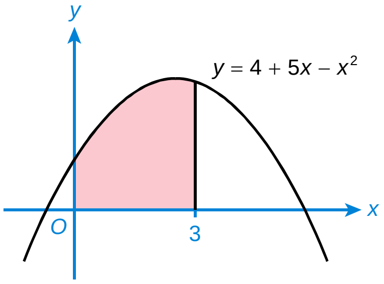
Chapter 9

Applications of Definite Integration

EXERCISE 9A

Level 1

1. Find the area of the region bounded by the curve $y = 4 + 5x - x^2$ and the x-axis from $x = 0$ to $x = 3$.



SOLUTION

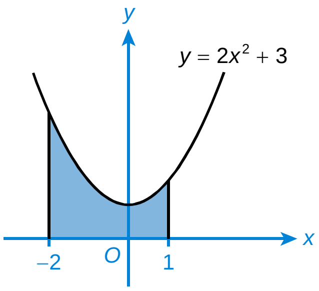
$$\text{Required area} = \int_0^3 \left|4 + 5x - x^2\right| dx$$

$$= \int_0^3 (4 + 5x - x^2) dx$$

$$= [4x + \frac{5}{2}x^2 - \frac{x^3}{3}]_0^3$$

$$= \underline{\underline{\frac{51}{2}}}$$

2. Find the area of the region bounded by the curve $y = 2x^2 + 3$ and the x-axis from $x = -2$ to $x = 1$.



SOLUTION

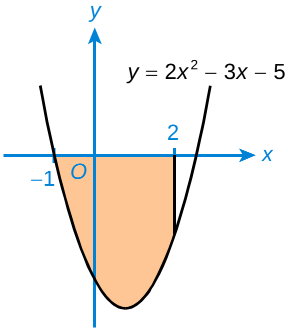
Required area $= \int_{-2}^{1} |2x^2 + 3| dx$

$= \int_{-2}^{1} (2x^2 + 3) dx$

$= [\frac{2}{3}x^3 + 3x]_{-2}^{1}$

$= \underline{\underline{15}}$

3. Find the area of the region bounded by the curve $y = 2x^2 - 3x - 5$ and the x-axis from $x = -1$ to $x = 2$.



SOLUTION

Required area $= \int_{-1}^{2} |2x^2 - 3x - 5| dx$

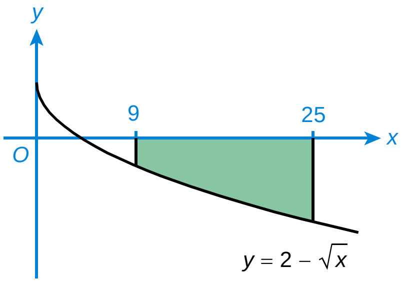
$= -\int_{-1}^{2} (2x^2 - 3x - 5) dx$

$= -[\frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x]_{-1}^{2}$

$= -(-\frac{27}{2})$

$= \underline{\underline{\frac{27}{2}}}$

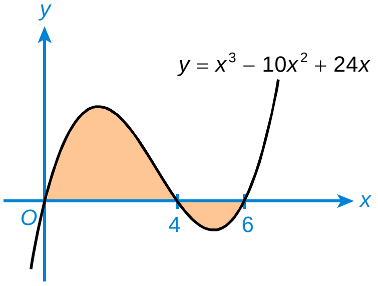
4. Find the area of the region bounded by the curve $y = 2 - \sqrt{x}$ and the x-axis from $x = 9$ to $x = 25$.



SOLUTION

$$\text{Required area} = \int_9^{25} \left|2-\sqrt{x}\right| dx$$
$$= -\int_9^{25} (2-\sqrt{x})dx$$
$$= -[2x - \frac{2}{3}x^{\frac{3}{2}}]_9^{25}$$
$$= -(-\frac{100}{3})$$
$$= \underline{\underline{\frac{100}{3}}}$$

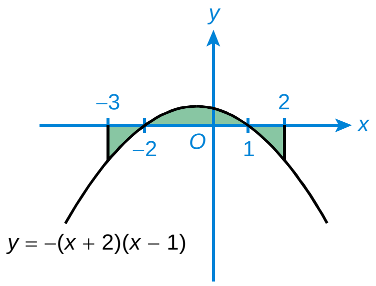
5. Find the area of the region bounded by the curve $y = x^3 - 10x^2 + 24x$ and the x-axis from $x = 0$ to $x = 6$.



SOLUTION

$$\text{Required area} = \int_0^{6} \left|x^3 - 10x^2 + 24x\right| dx$$
$$= \int_0^{4} \left|x^3 - 10x^2 + 24x\right| dx + \int_4^{6} \left|x^3 - 10x^2 + 24x\right| dx$$
$$= \int_0^{4} (x^3 - 10x^2 + 24x)dx - \int_4^{6} (x^3 - 10x^2 + 24x)dx$$
$$= [\frac{x^4}{4} - \frac{10}{3}x^3 + 12x^2]_0^4 - [\frac{x^4}{4} - \frac{10}{3}x^3 + 12x^2]_4^6$$
$$= \frac{128}{3} - (-\frac{20}{3})$$
$$= \underline{\underline{\frac{148}{3}}}$$

6. Find the area of the region bounded by the curve $y=-(x+2)(x-1)$ and the x-axis from $x=-3$ to $x=2$.



SOLUTION

$$\text{Required area} = \int_{-3}^{2} |-(x+2)(x-1)|\,dx$$

$$= \int_{-3}^{-2} |-(x+2)(x-1)|\,dx + \int_{-2}^{1} |-(x+2)(x-1)|\,dx + \int_{1}^{2} |-(x+2)(x-1)|\,dx$$

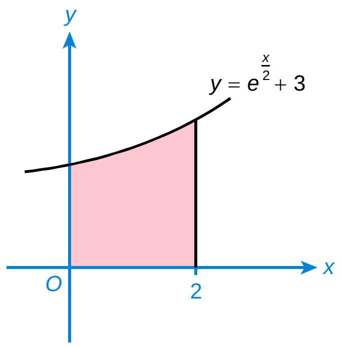
$$= -\int_{-3}^{-2} (-x^2-x+2)\,dx + \int_{-2}^{1} (-x^2-x+2)\,dx - \int_{1}^{2} (-x^2-x+2)\,dx$$

$$= -[-\frac{x^3}{3}-\frac{x^2}{2}+2x]_{-3}^{-2} + [-\frac{x^3}{3}-\frac{x^2}{2}+2x]_{-2}^{1} - [-\frac{x^3}{3}-\frac{x^2}{2}+2x]_{1}^{2}$$

$$= -(-\frac{11}{6}) + \frac{9}{2} - (-\frac{11}{6})$$

$$= \underline{\underline{\frac{49}{6}}}$$

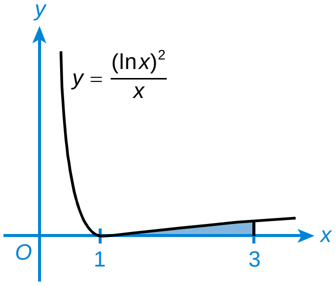
7. Find the area of the region bounded by the curve $y=e^{\frac{x}{2}}+3$ and the x-axis from $x=0$ to $x=2$.



SOLUTION

$$\text{Required area} = \int_0^2 \left| e^{\frac{x}{2}} + 3 \right| dx$$
$$= \int_0^2 (e^{\frac{x}{2}} + 3) dx$$
$$= \int_0^2 e^{\frac{x}{2}} dx + \int_0^2 3dx$$
$$= \int_0^2 2e^{\frac{x}{2}} d(\frac{x}{2}) + \int_0^2 3dx$$
$$= [2e^{\frac{x}{2}}]_0^2 + [3x]_0^2$$
$$= (2e - 2) + 6$$
$$= \underline{\underline{2e + 4}}$$

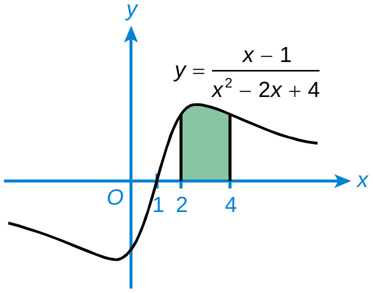
8. Find the area of the region bounded by the curve $y = \frac{(\ln x)^2}{x}$, x-axis and the line $x = 3$.



SOLUTION

$$\text{Required area} = \int_1^3 \left| \frac{(\ln x)^2}{x} \right| dx$$
$$= \int_1^3 \frac{(\ln x)^2}{x} dx$$
$$= \int_1^3 (\ln x)^2 d(\ln x)$$
$$= [\frac{1}{3}(\ln x)^3]_1^3$$
$$= \underline{\underline{\frac{1}{3}(\ln 3)^3}}$$

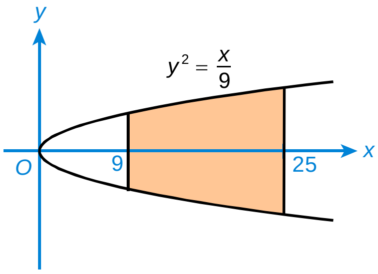
9. Find the area of the region bounded by the curve $y = \dfrac{x-1}{x^2-2x+4}$ and the x-axis from $x = 2$ to $x = 4$.



SOLUTION

$$\text{Required area} = \int_2^4 \left|\frac{x-1}{x^2-2x+4}\right| dx$$
$$= \int_2^4 \frac{x-1}{x^2-2x+4} dx$$
$$= \frac{1}{2}\int_2^4 \frac{2x-2}{x^2-2x+4} dx$$
$$= \frac{1}{2}\int_2^4 \frac{1}{x^2-2x+4} d(x^2-2x+4)$$
$$= \frac{1}{2}[\ln(x^2-2x+4)]_2^4$$
$$= \frac{1}{2}(\ln 12 - \ln 4)$$
$$= \frac{1}{2}\ln 3$$

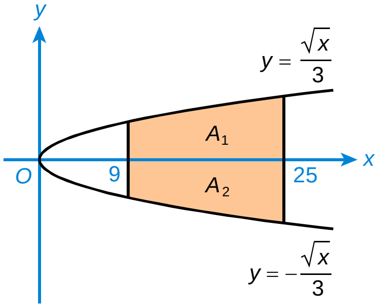
10. Find the area of the region bounded by the curve $y^2 = \dfrac{x}{9}$ and the lines $x = 9$ and $x = 25$.



SOLUTION

Rewrite the equation $y^2 = \dfrac{x}{9}$ as $y = \pm\sqrt{\dfrac{x}{9}}$, i.e. $y = \pm\dfrac{\sqrt{x}}{3}$. As shown in the figure, the curve $y^2 = \dfrac{x}{9}$ is split into two parts, $y = \dfrac{\sqrt{x}}{3}$ and $y = -\dfrac{\sqrt{x}}{3}$. The required area is the sum of A_1 and A_2.

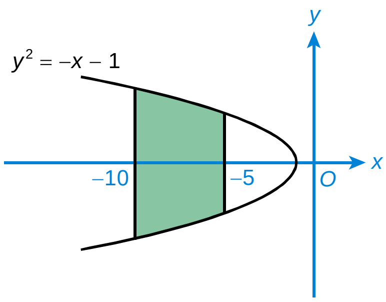
Since the curve $y^2 = \dfrac{x}{9}$ is symmetrical about the x-axis, $A_1 = A_2$.



$$A_1=\int_9^{25}\frac{\sqrt{x}}{3}\,dx$$
$$=[\frac{2}{9}x^{\frac{3}{2}}]_9^{25}$$
$$=\frac{196}{9}$$

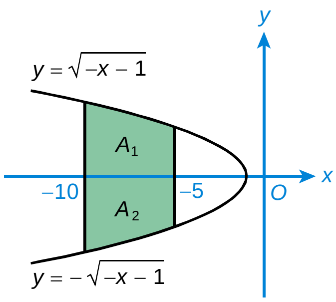
$\therefore$ Required area $=A_1+A_2$
$$=2(\frac{196}{9})$$
$$=\underline{\underline{\frac{392}{9}}}$$

11. Find the area of the region bounded by the curve $y^2=-x-1$ and the lines $x=-10$ and $x=-5$.



SOLUTION

Rewrite the equation $y^2=-x-1$ as $y=\pm\sqrt{-x-1}$. As shown in the figure, the curve $y^2=-x-1$ is split into two parts, $y=\sqrt{-x-1}$ and $y=-\sqrt{-x-1}$. The required area is the sum of A_1 and A_2. Since the curve $y^2=-x-1$ is symmetrical about the x-axis, $A_1=A_2$.



$$A_1 = \int_{-10}^{-5} \sqrt{-x-1}\,dx$$

$$= -\int_{-10}^{-5} \sqrt{-x-1}\,d(-x-1)$$

$$= -[\frac{2}{3}(-x-1)^{\frac{3}{2}}]_{-10}^{-5}$$

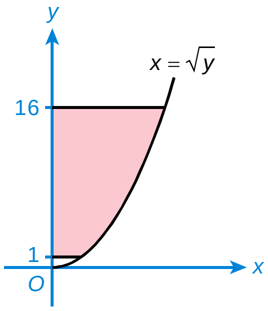
$$= \frac{38}{3}$$

$\therefore$ Required area $= A_1 + A_2$

$$= 2(\frac{38}{3})$$

$$= \underline{\underline{\frac{76}{3}}}$$

12. Find the area of the region bounded by the curve $x = \sqrt{y}$ and the y-axis from $y = 1$ to $y = 16$.



SOLUTION

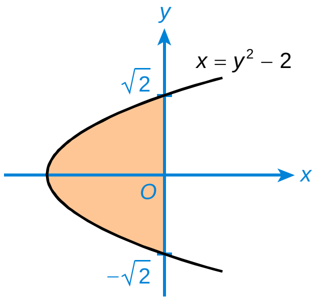
Required area $= \int_1^{16} \left|\sqrt{y}\right| dy$

$$= \int_1^{16} \sqrt{y}\,dy$$

$$= [\frac{2}{3}y^{\frac{3}{2}}]_1^{16}$$

$$= \underline{\underline{42}}$$

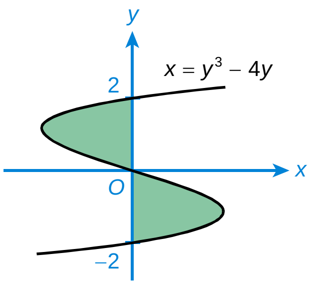
13. Find the area of the region bounded by the curve $x = y^2 - 2$ and the y-axis from $y = -\sqrt{2}$ to $y = \sqrt{2}$.



SOLUTION

$$\begin{aligned}\text{Required area} &= \int_{-\sqrt{2}}^{\sqrt{2}} \left|y^2 - 2\right| dy \\ &= -\int_{-\sqrt{2}}^{\sqrt{2}} (y^2 - 2)dy \\ &= -\left[\frac{y^3}{3} - 2y\right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= -\left(-\frac{8\sqrt{2}}{3}\right) \\ &= \underline{\underline{\frac{8\sqrt{2}}{3}}}\end{aligned}$$

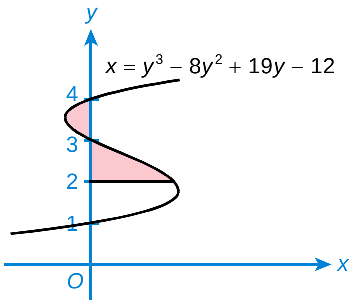
14. Find the area of the region bounded by the curve $x = y^3 - 4y$ and the y-axis from $y = -2$ to $y = 2$.



SOLUTION

$$\begin{aligned}\text{Required area} &= \int_{-2}^{2} \left|y^3 - 4y\right| dy \\ &= \int_{-2}^{0} \left|y^3 - 4y\right| dy + \int_{0}^{2} \left|y^3 - 4y\right| dy \\ &= \int_{-2}^{0} (y^3 - 4y)dy - \int_{0}^{2} (y^3 - 4y)dy \\ &= \left[\frac{y^4}{4} - 2y^2\right]_{-2}^{0} - \left[\frac{y^4}{4} - 2y^2\right]_{0}^{2} \\ &= 4 - (-4) \\ &= \underline{\underline{8}}\end{aligned}$$

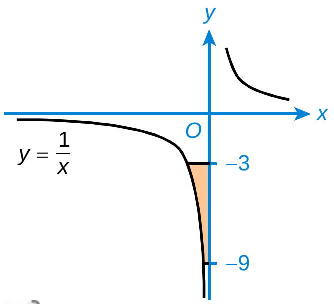
15. Find the area of the region bounded by the curve $x = y^3 - 8y^2 + 19y - 12$ and the y-axis from $y = 2$ to $y = 4$.



SOLUTION

$$\text{Required area} = \int_2^4 \left| y^3 - 8y^2 + 19y - 12 \right| dy$$
$$= \int_2^3 \left| y^3 - 8y^2 + 19y - 12 \right| dy + \int_3^4 \left| y^3 - 8y^2 + 19y - 12 \right| dy$$
$$= \int_2^3 (y^3 - 8y^2 + 19y - 12)\, dy - \int_3^4 (y^3 - 8y^2 + 19y - 12)\, dy$$
$$= \left[\frac{y^4}{4} - \frac{8}{3}y^3 + \frac{19}{2}y^2 - 12y\right]_2^3 - \left[\frac{y^4}{4} - \frac{8}{3}y^3 + \frac{19}{2}y^2 - 12y\right]_3^4$$
$$= \frac{13}{12} - \left(-\frac{5}{12}\right)$$
$$= \underline{\underline{\frac{3}{2}}}$$

16. Find the area of the region bounded by the curve $y = \frac{1}{x}$ and the y-axis from $y = -9$ to $y = -3$.

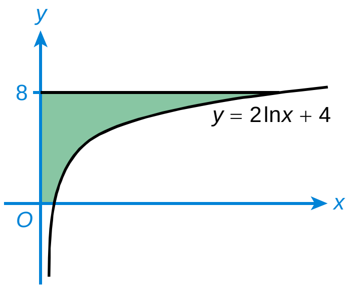


SOLUTION

Rewrite the equation $y = \frac{1}{x}$ as $x = \frac{1}{y}$.

$$\text{Required area} = \int_{-9}^{-3} \left| \frac{1}{y} \right| dy$$
$$= -\int_{-9}^{-3} \frac{1}{y}\, dy$$
$$= -[\ln|y|]_{-9}^{-3}$$
$$= -(\ln 3 - \ln 9)$$
$$= \underline{\underline{\ln 3}}$$

17. Find the area of the region bounded by the curve $y=2\ln x+4$, the x-axis, the y-axis and the line $y=8$.

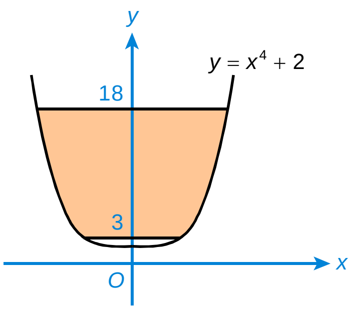


SOLUTION

Rewrite the equation $y=2\ln x+4$ as $x=e^{\frac{y-4}{2}}$.

