A Detailed Analysis and Derivation of the Black-Scholes Pricing Formula

The Markov Property

A sensible way to introduce the Markov property is through a sequence of random variables Z_i, which can take one of two values from the set $\{1, -1\}$. This is known as a *coin toss*. We can calculate the expectations of Z_i:

$$\mathrm{E}(\mathrm{Z_i}) = 0,$$

$$E\left(Z_i^2\right) = 1,$$

$$E(Z_i Z_k) = 0$$

The key point is that the *expectation of Z_i has no dependence on any previous values within the sequence*. Let us take the partial sums of our random variables within our coin toss, which we will denote by Si:

$$\mathrm{S_i} = \sum_{\{\mathrm{k=1}\}}^{\mathrm{i}} \mathrm{Zi}$$

We can now calculate the expectations of our partial sums, using the linearity of the expectation operator:

$$E(S_i) \; = \; 0,$$

$$E\left(S_i^2\right) = \; E(Z_1^2 \; + \; 2\, Z_1\, Z_2 \; + \; \ldots) \; = \; i$$

P(future | present, past) = P(future | present, ~~past~~)

Markov property

We see that, again, there is no dependence on the expectation of Si of any previous value within the sequence of partial sums. We can extend this to discuss *conditional expectation*. Conditional expectation is the expectation of a random variable with respect to some conditional probability distribution. Hence, we can ask that if $i = 4$ (i.e. we carry out four coin tosses), what does this mean for the expectation of S5?

$$E(S_5|Z_1, Z_2, Z_3, Z_4) = S_4$$

That is, the expected value of Si is only dependent upon the previous value Si−1, not on any values prior to that. This is known as the Markov Property. Essentially, there is no memory of past events beyond the point our variable is currently at within the sequence. Nearly all financial models discussed in these articles will possess the Markov property.

The Martingale Property

An additional property that holds for our sequence of partial sums is the *Martingale property*. It states that the conditional expectation of the sequence of partial sums, Si is simply the current value:

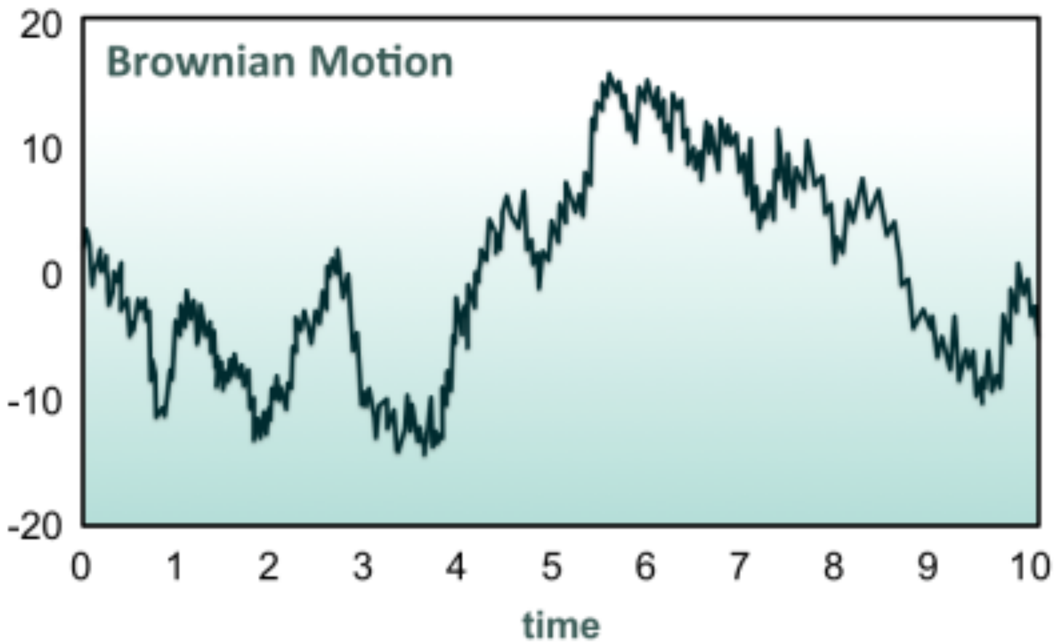
$$E(S_i|S_k, k < i) = S_k$$

Essentially, the martingale property ensures that in a "fair game", knowledge of the past will be of no use in predicting future winnings.

