Data meets PDEs: New approaches to parameter recovery and data assimilation in the Navier-Stokes equations

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Outline

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The Incompressible Navier-Stokes Equations

Claude L.M.H. Navier

George G. Stokes

Unknowns

$\vec{u}$:= Velocity (vector)
p := Pressure (scalar)

Parameters

ν := Kinematic viscosity

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{Acceleration} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{Advection} = \underbrace{-\nabla p}_{\substack{Pressure\\Gradient}} + \underbrace{\nu\triangle\vec{u}}_{\substack{Viscous\\Diffusion}} + \underbrace{\vec{f}}_{\substack{Body\\Force}}$$

Incompressibility

$$\nabla\cdot\vec{u} = 0$$

Initial Data

$$\vec{u}(t_0) = \vec{u}_0$$

Data Assimilation

$$\begin{cases} \frac{du}{dt} &= F(u), \\ u(t_0) &= u_0. \end{cases}$$

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Observation

- Imagine we even have a *perfect* solver.

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Use incoming data observations to force simulations toward the true solution.

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- Classical approach: Kalman filter

Outline

Kalman Filter

$$\begin{cases} \frac{du}{dt} &= Au, \\ u(t_0) &= u_0, \\ Y(t) &= Hu(t) + \eta(t) \end{cases}$$

Suppose there exist S.P.D. operators Q_0 and R such that:

$$(Q_0 u_0, u_0) \leq 1, \quad \int_0^T E(R\eta, \eta)\, dt \leq 1.$$

Optimal state estimator $\hat{u}$ is given by:

$$\begin{cases} \frac{d\hat{u}}{dt} = Au + PH^T R(Y - Hu), \\ \hat{u}(t_0) = 0, \\ \frac{dP}{dt} = AP + PA^* - PH^* RHP, \\ P(t_0) = Q_0^{-1}. \end{cases}$$

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Problems

- Can be expensive
- Assumes linearity ....
- Matrix Ricatti equation for P.
- Multiphysics?

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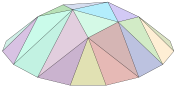

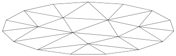

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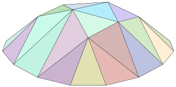
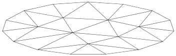
Linear Feedback Control (Azouani, Olson, Titi, [AOT] 2014)

$$\begin{cases} \frac{dv}{dt} &= F(v) + \mu(I_h(u) - I_h(v)), \quad \mu > 0, \\ v(t_0) &= v_0. \end{cases}$$





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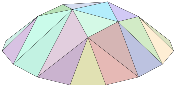
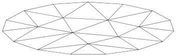
$$\|u - \mathrm{I}_h(u)\|_{L^2} \leq c_1 h \|\nabla u\|_{L^2} \quad \text{(Type I)},$$

$$\|u - \mathrm{I}_h(u)\|_{L^2} \leq c_2 h \|\nabla u\|_{L^2} + c_3 h^2 \|\triangle u\|_{L^2} \quad \text{(Type II)}.$$





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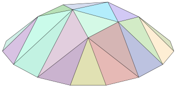

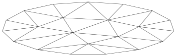

Theorem (Azouani, Olson, Titi, 2014)

*] Let F be given by 2D Navier-Stokes equations. For μ, $1/h$ sufficiently large, $\|u(t) - v(t)\|_{H^1} \to 0$ exponentially fast as $t \to \infty$ for **any** $v_0 \in L^2$, $\nabla \cdot v_0 = 0$.*

2D Navier-Stokes Equations [AOT argument]

Video

2D Navier-Stokes
with AOT data assimilation

Video credit: Masakazu Gesho (Gesho, Olson, Titi, 2015)

Idea behind L^2 convergence

$$\boxed{F(u) = N(u) + \nu\triangle u}$$

$$\begin{cases} \frac{\partial u}{\partial t} & = N(u) + \nu\triangle u \\ \frac{\partial v}{\partial t} & = N(v) + \nu\triangle v + \mu(I_h(u) - I_h(v)) \end{cases}$$

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- Prove global-in-time existence and uniqueness results for this system.
- Let $w = u - v$, then

$$\frac{\partial w}{\partial t} = N(u) - N(v) + \nu\triangle w + \big(\mu w - \mu I_h(w)\big) - \mu w.$$

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- Take (justified) inner product with w:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w\|^2 + \nu\|\nabla w\|^2 &= (N(u) - N(v), w) + \mu(w - I_h(w), w) - \mu\|w\|^2 \\ &\le (N(u) - N(v), w) + \mu\|w - I_h(w)\|\|w\| - \mu\|w\|^2 \\ &\le (N(u) - N(v), w) + \frac{\mu}{2}\|w - I_h(w)\|^2 - \frac{\mu}{2}\|w\|^2 \\ &\le (N(u) - N(v), w) + \frac{\mu c_1^2}{2}h^2\|\nabla w\|^2 - \frac{\mu}{2}\|w\|^2 \end{aligned}$$

Idea behind L^2 convergence [AOT argument]

We have:

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- For 2D Navier-Stokes: $(N(u) - N(v), w) \leq \frac{C}{2\nu}\|\nabla u\|^2\|w\|^2 + \frac{\nu}{2}\|\nabla w\|^2$.

