

CONTINUOUS DATA ASSIMILATION FOR THE MAGNETOHYDRODYNAMIC EQUATIONS IN 2D USING ONE COMPONENT OF THE VELOCITY AND MAGNETIC FIELDS

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ABSTRACT. We propose several continuous data assimilation (downscaling) algorithms based on feedback control for the 2D magnetohydrodynamic (MHD) equations. We show that for sufficiently large choices of the control parameter and resolution and assuming that the observed data is error-free, the solution of the controlled system converges exponentially (in L^2 and H^1 norms) to the reference solution independently of the initial data chosen for the controlled system. Furthermore, we show that a similar result holds when controls are placed only on the horizontal (or vertical) variables, or on a *single* Elsässer variable, under more restrictive conditions on the control parameter and resolution. Finally, using the data assimilation system, we show the existence of *abridged* determining modes, nodes and volume elements.

1. INTRODUCTION

In the study of solar storms, space weather forecasting, earth's geodynamo, and other areas, predicting the motion of fluids with magnetic properties is a central concern. The governing equations are often taken to be the magnetohydrodynamic (MHD) equations, or some modification of them. These equations are notoriously difficult to solve both analytically and computationally. Moreover, accurately initializing the system is challenging due to the sparsity of the available data. Fortunately, data is often given not just at a single time, but can be streaming in (e.g., from devices monitoring space plasma dynamics), or given in history (e.g., from surface geomagnetic observations, which in the earth can be traced back up to 7000 years [BGJ89, C JL00, SOL02]). This situation is similar to the problem of weather prediction on earth. Therefore the techniques of data assimilation, which were developed in weather prediction, have been applied to the MHD equations in recent years (see, e.g., [BRB02, CRB07, FEA07, FHJ⁺10, GDZGP00, L JL14, MDMB06, SML16, STK07, TRT⁺08]). It has also been speculated in [ASZL15] that data assimilation for magnetohydrodynamics may be useful in liquid sodium experiments modeling the Earth's core.

Data assimilation has been the subject of a very large body of work. Classically, these techniques are based on linear quadratic estimation, also known as the Kalman Filter. The Kalman Filter has the drawback of assuming that the underlying system and any corresponding observation models are linear. It also assumes that measurement noise is Gaussian distributed. This has been somewhat corrected via modifications, such as the Extended Kalman Filter and the Unscented Kalman Filter. For more about the Kalman Filter and its modifications, see, e.g., [Dal93, Kal03, LSZ15], and the references therein. Recently, a promising new approach to data assimilation was pioneered by Azouani, Olson, and Titi in [AOT14, AT14] (see also [CKT01, HOT11, OT03] for early ideas in this direction). This new approach is based on feedback control at the PDE level. The first works in this area assumed noise-free observations, but [BOT15] adapted the method to the case of noisy data, and [FMT16] adapted it to the case where measurements are obtained discretely

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in time and may be contaminated by systematic errors. Computational experiments on this technique were carried out in the cases of the 2D Navier-Stokes equations [GOT], the 2D Bénard convection equations [ATK⁺15], and the 1D Kuramoto-Sivashinsky equations [LT]. In addition to the results discussed here, a large amount of recent literature has built upon this idea; see, e.g., [ANLT16, FJT15, FLT16a, FLT16b, FLT16c, FDKT14b, FDKT14a, GHKVZ14, JST15, MTT16].

In the present work, we adapt the approach of [AOT14, AT14, FLT16a] to the 2D MHD equations. In Theorem 3.1, we show that solutions of the feedback-controlled system converge exponentially in the L^2 -norm to solutions of the MHD system when feedback control is applied to all variables (here, we use Elsässer variables for simplicity). This convergence holds under certain conditions on the spacing of the data and the weight given to the feedback control. Moreover, in Theorems 3.2 and 3.3, we establish *abridged data assimilation*, i.e., we show that feedback control need only be applied to a reduced set of the variables (horizontal variables or a single Elsässer variable, respectively) to obtain exponential convergence, at the cost of more restrictive conditions on the data resolution h and control weight μ . In Theorem 3.4, we establish exponential convergence in the H^1 -norm. Next, in Theorem 3.6, we show that if one makes weaker assumptions on the data interpolation function, and if feedback control is applied only to horizontal variables, then exponential convergence in the H^1 norm holds as well. Finally, in Section 3.3, we establish a rigorous connection between data assimilation and the concept of determining quantities, first introduced in [FP67], and further studied in [FT84, CJT95, JT93, JT92a, JT92b].

We now describe the general idea of the data assimilation scheme we use for the 2D MHD equations, based on the idea of feedback control, that was developed by Azouni, Olson and Titi in [AOT14, AT14] in the context of the 2D Navier-Stokes equations. In the study of a dynamical system in the form,

$$\frac{d}{dt}Y = F(Y),$$

subject to certain boundary conditions, one normally tries to show that unique solutions will arise given any initial value

$$Y(0) = Y_0,$$

in a certain space, and that the solution will change in a continuous way with respect to a change in the initial value.

