{4(0.12)x} + 4e^{2(0.12)x} + 3$$

$$= e^{0.48x} + 4e^{0.24x} + 3$$

$$\int_0^1 f'(x) dx = \int_0^1 (e^{0.48x} + 4e^{0.24x} + 3) dx$$

$$= \left[\frac{(e^{0.48})^x}{\ln e^{0.48}} + \frac{4(e^{0.24})^x}{\ln e^{0.24}} + 3x \right]_0^1$$

$$= \left[\frac{e^{0.48x}}{0.48} + \frac{4e^{0.24x}}{0.24} + 3x \right]_0^1$$

$$= \left[\frac{e^{0.48(1)}}{0.48} + \frac{4e^{0.24(1)}}{0.24} + 3(1) \right] - \left[\frac{e^{0.48(0)}}{0.48} + \frac{4e^{0.24(0)}}{0.24} + 3(0) \right]$$

$$= \underline{\underline{8.8}} \quad (\text{corr. to 2 sig. fig.})$$

$$(c) \quad f(1) - f(0) = \int_0^1 f'(x) dx$$

$$f(1) = \int_0^1 f'(x) dx + f(0)$$

$$= 8.80 + 5$$

$$= \underline{\underline{14}} \quad (\text{corr. to 2 sig. fig.})$$

EXERCISE 8B

Level 1

53. Evaluate $\int_0^1 (2-3x)^4 dx$.

 SOLUTION

$$\begin{aligned}\int_0^1 (2-3x)^4 dx &= -\frac{1}{3} \int_0^1 (2-3x)^4 \cdot (-3) dx \\&= -\frac{1}{3} \int_0^1 (2-3x)^4 d(2-3x) \\&= -\frac{1}{3} \left[\frac{(2-3x)^5}{5} \right]_0^1 \\&= -\frac{1}{3} \left\{ \frac{[2-3(1)]^5}{5} - \frac{[2-3(0)]^5}{5} \right\} \\&= \underline{\underline{\frac{11}{5}}}\end{aligned}$$

54. Evaluate $\int_1^2 \frac{dx}{(4x-1)^3}$.

 SOLUTION

$$\begin{aligned}\int_1^2 \frac{dx}{(4x-1)^3} &= \frac{1}{4} \int_1^2 \frac{4dx}{(4x-1)^3} \\&= \frac{1}{4} \int_1^2 (4x-1)^{-3} d(4x-1) \\&= \frac{1}{4} \left[\frac{(4x-1)^{-2}}{-2} \right]_1^2 \\&= \frac{1}{4} \left\{ \frac{[4(2)-1]^{-2}}{-2} - \frac{[4(1)-1]^{-2}}{-2} \right\} \\&= \underline{\underline{\frac{5}{441}}}\end{aligned}$$

55. Evaluate $\int_3^{11} \sqrt{2x+3} \, dx$.

SOLUTION

$$\begin{aligned}
 \int_3^{11} \sqrt{2x+3} \, dx &= \frac{1}{2} \int_3^{11} \sqrt{2x+3} \cdot 2 \, dx \\
 &= \frac{1}{2} \int_3^{11} (2x+3)^{\frac{1}{2}} d(2x+3) \\
 &= \frac{1}{2} \left[\frac{2(2x+3)^{\frac{3}{2}}}{\frac{3}{2}} \right]_3^{11} \\
 &= \frac{1}{2} \left\{ \frac{2[2(11)+3]^{\frac{3}{2}}}{3} - \frac{2[2(3)+3]^{\frac{3}{2}}}{3} \right\} \\
 &= \underline{\underline{\frac{98}{3}}}
 \end{aligned}$$

56. Evaluate $\int_2^{14} \frac{dx}{2x-1}$.

SOLUTION

$$\begin{aligned}
 \int_2^{14} \frac{dx}{2x-1} &= \frac{1}{2} \int_2^{14} \frac{2dx}{2x-1} \\
 &= \frac{1}{2} \int_2^{14} \frac{d(2x-1)}{2x-1} \\
 &= \frac{1}{2} [\ln|2x-1|]_2^{14} \\
 &= \frac{1}{2} \{ \ln[2(14)-1] - \ln[2(2)-1] \} \\
 &= \frac{1}{2} (\ln 27 - \ln 3) \\
 &= \frac{1}{2} \ln 9 \\
 &= \underline{\underline{\ln 3}}
 \end{aligned}$$

57. Evaluate $\int_0^2 e^{2x+1} dx$.

SOLUTION

$$\begin{aligned}\int_0^2 e^{2x+1} dx &= \frac{1}{2} \int_0^2 e^{2x+1} \cdot 2 dx \\&= \frac{1}{2} \int_0^2 e^{2x+1} d(2x+1) \\&= \frac{1}{2} [e^{2x+1}]_0^2 \\&= \frac{1}{2} [e^{2(2)+1} - e^{2(0)+1}] \\&= \frac{e^5 - e}{2}\end{aligned}$$

58. Evaluate $\int_2^3 2x(x^2 - 4)^4 dx$.

SOLUTION

$$\begin{aligned}\int_2^3 2x(x^2 - 4)^4 dx &= \int_2^3 (x^2 - 4)^4 \cdot 2x dx \\&= \int_2^3 (x^2 - 4)^4 d(x^2 - 4) \\&= \left[\frac{(x^2 - 4)^5}{5} \right]_2^3 \\&= \frac{(3^2 - 4)^5}{5} - \frac{(2^2 - 4)^5}{5} \\&= \underline{\underline{625}}\end{aligned}$$

59. Evaluate $\int_1^3 \frac{x dx}{3x^2 + 1}$.

SOLUTION

$$\begin{aligned}
 \int_1^3 \frac{x dx}{3x^2 + 1} &= \frac{1}{6} \int_1^3 \frac{6x dx}{3x^2 + 1} \\
 &= \frac{1}{6} \int_1^3 \frac{d(3x^2 + 1)}{3x^2 + 1} \\
 &= \frac{1}{6} [\ln(3x^2 + 1)]_1^3 \\
 &= \frac{1}{6} \{ \ln[3(3)^2 + 1] - \ln[3(1)^2 + 1] \} \\
 &= \frac{1}{6} (\ln 28 - \ln 4) \\
 &= \underline{\underline{\frac{1}{6} \ln 7}}
 \end{aligned}$$

60. Evaluate $\int_3^{10} \frac{2x + 3}{\sqrt{x^2 + 3x - 2}} dx$.

SOLUTION

$$\begin{aligned}
 \int_3^{10} \frac{2x + 3}{\sqrt{x^2 + 3x - 2}} dx &= \int_3^{10} (x^2 + 3x - 2)^{-\frac{1}{2}} d(x^2 + 3x - 2) \\
 &= [2(x^2 + 3x - 2)^{\frac{1}{2}}]_3^{10} \\
 &= 2[10^2 + 3(10) - 2]^{\frac{1}{2}} - 2[3^2 + 3(3) - 2]^{\frac{1}{2}} \\
 &= 2\sqrt{128} - 8 \\
 &= \underline{\underline{16\sqrt{2} - 8}}
 \end{aligned}$$

61. Evaluate $\int_4^8 \frac{x+3}{16-6x-x^2} dx$.

SOLUTION

$$\begin{aligned}\int_4^8 \frac{x+3}{16-6x-x^2} dx &= -\frac{1}{2} \int_4^8 \frac{-2x-6}{16-6x-x^2} dx \\&= -\frac{1}{2} \int_4^8 \frac{1}{16-6x-x^2} d(16-6x-x^2) \\&= -\frac{1}{2} [\ln|16-6x-x^2|]_4^8 \\&= -\frac{1}{2} [\ln|16-6(8)-8^2| - \ln|16-6(4)-4^2|] \\&= -\frac{1}{2} (\ln 96 - \ln 24) \\&= -\frac{1}{2} \ln 4 \\&= \underline{\underline{-\ln 2}}\end{aligned}$$

62. Evaluate $\int_0^1 (x+1)e^{x^2+2x-3} dx$.

SOLUTION

$$\begin{aligned}\int_0^1 (x+1)e^{x^2+2x-3} dx &= \frac{1}{2} \int_0^1 e^{x^2+2x-3} \cdot (2x+2) dx \\&= \frac{1}{2} \int_0^1 e^{x^2+2x-3} d(x^2+2x-3) \\&= \frac{1}{2} [e^{x^2+2x-3}]_0^1 \\&= \frac{1}{2} [e^{1^2+2(1)-3} - e^{0^2+2(0)-3}] \\&= \frac{1}{2} (e^0 - e^{-3}) \\&= \underline{\underline{\frac{1}{2}(1-e^{-3})}}\end{aligned}$$

