Comparison of Integrator

In the physics problem, There are many ordinary differential equations (ODE). For example, Equation of Motion, etc,. And if we want to solve these equations numerically, the integrator is necessary to do that. So, the integrator means the algorithm that solving ODE numericcally.

There are many types of integrator. In this case, I will ignore the integrator that solve by using function such as Heun-Method or Runge-Kutta Method because N-Body Problem algorithm is computing forces exerted on each body directly easier than substituting values in variables of functions. In this article, I will compare the error that be produced from following integrator.

- Explicit Euler Method
- Semi-Implicit Euler Method
- Verlet Integration
- Stormer–Verlet method

The system I'm using to test the integrator is planetary orbit. I'm using 5 planets moving around the center (or the sun). The integrator that makes the all of planetary orbit to be perfect circles and conserve total energy will be determined as the best integrator.

Explicit Euler Method

Explicit Euler Method is the easiest integrator. This Integrator updates position before velocity.

$$x_{i+1} = x_i + v_i \Delta t$$
$$v_{i+1} = v_i + \frac{F(x_{i+1})}{m} \Delta t$$

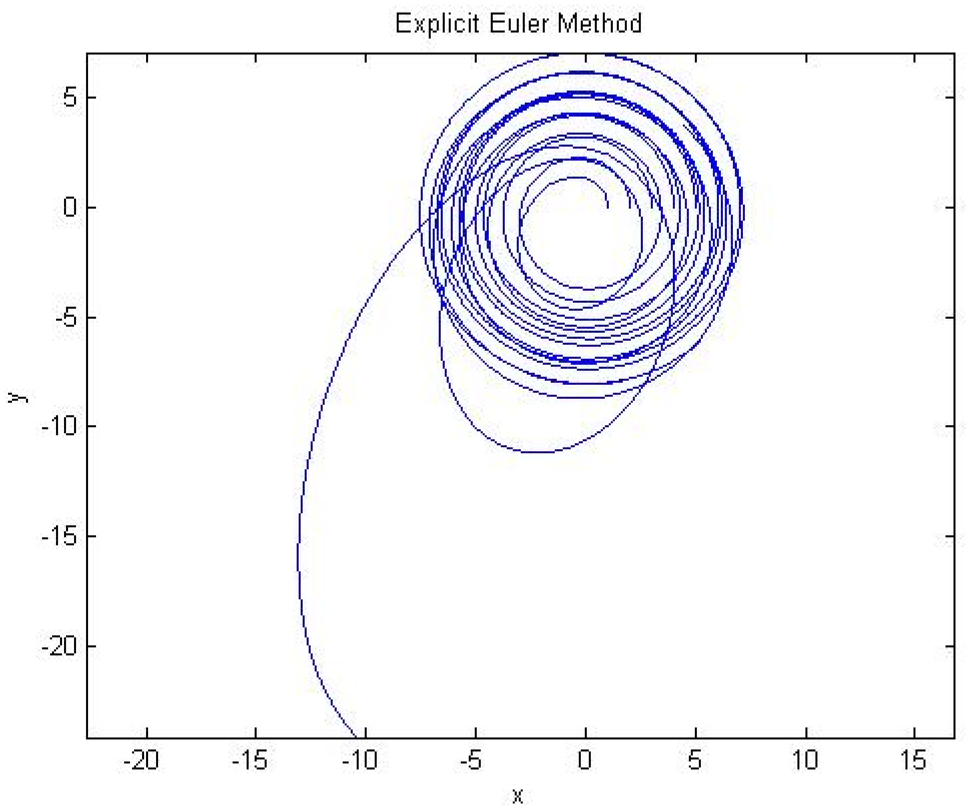

figure 1

From the figure 1, The Euler Method doesn't make the orbit of planet to be perfect circle especially the inner-most planet. The orbit is getting oval and then becoming unbound (It means that the energy is not conserved but increase over time) because this planet moving fastest. so if the system is changing too fast, The Explicit Euler Method will generate too much error.

