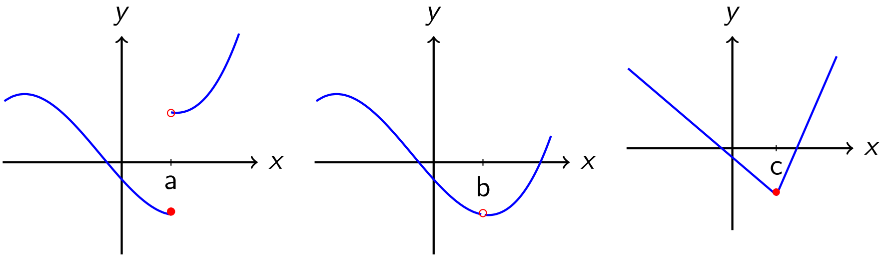
Differentiability

Definition. A function $f(x)$ is differentiable at x if

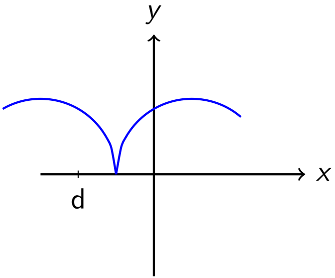
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

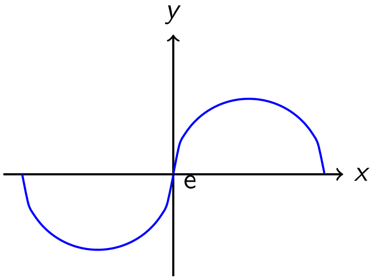
exists. If $f(x)$ is differentiable at $x=a$ then its graph has a non-vertical tangent line at $x=a$. What can go wrong?



(a), (b) discontinuities; (c) “corner”

What can go wrong?(cont.)





(d) = cusp;

(e)=vertical asymptote.

More non-differentiability

Example. Is $f(x) = x + |x|$ differentiable at $x = 0$?

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$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Then everything is well everywhere, except at $x = 0$.

$$\lim_{h \to 0+} \frac{f(h) - f(0)}{h} = 2$$

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While

$$\lim_{h \to 0-} \frac{f(h) - f(0)}{h} = 0$$

Since the side limits do not agree, the limit does not exist. Hence, f is not differentiable at $x = 0$.

Clicker question #1

Which of the functions below are differentiable?

(A) $|x|^2$

(B) $|x|^2 - 2|x|$

(C) $|x + 5|$

(D) $\sqrt{(x+2)^2}$

(E) $3 - |x|$

Differentiable $\implies$ continuous

Theorem. If $f(x)$ is differentiable at $x = a$ then f is continuous at $x = a$

The converse is not true! There are continuous functions which are not differentiable (see graphs (c), (d), (e) above).

Second order derivatives

The derivative of the derivative function is the second order derivative:

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

provided the limit exists.

Example. Find $f''(x)$ for $f(x) = 2x^2 + 3x$.

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Example. Find $f''(x)$ for $f(x) = 2x^2 + 3x$.
Last time we showed that $f'(x) = 4x + 3$, so we compute

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{4(x+h) + 3 - 4x - 3}{h} \\ &= 4. \end{aligned}$$

Notation

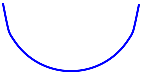
If $y = f(x)$, then

$$f''(x) = y'' = \frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[f'(x)]...$$

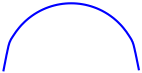
Concavity and convexity

For f a function

- the sign of f gives the monotonicity (increasing vs. decreasing)
- the sign of f'' gives the concavity/convexity as follows
  1. If $(f')' > 0$ then the slope of f' is increasing; hence, the shape is concave up/convex (the graph "holds water")
  2. If $(f')' < 0$ then the slope of f' is decreasing; hence, the shape is concave down (the graph " does not hold water")



Concave up (convex)

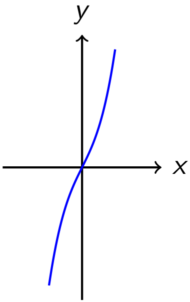


Concave down

Analyzing concavity

For $f(x) = x^3 + 2x$ we show that $f'(x) = 3x^2 + 2$ and $f''(x) = 6x$. Hence

- f is increasing everywhere as $f'(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$
- f is concave up whenever $f''(x) = 6x > 0$ (hence for $x > 0$) and it is concave down whenever $f''(x) = 6x < 0$ (for $x < 0$).



Clicker question #2

If the derivative of f is $f'(x) = 2x^2 + 4x$, on which interval is the function f concave up?

(A) $(-\infty, \infty)$

(B) $(-\infty, 0)$

(C) $(0, \infty)$

(D) $(-1, \infty)$

(E) $(2, \infty)$

Acceleration

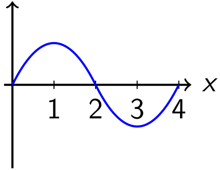
Assume that $y = s(t)$ gives the position at time t of an object. Then

- $v(t) = \dfrac{dy}{dt} = s'(t)$ is the velocity at time t (units of length/time)
- $a(t) = \dfrac{d^2y}{dt^2} = v'(t) = s''(t)$ provides the acceleration at time t (units of length/time^2).

Position, velocity, acceleration

Suppose that an object's **velocity** is given by the graph:

$y = v(t)$



- On what intervals is the acceleration positive?
- On what intervals is the object's position increasing?
- Where is the function $s(t)$ increasing/decreasing, concave up/concave down?
- Suppose that $s(0) = 1$. Sketch a possible graph for s.

Wrapping up

For next time

- Work on the suggested problems from sections 2.5 and 2.6 (from syllabus and webwork).
- Read sections 3.1 and 3.2 (derivatives of polynomials and exponentials).