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• Choose h so that $\mu c_1^2 h^2 \leq \nu$. Then

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• For 2D NSE, there exists $T > 0$ such that $\limsup_{t\to\infty} \int_t^{t+T} \|\nabla u(s)\|^2\,ds$ is bounded.
• Choose μ large enough, use uniform Grönwall inequality to show $\|w\| \to 0$ exponentially fast in time.

Outline

A Brief Recent History: Up to 2014

- Nudging (Hoke, 1974; Hoke, Anthes, 1976)
- Stabilization of NSE steady states (Cao, Kevrekidis, Titi, 2001)
- Determining modes (Olson, Titi, 2003)
- Lorenz (Hayden, Olson, Titi, 2011)
- Reaction-diffusion (Azouani, Titi, 2014)

A Brief Recent History: AOT Algorithm

- 2D NSE (Azouani, Olson, Titi, 2014)
- 2D simulations (Gesho, Olson, Titi, 2015)
- Stochastic noisy data (Bessiah, Olson, Titi, 2015).
- 2D Abridged (Farhat,Lunasin, Titi, 2016)
- Discrete in time data (Foias, Mondaini, Titi, 2016)
- 3D NS-α (Albanez, Nussenzveig, Lopes, Titi, 2016)
- Higher-order and Gevrey convergence (Biswas, Martinez, 2017)
- Statistical solutions (Biswas, Foias, Mondaini, Titi, 2017)
- Postprocessing Galerkin method (Mondaini, Titi, 2017)
- Fully Discrete Case (Mondaini, Titi, 2018) (L., Rebholz, Zerfas 2018)

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3D Navier-Stokes Equations

Observation

- For 3D Navier-Stokes, simulation is our only recourse.

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Analytical Results

- 3D Navier-Stokes-α (Albanez, Nussenzveig, Lopes, Titi, 2016)
- 3D Leray-α (Farhat, Lunasin, Titi, 2017)
- 3D Navier-Stokes-Voigt-α with physical data (L., Pei) (preprint)

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Computational Results

- 3D Navier-Stokes (Leoni, Mazzino, Biferale, 2018)
- 3D Isotropic turbulence (L., Pei) (preprint)

Algorithm: BDF2 with grad-div stabilzation (A. L., L. Rebholz, C. Zarfas, 2018)

Find $(v_h^{n+1}, q_h^{n+1}) \in (X_h, Q_h)$ for $n = 0, 1, 2, \ldots$, satisfying

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(3v_h^{n+1} - 4v_h^n + v_h^{n-1}, \chi_h\right) \\
&\quad + ((2v_h^n - v_h^{n-1}) \cdot \nabla v_h^{n+1}, \chi_h) - (q_h^{n+1}, \nabla \cdot \chi_h) \\
&\quad + \gamma(\nabla \cdot v_h^{n+1}, \nabla \cdot \chi_h) \\
&\quad + \nu(\nabla v_h^{n+1}, \nabla \chi_h) + \mu(I_h(v_h^{n+1} - u^{n+1}), \chi_h) \\
&= (f^{n+1}, \chi_h), \\
&(\nabla \cdot v_h^{n+1}, r_h) = 0,
\end{aligned}$$

for all $(\chi_h, r_h) \in X_h \times Q_h$ (appropriate finite element spaces).

