Calculus 1
Parametric Equations (page 134)

Why Parametric?

Up to this point, you have been representing a graph by a single equation involving two variables such as x and y. However, some items are not really well described by such equations.

In this section, you will study situations in which it is useful to introduce a **third** *variable* to represent a curve in the plane.

Consider the path of an object that is propelled into the air at an angle of 45°. In a physics class you might learn that this object would follow a parabolic path. That is

$$y = -\frac{x^2}{72} + x$$

However, this equation does not tell the whole story. Although it does tell you where the object has been, it does not tell you **when** the object was at a given point (x, y) on the path.
To determine this time, you can introduce a third variable t, called a parameter. It is possible to write both x and y as functions of t to obtain the parametric equations

$$x = 24\sqrt{2}t$$
$$y = -16t^2 + 24\sqrt{2}t$$

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$x = 24\sqrt{2}t$ The Parametric Equation for x

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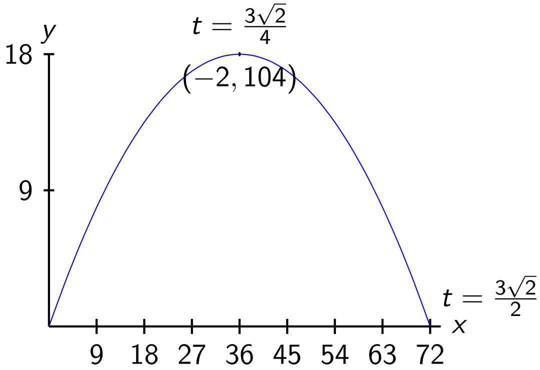

From this set of equations you can determine that at time $t = 0$, the object is at the point $(0, 0)$.
Similarly, at time $t = 1$, the object is at the point

$$(24\sqrt{2}, 24\sqrt{2} - 16)$$

and so on.

Clicker

$$
\begin{aligned}
y &= -\frac{x^2}{72} + x \\
x &= 24\sqrt{2}t \text{ The Parametric Equation for } x \\
y &= -16t^2 + 24\sqrt{2}t \text{ The Parametric Equation for } x
\end{aligned}
$$

After $t = 3$ seconds, what is the position of the object If the origin represents the initial position?

a) $\left(3, \frac{23}{8}\right)$
b) $\left(\frac{23}{8}, 3\right)$
c) $\left(72\sqrt{2}, 72\sqrt{2} - 144\right)$
d) $\left(72\sqrt{2} - 144, 72\sqrt{2}\right)$

Another Example

Sketch the curve given by the parametric equations

$$x = t^2 - 4$$
$$y = t/2$$

on the domain $-2 \leq t \leq 3$.

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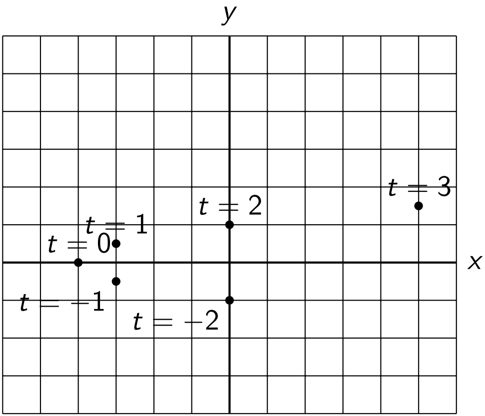
on the domain $-2 \le t \le 3$. Using values of t in the interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2

Why Parametric?

By plotting these points in the order of increasing t, you obtain the curve shown



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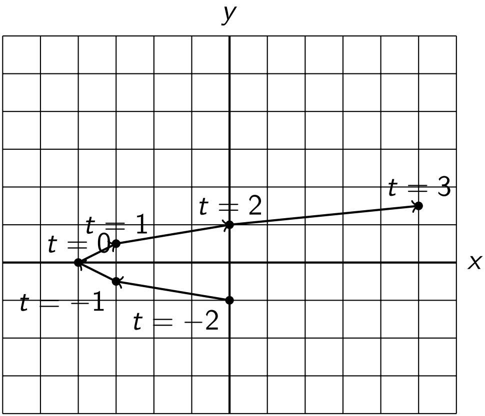

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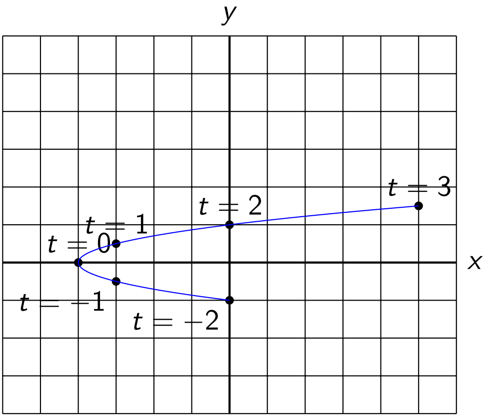

Clicker

Match the given parameterizations with the appropriate geometric description of the graph. Select the answer that matches I on your clicker:

a) $x = \sin(t)$ $y = -t$

b) $x = |t|$ $y = t$

c) $x = \sin(t)$ $y = \cos(t)$

d) $x = 3t^3$ $y = t^3$

e) $x = |3t|$ $y = 3t$

I) Line

II) Circle

III) Right Angle

IV) Acute Angle

V) Wave

Describe the motion of the particle whose coordinates at time t are expressed with the parametric equations: $x = \cos t$ and $y = \sin t$.