63. Evaluate $\int_0^2 \frac{x^2}{x^3 + 4} dx$.

SOLUTION

$$\begin{aligned}
 \int_0^2 \frac{x^2}{x^3 + 4} dx &= \frac{1}{3} \int_0^2 \frac{1}{x^3 + 4} \cdot 3x^2 dx \\
 &= \frac{1}{3} \int_0^2 \frac{1}{x^3 + 4} d(x^3 + 4) \\
 &= \frac{1}{3} [\ln|x^3 + 4|]_0^2 \\
 &= \frac{1}{3} [\ln(2^3 + 4) - \ln(0^3 + 4)] \\
 &= \frac{1}{3} (\ln 12 - \ln 4) \\
 &= \underline{\underline{\frac{1}{3} \ln 3}}
 \end{aligned}$$

64. Evaluate $\int_4^9 \frac{(2\sqrt{x} - 3)^4}{\sqrt{x}} dx$.

SOLUTION

$$\begin{aligned}
 \int_4^9 \frac{(2\sqrt{x} - 3)^4}{\sqrt{x}} dx &= \int_4^9 (2\sqrt{x} - 3)^4 \cdot x^{-\frac{1}{2}} dx \\
 &= \int_4^9 (2\sqrt{x} - 3)^4 d(2\sqrt{x} - 3) \\
 &= \left[\frac{1}{5} (2\sqrt{x} - 3)^5 \right]_4^9 \\
 &= \frac{1}{5} (2\sqrt{9} - 3)^5 - \frac{1}{5} (2\sqrt{4} - 3)^5 \\
 &= \underline{\underline{\frac{242}{5}}}
 \end{aligned}$$

65. Evaluate $\int_1^9 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx$.

SOLUTION

$$\begin{aligned}\int_1^9 \frac{2e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^9 4e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx \\&= \int_1^9 4e^{\sqrt{x}} d(\sqrt{x}) \\&= [4e^{\sqrt{x}}]_1^9 \\&= 4e^{\sqrt{9}} - 4e^{\sqrt{1}} \\&= \underline{\underline{4e^3 - 4e}}\end{aligned}$$

66. Evaluate $\int_0^{\ln 2} e^x (e^x - 1)^{\frac{3}{2}} dx$.

SOLUTION

$$\begin{aligned}\int_0^{\ln 2} e^x (e^x - 1)^{\frac{3}{2}} dx &= \int_0^{\ln 2} (e^x - 1)^{\frac{3}{2}} \cdot e^x dx \\&= \int_0^{\ln 2} (e^x - 1)^{\frac{3}{2}} d(e^x - 1) \\&= \left[\frac{2(e^x - 1)^{\frac{5}{2}}}{5} \right]_0^{\ln 2} \\&= \frac{2(e^{\ln 2} - 1)^{\frac{5}{2}}}{5} - \frac{2(e^0 - 1)^{\frac{5}{2}}}{5} \\&= \underline{\underline{\frac{2}{5}}}\end{aligned}$$

67. Evaluate $\int_1^{e^4} \frac{\sqrt{\ln x}}{x} dx$.

SOLUTION

$$\begin{aligned}
 \int_1^{e^4} \frac{\sqrt{\ln x}}{x} dx &= \int_1^{e^4} (\ln x)^{\frac{1}{2}} \cdot \frac{1}{x} dx \\
 &= \int_1^{e^4} (\ln x)^{\frac{1}{2}} d(\ln x) \\
 &= \left[\frac{2(\ln x)^{\frac{3}{2}}}{3} \right]_1^{e^4} \\
 &= \frac{2(\ln e^4)^{\frac{3}{2}}}{3} - \frac{2(\ln 1)^{\frac{3}{2}}}{3} \\
 &= \frac{16}{3}
 \end{aligned}$$

68. Evaluate $\int_1^e \frac{3 + \ln x^2}{x} dx$.

SOLUTION

$$\begin{aligned}
 \int_1^e \frac{3 + \ln x^2}{x} dx &= \int_1^e \frac{3 + 2 \ln x}{x} dx \\
 &= \int_1^e \left(\frac{3}{x} + \frac{2 \ln x}{x} \right) dx \\
 &= \int_1^e \frac{3 dx}{x} + \int_1^e 2 \ln x \cdot \frac{1}{x} dx \\
 &= \int_1^e \frac{3 dx}{x} + \int_1^e 2 \ln x d(\ln x) \\
 &= [3 \ln |x|]_1^e + [(\ln x)^2]_1^e \\
 &= 3(\ln e - \ln 1) + [(\ln e)^2 - (\ln 1)^2] \\
 &= 4
 \end{aligned}$$

69. Prove that $\int_5^{20} \frac{\ln \sqrt{2x}}{x} dx = 2(\ln 2)^2 + (\ln 2)(\ln 5)$.

PROOF

$$\begin{aligned} \int_5^{20} \frac{\ln \sqrt{2x}}{x} dx &= \int_5^{20} \frac{\frac{1}{2} \ln 2x}{x} dx \\ &= \frac{1}{2} \int_5^{20} \ln 2x \cdot \frac{1}{x} dx \\ &= \frac{1}{2} \int_5^{20} (\ln 2x) d(\ln 2x) \\ &= \frac{1}{2} \left[\frac{1}{2} (\ln 2x)^2 \right]_5^{20} \\ &= \frac{1}{4} \{ [\ln 2(20)]^2 - [\ln 2(5)]^2 \} \\ &= \frac{1}{4} [(\ln 40)^2 - (\ln 10)^2] \\ &= \frac{1}{4} (\ln 40 - \ln 10)(\ln 40 + \ln 10) \\ &= \frac{1}{4} \ln 4 \cdot \ln 400 \\ &= \frac{1}{4} (2 \ln 2)(2 \ln 20) \\ &= \ln 2 \cdot \ln 20 \\ &= (\ln 2)(2 \ln 2 + \ln 5) \\ &= 2(\ln 2)^2 + (\ln 2)(\ln 5) \end{aligned}$$

Level 2

70. Evaluate $\int_{10}^{100} \frac{\log x^2}{x} dx$.

SOLUTION

$$\begin{aligned} \int_{10}^{100} \frac{\log x^2}{x} dx &= \ln 10 \int_{10}^{100} \frac{2 \log x}{x \ln 10} dx \\ &= 2 \ln 10 \int_{10}^{100} \log x d(\log x) \\ &= 2(\ln 10) \left[\frac{1}{2} (\log x)^2 \right]_{10}^{100} \\ &= 2(\ln 10) \left[\frac{1}{2} (\log 100)^2 - \frac{1}{2} (\log 10)^2 \right] \\ &= 2(\ln 10) \left(2 - \frac{1}{2} \right) \\ &= \underline{\underline{3 \ln 10}} \end{aligned}$$

71. Evaluate $\int_0^1 x^3 3^{x^4} dx$.

SOLUTION

$$\begin{aligned}\int_0^1 x^3 3^{x^4} dx &= \frac{1}{4} \int_0^1 3^{x^4} \cdot 4x^3 dx \\ &= \frac{1}{4} \int_0^1 3^{x^4} d(x^4) \\ &= \frac{1}{4} \left[\frac{3^{x^4}}{\ln 3} \right]_0^1 \\ &= \frac{1}{4} \left(\frac{3^{1^4}}{\ln 3} - \frac{3^{0^4}}{\ln 3} \right) \\ &= \frac{1}{2 \ln 3}\end{aligned}$$

72. Evaluate $\int_1^2 \frac{10^x}{10^x + 5} dx$.

SOLUTION

$$\begin{aligned}\int_1^2 \frac{10^x}{10^x + 5} dx &= \frac{1}{\ln 10} \int_1^2 \frac{10^x \ln 10}{10^x + 5} dx \\ &= \frac{1}{\ln 10} \int_1^2 \frac{d(10^x + 5)}{10^x + 5} \\ &= \frac{1}{\ln 10} [\ln(10^x + 5)]_1^2 \\ &= \frac{1}{\ln 10} [\ln(10^2 + 5) - \ln(10^1 + 5)] \\ &= \frac{1}{\ln 10} (\ln 105 - \ln 15) \\ &= \frac{\ln 7}{\ln 10}\end{aligned}$$

73. Evaluate $\int_{\frac{1}{2}}^1 x(2x-1)^{99} dx$.

SOLUTION

Let $u = 2x - 1$. Then $du = 2dx$.

