The Orchestra of Partial Differential Equations

Adam Larios

University of Nebraska Lincoln

7 March 2019

Landscape Seminar

Outline

1. PDEs at UNL
2. The Simplest Partial Differential Equation
3. The Diffusion Equation and Fourier Series
4. Putting it all together
5. Instability and Energy Cascades

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1. PDEs at UNL
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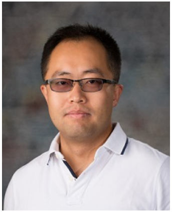
PDEs at UNL

Professors

Postdocs

PDEs at UNL

Grad Students

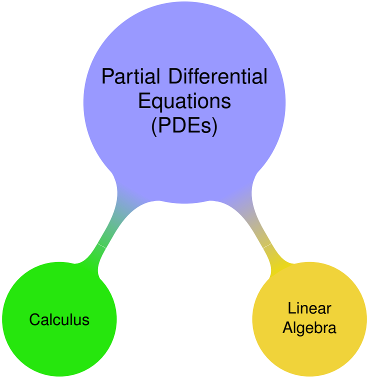


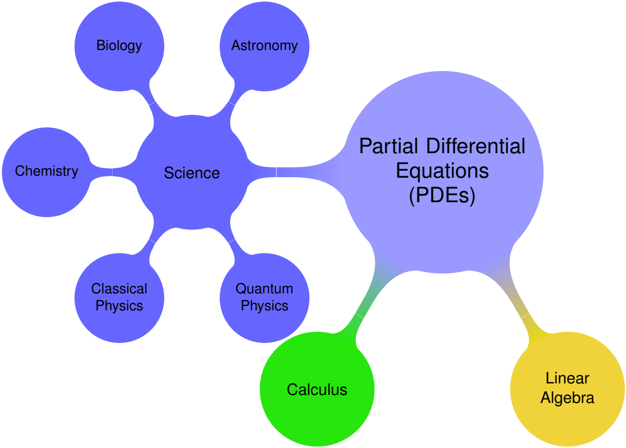





Grad Students


Partial Differential Equations (PDEs)
Calculus
Linear Algebra


Biology
Astronomy
Chemistry
Science
Partial Differential Equations (PDEs)
Classical Physics
Quantum Physics
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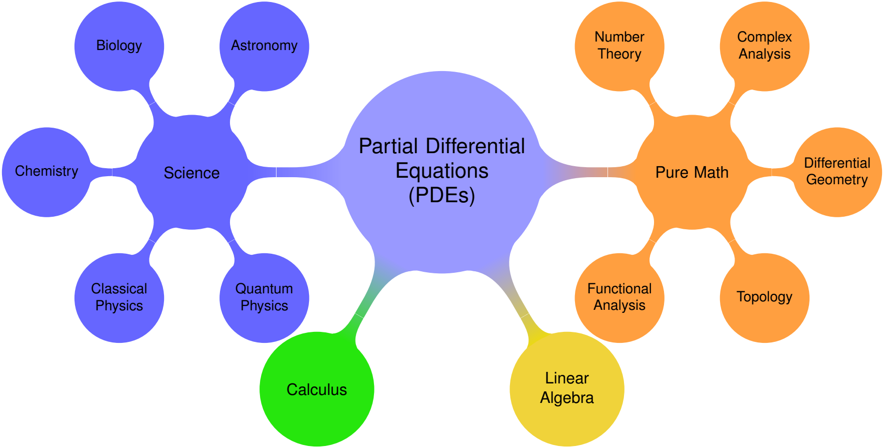
Partial Differential Equations (PDEs)
Science
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Pure Math
Number Theory
Complex Analysis
Differential Geometry
Functional Analysis
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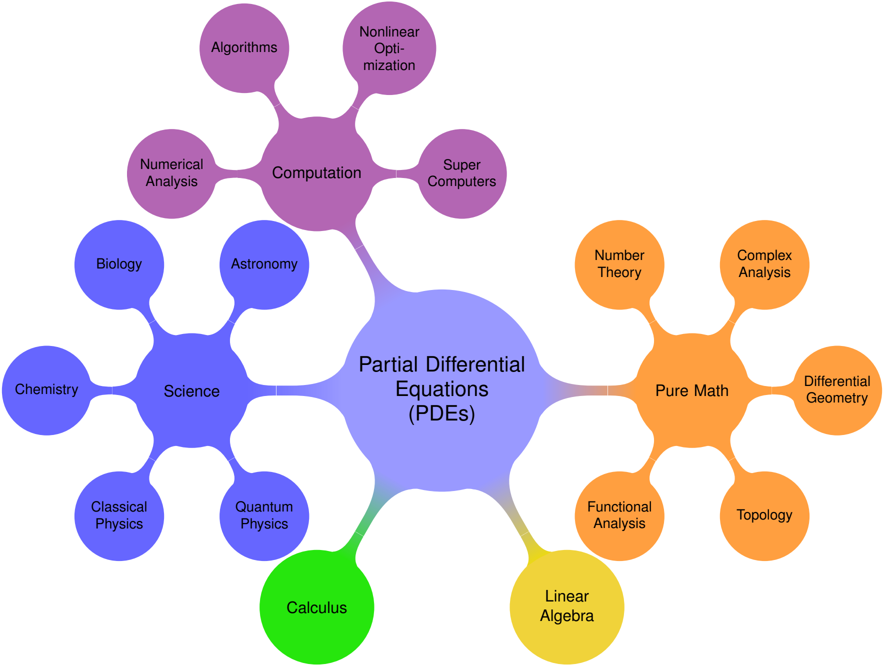
Algorithms
Nonlinear Opti- mization
Numerical Analysis
Computation
Super Computers
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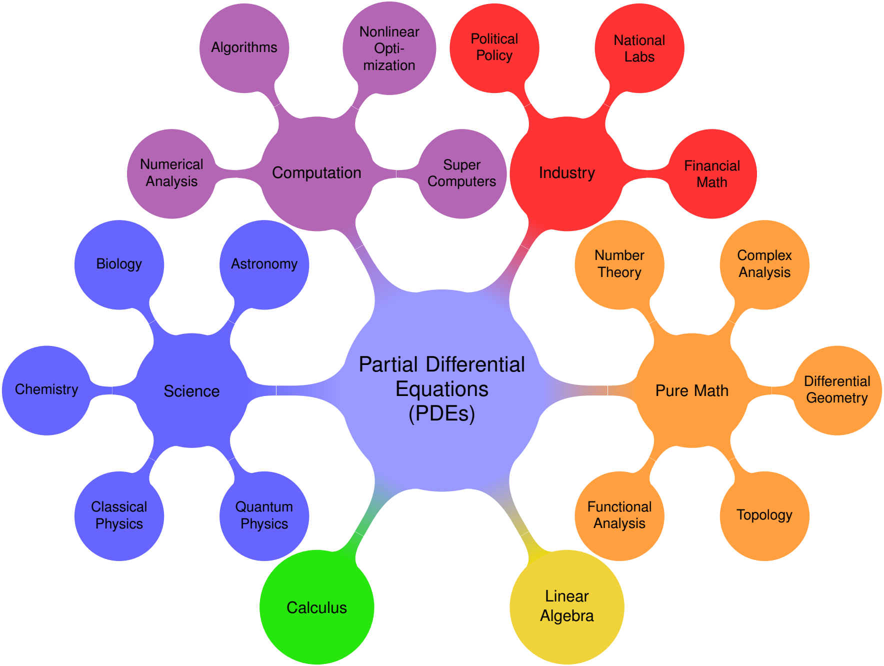
Partial Differential Equations (PDEs)
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Some Famous Linear PDEs I