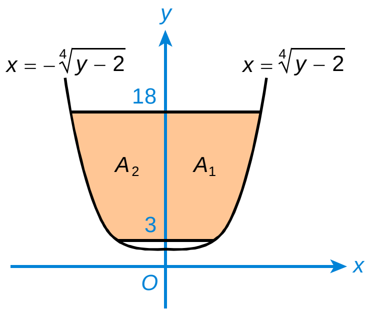
$$\begin{aligned}\text{Required area} &= \int_0^8 \left| e^{\frac{y-4}{2}} \right| dy \\ &= \int_0^8 e^{\frac{y-4}{2}}\, dy \\ &= \int_0^8 2e^{\frac{y-4}{2}}\, d\left(\frac{y-4}{2}\right) \\ &= \left[2e^{\frac{y-4}{2}}\right]_0^8 \\ &= \underline{\underline{2e^2-2e^{-2}}}\end{aligned}$$

18. Find the area of the region bounded by the curve $y=x^4+2$ and the lines $y=3$ and $y=18$.



SOLUTION

Rewrite the equation $y=x^4+2$ as $x=\pm\sqrt[4]{y-2}$. As shown in the figure, the curve $y=x^4+2$ is split into two parts, $x=\sqrt[4]{y-2}$ and $x=-\sqrt[4]{y-2}$. The required area is the sum of A_1 and A_2. Since the curve $y=x^4+2$ is symmetrical about the y-axis, $A_1=A_2$.



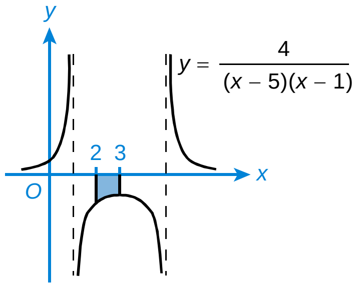
$$A_1 = \int_3^{18} \sqrt[4]{y-2}\,dy$$
$$= \int_3^{18} \sqrt[4]{y-2}\,d(y-2)$$
$$= [\frac{4}{5}(y-2)^{\frac{5}{4}}]_3^{18}$$
$$= \frac{124}{5}$$

$\therefore$ Required area $= A_1 + A_2$
$$= 2(\frac{124}{5})$$
$$= \underline{\underline{\frac{248}{5}}}$$

Level 2

19. (a) If $\dfrac{4}{(x-5)(x-1)} \equiv \dfrac{A}{x-5} + \dfrac{B}{x-1}$, find the values of constants A and B.

(b) Find the area of the region bounded by the curve $y = \dfrac{4}{(x-5)(x-1)}$ and the x-axis from $x = 2$ to $x = 3$.



SOLUTION

(a) $$\frac{4}{(x-5)(x-1)} \equiv \frac{A}{x-5} + \frac{B}{x-1}$$
$$\equiv \frac{A(x-1) + B(x-5)}{(x-5)(x-1)}$$

i.e. $A(x-1)+B(x-5)\equiv 4$

$(A+B)x-(A+5B)\equiv 4$

$$\therefore \begin{cases} A+B=0 \text{ .... (1)} \\ -(A+5B)=4 \text{ .... (2)} \end{cases}$$

$$(1)+(2):\ (A+B)-(A+5B)=0+4$$
$$-4B=4$$
$$B=-1$$

Substitute $B=-1$ into (1),

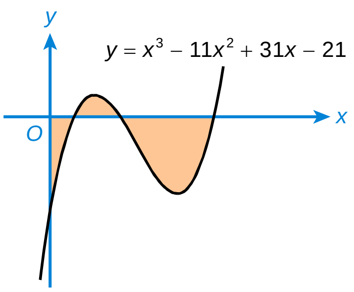
$$A+(-1)=0$$
$$A=1$$

$\therefore$ $\underline{\underline{A=1,\ B=-1}}$

(b) Required area $= \int_2^3 \left| \dfrac{4}{(x-5)(x-1)} \right| dx$

$$= -\int_2^3 \frac{4}{(x-5)(x-1)} dx$$
$$= -\int_2^3 \left(\frac{1}{x-5} + \frac{-1}{x-1}\right) dx$$
$$= -\int_2^3 \frac{1}{x-5} d(x-5) + \int_2^3 \frac{1}{x-1} d(x-1)$$
$$= -[\ln|x-5|]_2^3 + [\ln|x-1|]_2^3$$
$$= -(\ln 2 - \ln 3) + (\ln 2 - \ln 1)$$
$$= \underline{\underline{\ln 3}}$$

20. Find the total area of the shaded regions in the following figure.

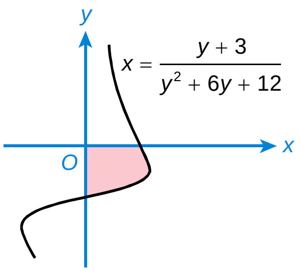


SOLUTION

By solving the simultaneous equations $y = x^3 - 11x^2 + 31x - 21$ and $y = 0$, the points of intersection (1, 0), (3, 0) and (7, 0) can be obtained.

$$\begin{aligned}\text{Required area} &= \int_0^7 |x^3-11x^2+31x-21|dx \\ &= \int_0^1 |x^3-11x^2+31x-21|dx + \int_1^3 |x^3-11x^2+31x-21|dx + \int_3^7 |x^3-11x^2+31x-21|dx \\ &= -\int_0^1 (x^3-11x^2+31x-21)dx + \int_1^3 (x^3-11x^2+31x-21)dx - \int_3^7 (x^3-11x^2+31x-21)dx \\ &= -[\frac{x^4}{4}-\frac{11}{3}x^3+\frac{31}{2}x^2-21x]_0^1 + [\frac{x^4}{4}-\frac{11}{3}x^3+\frac{31}{2}x^2-21x]_1^3 - [\frac{x^4}{4}-\frac{11}{3}x^3+\frac{31}{2}x^2-21x]_3^7 \\ &= -(-\frac{107}{12}) + \frac{20}{3} - (-\frac{128}{3}) \\ &= \underline{\underline{\frac{233}{4}}}\end{aligned}$$

21. Find the area of the region bounded by the curve $x=\dfrac{y+3}{y^2+6y+12}$, the x-axis and the y-axis in the fourth quadrant.

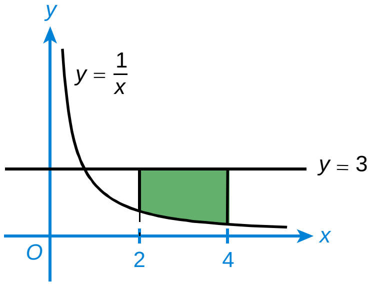


SOLUTION

By solving the simultaneous equations $x=\dfrac{y+3}{y^2+6y+12}$ and $x=0$, the point of intersection (0, −3) can be obtained.

$$\begin{aligned}\text{Required area} &= \int_{-3}^0 \left|\frac{y+3}{y^2+6y+12}\right|dy \\ &= \int_{-3}^0 \frac{y+3}{y^2+6y+12}dy \\ &= \frac{1}{2}\int_{-3}^0 \frac{2y+6}{y^2+6y+12}dy \\ &= \frac{1}{2}\int_{-3}^0 \frac{1}{y^2+6y+12}d(y^2+6y+12) \\ &= \frac{1}{2}[\ln(y^2+6y+12)]_{-3}^0 \\ &= \frac{1}{2}(\ln 12-\ln 3) \\ &= \frac{1}{2}\ln 4 \\ &= \underline{\underline{\ln 2}}\end{aligned}$$

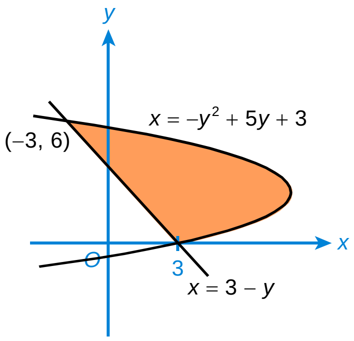
22. Find the area of the region bounded by the curve $y=\frac{1}{x}$ and the line $y=3$ from $x=2$ to $x=4$.



SOLUTION

$$\text{Required area} = \int_2^4 (3-\frac{1}{x})dx$$
$$= [3x - \ln|x|]_2^4$$
$$= (12-\ln 4)-(6-\ln 2)$$
$$= \underline{\underline{6-\ln 2}}$$

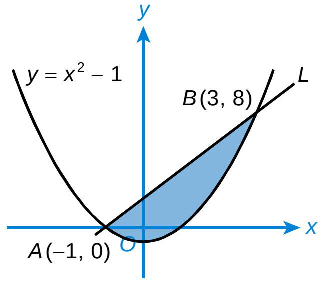
23. Find the area of the region bounded by the curve $x=-y^2+5y+3$ and the line $x=3-y$.



SOLUTION

$$\text{Required area} = \int_0^6 [(-y^2+5y+3)-(3-y)]dy$$
$$= \int_0^6 (-y^2+6y)dy$$
$$= [-\frac{y^3}{3}+3y^2]_0^6$$
$$= \underline{\underline{36}}$$

24. In the figure, the straight line L and the curve $y = x^2 - 1$ intersect at $A(-1, 0)$ and $B(3, 8)$.



(a) Find the equation of L.

(b) Find the area of the shaded region.

SOLUTION

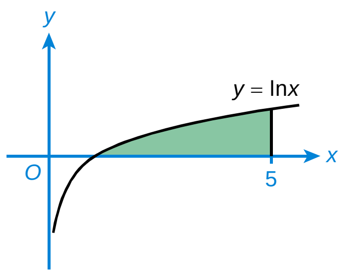
(a) The equation of L is

$$y - 0 = \frac{8-0}{3-(-1)}[x-(-1)]$$
$$y = 2(x+1)$$
$$\underline{\underline{y = 2x + 2}}$$

(b) Required area $= \int_{-1}^{3} [(2x+2)-(x^2-1)]dx$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\,dx$$
$$= [-\frac{x^3}{3} + x^2 + 3x]_{-1}^{3}$$
$$= \underline{\underline{\frac{32}{3}}}$$

25. Find the area of the region bounded by the curve $y = \ln x$, the x-axis and the line $x = 5$.



$A = \int_1^5 (\ln x)dx$
$= [x\ln x - x]_1^5$
$= 5\ln 5 - 4$

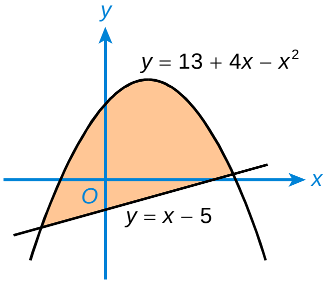
SOLUTION

Rewrite the equation $y = \ln x$ as $x = e^y$.

When $x = 5$, $y = \ln 5$.

$$\text{Required area} = \int_0^{\ln 5} \left|5 - e^y\right| dy$$
$$= \int_0^{\ln 5} (5 - e^y)dy$$
$$= [5y - e^y]_0^{\ln 5}$$
$$= (5\ln 5 - 5) - (-1)$$
$$= \underline{\underline{5\ln 5 - 4}}$$

26. Find the area of the region bounded by the curve $y = 13 + 4x - x^2$ and the line $y = x - 5$.

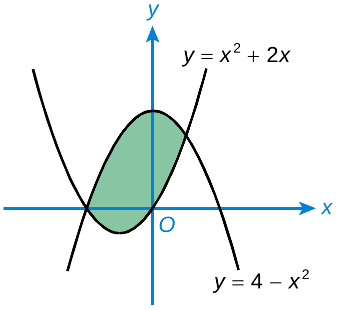


SOLUTION

By solving the simultaneous equations $y = 13 + 4x - x^2$ and $y = x - 5$, the points of intersection (−3, −8) and (6, 1) can be obtained.

$$\text{Required area} = \int_{-3}^{6} [(13 + 4x - x^2) - (x - 5)]dx$$
$$= \int_{-3}^{6} (18 + 3x - x^2)dx$$
$$= [18x + \frac{3}{2}x^2 - \frac{x^3}{3}]_{-3}^{6}$$
$$= \underline{\underline{\frac{243}{2}}}$$

27. Find the area of the region bounded by the curves $y = x^2 + 2x$ and $y = 4 - x^2$.



SOLUTION

By solving the simultaneous equations $y = x^2 + 2x$ and $y = 4 - x^2$, the points of intersection (−2, 0) and (1, 3) can be obtained.

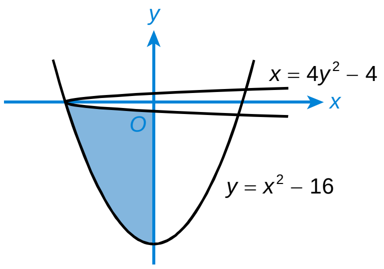
$$\text{Required area} = \int_{-2}^{1} [(4-x^2)-(x^2+2x)]dx$$
$$= \int_{-2}^{1} (4-2x-2x^2)dx$$
$$= [4x - x^2 - \frac{2}{3}x^3]_{-2}^{1}$$
$$= \underline{\underline{9}}$$

28. Find the area of the shaded region in the following figure.

A = ∫0_-1 (4y^2 - 4)dy
= 8/3

B = ∫0_-16 (y+16)^0.5 dy
= 128/3

Area = B - A = 40

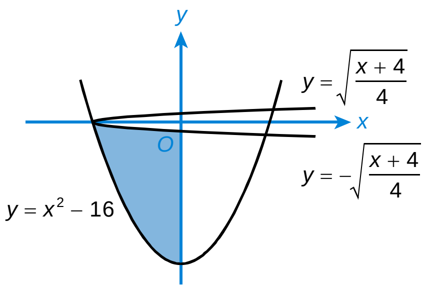


SOLUTION

By solving the simultaneous equations $y = x^2 - 16$ and $y = 0$, the points of intersection (–4, 0) and (4, 0) can be obtained.

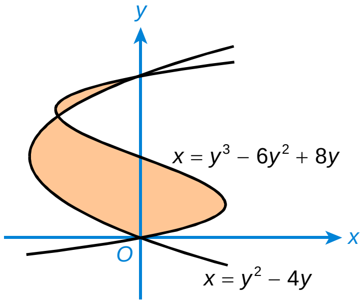
Rewrite the equation $x = 4y^2 - 4$ as $y = \pm\sqrt{\frac{x+4}{4}}$.

As shown in the figure, the curve $x = 4y^2 - 4$ is split into two parts, $y = \sqrt{\frac{x+4}{4}}$ and $y = -\sqrt{\frac{x+4}{4}}$.



$$\text{Required area} = \int_{-4}^{0} [-\sqrt{\frac{x+4}{4}} - (x^2-16)]dx$$
$$= -\int_{-4}^{0} \sqrt{\frac{x+4}{4}}dx - \int_{-4}^{0} (x^2-16)dx$$
$$= -4\int_{-4}^{0} \sqrt{\frac{x+4}{4}}d(\frac{x+4}{4}) - \int_{-4}^{0} (x^2-16)dx$$
$$= -4[\frac{2}{3}(\frac{x+4}{4})^{\frac{3}{2}}]_{-4}^{0} - [\frac{x^3}{3} - 16x]_{-4}^{0}$$
$$= -4(\frac{2}{3}) - (-\frac{128}{3})$$
$$= \underline{\underline{40}}$$

29. Find the area of the region bounded by the curves $x = y^2 - 4y$ and $x = y^3 - 6y^2 + 8y$.

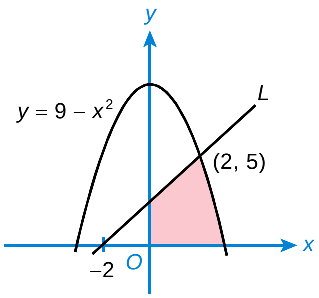


SOLUTION

By solving the simultaneous equations $x = y^2 - 4y$ and $x = y^3 - 6y^2 + 8y$, the points of intersection (0, 0), (–3, 3) and (0, 4) can be obtained.

$$\begin{aligned}
\text{Required area} &= \int_0^3 [(y^3 - 6y^2 + 8y) - (y^2 - 4y)]dy + \int_3^4 [(y^2 - 4y) - (y^3 - 6y^2 + 8y)]dy \\
&= \int_0^3 (y^3 - 7y^2 + 12y)dy + \int_3^4 (-y^3 + 7y^2 - 12y)dy \\
&= [\frac{y^4}{4} - \frac{7}{3}y^3 + 6y^2]_0^3 + [-\frac{y^4}{4} + \frac{7}{3}y^3 - 6y^2]_3^4 \\
&= \frac{45}{4} + \frac{7}{12} \\
&= \underline{\underline{\frac{71}{6}}}
\end{aligned}$$

30. In the figure, the straight line L cuts the x-axis at $x = -2$ and intersects the curve $y = 9 - x^2$ at (2, 5). Find the area of the region bounded by the x-axis, the y-axis, the straight line L and the curve in the first quadrant.



SOLUTION

The equation of L is

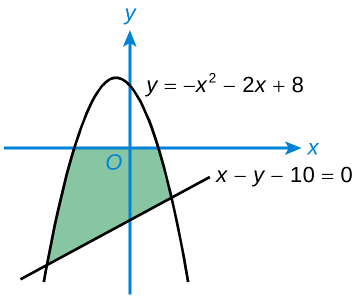
$$y - 0 = \frac{5-0}{2-(-2)}[x - (-2)]$$

$$y = \frac{5}{4}(x + 2)$$

By solving the simultaneous equations $y = 9 - x^2$ and $y = 0$, the points of intersection (−3, 0) and (3, 0) can be obtained.

$$\text{Required area} = \int_0^2 \frac{5}{4}(x+2)dx + \int_2^3 (9-x^2)dx$$
$$= \frac{5}{4}[\frac{1}{2}x^2 + 2x]_0^2 + [9x - \frac{1}{3}x^3]_2^3$$
$$= \frac{5}{4}(6) + \frac{8}{3}$$
$$= \underline{\underline{\frac{61}{6}}}$$

31. Find the area of the region bounded by the curve $y = -x^2 - 2x + 8$, the x-axis and the line $x - y - 10 = 0$.



SOLUTION

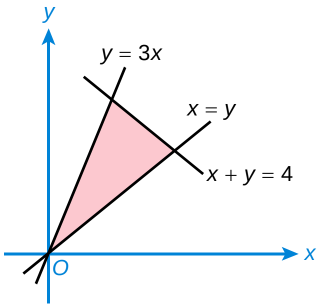
By solving the simultaneous equations $y = -x^2 - 2x + 8$ and $x - y - 10 = 0$, the points of intersection (−6, −16) and (3, −7) can be obtained.

By solving the simultaneous equations $y = -x^2 - 2x + 8$ and $y = 0$, the points of intersection (−4, 0) and (2, 0) can be obtained.

Rewrite the equation $x - y - 10 = 0$ as $y = x - 10$.

$$\text{Required area} = \int_{-6}^{3} [(-x^2 - 2x + 8) - (x - 10)]dx - \int_{-4}^{2} \left| -x^2 - 2x + 8 \right| dx$$
$$= \int_{-6}^{3} (-x^2 - 3x + 18)dx - \int_{-4}^{2} (-x^2 - 2x + 8)dx$$
$$= [-\frac{x^3}{3} - \frac{3}{2}x^2 + 18x]_{-6}^{3} - [-\frac{x^3}{3} - x^2 + 8x]_{-4}^{2}$$
$$= \frac{243}{2} - 36$$
$$= \underline{\underline{\frac{171}{2}}}$$

32. Find the area of the region bounded by the lines $x+y=4$, $y=3x$ and $x=y$.



SOLUTION

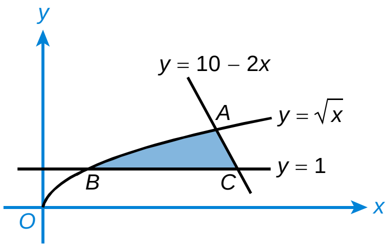
By solving the simultaneous equations $y=3x$ and $x+y=4$, the point of intersection (1, 3) can be obtained.