Theorem (A. L., L. Rebholz, C. Zarfas, 2018)(Convergence)

Suppose $u \in L^\infty(0,\infty;H^1(\Omega))$ *and* $u_t, u_{tt} \in L^\infty(0,\infty;L^2(\Omega))$, *and* Δt *satisfies*

$$0 \;<\; \Delta t \;\le C\nu(\|\nabla u^n\|_{L^3}^2 + \|u^n\|_{L^\infty}^2 + h^{2k}|u|_{k+1}^2)^{-1}$$

and μ *satisfies*

$$C\nu^{-1}(\|\nabla u^n\|^2 + \|u^n\|_{L^\infty}^2 + Ch^{2k}|u|_{k+1}^2) \;\le\; \mu \;\le\; \frac{\nu}{2}C_I^{-2}h^{-2},$$

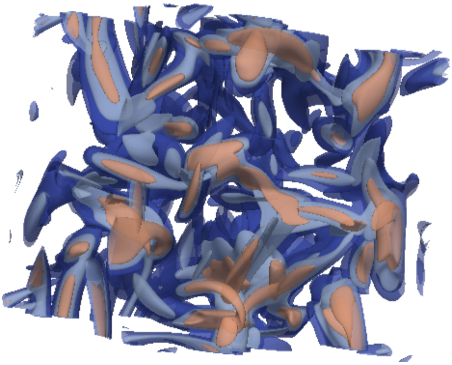
where h *is chosen sufficiently small so that this inequality holds. Let*

$$\lambda := C_P^{-2}\left(\frac{\nu}{2} - C\mu^{-1}(\|\nabla u^n\|_{L^3}^2 + \|u^n\|_{L^\infty}^2 + h^{2k}|u|_{k+1}^2)\right) + \left(\frac{\mu}{2} - C\mu^2\nu^{-1}h^2\right),$$

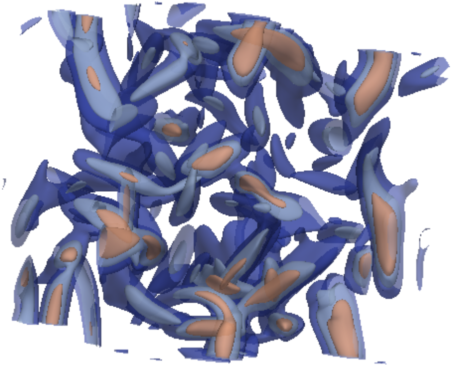
then for any time t^n, $n = 0, 1, 2, \ldots$, *it holds that*

$$\|v_h^n - u(t_n)\|^2 \le C(\Delta t^5 + h^{2k})\frac{1}{2\lambda\Delta t} + \|v_h^0 - P_{V_h}u(0)\|^2(1 + 2\lambda\Delta t)^{-(n+1)}.$$

3D Navier-Stokes simulations



NSE, $t = 0.05$



DA, $t = 0.05$

Data assimilation at resolution 256^3, passing only wave modes $\widehat{\boldsymbol{u}}_{\mathbf{k}}$ with $|\mathbf{k}| \leq 9$. (Level surfaces of vorticity magnitude: $|\nabla \times \boldsymbol{u}|$.) (A.L., Y. Pei, 2018 preprint)

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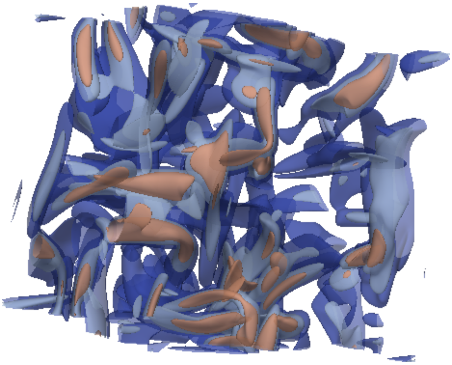
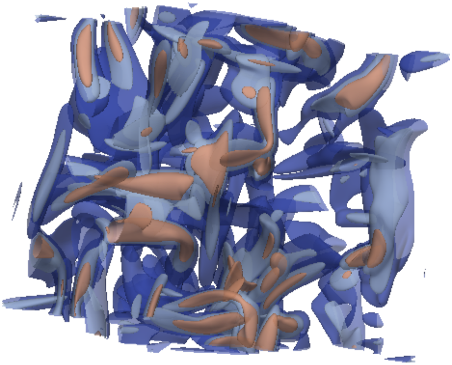


NSE, $t = 0.4$



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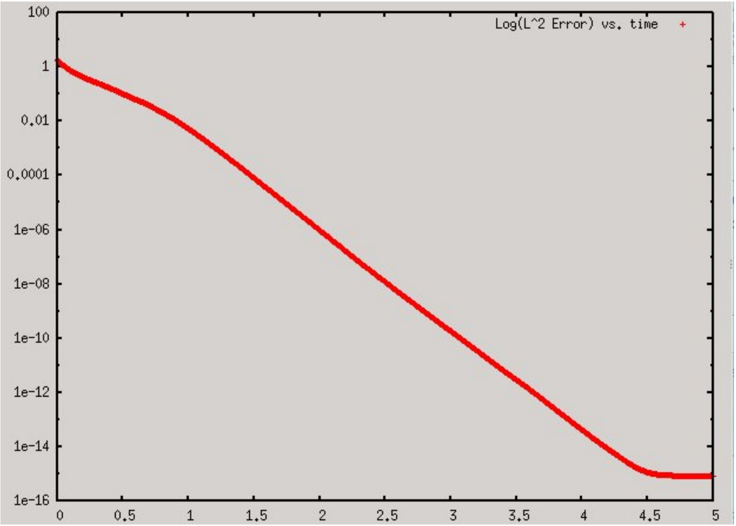

