Math 107-Lecture 21

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Announcements

- Quiz on Thursday will be about the alternating series test and the ratio test.

The direct comparison test (DCT)

Assume that we have $0 \leq a_n \leq b_n$ for all $n \geq 1$. Then

- If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges (because it is smaller)

  Example: $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{2n^2+1}$ converges because $\displaystyle\frac{e^{1/n}}{2n^2+1} \leq \frac{2}{2n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (Why?).

- If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges (because it is larger)

  Example: $\displaystyle\sum_{n=1}^{\infty} \frac{3-\cos(n)}{n^{1/e}-1}$ diverges because $\displaystyle\frac{3-\cos(n)}{n^{1/e}-1} \geq \frac{2}{n^{1/e}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{1/e}}$ diverges (Why?).

The limit comparison test (LCT)

Assume that $a_n, b_n > 0$ for all $n \geq 1$ and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0, \infty)$$

In other words, the two sequences have similar behavior at infinity.
Then

$$\sum_{n=1}^{\infty} a_n (\mathrm{C}) \iff \sum_{n=1}^{\infty} b_n (\mathrm{C}).$$

Example: We have $\displaystyle\sum_{n=1}^{\infty} \frac{n\sqrt{n^4+7}+1}{n^{20}+n^{3/2}}$ converges because

$$\lim_{n\to\infty} \frac{\frac{n\sqrt{n^4+7}+1}{n^{20}+n^{3/2}}}{\frac{n^3}{n^{20}}} = 1 \in (0, \infty)$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^{20}}$ converges.

Clicker question #1

What can we say about

$$\sum_{n=5}^{\infty} \frac{n-2}{n^3+n+1}$$

- The series converges by the divergence test
- The series diverges because of the divergence test
- The series converges by the comparison test
- The series diverges - by the comparison test
- We can not conclude convergence/divergence with the methods we learned so far.

Alternating series

Alternating Series Test (AST): Assume we have a sequence such that

$$c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0, \text{ and } \lim_{n\to\infty} c_n = 0.$$

Then the series $\sum_{n=0}^{\infty}(-1)^n c_n$ converges.

Example 1: The series

$$\sum_{n=0}^{\infty}(-1)^n \frac{3}{n}$$

is convergent by AST.

Example 2: However, for the series

$$\sum_{n=0}^{\infty}(-1)^n \frac{\sin n}{n}$$

AST is inconclusive (Why?).

Absolute convergence

- If the series of absolute values $\displaystyle\sum_{n=n_0}^{\infty} |a_n|$ converges, then the original series $\displaystyle\sum_{n=n_0}^{\infty} a_n$ also converges and we say it converges absolutely.

  Example of absolute convergence: The series

  $$\sum_{n=0}^{\infty}(-1)^n \frac{2}{n^3}$$

  is convergent by AST; the series of absolute values $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^3}$ also converges (Why?).

Conditional convergence

- If the series of absolute values $\displaystyle\sum_{n=n_0}^{\infty} |a_n|$ diverges, but the original series $\displaystyle\sum_{n=n_0}^{\infty} a_n$ converges, we say it converges conditionally.

  Example of conditional convergence: The series

$$\sum_{n=0}^{\infty}(-1)^n \frac{3}{n}$$

is convergent by AST, however the series of absolute values $\displaystyle\sum_{n=0}^{\infty} \frac{3}{n}$ is divergent (Why?).

The ratio test

For situations when factorials are present, we have the following tool:
The ratio test. Let

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$$

- If $\rho < 1$ then the series converges absolutely.
- If $\rho > 1$ then the series diverges.
- If $\rho = 1$ then the test is *inconclusive*.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n!}$

The root test

For situations when we have terms raised to power n, we employ The root test. Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

- If $\rho < 1$ then the series converges absolutely.
- If $\rho > 1$ then the series diverges.
- If $\rho = 1$ then the test is *inconclusive*.

Example: $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n+1}{2n}\right)^n$

More examples

- Determine if the series converges and, if it does, whether it converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

- Use the comparison and limit comparison tests to determine if the series below converges

$$\sum_{n=5}^{\infty} \frac{n-2}{n^3+n+1}$$

- Use the ratio test to study the series

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

Clicker question #2

If $\quad b_n = (2n)! \quad$ then what is $\quad \dfrac{b_{n+1}}{b_n} \quad$?

- $4n^2 + 6n + 2$
- $2n + 1$
- $2n + 2$
- $\dfrac{(2n!) + 1}{(2n!)} = 1 + \dfrac{1}{(2n!)}$
- Neither of these