By solving the simultaneous equations $x=y$ and $x+y=4$, the point of intersection (2, 2) can be obtained.

Rewrite the equations as $y=4-x$, $y=3x$ and $y=x$.

$$\begin{aligned}\text{Required area} &= \int_0^1 (3x-x)dx+\int_1^2 [(4-x)-x]dx \\ &= \int_0^1 2x\,dx+\int_1^2 (4-2x)dx \\ &= [x^2]_0^1+[4x-x^2]_1^2 \\ &= 1+1 \\ &= \underline{\underline{2}}\end{aligned}$$

33. In the figure, the curve $y=\sqrt{x}$, the lines $y=10-2x$ and $y=1$ intersect at A, B and C.



(a) Find the coordinates of A, B and C.

(b) Find the area of the shaded region.

SOLUTION

(a) $\begin{cases} y=10-2x \text{ ....(1)} \\ y=\sqrt{x} \text{ ....(2)} \end{cases}$

Substitute (1) into (2),

$$10-2x=\sqrt{x}$$

$$(10-2x)^2=x$$

$$100-40x+4x^2=x$$

$$4x^2-41x+100=0$$

$$(4x-25)(x-4)=0$$

$$x=\frac{25}{4} \text{ or } x=4$$

when $x=\frac{25}{4}$,

$$y=10-2(\frac{25}{4})$$

$$=-\frac{5}{2} \text{ (rejected)}$$

When $x=4$,

$$y=10-2(4)$$

$$=2$$

$\therefore$ The coordinates of A are (4, 2).

$$\begin{cases} y=\sqrt{x} \text{ .... (2)} \\ y=1 \text{ .... (3)} \end{cases}$$

Substitute (2) into (3),

$$\sqrt{x}=1$$

$$x=1$$

$\therefore$ The coordinates of B are (1, 1).

$$\begin{cases} y=10-2x \text{ .... (1)} \\ y=1 \text{ .... (3)} \end{cases}$$

Substitute (3) into (1),

$$1=10-2x$$

$$x=\frac{9}{2}$$

$\therefore$ The coordinates of C are $(\frac{9}{2}, 1)$.

(b) Required area $= \int_1^4 (\sqrt{x}-1)dx + \int_4^{\frac{9}{2}} [(10-2x)-1]dx$

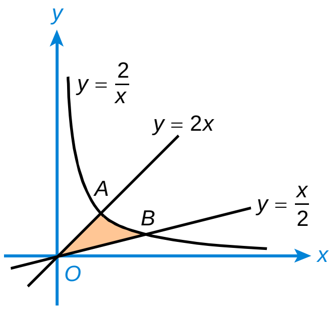
$$= \int_1^4 (\sqrt{x}-1)dx + \int_4^{\frac{9}{2}} (9-2x)dx$$

$$= [\frac{2}{3}x^{\frac{3}{2}} - x]_1^4 + [9x - x^2]_4^{\frac{9}{2}}$$

$$= \frac{5}{3} + \frac{1}{4}$$

$$= \underline{\underline{\frac{23}{12}}}$$

34. In the figure, the lines $y = 2x$ and $y = \frac{x}{2}$ intersect the curve $y = \frac{2}{x}$ at A and B respectively.



(a) Find the coordinates of A and B.

(b) Find the area of the shaded region.

SOLUTION

(a) $\begin{cases} y = \dfrac{2}{x} \text{ ....(1)} \\ y = 2x \text{ ....(2)} \end{cases}$

Substitute (2) into (1),

$$2x = \frac{2}{x}$$

$$x^2 = 1$$

$$x = 1 \quad \text{or} \quad -1 \text{ (rejected)}$$

When $x = 1$, $y = \frac{2}{1}$

$$= 2$$

$\therefore$ $\underline{\underline{\text{The coordinates of } A \text{ are } (1, 2).}}$

$$\begin{cases} y = \dfrac{2}{x} \text{..................(1)} \\ y = \dfrac{x}{2} \text{..................(3)} \end{cases}$$

Substitute (3) into (1),

$$\frac{x}{2} = \frac{2}{x}$$

$$x^2 = 4$$

$$x = 2 \quad \text{or} \quad -2 \text{ (rejected)}$$

When $x = 2$, $y = \dfrac{2}{2}$

$= 1$

$\therefore$ The coordinates of B are (2, 1).

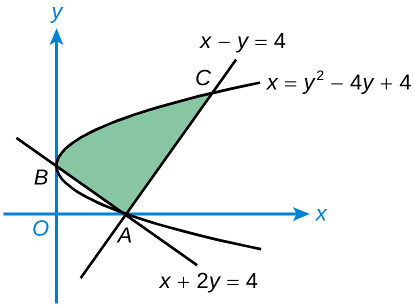
(b) Required area $= \int_0^1 (2x - \frac{x}{2})dx + \int_1^2 (\frac{2}{x} - \frac{x}{2})dx$

$$= [x^2 - \frac{x^2}{4}]_0^1 + [2\ln|x| - \frac{x^2}{4}]_1^2$$

$$= \frac{3}{4} + (2\ln 2 - \frac{3}{4})$$

$$= \underline{\underline{2\ln 2}}$$

35. In the figure, the lines $x + 2y = 4$ and $x - y = 4$ intersect the curve $x = y^2 - 4y + 4$ at points A, B and points A, C respectively. Find the area of the region bounded by AB, AC and the curve $x = y^2 - 4y + 4$.



SOLUTION

By solving the simultaneous equations $x + 2y = 4$ and $x = y^2 - 4y + 4$, the points of intersection $A(4, 0)$ and $B(0, 2)$ can be obtained.

By solving the simultaneous equations $x - y = 4$ and $x = y^2 - 4y + 4$, the points of intersection $A(4, 0)$ and $C(9, 5)$ can be obtained.

Rewrite the equations as $x = 4 - 2y$ and $x = 4 + y$.

$$\text{Required area} = \int_0^2 [(4+y)-(4-2y)]dy + \int_2^5 [(4+y)-(y^2-4y+4)]dy$$

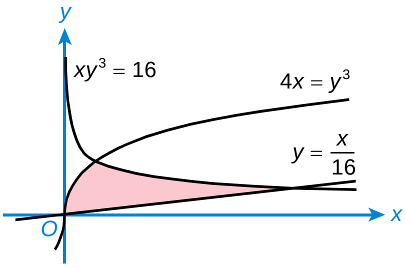
$$= \int_0^2 3y\,dy + \int_2^5 (-y^2+5y)dy$$

$$= [\frac{3}{2}y^2]_0^2 + [-\frac{y^3}{3}+\frac{5}{2}y^2]_2^5$$

$$= 6 + \frac{27}{2}$$

$$= \underline{\underline{\frac{39}{2}}}$$

36. Find the area of the region bounded by the curves $xy^3 = 16$, $4x = y^3$ and the line $y = \frac{x}{16}$ in the first quadrant.



SOLUTION

By solving the simultaneous equations $xy^3 = 16$ and $4x = y^3$, the points of intersection (2, 2) and (−2, −2) can be obtained.

By solving the simultaneous equations $xy^3 = 16$ and $y = \frac{x}{16}$, the points of intersection (16, 1) and (−16, −1) can be obtained.

Rewrite the equations as $x = \frac{16}{y^3}$, $x = \frac{y^3}{4}$ and $x = 16y$.

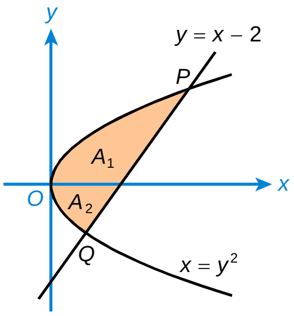
$$\text{Required area} = \int_0^1 (16y - \frac{y^3}{4})dy + \int_1^2 (\frac{16}{y^3} - \frac{y^3}{4})dy$$

$$= [8y^2 - \frac{y^4}{16}]_0^1 + [-\frac{8}{y^2} - \frac{y^4}{16}]_1^2$$

$$= \frac{127}{16} + \frac{81}{16}$$

$$= \underline{\underline{13}}$$

37. In the figure, the shaded region is bounded by the curve $x = y^2$ and the line $y = x - 2$. The x-axis divides the shaded region into A_1 and A_2.



(a) Find the coordinates of P and Q.

(b) Find the ratio of the area of A_1 to that of A_2.

SOLUTION

(a) $\begin{cases} x = y^2 \text{ ....(1)} \\ y = x - 2 \text{ ....(2)} \end{cases}$

Substitute (2) into (1),

$$x = (x-2)^2$$
$$x = x^2 - 4x + 4$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 1$$
$$x = 4 \text{ or } x = 1$$

When $x = 4$,

$y = 4 - 2$

$= 2$

$\therefore$ $\underline{\underline{\text{The coordinates of } P \text{ are } (4, 2).}}$

When $x = 1$,

$y = 1 - 2$

$= -1$

$\therefore$ $\underline{\underline{\text{The coordinates of } Q \text{ are } (1, -1).}}$

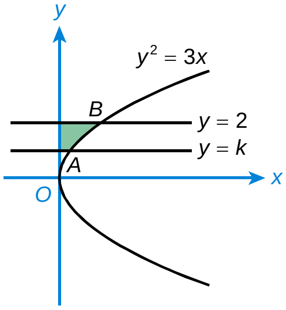
(b) Rewrite the equation $y = x - 2$ as $x = y + 2$.

$$\text{Area of } A_1 = \int_0^2 [(y+2) - y^2]\,dy$$
$$= [\frac{y^2}{2} + 2y - \frac{y^3}{3}]_0^2$$
$$= \frac{10}{3}$$

$$\text{Area of } A_2 = \int_{-1}^{0} [(y+2) - y^2]dy$$
$$= [\frac{y^2}{2} + 2y - \frac{y^3}{3}]_{-1}^{0}$$
$$= \frac{7}{6}$$

$\therefore$ Ratio of the area of A_1 to that of $A_2 = \frac{10}{3} : \frac{7}{6}$
$= \underline{\underline{20:7}}$

38. In the figure, the straight lines $y = k$ (where $0 < k < 2$) and $y = 2$ intersect the curve $y^2 = 3x$ at A and B respectively.



(a) **(i)** Express the coordinates of A in terms of k.

(ii) Find the coordinates of B.

(b) If the area of the shaded region is $\frac{37}{72}$, find the value of k.

SOLUTION

(a) **(i)** $\begin{cases} y^2 = 3x \text{ ....(1)} \\ y = k \text{ ....(2)} \end{cases}$

Substitute (2) into (1), $k^2 = 3x$

$$x = \frac{k^2}{3}$$

$\therefore$ $\underline{\underline{\text{The coordinates of } A \text{ are } (\frac{k^2}{3}, k).}}$

(ii) $\begin{cases} y^2 = 3x \text{ ....(1)} \\ y = 2 \text{ ....(3)} \end{cases}$

Substitute (3) into (1), $2^2 = 3x$

$$x = \frac{4}{3}$$

$\therefore$ $\underline{\underline{\text{The coordinates of } B \text{ are } (\frac{4}{3}, 2).}}$

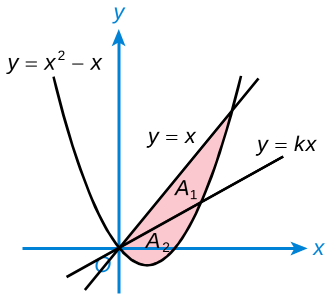
(b) Rewrite the equation $y^2 = 3x$ as $x = \frac{y^2}{3}$.

$$\text{Area of the shaded region} = \int_k^2 \left|\frac{y^2}{3}\right| dy$$
$$= \int_k^2 \frac{y^2}{3} dy$$
$$= [\frac{y^3}{9}]_k^2$$
$$= \frac{1}{9}(8 - k^3)$$

$\because$ Area of the shaded region $= \frac{37}{72}$

$$\therefore \quad \frac{1}{9}(8 - k^3) = \frac{37}{72}$$
$$8 - k^3 = \frac{37}{8}$$
$$k^3 = 8 - \frac{37}{8}$$
$$k^3 = \frac{27}{8}$$
$$k = \underline{\underline{\frac{3}{2}}}$$

39. In the figure, the shaded region is bounded by the curve $y = x^2 - x$ and the line $y = x$.



(a) Find the area of the shaded region.

(b) If the straight line $y = kx$ $(0 < k < 1)$ divides the shaded region into A_1 and A_2, where the areas of A_1 and A_2 are equal, find the value of k.

SOLUTION

(a) By solving the simultaneous equations $y = x^2 - x$ and $y = x$, the points of intersection (0, 0) and (2, 2) can be obtained.

$$\begin{aligned}\text{Required area} &= \int_0^2 [x-(x^2-x)]dx \\ &= \int_0^2 (2x-x^2)\,dx \\ &= [x^2-\frac{x^3}{3}]_0^2 \\ &= \underline{\underline{\frac{4}{3}}}\end{aligned}$$

(b) By solving the simultaneous equations $y=x^2-x$ and $y=kx$, the points of intersection (0, 0) and $(k+1,\ k^2+k)$ can be obtained.

$$\begin{aligned}\text{Area of } A_2 &= \int_0^{k+1} [kx-(x^2-x)]dx \\ &= \int_0^{k+1} [(k+1)x-x^2]dx \\ &= [\frac{k+1}{2}x^2-\frac{x^3}{3}]_0^{k+1} \\ &= \frac{k+1}{2}(k+1)^2-\frac{(k+1)^3}{3} \\ &= \frac{(k+1)^3}{6}\end{aligned}$$

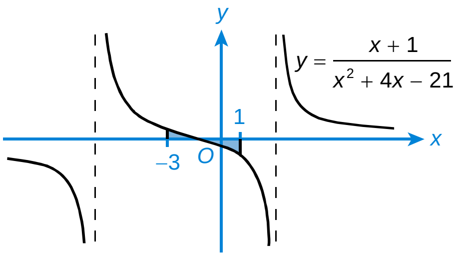
$$\begin{aligned}\text{Area of } A_1+\text{Area of } A_2 &= \frac{4}{3} \quad \text{[From the result of (a)]} \\ 2\times\text{Area of } A_2 &= \frac{4}{3} \\ \text{Area of } A_2 &= \frac{2}{3}\end{aligned}$$

$$\begin{aligned}\therefore \quad \frac{(k+1)^3}{6} &= \frac{2}{3} \\ (k+1)^3 &= 4 \\ k &= \underline{\underline{\sqrt[3]{4}-1}}\end{aligned}$$

Level 3

40. (a) If $\frac{5x+5}{(x-3)(x+7)} \equiv \frac{A}{x-3}+\frac{B}{x+7}$, find the values of constants A and B.

(b) Find the area of the region bounded by the curve $y=\frac{x+1}{x^2+4x-21}$ and the x-axis from $x=-3$ to $x=1$.



SOLUTION

(a) $\dfrac{5x+5}{(x-3)(x+7)} \equiv \dfrac{A}{x-3}+\dfrac{B}{x+7}$

$\equiv \dfrac{A(x+7)+B(x-3)}{(x-3)(x+7)}$

i.e. $A(x+7)+B(x-3) \equiv 5x+5$

$(A+B)x+(7A-3B) \equiv 5x+5$

$\therefore \begin{cases} A+B=5 \text{..........................(1)} \\ 7A-3B=5 \text{.....................(2)} \end{cases}$

$(1)\times 3$: $3A+3B=15$..................(3)

$(2)+(3)$: $(7A-3B)+(3A+3B)=5+15$

$10A=20$

$A=2$

Substitute $A=2$ into (1),

$2+B=5$

$B=3$

$\therefore \quad \underline{\underline{A=2,\ B=3}}$

(b) By solving the simultaneous equations $y=\dfrac{x+1}{x^2+4x-21}$ and $y=0$, the point of intersection (−1, 0) can be obtained.

Required area

$$=\int_{-3}^{1}\left|\frac{x+1}{x^2+4x-21}\right|dx$$

$$=\int_{-3}^{-1}\left|\frac{x+1}{x^2+4x-21}\right|dx+\int_{-1}^{1}\left|\frac{x+1}{x^2+4x-21}\right|dx$$

$$=\int_{-3}^{-1}\frac{x+1}{x^2+4x-21}dx-\int_{-1}^{1}\frac{x+1}{x^2+4x-21}dx$$

$$=\frac{1}{5}\int_{-3}^{-1}\frac{5x+5}{(x-3)(x+7)}dx-\frac{1}{5}\int_{-1}^{1}\frac{5x+5}{(x-3)(x+7)}dx$$

$$=\frac{1}{5}\int_{-3}^{-1}\left(\frac{2}{x-3}+\frac{3}{x+7}\right)dx-\frac{1}{5}\int_{-1}^{1}\left(\frac{2}{x-3}+\frac{3}{x+7}\right)dx$$ [From the result of **(a)**]

$$=\frac{1}{5}\int_{-3}^{-1}\frac{2}{x-3}d(x-3)+\frac{1}{5}\int_{-3}^{-1}\frac{3}{x+7}d(x+7)-\frac{1}{5}\int_{-1}^{1}\frac{2}{x-3}d(x-3)-\frac{1}{5}\int_{-1}^{1}\frac{3}{x+7}d(x+7)$$

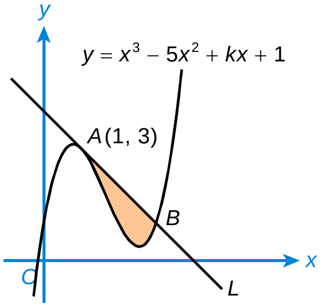
$$=\frac{2}{5}[\ln|x-3|]_{-3}^{-1}+\frac{3}{5}[\ln|x+7|]_{-3}^{-1}-\frac{2}{5}[\ln|x-3|]_{-1}^{1}-\frac{3}{5}[\ln|x+7|]_{-1}^{1}$$

$$=\frac{2}{5}\ln\frac{2}{3}+\frac{3}{5}\ln\frac{3}{2}-\frac{2}{5}\ln\frac{1}{2}-\frac{3}{5}\ln\frac{4}{3}$$

$$=\frac{2}{5}\ln 2-\frac{2}{5}\ln 3+\frac{3}{5}\ln 3-\frac{3}{5}\ln 2+\frac{2}{5}\ln 2-\frac{6}{5}\ln 2+\frac{3}{5}\ln 3$$

$$=\underline{\underline{-\ln 2+\frac{4}{5}\ln 3}}$$

41. In the figure, the straight line L is the tangent to the curve $y = x^3 - 5x^2 + kx + 1$ at the point $A(1, 3)$, where k is a constant. It is known that L intersects the curve again at point B.



(a) Find the value of k.
(b) Find the equation of L.
(c) Find the coordinates of B.
(d) Find the area of the shaded region.