Semi-Implicit Euler Method

Semi-Implicit Euler Method has nickname "Euler-Cromer Method". This Integrator updates velocity before position. It's samilar to Euler method except switching the first and second line.

$$v_{i+1} = v_i + \frac{F(x_i)}{m}\Delta t$$
$$x_{i+1} = x_i + v_{i+1}\Delta t$$

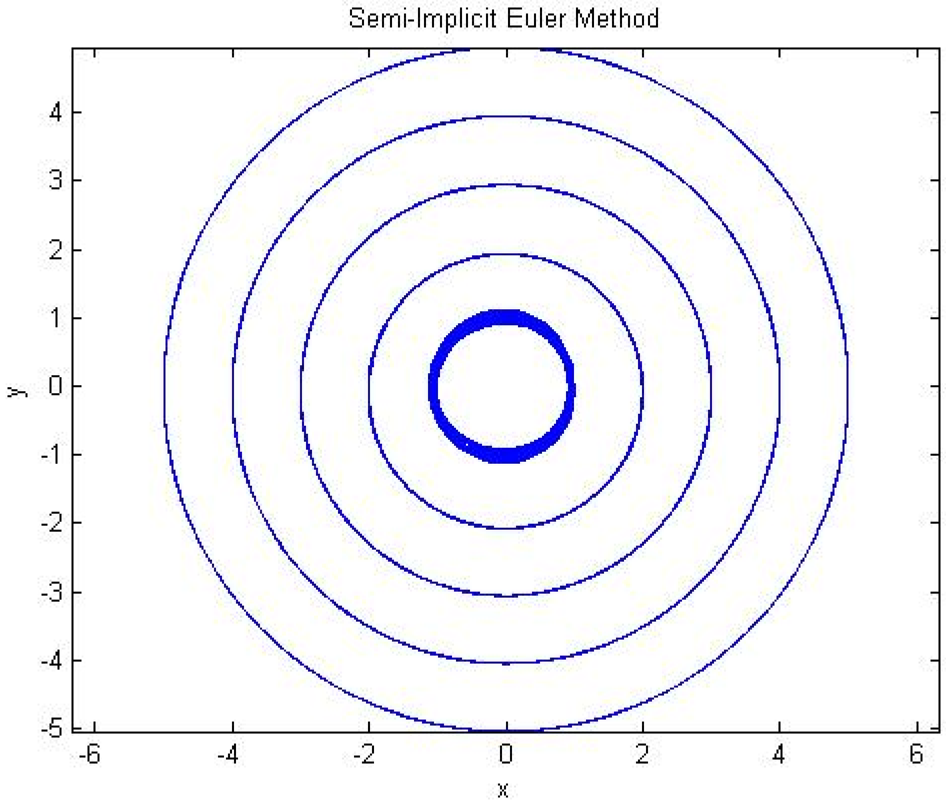

figure 2

This Method is better than Euler Method and the inner-most planet orbit is circle, but not perfect because of a little fluctuation in radius (and total energy may be fluctuate).

Verlet Integration

Verlet Method can update only position (or only velocity, but if update only velocity, the method is called "velocity verlet"). This method is inconvenient to use because it is not self-starting, need 2 initial conditions of position. so other integrator is needed to find initial condition.

$$x_{i+1} = 2x_i - x_{i-1} + \frac{F(x_i)}{m}\Delta t^2$$
$$v_i = \frac{x_{i+1} - x_{i-1}}{2\Delta t}$$

