The Connection Between the Navier-Stokes Equations, Dynamical Systems, and Turbulence Theory

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INTRODUCTION

A basic tenet of theoretical physics is that Nature is the complicated display of the repetitive interplay of some (mathematically) simple laws. Much human energy, ingenuity and glamour was devoted to the search and the explication of such laws, usually describing physical reality off the reach of our direct experience. Satisfactory theories for these "simple" and "extreme" phenomena seem to have been attained several times in this century. However, numerous physical phenomena occuring at the scale of our daily experience and hourly impinging on our lives remain still beyond our present theoretical understanding. Among these phenomena the most conspicuous are those displaying turbulence.

What is turbulence? Loosely speaking, turbulence is an irreversible disorder, statistically "organized" according to several time and space scales, of mostly fast moving fluid flows. There are three major distinct but not mutually contradictory theoretical views on turbulence, namely:

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a) The statistical view, in which the turbulence is considered as the observed behaviour of the evolution of statistical distributions of flows instead of the evolution of one individual flow.

b) The viewpoint of regularity breakdown, in which the turbulence is considered to result from the blow-up of the vorticity in finite time, albeit necessarily on a set of small Hausdorff dimension. This might express the fact that the continuum mechanics model could become invalid at least intermittently in time.

c) The dynamical systems view, in which the turbulence is a phenomenologic perception of the long time complicated behaviour of the individual flows.

Among the most influential proponents of these views were: O. Reynolds [1], G. I. Taylor [2], T. von Kármán [3], and A. N. Kolmogorov [4] for a); J. Leray [5] and B. Mandelbrot [6] for b); and L. L. Landau [7], E. Hopf [8] and D. Ruelle-R. Takens [9] for c). Although a) seems to be the view of most aeronautical engineers, mathematicians and theoretical physicists seem to hold the other views (especially c)). There are two reasons for this fact. First, b) and c) have inspired interesting mathematical research ([10], [11], [12], [13], [14], [15], [16],...) and some illuminating mathematical metaphors for the onset of turbulence under very stringent physical conditions ([17], [18], [19], [20], [21],...). Notice however that the phenomenon of turbulence cannot be equated with that of the onset of turbulence.

It is premature to exclusively adopt one of the views a), b), or c), but it is reasonable to test them by trying to establish rigorous mathematical facts based on the Navier-Stokes equations, the interpretation of which are consistent with any of a), b), and c). In this paper we shall present our contribution to this program from the dynamical systems point of view c). Moreover we prove in Section 3 a regularity and backward uniqueness theorem on an open dense subset of the universal attractor of the 3D Navier-Stokes equations. As far as we know this theorem is new and is the first of its kind.

In dedicating this paper to the sixtieth anniversary of Professor John Nohel, the authors express to him their warm and high appreciation.

2. THE NAVIER-STOKES EQUATIONS AND THEIR ABSTRACT FRAMEWORK

In this Section we shall recall some basic facts on the Navier-Stokes equations which concern a larger class of related dissipative partial differential equations (see for instance [22], [23]). First we consider the Navier-Stokes equations.

$$\begin{aligned} &u_t + (u\cdot\nabla)u = \nu\Delta u - \nabla p + f \\ &\nabla\cdot u = 0 \end{aligned} \qquad \text{in} \quad \Omega\times(0,\infty) \subset \mathbb{R}^{n+1} \qquad (2.1)$$

where Ω is an open connected bounded set $\subset \mathbb{R}^n$ $(n = 2 \text{ or } 3)$ such that its boundary Γ is a compact manifold of class $\mathscr{C}^2$ of dimension $n - 1$, $\Gamma = \partial\overline{\Omega}$ or $\Omega = \prod_{j=1}^{n} [a_j, b_j]$. In the first case (2.1) is supplemented with homogeneous boundary conditions

$$u|_\Gamma = 0 \qquad (2.2)$$

while in the second case it is supplemented with the periodic and zero mean conditions

$$u|_{x_j=a_j} = u|_{x_j=b_j} \quad (1 \le j \le n) \quad , \quad \int_\Omega u = 0 \quad . \tag{2.2}$$