- L^2 convergence in time. Resolution: 1024^3. (A.L., Y. Pei, 2018 preprint)

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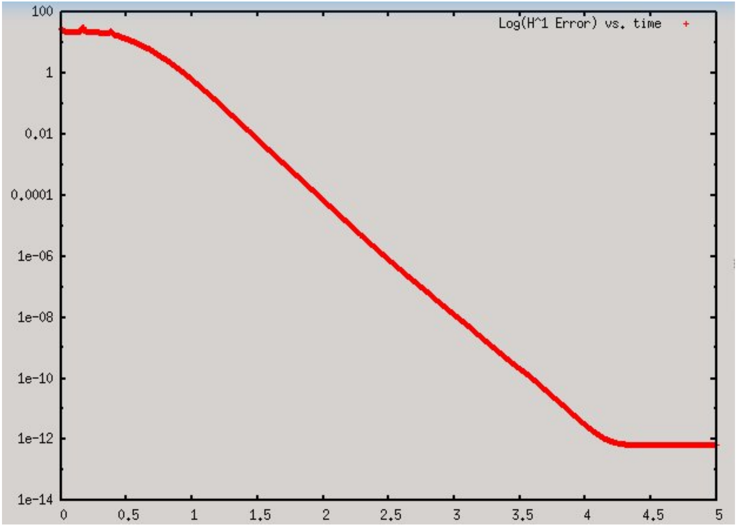

- H^1 convergence in time. Resolution: 1024^3. (A.L., Y. Pei, 2018 preprint)

Outline

Data assimilation for the Voigt turbulence model

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\triangle\boldsymbol{u} + \mathbf{f}, \\ \nabla\cdot\boldsymbol{u} = 0, \end{cases}$$

$$\begin{cases} (I-\alpha^2\triangle)\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla q = \nu\triangle\boldsymbol{v} + \mathbf{f} + \mu(I_h(\boldsymbol{u}) - I_h(\boldsymbol{v})), \\ \nabla\cdot\boldsymbol{v} = 0, \end{cases}$$

Theorem (A. L., Y. Pei)(Preprint)

Suppose the data are smooth enough. Let $h,\mu,\alpha>0$ be such that

$$h < C\lambda_1^{-1/2}G^{-1}, \quad \frac{\mu}{2} - C\nu\lambda_1 G^2 := M_1 > 0, \quad \textit{and} \quad \alpha^2 < \frac{\nu}{M_1}.$$

Then, for any (admissible) initial data $\boldsymbol{v}_0$,

$$\lim_{t\to\infty}\|\boldsymbol{u}(t)-\boldsymbol{v}(t)\|_{L^2} \le C\alpha^2, \quad \textit{and} \quad \lim_{t\to\infty}\|\nabla\boldsymbol{u}(t)-\nabla\boldsymbol{v}(t)\|_{L^2} \le C\alpha.$$

Outline

1. Continuous Data Assimilation and Navier-Stokes
   - Introduction
   - Kalman Filter and AOT Algorithm
   - Recent History
2. Extensions of AOT
   - 3D
   - Data assimilation + turbulence modeling
   - Multi-physics
3. A Few Variations
   - Variation 1: Nonlinearities
   - Variation 2: Moving Nodes
   - Variation 3: Parameter Recovery

A Brief Recent History: AOT in Geophysical Models

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu\triangle\vec{u} + \vec{\mathbf{e}}_d\theta$$

$$\nabla\cdot\vec{u} = 0$$

$$\frac{\partial \theta}{\partial t} + (\vec{u}\cdot\nabla)\theta = \kappa\triangle\theta + u_d$$

- 2D Bénard, velocity (Farhat, Jolly, Titi, 2015)
- Downscaling Bénard (Altaf, Titi, Knio, Zhao, Mc Cabe, Hoteit, 2015)
- Charney Conjecture, 3D Planetary Geostrophic (Farhat,Lunasin, Titi, 2016)
- 3D Bénard, Porus Media (Farhat, Lunasin, Titi, 2016)
- 3D Brinkman–Forchheimer-extended Darcy (Markowich, Titi, Trabelsi, 2016)
- 2D Bénard, horizontal velocity (Farhat, Lunasin, Titi, 2017)
- Subcritical 2D SQG (Jolly, Martinez, Titi, 2017)