When $x = \frac{1}{2}$, $u = 0$;

when $x = 1$, $u = 1$.

$$\begin{aligned}
 \therefore \int_{\frac{1}{2}}^1 x(2x-1)^{99} dx &= \int_{\frac{1}{2}}^1 \frac{1}{2} x(2x-1)^{99} \cdot 2 dx \\
 &= \int_0^1 \frac{1}{2} \left(\frac{u+1}{2} \right) \cdot u^{99} du \\
 &= \int_0^1 \left(\frac{u^{100}}{4} + \frac{u^{99}}{4} \right) du \\
 &= \left[\frac{u^{101}}{404} + \frac{u^{100}}{400} \right]_0^1 \\
 &= \left(\frac{1^{101}}{404} + \frac{1^{100}}{400} \right) - \left(\frac{0^{101}}{404} + \frac{0^{100}}{400} \right) \\
 &= \frac{201}{40\,400}
 \end{aligned}$$

74. Evaluate $\int_0^3 (x+2)\sqrt{4-x} \, dx$.

 **SOLUTION**

Let $u = 4 - x$. Then $du = -dx$.

When $x = 0$, $u = 4$;

when $x = 3$, $u = 1$.

$$\begin{aligned}
 \therefore \int_0^3 (x+2)\sqrt{4-x} \, dx &= -\int_4^1 (x+2)\sqrt{4-x} \cdot (-1) dx \\
 &= -\int_4^1 (6-u)\sqrt{u} \, du \\
 &= \int_1^4 (6u^{\frac{1}{2}} - u^{\frac{3}{2}}) \, du \\
 &= \left[4u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \right]_1^4 \\
 &= \left[4(4)^{\frac{3}{2}} - \frac{2}{5}(4)^{\frac{5}{2}} \right] - \left[4(1)^{\frac{3}{2}} - \frac{2}{5}(1)^{\frac{5}{2}} \right] \\
 &= \frac{78}{5}
 \end{aligned}$$

75. Evaluate $\int_2^7 \frac{x+3}{\sqrt{x+2}} \, dx$.

 **SOLUTION**

Let $u = x + 2$. Then $du = dx$.

When $x = 2$, $u = 4$;

when $x = 7$, $u = 9$.

$$\begin{aligned}
\therefore \int_2^7 \frac{x+3}{\sqrt{x+2}} dx &= \int_4^9 \frac{u+1}{\sqrt{u}} du \\
&= \int_4^9 \left(\sqrt{u} + \frac{1}{\sqrt{u}} \right) du \\
&= \left[\frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \right]_4^9 \\
&= \left[\frac{2}{3} (9)^{\frac{3}{2}} + 2(9)^{\frac{1}{2}} \right] - \left[\frac{2}{3} (4)^{\frac{3}{2}} + 2(4)^{\frac{1}{2}} \right] \\
&= \underline{\underline{\frac{44}{3}}}
\end{aligned}$$

76. Evaluate $\int_0^1 x^3 \sqrt{1-x^2} dx$.

SOLUTION

Let $u = 1 - x^2$. Then $du = -2x dx$.

When $u = 1 - x^2$, $x^2 = 1 - u$.

When $x = 0$, $u = 1$;

when $x = 1$, $u = 0$.

$$\begin{aligned}
\therefore \int_0^1 x^3 \sqrt{1-x^2} dx &= -\frac{1}{2} \int_1^0 x^2 \sqrt{1-x^2} \cdot (-2x) dx \\
&= -\frac{1}{2} \int_1^0 (1-u) \sqrt{u} du \\
&= \frac{1}{2} \int_0^1 (u^{\frac{1}{2}} - u^{\frac{3}{2}}) du \\
&= \frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} - \frac{2}{5} u^{\frac{5}{2}} \right]_0^1 \\
&= \frac{1}{2} \left\{ \left[\frac{2}{3} (1)^{\frac{3}{2}} - \frac{2}{5} (1)^{\frac{5}{2}} \right] - \left[\frac{2}{3} (0)^{\frac{3}{2}} - \frac{2}{5} (0)^{\frac{5}{2}} \right] \right\} \\
&= \underline{\underline{\frac{2}{15}}}
\end{aligned}$$

77. Evaluate $\int_0^4 \frac{x^3}{\sqrt{5x^2+1}} dx$.

SOLUTION

Let $u = 5x^2 + 1$. Then $du = 10x dx$.

When $u = 5x^2 + 1$, $x^2 = \frac{u-1}{5}$.

When $x = 0$, $u = 1$;

when $x = 4$, $u = 81$.

$$\begin{aligned} \therefore \int_0^4 \frac{x^3}{\sqrt{5x^2+1}} dx &= \int_0^4 \frac{x^2}{10\sqrt{5x^2+1}} \cdot 10x dx \\ &= \int_1^{81} \frac{u-1}{5} \cdot \frac{1}{10\sqrt{u}} du \\ &= \frac{1}{50} \int_1^{81} \left(\sqrt{u} - \frac{1}{\sqrt{u}} \right) du \\ &= \frac{1}{50} \left[\frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right]_1^{81} \\ &= \frac{1}{50} \left\{ \left[\frac{2}{3} (81)^{\frac{3}{2}} - 2(81)^{\frac{1}{2}} \right] - \left[\frac{2}{3} (1)^{\frac{3}{2}} - 2(1)^{\frac{1}{2}} \right] \right\} \\ &= \frac{704}{75} \end{aligned}$$

78. Evaluate $\int_9^{16} \frac{\sqrt{x}}{1+\sqrt{x}} dx$.

SOLUTION

Let $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} dx$.

When $u = 1 + \sqrt{x}$, $x = (u-1)^2$.

When $x = 9$, $u = 4$;

when $x = 16$, $u = 5$.

$$\begin{aligned}
\therefore \int_9^{16} \frac{\sqrt{x}}{1+\sqrt{x}} dx &= \int_9^{16} \frac{2x}{1+\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} dx \\
&= \int_4^5 \frac{2(u-1)^2}{u} du \\
&= \int_4^5 \frac{2u^2 - 4u + 2}{u} du \\
&= \int_4^5 \left(2u - 4 + \frac{2}{u}\right) du \\
&= [u^2 - 4u + 2\ln|u|]_4^5 \\
&= [5^2 - 4(5) + 2\ln 5] - [4^2 - 4(4) + 2\ln 4] \\
&= (5 + 2\ln 5) - 2\ln 4 \\
&= \underline{\underline{5 + 2\ln 5 - 4\ln 2}}
\end{aligned}$$

79. Evaluate $\int_0^4 \frac{\sqrt{x}-4}{\sqrt{x}+4} dx$.

SOLUTION

Let $u = \sqrt{x} + 4$. Then $du = \frac{1}{2\sqrt{x}} dx$.

When $u = \sqrt{x} + 4$, $\sqrt{x} = u - 4$.

When $x = 0$, $u = 4$;

when $x = 4$, $u = 6$.

$$\begin{aligned}
\therefore \int_0^4 \frac{\sqrt{x}-4}{\sqrt{x}+4} dx &= \int_4^6 \frac{2\sqrt{x}(\sqrt{x}-4)}{\sqrt{x}+4} \cdot \frac{1}{2\sqrt{x}} dx \\
&= \int_4^6 \frac{2(u-4)(u-4-4)}{u} du \\
&= \int_4^6 \frac{2u^2 - 24u + 64}{u} du \\
&= \int_4^6 \left(2u - 24 + \frac{64}{u}\right) du \\
&= [u^2 - 24u + 64\ln|u|]_4^6 \\
&= [6^2 - 24(6) + 64\ln 6] - [4^2 - 24(4) + 64\ln 4] \\
&= (-108 + 64\ln 6) - (-80 + 64\ln 4) \\
&= \underline{\underline{-28 + 64\ln \frac{3}{2}}}
\end{aligned}$$

80. Evaluate $\int_0^1 \frac{x[\ln(1+5x^2)]^2}{1+5x^2} dx$.

[Hint: Let $u = \ln(1+5x^2)$.]

SOLUTION

Let $u = \ln(1+5x^2)$. Then $du = \frac{10x}{1+5x^2} dx$.