In both cases let V be the space of the divergence free $u \in H^1(\Omega)^n$ satisfying the respective boundary conditions, let H be the closure of V in $L^2(\Omega)^n$ and let P denote the orthogonal projection of $L^2(\Omega)^n$ on H . Introducing the operators

$$Au = -P\Delta u \ , \ B(u,v) = P[(u\cdot\nabla)v] \ , \ u,v \in V\cap H^2(\Omega)^n \ , \tag{2.3}$$

the equations (2.1)-(2.2), (2.2)$'$ become a particular case of the abstract dissipative evolution equations of the type

$$\frac{du}{dt} + \nu Au + R(u) = 0 \tag{2.4}$$

$$R(u) = B(u,u) + C(u) - f \quad , \tag{2.5}$$

where $A : \mathfrak{D}_A \to H$ is a self-adjoint operator, with compact inverse, in a Hilbert space H , B and C are bilinear, resp. linear, operators from $\mathfrak{D}_A$ into H , $\nu > 0$, and $f \in H$ are fixed. The operators B and C satisfy usually conditions of the form

$$\begin{aligned} |B(u,v)| &\le c_1|A^\alpha u||A^\beta v| && u,v \in \mathfrak{D}_A \\ |C(u)| &\le c_2|A^\gamma u| && u \in \mathfrak{D}_A \end{aligned} \tag{2.6}$$

where $|\cdot|$ denotes the norm in H , α , β , $\gamma \in [0,1)$ and $\alpha + \beta$ is usually fixed. For instance, for the equations (2.1)-(2.2), (2.2)$'$, $\alpha + \beta = 1/2 + n/4$ and $C = 0$. (Here and in the sequel $c_1, c_2, \ldots$ will denote adequate absolute constants.)

Many other equations more or less related to the Navier-Stokes equations can be easily given under appropriate boundary conditions the form (2.4). We mention the Boussinesq equations, the Kuramoto-Sivashinsky equations, the (local or nonlocal) Burgers equations, etc. We only notice that the special form (2.5) of $R(u)$ is not a necessary feature of (2.4). Essentially we need that

$$\left|\frac{\partial R(u)}{\partial u} v\right| \leq c_1' \, |A^{\alpha}u||A^{\beta}v| \qquad (u,v \in \mathcal{D}_A) \quad , \tag{2.6'}$$

where $\partial R(u)/\partial u$ is a Gateaux type derivative and α , β are as in (2.6). In this way the Kahn-Hilliard equations and some diffusion-reaction equations can also be given a similar although a bit more complicated form. However, we shall consider here only the case (2.4), (2.5), (2.6), supplemented with the condition

$$(B(u,v),v) = 0 \quad , \quad u,v \in \mathcal{D}_A \tag{2.7}$$

which is verified by (2.1)-(2.2), (2.2)$'$; moreover in the case $n = 2$ and (2.1)-(2.2)$'$, $B(u,u)$ also verifies

$$(B(u,u),Au) = 0 \qquad u \in \mathcal{D}_A \ . \tag{2.7'}$$

(The relations (2.7) and (2.7)$'$ reflect the energy, resp. enstrophy, conserving property of the inertial forces in (2.1).)