A Brief Recent History: AOT in Geophysical Models

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu\triangle\vec{u} + \vec{\mathbf{e}}_d\theta$$

$$\nabla\cdot\vec{u} = 0$$

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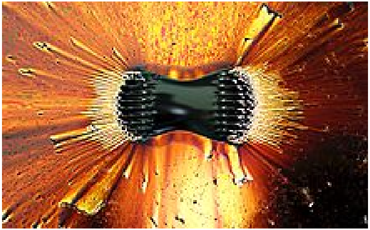
A Brief Recent History: Magnetohydrodynamics (MHD)

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu\triangle\vec{u} + (\vec{B}\cdot\nabla)\vec{B}$$

$$\nabla\cdot\vec{u} = 0$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{u}\cdot\nabla)\vec{B} = (\vec{B}\cdot\nabla)\vec{u} + \eta\triangle\vec{B}$$



Pre-AOT Data Assimilation

- Mantle circulation (Bunge, Richards, Baumgardner, 2002)
- MHD (Mendoza, DeMoor, Bernstein, 2006)
- 1D MHD (Sun, Tangborn, Kuang, 2007)
- Geomagnetic data assimilation (Fournier, Eymin, Thierry, 2007)
- Constrained MHD (Teixeira, Ridley, Torres, Aguirre, Bernstein, 2008)
- Data assimilation in geomagnetism (Fournier, Hulot, Jault, Kuang, Tangborn, Gillet, Canet, Aubert, L'huillier, 2010)
- MHD Dynamo (Li, Jackson, Livermore, 2014)

AOT for 2D MHD

MHD in Elsässer variables

$P = \vec{u} + \vec{B}$, $M = \vec{u} - \vec{B}$,

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$P = \vec{u} + \vec{B}$, $M = \vec{u} - \vec{B}$, $\alpha := \frac{1}{2}(\nu + \eta)$, $\beta := \frac{1}{2}(\nu - \eta)$.

$$\begin{aligned} \partial_t P - \alpha \triangle P - \beta \triangle M + (M \cdot \nabla) P &= -\nabla \Pi + f, \\ \partial_t M - \alpha \triangle M - \beta \triangle P + (P \cdot \nabla) M &= -\nabla \Pi + g, \\ \nabla \cdot P = 0, \quad \nabla \cdot M &= 0. \end{aligned}$$

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Data assimilation scheme

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Theorem (A. Biswas, J. Hudson, A. L., Y. Pei)(accepted-Asymptotic Analysis)

Let I_h be a Type I interpolant and suppose

$$\mu > \frac{\pi^2(c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2, \text{ and } \quad h < C(\alpha - \beta)^{\frac{1}{2}} \mu^{-\frac{1}{2}} \sim G^{-1}.$$

Then $\|P(t) - P'(t)\|_{H^1} + \|M(t) - M'(t)\|_{H^1} \to 0$ exponentially as $t \to \infty$.

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$$\mu > 32\pi^2 c^2 (\alpha - \beta) \left(\tilde{c} + 2 \ln G + C G^4 \right) G^2, \text{ and } \quad h < C(\alpha - \beta)^{\frac{1}{2}} \mu^{-\frac{1}{2}} \sim G^{-3}.$$

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Theorem (A. Biswas, J. Hudson, A. L., Y. Pei)(accepted-Asymptotic Analysis)

Let I_h be a Type I interpolant and suppose

$$\mu > \frac{\pi^2 c_L^4 G^2 (4 + (\alpha - \beta)^2 G^2)^2}{16(\alpha - \beta)}, \quad \textit{and} \quad h < C(\alpha - \beta)^{\frac{1}{2}} \mu^{-\frac{1}{2}} \sim G^{-3}.$$

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Outline

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Nonlinear Data Assimilation

Variation 1: Nonlinearity

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$$\begin{cases} \frac{dv}{dt} &= F(v) + \mu\left(I_h(u) - I_h(v)\right), \quad \mu > 0, \\ v(t_0) &= v_0. \end{cases}$$

Nonlinear Data Assimilation

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Nonlinear Data Assimilation

Variation 1: Nonlinearity $\boxed{\text{KSE: } u_t = -uu_x - u_{xx} - u_{xxxx} =: F(u)}$

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$$\boxed{0 < \gamma < 1}$$

$$\mathcal{N}(x) = \mathcal{N}_1(x) := \begin{cases} x|x|^{-\gamma}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

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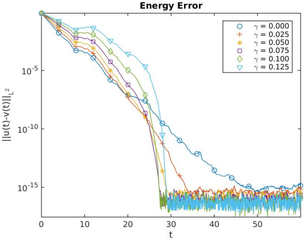

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Error in L^2-norm vs. time (1D KSE).