When $x=0$, $u=0$;

when $x=1$, $u=\ln 6$.

$$\begin{aligned} \therefore \int_0^1 \frac{x[\ln(1+5x^2)]^2}{1+5x^2} dx &= \int_0^1 \frac{[\ln(1+5x^2)]^2}{10} \cdot \frac{10x}{1+5x^2} dx \\ &= \int_0^{\ln 6} \frac{u^2}{10} du \\ &= \left[\frac{u^3}{30} \right]_0^{\ln 6} \\ &= \frac{(\ln 6)^3}{30} - \frac{0^3}{30} \\ &= \frac{(\ln 6)^3}{30} \end{aligned}$$

81. Evaluate $\int_0^1 e^x e^{e^x} dx$.

SOLUTION

Let $u = e^{e^x}$. Then $du = e^{e^x} e^x dx$.

When $x=0$, $u=e$;

when $x=1$, $u=e^e$.

$$\begin{aligned} \therefore \int_0^1 e^x e^{e^x} dx &= \int_e^{e^e} du \\ &= [u]_e^{e^e} \\ &= \underline{\underline{e^e - e}} \end{aligned}$$

82. Evaluate $\int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}} dx$.

SOLUTION

Let $u = e^x - e^{-x}$. Then $du = (e^x + e^{-x}) dx$.

When $x = \ln 2$, $u = \frac{3}{2}$;

when $x = \ln 4$, $u = \frac{15}{4}$.

$$\begin{aligned} \therefore \int_{\ln 2}^{\ln 4} \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}} dx &= \int_{\frac{3}{2}}^{\frac{15}{4}} \frac{du}{\sqrt{u}} \\ &= [2\sqrt{u}]_{\frac{3}{2}}^{\frac{15}{4}} \\ &= 2\sqrt{\frac{15}{4}} - 2\sqrt{\frac{3}{2}} \\ &= \underline{\underline{\sqrt{15} - \sqrt{6}}} \end{aligned}$$

83. Evaluate $\int_0^{\ln 2} \frac{e^x dx}{(e^x + 1) \ln(e^x + 1)}$.

[Hint: Let $u = \ln(e^x + 1)$.]

SOLUTION

Let $u = \ln(e^x + 1)$. Then $du = \frac{e^x dx}{e^x + 1}$.

When $x = 0$, $u = \ln 2$;

when $x = \ln 2$, $u = \ln 3$.

$$\begin{aligned} \therefore \int_0^{\ln 2} \frac{e^x dx}{(e^x + 1) \ln(e^x + 1)} &= \int_{\ln 2}^{\ln 3} \frac{du}{u} \\ &= [\ln|u|]_{\ln 2}^{\ln 3} \\ &= \ln(\ln 3) - \ln(\ln 2) \\ &= \underline{\underline{\ln\left(\frac{\ln 3}{\ln 2}\right)}} \end{aligned}$$

84. Evaluate $\int_{\ln 2}^{\ln 5} \frac{(e^x + 5)e^x}{e^x + 1} dx$.

SOLUTION

Let $u = e^x + 1$. Then $du = e^x dx$.

When $x = \ln 2$, $u = 3$;

when $x = \ln 5$, $u = 6$.

$$\begin{aligned} \therefore \int_{\ln 2}^{\ln 5} \frac{(e^x + 5)e^x}{e^x + 1} dx &= \int_3^6 \frac{u + 4}{u} du \\ &= \int_3^6 \left(1 + \frac{4}{u}\right) du \\ &= [u + 4 \ln |u|]_3^6 \\ &= (6 + 4 \ln 6) - (3 + 4 \ln 3) \\ &= \underline{\underline{3 + 4 \ln 2}} \end{aligned}$$

85. (a) Evaluate $\int_0^3 \frac{e^x}{1 + 3e^x} dx$.

(b) Hence, or otherwise, evaluate $\int_0^3 \frac{1}{1 + 3e^x} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_0^3 \frac{e^x}{1 + 3e^x} dx &= \frac{1}{3} \int_0^3 \frac{3e^x}{1 + 3e^x} dx \\ &= \frac{1}{3} \int_0^3 \frac{1}{1 + 3e^x} d(1 + 3e^x) \\ &= \frac{1}{3} [\ln |1 + 3e^x|]_0^3 \\ &= \frac{1}{3} [\ln(1 + 3e^3) - \ln(1 + 3e^0)] \\ &= \underline{\underline{\frac{1}{3} \ln \frac{1 + 3e^3}{4}}} \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_0^3 \frac{1}{1+3e^x} dx &= \int_0^3 \frac{(1+3e^x) - 3e^x}{1+3e^x} dx \\
 &= \int_0^3 \left(1 - \frac{3e^x}{1+3e^x}\right) dx \\
 &= \int_0^3 dx - 3 \int_0^3 \frac{e^x}{1+3e^x} dx \\
 &= [x]_0^3 - 3 \left(\frac{1}{3} \ln \frac{1+3e^3}{4} \right) \quad [\text{From the result of (a)}] \\
 &= \underline{\underline{3 - \ln \frac{1+3e^3}{4}}}
 \end{aligned}$$

86. (a) Evaluate $\int_4^8 \frac{x-4}{x^2-8x+32} dx$.

(b) Hence, or otherwise, evaluate $\int_4^8 \frac{x^2 dx}{x^2-8x+32}$.

 **SOLUTION**

$$\begin{aligned}
 \text{(a)} \quad \int_4^8 \frac{x-4}{x^2-8x+32} dx &= \frac{1}{2} \int_4^8 \frac{2x-8}{x^2-8x+32} dx \\
 &= \frac{1}{2} \int_4^8 \frac{1}{x^2-8x+32} d(x^2-8x+32) \\
 &= \frac{1}{2} [\ln(x^2-8x+32)]_4^8 \\
 &= \frac{1}{2} \{ \ln[8^2-8(8)+32] - \ln[4^2-8(4)+32] \} \\
 &= \frac{1}{2} (\ln 32 - \ln 16) \\
 &= \underline{\underline{\frac{1}{2} \ln 2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_4^8 \frac{x^2 dx}{x^2-8x+32} &= \int_4^8 \frac{(x^2-8x+32) + (8x-32)}{x^2-8x+32} dx \\
 &= \int_4^8 \left(1 + \frac{8x-32}{x^2-8x+32}\right) dx \\
 &= \int_4^8 dx + 8 \int_4^8 \frac{x-4}{x^2-8x+32} dx \\
 &= [x]_4^8 + 8 \left(\frac{1}{2} \ln 2 \right) \quad [\text{From the result of (a)}] \\
 &= (8-4) + 4 \ln 2 \\
 &= \underline{\underline{4 + 4 \ln 2}}
 \end{aligned}$$

87. (a) Evaluate $\int_1^4 \frac{e^{3x} + 1}{e^{3x} + 3x + 1} dx$.

(b) Hence, or otherwise, evaluate $\int_1^4 \frac{x dx}{e^{3x} + 3x + 1}$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_1^4 \frac{e^{3x} + 1}{e^{3x} + 3x + 1} dx &= \int_1^4 \frac{3e^{3x} + 3}{3(e^{3x} + 3x + 1)} dx \\ &= \frac{1}{3} \int_1^4 \frac{1}{e^{3x} + 3x + 1} d(e^{3x} + 3x + 1) \\ &= \frac{1}{3} [\ln |e^{3x} + 3x + 1|]_1^4 \\ &= \frac{1}{3} \{ \ln[e^{3(4)} + 3(4) + 1] - \ln[e^{3(1)} + 3(1) + 1] \} \\ &= \frac{1}{3} \ln \frac{e^{12} + 13}{e^3 + 4} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_1^4 \frac{x dx}{e^{3x} + 3x + 1} &= \int_1^4 \frac{3x dx}{3(e^{3x} + 3x + 1)} \\ &= \frac{1}{3} \int_1^4 \frac{(e^{3x} + 3x + 1) - (e^{3x} + 1)}{e^{3x} + 3x + 1} dx \\ &= \frac{1}{3} \int_1^4 \left(1 - \frac{e^{3x} + 1}{e^{3x} + 3x + 1} \right) dx \\ &= \frac{1}{3} \int_1^4 dx - \frac{1}{3} \int_1^4 \frac{e^{3x} + 1}{e^{3x} + 3x + 1} dx \\ &= \frac{1}{3} [x]_1^4 - \frac{1}{3} \left(\frac{1}{3} \ln \frac{e^{12} + 13}{e^3 + 4} \right) \quad [\text{From the result of (a)}] \\ &= \frac{1}{3} (4 - 1) - \frac{1}{9} \ln \frac{e^{12} + 13}{e^3 + 4} \\ &= 1 - \frac{1}{9} \ln \frac{e^{12} + 13}{e^3 + 4} \end{aligned}$$

88. (a) If $\frac{11}{(3x-4)(2x+1)} \equiv \frac{A}{3x-4} + \frac{B}{2x+1}$, find the values of constants A and B .