The problem arises in practice that the initial value may not be known exactly, but it may approximate the true initial value of a given observable, for example the temperature, which we'd like to predict the value of in the future. The continuous dependence on initial data addresses this issue, in that if the initial approximation is close enough to the true value, then the solution we obtain will accurately approximate the true value of the observable for some period of time. However, usual theory shows that the length of time the approximation is guaranteed to be good is short, in that the error may grow exponentially in time. Also, the initial measurement may need to give a very close approximation to the true initial value, but in practice measurements may only be available on a coarse grid, limiting the accuracy of the initial approximation and thus limiting both the accuracy the solution can be guaranteed to have, as well as the duration for which this accuracy can be guaranteed.

Data assimilation is the method where, to compensate for this lower bound on the accuracy of the measured initial condition, measurements are taken of the observable as time goes on (over the same possibly coarse grid on which the initial value is approximated) and fed back into the differential equation (giving a different equation, called the data assimilation equation) in such a way that the solution will become a better approximation as time goes on. This gives us the accuracy we need to apply the continuous dependence on initial data and say the prediction will be accurate for some duration from that time onwards.

The data assimilation algorithm (the way measurements are introduced to the differential equation) can take different forms, but the one we consider here was first introduced by Azouani, Olson, and Titi in

[AOT14, AT14]. Given that the true value of the observable at time t is $Y(t)$, then the data assimilation equation will be:

$$\begin{aligned}\frac{d}{dt}\tilde{Y} &= F(\tilde{Y}) - \mu(I_h(Y) - I_h(\tilde{Y})) \\ &= F(\tilde{Y}) - \mu I_h(Y - \tilde{Y}),\end{aligned}$$

where the second equality in the above equation follows because we'll assume the interpolant operator, I_h , is linear. Here, μ will be an adequately chosen tuning parameter. In addition, we will assume that for all $u \in H^1$, I_h satisfies one of the following:

$$\|u - I_h(u)\|_{L^2} \leq c_1 h \|\nabla u\|_{L^2}, \quad (1)$$

or

$$\|u - I_h(u)\|_{L^2} \leq c_2 h \|\nabla u\|_{L^2} + c_3 h^2 \|\Delta u\|_{L^2}. \quad (2)$$

Many relevant examples of operators satisfy one of these two conditions, including the projection onto the low modes, finite volume element operators, and nodal interpolant operators. For more information, see, e.g. [LT, FT91, AOT14].

We consider the 2D MHD equations for a fluid and magnetic field under periodic boundary conditions and with zero space average. Let u , b , and p represent the fluid velocity, magnetic field, and fluid pressure, respectively, and let the spatial domain be $[0, L]^2$. The system can be written as (see, e.g., [Dav01]):

$$\begin{aligned}\partial_t u - \nu \Delta u + (u \cdot \nabla) u - \frac{1}{\rho_0 \mu_0} (b \cdot \nabla) b &= -\frac{1}{\rho_0} \nabla \left(p + \frac{1}{2\mu_0} |b|^2 \right) + f_1, \\ \partial_t b - \lambda \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u &= g_1, \\ \nabla \cdot b &= 0, \quad \nabla \cdot u = 0.\end{aligned}$$

Here, $\nu > 0$ is the kinematic fluid viscosity, ρ_0 is the fluid density, $\mu_0 := 4\pi \times 10^{-7} H/m$ is the permeability of free space, $\lambda = (\mu_0 \sigma)^{-1} > 0$ is the magnetic diffusivity, and σ is the electrical conductivity of the fluid. We impose initial conditions $u(0, x, y) = u_0(x, y)$ and $b(0, x, y) = b_0(x, y)$ in an appropriate function space, and allow for time-dependent forcing functions, denoted above by f_1 and g_1 .

Our analyses will have to take into account the amount of energy being added to the system by the forcing functions, so to this end we define the Grashof number, G , to be

$$G := \frac{1}{\lambda_1} \limsup_{t \rightarrow \infty} \left(\max \left\{ \frac{1}{\nu^2} \|f_1(t)\|_{L^2}, \quad \frac{1}{\lambda^2 \sqrt{\rho_0 \mu_0}} \|g_1(t)\|_{L^2} \right\} \right).$$

where $\lambda_1 := \frac{4\pi^2}{L^2}$ is the smallest eigenvalue of the Stokes operator on the space of functions with space average zero on $[0, L]^2$ under periodic boundary conditions [FMRT01].