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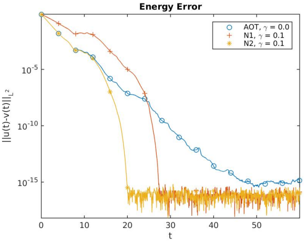

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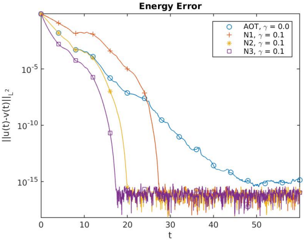

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Outline

Sweeping Probe Measurements

Allen-Cahn Reaction Diffusion Equation

$$u_t = \nu u_{xx} + u - u^3$$

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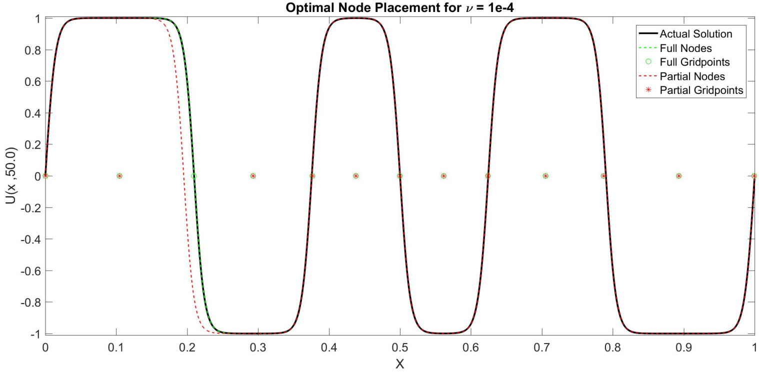
Time Dependant Assimilation Points

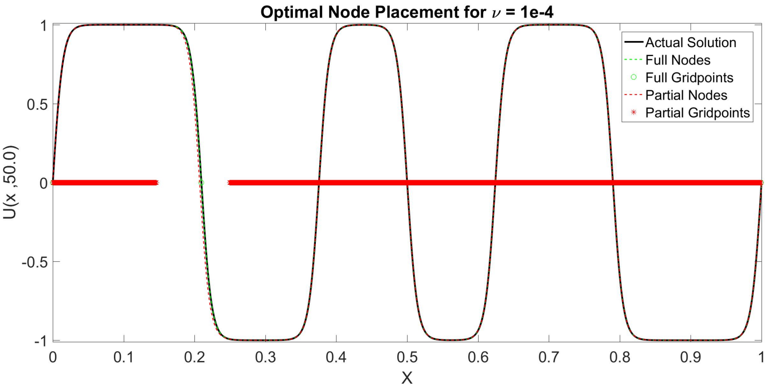
$$v_t = \nu v_{xx} + v - v^3 + \mu I_h(u - v, t)$$

Node Placement Difficulty



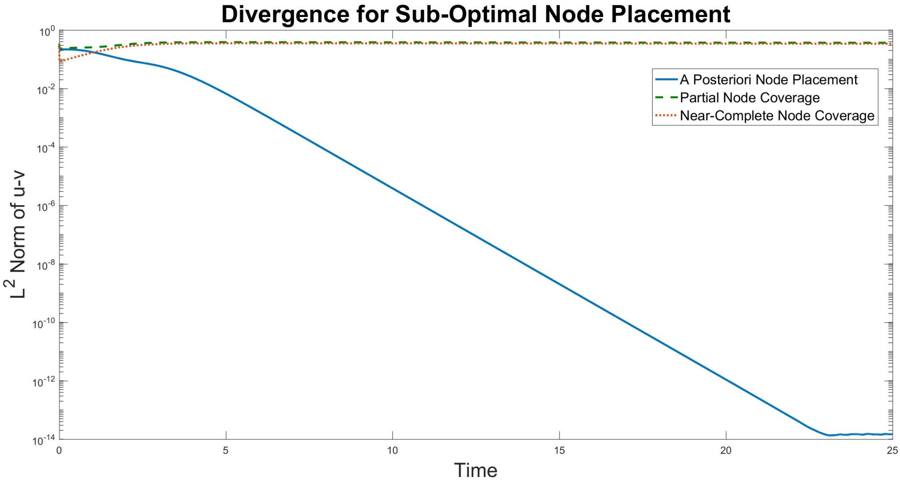
Sparse (a posteriori) node placement
Removing *one* node gives large error