(b) Hence evaluate $\int_2^5 \frac{11}{(3x-4)(2x+1)} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \frac{11}{(3x-4)(2x+1)} &\equiv \frac{A}{3x-4} + \frac{B}{2x+1} \\ &\equiv \frac{A(2x+1) + B(3x-4)}{(3x-4)(2x+1)} \end{aligned}$$

$$\text{i.e.} \quad A(2x+1) + B(3x-4) \equiv 11$$

$$(2A+3B)x + (A-4B) \equiv 11$$

$$\therefore \begin{cases} 2A+3B=0 & \dots\dots\dots(1) \\ A-4B=11 & \dots\dots\dots(2) \end{cases}$$

$$(1) - 2 \times (2):$$

$$(2A+3B) - 2(A-4B) = 0 - 2(11)$$

$$11B = -22$$

$$B = -2$$

Substitute $B = -2$ into (1),

$$2A + 3(-2) = 0$$

$$A = 3$$

$$\therefore \underline{\underline{A=3, B=-2}}$$

$$\text{(b)} \quad \int_2^5 \frac{11}{(3x-4)(2x+1)} dx = \int_2^5 \left(\frac{3}{3x-4} + \frac{-2}{2x+1} \right) dx \quad [\text{From the result of (a)}]$$

$$= \int_2^5 \frac{1}{3x-4} \cdot 3 dx - \int_2^5 \frac{1}{2x+1} \cdot 2 dx$$

$$= \int_2^5 \frac{1}{3x-4} d(3x-4) - \int_2^5 \frac{1}{2x+1} d(2x+1)$$

$$= [\ln|3x-4|]_2^5 - [\ln|2x+1|]_2^5$$

$$= (\ln 11 - \ln 2) - (\ln 11 - \ln 5)$$

$$= \ln 5 - \ln 2$$

$$= \underline{\underline{\ln \frac{5}{2}}}$$

89. (a) If $\frac{17-2x}{(2x-3)(5-x)} \equiv \frac{A}{2x-3} + \frac{B}{5-x}$, find the values of constants A and B .

(b) Hence evaluate $\int_2^4 \frac{17-2x}{(2x-3)(5-x)} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \frac{17-2x}{(2x-3)(5-x)} &\equiv \frac{A}{2x-3} + \frac{B}{5-x} \\ &\equiv \frac{A(5-x) + B(2x-3)}{(2x-3)(5-x)} \end{aligned}$$

$$\text{i.e.} \quad A(5-x) + B(2x-3) \equiv 17-2x$$

$$(2B-A)x + (5A-3B) \equiv 17-2x$$

$$\therefore \begin{cases} 2B-A = -2 & \text{..... (1)} \\ 5A-3B = 17 & \text{..... (2)} \end{cases}$$

$$\text{From (1), } A = 2B + 2 \text{ (3)}$$

Substitute (3) into (2),

$$5(2B+2) - 3B = 17$$

$$7B = 7$$

$$B = 1$$

Substitute $B=1$ into (3),

$$A = 2(1) + 2$$

$$= 4$$

$$\therefore \underline{\underline{A=4, B=1}}$$

$$\begin{aligned} \text{(b)} \quad \int_2^4 \frac{17-2x}{(2x-3)(5-x)} dx &= \int_2^4 \left(\frac{4}{2x-3} + \frac{1}{5-x} \right) dx \\ &= \int_2^4 \frac{2}{2x-3} \cdot 2 dx - \int_2^4 \frac{1}{5-x} \cdot (-1) dx \\ &= 2 \int_2^4 \frac{1}{2x-3} d(2x-3) - \int_2^4 \frac{1}{5-x} d(5-x) \\ &= 2[\ln|2x-3|]_2^4 - [\ln|5-x|]_2^4 \\ &= 2(\ln 5 - \ln 1) - (\ln 1 - \ln 3) \\ &= \underline{\underline{2\ln 5 + \ln 3}} \end{aligned}$$

90. It is given that $k > 0$.

(a) Express the value of $\int_0^k \frac{dx}{\sqrt{x+1}}$ in terms of k .

(b) If $\int_0^k \frac{dx}{\sqrt{x+1}} = 4$, find the value of k .

SOLUTION

$$\begin{aligned}
 \text{(a)} \quad \int_0^k \frac{dx}{\sqrt{x+1}} &= \int_0^k (x+1)^{-\frac{1}{2}} d(x+1) \\
 &= [2(x+1)^{\frac{1}{2}}]_0^k \\
 &= 2(k+1)^{\frac{1}{2}} - 2(0+1)^{\frac{1}{2}} \\
 &= \underline{\underline{2\sqrt{k+1} - 2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_0^k \frac{dx}{\sqrt{x+1}} &= 4 \\
 2\sqrt{k+1} - 2 &= 4 \\
 2\sqrt{k+1} &= 6 \\
 \sqrt{k+1} &= 3 \\
 k+1 &= 9 \\
 k &= \underline{\underline{8}}
 \end{aligned}$$

91. It is given that $k > -1$.

(a) Express the value of $\int_{-1}^k \frac{dx}{4x+5}$ in terms of k .

(b) If $\int_{-1}^k \frac{dx}{4x+5} = \frac{1}{2}$, find the value of k .

SOLUTION

$$\begin{aligned}
 \text{(a)} \quad \int_{-1}^k \frac{dx}{4x+5} &= \frac{1}{4} \int_{-1}^k \frac{4dx}{4x+5} \\
 &= \frac{1}{4} \int_{-1}^k \frac{d(4x+5)}{4x+5} \\
 &= \frac{1}{4} [\ln|4x+5|]_{-1}^k \\
 &= \frac{1}{4} \{\ln(4k+5) - \ln[4(-1)+5]\} \\
 &= \underline{\underline{\frac{1}{4} \ln(4k+5)}}
 \end{aligned}$$

$$(b) \quad \int_{-1}^k \frac{dx}{4x+5} = \frac{1}{2}$$

$$\frac{1}{4} \ln(4k+5) = \frac{1}{2}$$

$$\ln(4k+5) = 2$$

$$4k+5 = e^2$$

$$k = \frac{e^2 - 5}{4}$$

92. It is given that $k < -\frac{1}{2}$.

(a) Express the value of $\int_k^{-\frac{1}{2}} \frac{dx}{e^{2x+1}}$ in terms of k .

(b) If $\int_k^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} = 1$, find the value of k .

 **SOLUTION**

$$\begin{aligned} (a) \quad \int_k^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} &= \int_k^{-\frac{1}{2}} e^{-2x-1} dx \\ &= -\frac{1}{2} \int_k^{-\frac{1}{2}} e^{-2x-1} \cdot (-2) dx \\ &= -\frac{1}{2} \int_k^{-\frac{1}{2}} e^{-2x-1} d(-2x-1) \\ &= -\frac{1}{2} [e^{-2x-1}]_k^{-\frac{1}{2}} \\ &= -\frac{1}{2} [e^{-2(-\frac{1}{2})-1} - e^{-2k-1}] \\ &= \frac{1}{2} (e^{-2k-1} - 1) \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int_k^{-\frac{1}{2}} \frac{dx}{e^{2x+1}} = 1 \\
 & \frac{1}{2}(e^{-2k-1} - 1) = 1 \\
 & e^{-2k-1} - 1 = 2 \\
 & e^{-2k-1} = 3 \\
 & -2k - 1 = \ln 3 \\
 & k = \underline{\underline{\frac{-1 - \ln 3}{2}}}
 \end{aligned}$$

93. Suppose $f(x)$ is a continuous function and that $\int_0^4 f(x) dx = 10$. Evaluate $\int_0^2 f(2u) du$.

[Hint: Let $u = \frac{x}{2}$.]

SOLUTION

Let $u = \frac{x}{2}$. Then $du = \frac{1}{2} dx$.

When $u = 0$, $x = 0$;

when $u = 2$, $x = 4$.

$$\begin{aligned}
 \therefore \int_0^2 f(2u) du &= \int_0^4 f(x) \cdot \frac{1}{2} dx \\
 &= \frac{1}{2} \int_0^4 f(x) dx \\
 &= \frac{1}{2}(10) \\
 &= \underline{\underline{5}}
 \end{aligned}$$

94. Suppose $g(x)$ is a continuous function and that $\int_4^7 g(x) dx = 12$. Evaluate $\int_2^3 g(3u - 2) du$.

[Hint: Let $u = \frac{x+2}{3}$.]

