1. State Equation

$\beta_t = \beta_{t-1} + w_t$, where $w_t \sim N(0, Q)$

* Here y_t (OJIH price) depends on x_t (SPY price) based on time variance

2. Measurment Equation

$y_t = \beta_t x_t + \epsilon_t$, where $\epsilon_t \sim N(0, R)$

Predicted State

$\beta_{t|t-1} = \beta_{t-1|t-1}$

Predicted Covariance

$P_{t|t-1} = P_{t-1|t-1} + Q$

Kalman Gain

$$k_t = \frac{P_{t|t-1} x_t}{x_t^2 P_{t|t-1} + R}$$

Updated Covariance

$\underline{P_{t|t}} = (1 - k_t x_t) P_{t|t-1}$

Substitute the $P_{t|t}$ to the Kalman Gain Equatio

$$\beta_t = \beta_{t-1} + \frac{(P_{t-1} + Q) x_t}{x_t^2 (P_{t-1} + Q) + R} (y_t - \beta_{t-1} x_t)$$

$y_t - \beta_{t-1} x_t$ is innovation term its the residual error between observed y_t and the predicted value using prior beta

Observation Eq.

$$y_t = \beta_t x_t + \epsilon_t, \quad \epsilon_t \sim N(0, R)$$

y_t: this is the observed dependent variable at time t. (Return of SPY)
x_t: this is the independent variable at time t. (Return of DJIA)
β_t: time-varying hedge ratio, how many units of x_t we need to hedge one unit of y_t
ϵ_t: Measument noise or error, Normal variance, with zero mean and variance R

State Eq.

$$\beta_t = \beta_{t-1} + n_t, \quad n_t \sim N(0, Q)$$

The hedge ratio is random walk (follows no specific pattern). The current $\beta_t = \beta_{t-1}$ + random noise

Prediction State: $\hat{\beta}_{t|t-1} = \hat{\beta}_{t-1|t-1}$ since state Eq. is random walk our best prediction of the current state is prev. fitered Estimate ($\hat{\beta}_{t|t-1}$)

Preddicted Error Covariance: $P_{t|t-1} = P_{t-1|t-1} + Q$
$P_{t-1|t-1}$ = error covarionce of prev. state Estimate
Q: We add to account for uncertanity introduced by random walk

Innovation (Measurement Residual):
Residual: $v_t = y_t - x_t \beta_{t|t-1}$ is the difference b/w actual y_t and predicted $x_t \beta_{t|t-1}$
Covariance: $S_t = x_t^2 P_{t|t-1} + R$ represents total uncertanity in our v_t
$x_t^2 P_{t|t-1}$: comes from propgating uncertanity of our state Prediction into Measurment
R: is variance of measument noise

Kalman Gain: $K_t = \frac{P_{t|t-1} x_t}{S_t} = \frac{P_{t|t-1} x_t}{x^2 P_{t|t-1} + R}$ tells us how much weight to give to the innovation when updating our estimate.
If $P_{t|t-1}$ (Prediction Uncertanity) is large relative to R (measurment noise), Gain in K_t will be high -> meaning we trust new measurment more
If R is large, we rely on prev. state