Calculus 1
Day 11 3.3 The Quotient Rule

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Why do we need a rule

Today we will consider functions of the form $\dfrac{f(x)}{g(x)}$.

For example, let us consider $F(x) = \dfrac{4x^4}{x^2}$.

$$\frac{d}{dx}\left(\frac{4x^4}{x^2}\right) = \frac{d}{dx}(4x^2) = 8x$$

What if we had tried to just take the derivative of each?

$$\frac{\frac{d}{dx}(4x^4)}{\frac{d}{dx}(x^2)} = \frac{16x^3}{2x} = 8x^2$$

Finding a rule

Let's consider the function $Q(x) = \dfrac{f(x)}{g(x)}$. Of course we have to avoid points where $g(x) = 0$. Rewrite as $f(x) = Q(x)g(x)$ and use the product rule.

$$f'(x) = Q'(x)g(x) + Q(x)g'(x)$$

Replace $Q(x) = f(x)/g(x)$:

$$f'(x) = Q'(x)g(x) + \frac{f(x)}{g(x)}g'(x)$$

Then solve for $Q'(x)$:

$$Q'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

Multiply top and bottom to get:

$$Q'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Quotient Rule

Quotient Rule

If $u = f(x)$ and $v = g(x)$ are differentiable, then

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

or equivalently,

$$\frac{dv}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}.$$

In words:
The derivative of a quotient is the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all over the denominator squared.

Example

Consider the function

$$F(x) = \frac{4\sin(x)}{x^2+1}.$$

Find the derivative $F'(x)$.

Here $F(x) = \dfrac{f(x)}{g(x)}$, thus we use the quotient rule:

$$f'(x) = \frac{(4\cos(x))(x^2+1) - (4\sin(x))(2x)}{(x^2+1)^2}.$$

Clicker Question

Differentiate the function

$$g(x) = \frac{5x^2}{x^3+1}.$$

a) $\dfrac{10x}{3x^2}$

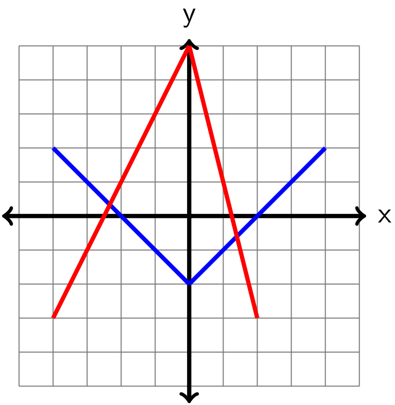
b) $\dfrac{-5x^4+10x}{(x^3+1)^2}$

c) $\dfrac{10x-15x^4}{(x^3+1)}$

d) $\dfrac{10x(x^3+1)+5x^2(3x^2)}{(x^3+1)^2}$

Clicker Question

If $h(x) = \dfrac{f(x)}{g(x)}$, what is $h'(1)$? In the graph $f(x)$ is blue and $g(x)$ is red.



a) Not enough information

b) $h'(1) = 5$

c) $h'(1) = -3$

d) $h'(1) = -4$

Derivation of Derivative of Tangent

We have already seen that

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2(x)}.$$

Now we want to use the quotient rule to derive it:

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$= \frac{\cos(x)\cos(x) - (\sin(x))(-\sin(x))}{(\cos(x))^2}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

Derivative of Cotangent

Show that:

$$\frac{d}{dx}(\cot(x)) = \frac{-1}{\sin^2(x)}$$

Examples

For each of the following, calculate the derivative.

1. $\frac{3x^2+3x}{x+1}$
2. $$\frac{x^3+1}{e^x}$$
3. $$\frac{4^x}{2x+1}$$