Laplace equation	$-\nabla^2 u = -\sum_{i=1}^{n} u_{x_i x_i} = 0$
Helmholtz's (or eigenvalue) equation	$-\nabla^2 u = \lambda u$
Linear transport equation	$u_t + \sum_{i=1}^{n} b^i u_{x_i} = u_t + u \cdot \nabla b = 0$
Liouville's equation	$u_t + \sum_{i=1}^{n} (b^i u)_{x_i} = u_t + \nabla \cdot (bu) = 0$
Heat (or diffusion) equation	$u_t = \nu \nabla^2 u$
Schrödinger's equation	$u_t = i \nabla^2 u$
Kolmogorov's equation	$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0$
Fokker-Planck equation	$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} + \sum_{i=1}^{n} (b^i u)_{x_i} = 0$

Some Famous Linear PDEs II

Wave equation	$u_{tt} - c^2\nabla^2 u = 0$
Klein-Gordon equation	$u_{tt} - c^2\nabla^2 u + m^2 u = 0$
Telegraph equation	$u_{tt} + 2du_t - c^2 u_{xx} = 0$
General wave equation	$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0$
Linear KdV	$u_t + u_{xxx} = 0$
Beam equation	$u_t + u_{xxxx} = 0$

Some Famous Nonlinear PDEs

Eikonal equation	$|\nabla u| = 1$
Nonlinear Poisson equation	$-\nabla^2 u = f(u)$
p-Laplacian equation	$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$
Minimal surface equation	$\nabla \cdot \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}} \right)$
Monge-Ampère equation	$\det(\nabla \nabla u) = f$
Hamilton-Jacobi equation	$u_t + H(\nabla u, x) = 0$
Scalar conservation law	$u_t + \nabla \cdot (\mathbf{F}(u)) = 0$
Inviscid Burgers' equation	$u_t + u u_x = 0.$
Scalar reaction-diffusion equation	$u_t - \nabla^2 u = f(u)$
Porous medium equation	$u_t - \nabla^2 (u^\gamma) = 0$
Nonlinear wave equation	$u_{tt} - \nabla^2 u + f(u)$
Korteweg-deVries (KdV) equation	$u_t + u u_x + u_{xxx} = 0$
Nonlinear Schrödinger equation	$i u_t + \nabla^2 u f(|u|^2) u$

Systems of partial differential equations.

- Equilibrium equations of linear elasticity

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{0}$$

- Evolution equations of linear elasticity

$$\boldsymbol{u}_{tt} - \mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \mathbf{0}$$

- Maxwell's equations

$$\begin{cases} \mathbf{E}_t + \mu_0\mathbf{J} = \nabla\times\mathbf{B} \\ \mathbf{B}_t = -\nabla\times\mathbf{E} \\ \nabla\cdot\mathbf{E} = q \\ \nabla\cdot\mathbf{B} = 0 \end{cases}$$

Nonlinear Systems.

- System of conservation laws

$$\boldsymbol{u}_t + \nabla \cdot (\mathbf{F}(\boldsymbol{u})) = 0$$

- Reaction-diffusion system

$$\boldsymbol{u}_t - \nu \nabla^2 \boldsymbol{u} = \mathbf{f}(\boldsymbol{u})$$

- Euler's equations for incompressible, inviscid flow

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

- Navier-Stokes equations for incompressible, viscous flow

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \nabla^2 \boldsymbol{u} = -\nabla p + \mathbf{f} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

Modern-Day Fact: Most equations can only be solved with computers

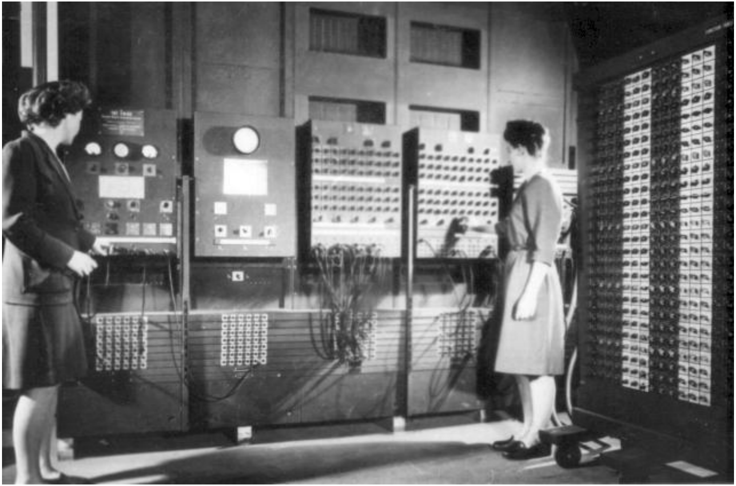
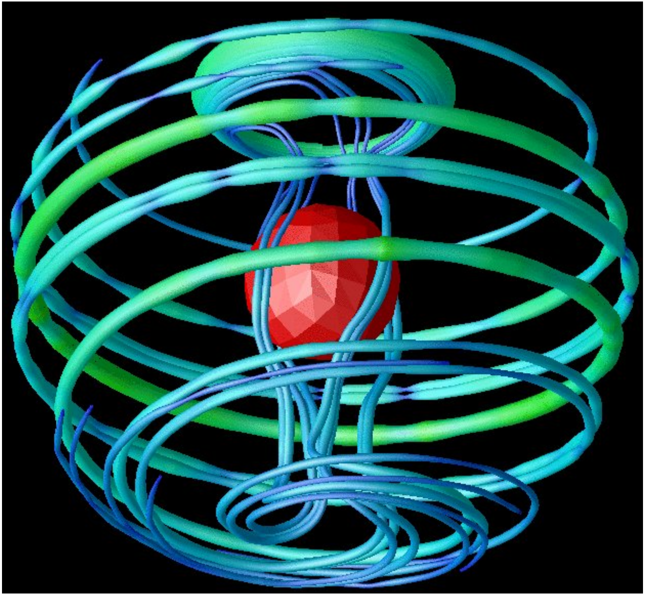


