AIP | Physics of Fluids

Singular front formation in a model for quasigeostrophic flow

Peter Constantin, Andrew J. Majda, and Esteban G. Tabak

Citation: Physics of Fluids **6**, 9 (1994); doi: 10.1063/1.868050
View online: http://dx.doi.org/10.1063/1.868050
View Table of Contents: http://scitation.aip.org/content/aip/journal/pof2/6/1?ver=pdfcov
Published by the AIP Publishing

Singular front formation in a model for quasigeostrophic flow

Peter Constantin
Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Andrew J. Majda
Department of Mathematics and Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544

Esteban G. Tabak
Program in Applied and Computational Mathematics, Princeton University, Princeton, New Jersey 08544

(Received 4 August 1993; accepted 17 September 1993)

A two-dimensional model for quasigeostrophic flow which exhibits an analogy with the three-dimensional incompressible Euler equations is considered. Numerical experiments show that this model develops sharp fronts without the need to explicitly incorporate any ageostrophic effect. Furthermore, these fronts appear to become singular in finite time. The numerical evidence for singular behavior survives the tests of rigorous mathematical criteria.

We study a two-dimensional (2-D) model for quasigeostrophic flow. Our motivation is twofold. On the one hand, we would like to know how much of the behavior of real flows is captured by this model; in particular, whether it predicts the formation of sharp fronts associated with boundaries between air masses in the atmosphere. On the other hand, we would like to exploit a formal analogy between the equations governing this model and those describing incompressible inviscid flows in three dimensions to gain insight into the latter. We will concern ourselves with the existence of finite time singularities, which are associated in the quasigeostrophic case with the formation of fronts.

The equations under study are

$$\frac{D\theta}{Dt}=\frac{\partial\theta}{\partial t}+v\cdot\nabla\theta=0,$$

where

$$(v_1,v_2)=\left(-\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right) \tag{1}$$

and

$$(-\Delta)^{1/2}\psi=-\theta\ \ (|k|\hat{\psi}(k)=-\hat{\theta}(k)).$$

The variable θ represents the potential temperature, v the flow velocity and ψ the streamfunction, which can be identified with the pressure. These equations are derived[1] for the evolution of temperature on the 2-D boundary of a half-space with small Rossby and Ekman numbers and constant potential vorticity. They have been investigated in various contexts.[2,3]

For simplicity, we will take θ, v and ψ to be spatially periodic with period 2π and zero average.

We can differentiate (1) and write the resulting equations in the form

$$\frac{D\nabla\theta^{\perp}}{Dt}=\nabla v\cdot\nabla\theta^{\perp},$$

where

$$\nabla\theta^{\perp}=(-\theta_y,\theta_x) \tag{2}$$

and

$$\widehat{\nabla v}(k)=-i\frac{k}{|k|}\otimes\widehat{\nabla\theta^{\perp}}(k),$$

where "hat" denotes Fourier transform. By identifying the $\nabla\theta^{\perp}$ from (2) with the vorticity vector ω, we note the analogy between (2) and the vorticity formulation of the three-dimensional (3-D) Euler equations

$$\frac{D\omega}{Dt}=\omega\cdot\nabla v,$$

where

$$\widehat{\nabla v}(k)=i\frac{(k\times\hat{\omega}(k))\otimes k}{|k|^2}. \tag{3}$$

The quantities $\nabla\theta^{\perp}$ and ω in (2) and (3) amplify in the same fashion through a quadratic nonlinear interaction with ∇v. In both cases, the "rate of strain" ∇v is related to the strained variable, respectively $\nabla\theta^{\perp}$ and ω, by a nondimensional singular integral operator, i.e. with a Fourier symbol homogeneous of degree zero. Whether the 3-D Euler equations develop finite time singularities from smooth initial data is an important and highly controversial open problem.[4-6] We may ask the same question for the equations in (2). These equations should be thought of in this context as a 2-D member of a hierarchy of problems that has the one-dimensional (1-D) member proposed in Ref. 7, for which the development of finite time singularities from general initial data has been established.

