Math 107-Lecture 6

Dr. Adam Larios

University of Nebraska-Lincoln

Plan for today

1. Review trigonometric substitution.
2. Numerical integration

Motivation for trigonometric substitution

Recall the trigonometric identities:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ 1 + \tan^2\theta &= \sec^2\theta \end{aligned}$$

Or, after a few algebraic manipulations (ignore the absolute values for now):

$$\begin{aligned} a\sin\theta &= \sqrt{a^2 - (a\cos\theta)^2} \\ a\sec\theta &= \sqrt{a^2 + (a\tan\theta)^2} \\ a\tan\theta &= \sqrt{(a\sec\theta)^2 - a^2} \end{aligned}$$

Trigonometric substitutions

Expression (in the integrand)	Substitution	$dx, d\theta$ relation
$a^2 - x^2$	$x = a\sin\theta$	$dx = a\cos\theta d\theta$
$a^2 + x^2$	$x = a\tan\theta$	$dx = a\sec^2\theta d\theta$
$x^2 - a^2$	$x = a\sec\theta$	$dx = -a\tan\theta\sec\theta d\theta$

Examples

Example 1. Show that

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Solution. According to the table above we need to use the substitution

$$x = \sin\theta, \quad (\text{ so } \theta = \arcsin x), \quad dx = \cos\theta d\theta.$$

With this substitution the integral becomes

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\cos\theta} \cdot \cos\theta\, d\theta = \theta + C = \arcsin x + C$$

Clicker question #1

What is

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx \qquad ?$$

- $(\arctan x)^{3/2} + C$
- $\arctan(x^{3/2}) + C$
- $\dfrac{-1}{2(1+x^2)^{1/2}} + C$
- $\dfrac{x}{\sqrt{1+x^2}} + C$
- $x = \ln[(1+x^2)^2] + C$

Why do we need it?

- Many functions do not have antiderivatives that can be written in terms of elementary functions; i.e. we can not get rid of the integration sign to write the function. Examples:

$$\int e^{x^2}dx, \quad \int e^{1/x}dx, \quad \int \sin(x^2)dx, \quad \int \ln(x)\cos(x)dx\ldots$$

- Many functions require **a lot of work** to find the exact answer, but we often just need a rough estimate.

What to do?

Approximate!

The importance of good estimates ...

"... Of course it's 'a lot higher than my original estimate'. If I'd given you an accurate estimate, you'd never have given me the job."

Approximation of definite integrals

Recall $\int_a^b f(x)dx$ can be approximated by

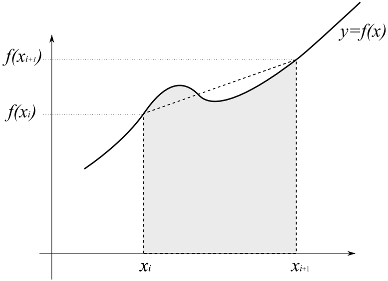
- left Riemann sums : $\approx (\Delta x) \cdot [f(x_0) + f(x_1) + ... f(x_{n-1})]$
- right Riemann sums: $\approx (\Delta x) \cdot [f(x_1) + f(x_2) + ... f(x_n)]$

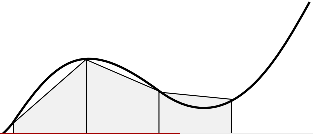
Even for linear functions these are not very good (exact) approximations! (Why?)

A better way?

We will continue to subdivide $[a, b]$ into n subintervals, but we seek a better way to approximate the area under the graph.

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \frac{\Delta x}{2}\left[f(x_i) + f(x_{i+1})\right].$$





The trapezoidal rule

For 3 subintervals we have

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}(f(x_0)+f(x_1))+\frac{\Delta x}{2}(f(x_1)+f(x_2))+\frac{\Delta x}{2}(f(x_2)+f(x_3))$$

Generalizing the above formula for n subintervals, we obtain

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+\ldots+2f(x_{n-1})+f(x_n)\right]$$

Examples of numerical integration/approximation

Example. Use the trapezoidal rule on $n = 5$ intervals to approximate $\displaystyle\int_0^2 e^{x^2}\,dx$.

We have that $\Delta x = \dfrac{2}{5} = 0.4$, so

$$x_0 = 0,\ x_1 = 0.4,\ x_2 = 0.8,\ x_3 = 1.2,\ x_4 = 1.6,\ x_5 = 2.$$

The trapezoidal approximation is given by

$$\int_0^2 e^{x^2}\,dx \approx 0.2\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right]$$

Hence

$$\int_0^2 e^{x^2}\,dx \approx 0.2\left[e^0 + 2e^{0.16} + 2e^{0.64} + 2e^{1.44} + 2e^{2.56} + e^4\right] \approx 19.21$$

Clicker question #2

What is the trapezoidal approximation for $\displaystyle\int_0^\pi \sin(x)dx$ with $n = 4$?

- $\pi\left[\sin 0 + \sin(\pi/4) + \sin(\pi/2) + \sin(3\pi/4)\right]$
- $\dfrac{\pi}{4}\left[\sin 0 + 2\sin(\pi/4) + 2\sin(\pi/2) + \sin(3\pi/4)\right]$
- $\dfrac{\pi}{8}\left[2\sin 0 + 2\sin(\pi/4) + 2\sin(\pi/2) + 2\sin(3\pi/4)\right]$
- $\dfrac{\pi}{8}\left[\sin 0 + 2\sin(\pi/4) + 2\sin(\pi/2) + 2\sin(3\pi/4) + \sin(\pi)\right]$
- $\dfrac{\pi}{8}\left[\sin 0 + 2\sin(\pi/4) + 2\sin(\pi/2) + \sin(3\pi/4)\right]$

Wrapping up:

- Today we reviewed exact integration and covered numerical integration (7.5).
- For next time finish working all suggested problems from section 7.5.
- Webwork on Trigonometric Substitutions is due tomorrow, 02/01 at 10:00 pm CST.
- For Thursday read Section 7.6 – Improper integrals.