Note that we have constructed G to be dimensionless. We will also non-dimensionalize the system so that we can later reformulate it in terms of the Elsässer variables. Let U be a reference velocity and use L as a reference length. We denote the dimensionless fluid Reynolds number and the dimensionless magnetic Reynolds number by $Re := UL/\nu$ and $Rm := UL/\lambda$, respectively. In non-dimensional form, the system can be written as:

$$\partial_t u - \frac{1}{Re} \Delta u + (u \cdot \nabla) u - (b \cdot \nabla) b = -\nabla \pi + f_1, \quad (3a)$$

$$\partial_t b - \frac{1}{Rm} \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u = g_1, \quad (3b)$$

$$\nabla \cdot b = 0, \quad \nabla \cdot u = 0. \quad (3c)$$

with the initial conditions $u(0, x, y) = u_0(x, y)$ and $b(0, x, y) = b_0(x, y)$, and where π is the (non-dimensionalized) sum of the fluid and magnetic pressures, and u , b , u_0 , b_0 , f_1 , and g_1 have been replaced by their appropriate non-dimensional versions. Note the bilinearity in (u, b) on the left-hand side of (3b) allows for the

important fact that the four non-linear terms in (3) can be written with coefficients ± 1 . We will denote the non-dimensionalized spatial domain by

$$\Omega := [0, 1]^2 \subset \mathbb{R}^2.$$

Short-time existence and uniqueness of solutions to (11) was proven in a slightly more general context in [Ale82] (see also [Sch88, Sec93]). For a derivation and physical discussion of the MHD equations, see, e.g., [Cha61]. For an overview of the classical and recent mathematical results pertaining to the MHD equations, see, e.g., [DL72a, Dav01].

2. PRELIMINARIES

We begin by listing the notations that we use all through this paper.

Notation.

For a matrix A , we denote $|A|^2 := \sum_{i,j} |A_{i,j}|^2$. We denote the standard L^2 inner-product and norm by $\langle u, v \rangle := \int_{\Omega} u \cdot v \, dxdy$ and $\|u\|_{L^2} := \left(\int_{\Omega} |u|^2 \, dxdy \right)^{\frac{1}{2}}$, respectively (note that the integral is taken over the non-dimensionalized domain, Ω , so $\|u\|_{L^2}$ has the same units as u). We also denote $\|u\|_{H^1} := \|\nabla u\|_{L^2}$, which is equivalent to the standard H^1 norm, due to the Poincaré inequality (6).

Inequalities.

We recall some standard inequalities. Here $\epsilon > 0$, $a, b \geq 0$, and u, v , and w are divergence-free periodic functions, with sufficient regularity to make all the norms involved finite.

We will frequently use the following forms of Young's inequality and Hölder's inequality:

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2 \quad (4)$$

$$\left| \int_{\Omega} uvw \, dxdy \right| \leq \|u\|_{L^2} \|v\|_{L^4} \|w\|_{L^4} \quad (5)$$

We also recall the following version of Poincaré's inequality, valid for periodic functions with zero space average on Ω :

$$\|\nabla u\|_{L^2} \geq 2\pi \|u\|_{L^2} \quad (6)$$

The following inequality due to Ladyzhenskaya will be used to bound the nonlinear terms for the cases where we have measurements on all the components and when we only measure one Elsässer variable:

$$\|u\|_{L^4}^2 \leq c_L \|u\|_{L^2} \|\nabla u\|_{L^2} \quad (7)$$

The next two inequalities are extensions of the Brezis-Gallouet and are due to Titi [Tit87]. They will be necessary to bound the nonlinear terms in the case of measuring only one component of the reference velocity and magnetic fields:

$$\left| \int_{\Omega} u \partial_i v w \, dxdy \right| \leq c_B \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|w\|_{L^2} \left(1 + \ln \left(\frac{\|\nabla w\|_{L^2}}{2\pi \|w\|_{L^2}} \right) \right)^{1/2}, \quad (8)$$

$$\left| \int_{\Omega} u \partial_i v \Delta w \, dxdy \right| \leq c_T \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \left(\frac{\|\Delta z\|_{L^2}}{2\pi \|\nabla z\|_{L^2}} \right) \right)^{1/2}, \quad (9)$$

where in (9), z can be u or v .

The following generalization of the Grönwall Lemma will be useful, which was first shown by Foias et al. in [FMRT83]. For a proof of an even more general version due to Jones and Titi, see [FMRT01].

Proposition 2.1 (Generalized Gronwall Inequality). *Let $\psi : [0, \infty) \rightarrow \mathbb{R}$ be a locally integrable function such that for some $T > 0$ the following two conditions hold:*

$$\liminf_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \psi(s) ds > 0, \quad (10a)$$

$$\limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \psi^-(s) ds < \infty, \quad (10b)$$

where $\psi^-(t) := \max\{0, -\psi(t)\}$. Then if $Y : [0, \infty) \rightarrow [0, \infty)$ is absolutely continuous and for almost all t ,

$$\frac{d}{dt} Y + \psi Y \leq \phi, \quad (10c)$$

where $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$, then $Y(t) \rightarrow 0$ as well. Furthermore, if $\phi \equiv 0$ then $Y(t) \rightarrow 0$ exponentially as $t \rightarrow \infty$.

Next, in order to simplify our calculations we will reformulate the MHD equations in terms of new variables which we call v and w , in such a way as to symmetrize the system.