Finally let us recall that the initial value problem for (2.4) has always a (weak) solution on $[0,\infty)$. By a (weak) solution on a time interval σ we mean a function $u(\cdot) \in C(\sigma;H_{weak})$ satisfying in V' the integral form of (2.4), namely

$$u(t) = u(t_o) + \int_{t_o}^{t} \left[\nu Au(\tau) + R(u(\tau))\right] d\tau \quad (t_o < t\ ,\ t_o,\ t \in \sigma),$$

and satisfying the energy inequality

$$\frac{1}{2}\,|u(t')|^2 + \nu\int_{t}^{t'} \|u(t)\|^2 dt \le \frac{1}{2}\,|u(t)|^2 + \int_{t}^{t'} (f,u(t))\, dt \quad (2.8)$$

for all $t' \in \sigma$, $t' \ge t$ and t a. e. in σ , and such that

$$u \in L^{\infty}(\sigma;H) \cap L^2_{loc}(\sigma;V) \quad . \quad (2.9)$$

For every weak solution in σ there is an open set σ_u of total measure in σ such that u is regular on the intervals of σ_u . The (weak) solution is called regular on σ if moreover $u \in C(\sigma;V)$. This solution is then an analytic $\mathcal{D}_A$ -valued function (see for instance [14]).

In all the mentioned cases except those involving the 3D Navier-Stokes, u is uniquely determined by u_o and $\sigma_u = (0,\infty)$. In this case as well as in the case when the value $u(t)$ is uniquely determined by u_o we shall also denote it by $S(t,u_o)$. (Thus $S(0,u_o) = u_o$.) The regularity break-down viewpoint b) is based on the assumption that for the 3D Navier-Stokes equations there exist weak solutions on $(0,\infty)$ which are not regular.

3. THE UNIVERSAL ATTRACTOR

Let X denote the set of those $u_o \in H$ for which there exists a weak solution on $\sigma = (-\infty,\infty)$ of (2.4), H-bounded on $(-\infty,\infty)$, such that $u(0) = u_o$; $X \neq \phi$, since (2.4) has stationary solutions. From (2.7), (2.9) it follows readily

$$X \subset \left\{ u \in H : |u| \leq \frac{|A^{-1/2} f|}{\nu \lambda_1^{1/2}} \right\} . \qquad (3.1)$$

The definition obviously implies that if $u_o \in X$ there exists at least one solution $u(t)$ such that $u(0) = u_o$ and $u(t) \in X$ for all t ; thus in particular

$$S(t,u) \in X \quad \text{whenever} \quad u \in X , \quad -\infty < t < \infty \qquad (3.2)$$

and $S(t,u)$ makes sense. Moreover for every weak solution $u = u(\cdot)$ of (2.4) in $(0 , \infty)$, we have

$$u(t) \to X \quad \text{in} \quad H_{weak} \quad \text{for} \quad t \to \infty \qquad (3.3)$$

(i.e. every weak neighbourhood of X will eventually contain $u(t)$). This can be shown by a straightforward argument based on the following

<u>Lemma.</u> Let $\{u_j(\cdot)\}$ be a sequence of weak solutions on $\sigma = (a,b)$ bounded in H. Then there exists a subsequence $\{u_{j'}(\cdot)\}$ and a weak solution $u(\cdot)$ on σ such that $u_{j'}(t) \to u(t)$ in H_{weak} uniformly on any compact sub-

interval $\subset \mathcal{O}$. Moreover if $u(\cdot)$ is regular on $\mathcal{O}_1 = (a_1,b_1)$ $(\subset \mathcal{O})$ then $u_{j'}(t) \to u(t)$ in V , uniformly on any compact interval $\subset \mathcal{O}_1$.

For a proof we refer to [24], Ch. 1.

This Lemma also yields easily

$$X \text{ is compact in } H_{weak} . \tag{3.4}$$

Because of all these simple properties (3.1-4) we call X the <u>universal</u> <u>attractor</u> of the equation (2.4).

A first basic purpose of the dynamical systems approach should be the understanding of the action of $S(\cdot,\cdot)$ on $(-\infty,\infty) \times X$ in this general framework.