SOLUTION

(a) Substitute (1, 3) into $y = x^3 - 5x^2 + kx + 1$,

$$3 = 1^3 - 5(1)^2 + k(1) + 1$$
$$k = \underline{\underline{6}}$$

(b) $$y = x^3 - 5x^2 + 6x + 1$$
$$\frac{dy}{dx} = 3x^2 - 10x + 6$$

Slope of the tangent at $A(1, 3) = \left.\frac{dy}{dx}\right|_{x=1}$
$$= 3(1)^2 - 10(1) + 6$$
$$= -1$$

The equation of L is
$$y - 3 = -(x - 1)$$
$$\underline{\underline{x + y - 4 = 0}}$$

(c) $$\begin{cases} x + y - 4 = 0 \text{ ....} (1) \\ y = x^3 - 5x^2 + 6x + 1 \text{ ....} (2) \end{cases}$$

From (1), $y = -x + 4$ ....(3)

Substitute (3) into (2),
$$-x + 4 = x^3 - 5x^2 + 6x + 1$$
$$x^3 - 5x^2 + 7x - 3 = 0$$
$$(x - 3)(x - 1)^2 = 0$$
$$x = 3 \quad \text{or} \quad x = 1 \text{ (rejected)}$$

When $x=3$,
$y=-3+4$
$=1$
$\therefore$ The coordinates of B are (3, 1).

(d) Rewrite the equation $x+y-4=0$ as $y=-x+4$.

$$\text{Required area} = \int_1^3 [(-x+4)-(x^3-5x^2+6x+1)]dx$$
$$= \int_1^3 (-x^3+5x^2-7x+3)dx$$
$$= [-\frac{x^4}{4}+\frac{5}{3}x^3-\frac{7}{2}x^2+3x]_1^3$$
$$= \frac{4}{3}$$

EXERCISE 9B

Level 1

42. For the following table, find an approximate value of $\int_a^b f(x)dx$ by using the trapezoidal rule with 8 subintervals. (Give your answer correct to 2 decimal places.)

x	$a=2$	2.375	2.75	3.125	3.5	3.875	4.25	4.625	$b=5$
$f(x)$	20.09	29.22	42.52	61.87	90.02	130.97	190.57	277.27	403.43

SOLUTION

$\Delta x = 0.375$

$$\int_a^b f(x)dx \approx \frac{0.375}{2}[f(2)+2f(2.375)+2f(2.75)+2f(3.125)+2f(3.5)+2f(3.875)+2f(4.25)+2f(4.625)+f(5)]$$
$$= \frac{0.375}{2}[20.09+2(29.22)+2(42.52)+2(61.87)+2(90.02)+2(130.97)+2(190.57)+2(277.27)+403.43]$$
$$= 387.83 \text{ (corr. to 2 d.p.)}$$

43. For the following table, find an approximate value of $\int_1^4 f(x)dx$ by using the trapezoidal rule with 6 subintervals.

x	1	1.5	2	2.5	3	3.5	4
$f(x)$	0.500 0	0.400 0	0.333 3	0.285 7	0.250 0	0.222 2	0.200 0

SOLUTION

$\Delta x = 0.5$

$$\int_1^4 f(x)dx \approx \frac{0.5}{2}[f(1)+2f(1.5)+2f(2)+2f(2.5)+2f(3)+2f(3.5)+f(4)]$$

$$= \frac{0.5}{2}[0.5000+2(0.4000)+2(0.3333)+2(0.2857)+2(0.2500)+2(0.2222)+0.2000]$$

$$= \underline{\underline{0.9206}}$$

44. For the following table, find an approximate value of $\int_{-3}^{-1} f(x)dx$ by using the trapezoidal rule with 5 subintervals. (Give your answer correct to 3 decimal places.)

x	−3	−2.6	−2.2	−1.8	−1.4	−1
$f(x)$	−0.444	−0.379	−0.298	−0.198	−0.082	0

SOLUTION

$\Delta x = 0.4$

$$\int_{-3}^{-1} f(x)dx \approx \frac{0.4}{2}[f(-3)+2f(-2.6)+2f(-2.2)+2f(-1.8)+2f(-1.4)+f(-1)]$$

$$= \frac{0.4}{2}[-0.444+2(-0.379)+2(-0.298)+2(-0.198)+2(-0.082)+0]$$

$$= \underline{\underline{-0.472}} \text{ (corr. to 3 d.p.)}$$

45. For the following table, find an approximate value of $\int_{-1}^{2} f(x)dx$ by using the trapezoidal rule with 5 subintervals. (Give your answer correct to 4 decimal places.)

x	−1	−0.4	0.2	0.8	1.4	2
$f(x)$	1.550 7	1.273 9	0.826 5	0.103 5	−1.064 9	−2.953 0

SOLUTION

$\Delta x = 0.6$

$$\int_{-1}^{2} f(x)dx \approx \frac{0.6}{2}[f(-1)+2f(-0.4)+2f(0.2)+2f(0.8)+2f(1.4)+f(2)]$$

$$= \frac{0.6}{2}[1.5507+2(1.2739)+2(0.8265)+2(0.1035)+2(-1.0649)+(-2.9530)]$$

$$= \underline{\underline{0.2627}} \text{ (corr. to 4 d.p.)}$$

46. For the following table, find an approximate value of $\int_{-1}^{1} f(x)dx$ by using the trapezoidal rule with 8 subintervals. (Give your answer correct to 2 decimal places.)

x	−1	−0.75	−0.5	−0.25	0	0.25	0.5	0.75	1
$f(x)$	27.00	34.33	42.88	52.73	64.00	76.77	91.13	107.17	125.00

SOLUTION

$\Delta x = 0.25$

$$\int_{-1}^{1} f(x)dx \approx \frac{0.25}{2}[f(-1)+2f(-0.75)+2f(-0.5)+2f(-0.25)+2f(0)+2f(0.25)+2f(0.5)$$
$$+2f(0.75)+f(1)]$$

$$= \frac{0.25}{2}[27.00+2(34.33)+2(42.88)+2(52.73)+2(64.00)+2(76.77)+2(91.13)$$
$$+2(107.17)+125.00]$$

$$= \underline{\underline{136.25}} \text{ (corr. to 2 d.p.)}$$

47. For the following table, find an approximate value of $\int_{0.75}^{3} f(x)dx$ by using the trapezoidal rule with 6 subintervals. (Give your answer correct to 3 decimal places.)

x	0	0.375	0.75	1.125	1.5	1.875	2.25	2.625	3
$f(x)$	0.693	0.865	1.012	1.139	1.253	1.355	1.447	1.531	1.609

SOLUTION

$\Delta x = 0.375$

$$\int_{0.75}^{3} f(x)dx \approx \frac{0.375}{2}[f(0.75)+2f(1.125)+2f(1.5)+2f(1.875)+2f(2.25)+2f(2.625)+f(3)]$$

$$= \frac{0.375}{2}[1.012+2(1.139)+2(1.253)+2(1.355)+2(1.447)+2(1.531)+1.609]$$

$$= \underline{\underline{3.013}} \text{ (corr. to 3 d.p.)}$$

48. Use the trapezoidal rule with 4 subintervals to estimate $\int_0^2 \sqrt{4-x^2}\,dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x)=\sqrt{4-x^2}$.

$$\Delta x=\frac{2-0}{4}=0.5$$

$$\int_0^2 \sqrt{4-x^2}\,dx \approx \frac{0.5}{2}[f(0)+2f(0.5)+2f(1)+2f(1.5)+f(2)]$$
$$=\frac{0.5}{2}(\sqrt{4-0^2}+2\sqrt{4-0.5^2}+2\sqrt{4-1^2}+2\sqrt{4-1.5^2}+\sqrt{4-2^2})$$
$$=\underline{\underline{2.9957}} \text{ (corr. to 4 d.p.)}$$

49. Use the trapezoidal rule with 6 subintervals to estimate $\int_0^3 \frac{dx}{x^2+1}$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x)=\frac{1}{x^2+1}$.

$$\Delta x=\frac{3-0}{6}=0.5$$

$$\int_0^3 \frac{dx}{x^2+1} \approx \frac{0.5}{2}[f(0)+2f(0.5)+2f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)]$$
$$=\frac{0.5}{2}\left(\frac{1}{0^2+1}+\frac{2}{0.5^2+1}+\frac{2}{1^2+1}+\frac{2}{1.5^2+1}+\frac{2}{2^2+1}+\frac{2}{2.5^2+1}+\frac{1}{3^2+1}\right)$$
$$=\underline{\underline{1.2478}} \text{ (corr. to 4 d.p.)}$$

50. Use the trapezoidal rule with 5 subintervals to estimate $\int_1^2 \frac{x-1}{x(x+1)}dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x)=\frac{x-1}{x(x+1)}$.

$\Delta x = \dfrac{2-1}{5}$

$= 0.2$

$$\int_1^2 \frac{x-1}{x(x+1)}dx \approx \frac{0.2}{2}[f(1)+2f(1.2)+2f(1.4)+2f(1.6)+2f(1.8)+f(2)]$$

$$= \frac{0.2}{2}[\frac{1-1}{(1)(1+1)}+\frac{2(1.2-1)}{(1.2)(1.2+1)}+\frac{2(1.4-1)}{(1.4)(1.4+1)}+\frac{2(1.6-1)}{(1.6)(1.6+1)}+\frac{2(1.8-1)}{(1.8)(1.8+1)}+\frac{2-1}{2(2+1)}]$$

$= \underline{\underline{0.1162}}$ (corr. to 4 d.p.)

51. Use the trapezoidal rule with 8 subintervals to estimate $\int_1^2 x\ln x\,dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x) = x\ln x$.

$\Delta x = \dfrac{2-1}{8}$

$= 0.125$

$$\int_1^2 x\ln x\,dx \approx \frac{0.125}{2}[f(1)+2f(1.125)+2f(1.25)+2f(1.375)+2f(1.5)+2f(1.625)+2f(1.75)+2f(1.875)+f(2)]$$

$$= \frac{0.125}{2}[1\ln 1+2(1.125)\ln 1.125+2(1.25)\ln 1.25+2(1.375)\ln 1.375+2(1.5)\ln 1.5+2(1.625)\ln 1.625+2(1.75)\ln 1.75+2(1.875)\ln 1.875+2\ln 2]$$

$= \underline{\underline{0.6372}}$ (corr. to 4 d.p.)

52. Use the trapezoidal rule with 10 subintervals to estimate $\int_0^3 \ln(x^2+1)dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x) = \ln(x^2+1)$.

$\Delta x = \dfrac{3-0}{10}$

$= 0.3$

$$\int_0^3 \ln(x^2+1)dx \approx \frac{0.3}{2}[f(0)+2f(0.3)+2f(0.6)+2f(0.9)+2f(1.2)+2f(1.5)$$
$$+2f(1.8)+2f(2.1)+2f(2.4)+2f(2.7)+f(3)]$$
$$=\frac{0.3}{2}[\ln(0^2+1)+2\ln(0.3^2+1)+2\ln(0.6^2+1)+2\ln(0.9^2+1)$$
$$+2\ln(1.2^2+1)+2\ln(1.5^2+1)+2\ln(1.8^2+1)+2\ln(2.1^2+1)$$
$$+2\ln(2.4^2+1)+2\ln(2.7^2+1)+\ln(3^2+1)]$$
$$=\underline{\underline{3.4103}} \text{ (corr. to 4 d.p.)}$$

53. Use the trapezoidal rule with 5 subintervals to estimate $\int_0^4 e^{\frac{x^2}{10}}dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x)=e^{\frac{x^2}{10}}$.

$$\Delta x=\frac{4-0}{5}=0.8$$

$$\int_0^4 e^{\frac{x^2}{10}}dx \approx \frac{0.8}{2}[f(0)+2f(0.8)+2f(1.6)+2f(2.4)+2f(3.2)+f(4)]$$
$$=\frac{0.8}{2}(e^{\frac{0^2}{10}}+2e^{\frac{0.8^2}{10}}+2e^{\frac{1.6^2}{10}}+2e^{\frac{2.4^2}{10}}+2e^{\frac{3.2^2}{10}}+e^{\frac{4^2}{10}})$$
$$=\underline{\underline{7.9181}} \text{ (corr. to 4 d.p.)}$$

54. Use the trapezoidal rule with 8 subintervals to estimate $\int_0^6 \frac{e^x}{e^x+4}dx$. (Give your answer correct to 4 decimal places.)

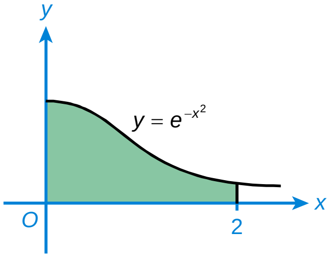
SOLUTION

Let $f(x)=\frac{e^x}{e^x+4}$.

$$\Delta x=\frac{6-0}{8}=0.75$$

$$\int_0^6 \frac{e^x}{e^x+4}dx \approx \frac{0.75}{2}[f(0)+2f(0.75)+2f(1.5)+2f(2.25)+2f(3)+2f(3.75)$$
$$+2f(4.5)+2f(5.25)+f(6)]$$
$$=\frac{0.75}{2}(\frac{e^0}{e^0+4}+\frac{2e^{0.75}}{e^{0.75}+4}+\frac{2e^{1.5}}{e^{1.5}+4}+\frac{2e^{2.25}}{e^{2.25}+4}+\frac{2e^3}{e^3+4}+\frac{2e^{3.75}}{e^{3.75}+4}+\frac{2e^{4.5}}{e^{4.5}+4}$$
$$+\frac{2e^{5.25}}{e^{5.25}+4}+\frac{e^6}{e^6+4})$$
$$=\underline{\underline{4.3934}} \text{ (corr. to 4 d.p.)}$$

55. The figure shows a sketch of $y = e^{-x^2}$, where $x \geq 0$.



Use the trapezoidal rule with 5 subintervals to estimate the area of the shaded region. (Give your answer correct to 4 decimal places.)

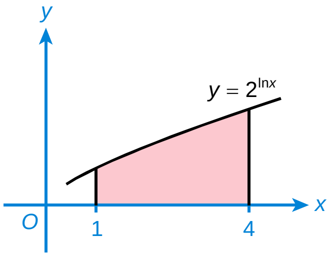
SOLUTION

Let $f(x) = e^{-x^2}$.

$$\Delta x = \frac{2-0}{5} = 0.4$$

$$\begin{aligned}\text{Required area} &= \int_0^2 e^{-x^2}\,dx \\ &\approx \frac{0.4}{2}[f(0) + 2f(0.4) + 2f(0.8) + 2f(1.2) + 2f(1.6) + f(2)] \\ &= \frac{0.4}{2}(e^{-0^2} + 2e^{-0.4^2} + 2e^{-0.8^2} + 2e^{-1.2^2} + 2e^{-1.6^2} + e^{-2^2}) \\ &= \underline{\underline{0.8811}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

56. The figure shows a sketch of $y = 2^{\ln x}$.



Use the trapezoidal rule with 8 subintervals to estimate the area of the shaded region. (Give your answer correct to 4 decimal places.)

SOLUTION

Let $f(x) = 2^{\ln x}$.

$$\Delta x = \frac{4-1}{8} = 0.375$$

$$\text{Required area} = \int_1^4 2^{\ln x}\,dx$$

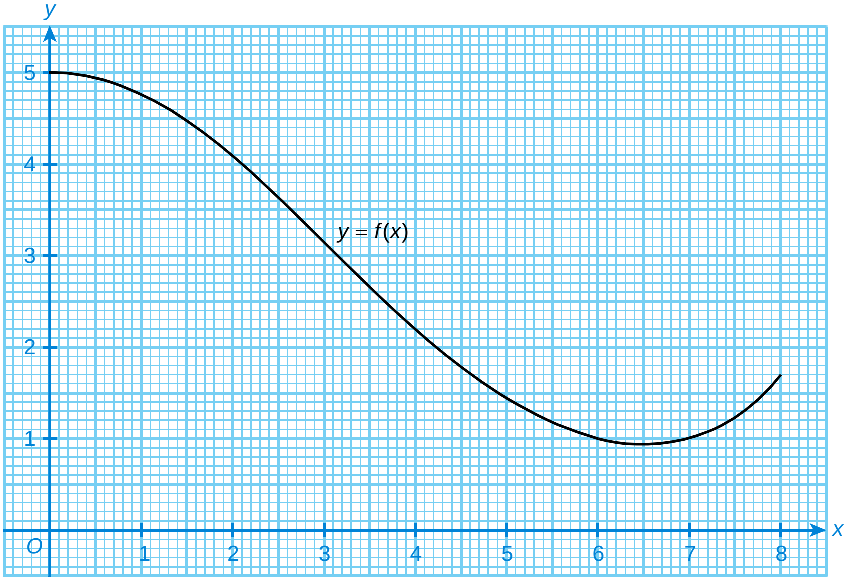
$$\approx \frac{0.375}{2}[f(1)+2f(1.375)+2f(1.75)+2f(2.125)+2f(2.5)+2f(2.875)$$
$$+2f(3.25)+2f(3.625)+f(4)]$$

$$= \frac{0.375}{2}[2^{\ln 1}+2(2^{\ln 1.375})+2(2^{\ln 1.75})+2(2^{\ln 2.125})+2(2^{\ln 2.5})+2(2^{\ln 2.875})$$
$$+2(2^{\ln 3.25})+2(2^{\ln 3.625})+2^{\ln 4}]$$

$$= \underline{\underline{5.5822}} \text{ (corr. to 4 d.p.)}$$

Level 2

57. The following figure shows the graph of $y = f(x)$ from $x = 0$ to $x = 8$. Estimate $\int_0^8 f(x)dx$ by using the trapezoidal rule with 4 subintervals.

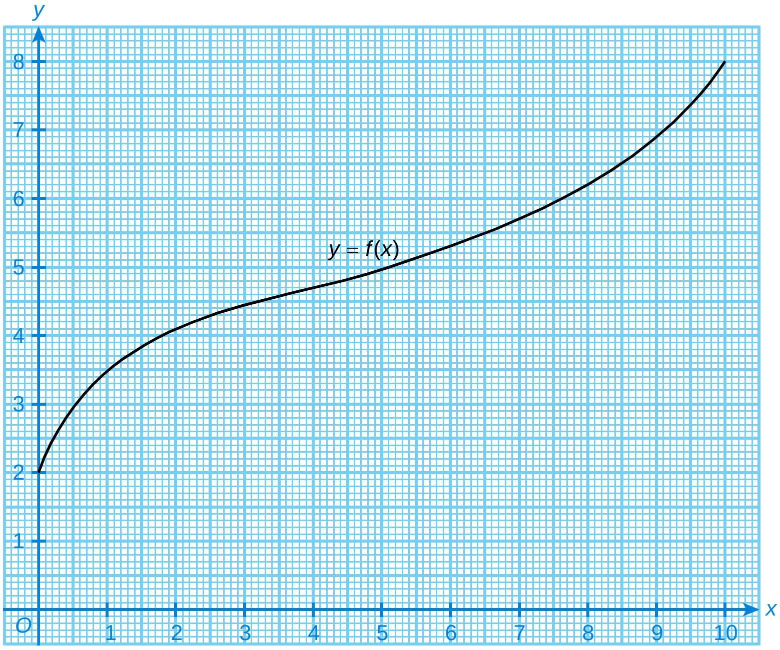


SOLUTION

$$\Delta x = \frac{8-0}{4}$$
$$= 2$$

$$\int_0^8 f(x)dx \approx \frac{2}{2}[f(0)+2f(2)+2f(4)+2f(6)+f(8)]$$
$$= 5+2(4.1)+2(2.2)+2(1)+1.7$$
$$= \underline{\underline{21.3}}$$

58. The following figure shows the graph of $y = f(x)$ from $x = 0$ to $x = 10$. Estimate $\int_0^{10} f(x)dx$ by using the trapezoidal rule with 5 subintervals.