These properties will be of fundamental importance in regard to defining Brownian motion, which will later be used as a model for an asset price path.

The Brownian Motion and Weiner Process

In the previous discussion on the Markov and Martingale properties, a discrete coin toss experiment was carried out, with an arbitrary number of time steps. The current goal is to work towards a continuous-time random walk, which will provide a more sophisticated model for the time-varying price of assets. In order to achieve this, the number of time steps will need to be increased. However, the manner in which they are increased must occur in a specific fashion, so as to avoid a nonsensical (infinite) result.



Consider a continuous real-valued time interval $[0, T]$, $T > 0$. In this interval N coin tosses will be carried out, which each take a time T/N and hence are spaced equally. Concurrently, the payoff returned from each coin toss will be modified. The sequence of discrete random variables representing the coin toss is $Zi \in \{-1,1\}$. A further sequence of discrete random variables, $\widetilde{Z}_\iota \in T/N, -T/N$, can be defined. The definition of such a sequence of DRVs is used to provide a very specific *quadratic variation* of the coin toss.

The quadratic variation of a sequence of DRVs is defined to be simply the sum of the squared differences of the current and previous terms:

$$\sum_{k=1}^{i}(S_k - S_{k-1})^2$$

For Zi, the previous coin toss random variable sequence, the quadratic variation is given by:

$$\sum_{k=1}^{i}(S_k - S_{k-1})^2 = i$$

For $\widetilde{Z}_\iota$, the quadratic variation of the partial sums $\widetilde{S}_\iota$is:

$$\sum_{k=1}^{N}\left(\widetilde{S_k} - \widetilde{S_{k-1}}\right)^2 = N \times \left(\sqrt{\frac{T}{N}}\right)^2 = T$$

Thus, by construction, the quadratic variation of the amended coin toss $\widetilde{Z}_\iota$ is simply the total duration of all tosses, T.

Importantly, note that both the Markov and Martingale properties are retained by $\widetilde{Z}_\iota$. As $N \to \infty$ the random walk coin toss does not diverge. If the value of the asset at time t, with t∈[0,T], is given by S(t), then its conditional expectation at the end of the interval, given that S(0)=0, is $E(S(T)) = 0$ with a variance of $E(S(T)^2) = T$.

Although the technical details will not be discussed, as the number of steps N becomes infinite, the *Wiener process* is obtained, more commonly called a *standard Brownian motion*, which will be denoted by B(t). Formally, the definition is given by:

Definition (Wiener Process/Standard Brownian Motion)
A sequence of random variables $B(t)$ is a *Brownian motion* if $B(0) = 0$, and for all t, s such that $s < t$, $B(t) - B(s)$ is normally distributed with variance t−s and the distribution of $B(t) - B(s)$ is independent of $B(r)$ for $r \leq s$.

Properties of Brownian Motion

Standard Brownian motion has some interesting properties. In particular:

- Brownian motions are **finite**. The construction of $Z{\sim}i$ was chosen carefully in order that in the limit of large N, B was both finite and non-zero.
- Brownian motions have **unbound variation**. This means that if the sign of all negative gradients were switched to positive, then B would hit infinity in an arbitrarily short time period.
- Brownian motions are **continuous**. Although Brownian motions are continuous everywhere, they are differentiable nowhere. Essentially this means that a Brownian motion has fractal geometry. This has important implications regarding the choice of calculus methods when Brownian motions are to be manipulated.

- Brownian motions satisfy both the **Markov** and **Martingale** properties. The conditional distribution of B(t) given information until $s < t$ is dependent only on $B(s)$ and, given information until s<t, the conditional expectation of $B(t)$ is $B(s)$.
- Brownian motions are **strongly normally distributed**. This means that, for $s < t,\ s, t \in [0, T]$, that $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$.

Brownian motions are a fundamental component in the construction of stochastic differential equations, which will eventually allow derivation of the famous Black-Scholes equation for contingent claims pricing.

The Stochastic Differential Equations

The previous discussion on Brownian motion and the Wiener Process we introduced the *standard Brownian motion*, as a means of modelling asset price paths. However, a standard Brownian motion has a non-zero probability of being negative. This is clearly not a property shared by real-world assets - stock prices cannot be less than zero. Hence, although the stochastic nature of a Brownian motion for our model should be retained, it is necessary to adjust exactly how that randomness is distributed. In particular, the concept of *geometric Brownian motion* (GBM) will now be introduced, which will solve the problem of negative stock prices.