In the 2D case (i.e. $n = 2$), the universal attractor satisfies

$$X \text{ is bounded in } V \text{ and } S(t,X) = X \quad , \quad \forall t \quad . \tag{3.5}$$

Moreover, whenever (3.5) holds, X is included in $\mathcal{D}_A$ and is of finite dimension, i.e. homeomorphic to a compact subset of some $\mathbb{R}^N$ [13], [25], [24]. These properties are necessary if the dynamical systems approach c) is mathematically consistent with the Navier-Stokes equations. Therefore the following open question is basic for both approaches b) and c).

<u>Question 1.</u> Is the universal attractor X included in V even in the 3D case (i.e. $n = 3$)?

It is noteworthy that

$$X \subset V \iff (3.5) \quad . \tag{3.6}$$

The only nontrivial part of (3.6) is

$$X \subset V \implies X \text{ is bounded in } V . \tag{3.7}$$

<u>Proof.</u> If $X \subset V$ but is unbounded in V then there exists $\{u_j\}_{j=0}^{\infty} \subset X$, $u_j \to u_o$ in H_{weak} and $\|u_j\| \to \infty$. By (3.6), there exists $\{v_j\}_{j=0}^{\infty} \subset X$ such that $S(1,v_j) = u_j$, $\forall j \geq 0$. Using the Lemma above, we can also assume that $S(t,v_j) \to u(t)$ in H_{weak} for all $t \in (0,2)$. In particular $u(1) = u_o$ and since $u(t) \in X \subset V$ for all $t \in (0,2)$ we have also, by using once again the Lemma, that $u_j = S(1,v_j) \to u_o$ in V, contradicting the choice of $\{u_j\}_{j=0}^{\infty}$. Thus X must be bounded in V.

It is easy to see that $X \cap V$ is always dense in X (with the topology of H_{weak}). This is not surprising since for any ball $K \subset H$, $K \cap V$ is dense and also meager in K. Therefore the next result is a bit surprising and yields an almost positive answer to Question 1.

<u>Theorem 1.</u> $X \cap V$ contains an open dense subset X_{reg} of X (with the topology of H_{weak}). Moreover for any $u_o \in X_{reg}$ there exists an interval $\sigma_{u_o} = (-a,a)$ such that for any weak solution $u(\cdot)$ satisfying $u(t) \in X$, $\forall t$, we have that $u(\cdot)|\sigma_{u_o}$ is uniquely determined and regular.

<u>Sketch of the Proof.</u> Let X_{reg} be the set of the $u_o \in X$ satisfying the properties given in the statement. We must show that X_{reg} is open and dense in X endowed with the weak topology of H. Using the time analyticity of the regular solutions it is easy to show that if $u_o^o \notin X_{reg}$ then there exists $\{u_j^o\}_{j=1}^{\infty} \subset X$ such that $u_j^o \to u_o^o$ in H_{weak} and

$\|u_j^o\| \to \infty$ (for $j \to \infty$). Obviously this property is also valid for u_o^o in the closure of $X \setminus X_{reg}$. If moreover $u_o^o \in X_{reg}$, and $\{u_j^o\}_{j=1}^{\infty}$ is as above we can, by virtue of the above Lemma, assume that there exist X-valued weak solutions $u_j(\cdot)$ on $(-\infty,\infty)$, $u_j(0) = u_j^o$, $j = 0,1,\ldots$ such that $u_j(t) \to u_o(t)$ in H_{weak} for all $t \in (-\infty,\infty)$. But on the interval $(-a^o,a^o)$ associated to u_o^o described in the statement, $u_o(\cdot)$ is regular. Therefore the Lemma implies that $u_j^o = u_j(0) \to u_o(0) = u_o^o$ in V, contradicting $\|u_j^o\| \to \infty$. Thus $u_o^o \notin X_{reg}$ and $X \setminus X_{reg}$ is closed in X; hence X_{reg} is open in X.

The density of X_{reg} in X follows easily from the following fact:

Let $u(\cdot)$ be a weak solution on $(-\infty,\infty)$, $u(t) \in X$, $\forall t$, regular on (a,b). Then $u(c) \in \overline{X}_{reg}$ for any $c \in (a,b)$.