Figure: Programmers working on ENIAC, c. 1946

Numerical Solutions of PDEs



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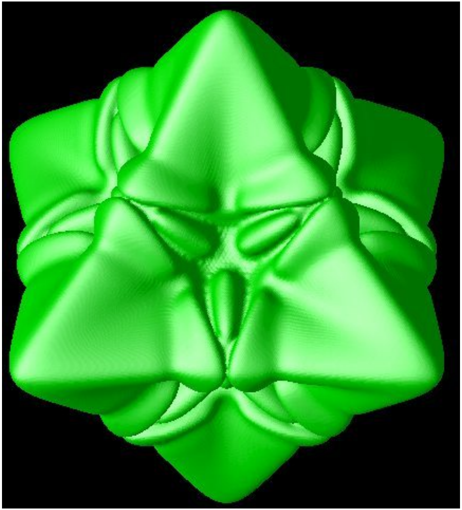

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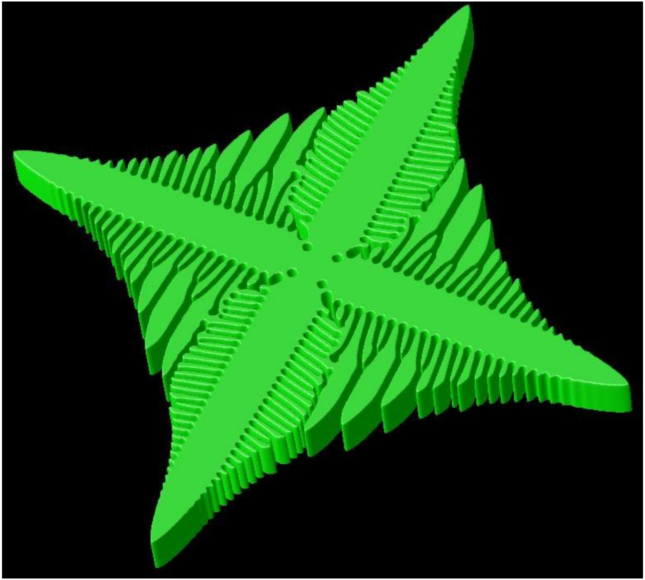

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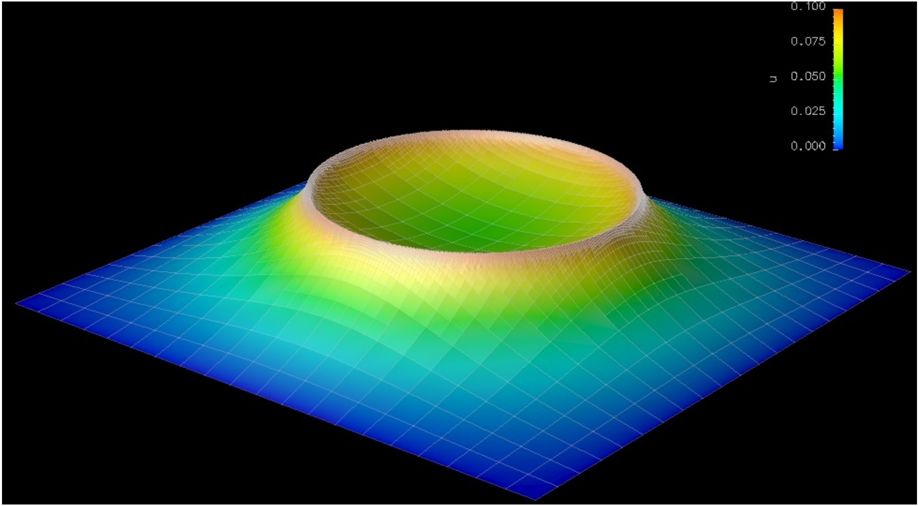

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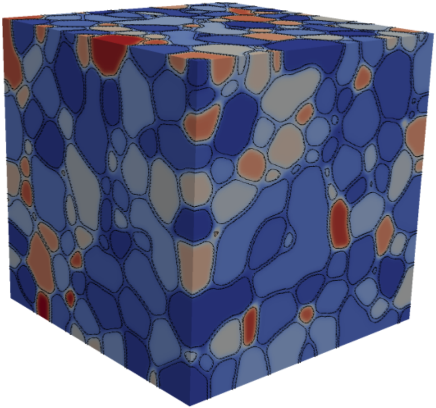

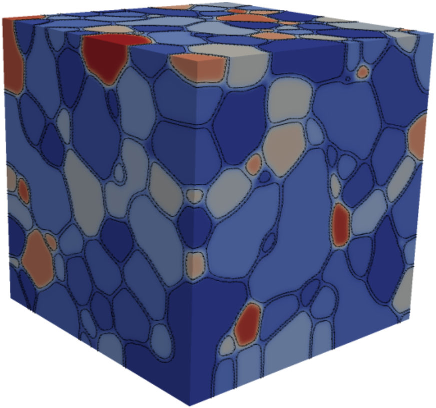

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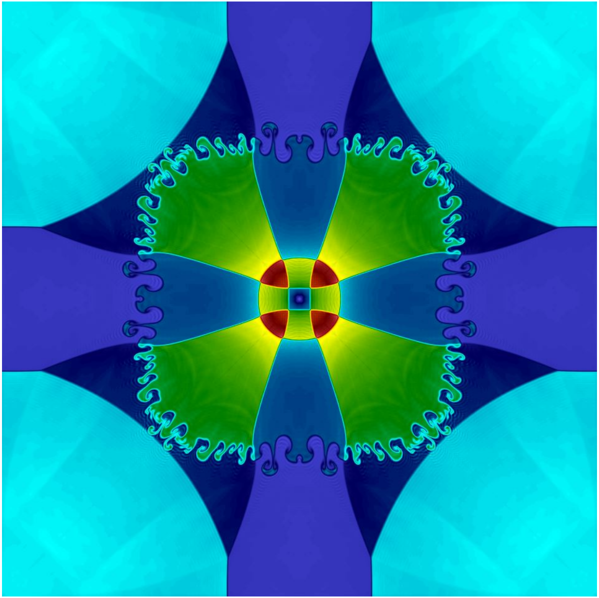