However, before the geometric Brownian motion is considered, it is necessary to discuss the concept of a *Stochastic Differential Equation* (SDE). This will allow us to formulate the GBM and solve it to obtain a function for the asset price path.

Stochastic Differential Equations

Now that we have defined Brownian motion, we can utilise it as a building block to start constructing *stochastic differential equations* (SDE). We need SDE in order to discuss how functions $f = f(S)$ and their derivatives with respect to S behave, where S is a stock price determined by a Brownian motion.

Some of the rules of ordinary calculus do not work as expected in a stochastic world. We need to modify them to take into account both the random behaviour of Brownian motion as well as its non-differentiable nature. We will begin by discussing *stochastic integrals*, which will lead us naturally to the concept of an SDE.

Definition (Stochastic Integral)
A *stochastic integral* of the function $f = f(t)$ is a function $W = W(t),\ t \in [0, T]$ given by:

$$W(t) = \int_0^t f(s)dB(s) = \lim_{n\to\infty} \sum_{k=1}^{N} f(t_{k-1})\big(B(t_k) - B(t_{k-1})\big)$$

where $t_k = kt/N$.

Note that the function f is *non-anticipatory*, in the sense that it is evaluated within the summation sign at time $t_k - 1$. This means that it has no information as to what the random variable at $X(t_k)$ is. Supposing that f represented some portfolio allocation based on B, then if it were not evaluated at t_k−1, but rather at t_k, we would be able to anticipate the future and modify the portfolio accordingly.

The previous expression provided for W(t) is an integral expression and thus is well-defined for a non-differentiable variable, B(t), due the property of finiteness as well as the chosen mean and variance. However, we wish to be able to write it in differential form:

$$dW \ = \ f(t)dB$$

One can consider the term dB as being a normally distributed random variable with zero mean and variance dt. The formal definition is provided:

Definition (Stochastic Differential Equation)
Let B(t) be a Brownian motion. If W(t) is a sequence of random variables, such that for all t,

$$W(t+\delta t) - W(t) - \delta t \mu\big(t, W(t)\big) - \sigma\big(t, B(t)\big)\big(B(t+\delta t) - B(t)\big)$$

is a random variable with mean and variance that are $o(\delta t)$, then:

$$dW = \mu\big(t, W(t)\big)dt + \sigma\big(t, W(t)\big)dB$$

is a *stochastic differential equation* for $W(t)$.

A sequence of random variables given by the above is termed an *Ito drift-diffusion process*, or simply an *Ito process* or a *stochastic process*.

It can be seen that μ and σ are both functions of t and W. μ has the interpretation of a non-stochastic *drift* coefficient, while σ represents the coefficient of *volatility* - it is multiplied by the stochastic dB term. Hence, stochastic differential equations have both a non-stochastic and stochastic component.

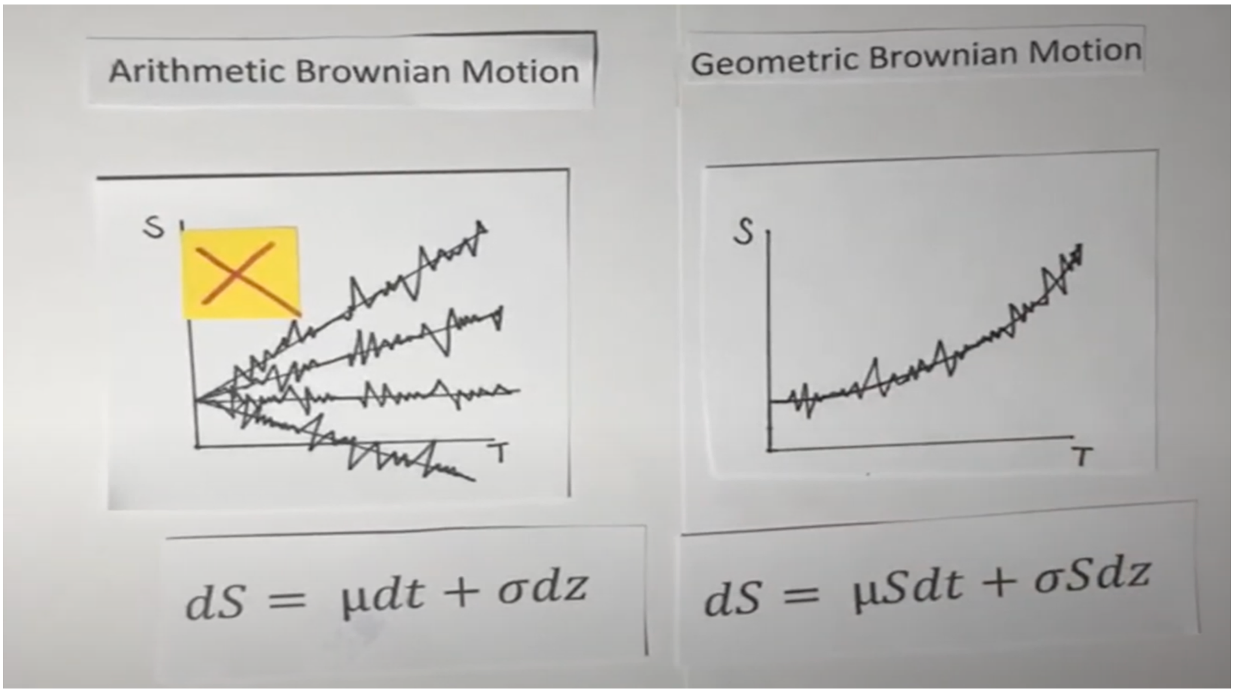
In the following section on geometric Brownian motion, a stochastic differential equation will be utilised to model asset price movements.