Assuming that $u(c) \notin \overline{X}_{reg}$ for some $c \in (a,b)$, we notice first that there exist $d \in (a,c)$ such that for any X-valued weak solution $v(\cdot)$ on $(-\infty,\infty)$, $v(c) = u(c)$, we have $v(t) \notin \overline{X}_{reg}$, $\forall t \in (d,c]$. (This is a consequence of the equicontinuity of all weak solutions on $(-\infty,\infty)$ with values in X, when viewed as V'-valued.) Therefore if $c_o = c > c_1 > c_2 > \ldots > c_k > c_{k+1} = d$ there must exist a weak solution $v(\cdot)$ on $(-\infty,\infty)$ such that $v(c) = u(c)$ and which is not regular on any interval (c_{j+1},c_j). It follows that the length of any interval of regularity of $v(\cdot)$ in (d,c) is $\leq 2\delta$ where $\delta = \max_{0\leq j\leq k} (c_j - c_{j+1})$. It is well-known that if $\ell_1,\ell_2,\ldots$ are the length of these intervals then

$\Sigma\sqrt{\ell_j} \le C$ where the constant C is independent of the X-valued weak solution (see for instance [25]). It follows

$$C\sqrt{2\delta} \ge \Sigma\,\ell_j \ge c - d - 2\delta$$

Letting $\delta \to 0$ we obtain a contradiction.

This finishes the proof.

4. ESTIMATES OF THE FRACTAL DIMENSION

Let Y be any compact subset of H. The fractal dimension (capacity) $d_M(Y)$ of Y is defined by

$$d_M(Y) = \overline{\lim_{\varepsilon\to 0}} \frac{\log N_\varepsilon(Y)}{\log\frac{1}{\varepsilon}}$$

where $N_\varepsilon(Y)$ is the smallest number of balls of radii $\le \varepsilon$ covering Y. This dimension, much emphasized by B. Mandelbrot [6] is larger than the classic Hausdorff dimension $d_H(Y)$. It can be ∞ even if the latter one is 0 [26]. Therefore $d_M(Y) < \infty$ is a finer condition of finite dimensionality than $d_H(Y) < \infty$. Moreover if X is the universal attractor of (2.4), $d_M(X)$ can be viewed as a lower bound for the number N of asymptotic degrees of freedom of the phenomena described by the equation (2.4). Indeed the number of real parameters necessary to describe for $t \to \infty$ the asymptotic behaviour of (2.4) must obviously exceed $d_M(X)$. (The reason why the number $N_H \sim 2d_H(X) + 1$ of parameters provided by R. Mañe's theorem (see [16]) may be insufficient is due to the fact that the homeomorphism from a compact set K in $\mathbb{R}^{N_H}$

to X may not be a Hölder function from K to X.)
Moreover a simple heuristic argument (due to L.L. Landau-E.M. Lifshitz [27], §32) yields the following estimate for the number N : Define the maximal mean dissipation of energy ϵ by

$$\begin{aligned}\epsilon &= \lim_{t\to\infty} \sup \frac{\nu}{t}\int_0^t \frac{1}{\mathrm{vol}\Omega}\|u(\tau)\|^2 d\tau \\ &= \sup \lim_{t\to\infty} \sup \frac{\nu}{t}\int_0^t \frac{1}{\mathrm{vol}\Omega}\int_\Omega |\nabla u(x,\tau)|^2 dx\, d\tau\end{aligned} \tag{4.1}$$

where the supremum is taken over all weak solutions $u(\cdot)$ on $(-\infty,\infty)$ such that $u(t) \in X$, $\forall t$. The Kolmogorov dissipation length ℓ_d is the only length of the form $\epsilon^\alpha \nu^\beta$, i.e. $\ell_d = (\nu^3/\epsilon)^{1/4}$. It is assumed that the turbulent eddies of linear size $< \ell_d$ are quickly killed by the viscosity of the fluid. Thus a grid with mesh $\sim \ell_d$ will be able to monitor the longer-lasting eddies. Therefore