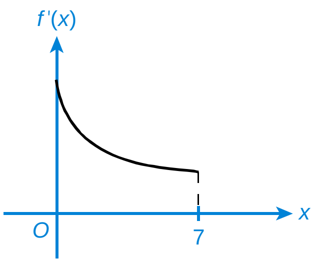


SOLUTION

$$\Delta x = \frac{10-0}{5} = 2$$

$$\begin{aligned}\int_0^{10} f(x)dx &\approx \frac{2}{2}[f(0)+2f(2)+2f(4)+2f(6)+2f(8)+f(10)]\\ &= 2+2(4.1)+2(4.7)+2(5.3)+2(6.2)+8\\ &= \underline{\underline{50.6}}\end{aligned}$$

59. It is given that $f'(x) = \dfrac{6}{\ln(x+2)}$, where $x \geq 0$ and $f(0) = 3$. The figure shows a sketch of $f'(x)$ against x.



(a) Estimate the value of $f(7)$ by using the trapezoidal rule with 7 subintervals. (Give your answer correct to 4 decimal places.)

(b) From the figure, determine whether the approximate value obtained in **(a)** is an underestimate or an overestimate.

SOLUTION

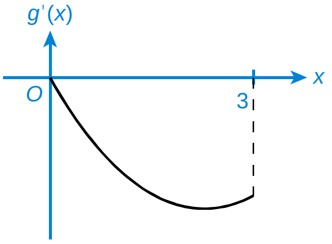
(a) $\Delta x = \dfrac{7-0}{7} = 1$

$$\int_0^7 f'(x)dx$$
$$\approx \frac{1}{2}[f'(0)+2f'(1)+2f'(2)+2f'(3)+2f'(4)+2f'(5)+2f'(6)+f'(7)]$$
$$= \frac{1}{2}\left[\frac{6}{\ln(0+2)}+\frac{2\times 6}{\ln(1+2)}+\frac{2\times 6}{\ln(2+2)}+\frac{2\times 6}{\ln(3+2)}+\frac{2\times 6}{\ln(4+2)}+\frac{2\times 6}{\ln(5+2)}+\frac{2\times 6}{\ln(6+2)}+\frac{6}{\ln(7+2)}\right]$$
$$= 28.5284 \text{ (corr. to 4 d.p.)}$$

$$f(7)-f(0) = \int_0^7 f'(x)dx$$
$$f(7)-3 \approx 28.5284$$
$$f(7) = \underline{\underline{31.5284}} \text{ (corr. to 4 d.p.)}$$

(b) Since the graph of $f'(x)$ is concave upwards on the interval $0 \le x \le 7$, the approximate value obtained in **(a)** is an overestimate.

60. It is given that $g'(x) = -2xe^{-\frac{x}{2}}$, where $x \ge 0$ and $g(0) = 10$. The figure shows a sketch of $g'(x)$ against x.



(a) Estimate the value of $g(3)$ by using the trapezoidal rule with 4 subintervals. (Give your answer correct to 4 decimal places.)

(b) From the figure, determine whether the approximate value obtained in **(a)** is an underestimate or an overestimate.

SOLUTION

(a) $\Delta x = \dfrac{3-0}{4} = 0.75$

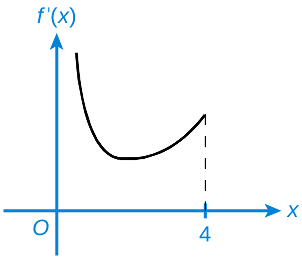
x	0	0.75	1.5	2.25	3
$g'(x)$	0	−1.030 93	−1.417 10	−1.460 94	−1.338 78

$$\int_0^3 g'(x)dx \approx \frac{0.75}{2}[g'(0)+2g'(0.75)+2g'(1.5)+2g'(2.25)+g'(3)]$$
$$= -3.4338 \text{ (corr. to 4 d.p.)}$$

$$g(3)-g(0)=\int_0^3 g'(x)dx$$
$$g(3)-10\approx -3.4338$$
$$g(3)=\underline{\underline{6.5662}} \text{ (corr. to 4 d.p.)}$$

(b) Since the graph of $g'(x)$ is concave upwards on the interval $0\le x\le 3$, the approximate value obtained in **(a)** is an overestimate.

61. It is given that $f'(x)=\dfrac{e^x}{x^2}$, where $x>0$ and $f(4)=20$. The figure shows a sketch of $f'(x)$ against x.



(a) Estimate the value of $f(1)$ by using the trapezoidal rule with 6 subintervals. (Give your answer correct to 4 decimal places.)

(b) From the figure, determine whether the approximate value of $f(1)$ obtained in **(a)** is an underestimate or an overestimate.

SOLUTION

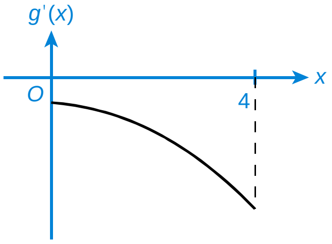
(a) $\Delta x=\dfrac{4-1}{6}=0.5$

$$\int_1^4 f'(x)dx\approx\frac{0.5}{2}[f'(1)+2f'(1.5)+2f'(2)+2f'(2.5)+2f'(3)+2f'(3.5)+f'(4)]$$
$$=\frac{0.5}{2}\left(\frac{e^1}{1^2}+\frac{2e^{1.5}}{1.5^2}+\frac{2e^2}{2^2}+\frac{2e^{2.5}}{2.5^2}+\frac{2e^3}{3^2}+\frac{2e^{3.5}}{3.5^2}+\frac{e^4}{4^2}\right)$$
$$=6.8943 \text{ (corr. to 4 d.p.)}$$

$$f(4)-f(1)=\int_1^4 f'(x)dx$$
$$20-f(1)\approx 6.8943$$
$$f(1)=\underline{\underline{13.1057}} \text{ (corr. to 4 d.p.)}$$

(b) Since the graph of $f'(x)$ is concave upwards on the interval $1\le x\le 4$, the approximate value of $\int_1^4 f'(x)dx$ is an overestimate, while the approximate value of $f(1)$ obtained in **(a)** is an underestimate.

62. It is given that $g'(x) = -e^{\frac{x^2}{10}}$, where $x \geq 0$ and $g(4) = -3$. The figure shows a sketch of $g'(x)$ against x.



(a) Estimate the value of $g(3)$ by using the trapezoidal rule with 5 subintervals. (Give your answer correct to 4 decimal places.)

(b) From the figure, determine whether the approximate value of $g(3)$ obtained in **(a)** is an underestimate or an overestimate.

SOLUTION

(a) $\Delta x = \frac{4-3}{5} = 0.2$

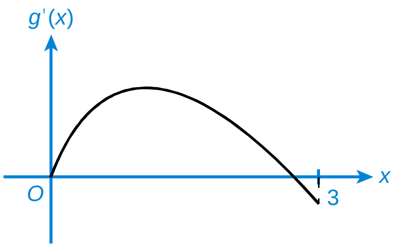
x	3	3.2	3.4	3.6	3.8	4
$g'(x)$	−2.459 60	−2.784 31	−3.177 20	−3.654 65	−4.237 61	−4.953 03

$$\int_3^4 g'(x)dx \approx \frac{0.2}{2}[g'(3) + 2g'(3.2) + 2g'(3.4) + 2g'(3.6) + 2g'(3.8) + g'(4)]$$
$$= -3.5120 \text{ (corr. to 4 d.p.)}$$

$$g(4) - g(3) = \int_3^4 g'(x)dx$$
$$-3 - g(3) \approx -3.5120$$
$$g(3) = \underline{\underline{0.5120}} \text{ (corr. to 4 d.p.)}$$

(b) Since the graph of $g'(x)$ is concave downwards on the interval $3 \leq x \leq 4$, the approximate value of $\int_3^4 g'(x)dx$ is an underestimate, while the approximate value of $g(3)$ obtained in **(a)** is an overestimate.

63. It is given that $g'(x) = 2kx - x\ln x$, where $x > 0$, k is a constant and $g(1) = g'(1) = 1$. The figure shows a sketch of $g'(x)$ against x.



(a) Find the value of k.

(b) Estimate the value of $g(2)$ by using the trapezoidal rule with 5 subintervals. (Give your answer correct to 4 decimal places.)

(c) From the figure, determine whether the value of $g(2)$ is greater than 1.85.

SOLUTION

(a) $g'(1) = 2k(1) - 1\ln 1$

$= 2k$

$\because \quad g'(1) = 1$

$\therefore \quad 2k = 1$

$k = \underline{\underline{0.5}}$

(b) From **(a)**, $g'(x) = 2(0.5)x - x\ln x$

$= x - x\ln x$

$\Delta x = \dfrac{2-1}{5} = 0.2$

x	1	1.2	1.4	1.6	1.8	2
$g'(x)$	1	0.981 21	0.928 94	0.847 99	0.741 98	0.613 71

$$\int_1^2 g'(x)dx \approx \frac{0.2}{2}[g'(1) + 2g'(1.2) + 2g'(1.4) + 2g'(1.6) + 2g'(1.8) + g'(2)]$$

$$= 0.8614 \text{ (corr. to 4 d.p.)}$$

$$g(2) - g(1) = \int_1^2 g'(x)dx$$

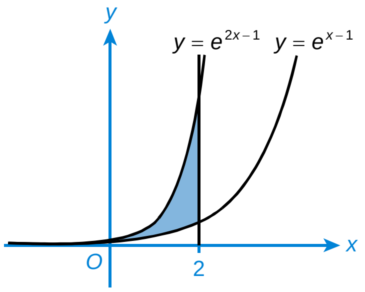
$$g(2) - 1 \approx 0.8614$$

$$g(2) = \underline{\underline{1.8614}} \text{ (corr. to 4 d.p.)}$$

(c) Since the graph of $g'(x)$ is concave downwards on the interval $1 \le x \le 2$, the approximate value obtained in **(b)** is an underestimate, i.e. the value of $g(2)$ is greater than 1.861 4.

$\therefore$ The value of $g(2)$ is greater than 1.85.

64. Use the trapezoidal rule with 5 subintervals to estimate the area of the region bounded by the curves $y = e^{2x-1}$ and $y = e^{x-1}$ from $x = 0$ to $x = 2$. (Give your answer correct to 4 decimal places.)



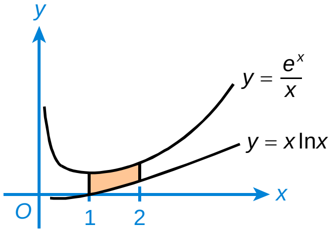
SOLUTION

Let $f(x)=e^{2x-1}-e^{x-1}$.

$$\Delta x=\frac{2-0}{5}$$
$$=0.4$$

$$\text{Required area}=\int_0^2(e^{2x-1}-e^{x-1})dx$$
$$\approx\frac{0.4}{2}[f(0)+2f(0.4)+2f(0.8)+2f(1.2)+2f(1.6)+f(2)]$$
$$=\frac{0.4}{2}\{[e^{2(0)-1}-e^{0-1}]+2[e^{2(0.4)-1}-e^{0.4-1}]+2[e^{2(0.8)-1}-e^{0.8-1}]$$
$$+2[e^{2(1.2)-1}-e^{1.2-1}]+2[e^{2(1.6)-1}-e^{1.6-1}]+[e^{2(2)-1}-e^{2-1}]\}$$
$$=\underline{\underline{7.9975}}\text{ (corr. to 4 d.p.)}$$

65. Use the trapezoidal rule with 5 subintervals to estimate the area of the shaded region bounded by the curves $y=\frac{e^x}{x}$ and $y=x\ln x$ from $x=1$ to $x=2$. (Give your answer correct to 4 decimal places.)



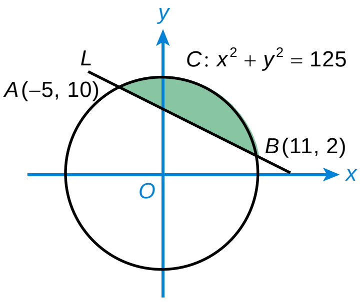
SOLUTION

Let $f(x)=\frac{e^x}{x}-x\ln x$.

$$\Delta x=\frac{2-1}{5}$$
$$=0.2$$

$$\text{Required area}=\int_1^2\left(\frac{e^x}{x}-x\ln x\right)dx$$
$$\approx\frac{0.2}{2}[f(1)+2f(1.2)+2f(1.4)+2f(1.6)+2f(1.8)+f(2)]$$
$$=\frac{0.2}{2}[(\frac{e^1}{1}-1\ln 1)+2(\frac{e^{1.2}}{1.2}-1.2\ln 1.2)+2(\frac{e^{1.4}}{1.4}-1.4\ln 1.4)+2(\frac{e^{1.6}}{1.6}-1.6\ln 1.6)$$
$$+2(\frac{e^{1.8}}{1.8}-1.8\ln 1.8)+(\frac{e^2}{2}-2\ln 2)]$$
$$=\underline{\underline{2.4267}}\text{ (corr. to 4 d.p.)}$$

66. In the figure, the straight line L and the circle $C: x^2 + y^2 = 125$ intersect at $A(-5, 10)$ and $B(11, 2)$.



(a) Find the equation of L.

(b) Use the trapezoidal rule with 4 subintervals to estimate the area of the minor segment. (Give your answer correct to 2 decimal places.)

SOLUTION

(a) The equation of L is

$$y - 2 = \frac{10-2}{-5-11}(x-11)$$

$$y - 2 = -\frac{1}{2}(x-11)$$

$$-2y + 4 = x - 11$$

$$\underline{\underline{x + 2y - 15 = 0}}$$

(b) Rewrite C and L as $y = \pm\sqrt{125-x^2}$ and $y = -\frac{x}{2} + \frac{15}{2}$ respectively.

$$\text{Required area} = \int_{-5}^{11} [\sqrt{125-x^2} - (-\frac{x}{2} + \frac{15}{2})]dx$$

$$= \int_{-5}^{11} (\sqrt{125-x^2} + \frac{x}{2} - \frac{15}{2})dx$$

Let $f(x) = \sqrt{125-x^2} + \frac{x}{2} - \frac{15}{2}$.

$$\Delta x = \frac{11-(-5)}{4} = 4$$

$$\text{Required area} = \int_{-5}^{11} (\sqrt{125-x^2} + \frac{x}{2} - \frac{15}{2})dx$$

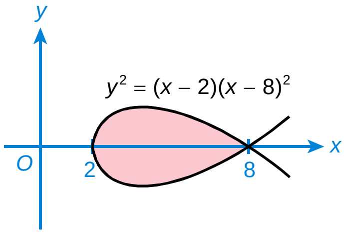
$$\approx \frac{4}{2}[f(-5) + 2f(-1) + 2f(3) + 2f(7) + f(11)]$$

$$= \frac{4}{2}\{[\sqrt{125-(-5)^2} + \frac{-5}{2} - \frac{15}{2}] + 2[\sqrt{125-(-1)^2} + \frac{-1}{2} - \frac{15}{2}]$$

$$+ 2(\sqrt{125-3^2} + \frac{3}{2} - \frac{15}{2}) + 2(\sqrt{125-7^2} + \frac{7}{2} - \frac{15}{2}) + (\sqrt{125-11^2} + \frac{11}{2} - \frac{15}{2})\}$$

$$= \underline{\underline{50.49}} \text{ (corr. to 2 d.p.)}$$

67. The figure shows a sketch of $y^2 = (x-2)(x-8)^2$.



(a) Use the trapezoidal rule with 4 subintervals to estimate $\int_2^8 \sqrt{(x-2)(x-8)^2}\,dx$.

(b) Hence estimate the area of the shaded region.

(Give your answers correct to 4 decimal places.)

SOLUTION

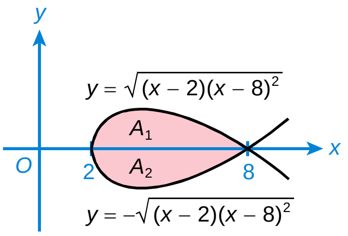
(a) Let $f(x) = \sqrt{(x-2)(x-8)^2}$.

$$\Delta x = \frac{8-2}{4} = 1.5$$

$$\begin{aligned}\int_2^8 \sqrt{(x-2)(x-8)^2}\,dx &\approx \frac{1.5}{2}[f(2)+2f(3.5)+2f(5)+2f(6.5)+f(8)]\\ &= \frac{1.5}{2}[\sqrt{(2-2)(2-8)^2}+2\sqrt{(3.5-2)(3.5-8)^2}+2\sqrt{(5-2)(5-8)^2}\\ &\quad +2\sqrt{(6.5-2)(6.5-8)^2}+\sqrt{(8-2)(8-8)^2}]\\ &= \underline{\underline{20.8342}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

(b) Rewrite $y^2 = (x-2)(x-8)^2$ as $y = \pm\sqrt{(x-2)(x-8)^2}$.

As shown in the figure, the curve $y^2 = (x-2)(x-8)^2$ is split into two parts, $y = \sqrt{(x-2)(x-8)^2}$ and $y = -\sqrt{(x-2)(x-8)^2}$. The required area is the sum of A_1 and A_2. Since the curve $y^2 = (x-2)(x-8)^2$ is symmetrical about the x-axis, $A_1 = A_2$.



$$\begin{aligned}\therefore \quad \text{Required area} &= A_1 + A_2\\ &= 2A_1\\ &= 2\int_2^8 \sqrt{(x-2)(x-8)^2}\,dx\\ &= 2 \times 20.83423 \quad [\text{From the result of } \textbf{(a)}]\\ &= \underline{\underline{41.6685}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

68. **(a)** Evaluate $\int_0^5 \frac{dx}{x+1}$.

(b) According to the result of **(a)**, use the trapezoidal rule with 5 subintervals to estimate the value of $\ln 6$. (Give your answer correct to 4 decimal places.)

SOLUTION

(a)
$$\begin{aligned}\int_0^5 \frac{dx}{x+1} &= \int_0^5 \frac{d(x+1)}{x+1} \\ &= [\ln|x+1|]_0^5 \\ &= \ln 6 - \ln 1 \\ &= \underline{\underline{\ln 6}}\end{aligned}$$

(b) Let $f(x) = \frac{1}{x+1}$.

$$\begin{aligned}\Delta x &= \frac{5-0}{5} \\ &= 1\end{aligned}$$

$$\begin{aligned}\ln 6 &= \int_0^5 \frac{dx}{x+1} \\ &\approx \frac{1}{2}[f(0)+2f(1)+2f(2)+2f(3)+2f(4)+f(5)] \\ &= \frac{1}{2}\left(\frac{1}{0+1}+\frac{2}{1+1}+\frac{2}{2+1}+\frac{2}{3+1}+\frac{2}{4+1}+\frac{1}{5+1}\right) \\ &= \underline{\underline{1.8667}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

69. It is given that $\int_0^2 \frac{1}{x+2}dx = \ln k$, where k is a constant.

(a) Find the value of k.

(b) Use the trapezoidal rule with 5 subintervals to estimate the value of $\ln k$. (Give your answer correct to 4 decimal places.)

