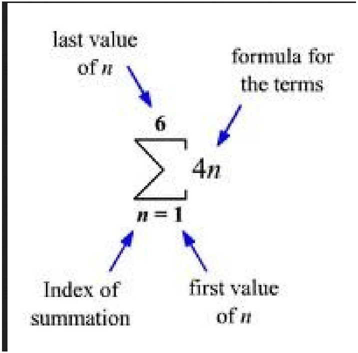
# Calculus 1
# The Definite Integral

# Sigma Notation: Introduction

Notice that in the last section, we added together several very similar terms (area of rectangles) when we used the LHS and RHS. To write these out again and again can be very tedious and time consuming. Since we are mathematicians, we will develop a more efficient way to write these sums. Some of you may already know this idea, **summation notation**. It uses the capital Greek letter for sum, sigma. Let's see how it works.

The capital Sigma has two rules that make it work.

1. The index counter always moves in the positive direction and it always increases by 1 unit. Note here "$n$" never goes down in value and always increases by 1 unit for the next term.
2. Each term found by a value of "$n$" is added to the previous term that was calculated by "$n - 1$".



## Examples

Write out the sum of the first 100 integers using sigma notation.

$$1 + 2 + 3 + \cdots + 100 = \sum_{n=1}^{100} n$$

# Examples

Write out the sum of the first 100 integers using sigma notation.

$$1 + 2 + 3 + \cdots + 100 = \sum_{n=1}^{100} n$$

Write out the sum of the squares integers between 17 and 1000 using sigma notation.

$$17^2 + 18^2 + 19^2 + \cdots + 1000^2 = \sum_{i=17}^{100} i^2$$

## Clicker Question

Use Sigma notation to write the sum $2 + 4 + 6 + 8$.

(a) $\sum_{i=2}^{8} i$
(b) $\sum_{i=1}^{4} i$
(c) $\sum_{i=1}^{4} 2i$
(d) $\sum_{i=4}^{1} 2i$

# Clicker Question

Use Sigma notation to write the sum $1^2 + 3^2 + 5^2 + 7^2$.

(a) $\sum_{i=1}^{7} i^2$
(b) $\sum_{i=1}^{4} (2i+1)^2$
(c) $\sum_{i=1}^{4} (2i)^2$
(d) $\sum_{i=1}^{7} (2i)^2$
(e) $\sum_{i=0}^{3} (2i+1)^2$

# Here are two more examples of sigma notation.

Starting at a negative index.

$$\sum_{i=-2}^{4} i^2 = (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 + 4^2$$

Fractional indices (rarely used, but possible).

$$\sum_{i=1/2}^{7/2} (3i+1) = \left[3\left(\tfrac{1}{2}+1\right)\right] + \left[3\left(\tfrac{3}{2}+1\right)\right] + \left[3\left(\tfrac{5}{2}+1\right)\right] + \left[3\left(\tfrac{7}{2}+1\right)\right] = 12$$

It is better to write this as

$$\sum_{i=1}^{3} (3\tfrac{2i+1}{2} + 1)$$

# Sigma notation for approximating area under curves

Suppose $f(t)$ is a continuous function for $a \leq t \leq b$. We divide the interval from $a$ to $b$ into $n$ equal subdivisions, and we call the width of an individual subdivision $\Delta t$, so

$$\Delta t = \frac{b-a}{n}$$

Let $t_0$, $t_1$, $t_2$, . . . , tn be endpoints of the subdivisions. Both the left-hand and right-hand sums can be written more compactly using sigma, or summation, notation. The symbol $\sum$ is a capital sigma, or Greek letter "S." We write

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_n)\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$$

The $\sum$ tells us to add terms of the form $f(t_i)\Delta t$. The "$i = 1$" at the base of the sigma sign tells us to start at $i = 1$, and the "$n$" at the top tells us to stop at $i = n$. In the left-hand sum we start at $i = 0$ and stop at

# Clicker Question

In the figures below, as the number of rectangles (of equal width) goes to infinity which of the following statements is true? Consider only continuous curves. Note that the shaded regions are the short or long amounts.

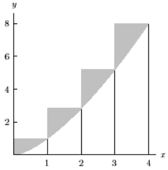



Figure 5.4: $A1 = Shaded\ Region$

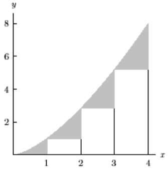



Figure 5.5: $A2 = Shaded\ Region$

(a) A1 goes to zero but A2 does not. The lower rectangular area does not equal the upper rectangular area.