We assume, without loss of generality, that $\frac{1}{Re} \geq \frac{1}{Rm}$, and denote the Elsässer variables [Els50] by $v = u + b$ and $w = u - b$ (if $\frac{1}{Re} < \frac{1}{Rm}$ then we would denote $w = b - u$ and proceed similarly).

Then we can derive evolution equations for v and w by considering both the sum and difference of (3a) and (3b) and obtain the following system:

System 2.2.

$$\partial_t v - \alpha \Delta v + \beta \Delta w + (w \cdot \nabla) v = -\nabla \pi + f, \quad (11a)$$

$$\partial_t w - \alpha \Delta w + \beta \Delta v + (v \cdot \nabla) w = -\nabla \pi + g, \quad (11b)$$

$$\nabla \cdot v = 0, \quad \nabla \cdot w = 0, \quad (11c)$$

subject to the initial conditions $v(0) = v_0 := u_0 + b_0$ and $w(0) = w_0 := u_0 - b_0$.

Here we relabeled the forcing terms as $f := f_1 + g_1$ and $g := f_1 - g_1$, and we denote $\alpha := \frac{1}{Re} + \frac{1}{Rm}$ and $\beta := \frac{1}{Re} - \frac{1}{Rm}$. It will be important to note that $\alpha - \beta = \frac{2Rm}{ReRm} > 0$ and that $\alpha > 0$ and $\beta \geq 0$ (this last inequality is true by the assumption that $\frac{1}{Re} \geq \frac{1}{Rm}$, however if $\frac{1}{Re} < \frac{1}{Rm}$ then we would arrive at the above system except with a different sign on the pressure, and $\beta = \frac{1}{Rm} - \frac{1}{Re}$, so still we have $\beta \geq 0$, and in general we will have $\alpha - \beta = 2 \max\{\frac{1}{Re}, \frac{1}{Rm}\}$).

We note here that G can be expressed in terms of the forcing functions for the reformulated system:

$$G = \limsup_{t \rightarrow \infty} \left(\max\left\{ \frac{Re^2}{4\pi^2} \|f(t) + g(t)\|_{L^2}, \frac{Rm^2}{4\pi^2} \|f(t) - g(t)\|_{L^2} \right\} \right),$$

hence,

$$G \geq \frac{1}{\pi^2(\alpha - \beta)^2} \limsup_{t \rightarrow \infty} (\max\{\|f(t)\|_{L^2}, \|g(t)\|_{L^2}\}),$$

Now, we describe the data assimilation algorithms studied in this paper. Following the ideas of [AOT14, AT14] we incorporate measurements obtained from a fixed reference solution (of which we want to predict future values) through a damping term. This will “steer” the data assimilation solutions to the reference solution exponentially in time. In what sense we will have convergence depends on the type of interpolant I_h with which we take measurements.

The results are separated by the type of interpolant considered and by which measurements are recorded. We frame our results in terms of the Elsässer variables, not in terms of u and b . Also, we consider algorithms which require measurements taken only on the first components, u_1 and b_1 (which is the same as measuring

v_1 and w_1), by measuring all the components of u and b , or by measuring either the sum $u + b$ or the difference $u - b$ only.

In the following, let (v, w) be a fixed solution of (11), and we denote the data assimilation variables by \tilde{v} and \tilde{w} , which will approximate v and w respectively. I_h may satisfy either (1) or (2), and we will analyze each case separately.

First, we have the following algorithm which utilizes measurements taken on all components (so measuring u and b):

System 2.3.

$$\partial_t \tilde{v} - \alpha \Delta \tilde{v} + \beta \Delta \tilde{w} + (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla \tilde{\pi} + f + \mu I_h(v - \tilde{v}) \quad (12a)$$

$$\partial_t \tilde{w} - \alpha \Delta \tilde{w} + \beta \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{w} = -\nabla \tilde{\pi} + g + \mu I_h(w - \tilde{w}) \quad (12b)$$

$$\nabla \cdot \tilde{v} = 0, \quad \nabla \cdot \tilde{w} = 0, \quad (12c)$$

subject to the initial conditions $\tilde{v}(0) \equiv \tilde{w}(0) \equiv 0$.

Next, using measurements only on the first components of v and w (which is equivalent to measuring u_1 and b_1):

System 2.4.

$$\partial_t \tilde{v} - \alpha \Delta \tilde{v} + \beta \Delta \tilde{w} + (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla \tilde{\pi} + f + \mu I_h(v_1 - \tilde{v}_1)e_1 \quad (13a)$$

$$\partial_t \tilde{w} - \alpha \Delta \tilde{w} + \beta \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{w} = -\nabla \tilde{\pi} + g + \mu I_h(w_1 - \tilde{w}_1)e_1 \quad (13b)$$

$$\nabla \cdot \tilde{v} = 0, \quad \nabla \cdot \tilde{w} = 0, \quad (13c)$$

subject to the initial conditions $\tilde{v}(0) \equiv \tilde{w}(0) \equiv 0$.