SOLUTION

(a)
$$\begin{aligned}\int_0^2 \frac{1}{x+2}dx &= \int_0^2 \frac{1}{x+2}d(x+2) \\ &= [\ln|x+2|]_0^2 \\ &= \ln 4 - \ln 2 \\ &= \ln 2\end{aligned}$$

$\because \quad \int_0^2 \frac{1}{x+2}\,dx = \ln k$

$\therefore \quad \ln 2 = \ln k$

$k = \underline{\underline{2}}$

(b) Let $f(x) = \frac{1}{x+2}$.

$$\Delta x = \frac{2-0}{5} = 0.4$$

$$\begin{aligned}\ln k &= \int_0^2 \frac{1}{x+2}\,dx \\ &\approx \frac{0.4}{2}[f(0)+2f(0.4)+2f(0.8)+2f(1.2)+2f(1.6)+f(2)] \\ &= \frac{0.4}{2}\left(\frac{1}{0+2}+\frac{2}{0.4+2}+\frac{2}{0.8+2}+\frac{2}{1.2+2}+\frac{2}{1.6+2}+\frac{1}{2+2}\right) \\ &= \underline{\underline{0.6956}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

70. It given that $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$.

(a) Use the trapezoidal rule with 5 subintervals to estimate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

(b) Hence find an approximate value of π.

(Give your answers correct to 4 decimal places.)

SOLUTION

(a) Let $f(x) = \frac{1}{\sqrt{1-x^2}}$.

$$\Delta x = \frac{\frac{1}{2}-0}{5} = 0.1$$

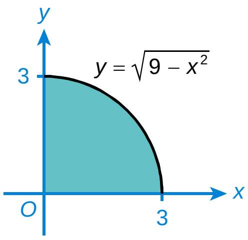
$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} &\approx \frac{0.1}{2}[f(0)+2f(0.1)+2f(0.2)+2f(0.3)+2f(0.4)+f(0.5)] \\ &= \frac{0.1}{2}\left(\frac{1}{\sqrt{1-0^2}}+\frac{2}{\sqrt{1-0.1^2}}+\frac{2}{\sqrt{1-0.2^2}}+\frac{2}{\sqrt{1-0.3^2}}+\frac{2}{\sqrt{1-0.4^2}}+\frac{1}{\sqrt{1-0.5^2}}\right) \\ &= \underline{\underline{0.5242}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

(b) $\because \quad \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$

$\therefore \quad 0.524\,24 = \frac{\pi}{6}$

$\pi = \underline{\underline{3.145\,4}}$ (corr. to 4 d.p.)

71. It is given that the graph of the curve $y = \sqrt{9-x^2}$ $(0 \le x \le 3)$ is $\frac{1}{4}$ of a circle in quadrant I, with the centre at the origin O and a radius of 3.



(a) Use the trapezoidal rule with 6 subintervals to estimate the area of the shaded region.

(b) Express the area of the shaded region in terms of π. Hence find an approximate value of π.

(Give your answers correct to 4 decimal places if necessary.)

SOLUTION

(a) Let $f(x) = \sqrt{9-x^2}$.

$\Delta x = \frac{3-0}{6} = 0.5$

$$\begin{aligned}\text{Required area} &= \int_0^3 \sqrt{9-x^2}\,dx \\ &\approx \frac{0.5}{2}[f(0)+2f(0.5)+2f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)] \\ &= \frac{0.5}{2}(\sqrt{9-0^2}+2\sqrt{9-0.5^2}+2\sqrt{9-1^2}+2\sqrt{9-1.5^2}+2\sqrt{9-2^2} \\ &\quad +2\sqrt{9-2.5^2}+\sqrt{9-3^2}) \\ &= \underline{\underline{6.889\,5}} \text{ (corr. to 4 d.p.)}\end{aligned}$$

(b) Area of the shaded region $= \frac{1}{4}\pi(3)^2$

$= \underline{\underline{\frac{9\pi}{4}}}$

$\therefore \quad \frac{9\pi}{4} \approx 6.889\,46$

$\pi = \underline{\underline{3.062\,0}}$ (corr. to 4 d.p.)

72. Let $g(x)=x^2e^{kx}$, where k is a constant. It is given that $g(x)$ attains its local extremum at $x=2$.

(a) Find the value of k.

(b) Use the trapezoidal rule with 6 subintervals to estimate $\int_1^3 g(x)dx$. (Give your answer correct to 4 decimal places.)

SOLUTION

(a) $g(x)=x^2e^{kx}$

$g'(x)=(2x)e^{kx}+x^2(ke^{kx})=2xe^{kx}+kx^2e^{kx}$

$\because$ $g(x)$ attains its local extremum at $x=2$.

$$\begin{aligned}\therefore\quad g'(2)&=0\\ 2(2)e^{k(2)}+k(2)^2e^{k(2)}&=0\\ 4e^{2k}(1+k)&=0\\ k+1&=0\\ k&=\underline{\underline{-1}}\end{aligned}$$

(b) From **(a)**, $g(x)=x^2e^{-x}$.

$\Delta x=\frac{3-1}{6}=\frac{1}{3}$

x	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3
$g(x)$	0.367 88	0.468 62	0.524 65	0.541 34	0.527 96	0.494 10	0.448 08

$$\int_1^3 g(x)dx\approx\frac{\frac{1}{3}}{2}[g(1)+2g(\frac{4}{3})+2g(\frac{5}{3})+2g(2)+2g(\frac{7}{3})+2g(\frac{8}{3})+g(3)]$$

$=\underline{\underline{0.9882}}$ (corr. to 4 d.p.)

Level 3

73. Figure I and Figure II show the sketches of the curves $y=\frac{1}{\sqrt{1-x^2}}$ $(0\le x\le\frac{1}{2})$ and $y=\frac{1}{1+x^2}$ $(0\le x\le\frac{1}{\sqrt{3}})$ respectively.

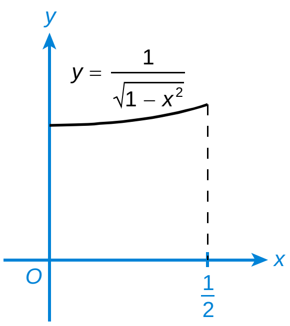


Figure I

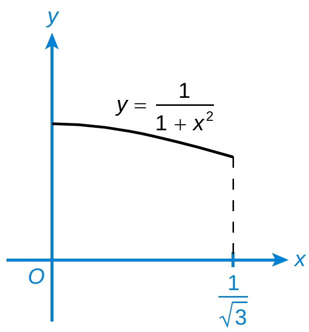


Figure II

(a) **(i)** Use the trapezoidal rule with 4 subintervals to estimate $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

(ii) From Figure I, determine whether the approximate value obtained in **(a)(i)** is an underestimate or an overestimate.

(b) **(i)** Use the trapezoidal rule with 4 subintervals to estimate $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+x^2}$.

(ii) From Figure II, determine whether the approximate value obtained in **(b)(i)** is an underestimate or an overestimate.

(c) It is given that $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+x^2} = \frac{\pi}{6}$. Hence find the range of values of π.

(Give your answers correct to 4 significant figures if necessary.)

SOLUTION

(a) **(i)** Let $f(x) = \frac{1}{\sqrt{1-x^2}}$.

$$\Delta x = \frac{\frac{1}{2}-0}{4} = \frac{1}{8}$$

$$\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} \approx \frac{\frac{1}{8}}{2}[f(0)+2f(\tfrac{1}{8})+2f(\tfrac{1}{4})+2f(\tfrac{3}{8})+f(\tfrac{1}{2})]$$

$$= \frac{1}{16}[\frac{1}{\sqrt{1-0^2}} + \frac{2}{\sqrt{1-(\frac{1}{8})^2}} + \frac{2}{\sqrt{1-(\frac{1}{4})^2}} + \frac{2}{\sqrt{1-(\frac{3}{8})^2}} + \frac{1}{\sqrt{1-(\frac{1}{2})^2}}]$$

$$= \underline{\underline{0.5246}} \text{ (corr. to 4 sig. fig.)}$$

(ii) Since the graph of $f(x)$ is concave upwards on the interval $0 \le x \le \frac{1}{2}$, the approximate value obtained in **(a)(i)** is an overestimate.

(b) **(i)** Let $g(x) = \frac{1}{1+x^2}$.

$$\Delta x = \frac{\frac{1}{\sqrt{3}}-0}{4} = \frac{1}{4\sqrt{3}}$$

$$\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+x^2} \approx \frac{\frac{1}{4\sqrt{3}}}{2}[g(0)+2g(\frac{1}{4\sqrt{3}})+2g(\frac{1}{2\sqrt{3}})+2g(\frac{3}{4\sqrt{3}})+g(\frac{1}{\sqrt{3}})]$$
$$= \frac{1}{8\sqrt{3}}[\frac{1}{1+0^2}+\frac{2}{1+(\frac{1}{4\sqrt{3}})^2}+\frac{2}{1+(\frac{1}{2\sqrt{3}})^2}+\frac{2}{1+(\frac{3}{4\sqrt{3}})^2}+\frac{1}{1+(\frac{1}{\sqrt{3}})^2}]$$
$$= \underline{\underline{0.5225}} \text{ (corr. to 4 sig. fig.)}$$

(ii) Since the graph of $g(x)$ is concave downwards on the interval $0 \le x \le \frac{1}{\sqrt{3}}$, the approximate value obtained in **(b)(i)** is an underestimate.

(c) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} \le 0.524\,60$ [From the result of **(a)**]

$\therefore \quad \frac{\pi}{6} \le 0.524\,60$

$\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+x^2} \ge 0.522\,47$ [From the result of **(b)**]

$\therefore \quad \frac{\pi}{6} \ge 0.522\,47$

$\therefore \quad 0.522\,47 \le \frac{\pi}{6} \le 0.524\,60$

$\underline{\underline{3.135 \le \pi \le 3.148}}$ (corr. to 4 sig. fig.)

74. **(a)** Evaluate $\int_0^{100} x^3 dx$.

(b) Hence find an approximate value of $1^3+2^3+3^3+\cdots+99^3+100^3$.

SOLUTION

(a) $\int_0^{100} x^3 dx = [\frac{x^4}{4}]_0^{100}$

$= \underline{\underline{25\,000\,000}}$

(b) Let $f(x)=x^3$.

$$\begin{aligned}1^3+2^3+\cdots+99^3+100^3 &= f(1)+f(2)+\cdots+f(99)+f(100)\\ &= \frac{1}{2}[f(0)+2f(1)+2f(2)+\cdots+2f(99)+f(100)]-\frac{1}{2}f(0)+\frac{1}{2}f(100)\\ &\approx \int_0^{100} x^3 dx-\frac{1}{2}(0)^3+\frac{1}{2}(100)^3\\ &= 25\,000\,000-0+500\,000 \quad \text{[From the result of (a)]}\\ &= \underline{\underline{25\,500\,000}}\end{aligned}$$

75. **(a)** Evaluate $\int_0^{100} x(x+1)\,dx$.

(b) Hence find an approximate value of $0\times1+2\times3+4\times5+\cdots+98\times99+100\times101$. (Give your answer correct to the nearest integer.)

SOLUTION

(a) $$\begin{aligned}\int_0^{100} x(x+1)\,dx &= \int_0^{100} (x^2+x)\,dx \\ &= [\frac{x^3}{3}+\frac{x^2}{2}]_0^{100} \\ &= \underline{\underline{\frac{1\,015\,000}{3}}}\end{aligned}$$

(b) Let $f(x)=x(x+1)$.

$$\begin{aligned}&0\times1+2\times3+4\times5+\cdots+98\times99+100\times101 \\ &= f(0)+f(2)+f(4)+\cdots+f(98)+f(100) \\ &= (\frac{1}{2})(\frac{2}{2})[f(0)+2f(2)+2f(4)+\cdots+2f(98)+f(100)]+\frac{1}{2}f(0)+\frac{1}{2}f(100) \\ &\approx \frac{1}{2}\int_0^{100} x(x+1)dx+\frac{1}{2}(0)(0+1)+\frac{1}{2}(100)(100+1) \\ &= \frac{1}{2}(\frac{1\,015\,000}{3})+0+5\,050 \quad [\text{From the result of } \mathbf{(a)}] \\ &= \underline{\underline{174\,217}} \text{ (corr. to the nearest integer)}\end{aligned}$$

EXERCISE 9C

Level 1

76. The rate of change of the weekly sales volume $S(t)$ of a mobile phone model after t weeks since its launch can be modelled by $S'(t)=3t^2-84t+500$ $(0\le t\le 25)$. If the weekly sales volume of the model is 0 when it is just launched, find the weekly sales volume of the model after 24 weeks since its launch.

SOLUTION

$$\begin{aligned}S(24)-S(0) &= \int_0^{24} (3t^2-84t+500)dt \\ S(24)-0 &= [t^3-42t^2+500t]_0^{24} \\ S(24) &= 1\,632\end{aligned}$$

$\therefore$ The weekly sales volume of the model is 1 632 after 24 weeks since its launch.

77. The rate of change of the number of pigs $P(t)$ infected with foot-and-mouth disease in a farm can be modelled by $P'(t) = 225e^{-0.15t}$, where t is the number of days elapsed since the spread of the disease. Find the number of pigs infected in the first week after the spread of the disease. (Give your answer correct to the nearest integer.)

SOLUTION

$$P(7) - P(0) = \int_0^7 225e^{-0.15t}\,dt$$
$$= 225\int_0^7 (-\frac{1}{0.15})e^{-0.15t}\,d(-0.15t)$$
$$= -1\,500[e^{-0.15t}]_0^7$$
$$= 975 \text{ (corr. to the nearest integer)}$$

$\therefore$ The number of pigs infected in the first week after the spread of the disease is 975.

78. The rate of change of the population $P(t)$ of a town after t years can be modelled by $P'(t) = 200e^{0.05t}$ $(0 \le t \le 20)$. Given that the current population of the town is 70 000, find the population of the town after 5 years. (Give your answer correct to 3 significant figures.)

SOLUTION

$$P(5) - P(0) = \int_0^5 200e^{0.05t}\,dt$$
$$P(5) - 70\,000 = 200\int_0^5 \frac{1}{0.05}e^{0.05t}\,d(0.05t)$$
$$P(5) = 4\,000[e^{0.05t}]_0^5 + 70\,000$$
$$= 71100 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The population of the town is 71 100 after 5 years.

79. The rate of change of the value $\$V(t)$ of a computer after t months can be modelled by $V'(t) = -\dfrac{3\,000}{t+1}$ $(0 \le t \le 36)$. If the initial value of the computer is \$15 000, find its value after 2 years. (Give your answer correct to 3 significant figures.)

SOLUTION

$$V(24) - V(0) = \int_0^{24} (-\frac{3\,000}{t+1})dt$$
$$V(24) - 15\,000 = \int_0^{24} (-\frac{3\,000}{t+1})d(t+1)$$
$$V(24) = [-3\,000\ln|t+1|]_0^{24} + 15\,000$$
$$= 5\,340 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The value of the computer is \$5 340 after 2 years.

80. Water is leaking from a crack in a tank. If the rate of decrease (in cm^3/s) of the volume of water in the tank after t seconds since the crack first appears can be modelled by $R(t)=\dfrac{50}{2+t}$ $(0\le t\le 20)$, find the decrease in the volume of water in the tank in the first 10 seconds. (Give your answer correct to 3 significant figures.)

SOLUTION

Let $V(t)$ (in cm^3) be the decrease in the volume of water in the tank after t seconds.

$$\begin{aligned}V(10)-V(0)&=\int_0^{10}\frac{50}{2+t}dt\\&=\int_0^{10}\frac{50}{2+t}d(2+t)\\&=[50\ln|2+t|]_0^{10}\\&=89.6\text{ (corr. to 3 sig .fig.)}\end{aligned}$$

$\therefore$ The decrease in the volume of water in the tank in the first 10 seconds is $89.6\,cm^3$.

81. The speed of a vehicle is $v(t)$ (in km/h) after travelling for t hours. The following table records some values of $v(t)$ corresponding to t.

t	0	0.5	1	1.5	2	2.5	3	3.5	4
$v(t)$	0	48	56	64	50	80	56	32	0

Estimate the total distance travelled by the vehicle in these 4 hours by using the trapezoidal rule with 8 subintervals.

SOLUTION

$\Delta t=0.5$

Let s m be the distance travelled by the vehicle in these 4 hours.

$$\begin{aligned}s&=\int_0^{60}v(t)dt\\&\approx\frac{0.5}{2}[v(0)+2v(0.5)+2v(1)+2v(1.5)+2v(2)+2v(2.5)+2v(3)+2v(3.5)+v(4)]\\&=\frac{0.5}{2}[0+2(48)+2(56)+2(64)+2(50)+2(80)+2(56)+2(32)+0]\\&=193\end{aligned}$$

$\therefore$ The distance travelled by the vehicle in these 4 hours is about 193 km.

82. The growth rate of a plant can be modelled by $H'(t)=\frac{1}{3}e^{0.25t}(12-t)$ $(0\le t\le 12)$, where $H(t)$ (in cm) is the height of the plant after t months since its germination. Use the trapezoidal rule with 4 subintervals to estimate the change in the height of the plant from the 9th month to the 12th month after its germination. (Give your answer correct to 3 significant figures.)

SOLUTION

$$\Delta t=\frac{12-9}{4}=0.75$$

t	9	9.75	10.5	11.25	12
$H'(t)$	9.488	8.583	6.902	4.163	0

$$\begin{aligned}H(12)-H(9)&=\int_9^{12}H'(t)dt\\&\approx\frac{0.75}{2}[H'(9)+2H'(9.75)+2H'(10.5)+2H'(11.25)+H'(12)]\\&=18.3\text{(corr. to 3 sig. fig.)}\end{aligned}$$

$\therefore$ The change in the height is about 18.3 cm.

83. The rate of change of the value $V(t)$ (in thousand dollars) of a flat can be modelled by $V'(t)=\frac{50}{t^2+10}+5$ $(0\le t\le 24)$, where t is the number of months elapsed since the beginning of January 2010. Use the trapezoidal rule with 4 subintervals to estimate the change in the value of the flat for the year from the beginning of April 2010 to the beginning of April 2011. (Give your answer correct to 4 significant figures.)

SOLUTION

$$\Delta t=\frac{15-3}{4}=3$$

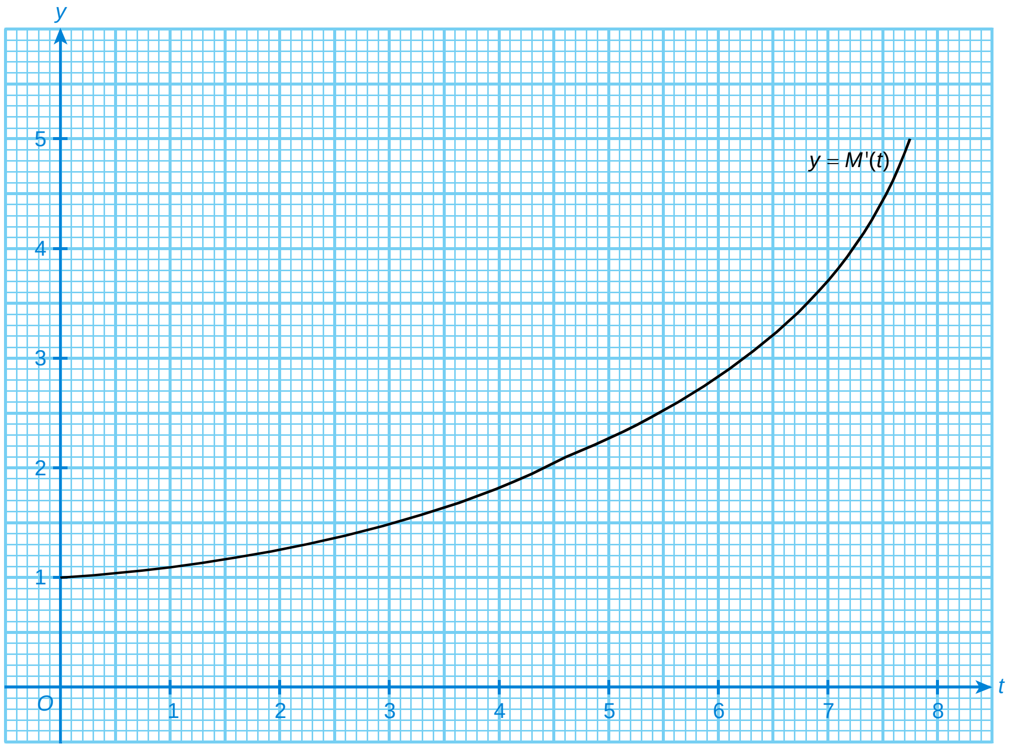
t	3	6	9	12	15
$V'(t)$	7.631 6	6.087 0	5.549 5	5.324 7	5.212 8

$$\begin{aligned}V(15)-V(3)&=\int_3^{15}V'(t)dt\\&\approx\frac{3}{2}[V'(3)+2V'(6)+2V'(9)+2V'(12)+V'(15)]\\&=70.15\text{ (corr. to 4 sig. fig.)}\end{aligned}$$

$\therefore$ The value of the flat increases by about 70.15 thousand dollars for the year from the beginning of April 2010 to the beginning of April 2011.