(b) A2 goes to zero but A1 does not. The lower rectangular area does not equal the upper rectangular area.

(c) A2 and A1 do not go to zero. The upper and lower rectangular areas are equal.

(d) A1 and A2 both go to zero. The upper and lower rectangular areas are equal.

(e) A1 and A2 both go to zero. The upper and lower rectangular areas are not equal.

# Answer

(d)

**Comment**

Note that all continuous curves have this property. Explore this idea by drawing arbitrary continuous curves!

# Clicker Question

For the following question consider continuous curves that are increasing and concave up. As the number of rectangles (of equal width) triples in Figures 5.4 and 5.5, the shaded areas represented by A1 and A2 decrease.

1. True
2. False
3. It depends on the graph.

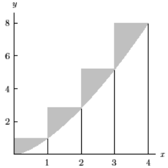



Figure 5.4: $A1 = \textit{Shaded Region}$

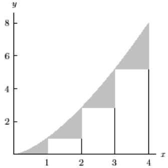



Figure 5.5: $A2 = \textit{Shaded Region}$

## Answer

(a). In both graphs, taking more rectangles of equal width will decrease the values of A1 and A2, with the area of the rectangles in both figures becoming closer to the area under the curve.

**Exercise**

What happens if we do not assume that the function is increasing and concave up?

# The Definite Integral

Suppose $f$ is continuous for $a \leq t \leq b$. The definite integral of $f$ from $a$ to $b$, written

$$\int_a^b f(t)\,dt$$

is the limit of the left-hand or right-hand sums with $n$ subdivisions of $a \leq t \leq b$ as $n$ gets arbitrarily large. In other words,

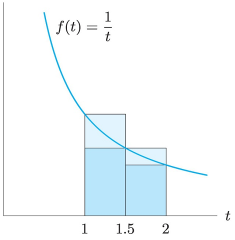
$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Left-hand sum}) = \lim_{n\to\infty} \left( \sum_{i=0}^{n-1} f(t_i)\Delta t \right)$$

and

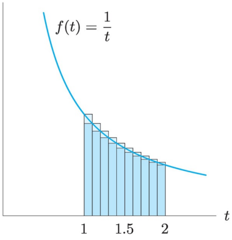
$$\int_a^b f(t)\,dt = \lim_{n\to\infty} (\text{Right-hand sum}) = \lim_{n\to\infty} \left( \sum_{i=1}^{n} f(t_i)\Delta t \right)$$

Each of these sums is called a *Riemann sum*, $f$ is called the *integrand*, and $a$ and $b$ are called the *limits of integration*.

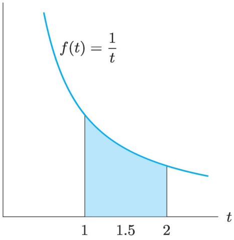
# Computing a Definite Integral




Approximating $\int_1^2 \frac{1}{t}\,dt$ with $n = 2$.




Approximating $\int_1^2 \frac{1}{t}\,dt$ with $n = 10$.




Shaded area is the exact value of $\int_1^2 \frac{1}{t}\,dt$.

When $n = 250$, a calculator or computer gives $0.6921 \leq \int_1^2 \frac{1}{t}\,dt \leq 0.6941$.
So, to two decimal places, we can say $\int_1^2 \frac{1}{t}\,dt \approx 0.69$.
The exact value is known to be $\int_1^2 \frac{1}{t}\,dt = \ln 2 = 0.693147\ldots$.

# Clicker Question

Below is the graph of $y = x^2$. The right-hand sum for eight equal divisions is given by...