Finally, only taking measurements on v (which would in practice still require recording measurements on both u and b):

System 2.5.

$$\partial_t \tilde{v} - \alpha \Delta \tilde{v} + \beta \Delta \tilde{w} + (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla \tilde{\pi} + f + \mu I_h(v - \tilde{v}) \quad (14a)$$

$$\partial_t \tilde{w} - \alpha \Delta \tilde{w} + \beta \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{w} = -\nabla \tilde{\pi} + g \quad (14b)$$

$$\nabla \cdot \tilde{v} = 0, \quad \nabla \cdot \tilde{w} = 0, \quad (14c)$$

subject to the initial conditions $\tilde{v}(0) \equiv \tilde{w}(0) \equiv 0$.

In the above we chose to make the initial conditions 0, but in fact the initial conditions may be chosen essentially arbitrarily, albeit in accordance with the existence theorems.

Remark 2.6. Here we first constructed the Elsässer variables from the original variables u and b after nondimensionalizing, and then proceeded to define the various data assimilation algorithms and variables. However, since the transformations were linear, if we were to define each data assimilation algorithm using the original variables, in the process defining data assimilation variables \tilde{u} and \tilde{b} , and then nondimensionalize and change to the Elsässer variables, we would arrive at the same systems above. So, all our results apply to the corresponding algorithms formulated in terms of the original variables.

Note also that although the results are framed in terms of the Elsässer variables, by the triangle inequality convergence of \tilde{v} to v and \tilde{w} to w implies convergence of \tilde{u} and \tilde{b} to u and b respectively.

3. STATEMENTS OF THE RESULTS

3.1. Results for Type 1 Interpolants.

Theorem 3.1. *Let (v, w) be a strong solution of (11) which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Let I_h satisfy (1), where*

$$h < \frac{\alpha - \beta}{\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1} \sim G^{-1}, \text{ and } \mu > \frac{\pi^2(c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2 \sim G^2.$$

Then there is a unique strong solution, (\tilde{v}, \tilde{w}) , of (12) corresponding to (v, w) which exists globally in time, and furthermore $\|v(t) - \tilde{v}(t)\|_{L^2} + \|w(t) - \tilde{w}(t)\|_{L^2} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

Theorem 3.2. *Let (v, w) be a strong solution of (11) which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Let I_h satisfy (1), where*

$$h < \frac{1}{4\sqrt{2}\pi c_1 c} (\tilde{c} + 2\ln G + CG^4)^{-\frac{1}{2}} G^{-1} \sim G^{-3}, \text{ and } \mu > 32\pi^2 c^2 (\alpha - \beta) (\tilde{c} + 2\ln G + CG^4) G^2 \sim G^6.$$

Then there is a unique strong solution, (\tilde{v}, \tilde{w}) , of (13) corresponding to (v, w) which exists globally in time, and furthermore $\|v(t) - \tilde{v}(t)\|_{L^2} + \|w(t) - \tilde{w}(t)\|_{L^2} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

Theorem 3.3. *Let (v, w) be a strong solution of (11) which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Let I_h satisfy (1), where*

$$h < \frac{4(\alpha - \beta)}{\pi c_1 c_L^2 G(4 + (\alpha - \beta)^2 G^2)} \sim G^{-3}, \text{ and } \mu > \frac{\pi^2 c_L^4 G^2 (4 + (\alpha - \beta)^2 G^2)^2}{16(\alpha - \beta)} \sim G^6.$$

Then there is a unique strong solution, (\tilde{v}, \tilde{w}) , of (14) corresponding to (v, w) which exists globally in time, and furthermore $\|v(t) - \tilde{v}(t)\|_{L^2} + \|w(t) - \tilde{w}(t)\|_{L^2} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

With the following theorems, we show *a priori* that the H^1 -norm of the difference of the solution to the interpolated system, (12) and (13), and that of the original system (11), tends to zero exponentially fast with respect to time.

Theorem 3.4. *Let (v, w) be a strong solution of (11) which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Let I_h satisfy (1), where*

$$h < \frac{\alpha - \beta}{2\sqrt{2}\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1} \sim G^{-1}, \text{ and } \mu > \frac{\pi^2(c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2 \sim G^2.$$

Then there is a unique strong solution, (\tilde{v}, \tilde{w}) , of (12) corresponding to (v, w) which exists globally in time, and furthermore $\|v(t) - \tilde{v}(t)\|_{H^1} + \|w(t) - \tilde{w}(t)\|_{H^1} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

Theorem 3.5. *Let (v, w) be a strong solution of (11) which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Let I_h satisfy (1), where*

$$h < (8\sqrt{2}\pi c_1 c)^{-1} (\tilde{c} + 2\ln G + CG^4)^{-\frac{1}{2}} G^{-1} \sim G^{-3}, \text{ and } \mu > 32\pi^2 c^2 (\alpha - \beta) (\tilde{c} + 2\ln G + CG^4) G^2 \sim G^6.$$

Then there is a unique strong solution, (\tilde{v}, \tilde{w}) , of (13) corresponding to (v, w) which exists globally in time, and furthermore $\|v(t) - \tilde{v}(t)\|_{H^1} + \|w(t) - \tilde{w}(t)\|_{H^1} \rightarrow 0$ exponentially as $t \rightarrow \infty$.

