Calculus 1
Section 6.4
Second Fundametal Theorem of Calculus

Set Up

You may remember

$$\int_a^b e^{-t^2} dt$$

this integrand has no antiderivative that is an elementary function. However, assuming F exists, we know from the Fundamental Theorem of Calculus (part 1)

$$F(b) - F(a) = \int_a^b e^{-t^2} dt.$$

Since this would be true for any a and b, let $a = 0$ and let $b = x$. Then we have

$$F(x) - F(0) = \int_0^x e^{-t^2} dt.$$

Now choose the antiderivative that satisfies $F(0) = 0$. Then we get

$$F(x) = \int_0^x e^{-t^2} dt$$

and this is a function that represents the antiderivative.

Construction of Antiderivative Using the Definite Integral

Theorem 6.2: Construction Theorem for Antiderivatives (Second Fundamental Theorem of Calculus)

If f is a continuous function on an interval, and if a is any number on that interval, then the function F defined on the interval as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)dt.$$

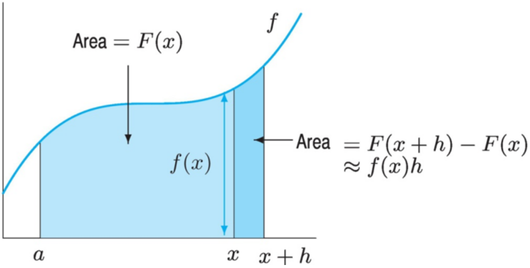
Proof of Theorem 6.2

Our task is to show that F, defined by this integral, is an antiderivative of f. That is we want to show $F'(x) = f(x)$. By the definition of the derivative:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

To gain some geometric insight, lets suppose f is positive and h is positive. Then we can visualize the following areas:

$F(x) = \int_a^x f(t)dt$ and $F(x+h) = \int_a^{x+h} f(t)dt$, so $F(x) - F(x+h) = \int_x^{x+h} f(t)dt$



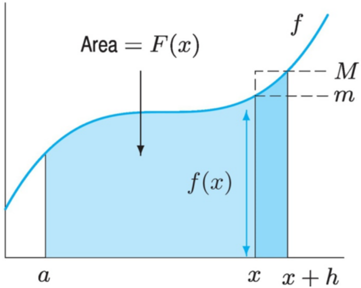
From this figure we see that $F(x+h) - F(x)$ is roughly the area of a rectangle of height $f(x)$ and width h, the darkly shaded area, so $F(x+h) - F(x) \approx f(x)h$ and thus $\dfrac{F(x+h) - F(x)}{h} \approx f(x)$.

Proof Continued

More precisely, we can use Theorem 5.4 on bounding integrals to conclude that

$$mh \leq \int_x^{x+h} f(t)dt \leq Mh$$

as shown in the figure below:



so

$$m \leq \frac{F(x+h) - F(x)}{h} \leq M$$

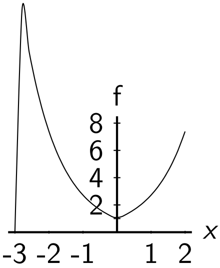
Since f is continuous both m and M approach the value of $f(x)$ as h approaches zero. Thus

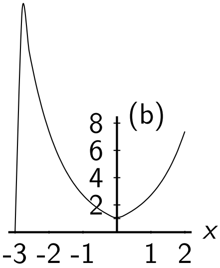
$$f(x) \leq \lim_{hto0} \frac{F(x+h) - F(x)}{h} \leq f(x)$$

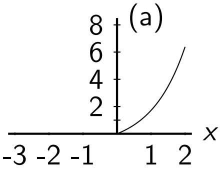
so $f(x) = F'(x)$

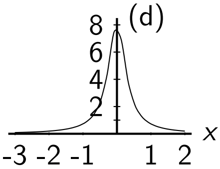
Clicker Question 1

The figure to the right is a graph of f . Which of the figures (a)–(d) is the graph of $\int_{-3}^{x} f(t)dt$ for $-3 < x < 2$?