SOLUTION

Let $u = \frac{x+2}{3}$. Then $du = \frac{1}{3} dx$.

When $u = 2$, $x = 4$;

when $u = 3$, $x = 7$.

$$\begin{aligned}
 \therefore \int_2^3 g(3u-2) du &= \int_4^7 g(x) \cdot \frac{1}{3} dx \\
 &= \frac{1}{3} \int_4^7 g(x) dx \\
 &= \frac{1}{3}(12) \\
 &= \underline{\underline{4}}
 \end{aligned}$$

95. It is given that $f(-x) = -f(x)$ and $\int_3^5 f(x) dx = 2$.

(a) Prove that $\int_0^a f(x) dx = -\int_{-a}^0 f(x) dx$.

(b) Hence evaluate $\int_{-3}^5 f(x) dx$.

SOLUTION

(a) Let $u = -x$. Then $du = -dx$.

When $x = 0$, $u = 0$;

when $x = a$, $u = -a$.

$$\begin{aligned}
 \therefore \int_0^a f(x) dx &= \int_0^{-a} f(-u) \cdot (-1) du \\
 &= \int_{-a}^0 f(-u) du \\
 &= \int_{-a}^0 [-f(u)] du \\
 &= -\int_{-a}^0 f(u) du \\
 &= -\int_{-a}^0 f(x) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } \int_{-3}^5 f(x) dx &= \int_{-3}^0 f(x) dx + \int_0^3 f(x) dx + \int_3^5 f(x) dx \\
 &= \int_{-3}^0 f(x) dx + \left[-\int_{-3}^0 f(x) dx\right] + 2 \quad [\text{From the result of (a)}] \\
 &= \underline{\underline{2}}
 \end{aligned}$$

96. It is given that $f(x) = f(-x)$ and $\int_0^6 f(x) dx = a$.

(a) Express the value of $\int_{-6}^0 f(x) dx$ in terms of a .

(b) Hence express the value of $\int_{-6}^6 [f(x) + 3x^2] dx$ in terms of a .

SOLUTION

(a) Let $u = -x$. Then $du = -dx$.

When $x = -6$, $u = 6$;

when $x = 0$, $u = 0$.

$$\begin{aligned} \therefore \int_{-6}^0 f(x) dx &= \int_6^0 f(-u) \cdot (-1) du \\ &= \int_0^6 f(-u) du \\ &= \int_0^6 f(u) du \\ &= \int_0^6 f(x) dx \\ &= \underline{\underline{a}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_{-6}^6 [f(x) + 3x^2] dx &= \int_{-6}^6 f(x) dx + \int_{-6}^6 3x^2 dx \\ &= \int_{-6}^0 f(x) dx + \int_0^6 f(x) dx + [x^3]_{-6}^6 \\ &= a + a + [6^3 - (-6)^3] \quad [\text{From the result of (a)}] \\ &= \underline{\underline{2a + 432}} \end{aligned}$$

97. (a) Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$.

(b) Hence, or otherwise, evaluate $\int_0^5 \sqrt{5-x} dx$.

SOLUTION

(a) Let $u = a - x$. Then $du = -dx$.

When $x = 0$, $u = a$;

when $x = a$, $u = 0$.

$$\begin{aligned}\therefore \int_0^a f(x) dx &= \int_a^0 f(a-u) \cdot (-1) du \\ &= \int_0^a f(a-u) du \\ &= \int_0^a f(a-x) dx\end{aligned}$$

$$\begin{aligned}\text{(b)} \quad \int_0^5 \sqrt{5-x} dx &= \int_0^5 \sqrt{5-(5-x)} dx \quad [\text{From the result of (a)}] \\ &= \int_0^5 x^{\frac{1}{2}} dx \\ &= \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^5 \\ &= \frac{2}{3} (5)^{\frac{3}{2}} - \frac{2}{3} (0)^{\frac{3}{2}} \\ &= \frac{10\sqrt{5}}{3}\end{aligned}$$

98. (a) Prove that $\int_a^b f(x) dx = \int_{a-b}^0 f(a-x) dx$.

(b) Hence, or otherwise, evaluate $\int_6^7 (x-3)(6-x)^{99} dx$.

SOLUTION

(a) Let $u = a - x$. Then $du = -dx$.

When $x = a$, $u = 0$;

when $x = b$, $u = a - b$.

$$\begin{aligned}\therefore \int_a^b f(x) dx &= \int_{a-b}^0 f(a-u) \cdot (-1) du \\ &= \int_{a-b}^0 f(a-u) du \\ &= \int_{a-b}^0 f(a-x) dx\end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_6^7 (x-3)(6-x)^{99} dx &= \int_{6-7}^0 [(6-x)-3][6-(6-x)]^{99} dx \quad [\text{From the result of (a)}] \\
 &= \int_{-1}^0 (3-x)x^{99} dx \\
 &= \int_{-1}^0 (3x^{99} - x^{100}) dx \\
 &= \left[\frac{3}{100} x^{100} - \frac{1}{101} x^{101} \right]_{-1}^0 \\
 &= \left[\frac{3}{100} (0)^{100} - \frac{1}{101} (0)^{101} \right] - \left[\frac{3}{100} (-1)^{100} - \frac{1}{101} (-1)^{101} \right] \\
 &= -\frac{403}{10100}
 \end{aligned}$$

99. (a) Prove that $\int_a^b f(x) dx = \int_0^{b-a} f(x+a) dx$.

(b) Hence, or otherwise, evaluate $\int_4^8 (8-2x)^{\frac{1}{3}} dx$.

SOLUTION

(a) Let $u = x - a$. Then $du = dx$.

When $x = a$, $u = 0$;

when $x = b$, $u = b - a$.

$$\begin{aligned}
 \therefore \int_a^b f(x) dx &= \int_0^{b-a} f(u+a) du \\
 &= \int_0^{b-a} f(x+a) dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_4^8 (8-2x)^{\frac{1}{3}} dx &= \int_0^{8-4} [8-2(x+4)]^{\frac{1}{3}} dx \quad [\text{From the result of (a)}] \\
 &= \int_0^4 (-2x)^{\frac{1}{3}} dx \\
 &= -\frac{1}{2} \int_0^4 (-2x)^{\frac{1}{3}} \cdot (-2) dx \\
 &= -\frac{1}{2} \int_0^4 (-2x)^{\frac{1}{3}} d(-2x) \\
 &= -\frac{1}{2} \left[\frac{3}{4} (-2x)^{\frac{4}{3}} \right]_0^4 \\
 &= -\frac{1}{2} \left\{ \frac{3}{4} [-2(4)]^{\frac{4}{3}} - \frac{3}{4} [-2(0)]^{\frac{4}{3}} \right\} \\
 &= \underline{\underline{-6}}
 \end{aligned}$$

Level 3

100. (a) Evaluate $\int_{-2}^2 \frac{2x+2}{x^2+2x+4} dx$.

(b) Hence, or otherwise, evaluate $\int_{-2}^2 \frac{3x^2+2x+8}{x^2+2x+4} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \int_{-2}^2 \frac{2x+2}{x^2+2x+4} dx &= \int_{-2}^2 \frac{d(x^2+2x+4)}{x^2+2x+4} \\ &= [\ln(x^2+2x+4)]_{-2}^2 \\ &= \ln[2^2+2(2)+4] - \ln[(-2)^2+2(-2)+4] \\ &= \ln 12 - \ln 4 \\ &= \underline{\underline{\ln 3}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_{-2}^2 \frac{3x^2+2x+8}{x^2+2x+4} dx &= \int_{-2}^2 \frac{(3x^2+6x+12)-(4x+4)}{x^2+2x+4} dx \\ &= \int_{-2}^2 \frac{3(x^2+2x+4)}{x^2+2x+4} dx - \int_{-2}^2 \frac{2(2x+2)}{x^2+2x+4} dx \\ &= 3 \int_{-2}^2 dx - 2 \int_{-2}^2 \frac{2x+2}{x^2+2x+4} dx \\ &= 3[x]_{-2}^2 - 2 \ln 3 \\ &= 3[2 - (-2)] - 2 \ln 3 \\ &= \underline{\underline{12 - 2 \ln 3}} \end{aligned}$$

101. (a) Evaluate $\int_0^6 \frac{x-3}{x^2-6x+18} dx$.

(b) Hence evaluate $\int_0^{\sqrt{6}} \frac{x^5 dx}{x^4-6x^2+18}$.