Level 2

84. The following shows the graph of the rate of change $M'(t)$ of the accumulated medical expenses $M(t)$ (in thousand dollars) of a family after t years since the beginning of 2010.



Use the trapezoidal rule with 5 subintervals to estimate the total medical expenses of the family for the six years from the beginning of 2011 to the end of 2016.

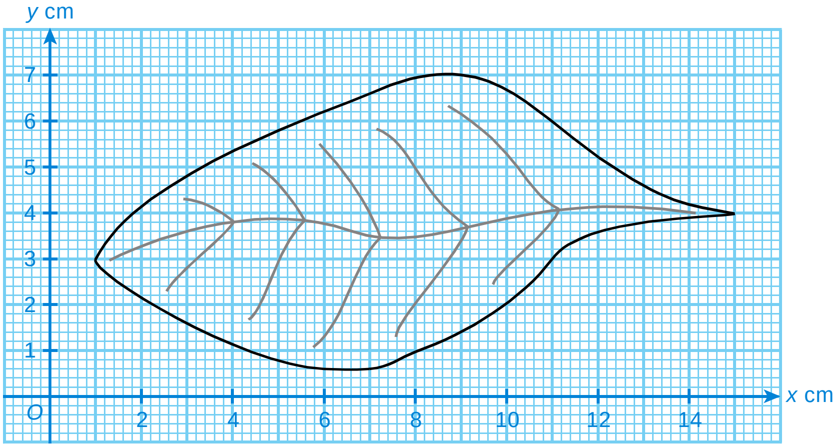
SOLUTION

$$\Delta t = \frac{7-1}{5} = 1.2$$

$$\begin{aligned} M(7) - M(1) &= \int_1^7 M'(t)dt \\ &\approx \frac{1.2}{2}[M'(1) + 2M'(2.2) + 2M'(3.4) + 2M'(4.6) + 2M'(5.8) + M'(7)] \\ &= \frac{1.2}{2}[1.1 + 2(1.3) + 2(1.6) + 2(2.1) + 2(2.7) + 3.7] \\ &= 12.12 \end{aligned}$$

$\therefore$ The total medical expenses of the family for the six years is about 12.12 thousand dollars.

85. The following shows a leaf.



Use the trapezoidal rule with 7 subintervals to estimate the area of the leaf.

SOLUTION

Suppose the curves representing the upper part and lower part of the boundary of the leaf are $y = f(x)$ and $y = g(x)$ respectively, and the area of the leaf is A cm^2.

$$\Delta x = \frac{15-1}{7} = 2$$

$$A = \int_1^{15} [f(x) - g(x)]dx$$

$$\approx \frac{2}{2}\{[f(1)-g(1)]+2[f(3)-g(3)]+2[f(5)-g(5)]+2[f(7)-g(7)]+2[f(9)-g(9)]$$
$$+2[f(11)-g(11)]+2[f(13)-g(13)]+[f(15)-g(15)]\}$$

$$= (3-3)+2(4.8-1.6)+2(5.8-0.8)+2(6.6-0.6)+2(7-1.4)+2(6-3)+2(4.6-3.8)+(4-4)$$

$$= 47.2$$

$\therefore$ The area of the leaf is about 47.2 cm^2.

86. The following table records the speeds $v(t)$ (in m/s) of Victor after riding a bicycle for t minutes.

t	0	2	4	6	8	10
$v(t)$	0	6.89	9.22	10.52	6.44	0

Use the trapezoidal rule with 5 subintervals to estimate the distance travelled by Victor in these 10 minutes. (Give your answer correct to the nearest 0.1 km.)

SOLUTION

Convert the given data into the following.

t (s)	0	120	240	360	480	600
$v(t)$ (m/s)	0	6.89	9.22	10.52	6.44	0

Let s m be the distance travelled by Victor in these 10 minutes.

$\Delta t = 120$

$$s = \int_0^{600} v(t)dt$$

$$\approx \frac{120}{2}[v(0) + 2v(120) + 2v(240) + 2v(360) + 2v(480) + v(600)]$$

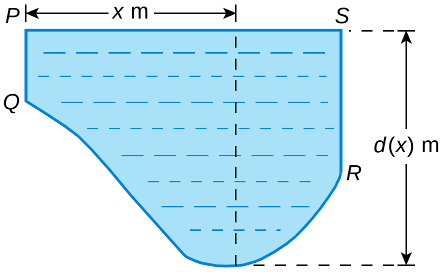
$$= \frac{120}{2}[0 + 2(6.89) + 2(9.22) + 2(10.52) + 2(6.44) + 0]$$

$$= 3\,968.4$$

$\therefore$ Distance travelled by Victor in these 10 minutes $= 3\,968.4$ m

$= \underline{\underline{4.0 \text{ km}}}$ (corr. to the nearest 0.1 km)

87. In the figure, $PQRS$ is the cross-section of a swimming pool. It is given that the length and width PS of the swimming pool are 50 m and 30 m respectively. Rita plans to estimate the capacity of the swimming pool. At the position of x m apart from point P on the pool side, the water depth measures $d(x)$ m.



The following table records some values of $d(x)$ corresponding to x.

x	0	5	10	15	20	25	30
$d(x)$	1.2	1.8	2.8	3.8	4.0	3.6	2.4

Use the trapezoidal rule to estimate the capacity of the swimming pool.

SOLUTION

Let A m^2 be the area of $PQRS$.

$\Delta x = 5$

$$A = \int_0^{30} d(x)dx$$

$$\approx \frac{5}{2}[d(0) + 2d(5) + 2d(10) + 2d(15) + 2d(20) + 2d(25) + d(30)]$$

$$= \frac{5}{2}[1.2 + 2(1.8) + 2(2.8) + 2(3.8) + 2(4.0) + 2(3.6) + 2.4]$$

$$= 89$$

$\therefore$ Capacity of the swimming pool $\approx 89 \times 50$ m^3

$= \underline{\underline{4\,450 \text{ m}^3}}$

88. The rate of change of the accumulated sales $S(t)$ (in thousand dollars) of a company after t months since the start of the business can be modelled by $S'(t)=\dfrac{900}{2+3e^{-0.075t}}$ $(0\le t\le 24)$.

(a) Let $u=2e^{0.075t}+3$, find $\dfrac{du}{dt}$.

(b) Find the total sales in the third half year. (Give your answer correct to 3 significant figures.)

SOLUTION

(a)
$$u=2e^{0.075t}+3$$
$$\frac{du}{dt}=2(0.075)e^{0.075t}$$
$$=\underline{\underline{0.15e^{0.075t}}}$$

(b) When $t=12$, $u=2e^{0.9}+3$;
when $t=18$, $u=2e^{1.35}+3$.
$$S(18)-S(12)=\int_{12}^{18}\frac{900}{2+3e^{-0.075t}}\,dt$$
$$=\int_{12}^{18}\frac{900}{0.15(2e^{0.075t}+3)}\cdot 0.15e^{0.075t}\,dt$$
$$=\int_{2e^{0.9}+3}^{2e^{1.35}+3}\frac{6\,000}{u}\,du$$
$$=6\,000[\ln|u|]_{2e^{0.9}+3}^{2e^{1.35}+3}$$
$$=1\,810 \text{ (corr. to 3 sig. fig.)}$$
$\therefore$ $\underline{\underline{\text{The total sales in the third half year is 1 810 thousand dollars.}}}$

89. The rate of change of the volume $V(t)$ (in m^3) of water in a tank in a building can be modelled by $V'(t)=4t-ke^{\frac{t}{4}}$ $(0\le t\le 12)$, where k is a constant, t is the number of hours elapsed since 6:00 a.m. and $V'(4)=0$.

(a) Find the value of k.

(b) Find the change in the volume of water in the tank from 8:00 a.m. to 2:00 p.m. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) $\because$ $V'(4)=0$

$\therefore$ $4(4)-ke^{\frac{4}{4}}=0$
$$ke=16$$
$$k=\underline{\underline{\frac{16}{e}}}$$

(b) From **(a)**, $V'(t) = 4t - (\frac{16}{e})e^{\frac{t}{4}} = 4t - 16e^{\frac{t}{4}-1}$

$$V(8) - V(2) = \int_2^8 (4t - 16e^{\frac{t}{4}-1})dt$$
$$= \int_2^8 4t\,dt - \int_2^8 16e^{\frac{t}{4}-1}dt$$
$$= \int_2^8 4t\,dt - \int_2^8 4\cdot 16e^{\frac{t}{4}-1}d(\frac{t}{4}-1)$$
$$= [2t^2]_2^8 - [64e^{\frac{t}{4}-1}]_2^8$$
$$= -15.2 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The volume of water in the tank decreases by 15.2 m^3 from 8:00 a.m. to 2:00 p.m.

90. The rates of change of the total cost $C(x)$ and the total revenue $R(x)$ (both in thousand dollars) of a company for producing and selling x thousand products can be modelled by $C'(x) = 2.5x$ $(0 \le x \le 5)$ and $R'(x) = \frac{50\ln(x+1)}{x+1}$ $(0 \le x \le 5)$.

(a) If all the products are sold, express the rate of change of the total profit in terms of x.

(b) It is known that a loss of \$10 000 would be made even if nothing is produced. Find the total profit for producing 2 000 products. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) Let $P(x)$ (in thousand dollars) be the total profit for selling x thousand products.

$$P(x) = R(x) - C(x)$$
$$P'(x) = R'(x) - C'(x) = \frac{50\ln(x+1)}{x+1} - 2.5x$$

$\therefore$ The rate of change of the total profit is $[\frac{50\ln(x+1)}{x+1} - 2.5x]$ thousand dollars/thousand products.

(b)
$$P(2) - P(0) = \int_0^2 [\frac{50\ln(x+1)}{x+1} - 2.5x]dx$$
$$P(2) - (-10) = \int_0^2 \frac{50\ln(x+1)}{x+1}dx - \int_0^2 2.5x\,dx$$
$$P(2) = \int_0^2 50\ln(x+1)d[\ln(x+1)] - \int_0^2 2.5x\,dx - 10$$
$$= [\frac{50}{2}[\ln(x+1)]^2]_0^2 - [\frac{2.5}{2}x^2]_0^2 - 10$$
$$= 15.2 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The total profit for producing 2 000 products is 15.2 thousand dollars.

91. A research discovers that the rate of change of the levels $L(t)$ of players for an online game can be modelled by $L'(t) = 3 - \alpha^{t+k}$ $(0 \le t \le 50)$, where t is the number of online hours spent by the player, α and k are constants and $\alpha > 0$.

(a) Express $\ln[3 - L'(t)]$ as a linear function of t.

(b) It is given that the slope and the intercept on the vertical axis of the graph of the linear function in **(a)** are 0.019 8 and –0.396 respectively. Find the values of α and k. (Give your answers correct to 3 significant figures if necessary.)

(c) Using the values of α and k obtained in **(b)**, find the level of a player who has spent 50 online hours if $L(0) = 0$. (Give your answer correct to the nearest integer.)

SOLUTION

(a)
$$L'(t) = 3 - \alpha^{t+k}$$
$$3 - L'(t) = \alpha^{t+k}$$
$$\ln[3 - L'(t)] = \ln \alpha^{t+k}$$
$$\ln[3 - L'(t)] = (t + k)\ln \alpha$$
$$\underline{\underline{\ln[3 - L'(t)] = (\ln \alpha)t + k\ln \alpha}}$$

(b)
$$\ln \alpha = \text{Slope}$$
$$= 0.0198$$
$$\alpha = \underline{\underline{1.02}} \text{ (corr. to 3 sig. fig.)}$$

$$k\ln \alpha = \text{Intercept on the vertical axis}$$
$$k(0.0198) = -0.396$$
$$k = \underline{\underline{-20}}$$

(c) From **(b)**, $L'(t) = 3 - 1.02^{t-20}$.

$$L(50) - L(0) = \int_0^{50} (3 - 1.02^{t-20})\,dt$$
$$L(50) - 0 = \int_0^{50} 3\,dt - \int_0^{50} 1.02^{t-20}\,dt$$
$$L(50) = \int_0^{50} 3\,dt - \int_0^{50} 1.02^{t-20}\,d(t - 20)$$
$$= [3t]_0^{50} - [\frac{1.02^{t-20}}{\ln 1.02}]_0^{50}$$
$$= 93 \text{ (corr. to the nearest integer)}$$

$\therefore$ <u>The level of a player who has spent 50 online hours is 93.</u>

92. The rate of change of the accumulated sales volume $F(t)$ (in thousand) of albums of a singer can be modelled by $F'(t) = Ate^{-kt^2} + \frac{t^2}{10}$ $(0 < t \le 15)$, where A and k are positive constants and t is the number of years elapsed since the beginning of 2010.

(a) Express $\ln[\frac{F'(t)}{t} - \frac{t}{10}]$ as a linear function of t^2.

(b) It is given that the slope of the graph of the linear function in **(a)** is −0.125 and the graph passes through (1, 1.484). Find the values of A and k. (Give your answers correct to 3 significant figures if necessary.)

(c) Using the value of k obtained in **(b)** and taking $A = 5$, find the total sales volume of albums of the singer from the beginning of 2011 to the beginning of 2013. (Give your answer correct to 3 significant figures.)

SOLUTION

(a)
$$F'(t) = Ate^{-kt^2} + \frac{t^2}{10}$$
$$F'(t) - \frac{t^2}{10} = Ate^{-kt^2}$$
$$\frac{F'(t)}{t} - \frac{t}{10} = Ae^{-kt^2}$$
$$\ln[\frac{F'(t)}{t} - \frac{t}{10}] = \ln A + \ln e^{-kt^2}$$
$$\underline{\underline{\ln[\frac{F'(t)}{t} - \frac{t}{10}] = \ln A - kt^2}}$$

(b) $-k = \text{Slope}$

$= -0.125$

$k = \underline{\underline{0.125}}$

∵ The graph passes through (1, 1.484).

∴ $1.484 = \ln A - (0.125)(1)^2$

$\ln A = 1.609$

$A = \underline{\underline{5.00}}$ (corr. to 3 sig. fig.)

(c) From **(b)**, $F'(t) = 5te^{-0.125t^2} + \frac{t^2}{10}$.

$$\begin{aligned} F(3)-F(1) &= \int_1^3 (5te^{-0.125t^2} + \frac{t^2}{10})dt \\ &= \int_1^3 5te^{-0.125t^2} dt + \int_1^3 \frac{t^2}{10} dt \\ &= 5\int_1^3 (-\frac{1}{0.25})e^{-0.125t^2} d(-0.125t^2) + \int_1^3 \frac{t^2}{10} dt \\ &= -20[e^{-0.125t^2}]_1^3 + [\frac{t^3}{30}]_1^3 \\ &= 12.0 \text{ (corr. to 3 sig. fig.)} \end{aligned}$$

∴ The total sales volume of albums of the singer from the beginning of 2011 to the beginning of 2013 is 12.0 thousand.

93. The rates of change of the accumulated expenditure $E(t)$ and the accumulated revenue $R(t)$ (both in million dollars) of a company can be modelled by $E'(t) = 7 + 0.4t$ $(0 \le t \le 3)$ and $R'(t) = 40te^{-0.2t}$ $(0 \le t \le 3)$ respectively, where t is the number of years of running the business.

(a) Find the total expenditure of the company in the first three years.

(b) Use the trapezoidal rule with 4 subintervals to estimate the total revenue of the company in the first three years. (Give your answer correct to 4 significant figures.)

(c) Hence estimate the average yearly profit made by the company in the first three years. (Give your answer correct to 4 significant figures.)

SOLUTION

(a)
$$\begin{aligned} E(3)-E(0) &= \int_0^3 (7+0.4t)dt \\ &= [7t + 0.2t^2]_0^3 \\ &= 22.8 \end{aligned}$$

∴ The total expenditure of the company in the first three years is 22.8 million dollars.

(b)
$$\begin{aligned} \Delta t &= \frac{3-0}{4} \\ &= 0.75 \end{aligned}$$

t	0	0.75	1.5	2.25	3
$R'(t)$	0	25.821	44.449	57.387	65.857

$$R(3)-R(0)=\int_0^3 R'(t)dt$$

$$\approx \frac{0.75}{2}[R'(0)+2R'(0.75)+2R'(1.5)+2R'(2.25)+R'(3)]$$

$$=120.4 \text{ (corr. to 4 sig. fig.)}$$

$\therefore$ The total revenue of the company in the first three years is about 120.4 million dollars.

(c) Average yearly profit in the first three years $\approx \dfrac{[R(3)-R(0)]-[E(3)-E(0)]}{3}$ million dollars

$$=\frac{120.44-22.8}{3} \text{ million dollars}$$

$$=\underline{\underline{32.55 \text{ million dollars}}} \text{ (corr. to 4 sig. fig.)}$$

94. **(a)** Use the trapezoidal rule with 6 subintervals to find an approximate value of $\int_0^3 e^{0.05x^2}dx$. (Give your answer correct to 3 decimal places.)

(b) The rate of change of the number of fish $P(t)$ (in thousand) in a pond after t years can be modelled by $P'(t)=2e^{0.05t^2}-\dfrac{3}{4}e^{0.4t}$ $(0\le t\le 4)$. Estimate the change in the number of fish in the pond in the first three years. (Give your answer correct to 2 decimal places.)

SOLUTION

(a) Let $f(x)=e^{0.05x^2}$.

$$\Delta x=\frac{3-0}{6}$$

$$=0.5$$

$$\int_0^3 e^{0.05x^2}dx \approx \frac{0.5}{2}[f(0)+2f(0.5)+2f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)]$$

$$=\frac{0.5}{2}[e^{0.05(0)^2}+2e^{0.05(0.5)^2}+2e^{0.05(1)^2}+2e^{0.05(1.5)^2}+2e^{0.05(2)^2}$$

$$+2e^{0.05(2.5)^2}+e^{0.05(3)^2}]$$

$$=\underline{\underline{3.528}} \text{ (corr. to 3 d.p.)}$$

(b) $P(3)-P(0)=\int_0^3 (2e^{0.05t^2}-\frac{3}{4}e^{0.4t})dt$

$=2\int_0^3 e^{0.05t^2}dt-\frac{3}{4}\int_0^3 e^{0.4t}dt$

$=2\int_0^3 e^{0.05t^2}dt-\frac{3}{4}\int_0^3 \frac{1}{0.4}e^{0.4t}d(0.4t)$

$=2(3.527\,7)-\frac{15}{8}[e^{0.4t}]_0^3$ [From the result of **(a)**]

$=2.71$ (corr. to 2 d.p.)