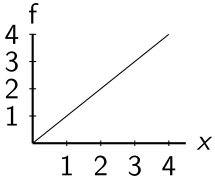


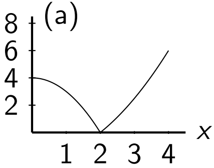
Solution

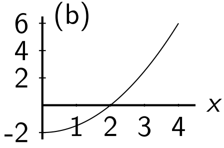
(b). Because $\int_{-3}^{-3} f(t)dt = 0$ the point $(-3, 0)$ is on the graph of $\int_{-3}^{x} f(t)dt$. Because the graph of f is positive and increasing, the graph of $\int_{-3}^{x} f(t)dt$ is increasing anc concave up.

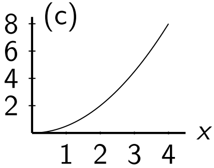
Clicker Question 2

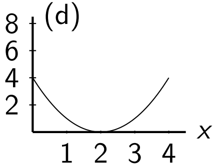
The figure to the right is a graph of f . Which of the figures (a)–(d) is the graph of $\int_2^x f(t)dt$ for $0 < x < 4$?











Solution

(b). Because $\int_2^2 f(t)dt = 0$ the point $(2, 0)$ is on the graph of $\int_2^x f(t)dt$.
Because $\int_2^0 f(t)dt = -\int_0^2 f(t)dt$ and because be the area under f from 0 to 2 is positive, the graph of $\int_2^0 f(t)dt$ is negative.

Using the Construction Theorem for Antiderivatives

Example

Construct a table of values for $Si(x)$ for $x = 0, 1, 2, 3$ where $Si(x) = \int_0^x \frac{\sin(t)}{t} dt$.

Using the Construction Theorem for Antiderivatives

Example

Construct a table of values for $Si(x)$ for $x = 0, 1, 2, 3$ where $Si(x) = \int_0^x \frac{\sin(t)}{t} dt$. Solution

Using numerical methods, we approximate the values of $Si(x)$ in the table below.

x	0	1	2	3
$Si(x)$	0	0.95	1.61	1.85

The reason the sine-integral has a name is that some scientist and engineers use it all the time (for example, in optics). For them, it is just another common function like sine and cosine. Its derivative is given by

$$\frac{d}{dx} Si(x) = \frac{\sin(x)}{x}.$$

Dirichlet proved that $\lim_{x\to\infty} Si(x) = \int_0^\infty \frac{\sin(t)}{t} dt = \frac{\pi}{2}$.

Some Examples

1. $$\frac{d}{dx}\left(\int_0^x \cos(t^2)dt\right) = \cos(x^2)$$
2. $$\frac{d}{dx}\left(\int_x^3 \ln(t^2)dt\right) = -\frac{d}{dx}\left(\int_3^x \ln(t^2)dt\right) = -\ln(x^2)$$
3. $$\frac{d}{dx}\left(Si(x^2)\right) = \frac{d}{dx}\left(\int_0^{x^2} \frac{\sin(t)}{t}dt\right) = \frac{\sin(x)}{x}(2x) = 2\sin(x)$$

A Harder Example

$$\frac{d}{dx}\left(\int_0^{x^2} \ln(t+1)dt\right) = ?$$

Solving this problem can be a little more challenging. Let

$$f(x) = \int_0^{x} \ln(t+1)dt$$

then we we are actually solving

$$\frac{d}{dx}f(x^2) = f'(x^2)2x = \ln(x^2+1)2x$$

More Examples

$$\frac{d}{dx}\left(\int_{1}^{\sin(x)} \cos(t^2)dt\right) = \cos(\sin^2(x))\cos(x)$$

$$\frac{d}{dx}\left(\int_{-x^2}^{x^2} e^t dt\right) = \frac{d}{dx}\left(\int_{-x^2}^{0} e^t dt\right) + \frac{d}{dx}\left(\int_{0}^{x^2} e^t dt\right)$$

$$= -\frac{d}{dx}\left(\int_{0}^{-x^2} e^t dt\right) + \frac{d}{dx}\left(\int_{0}^{x^2} e^t dt\right) = e^{-x^2}(2x) + e^{x^2}(2x)$$

Clicker Question 3

$$\frac{d}{dx}\left(\int_1^{\ln(x)} t^2\,dt\right)$$

(a) x^2

(b) $(\ln(x))^2$

(c) $\dfrac{\ln(x)}{x}$

(d) $\dfrac{(\ln(x))^2}{x}$

(e) $\dfrac{(\ln(t))^2}{t}$