In this Letter, we present numerical evidence that the equations in (2) do develop finite time singularities along very elongated structures, corresponding to fronts in the geophysical context. The formation of fronts is very fast and seems to occur for a wide range of initial conditions. An analytic treatment of these singularities and their consequences is the subject of current work.[8]

The numerical determination of the occurrence of singularities is a difficult task, since, close to the singular time, truncation errors become of the same order of the solution, no matter how fine a computational grid we choose. We would like, therefore, to have at least some rigorous criteria to make the detection of a false singularity less likely.

Two such mathematical criteria, similar to those found in Ref. 9 for the 3-D Euler equations, can be established rigorously for Eq. (1). These criteria state that, for the occurrence of any singularity in θ at time T_*, the maxima of both $\nabla\theta^{\perp}$ and the stretching rate α must blow up at time T_* in specific ways:

$$\lim_{t\to T_*}\int_0^t \max(|\nabla\theta^{\perp}|)\,dt=\infty\ ,\quad \lim_{t\to T_*}\int_0^t \max(\alpha)\,dt=\infty. \tag{4}$$

Here α is defined by

$$\alpha=\xi\cdot(\nabla v\cdot\xi)\ ,\ \text{where}\ \xi=\frac{\nabla\theta^{\perp}}{|\nabla\theta^{\perp}|}. \tag{5}$$

These conditions are necessary and sufficient for singularity formation. Of course, $|\nabla\theta^{\perp}|$ is the length element of the corresponding contour line.

A third rigorous constraint, more geometric in nature, can be stated as follows. If the contour line of θ which will pass through the singularity does not become wildly oscillatory, its curvature must tend to zero as t approaches T_*.[8]

We solved (1) numerically with a spectral collocation method, computing $v(\theta)$ in Fourier space and the product $v\cdot\nabla\theta$ in physical space, with an exponential filter of high frequencies.[10] As time advancing routine, we used a fourth-order Runge–Kutta method.[10]

Eigenfunctions of the Laplacian define steady exact solutions of (1). We need therefore to combine modes with wave vectors with two different lengths in order to get unsteady solutions. In the following example, we have chosen the simplest initial condition

$$\theta(x,y,0)=\cos(y)+\sin(x)\sin(y). \tag{6}$$

The contour lines of θ at times 0, 2, 4, 6 and 7 are plotted in Figs. 1 and 2 (we have used progressively refined grids from 128×128 to 256×256 to 512×512). We see the fast formation of two very elongated fronts, across which θ is creating a cusp. At the center of these fronts, ψ develops a slightly distorted saddle point. Such saddle points in the wind, denoted "deformation fields," have long been associated with frontogenesis.[1] These results show the development of sharp fronts completely within the quasi-geostrophic approximation. These candidate singular fronts also satisfy the geometric mathematical criterion mentioned earlier.

In Fig. 3, we show a log–log plot of the maximum value of $|\nabla\theta^{\perp}|$ versus time for grids with 256×256 and 512×512 points. By predicting the singular time at about $t=8.25$, we obtain an almost perfect fit with a straight line with slope $-7/4$, indicating a behavior of the form

$$\max|\nabla\theta^{\perp}|\approx\frac{c}{(8.25-t)^{7/4}}.$$

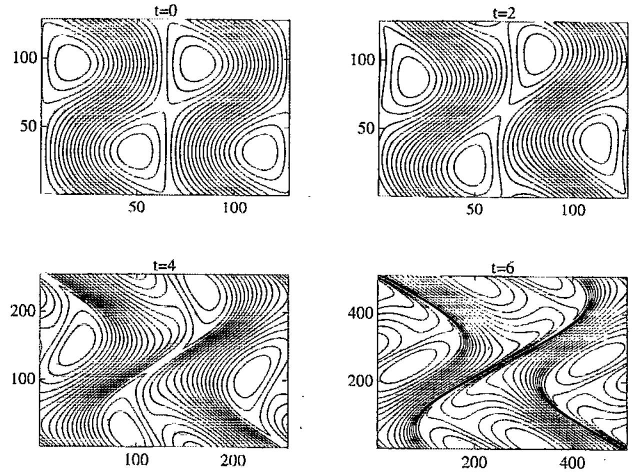

FIG. 1. Contour lines of θ exhibiting front formation, with initial data $\theta_0=\cos(y)+\sin(x)\sin(y)$. The resolution varies from 128×128 at the times $t=0$ and $t=2$, to 256×256 at $t=4$, to 512×512 at $t=6$.