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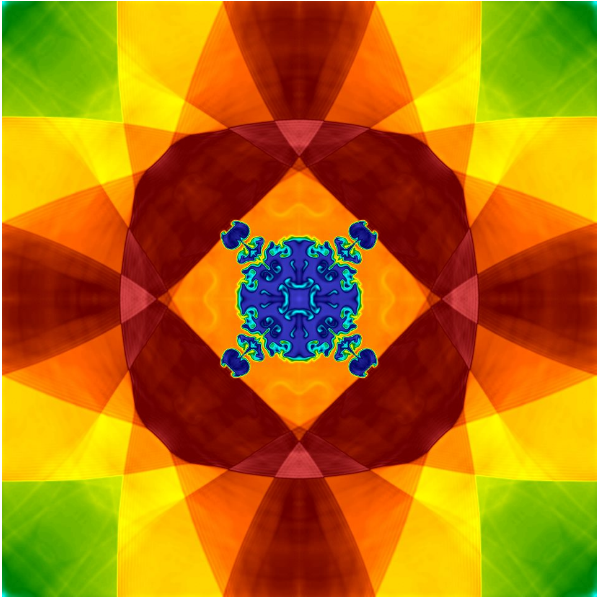
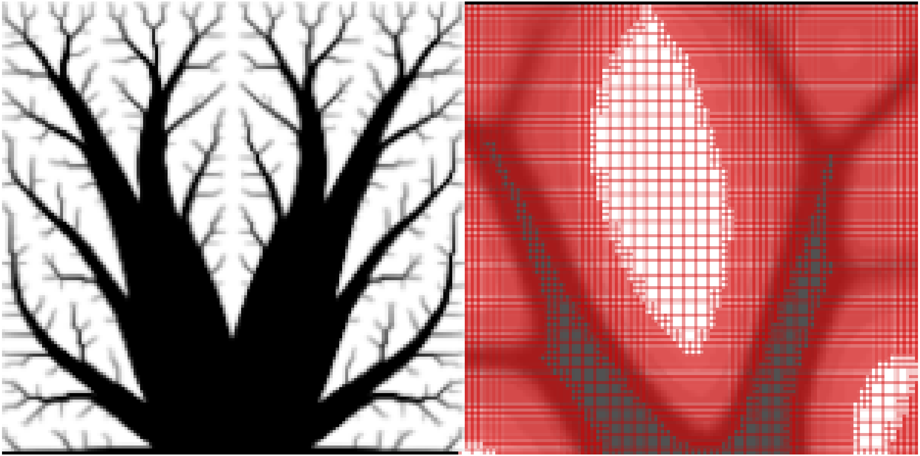


Numerical Solutions of PDEs



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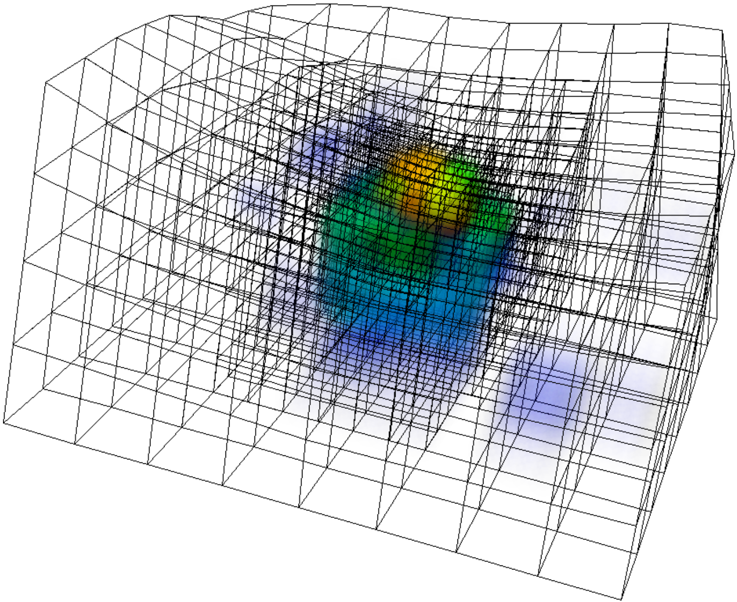

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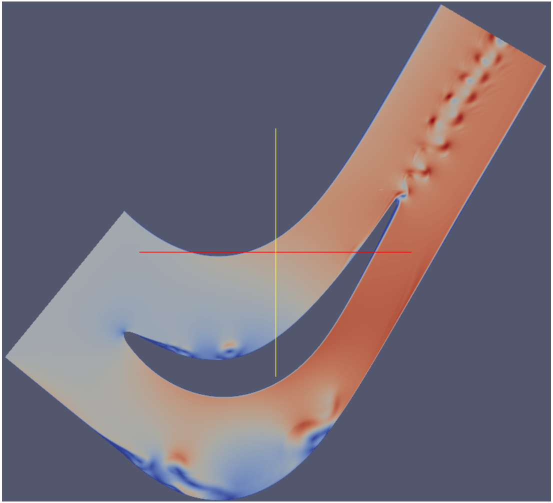

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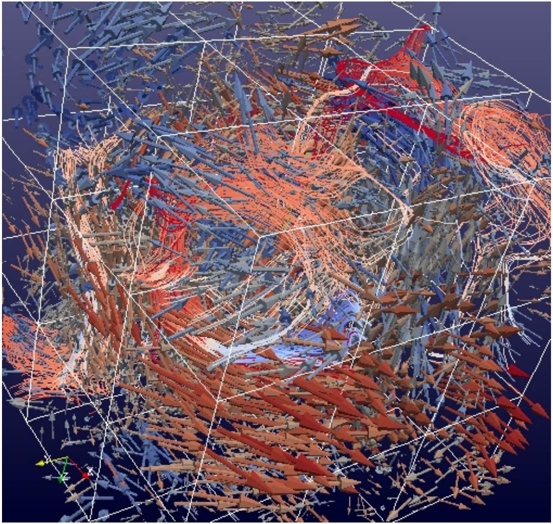

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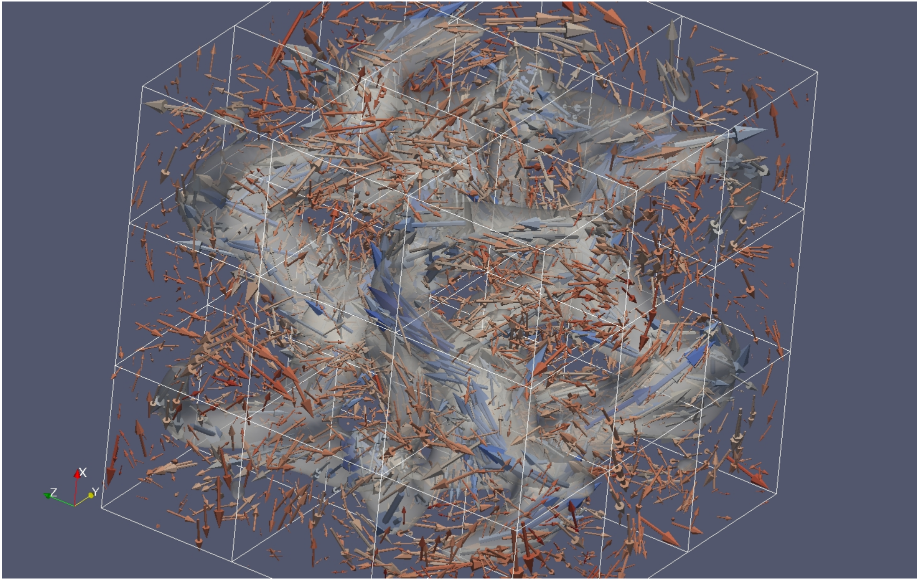

