Math 107: Calculus II

Section 6.2: Constructing Antiderivatives Analytically

University of Nebraska-Lincoln

Finding Antiderivatives

If $F'(x) = 0$ on an interval, then $F(x) = C$ on this interval, for some constant C.
Why is this true? If the derivative of a constant is 0, then the antiderivative of 0 is a constant.
We write the general antiderivative as an indefinite integral. If

$$F'(x) = f(x) \quad \rightarrow \quad \int f(x)\,dx = F(x) + C.$$

If F and G are both antiderivatives of f on an interval, then

$$G(x) - F(x) = C.$$

Theorem Properties of Antiderivatives: Sums and Constant Multiples.
In indefinite integral notation,

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$$

$$\int cf(x)dx = c \int f(x)dx$$

Antiderivatives for powers of x

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Also,

$$\int x dx = \frac{x^2}{2} + C$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int x^3 dx = \frac{x^4}{4} + C$$

So

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \neq -1.$$

Antiderivatives of other functions

Since $(e^x)' = e^x$ we have $\int e^x \, dx = e^x + C$

Since $(\sin x)' = \cos x$ we have $\int \cos x dx = \sin x + C$

Since $(\cos x)' = -\sin x$ we have $\int \sin x dx = -\cos x + C$

For $x > 0$, since $(\ln x)' = \frac{1}{x}$ we have $\int \frac{1}{x} dx = \ln x + C$.

For $x < 0$, since $(\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$ we have $\int \frac{1}{x} dx = \ln(-x) + C$.

Since $(\tan x)' = \sec^2 x$ we have $\int \sec^2(x) dx = \tan x + C$

Clicker Question

What is $\int x^2\,dx$?

A. $2x + C$
B. $2x$
C. $\dfrac{x^3}{3} + C$
D. $\dfrac{x^3}{3}$
E. $x^3 + C$.

Using Antiderivatives to Compute Definite Integrals

Compute the following integral by using the Fundamental Theorem of Calculus (FTC):

$$\int_0^1 5x^3 + \sin x \, dx.$$

To use the FTC we need an antiderivative F of $5x^3 + \sin x$; in other words, a function F such that $F'(x) = 5x^3 + \sin x$. Note that by the linearity theorem stated earlier, we have (as one choice) $F(x) = 5\frac{x^4}{4} - \cos x$. By FTC we have

$$\int_0^1 5x^3 + \sin x \, dx = (5\frac{x^4}{4} - \cos x)\Big|_0^1 = \frac{5}{4} - \cos 1 - (0 - \cos 0) = \frac{9}{4} - \cos 1.$$

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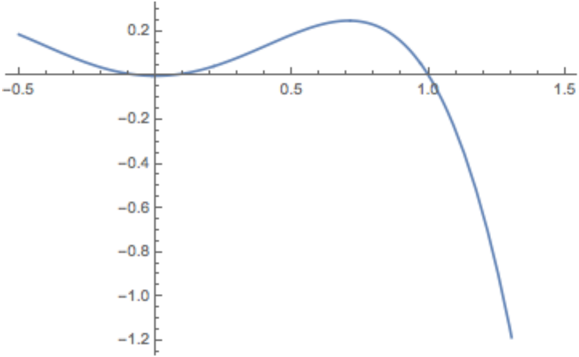
Remark. Note that we could have chosen $F(x) = 5\dfrac{x^4}{4} - \cos x - 5$, or $F(x) = 5\dfrac{x^4}{4} - \cos x + C$, for any C as the constants would have canceled when applying the FTC.

Example

Find the exact area enclosed by the curve $x^2(1-x)$ and the x-axis.



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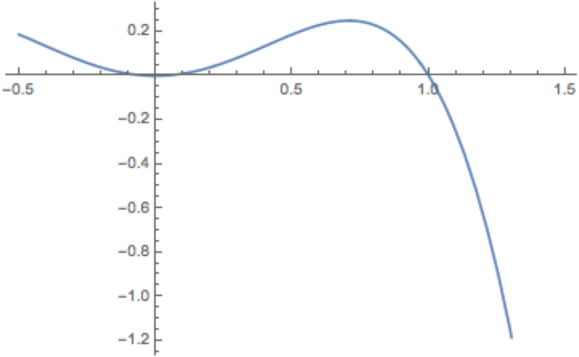
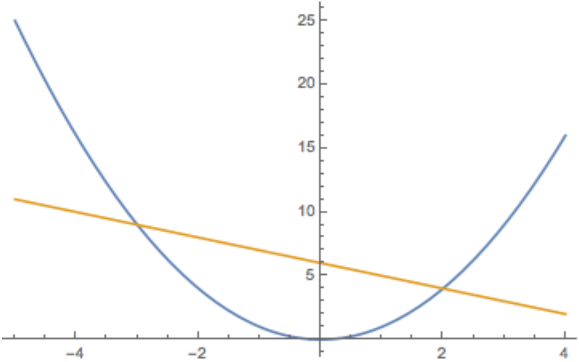


Answer: $\int_0^1 x^2(1-x), dx = \frac{1}{12} \approx 0.08333\ldots$

Another example

Find the exact area enclosed by the curves $y = x^2$ and $y = 6 - x$.



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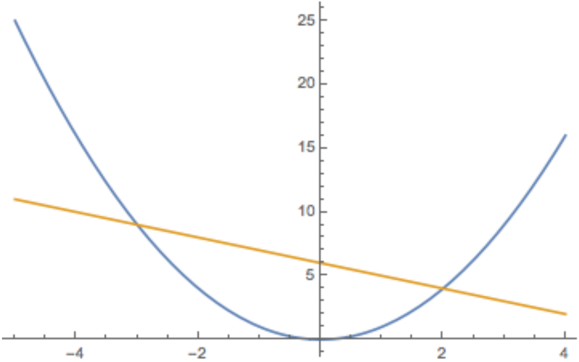

Answer: $\int_{-3}^{2} ((6 - x) - (x^2))\, dx = \frac{125}{6} \approx 20.8333\ldots$