The times at which the curve for each grid departs from the $t^{-7/4}$ behavior, roughly 5.5 and 7, correspond to those times at which the grids stop resolving the cusp in ω. This can be checked visually, for instance, from wiggles arising in the contour lines of θ and, more accurately, from the spectrum of $\nabla\theta^{\perp}$, whose energy at the "cutoff" frequency $N/2$ becomes abruptly significant precisely at those times. We have also checked a coarser grid with 128×128 points, with a "nonresolving" time, and corresponding departure from the straight line of Fig. 3, at about $t=4.5$.

In Fig. 4, we show a log–log plot of α, computed at the location of the maximum for $|\nabla\theta^{\perp}|$, versus $(8.25-t)$ for the same two grids. The -1 slope corresponds to the boundary of nonintegrability and to algebraic blowup of $\nabla\theta^{\perp}$. The staircase-like look of the plot is easy to understand. When the maximum $\nabla\theta^{\perp}$ jumps from one grid point to the next one, α experiences a sudden growth, followed

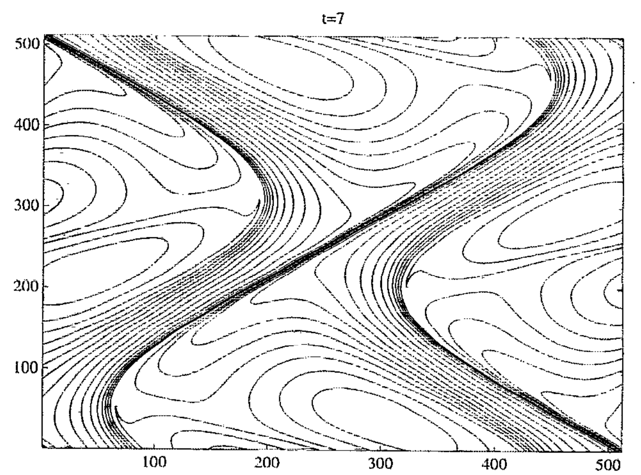

FIG. 2. Contour lines of θ at time $t=7$, just before the grid's resolution (512×512) fails. The corresponding streamfunction Ψ has a saddle point at the origin.

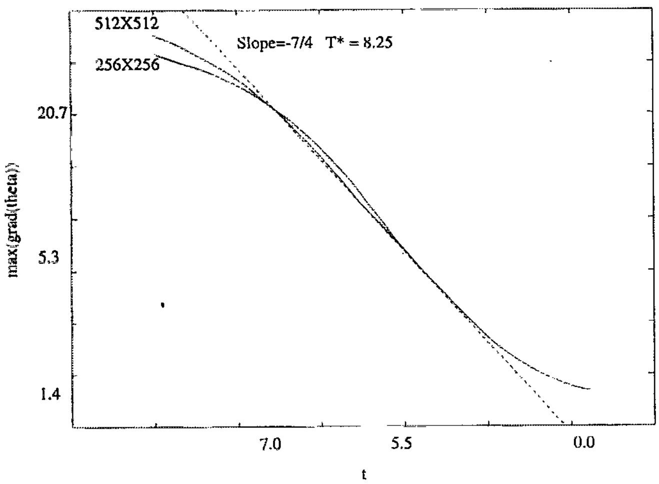

FIG. 3. Log–log plot of max($|\nabla\theta^{\perp}|$) versus $(8.25-t)$ for two grids and the fit with a straight line. The times marked are those at which each grid stops resolving the singularity.

by decay as the real maximum travels to the following node. The physical reason is that the strain is concentrated along a very thin neighborhood of the front and decays very fast away from it. Therefore, when we compute α at a point slightly farther away from the front than the real maximum of $\nabla\theta^{\perp}$, its value may decrease significantly. This mechanism also explains why the curves depart from