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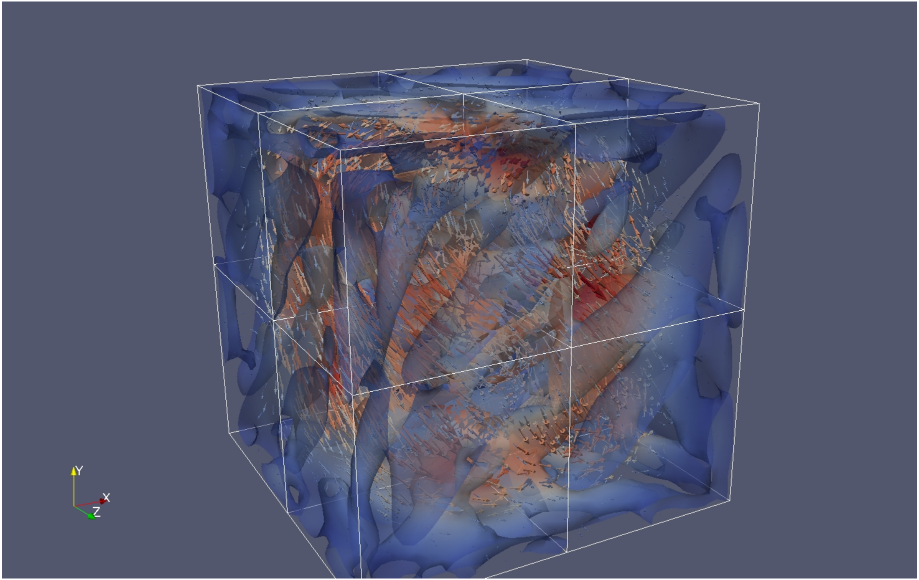


What do I study?

Turbulence: A Wide Range of Scales

Turbulence: A Wide Range of Scales

Turbulence: A Wide Range of Scales ($10^{-2}m$)

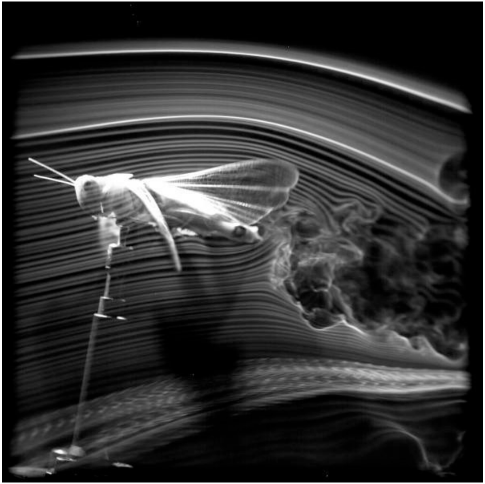
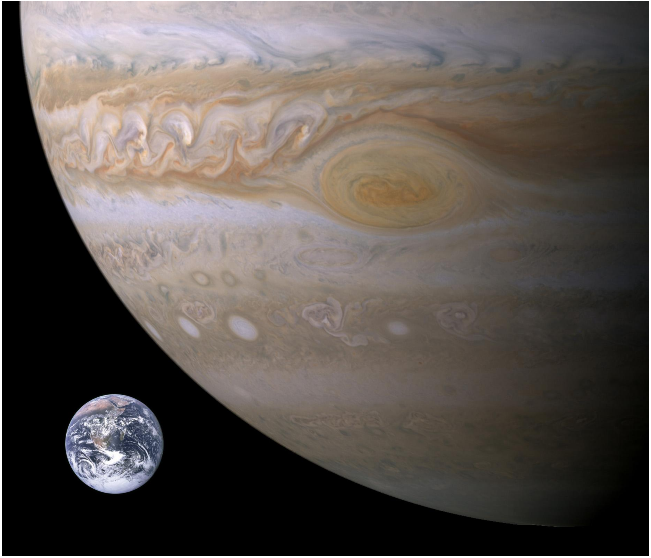
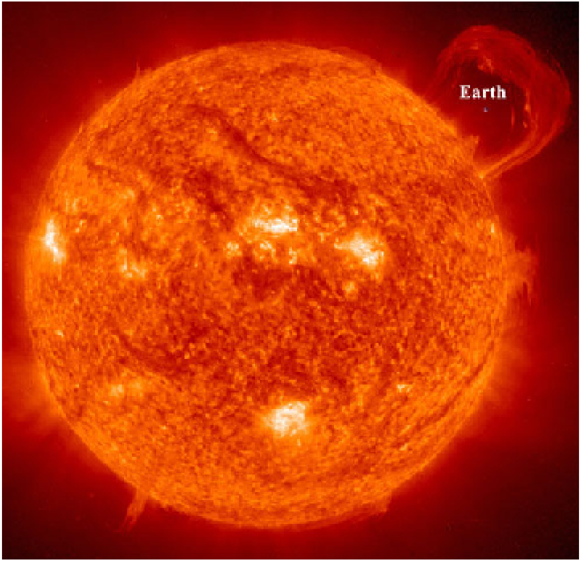


Image Credit: Animal Flight Group, Dept. of Zoology, Oxford University and Dr. John Young

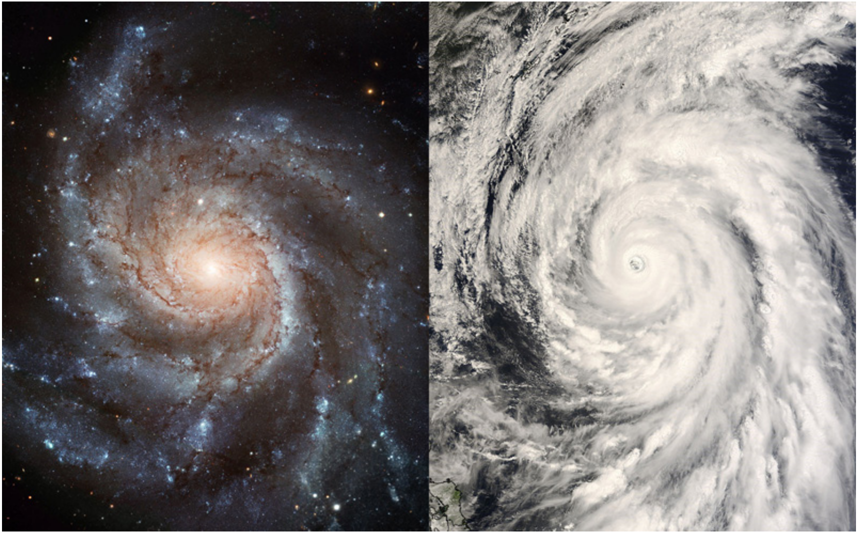
Turbulence: A Wide Range of Scales ($10^8 m$)