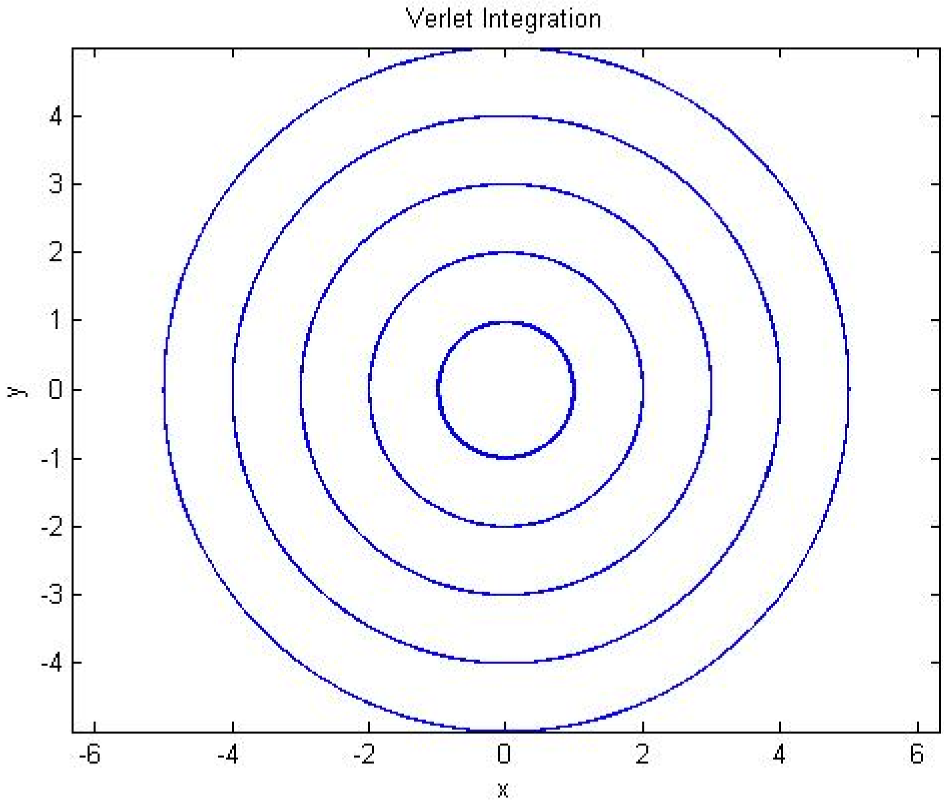

figure 3

Compare with the figure 1 and figure 2, this method is better than 2 integrators above. The inner-most planet may be has a little fluctuation in radius but less than Euler-Cromer Method.

Stormer–Verlet method

This method divides updating position into 2 times, before and after calculating force. the first updating, the position will be updated half time step by using previous step of velocity and the second updating, the position will be updated half time step by using present step of velocity.