Geometric Brownian Motion

The usual model for the time-evolution of an asset price S(t) is given by the geometric Brownian motion, represented by the following stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$$

Note that the coefficients μ and σ, representing the *drift* and *volatility* of the asset, respectively, are both constant in this model. In more sophisticated models they can be made to be functions of t, $S(t)$ and other stochastic processes.



The solution $S(t)$ can be found by the application of Ito's Lemma to the stochastic differential equation.

Dividing through by S(t) in the above equation leads to:

$$dS(t)/S(t) = \mu dt + \sigma dB(t)$$

Notice that the left hand side of this equation looks similar to the derivative of $log\ S(t)$. Applying Ito's Lemma to $log\ S(t)$ gives:

$$d\big(logS(t)\big) = \big(logS(t)\big)' \mu S(t)dt + \big(logS(t)\big)' \sigma S(t)dB(t) + \frac{1}{2}\big(logS(t)\big)'' \sigma^2 S(t)^2 dt$$

This becomes:

$$d\big(logS(t)\big) = \mu dt + \sigma dB(t) - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB(t)$$

This is an Ito drift-diffusion process. It is a standard Brownian motion with a drift term. Since the above formula is simply shorthand for an integral formula, we can write this as:

$$log\big(S(t)\big) - log\big(S(0)\big) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t)$$

Finally, taking the exponential of this equation gives:

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t)\right)$$

This is the solution the stochastic differential equation. In fact it is one of the only analytical solutions that can be obtained from stochastic differential equations.

Ito’s Lemma

Ito's Lemma is a key component in the Ito Calculus, used to determine the derivative of a time-dependent function of a stochastic process. It performs the role of the chain rule in a stochastic setting, analogous to the chain rule in ordinary differential calculus. Ito's Lemma is a cornerstone of quantitative finance and it is intrinsic to the derivation of the Black-Scholes equation for contingent claims (options) pricing.

It is necessary to understand the concepts of Brownian motion, stochastic differential equations and geometric Brownian motion before proceeding.

The Chain Rule

One of the most fundamental tools from ordinary calculus is the chain rule. It allows the calculation of the derivative of chained functional composition. Formally, if W(t) is a continuous function, and:

$$dW(t) = \mu\,(W(t), t)\,dt$$

Then the chain rule states:

$$d\left(f\left(W(t)\right)\right) = f'\left(W(t)\right)\mu(W(t), t)dt$$

When f has t as a direct dependent parameter also, we require additional terms and partial derivatives. In this instance, the chain rule is given by:

$$d\left(f(W(t), t)\right) = \left(\frac{\partial f}{\partial w}(W(t), t)\mu(W(t), t) + \frac{\partial f}{\partial t}(W(t), t)\right)dt$$

In order to model an asset price distribution correctly in a log-normal fashion, a stochastic version of the chain rule will be used to solve a stochastic differential equation representing geometric Brownian motion.

The primary task is now to correctly extend the ordinary calculus version of the chain rule to be able to cope with random variables.