SOLUTION

$$\begin{aligned}
 \text{(a)} \quad \int_0^6 \frac{x-3}{x^2-6x+18} dx &= \frac{1}{2} \int_0^6 \frac{2x-6}{x^2-6x+18} dx \\
 &= \frac{1}{2} \int_0^6 \frac{1}{x^2-6x+18} d(x^2-6x+18) \\
 &= \frac{1}{2} [\ln|x^2-6x+18|]_0^6 \\
 &= \frac{1}{2} \{ \ln[6^2-6(6)+18] - \ln[0^2-6(0)+18] \} \\
 &= \frac{1}{2} (\ln 18 - \ln 18) \\
 &= \underline{\underline{0}}
 \end{aligned}$$

(b) Let $u = x^2$. Then $du = 2x dx$.

When $x = 0$, $u = 0$;

when $x = \sqrt{6}$, $u = 6$.

$$\begin{aligned}
 \int_0^{\sqrt{6}} \frac{x^5 dx}{x^4-6x^2+18} &= \frac{1}{2} \int_0^{\sqrt{6}} \frac{x^4}{x^4-6x^2+18} \cdot 2x dx \\
 &= \frac{1}{2} \int_0^6 \frac{u^2}{u^2-6u+18} du \\
 &= \frac{1}{2} \int_0^6 \frac{(u^2-6u+18) + (6u-18)}{u^2-6u+18} du \\
 &= \frac{1}{2} \int_0^6 \left(1 + \frac{6u-18}{u^2-6u+18} \right) du \\
 &= \frac{1}{2} \int_0^6 du + 3 \int_0^6 \frac{u-3}{u^2-6u+18} du \\
 &= \frac{1}{2} [u]_0^6 + 3(0) \quad [\text{From the result of (a)}] \\
 &= \frac{1}{2} (6-0) \\
 &= \underline{\underline{3}}
 \end{aligned}$$

102. (a) If $\frac{x+2}{(x^2-2)(x+1)} \equiv \frac{Ax+B}{x^2-2} + \frac{C}{x+1}$, find the values of constants A , B and C .

(b) Hence evaluate $\int_3^4 \frac{x+2}{x^3+x^2-2x-2} dx$.

SOLUTION

$$\begin{aligned} \text{(a)} \quad \frac{x+2}{(x^2-2)(x+1)} &\equiv \frac{Ax+B}{x^2-2} + \frac{C}{x+1} \\ &\equiv \frac{(Ax+B)(x+1) + C(x^2-2)}{(x^2-2)(x+1)} \end{aligned}$$

$$\text{i.e.} \quad (Ax+B)(x+1) + C(x^2-2) \equiv x+2$$

$$Ax^2 + (A+B)x + B + Cx^2 - 2C \equiv x+2$$

$$(A+C)x^2 + (A+B)x + (B-2C) \equiv x+2$$

$$\therefore \begin{cases} A+C=0 & \dots\dots\dots (1) \\ A+B=1 & \dots\dots\dots (2) \\ B-2C=2 & \dots\dots\dots (3) \end{cases}$$

$$\text{From (1), } A = -C \dots\dots\dots (4)$$

Substitute (4) into (2),

$$-C + B = 1 \dots\dots\dots (5)$$

$$(3) - (5): (B-2C) - (-C+B) = 2-1$$

$$-C = 1$$

$$C = -1$$

Substitute $C = -1$ into (3),

$$B - 2(-1) = 2$$

$$B = 0$$

Substitute $C = -1$ into (4),

$$A = -(-1)$$

$$= 1$$

$$\therefore \underline{\underline{A=1, B=0, C=-1}}$$

$$\begin{aligned}
\text{(b)} \quad \int_3^4 \frac{x+2}{x^3+x^2-2x-2} dx &= \int_3^4 \frac{x+2}{(x^2-2)(x+1)} dx \\
&= \int_3^4 \left(\frac{x}{x^2-2} + \frac{-1}{x+1} \right) dx \quad [\text{From the result of (a)}] \\
&= \int_3^4 \frac{x}{x^2-2} dx - \int_3^4 \frac{1}{x+1} dx \\
&= \frac{1}{2} \int_3^4 \frac{2x}{x^2-2} dx - \int_3^4 \frac{1}{x+1} dx \\
&= \frac{1}{2} \int_3^4 \frac{1}{x^2-2} d(x^2-2) - \int_3^4 \frac{1}{x+1} d(x+1) \\
&= \frac{1}{2} [\ln|x^2-2|]_3^4 - [\ln|x+1|]_3^4 \\
&= \frac{1}{2} (\ln 14 - \ln 7) - (\ln 5 - \ln 4) \\
&= \frac{1}{2} \ln 2 - \ln 5 + 2 \ln 2 \\
&= \underline{\underline{\frac{5}{2} \ln 2 - \ln 5}}
\end{aligned}$$

103. (a) Evaluate $\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}} dx$.

(b) If $\frac{1}{u(u-1)} \equiv \frac{A}{u} + \frac{B}{u-1}$, find the values of constants A and B .

(c) Hence evaluate $\int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x-1)} dx$.

 SOLUTION

$$\begin{aligned}
\text{(a)} \quad \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}} dx &= \int_{\ln 2}^{\ln 3} \frac{1}{1-e^{-x}} d(1-e^{-x}) \\
&= [\ln|1-e^{-x}|]_{\ln 2}^{\ln 3} \\
&= \ln(1-e^{-\ln 3}) - \ln(1-e^{-\ln 2}) \\
&= \ln(1-e^{\ln \frac{1}{3}}) - \ln(1-e^{\ln \frac{1}{2}}) \\
&= \ln(1-\frac{1}{3}) - \ln(1-\frac{1}{2}) \\
&= \ln \frac{2}{3} - \ln \frac{1}{2} \\
&= \underline{\underline{\ln \frac{4}{3}}}
\end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{1}{u(u-1)} &\equiv \frac{A}{u} + \frac{B}{u-1} \\ &\equiv \frac{A(u-1) + Bu}{u(u-1)} \end{aligned}$$

$$\text{i.e. } A(u-1) + Bu \equiv 1$$

$$(A+B)u - A \equiv 1$$

$$\therefore \begin{cases} A+B=0 \\ -A=1 \end{cases}$$

$$\therefore \underline{\underline{A=-1, B=1}}$$

$$\begin{aligned} \text{(c)} \quad \int_{\ln 2}^{\ln 3} \frac{1}{e^x(e^x-1)} dx &= \int_{\ln 2}^{\ln 3} \left[\frac{-1}{e^x} + \frac{1}{e^x-1} \right] dx \quad [\text{From the result of (b)}] \\ &= -\int_{\ln 2}^{\ln 3} e^{-x} dx + \int_{\ln 2}^{\ln 3} \frac{1}{e^x-1} dx \\ &= \int_{\ln 2}^{\ln 3} e^{-x} d(-x) + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{e^x(e^x-1)} dx \\ &= [e^{-x}]_{\ln 2}^{\ln 3} + \int_{\ln 2}^{\ln 3} \frac{e^{-x}}{1-e^{-x}} dx \\ &= [e^{-\ln 3} - e^{-\ln 2}] + \ln \frac{4}{3} \quad [\text{From the result of (a)}] \\ &= \left(\frac{1}{3} - \frac{1}{2} \right) + \ln \frac{4}{3} \\ &= \underline{\underline{-\frac{1}{6} + \ln \frac{4}{3}}} \end{aligned}$$

104. (a) Prove that $\int_{-a}^0 f(x) dx = \int_0^a f(-x) dx$.

(b) Hence, or otherwise, evaluate $\int_{-2}^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx$.

 **SOLUTION**

(a) Let $u = -x$. Then $du = -dx$.

When $x = -a$, $u = a$;

when $x = 0$, $u = 0$.

$$\begin{aligned}
 \therefore \int_{-a}^0 f(x)dx &= \int_a^0 f(-u) \cdot (-1)du \\
 &= \int_0^a f(-u)du \\
 &= \int_0^a f(-x)dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_{-2}^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx &= \int_{-2}^0 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx + \int_0^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx \\
 &= \int_0^2 \frac{-x[e^{(-x)^2} - e^{-(-x)^2}]}{1+(-x)^2} dx + \int_0^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx \quad [\text{From the result of (a)}] \\
 &= -\int_0^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx + \int_0^2 \frac{x(e^{x^2} - e^{-x^2})}{1+x^2} dx \\
 &= \underline{\underline{0}}
 \end{aligned}$$

105. (a) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

(ii) Hence prove that $\int_0^a f(x)dx = \frac{1}{2} \int_0^a [f(x) + f(a-x)]dx$.