Node Placement Difficulty



All but a few nodes missing:
Error is still large

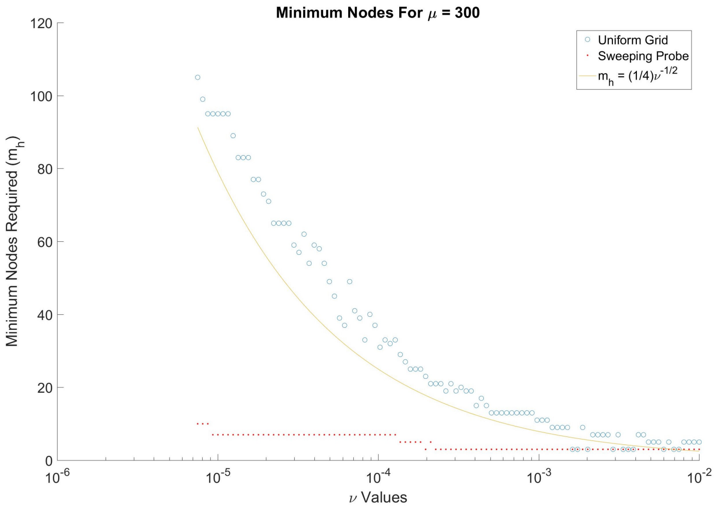
Node Placement Difficulty



Error comparison

Image: Dr. Trenton Franz with his Cosmic Ray Newton Rover.

Moving Nodes



Uniform vs. Moving Data Points

Car Animation

Simulaiton

Driving the car

Joint with: Collin Victor

Outline

2D Navier-Stokes Equations with unknown viscosity

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu_1 \triangle \boldsymbol{u} + \mathbf{f}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \end{cases}$$

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Problem

- Do we really know what ν_2 is?

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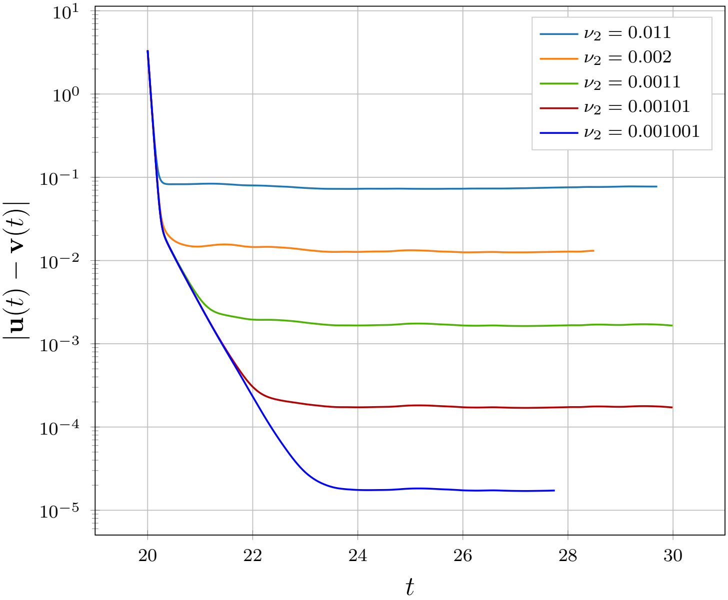
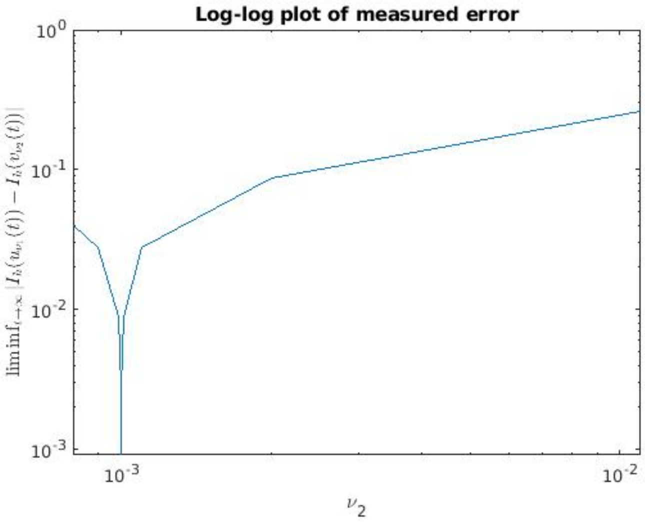
Theorem (E. Carlson, J. Hudson, A. L., 2018)(preprint)

For large enough μ (compared to ν_1) and small enough h (compared to ν_2), then $\limsup_{\to(t)\infty} \|\boldsymbol{v}(t) - \boldsymbol{u}(t)\|_{H^1} \leq C\frac{|\nu_1-\nu_2|}{\sqrt{\nu_2}}\|\boldsymbol{u}\|^2_{L^2(0,T;H^1)}$. Also sensitivity $\frac{\partial \boldsymbol{v}}{\partial \nu_2}$ is bounded in $L^\infty(0,T;L^2)$.