$$x_{i+\frac{1}{2}} = x_i + \frac{1}{2} v_i \Delta t$$

$$v_{i+1} = vi + \frac{F(x_{i+\frac{1}{2}})}{m} \Delta t$$

$$x_{i+1} = x_{i+\frac{1}{2}} + \frac{1}{2} v_{i+1} \Delta t$$

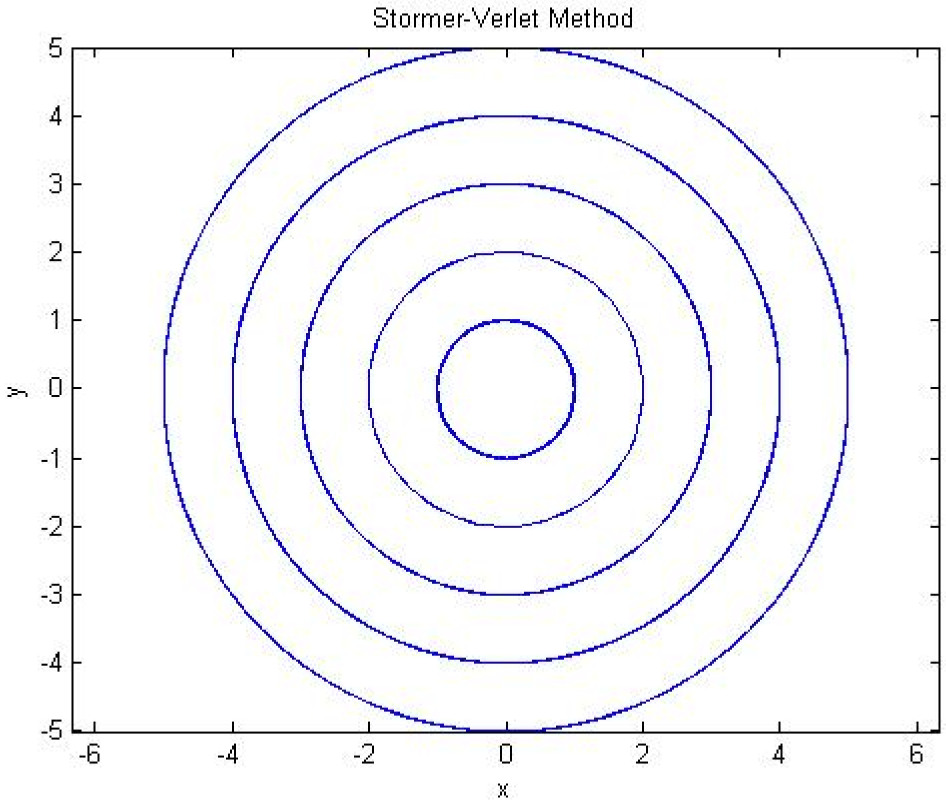

figure 4

The result of this method is resemble to verlet method. and fluctuation in radius is very small (may be smaller than verlet method). The orbit is almost perfect circle.