Theorem (Ito's Lemma)

Let B(t) be a Brownian motion and W(t) be an Ito drift-diffusion process which satisfies the stochastic differential equation:

$$dW(t) = \mu(W(t), t)dt + \sigma(W(t), t)dB(t)$$

If $f(w, t) \in C^2(R^2, R)$ then $f(W(t), t)$ is also an Ito drift-diffusion process, with its differential given by:

$$d\big(f(W(t), t)\big) = \frac{\partial f}{\partial t}(W(t), t)dt + f'(W(t), t)dW + \frac{1}{2}f''(W(t), t)dW(t)^2$$

$$With\ \ dW(t)^2\ given\ by:\ dt^2 = 0,\ dtdB(t) = 0\ and\ dB(t)^2 = dt.$$

Deriving the Black-Scholes

Now that we have derived Ito's Lemma, we are in a position to derive the Black-Scholes equation.

Suppose we wish to price a vanilla European contingent claim C, on a time-varying asset S, which is set to mature at T. We shall assume that S follows a geometric Brownian motion with mean growth rate of μ and volatility σ. r will represent the continuously compounding risk free interest rate. r, μ and σ are not functions of time, t, or the asset price S and so are fixed for the duration of the option's lifetime.

Since our option price, C, is a function of time t and the price of the asset S, we will use the notation $C = C(S,t)$ to represent the price of the option. Note that we are *assuming* at this stage that C exists and is well-defined. We will later show this to be a justified claim.

The first step is to utilise Ito's Lemma on the function $C(S,t)$ to give us a SDE:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(S,t)dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(S,t)dS^2$$

Our asset price is modelled by a geometric Brownian motion, the expression for which is recalled here. *Note that μ and σ are constant - i.e. not functions of S or t*:

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t)$$

We can substitute this expression into Ito's Lemma to obtain:

$$dC = \left(\frac{\partial C}{\partial t}(S,t) + \mu S\frac{\partial C}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)dt + \sigma S\frac{\partial C}{\partial S}(S,t)dX$$

The thrust of our derivation argument will essentially be to say that a fully hedged portfolio, with all risk eliminated, will grow at the risk free rate. Thus, we need to determine how our portfolio changes in time. Specifically, we are interested in the infinitesimal change of a mixture of a call option and a quantity of assets. The quantity will be denoted by Δ. Hence:

$$d(C+\Delta S)=\left(\frac{\partial C}{\partial t}(S,t)+\mu S\frac{\partial C}{\partial S}(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)+\Delta\mu S\right)dt$$
$$+\Delta S\left(\frac{\partial C}{\partial S}+\Delta\right)dX$$

This leads us to a choice for Δ which will eliminate the term associated with the randomness. If we set $\Delta = -\frac{\partial C}{\partial S}(S,t)$, we receive:

$$d(C+\Delta S)=\left(\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)\right)dt$$

Note that we have glossed over the issue of what the derivative of Δ is. We will return to this later.

This technique is known as *Delta-Hedging* and provides us with a portfolio that is free of randomness. This is how we can apply the argument that it should grow at the risk free rate, otherwise, as with our previous arguments, we would have an arbitrage opportunity. Hence the growth rate of our delta-hedged portfolio must be equal the continuously compounding risk free rate, r. Thus we are able to state that:

$$\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)=r\left(C-S\frac{\partial C}{\partial S}\right)$$

If we rearrange this equation, and using shorthand notation to drop the dependence on (S,t) we arrive at the famous Black-Scholes equation for the value of our contingent claim:

$$\frac{\partial C}{\partial t}+rS\frac{\partial C}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}-rC=0$$

Although we have derived the equation, we do not yet possess enough conditions in order to provide a unique solution. The equation is a second-order linear partial differential equation (PDE) and without boundary conditions (such as a payoff function for our contingent claim), we will not be able to solve it.

One payoff function we can use is that of a European call option struck at K. This has a payoff function at expiry, T, of:

$$C(S,T) = \max(S-K,0)$$

We are now in a position to solve the Black-Scholes equation.

