Math 107-Lecture 25

Dr. Adam Larios

University of Nebraska-Lincoln

Convergence/divergence tests

- **DCT:** $0 \leq a_n \leq b_n$; $\sum_{n=1}^{\infty} b_n$ (C)$\Rightarrow \sum_{n=1}^{\infty} a_n$ (C); $\sum_{n=1}^{\infty} a_n$ (D)$\Rightarrow \sum_{n=1}^{\infty} b_n$ (D)
- **LCT:** $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0, \infty)$ then $\sum_{n=1}^{\infty} a_n(C/D) \iff \sum_{n=1}^{\infty} b_n(C/D)$.
- **AST:** If $c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0$, $\lim_{n\to\infty} c_n = 0$ then $\sum_{n=0}^{\infty}(-1)^n c_n$ (C)
- **Ratio/Root** Let $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ (Ratio) $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (Root).
  $\rho < 1$ (AC); $\rho > 1$ (D); $\rho = 1$ *inconclusive*.

Taylor Polynomials

The Taylor polynomial of degree n for the function $f(x)$ near $x = a$ is the n-th degree polynomial

$$P_n(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

Example 1: The Taylor polynomial of fourth degree for $f(x) = \sin x$ about $x = 0$ is

$$P_3(x) = P_4(x) = x - \frac{x^3}{6}$$

Note: it is only a **third** degree polynomial. Why?

Example 2: The Taylor polynomial of fourth degree for $g(x) = 3x^4 + 2x^2 - \sqrt{\pi}$ is

$$Q_4(x) = 3x^4 + 2x^2 - \sqrt{\pi}.$$

More examples

Find the Taylor polynomials of order n for

- $\dfrac{1}{1-x}$ near $a = 0$
- $\dfrac{1}{1-(x-2)^2}$ near $a = 2$
- e^{2x} near $a = 0$
- $\sin(x^2)$ near $a = 0$

Clicker question #1

What is the 3rd-degree Taylor polynomial of $\ln(x)$ near $x = e$?

- $e + \frac{1}{e}x + \frac{1}{e^2}x^2 + \frac{1}{e^3}x^3$
- $\frac{1}{e} + \frac{1}{e}(x-e) - \frac{1}{e^2}(x-e)^2 + \frac{1}{e^3}(x-e)^3$
- $1 + \frac{1}{e}(x-e) - \frac{1}{e^2}(x-e)^2 + \frac{1}{e^3}(x-e)^3$
- $1 + \frac{1}{e}(x-e) - \frac{1}{e^2}(x-e)^2 + \frac{2}{e^3}(x-e)^3$
- $1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3$

Taylor Series

The Taylor series is "a Taylor polynomial of infinite order"; do not stop at degree n, but keep going.

$$
\begin{aligned}
f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \\
&\qquad + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n
\end{aligned}
$$

Does it make sense to add infinitely many terms? How do we know that we get the function back as the sum of the Taylor series?

Taylor series you must know

The series below are near $a = 0$ (also called Maclaurin series):

- $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots \qquad , -\infty < x < \infty$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots \qquad -\infty < x < \infty$
- $\ln(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots \qquad -1 < x \leq 1$
- $(1 + x)^p =$
  $1 + px + \dfrac{p(p-1)}{2!}x^2 + \dfrac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad -1 < x < 1$

Cool stuff to do with Taylor series

- inside the interval of convergence (Taylor series are power series!) we can add, subtract, differentiate, integrate series
- we can find sums of many kinds of power series
- we can do very good approximations of functions near the point where the series is computed

Application 1: computing limits, derivatives, integrals

- Find $\lim_{x \to 0} \dfrac{\sin x}{x}$
- Why is it that
$$\frac{d}{dx} e^x = e^x \quad ?$$
- Try to compute
$$\int_0^1 e^{x^2} \, dx.$$