Oscillation of Orbital Radius

For the 4 integrators above, now I am comparing the planetary orbital radius

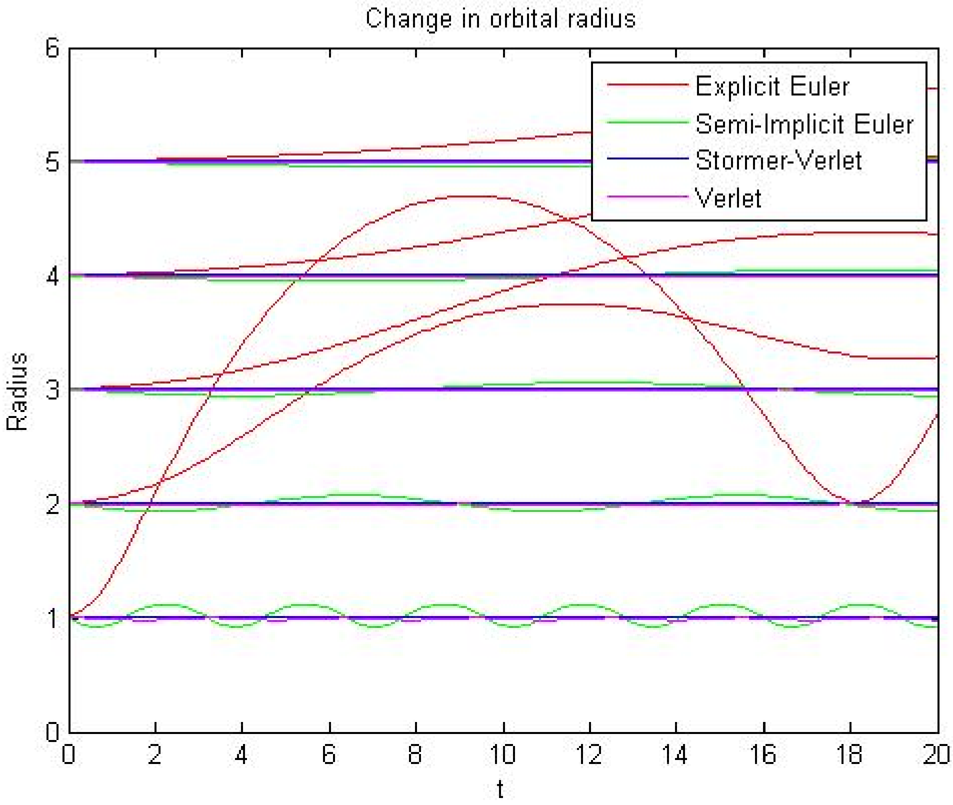

figure 5

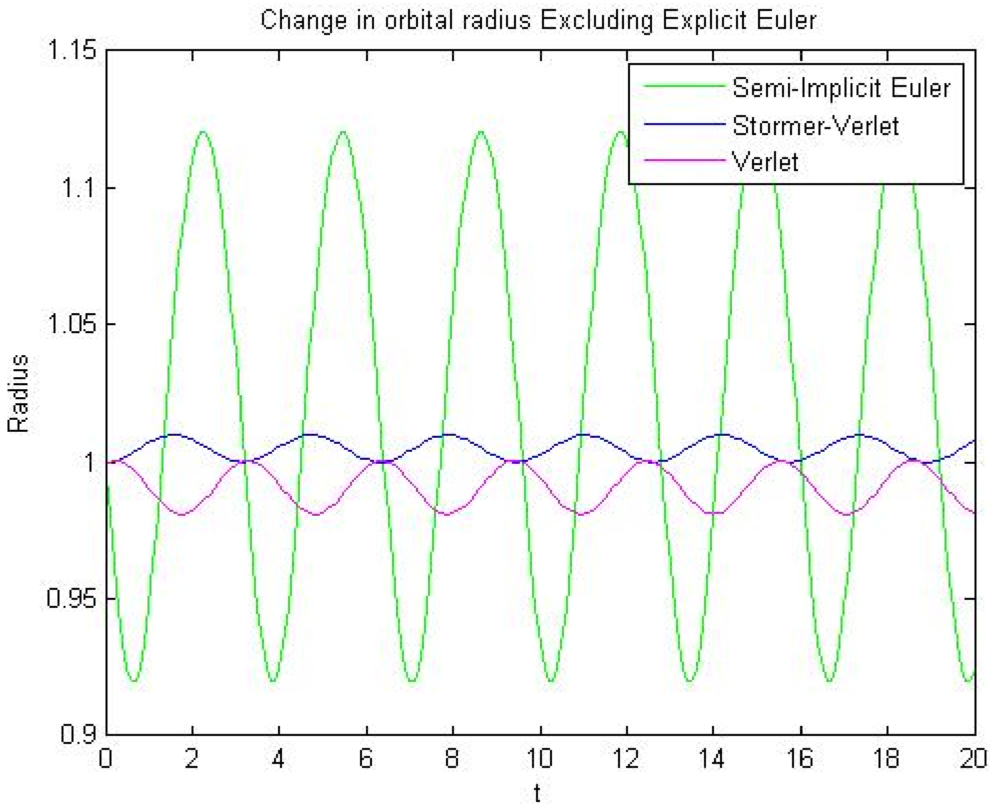

figure 6

In figure 5, the all of planetary orbit in Euler method has got greater, Euler-Cromer Method is oscillate between some value and the oscillation greater than verlet and Stormer–Verlet method, verlet and Stormer–Verlet method is not different in figure 5. So in figure 6 I'm removing the Euler method graph and plot only the orbital radius of inner-most planet.

In figure 6, We see that the orbital radius of verlet integration and Stormer-Verlet method is also oscillate between some value but the Stormer–Verlet method has got the smallest orbital radius oscillation.

Conservation of Energy

In figure 7, the energy from Euler method is not conserved, it is blowing up rapidly. The Euler-Cromer has an orbital radius oscillation in some value and greater than verlet integration and Stormer–Verlet method. The Verlet Integration and the Stormer–Verlet method are no difference in figure 7. So I'm also removing the Euler method.

In figure 8, we see that the oscillation in energy of Stormer-Verlet method is very small and it looks like a straight line.