$$N \sim c_3\left[\frac{\ell_o}{\ell_d}\right]^n \quad \text{where} \quad \ell_o \sim (\mathrm{vol}\Omega)^{1/n} , \tag{4.2}$$

and $n = 2,3$ is the dimension of Ω. The idea to connect the estimate (4.2) with that for $d_M(X)$ was first presented in [28] (see also [24] and [31]). It is a remarkable fact that for $n = 2$ one has

$$d_M(X) \le c_4\left[\frac{\ell_o}{\ell_d}\right]^2 .$$

However the following question is still open.

<u>Question 2.</u> Is the formula

$$d_M(X) \le c_5 \left[\frac{\ell_o}{\ell_d}\right]^3$$

true in the 3D case?

What is known now is the following: Assume that $X \subset V$ and define $\ell_d' = (\nu/\epsilon')^{1/4}$, with

$$\epsilon' = \sup \limsup_{t\to\infty} \nu \left[\frac{1}{t}\int_0^t \frac{1}{\mathrm{vol}\,\Omega}\int_\Omega |\nabla u(x,\tau)|^{5/2} dx\, d\tau\right]^{4/5} \tag{4.2$'$}$$

where the supremum is taken over the same family as in (4.1). Then

$$d_M(X) \le c_6 \left[\frac{\ell_o}{\ell_d'}\right]^3$$

(see [24], §4.5). The odd power in $(4.2)'$ is the best that can be obtained now using the methods of [26], [25], [31] and the Lieb-Thirring Sobolev-type inequality [32].

The physical meaning of the estimate of $d_M(X)$ leads to another open question. Indeed introducing the Reynolds number

$$Re = \frac{1}{\nu \ell_o} \sup \limsup_{t\to\infty} \left[\frac{1}{t}\int_0^t \frac{|u(\tau)|^2}{\mathrm{vol}\,\Omega} d\tau\right]^{1/2}$$

$$= \frac{1}{\nu \ell_o} \sup \limsup_{t\to\infty} \left[\frac{1}{t}\int_0^t \frac{1}{\mathrm{vol}\,\Omega}\int_\Omega |u(x,\tau)|^2 dx\, d\tau\right]^{1/2}$$

where the supremum is again taken over the same family as in (4.1), a classical argument in the conventional theory of turbulence yields that $\ell_o/\ell_d \sim Re^{3/4}$ and therefore $N \sim Re^{9/4}$ (see again [27], §32). Thus Question 2 has the following supplement

<u>Question 3.</u> Is the formula

$$d_M(X) \leq c_6 Re^{9/4} \tag{4.3}$$

true in the 3D case?

The best known result is the following ([24], §4.5): Assume $X \subset V$ and define

$$Re' = \frac{1}{\nu \ell_o} \sup \limsup_{t \to \infty} \left[\frac{1}{t} \int_o^t \frac{1}{\text{vol}\Omega} \int_\Omega |u(x,\tau)|^5 \, dx \, d\tau \right]^{1/5}$$

Then

$$d_M(X) \leq c_7 Re'^{3} \tag{4.4}$$

Every rigorous result which would bring (4.4) nearer to (4.3) will constitute an important contribution to the connection between the mathematical theory of fluid dynamics and the conventional theory of turbulence.