Error over time for different viscosities.



Large-time error for different viscosities.

An equation for the difference, $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$, $\quad I_h(\boldsymbol{w}) = I_h(\boldsymbol{u}) - I_h(\boldsymbol{v})$:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|I_h(\boldsymbol{w})|^2 + (\nu_1 - \nu_2)\left\langle I_h(A\boldsymbol{v}), I_h(\boldsymbol{w})\right\rangle &= -\mu|I_h(\boldsymbol{w})|^2 \\ &\quad + \left\langle I_h\left(\nu_1 A\boldsymbol{w} - B(\boldsymbol{w},\boldsymbol{v}) - B(\boldsymbol{u},\boldsymbol{w})\right), I_h(\boldsymbol{w})\right\rangle.\end{aligned}$$

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Idea: Drop them!

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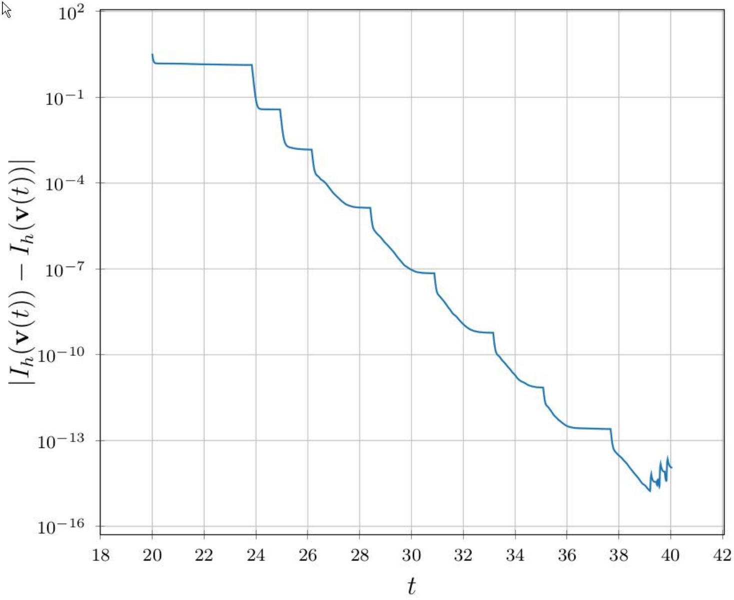
If we *do not know* ν_1, this gives us a way to update ν_1!

Algorithm

- Choose an initial guess ν^0.
- Given ν^n, run AOT data assmilation with $\nu_2 = \nu^n$ until error flatlines.
- Choose ν^{n+1} via
$$\nu^{n+1} = \nu^n - \mu \frac{|I_h(\boldsymbol{w})|}{\langle I_h(A\boldsymbol{v}), I_h(\boldsymbol{w})\rangle}$$


10^{2}
10^{-1}
10^{-4}
10^{-7}
10^{-10}
10^{-13}
10^{-16}
$|I_h(\mathbf{v}(t)) - I_h(\mathbf{v}(t))|$
18
20
22
24
26
28
30
32
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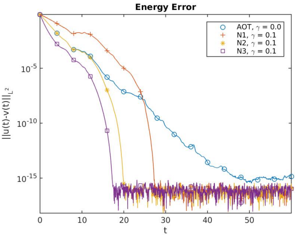
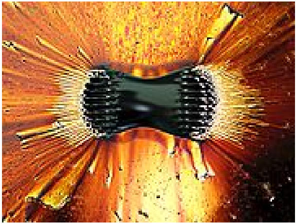
42
t

Outlook

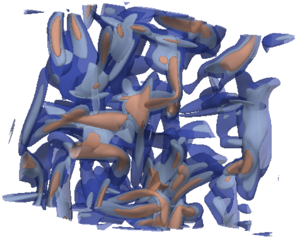
- *Proofs* that any of these silly ideas work?
- Is there any theory to decide how to choose the “best” nonlinearity $\mathcal{N}$?
-
- How does AOT data assimilation compare with, e.g., Kalman filters or 4DVAR?
- Implementation in real-world models?
  (Upcoming work on aquaplanet simulation.) (E. Carlson, A. L., Q. Hu)



1D KSE, Nonlinear



2D MHD



3D NSE

Thank you!