Turbulence: A Wide Range of Scales ($10^9 m$)



Turbulence: A Wide Range of Scales ($10^{21}m$)



The Incompressible Navier-Stokes Equations

Claude L.M.H. Navier

George G. Stokes

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{Acceleration} + \underbrace{(\vec{u} \cdot \nabla)\vec{u}}_{Advection} = \underbrace{-\nabla p}_{\substack{Pressure\\Gradient}} + \underbrace{\nu \nabla^2 \vec{u}}_{\substack{Viscous\\Diffusion}}$$

Continuity Equation (Divergence-Free Condition)

$$\nabla \cdot \vec{u} = 0$$

Unknowns

$\vec{u}$:= Velocity (vector)

p := Pressure (scalar)

Parameter

ν := Kinematic Viscosity

Problem (Leray 1933)

Existence and uniqueness of strong solutions in 3D for all time. ($1,000,000 Clay Millennium Prize Problem)

Outline

1 PDEs at UNL

2 The Simplest Partial Differential Equation

3 The Diffusion Equation and Fourier Series

4 Putting it all together

5 Instability and Energy Cascades

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} =$$

$$\text{position} = x = x(t)$$

$$\text{velocity} = v = v(t, x(t)) = \frac{dx}{dt}$$

$$\text{density} = \rho = \rho(t, x(t))$$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

position $= x = x(t)$

velocity $= v = v(t, x(t)) = \dfrac{dx}{dt}$

density $= \rho = \rho(t, x(t))$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}$$

$$= 0$$

position $= x = x(t)$

velocity $= v = v(t, x(t)) = \dfrac{dx}{dt}$

density $= \rho = \rho(t, x(t))$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} \\ &= \frac{\partial\rho}{\partial t} + \quad v\frac{\partial\rho}{\partial x} = 0\end{aligned}$$

$$\text{position} = x = x(t)$$

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The Simplest Partial Differential Equation

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Transport Equation

$$\rho_t + v\rho_x = 0$$

position $= x = x(t)$

velocity $= v = v(t, x(t)) = \dfrac{dx}{dt}$

density $= \rho = \rho(t, x(t))$

The Simplest Partial Differential Equation

$$\frac{d\rho}{dt} = 0$$

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \frac{dx}{dt}\frac{\partial\rho}{\partial x} \\ &= \frac{\partial\rho}{\partial t} + \quad v\frac{\partial\rho}{\partial x} = 0\end{aligned}$$

Transport Equation

$$\rho_t + v\rho_x = 0$$

Transport Equation in $\mathbb{R}^n$

$$\rho_t + (\vec{v}\cdot\nabla)\rho = 0$$

position $= x = x(t)$

velocity $= v = v(t, x(t)) = \dfrac{dx}{dt}$

density $= \rho = \rho(t, x(t))$

$$(\vec{v}\cdot\nabla)\rho = v_1\frac{\partial\rho}{\partial x} + v_2\frac{\partial\rho}{\partial y} + v_3\frac{\partial\rho}{\partial z}$$

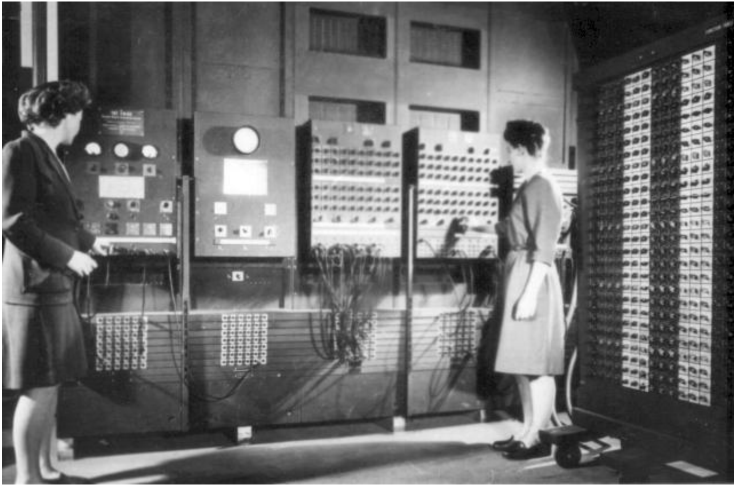

Figure: Programmers working on ENIAC, c. 1946

Idea

What about the water itself?
What if we set $\rho = v = u$?

Idea

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What if we set $\rho = v = u$?

Burgers' Equation

$$u_t + uu_x = 0$$

Burgers' Equation in $\mathbb{R}^n$

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = 0$$

Computer Time!

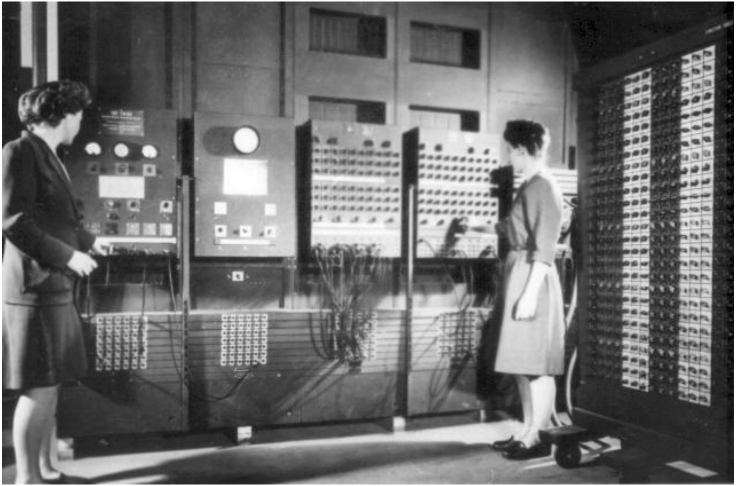

Figure: Programmers working on ENIAC, one of the first computers (c. 1946)

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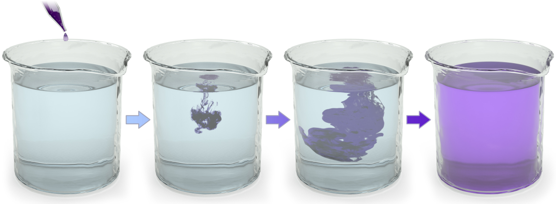
Diffusion Equation