(a) $0.5^2 + 1^2 + 1.5^2 + 2^2 + 2.5^2 + 3^2 + 3.5^2 + 4^2$

(b) $0.5(0.5) + 1(0.5) + 1.5(0.5) + 2(0.5) + 2.5(0.5) + 3(0.5) + 3.5(0.5) + 4(0.5)$

(c) $0.5^2(0.5) + 1^2(0.5) + 1.5^2(0.5) + 2^2(0.5) + 2.5^2(0.5) + 3^2(0.5) + 3.5^2(0.5) + 4^2(0.5)$

(d) $0^2(0.5) + 0.5^2(0.5) + 1^2(0.5) + 1.5^2(0.5) + 2^2(0.5) + 2.5^2(0.5) + 3^2(0.5) + 3.5^2(0.5)$

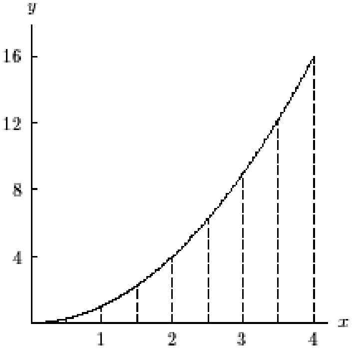


Figure 5.3

# Answer

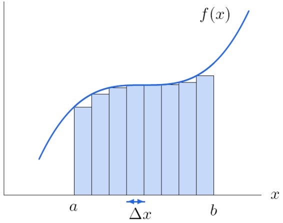
(c). The width of each rectangle is 0.5, and the x values are squared.

**Exercise**

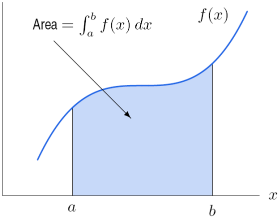
What is the left-hand sum? Are these sums under estimates or over estimates? Why?

# The Definite Integral as an Area

When $f(x)$ is positive and $a < b$,
The area under the graph of $f$ between $a$ and $b$ is $\int_a^b f(x)\,dx$.




Area of rectangles approximating the area under the curve.




Shaded area is the define integral $\int_a^b f(x)\,dx$.

## Clicker Question

What is represented by $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$.
This integral is asking us to find the area under the top half of a circle of radius 2 centered at the origin and the $x$-axis.

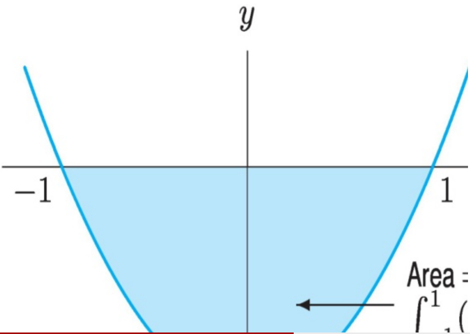
1. $\pi$
2. $2\pi$
3. $3\pi$
4. $4\pi$
5. Pumpkin Pie

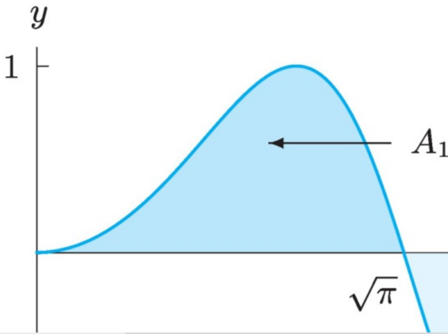
# Signed Area

When $f(x)$ is positive for some $x$ values and negative for others, and $a < b$,

$$\int_a^b f(x)\,dx$$

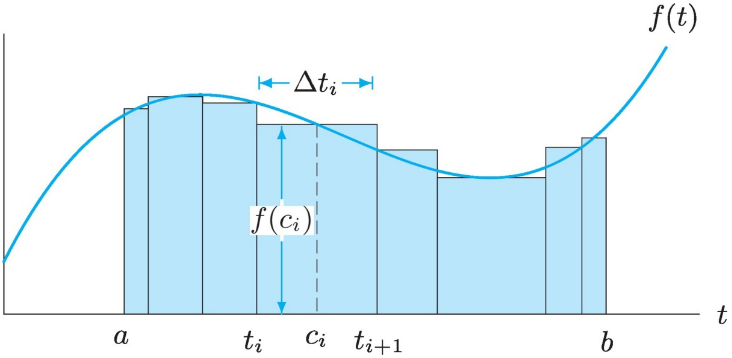
is the sum of areas above the $x$-axis, counted positively, and areas below the $x$-axis counted negatively.



# More General Riemann Sums

A general Riemann sum for f on the interval [a, b] is a sum of the form where $a = t_0 < t_1 < \; < t_n = b$, and, for $i = 1, ..., n$, $\Delta t_i = t_i - ti - 1$, and $t_{i-1} c_i t_i$




A general Riemann sum approximating $\int_a^b f(t)\,dt$.