Review of trigonometry

- A radian is the measure of the angle that cuts off a sector of the unit circle of radius 1. (i.e. if we walk 1 length unit along a circle of radius 1). The whole circle has $2\pi \approx 6.28$ radians (corresponding to a 360° angle.
- Put your calculator in radian mode!

Sine and Cosine functions

- The coordinates of a point on the unit circle are $x = \cos t$, $y = \sin t$, $t \in \mathbb{R}$. Otherwise, in a right triangle $\sin t = \dfrac{\text{opp. side}}{\text{hyp}}$ and $\cos t = \dfrac{\text{adj. side}}{\text{hyp}}$.
- Recall the Pythagorean formula (fundamental identity in trigonometry)

$$x^2 + y^2 = \sin^2 t + \cos^2 t = 1.$$

- The sine and cosine functions are 2π periodic i.e.

$$\sin t = \sin(t + 2\pi \cdot k); \quad \cos t = \cos(t + 2\pi \cdot k).$$

- Amplitude= half the distance between max. and min.
- Period = minimum change in argument to cycle through the range. If f has period P then $f(x) = f(x + P)$.

The tangent function

- Defined as

$$\tan t = \frac{\sin t}{\cos t} = \frac{\text{opp. side}}{\text{adj. side}}.$$

- Undefined at $t = \frac{\pi}{2} + k\pi$. (vertical asymptote)

We often need to solve equations of the form

$$\sin\theta = \tfrac{1}{2} \text{ or even } \sin\theta = 0.329748.$$

We will invert the trigonometric functions to obtain a value for the unknown angle.

- $\sin x = y \iff x = \arcsin y; \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ and $y \in [-1,1]$. (why?)
- $\cos x = y \iff x = \arccos y; \quad 0 \le x \le \pi$ and (still!) $y \in [-1,1]$.
- $\tan x = y \iff x = \arctan y; \quad -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ and $y \in (-\infty, \infty)$.

Clicker question

What is $\sin\alpha$ when $\cos\alpha = \dfrac{1}{3}$? $(\alpha \in (0, \frac{\pi}{2}))$

(A) $\dfrac{2\sqrt{2}}{3}$

(B) $\dfrac{2}{3}$

(C) $-\dfrac{2\sqrt{2}}{3}$

(D) $\dfrac{2}{3\sqrt{2}}$

(E) $\dfrac{8}{9}$

Continuity

Continuity of a Function on an Interval: Graphical Viewpoint

A continuous function has a graph which can be drawn without lifting the pencil from the paper. That is, a continuous graph has no breaks, jumps or holes.

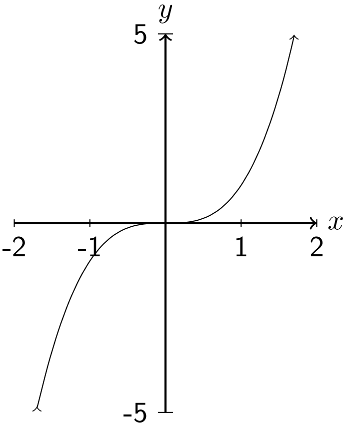
Examples of graphs

Intervals of Continuity

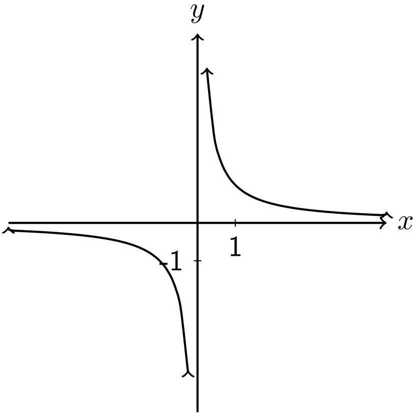
Examples

1. $f(x) = 4x^4 + 3x^3 + 3$ is continuous on any interval.
2. $g(x) = \dfrac{1}{x}$ is not defined at $x = 0$. It is continuous on any interval that does not include $x = 0$.

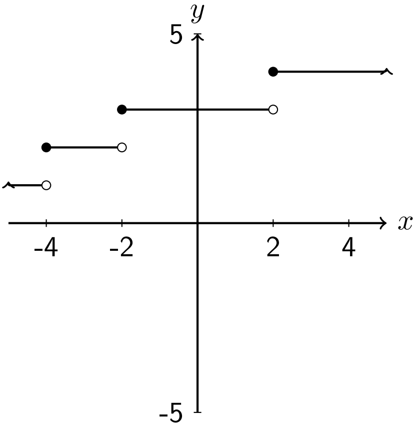
Intervals of Continuity



Intervals of Continuity



Intervals of Continuity



Question

Consider the function $f(x) = \dfrac{3x^2}{x^2 - 16}$, is $f(x)$ continuous on the interval $[-1, 1]$?

(a) Yes

(b) No

Intervals of Continuity

Examples

piecewise-defined function

A piecewise-defined function is one which is defined not by a single equation, but by two or more. Each equation is valid for some interval .

1. $$h(x) = \begin{cases} 4x^2 & \text{if } x \leq 1 \\ 3x+3 & \text{if } x > 1 \end{cases}$$

   is discontinuous at $x = 1$. It is continuous on any interval that does not include $x = 1$.

Question

The amount that a person pays in taxes is dependent on how much they earn. A single person taking the standard deduction can calculate how much they owe in taxes using the following piecewise-defined function, $T(x)$, where x represents their total income.

$$T(x) = \begin{cases} \frac{3}{20}(x - 6,300) & \text{if } 15,000 \leq x < 44,000, \\ \frac{1}{4}(x - 6,300) & \text{if } 44,000 \leq x \leq 98,000. \end{cases}$$

Is the tax function $T(x)$ continuous on the interval $[15000, 98000]$?

(a) Yes

(b) No

Question

Consider the piecewise function

$$D(x) = \begin{cases} 4x^2 - k & \text{if } x < 2, \\ kx + 1 & \text{if } 2 \leq x. \end{cases}$$

Find the value of k to make this function continuous for all x.

(a) $k = 3$
(b) $k = 1$
(c) any k works
(d) $k = 5$

Intermediate Value Theorem

Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Question

If $f(x)$ is a polynomial such that $f(-1) < 0$, $f(2) > 0$ and $f(4) < 0$ we can say that $f(x)$ has at least how many zeros?

(a) 1

(b) 2

(c) 3

(d) not enough information

Continuity of a Function at a point

Continuous Function

A function is continuous if nearby values of the independent variable give nearby values of the function. In practical work, continuity is important because it means that small errors in the independent variable lead to smalls errors in the function.

Example

Consider our tax problem

$$T(x) = \begin{cases} \frac{3}{20}(x - 6,300) & \text{if } 15,000 \leq x < 44,000, \\ \frac{1}{4}(x - 6,300) & \text{if } 44,000 \leq x \leq 98,000. \end{cases}$$

If you report your income as $43,999$ you would pay $T(43,999) = 5654.83$ in taxes. However, if you made an error and report your income as $44,000$ you would pay $T(44,000) = 9425$. So a small difference (only $\$1$) in your income results in a huge increase in what you have to pay in taxes.

If $f(x)$ is continuous at $x = c$, the values of $f(x)$ approach $f(c)$ as x approaches c.

Question

Is the function:

$$f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

continuous on $[-1, 1]$?

(a) Yes

(b) No