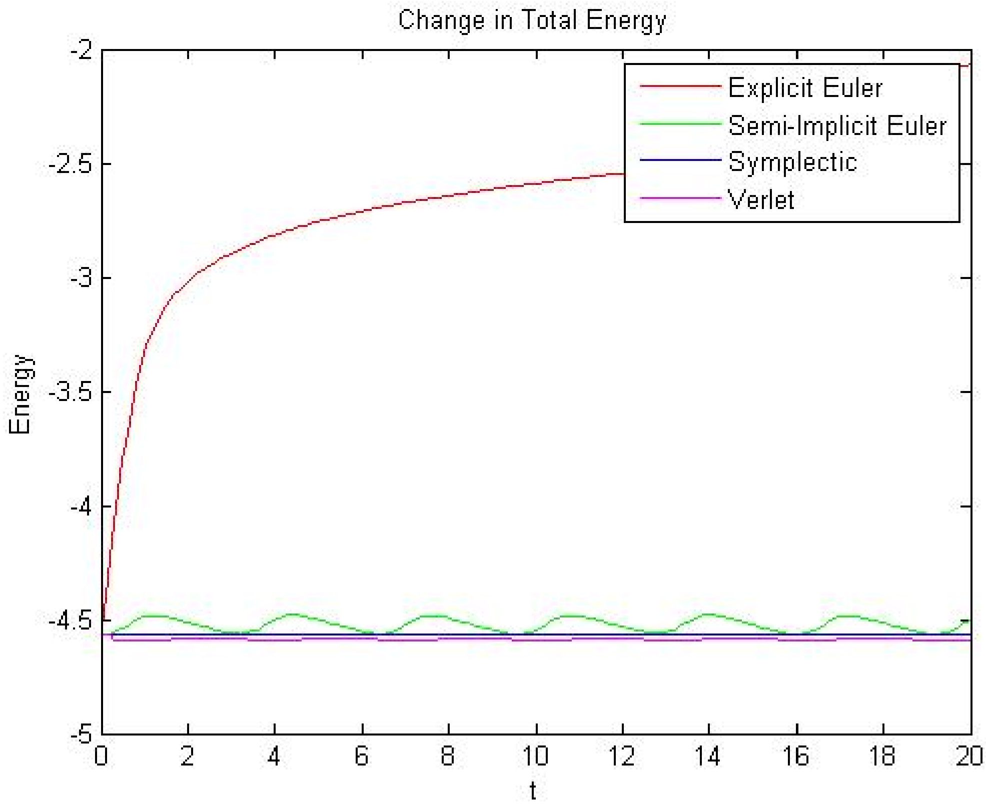

figure 7

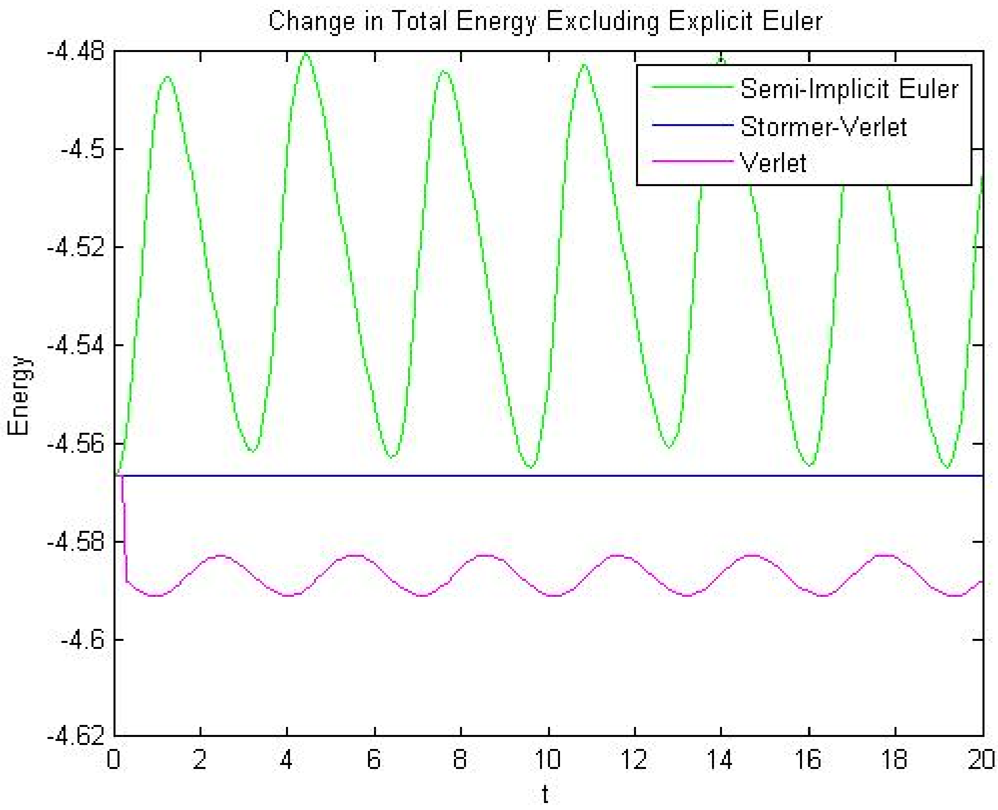

figure 8

Conclusion

The result shows that the best integrator here is **Stormer-Verlet method**. In the Future I will try to use an elliptic orbit to test integrator and see the changing of periapsis and apoapsis point.