(b) From the result of (a), evaluate $\int_0^3 \frac{dx}{e^{3-2x} + 1}$.

SOLUTION

(a) (i) Let $u = a - x$. Then $du = -dx$.

When $x = 0$, $u = a$;

when $x = a$, $u = 0$.

$$\begin{aligned}
 \therefore \int_0^a f(x)dx &= \int_a^0 f(a-u) \cdot (-1)du \\
 &= \int_0^a f(a-u)du \\
 &= \int_0^a f(a-x)dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \frac{1}{2} \int_0^a [f(x) + f(a-x)] dx &= \frac{1}{2} \int_0^a f(x) dx + \frac{1}{2} \int_0^a f(a-x) dx \\
 &= \frac{1}{2} \int_0^a f(x) dx + \frac{1}{2} \int_0^a f(x) dx \quad [\text{From the result of (a)(i)}] \\
 &= \int_0^a f(x) dx \\
 \therefore \int_0^a f(x) dx &= \frac{1}{2} \int_0^a [f(x) + f(a-x)] dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_0^3 \frac{dx}{e^{3-2x} + 1} &= \frac{1}{2} \int_0^3 \left[\frac{1}{e^{3-2x} + 1} + \frac{1}{e^{3-2(3-x)} + 1} \right] dx \quad [\text{From the result of (a)(ii)}] \\
 &= \frac{1}{2} \int_0^3 \left(\frac{1}{e^{3-2x} + 1} + \frac{1}{e^{2x-3} + 1} \right) dx \\
 &= \frac{1}{2} \int_0^3 \frac{(e^{2x-3} + 1) + (e^{3-2x} + 1)}{(e^{3-2x} + 1)(e^{2x-3} + 1)} dx \\
 &= \frac{1}{2} \int_0^3 \frac{e^{2x-3} + e^{3-2x} + 2}{1 + e^{3-2x} + e^{2x-3} + 1} dx \\
 &= \frac{1}{2} \int_0^3 \frac{e^{2x-3} + e^{3-2x} + 2}{e^{2x-3} + e^{3-2x} + 2} dx \\
 &= \frac{1}{2} \int_0^3 dx \\
 &= \frac{1}{2} [x]_0^3 \\
 &= \frac{3}{2}
 \end{aligned}$$

106. (a) Prove that $\int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx = \int_0^1 \frac{(1-x)^4}{x^4 + (1-x)^4} dx$.

[Hint: Let $u = 1 - x$.]

(b) Hence, or otherwise, evaluate $\int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx$.

SOLUTION

(a) Let $u = 1 - x$. Then $du = -dx$.

When $x = 0$, $u = 1$;

when $x = 1$, $u = 0$.

$$\begin{aligned}
 \therefore \int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx &= \int_1^0 \frac{(1-u)^4}{(1-u)^4 + u^4} \cdot (-1) du \\
 &= \int_0^1 \frac{(1-u)^4}{u^4 + (1-u)^4} du \\
 &= \int_0^1 \frac{(1-x)^4}{x^4 + (1-x)^4} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx &= \int_0^1 \frac{[x^4 + (1-x)^4] - (1-x)^4}{x^4 + (1-x)^4} dx \\
 &= \int_0^1 \left[1 - \frac{(1-x)^4}{x^4 + (1-x)^4} \right] dx \\
 &= \int_0^1 dx - \int_0^1 \frac{(1-x)^4}{x^4 + (1-x)^4} dx \\
 &= [x]_0^1 - \int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx \quad [\text{From the result of (a)}]
 \end{aligned}$$

$$2 \int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx = 1 - 0$$

$$\int_0^1 \frac{x^4}{x^4 + (1-x)^4} dx = \underline{\underline{\frac{1}{2}}}$$

107. (a) Prove that $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$.

(b) Hence, or otherwise, evaluate $\int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx$.

SOLUTION

(a) Let $u = a + b - x$. Then $du = -dx$.

When $x = a$, $u = b$;

when $x = b$, $u = a$.

$$\begin{aligned}
 \therefore \int_a^b f(x) dx &= \int_b^a f(a+b-u) \cdot (-1) du \\
 &= \int_a^b f(a+b-u) du \\
 &= \int_a^b f(a+b-x) dx
 \end{aligned}$$

$$(b) \int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx = \int_5^{10} \frac{(5+10-x)^{10}}{(5+10-x)^{10} + [15-(5+10-x)]^{10}} dx \quad [\text{From the result of (a)}]$$

$$= \int_5^{10} \frac{(15-x)^{10}}{(15-x)^{10} + x^{10}} dx$$

$$\therefore \int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx = \int_5^{10} \frac{[x^{10} + (15-x)^{10}] - (15-x)^{10}}{x^{10} + (15-x)^{10}} dx$$

$$= \int_5^{10} \left[1 - \frac{(15-x)^{10}}{x^{10} + (15-x)^{10}} \right] dx$$

$$= \int_5^{10} dx - \int_5^{10} \frac{(15-x)^{10}}{x^{10} + (15-x)^{10}} dx$$

$$= [x]_5^{10} - \int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx$$

$$2 \int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx = 10 - 5$$

$$\therefore \int_5^{10} \frac{x^{10}}{x^{10} + (15-x)^{10}} dx = \underline{\underline{\frac{5}{2}}}$$

108. Let $f(x)$ be a function defined on $x \geq 0$. It is given that $f'(x) = \frac{90}{2 + ae^{-kx}}$, where a and k are positive constants.

(a) (i) Express $\ln\left[\frac{90}{f'(x)} - 2\right]$ as a linear function of x .

(ii) It is given that the graph of the linear function obtained in (a)(i) passes through (1, 2.44) and the intercept on the vertical axis is 2.89. Find the values of a and k . (Give your answers correct to 2 significant figures if necessary.)

(b) (i) Prove that $f'(x) = \frac{90e^{kx}}{2e^{kx} + a}$.

(ii) It is given that $f(0) = 100$. Using the values of a and k obtained in (a)(ii), find the value of $f(5)$. (Give your answer correct to 3 significant figures.)

SOLUTION

$$(a) \quad (i) \quad f'(x) = \frac{90}{2 + ae^{-kx}}$$

$$2 + ae^{-kx} = \frac{90}{f'(x)}$$

$$\frac{90}{f'(x)} - 2 = ae^{-kx}$$

$$\ln\left[\frac{90}{f'(x)} - 2\right] = \ln(ae^{-kx})$$

$$\ln\left[\frac{90}{f'(x)} - 2\right] = \ln a - kx$$

(ii) $\ln a$ = Intercept on the vertical axis

$$= 2.89$$

$$a = e^{2.89}$$

$$= \underline{\underline{18}} \text{ (corr. to 2 sig. fig.)}$$

\therefore The graph passes through (1, 2.44).

$$\therefore 2.44 = 2.89 - k(1)$$

$$k = \underline{\underline{0.45}}$$

$$(b) \quad (i) \quad f'(x) = \frac{90}{2 + ae^{-kx}}$$

$$= \frac{90}{e^{-kx}(2e^{kx} + a)}$$

$$= \frac{90e^{kx}}{2e^{kx} + a}$$

$$(ii) \text{ From (a)(ii), } f'(x) = \frac{90e^{0.45x}}{2e^{0.45x} + 18}.$$

Let $u = 2e^{0.45x} + 18$. Then $du = 0.9e^{0.45x} dx$.

When $x = 0$, $u = 20$;

when $x = 5$, $u = 2e^{2.25} + 18$.

$$\begin{aligned}
 \int_0^5 f'(x) dx &= \int_0^5 \frac{90e^{0.45x}}{2e^{0.45x} + 18} dx \\
 &= \int_0^5 \frac{100}{2e^{0.45x} + 18} \cdot (0.9e^{0.45x}) dx \\
 &= \int_{20}^{2e^{2.25} + 18} \frac{100}{u} du \\
 &= 100[\ln|u|]_{20}^{2e^{2.25} + 18} \\
 &= 100[\ln(2e^{2.25} + 18) - \ln 20] \\
 &= 100\ln \frac{2e^{2.25} + 18}{20}
 \end{aligned}$$

$$\begin{aligned}
 f(5) - f(0) &= \int_0^5 f'(x) dx \\
 f(5) &= \int_0^5 f'(x) dx + f(0) \\
 &= 100\ln \frac{2e^{2.25} + 18}{20} + 100 \\
 &= \underline{\underline{161}} \text{ (corr. to 3 sig. fig.)}
 \end{aligned}$$