$\therefore$ The number of fish in the pond increases by about 2.71 thousand in the first three years.

Level 3

95. A particle moves along the x-axis from the origin. It is given that the velocity v m/s of the particle after t seconds can be modelled by $v=te^{-t}$ $(t\geq 0)$. Let x m and a m/s^2 be the displacement and the acceleration of the particle after t seconds respectively.

(a) Express a in terms of t.

(b) Find the maximum velocity of the particle.

(c) Use the trapezoidal rule with 4 subintervals to estimate the distance travelled by the particle from $t=1$ and $t=3$. (Give your answer correct to 4 decimal places.)

SOLUTION

(a) $a=\frac{dv}{dt}$

$=\frac{d}{dt}(te^{-t})$

$=(1)e^{-t}+t(-e^{-t})$

$=\underline{\underline{e^{-t}-te^{-t}}}$

(b) When $\frac{dv}{dt}=0$,

$e^{-t}-te^{-t}=0$

$(1-t)e^{-t}=0$

$1-t=0$

$t=1$

When $t=0$,

$v=(0)e^{-0}$

$=0$

When $t=1$,

$v=(1)e^{-1}$

$\quad =e^{-1}$

t	$t=0$	$0<t<1$	$t=1$	$t>1$
v	0		e^{-1}	
$\dfrac{dv}{dt}$		+	0	−

When $t=1$, v attains its maximum value.

$\therefore$ The maximum velocity of the particle is e^{-1} m/s.

(c) Let $f(t)=te^{-t}$.

$\Delta t=\dfrac{3-1}{4}$

$\quad =0.5$

Required distance $=\int_1^3 f(t)\,dt$

$$\approx \frac{0.5}{2}[f(1)+2f(1.5)+2f(2)+2f(2.5)+f(3)]$$

$$=\frac{0.5}{2}[(1)e^{-1}+2(1.5)e^{-1.5}+2(2)e^{-2}+2(2.5)e^{-2.5}+(3)e^{-3}]$$

$=\underline{\underline{0.5346}}$ (corr. to 4 d.p.)

96. The rate of change of the accumulated number of fires $N(t)$ (in thousand) broken out in a town after t years can be modelled by $N'(t)=4-a+(3a-2)e^{-kt}-(a+3)e^{-2kt}$ $(0\le t\le 10)$, where a and k are positive constants. It is given that the rate of change of the accumulated number of fires broken out is the greatest after 3 years and $N'(0)=1$.

(a) Find the values of a and k. (Give your answers correct to 4 decimal places if necessary.)

(b) Using the value of a obtained in **(a)** and taking $k=0.3$, find the number of fires broken out in the third year. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) $\because$ $N'(0)=1$

$\therefore$ $4-a+(3a-2)e^{-k(0)}-(a+3)e^{-2k(0)}=1$

$$4-a+(3a-2)-(a+3)=1$$

$$a=\underline{\underline{2}}$$

$$N'(t) = 4 - 2 + [3(2) - 2]e^{-kt} - (2+3)e^{-2kt}$$
$$= 2 + 4e^{-kt} - 5e^{-2kt}$$

$$N''(t) = 4(-k)e^{-kt} - 5(-2k)e^{-2kt}$$
$$= -4ke^{-kt} + 10ke^{-2kt}$$

$\because \quad N''(3) = 0$

$\therefore \quad -4ke^{-k(3)} + 10ke^{-2k(3)} = 0$

$$-2e^{-3k} + 5e^{-6k} = 0$$
$$e^{-3k}(5e^{-3k} - 2) = 0$$
$$5e^{-3k} - 2 = 0$$
$$e^{-3k} = \frac{2}{5}$$
$$-3k = \ln\frac{2}{5}$$
$$k = -\frac{1}{3}\ln\frac{2}{5}$$
$$= \underline{\underline{0.3054}} \text{ (corr. to 4 d.p.)}$$

(b) From **(a)**, $N'(t) = 2 + 4e^{-0.3t} - 5e^{-2(0.3)t}$

$$= 2 + 4e^{-0.3t} - 5e^{-0.6t}$$

$$N(3) - N(2) = \int_2^3 (2 + 4e^{-0.3t} - 5e^{-0.6t})\,dt$$
$$= \int_2^3 2\,dt + 4\int_2^3 e^{-0.3t}dt - 5\int_2^3 e^{-0.6t}dt$$
$$= \int_2^3 2\,dt + 4\int_2^3 \frac{e^{-0.3t}}{-0.3}\,d(-0.3t) - 5\int_2^3 \frac{e^{-0.6t}}{-0.6}\,d(-0.6t)$$
$$= [2t]_2^3 + \frac{4}{-0.3}[e^{-0.3t}]_2^3 - \frac{5}{-0.6}[e^{-0.6t}]_2^3$$
$$= 2.76 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ $\underline{\underline{\text{The number of fires broken out in the third year is 2.76 thousand.}}}$

97. A publishing company models the rate of change of the sales volume of a novel by the following formula:

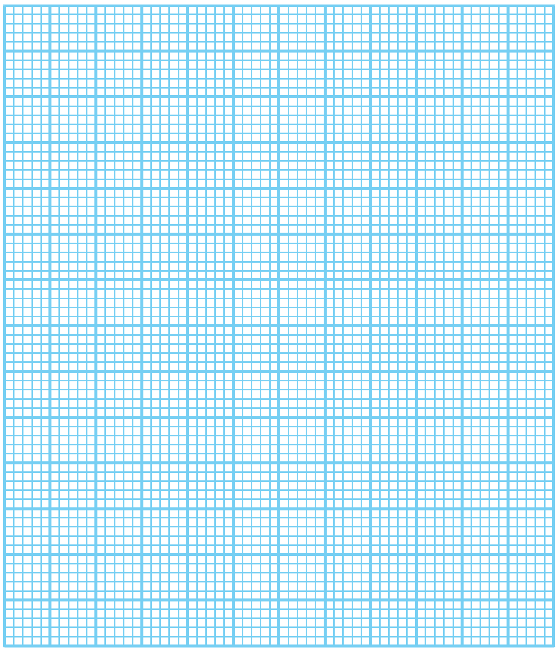
$$N'(t) = \frac{Ae^{-kt}}{t+4} \quad (0 \le t \le 12),$$

where $N(t)$ is the accumulated sales volume (in thousand) of the novel, t is the number of months elapsed since the novel is first sold, A and k are constants. The following table records some values of $N'(t)$ corresponding to t.

t	2	4	6	8
$N'(t)$	4.48	3.01	2.16	1.61

(a) Express $\ln[(t+4)N'(t)]$ as a linear function of t.

(b) Plot the graph of $\ln[(t+4)N'(t)]$ against t.



(c) Use the graph in **(b)** to find the values of A and k. (Give your answers correct to 2 significant figures if necessary.)

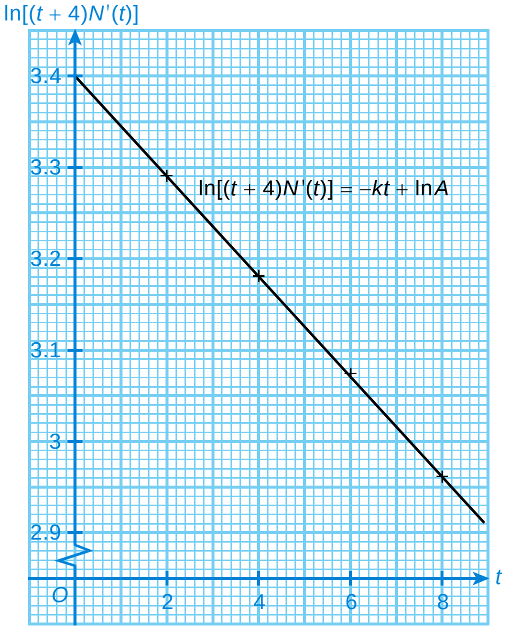
(d) By taking $A = 30$ and $k = 0.055$, use the trapezoidal rule with 6 subintervals to estimate the total sales volume of the novel for the three months from the beginning of the 4th month to the end of the 6th month. (Give your answer correct to 3 significant figures.)

SOLUTION

(a)
$$N'(t) = \frac{Ae^{-kt}}{t+4}$$
$$(t+4)N'(t) = Ae^{-kt}$$
$$\ln[(t+4)N'(t)] = \ln A + \ln e^{-kt}$$
$$\underline{\underline{\ln[(t+4)N'(t)] = -kt + \ln A}}$$

(b) Convert the given data into the following.

t	2	4	6	8
$\ln[(t+4)N'(t)]$	3.291	3.181	3.073	2.961



(c) From the graph in **(b)**,

$$-k = \text{Slope}$$
$$= \frac{2.961-3.291}{8-2}$$
$$= -0.055$$
$$k = \underline{\underline{0.055}}$$

$$\ln A = \text{Intercept on the vertical axis}$$
$$= 3.4$$
$$A = \underline{\underline{30}} \text{ (corr. to 2 sig. fig.)}$$

(d) $N'(t) = \dfrac{30e^{-0.055t}}{t+4}$

$$\Delta t = \frac{6-3}{6}$$
$$= 0.5$$

t	3	3.5	4	4.5	5	5.5	6
$N'(t)$	3.634	3.300	3.009	2.756	2.532	2.334	2.157

$$N(6)-N(3)=\int_3^6 N'(t)dt$$

$$\approx \frac{0.5}{2}[N'(3)+2N'(3.5)+2N'(4)+2N'(4.5)+2N'(5)+2N'(5.5)+N'(6)]$$

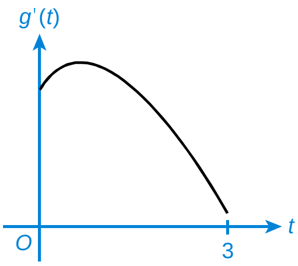
$$=8.41 \text{ (corr. to 3 sig. fig.)}$$

$\therefore$ The total sales volume for the three months from the beginning of the 4th month to the end of the 6th month is about 8.41 thousand.

98. (a) The rate of change of the accumulated sales volume $f(t)$ (in million) of brand A mobile phone can be modelled by $f'(t)=t^3-6t^2+9t+2$ $(0\le t\le 3)$, where t is the number of years elapsed since the beginning of 2010. Find the accumulated sales volume of the mobile phone for the three years from the beginning of 2010 to the end of 2012.

(b) The rate of change of the accumulated sales volume $g(t)$ (in million) of brand B mobile phone can be modelled by $g'(t)=5-2.7t\ln\frac{t+0.5}{2}$ $(0\le t\le 3)$, where t is the number of years elapsed since the beginning of 2010. Use the trapezoidal rule with 6 subintervals to estimate the accumulated sales volume of the mobile phone for the three years from the beginning of 2010 to the end of 2012. (Give your answer correct to 2 decimal places.)

(c) The following shows a sketch of $g'(t)$ against t, where $t\ge 0$.



According to the graph, Milton claims that the accumulated sales volume of brand B mobile phone is greater than that of brand A mobile phone for the three years from the beginning of 2010 to the end of 2012. Do you agree with him? Explain briefly.

SOLUTION

(a) $$f(3)-f(0)=\int_0^3(t^3-6t^2+9t+2)dt$$

$$=[\frac{t^4}{4}-2t^3+\frac{9}{2}t^2+2t]_0^3$$

$$=12.75$$

$\therefore$ The accumulated sales volume of brand A mobile phone for the three years from the beginning of 2010 to the end of 2012 is 12.75 million.

(b) $\Delta t = \dfrac{3-0}{6}$
$= 0.5$

t	0	0.5	1	1.5	2	2.5	3
$g'(t)$	5	5.936	5.777	5	3.795	2.263	0.467

$$g(3) - g(0) = \int_0^3 g'(t)dt$$
$$\approx \frac{0.5}{2}[g'(0) + 2g'(0.5) + 2g'(1) + 2g'(1.5) + 2g'(2) + 2g'(2.5) + g'(3)]$$
$$= 12.75 \text{ (corr. to 2 d.p.)}$$

∴ The accumulated sales volume of brand B mobile phone for the three years from the beginning of 2010 to the end of 2012 is 12.75 million.

(c) Since the graph of $g'(t)$ is concave downwards on the interval $0 \leq t \leq 3$, the approximate value obtained in **(b)** is an underestimate.

∴ Accumulated sales volume of brand B > 12.75 million
= Accumulated sales volume of brand A

∴ I agree with Milton.

99. An oil company estimates that the rate of change of the oil production rate $R(t)$ (in thousand barrels/year) after t years since the first extraction from an oil field can be modelled by $R'(t) = \dfrac{28t - 224}{t^2 - 16t + 80}$ $(0 \leq t \leq 8)$, where $R(0) = 30$.

(a) **(i)** Let $u = t^2 - 16t + 80$, find $\dfrac{du}{dt}$.

(ii) Find $R(t)$.

(b) **(i)** Use the trapezoidal rule with 4 subintervals to estimate the total oil production from the beginning of the 2nd year to the end of the 5th year. (Give your answer correct to 3 significant figures.)

(ii) Hence find the average yearly oil production in this period. (Give your answer correct to 3 significant figures.)

SOLUTION

(a) **(i)** $u = t^2 - 16t + 80$

$\dfrac{du}{dt} = 2t - 16$

(ii) $\because \quad R'(t)=\dfrac{28t-224}{t^2-16t+80}$

$\therefore \quad R(t)=\displaystyle\int \frac{28t-224}{t^2-16t+80}dt$

$=\displaystyle\int \frac{14}{t^2-16t+80}\cdot(2t-16)dt$

$=\displaystyle\int \frac{14}{u}du$

$=14\ln|u|+C$

$=14\ln(t^2-16t+80)+C$

$\because \quad R(0)=30$

$\therefore \quad 14\ln[0^2-16(0)+80]+C=30$

$C=30-14\ln 80$

$\therefore \quad R(t)=\underline{\underline{14\ln(t^2-16t+80)+30-14\ln 80}}$

(b) (i) Let $N(t)$ (in thousand barrels) be the total oil production in the first t years after the first extraction.

$\Delta t=\dfrac{5-1}{4}=1$

t	1	2	3	4	5
$R(t)$	27.09	23.97	20.64	17.17	13.72

$N(5)-N(1)=\displaystyle\int_1^5 R(t)\,dt$

$\approx \dfrac{1}{2}[R(1)+2R(2)+2R(3)+2R(4)+R(5)]$

$=82.2$ (corr. to 3 sig. fig.)

$\therefore$ <u>The total oil production from the beginning of the 2nd year to the end of the 5th year is about 82.2 thousand barrels.</u>

(ii) Average yearly oil production $=\dfrac{82.19}{4}$ thousand barrels

$=$ <u>20.5 thousand barrels</u> (corr. to 3 sig. fig.)

100. The rate of change $C'(t)$ of the total consumption of electricity $C(t)$ (in unit) of a city can be modelled by $C'(t)=Ae^{kt^2}$ $(0\le t\le 3)$, where t is the number of years elapsed since the beginning of 2010, A and k are constants.

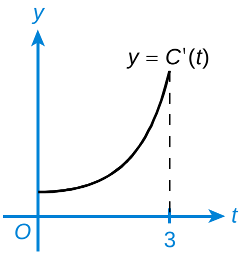
(a) It is given that $C'(0)=5$ and $C'(1)=6.1$. Find the values of A and k. (Give your answers correct to 1 decimal place if necessary.)

(b) **(i)** Using the value of k obtained in **(a)**, expand e^{kt^2} in ascending powers of t up to t^4.

(ii) Using the result of **(b)(i)**, estimate the total consumption of electricity of the city for the first two years from the beginning of 2010. (Give your answer correct to 1 decimal place.)

(iii) Determine whether the approximate value obtained in **(b)(ii)** is an underestimate or an overestimate.

(c) The following shows a sketch of $y=C'(t)$, where $t \ge 0$.



(i) Use the trapezoidal rule with 4 subintervals to estimate the total consumption of electricity of the city for the first two years from the beginning of 2010. (Give your answer correct to 1 decimal place.)

(ii) From the graph, determine whether the approximate value obtained in **(c)(i)** is an underestimate or an overestimate.

(d) Using the results of **(b)** and **(c)**, estimate the range of the total consumption of electricity of the city for the first two years from the beginning of 2010.

SOLUTION

(a) $\because$ $C'(0)=5$

$\therefore$ $Ae^{k(0)^2}=5$

$$A=\underline{\underline{5}}$$

$\because$ $C'(1)=6.1$

$\therefore$ $5e^{k(1)^2}=6.1$

$$e^k=\frac{6.1}{5}$$

$$k=\ln\frac{6.1}{5}$$

$$=\underline{\underline{0.2}} \text{ (corr. to 1 d.p.)}$$

(b) **(i)** $e^{kt^2}=e^{0.2t^2}$

$$=1+\frac{0.2t^2}{1!}+\frac{(0.2t^2)^2}{2!}+\cdots$$

$$=\underline{\underline{1+0.2t^2+0.02t^4+\cdots}}$$

(ii) $C(2)-C(0)=\int_0^2 5e^{0.2t^2}dt$

$$=5\int_0^2(1+0.2t^2+0.02t^4+\cdots)dt \quad \text{[From the result of } \textbf{(b)(i)}\text{]}$$

$$\approx 5\int_0^2(1+0.2t^2+0.02t^4)dt$$

$$=5[t+\frac{0.2}{3}t^3+\frac{0.02}{5}t^5]_0^2$$

$$=13.3 \text{ (corr. to 1 d.p.)}$$

∴ The total consumption of electricity of the city for the first two years from the beginning of 2010 is about 13.3 units.

(iii) When $r>0$, $\frac{(0.2t^2)^r}{r!}>0$.

$$5e^{0.2t^2}=5[1+\frac{0.2t^2}{1!}+\frac{(0.2t^2)^2}{2!}+\cdots]>5[1+\frac{0.2t^2}{1!}+\frac{(0.2t^2)^2}{2!}]$$

∴ $\int_0^2 5e^{0.2t^2}dt>5\int_0^2(1+0.2t^2+0.02t^4)dt$

∴ The approximate value obtained in **(b)(ii)** is an underestimate.

(c) **(i)** $\Delta t=\frac{2-0}{4}=0.5$

t	0	0.5	1	1.5	2
$C'(t)$	5	5.26	6.11	7.84	11.13

$C(2)-C(0)=\int_0^2 C'(t)dt$

$$\approx \frac{0.5}{2}[C'(0)+2C'(0.5)+2C'(1)+2C'(1.5)+C'(2)]$$

$$=13.6 \text{ (corr. to 1 d.p.)}$$

∴ The total consumption of electricity of the city for the first two years from the beginning of 2010 is about 13.6 units.

(ii) Since the graph of $y=C'(t)$ is concave upwards on the interval $0\le x\le 2$, the approximate value obtained in **(c)(i)** is an overestimate.

(d) From the result of **(b)**, total consumption of electricity ≥ 13.3 units.

From the result of **(c)**, total consumption of electricity ≤ 13.6 units.

∴ 13.3 units $\le$ Total consumption of electricity ≤ 13.6 units