Calculus 1
The Chain Rule (page 81)

Intuition Behind the Chain Rule

Imagine we are moving straight upward in a hot air balloon. Let y be our distance from the ground. The temperature, H, is changing as a function of altitude, so $H = f(y)$. How does our temperature change with time? Since temperature is a function of height, $H = f(y)$, and height is a function of time, $y = g(t)$, we can think of temperature as a composite function of time, $H = f(g(t))$, with f as the outside function and g as the inside function. The example suggests the following result:

$$\begin{aligned}&\text{Rate of Change of Composite Function}\\ =&\Big(\text{Rate of Change of Outside Function}\Big) \times \Big(\text{Rate of Chance of Inside Function}\Big)\end{aligned}$$

In Other Words If $H(t) = f(g(t))$ then

$$\frac{\Delta f}{\Delta t} = \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta t} \text{ or } \frac{df}{dt} = \frac{df}{dg} \cdot \frac{dg}{dt}$$

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$$\begin{aligned}
\frac{d}{dx}(f(g(x))) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.
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$$\begin{aligned}
\frac{d}{dx}(f(g(x))) &= \lim_{a\to b} \frac{f(a)-f(b)}{a-b} \cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(b) \cdot g'(x) \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}$$

The Chain Rule

The Chain Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

In words: The derivative of a composite function is the product of the derivatives of the outside and inside functions. The derivative of the outside function must be evaluated at the inside function.

For Example

$$\frac{d}{dx}\left(x^2+1\right)^{100} = 100\left(x^2+1\right)^{99} \cdot 2x$$

More Examples

- 3^{2x+3}
- $\sqrt{x^2 + 3x + 1}$

Clicker

If $f(x) = \left(x^3 + 3\right)^2$, what is $f'(x)$?

a) $f'(x) = 2(x^3 + 3)$
b) $f'(x) = 2(3x^2)$
c) $f'(x) = 2(x^3 + 3)3x^2$
d) $f'(x) = (3x^2)^2$

Clicker Question

Consider the function $h(x) = f(g(x))$. What is $h'(1)$?

x	-2	-1	0	1	2
$f(x)$	3	2	4	3	2
$f'(x)$	1	2	5	2	4
$g(x)$	2	3	-2	-1	5
$g'(x)$	0	2	3	2	5

1. $h'(1) = 8$
2. $h'(1) = 4$
3. $h'(1) = -4$
4. $h'(1) = 10$
5. None of the above

Clicker Question

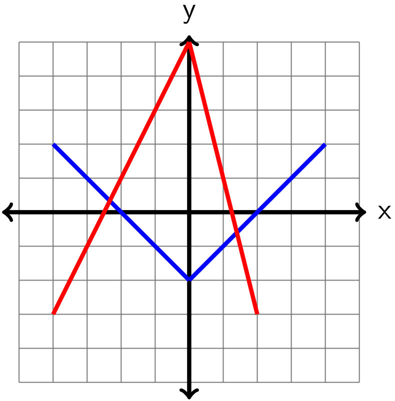
Consider the function $h(x) = f(g(x))$. What is $h'(0)$?

x	-2	-1	0	1	2
$f(x)$	3	2	4	3	2
$f'(x)$	1	2	5	2	4
$g(x)$	2	3	-2	-1	5
$g'(x)$	0	2	3	2	5

1. $h'(0) = 9$
2. $h'(0) = 3$
3. $h'(0) = -2$
4. $h'(0) = 5$
5. None of the above

Clicker Question

If $h(x) = f(g(x))$, what is $h'(1)$? In the graph $f(x)$ is blue and $g(x)$ is red.



a) Not enough information

b) $h'(1) = -1$

c) $h'(1) = 1$

d) $h'(1) = -4$

Examples

For each of the following, calculate the derivative (also see page 82).

1. $\sin(t^2)$
2. $t\cos(2t)$
3. $\sqrt{1+e^{\sqrt{3+x^2}}}$

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1. $\sin(t^2)$
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3. $\sqrt{1+e^{\sqrt{3+x^2}}}$

$$\frac{d}{dx}\sqrt{1+e^{\sqrt{3+x^2}}}$$
$$=\left(\frac{1}{2\sqrt{1+e^{\sqrt{3+x^2}}}}\right)\left(e^{\sqrt{3+x^2}}\right)\left(\frac{1}{2\sqrt{3+x^2}}\right)(2x)$$