3.2. Results for Type 2 Interpolants.

Theorem 3.6. *Let (v, w) be a strong solution of (11), which at time $t = 0$ has evolved enough so that Proposition 4.1 holds with $t_0 = 0$. Then $h \sim G^{-6}e^{-CG^4}$ and $\mu \sim (\alpha - \beta)G^{12}e^{2CG^4}$ can be chosen so that if I_h satisfies (2), then there is a unique strong solution (\tilde{v}, \tilde{w}) of (13) corresponding to (v, w) which exists globally in time, and $\|v(t) - \tilde{v}(t)\|_{H^1} + \|w(t) - \tilde{w}(t)\|_{H^1} \rightarrow 0$ exponentially as $t \rightarrow \infty$.*

Remark 3.7. A similar theorem holds for the case of measurement on all variables (although it is not a direct corollary, since the dynamical system involved is slightly different). However, in this case, we do not find much improvement in the restrictions on h and μ .

3.3. Determining Interpolants.

In order to prove that there are finitely many (say N) determining modes, one needs to show that if $(v^{(1)}, w^{(1)})$ and $(v^{(2)}, w^{(2)})$ are different solutions of (11) with possibly different forcing terms and initial data, then knowledge that $\|P_N(v^{(1)}, w^{(1)}) - P_N(v^{(2)}, w^{(2)})\|_{L^2} \rightarrow 0$ is sufficient to conclude that $\|(v^{(1)}, w^{(1)}) - (v^{(2)}, w^{(2)})\|_{L^2} \rightarrow 0$, where P_N denotes the projection onto the modes with magnitude at most N . In general, we replace P_N by a different operator, say I_h , and ask the question of whether the knowledge inherent in I_h is “determining”.

In the following theorems, we show that the data assimilation results we have obtained can be adapted to show that the interpolant operators, I_h , are determining. We do this by first generalizing the convergence results we developed in the previous theorems to allow for the evolution equations of the reference solution and the data assimilation solution to have different forcing terms, which converge in L^2 as $t \rightarrow \infty$, at the cost of losing the exponential rate of convergence of the solutions. We also allow for the reference solution to be perturbed by a function which decays in L^2 .

We illustrate the ideas for the algorithm studied in Theorem 3.1, i.e. with measurements taken on all variables and for I_h satisfying (1), but the results can be obtained for all the other cases as well. So, we can show that operators which satisfy (1) or (2) and use measurements on (v, w) , (v_1, w_1) , or v , are determining in the sense of convergence in L^2 and H^1 .

Theorem 3.8. *Let I_h satisfy (1) and let (v, w) be a reference solution of (11). Then if μ and h satisfy the hypotheses of Theorem 3.1, and if $\|\delta^{(1)}(t)\|_{L^2}, \|\delta^{(2)}(t)\|_{L^2} \rightarrow 0$ and $\|I_h(\epsilon^{(1)}(t))\|_{L^2}, \|I_h(\epsilon^{(2)}(t))\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$, then for any choice of \tilde{v}_0 and \tilde{w}_0 there are unique \tilde{v}, \tilde{w} and $\tilde{\pi}$ which satisfy the following modified version of (12):*

System 3.9.

$$\partial_t \tilde{v} - \alpha \Delta \tilde{v} + \beta \Delta \tilde{w} + (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla \tilde{\pi} + f + \delta^{(1)} + \mu I_h(v + \epsilon^{(1)} - \tilde{v}) \quad (15a)$$

$$\partial_t \tilde{w} - \alpha \Delta \tilde{w} + \beta \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{w} = -\nabla \tilde{\pi} + g + \delta^{(2)} + \mu I_h(w + \epsilon^{(2)} - \tilde{w}) \quad (15b)$$

$$\nabla \cdot \tilde{v} = 0, \quad (15c)$$

$$\nabla \cdot \tilde{w} = 0, \quad (15d)$$

subject to the initial conditions $\tilde{v}(0) = \tilde{v}_0, \tilde{w}(0) = \tilde{w}_0$,

and furthermore, $\|v - \tilde{v}\|_{L^2}, \|w - \tilde{w}\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$.

In the next theorem we illustrate the result that if an interpolant I_h satisfies the conditions for the generalized data assimilation theorem then I_h is determining, for the case of the generalized version of Theorem 3.1. Note that the projection onto the low modes, P_N , is an example of an interpolant operator I_h for which the theorem applies, provided that $h := \frac{1}{N} \lesssim G^{-1}$. Hence, the following theorem shows that there are finitely many determining modes for instance.