Diffusion

Diffusion Equation



Diffusion

Concentration $= \theta = \theta(x,t)$

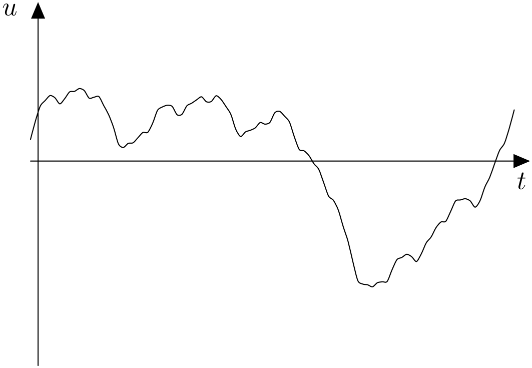
Diffusion Equation

$$\theta_t = \nu\theta_{xx}$$

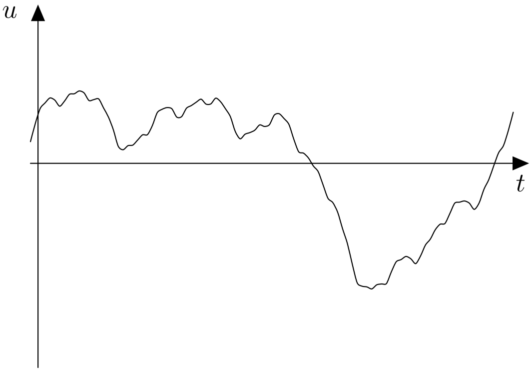
Diffusion Equation in $\mathbb{R}^3$

$$\theta_t = \nu(\theta_{xx} + \theta_{yy} + \theta_{zz}) = \nu\nabla^2\theta$$

Frequency

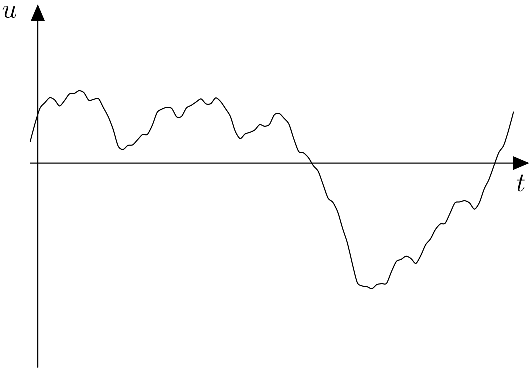


Frequency



$$u(t) = 0.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
$$+1.0\sin(1t) + 0.25\sin(4t) + 0.0625\sin(16t)$$

Frequency



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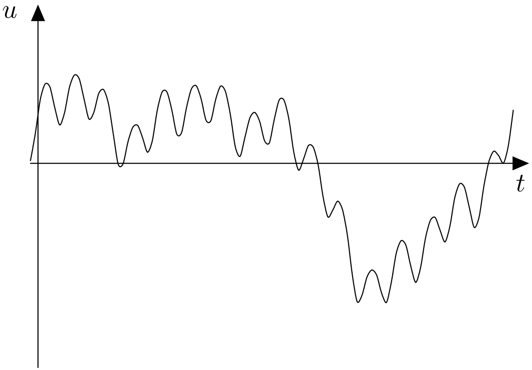

$$u(t) = 0.5\cos(2t) + 0.125\cos(8t) + 0.03125\cos(32t)$$
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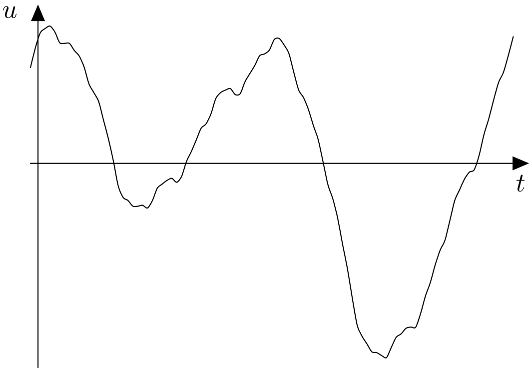
Frequency



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Fourier Series

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt) \right)$$

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$$u(t) = \sum_{k=0}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt) \right)$$

$$\boxed{\begin{aligned} e^{ikt} &= \cos(kt) + i\sin(kt) \\ \cos(kt) &= \frac{e^{ikt} + e^{-ikt}}{2} \\ \sin(kt) &= \frac{e^{ikt} - e^{-ikt}}{2i} \end{aligned}}$$

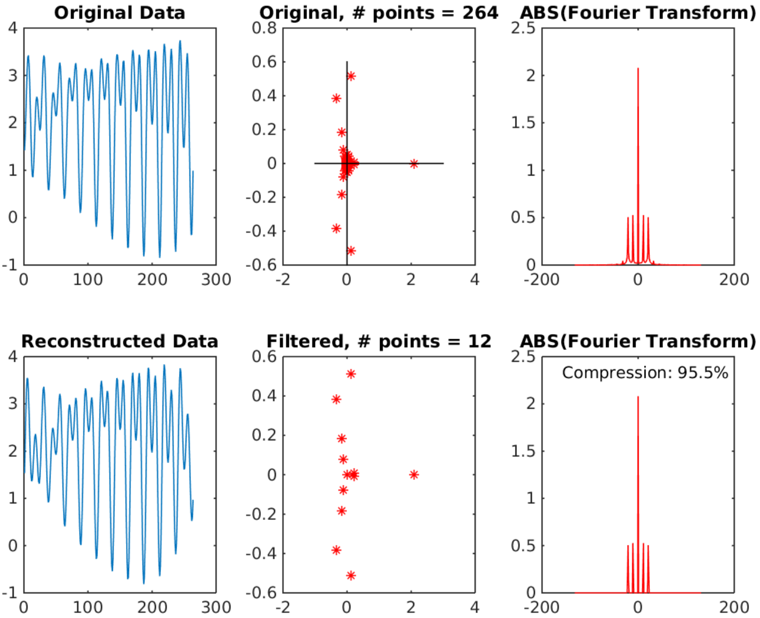
Fourier Series

$$u(t) = \sum_{k=0}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt)\right)$$

$$\boxed{\begin{aligned} e^{ikt} &= \cos(kt) + i\sin(kt) \\ \cos(kt) &= \frac{e^{ikt} + e^{-ikt}}{2} \\ \sin(kt) &= \frac{e^{ikt} - e^{-ikt}}{2i} \end{aligned}}$$

$$u(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$$

$$\begin{aligned} c_k &= \tfrac{1}{2}\left(a_k - ib_k\right), \quad k > 0, \\ c_k &= \tfrac{1}{2}\left(a_k + ib_k\right), \quad k < 0. \end{aligned}$$


Original Data
Original, # points = 264
ABS(Fourier Transform)
Reconstructed Data
Filtered, # points = 12
ABS(Fourier Transform)
Compression: 95.5%