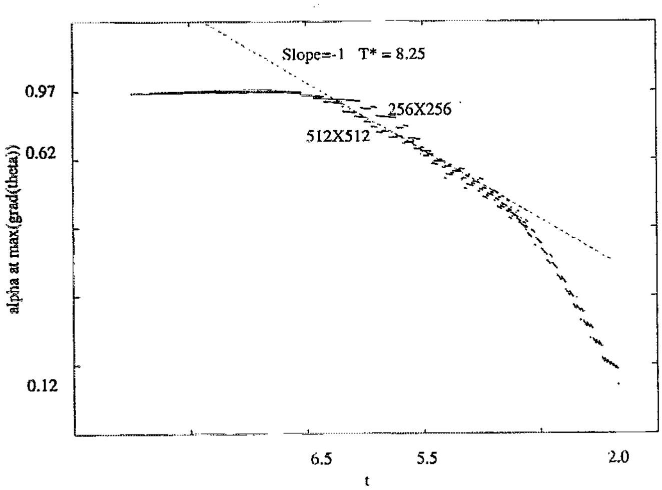

FIG. 4. Log–log plot of α at the location of maximum $|\nabla\theta^{\perp}|$ for two grids and the fit with the straight line $\alpha = 7/4 \lessdot 8.25 - t$.

a straight line slightly earlier than those of Fig. 3, and approach a constant basically independent of the grid size, the "far field" value of α. Still the fit with a $1/(T_{*}-t)$ growth, particularly for the finest grid, is remarkable.

We would like to emphasize that, although for clarity we have shown the results corresponding to only one initial condition, singular front formation occurs for a wide range of initial data.[8] Therefore frontogenesis appears to be a phenomenon robustly associated with Eq. (1).

We have presented numerical evidence that a simple quasigeostrophic model for thermal winds exhibits frontogenesis without explicitly including any ageostrophic effect. In addition to the geophysical significance of these fronts, the fact that they appear to become singular in a finite time suggests the possibility that integrodifferential equations of the type of the 3-D Euler equations might be associated with singular behavior. A recent interesting candidate singularity for 3-D Euler was provided in Ref. 6. The existence of such singularities would have important consequences for our understanding of turbulence.

ACKNOWLEDGMENTS

The authors thank Isaac Held and Ray Pierrehumbert for interesting conversations on the geophysical significance of the equations in (1), which inspired this investigation. A. Majda and E. Tabak are partially supported by research grants from NSF, ONR and DARPA, and P. Constantin is supported by NSF and DOE.

[1]J. Pedlosky, *Geophysical Fluid Dynamics* (Springer-Verlag, New York, 1987), pp. 345–368 and 653–670.
[2]R. T. Pierrehumbert, I. M. Held, and K. L. Swanson, "Spectra of local and nonlocal two dimensional turbulence," Preprint, December 1992.
[3]P. Constantin, "Regularity results for incompressible fluids," Proceedings of the Workshop on The Earth Climate as a Dynamical System, Argonne National Laboratory, 25–26 September, 1992, ANL/MCS-TM-170. Also "Geometric statistics in turbulence" submitted to SIAM Rev., May 1993.
[4]A. Chorin, "The evolution of a turbulent vortex," Commun. Math. Phys. **83**, 517 (1982).
[5]A. Majda, "Vorticity, turbulence and acoustics in fluid flow," SIAM Rev. **33**, 349 (1991).
[6]R. Kerr, "Evidence for a singularity of the three-dimensional, incompressible Euler equations," Phys. Fluids A **5**, 1725 (1993).
[7]P. Constantin, P. D. Lax, and A. Majda, "A simple one-dimensional model for the three-dimensional vorticity equation," Commun. Pure Appl. Math., **38**, 715 (1985).
[8]P. Constantin, A. Majda, and E. G. Tabak, "Singular front formation in a two-dimensional model for quasigeostrophic flow," in preparation.
[9]T. Beale, T. Kato, and A. Majda, "Remarks on breakdown of smooth solutions for the three-dimensional Euler equations," Commun. Math. Phys. **94**, 61 (1984).
[10]W. E and C. Shu, "Effective equations and inverse cascade theory for Kolmogorov flows," Phys. Fluids A **5**, 998 (1993).