Theorem 3.10. *Let $(v^{(1)}, w^{(1)})$ and $(v^{(2)}, w^{(2)})$ be solutions of (11) with forcing terms $f^{(1)}, g^{(1)}$ and $f^{(2)}, g^{(2)}$ respectively, and suppose that $\|f^{(1)} - f^{(2)}\|_{L^2}, \|g^{(1)} - g^{(2)}\|_{L^2} \rightarrow 0$.*

Let \mathbf{I}_h satisfy (1) where

$$h < \frac{\alpha - \beta}{\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1}, \text{ and}$$

$$G := \frac{1}{\pi^2(\alpha - \beta)^2} \limsup_{t \rightarrow \infty} \left(\max\{\|f^{(1)}(t)\|_{L^2}, \|g^{(1)}(t)\|_{L^2}\} \right) = \frac{1}{\pi^2(\alpha - \beta)^2} \limsup_{t \rightarrow \infty} \left(\max\{\|f^{(2)}(t)\|_{L^2}, \|g^{(2)}(t)\|_{L^2}\} \right),$$

and suppose that $\|\mathbf{I}_h(v^{(1)}(t) - v^{(2)}(t))\|_{L^2}, \|\mathbf{I}_h(w^{(1)}(t) - w^{(2)}(t))\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$.

Then $\|v^{(1)}(t) - v^{(2)}(t)\|_{L^2}, \|w^{(1)}(t) - w^{(2)}(t)\|_{L^2} \rightarrow 0$ as well.

4. PROOFS OF THE RESULTS

Before we get to the main proofs, we first state the following bounds for the reference solution to the MHD system. Moreover, we prove (16), which follows standard arguments from the Navier-Stokes theory (see, e.g., [CF88, Tem01]). The proofs of (17) and (18) can be obtained by modifying the corresponding proofs from the Navier-Stokes theory in a similar way (see, e.g. [DL72b, ST83] for more details on (17) and the appendix of [FLT16a] for (18)).

Proposition 4.1 (Upper Bounds on Solutions of the MHD). *Let $T > 0$ and let (v, w) be a solution of (11). Then there is a $t_0 > 0$ and constants $c_M > 0$ and $C = \frac{81}{4}c_L^8$ such that for all $t > t_0$,*

$$\int_t^{t+T} (\|\nabla v(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2) ds \leq (1 + T\pi^2(\alpha - \beta))(\alpha - \beta)G^2, \quad (16)$$

$$\|\nabla v(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2 \leq 10\pi^2(\alpha - \beta)^2 G^2 e^{CG^4}. \quad (17)$$

$$\|\Delta v(t)\|_{L^2}^2 + \|\Delta w(t)\|_{L^2}^2 \leq c_M(\alpha - \beta)^2 G^2 \left(1 + \left(1 + G^2 e^{CG^4}\right) \left(1 + e^{CG^4} + G^4 e^{CG^4}\right)\right). \quad (18)$$

Proof of (16). We provide only a formal proof here. A rigorous proof can be carried out by, e.g., first proving the bounds at the level of finite-dimensional Galerkin truncation, and then passing to a limit.

Taking a (formal) inner-product of (11a) with v , and of (11b) with w , using (11c) and adding the results, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|v\|_{L^2}^2 + \|w\|_{L^2}^2) + (\alpha - \beta) (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \\ & \leq \langle f, v \rangle + \langle g, w \rangle \leq \|f\|_{L^2} \|v\|_{L^2} + \|g\|_{L^2} \|w\|_{L^2} \\ & \leq \frac{1}{8\pi^2(\alpha - \beta)} (\|f\|_{L^2}^2 + \|g\|_{L^2}^2) + \frac{(\alpha - \beta)}{2} 4\pi^2 (\|v\|_{L^2}^2 + \|w\|_{L^2}^2) \\ & \leq \frac{1}{8\pi^2(\alpha - \beta)} (\|f\|_{L^2}^2 + \|g\|_{L^2}^2) + \frac{(\alpha - \beta)}{2} (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2), \end{aligned}$$

where we used the Poincaré inequality. Therefore, after collecting terms,

$$\frac{d}{dt} (\|v\|_{L^2}^2 + \|w\|_{L^2}^2) + (\alpha - \beta) (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \leq \frac{1}{4\pi^2(\alpha - \beta)} (\|f\|_{L^2}^2 + \|g\|_{L^2}^2), \quad (19)$$

and by using the Poincaré inequality on the left hand side,

$$\frac{d}{dt} (\|v\|_{L^2}^2 + \|w\|_{L^2}^2) + 4\pi^2(\alpha - \beta) (\|v\|_{L^2}^2 + \|w\|_{L^2}^2) \leq \frac{1}{4\pi^2(\alpha - \beta)} (\|f\|_{L^2}^2 + \|g\|_{L^2}^2). \quad (20)$$