Multi-dimensional Fourier series

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k e^{ikx}$$

$$u(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^n} \widehat{u}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

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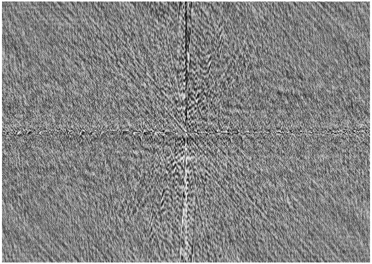
$$u(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^n} \widehat{u}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}$$

$$\widehat{u}_{\vec{k}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\vec{x})e^{-i\vec{k}\cdot\vec{x}}\,d\vec{x}$$

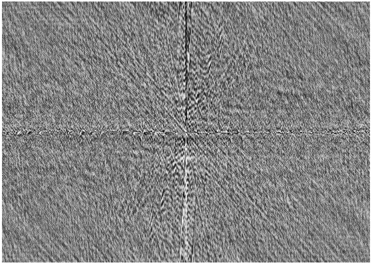
(a) Original image

(a) Original image

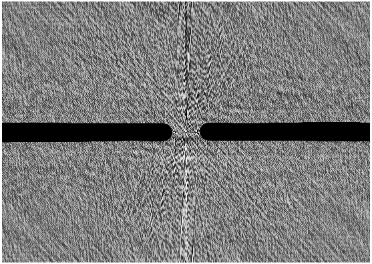


(b) Fourier transform (magnitude)

(a) Original image

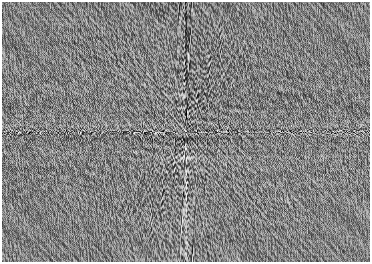


(b) Fourier transform (magnitude)



(c) Zero-out Fourier coeffcients

(a) Original image



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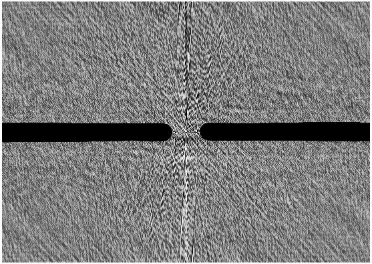

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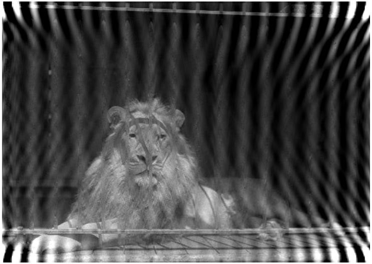

(d) Inverse Fourier transform

Derivatives

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Idea

Can the Fourier transform be used to understand differential equations?

Diffusion Equation and the Fourier Transform

$$\boxed{u_t = \nu u_{xx}}$$

$$u(x,t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t) e^{ikx}$$

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Computer Time Again!

Figure: Technician working on a Cray supercomputer. (c. 1986)

Outline

1. PDEs at UNL
2. The Simplest Partial Differential Equation
3. The Diffusion Equation and Fourier Series
4. Putting it all together
5. Instability and Energy Cascades

Burgers Equation: Well-posedness

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Take derivative $w = u_x$.

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$$w_t + w^2 + uw_x = \nu w_{xx}$$

If w is at a *maximum* . . .

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So w $(= u_x)$ is decreasing at any maximum!

Burgers Equation: Higher dimensions

n-Dimensional Burgers

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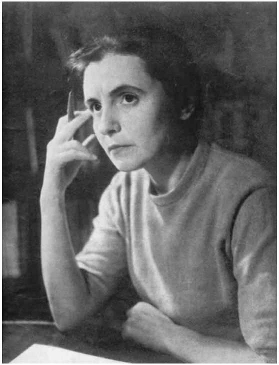
Olga Ladyzhenskaya: Maximum Principle for $\frac{1}{2}|\vec{u}|^2$.



Back to Navier-Stokes

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = \nu\nabla^2\vec{u} - \nabla p \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

When do we have no singularity?

	$\nu > 0$	$\nu = 0$
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Navier-Stokes 2D		
Navier-Stokes 3D		

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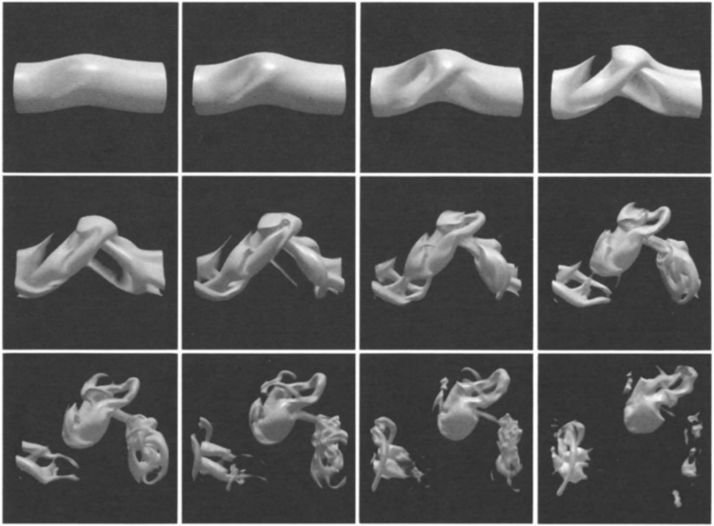

Image Credit: J. Bell, D. Marcus, *Comm. Math. Phys.* **147**, 371-394 (1992).

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Nonlinear Equations

Burgers Equation [Shock Waves, Traffic]

$$u_t + uu_x = \nu u_{xx}$$

Navier-Stokes Equations [Incompressible Fluids]

$$\begin{cases} \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = \nu \nabla^2 \vec{u} - \nabla p \\ \nabla \cdot \vec{u} = 0 \end{cases}$$

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Computer Time Once More! (Kuramoto-Sivashinsky)

Figure: Hopper Cray XE6 at NERSC, named after American computer scientist Dr. Grace Hopper, 1906-1992.

The Incompressible Navier-Stokes Equations

Claude L.M.H. Navier

George G. Stokes

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{Acceleration} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{Advection} = \underbrace{-\nabla p}_{\substack{Pressure\\Gradient}} + \underbrace{\nu\nabla^2\vec{u}}_{\substack{Viscous\\Diffusion}}$$

Continuity Equation (Divergence-Free Condition)

$$\nabla\cdot\vec{u} = 0$$

Unknowns
$\vec{u}$:= Velocity (vector)
p := Pressure (scalar)

Parameter
ν := Kinematic Viscosity

Thank you!