Black Scholes Pricing formula

It has already been outlined that the reader is to be familiar with the Black-Scholes formula for the pricing of European Vanilla Calls and Puts. For completeness, the price of a European Vanilla Call, $C(S,t)$ is given below, where S is the underlying asset, K is the strike price, r is the risk-free rate, T is the time to maturity and σ is the (constant) volatility of the underlying S (N is described below):

$$C(S,t) = SN(d_1) - Ke^{-rT}N(d_2)$$

With d_1 and d_2 defined as follows:

$$d_1 = \frac{log(S/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

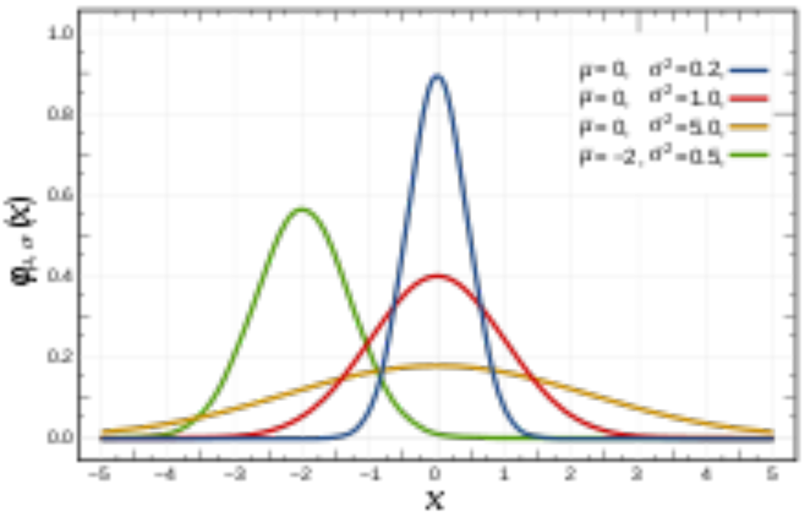
$$d_2 = d_1 - \sigma\sqrt{T}$$

Thanks to Put-Call Parity, we are also able to price a European Vanilla Put $P(S,t)$ with the following formula:

$$P(S,t) = Ke^{-rT} - S + C(S,t) = Ke^{-rT} - S + \left(SN(d_1) - Ke^{-rT}N(d_2)\right)$$

The remaining function we have yet to describe is N. This is the cumulative distribution function of the standard normal distribution. The formula for N is given by:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$



In addition, we would like to have closed form solutions for the "Greeks", which are the option price sensitivities to the underlying variables and parameters. For this reason we also need the formula for the probability density function of the standard normal distribution which is given below:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Code Snippets To Calculate the Black-Scholes Pricing Value:

```
From math import exp, log, sqrt, pi
from statistics import norm_pdf, norm_cfd


def norm_pdf(x):
    """
    Standard normal probability density function
    """
    return (1.0/((2*pi)**0.5))*exp(-0.5*x*x)


def norm_cdf(x):
    """
    An approximation to the cumulative distribution
    function for the standard normal distribution:
    N(x) = \frac{1}{sqrt(2*\pi)} \int^x_{-\infty} e^{-
\frac{1}{2}s^2} ds
    """
    k = 1.0/(1.0+0.2316419*x)
    k_sum = k * (0.319381530 + k * (-0.356563782 + \
        k * (1.781477937 + k * (-1.821255978 + 1.330274429 * k))))

    if x >= 0.0:
        return (1.0 - (1.0 / ((2 * pi)**0.5)) * exp(-0.5 * x * x) *
k_sum)
    else:
        return 1.0 - norm_cdf(-x)
```

```python
def d_j(j, S, K, r, v, T):
    """
    d_j = \frac{log(\frac{S}{K})+(r+(-1)^{j-1} \frac{1}{2}v^2)T}{v
sqrt(T)}
    """
    return (log(S/K) + (r + ((-1)**(j-1))*0.5*v*v)*T)/(v*(T**0.5))



def vanilla_call_price(S, K, r, v, T):
    """
    Price of a European call option struck at K, with
    spot S, constant rate r, constant vol v (over the
    life of the option) and time to maturity T
    """
    return S * norm_cdf(d_j(1, S, K, r, v, T)) - \
        K*exp(-r*T) * norm_cdf(d_j(2, S, K, r, v, T))

def vanilla_put_price(S, K, r, v, T):
    """
    Price of a European put option struck at K, with
    spot S, constant rate r, constant vol v (over the
    life of the option) and time to maturity T
    """
    return -S * norm_cdf(-d_j(1, S, K, r, v, T)) + \
        K*exp(-r*T) * norm_cdf(-d_j(2, S, K, r, v, T))
```