It is interesting that the estimates for the Hausdorff dimension of the attractors of dissipative partial differential systems, including the Navier-Stokes equations, do not improve upon ours, when expressed in the natural physical quantities introduced above (see for instance [29], [30] or

[15]). Therefore it would be interesting to prove that for the universal attractor X of the 2D Navier-Stokes equations one has

$$d_M(X) \le c_8 d_H(X) \quad . \tag{4.5}$$

5. INERTIAL MANIFOLDS AND INERTIAL FORMS

An *inertial form* of a dissipative partial differential equation, in particular of an equation of the form (2.4), is an ordinary differential system

$$\frac{dv}{dt} + N_I(v) = 0 \tag{5.1}$$

in an open bounded domain $\mathcal{O}_I$ in $\mathbb{R}^I$ and a map $\phi_I : \mathcal{O}_I \to V$ satisfying the following conditions:

(i) N_I and ϕ_I are Lipschitz functions,

(ii) $\mathcal{O}_I$ is invariant to (5.1) and there exists a compact set $X_I \subset \mathcal{O}_I$ attracting all solutions of (5.1),

(iii) For every solution $v(\cdot)$ of (5.1), $u(\cdot) = \phi_I(v(\cdot))$ is a solution of (2.4),

(iv) Every solution of (2.4) is exponentially attracted in H (or equivalently in V) to $\phi_I(\mathcal{O}_I)$.

An *inertial manifold* of (2.4) is a Lipschitz manifold which is of the form $\phi_I(\mathcal{O}_I)$ for an appropriate inertial form. Also the existence (resp. determination) of an inertial form provides a good theoretical (resp. practical) base for the numerical simulation of the long time behaviour of a dissipative partial differential equation, provided the dimension I is not too large. It is also clear that the universal attractor X lies on any inertial manifold or more generally that $\phi_I(X_I) \supset X$ for any inertial form. Thus I must be $\geq d_M(X)$. It is also clear that the existence of an inertial form will constitute a sound mathematical framework for the dynamical systems approach to turbulence. Therefore a first question to be asked is:

<u>Question 4.</u> Which equations of the form (2.4) have inertial forms?

Presently the best answers to this question seem to be of the following kind:

<u>Theorem 2.</u> There exist $\delta,\eta,\mu \in (0,1]$ depending on the constants α,β,γ appearing in (2.6)-(2.6)$'$ such that if

$$\mu\nu^{\eta}(\Lambda_{m+1}^{\delta} - \Lambda_{m}^{\delta}) \geq 1 \quad , \tag{5.2}$$

where $0 < \Lambda_1 \leq \Lambda_2 \leq \ldots$ are the distinct eigenvalues of A, then (2.4) has an inertial manifold.

For a hint on the proof of this type of results we refer to [33], [34].

It can be shown that the hypotheses of the theorem apply to many interesting partial differential equations. In particular this is true for the Kuramoto-Sivashinsky equation

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \quad , \tag{5.3}$$

where u is odd and L -periodic, $L > 0$. It was shown that (5.2) has inertial manifolds of dimension $I \leq c_9 L^{3.5}$ [34]. However for the universal attractor X of (5.3) one has $d_M(X) \leq c_{10} L^{1.5}$ [35]. These results leave the following question still open:

Question 5. Does the Kuramoto-Sivashinsky equation (5.3) satisfy the condition

$$c_{11}L \leq d_M(X) \leq I \leq c_{12}L \quad , \tag{5.4}$$

where I stands for the dimension of an appropriate inertial manifold of (5.3)?

A conjecture due to Y. Pomeau based on some strong numerical evidence suggests that (5.4) is true.

Theorem 2 applies also to reaction-diffusion equations with no restriction on the diffusion coefficient ν . Remarkably, an explicit inertial form for systems of such equations with diffusion coefficients restricted to an adequate range was given earlier in [36]. However Theorem 2 does not apply to the Navier-Stokes equations, even if $n = 2$. Indeed in this case $\delta = \frac{1}{2}$ and the condition (5.2) is never satisfied if ν is small enough. Therefore we finish with a particular version of Question 4, which we hope will turn out to be true.

QUESTION 6. Do the 2D Navier-Stokes equations always have inertial manifolds?

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