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Solution:

Since $(\cos t)^2 + (\sin t)^2 = 1$, we have $x^2 + y^2 = 1$. That is, at any time t the particle is at a point (x, y) on the unit circle $x^2 + y^2 = 1$. We plot points at different times to see how the particle moves on the circle. The particle completes one full trip counterclockwise around the circle every 2π units of time. Notice how the x-coordinate goes back and forth from -1 to 1 while the y-coordinate goes up and down from -1 to 1. The two motions combine to trace out a circle.

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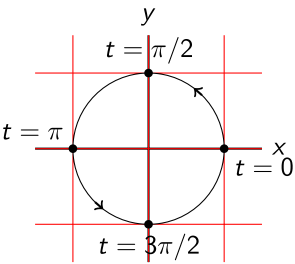

t	0	$\pi/2$	π	$3\pi/2$	2π
x	1	0	-1	0	1
y	0	1	0	-1	0

Representing Motion in the Plane

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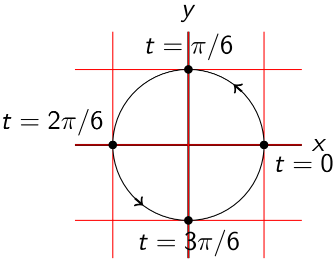
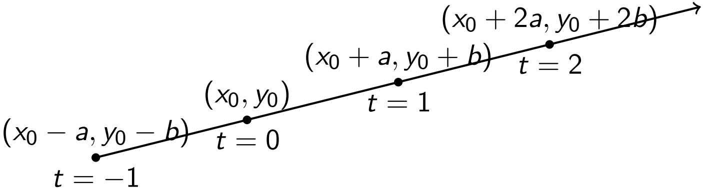
Solution: Since $(\cos 3t)^2 + (\sin 3t)^2 = 1$, we have $x^2 + y^2 = 1$. so we still have a circle. However, now the particle is traveling three times as fast!



t	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$
x	1	0	-1	0	1
y	0	1	0	-1	0

Motion in a Straight Line

An object moves with constant speed along a straight line through the point (x_0, y_0). Both the x and y-coordinates have a constant rate of change. Let $a = dx/dt$ and $b = dy/dt$. Then at time t the object has coordinates $x = x_0 + at$, $y = y_0 + bt$. Notice that a represents the change in x in one unit of time, and b represents the change in y. Thus the line has slope $m = b/a$.



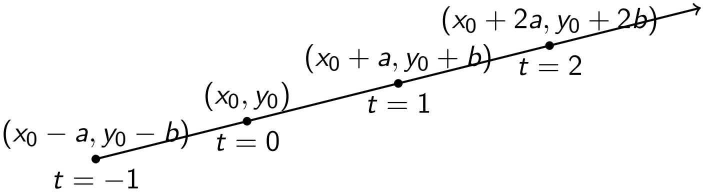
Motion in a Straight Line

Parametric Equations for the tangent line

An object moving along a line through the point (x_0, y_0), with $dx/dt = a$ and $dy/dt = b$, has parametric equations

$$x = x_0 + at,\ y = y_0 + bt.$$

The slope of the line is $m = b/a$.



Example

Find the tangent line at the point $(1, 2)$ to the curve defined by the parametric equations $x = t^3$, $y = 2t$.

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Solution:

At time $t = 1$ the particle is at the point $(1, 2)$. The velocity in the x-direction at time t is $v_x = dx/dt = 3t^2$, and the velocity in the y-direction is $v_y = dy/dt = 2$. So at $t = 1$ the velocity in the x-direction is 3 and the velocity in the y-direction is 2. Thus the tangent line has parametric equations

$$x = 1 + 3t, \ y = 2 + 2t.$$

Clicker

The equation of the line tangent to the curve $x = \sin 3t$, $y = \sin t + 1$ at the point where $t = 0$ is

a) $y = 3x$

b) $y = 3x + 1$

c) $y = (1/3)x$

d) $y = (1/3)x + 1$

e) None of the above

Speed and Velocity

Instantaneous Speed

The instantaneous speed of a moving object is defined to be

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The quantity $v_x = dx/dt$ is the instantaneous velocity in the x-direction; $v_y = dy/dt$ is the instantaneous velocity in the y-direction. The velocity vector $\mathbf{v}$ is written

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}.$$

Question: Why might the formula for instantaneous speed seem like a perfectly reasonable formula?

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Answer: The Pythagorean Theorem

Example

A particle moves along a curve in the xy-plane with $x(t) = 2t + e^t$ and $y(t) = 3t - 4$, where t is time. Find the velocity vector and speed of the particle when $t = 1$.

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Solution:

Differentiating gives $dx/dt = 2 + e^t$, $dy/dt = 3$. When $t = 1$ we have $v_x = 2 + e$, $v_y = 3$. So the velocity vector is

$$\mathbf{v} = (2 + e)\mathbf{i} + 3\mathbf{j}$$

The speed at $t = 1$ is

$$v = \sqrt{(2 + e)^2 + 3^2} \approx 5.591$$

Slope and Concavity of Parametric Curves

Slope of curve

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

A particle moves in the xy-plane so that its position at time t is given by $x = \sin t$, $y = \cos(2t)$ for $0 \leq t < 2\pi$.

a) At what time does the particle first touch the x-axis? What is the speed of the particle at that time?

b) Is the particle ever at rest?

c) Discuss the concavity of the graph.