Then by Grönwall's inequality,

$$\|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 \leq (\|v(0)\|_{L^2}^2 + \|w(0)\|_{L^2}^2) e^{-4\pi^2(\alpha - \beta)t} + \frac{1}{16\pi^4(\alpha - \beta)^2} \operatorname{ess\,sup}_{s \in [0, t]} (\|f(s)\|_{L^2}^2 + \|g(s)\|_{L^2}^2). \quad (21)$$

Let $t_* > 0$ be large enough so that

$$\operatorname{ess\,sup}_{t \geq t_*} \|f(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2 \leq 2 \limsup_{t \rightarrow \infty} (\|f(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2), \quad (22)$$

and choose $t_0 > t_*$ so that

$$(\|v(t_*)\|_{L^2}^2 + \|w(t_*)\|_{L^2}^2) e^{-4\pi^2(\alpha-\beta)(t_0-t_*)} \leq \frac{3}{8\pi^4(\alpha-\beta)^2} \limsup_{t \rightarrow \infty} (\|f(t)\|_{L^2}^2 + \|g(t)\|_{L^2}^2).$$

Then by using Grönwall's inequality again on (20) with initial time t_* , we see that for all $t > t_0$,

$$\begin{aligned} \|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 &\leq (\|v(t_*)\|_{L^2}^2 + \|w(t_*)\|_{L^2}^2) e^{-4\pi^2(\alpha-\beta)(t-t_*)} + \frac{1}{16\pi^4(\alpha-\beta)^2} \operatorname{ess\,sup}_{s \in [t_*, t]} (\|f(s)\|_{L^2}^2 + \|g(s)\|_{L^2}^2) \\ &\leq \frac{1}{2\pi^4(\alpha-\beta)^2} \limsup_{s \rightarrow \infty} (\|f(s)\|_{L^2}^2 + \|g(s)\|_{L^2}^2). \end{aligned} \quad (23)$$

Next, integrating (19) on $[t, t+T]$, and using (22),

$$\begin{aligned} &\|v(t+T)\|_{L^2}^2 + \|w(t+T)\|_{L^2}^2 + (\alpha-\beta) \int_t^{t+T} (\|\nabla v(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2) ds \\ &\leq \|v(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2 + \frac{T}{2\pi^2(\alpha-\beta)} \limsup_{s \rightarrow \infty} (\|f(s)\|_{L^2}^2 + \|g(s)\|_{L^2}^2). \end{aligned}$$

Thus, using (23), for $t > t_0$,

$$\int_t^{t+T} (\|\nabla v(s)\|_{L^2}^2 + \|\nabla w(s)\|_{L^2}^2) ds \leq (1 + \pi^2(\alpha-\beta)T)(\alpha-\beta) \limsup_{s \rightarrow \infty} \frac{\|f(s)\|_{L^2}^2 + \|g(s)\|_{L^2}^2}{2\pi^4(\alpha-\beta)^4}, \quad (24)$$

which implies (16). \square

4.1. Proofs of L^2 Convergence Results with Type 1 Interpolants.

Before we get to the proofs of the main theorems, we first collect the various estimates needed for the bilinear term in the following lemma.

Lemma 4.2. *Let $u, v, w \in H^1$ be divergence free. Then the following inequalities hold for any $\epsilon, \delta > 0$:*

$$(a) \quad \left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx dy \right| \leq \frac{c_L \delta}{4} \|\nabla u\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla w\|_{L^2}^2 + \frac{c_L \delta}{4} \|\nabla v\|_{L^2}^2 \|u\|_{L^2}^2 + \frac{c_L^2}{8\epsilon \delta^2} \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2, \quad (25)$$

or

$$\left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx dy \right| \leq \frac{c_L \delta}{4} \|\nabla w\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla u\|_{L^2}^2 + \frac{c_L \delta}{4} \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2 + \frac{c_L^2}{8\epsilon \delta^2} \|\nabla v\|_{L^2}^2 \|u\|_{L^2}^2, \quad (26)$$

$$\begin{aligned} (b) \quad \left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx dy \right| &\leq c\delta \|\nabla u\|_{L^2}^2 + c\delta \|\nabla w\|_{L^2}^2 + \frac{c}{\delta} \|\nabla v\|_{L^2}^2 (\|u_1\|_{L^2}^2 + \|w_1\|_{L^2}^2) \\ &\quad + \frac{c}{\delta} \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \|u_1\|_{L^2}^2 + \frac{c}{\delta} \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla w_1\|_{L^2}}{2\pi \|w_1\|_{L^2}} \right) \|w_1\|_{L^2}^2. \end{aligned} \quad (27)$$

Proof. To show (25), we first apply (5) and (4) then (7) and (4):

$$\begin{aligned}
\left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dxdy \right| &\leq \int_{\Omega} |u| |\nabla v| |w| \, dxdy \leq \|\nabla v\|_{L^2} \|u\|_{L^4} \|w\|_{L^4} \\
&\leq \frac{\delta}{2} \|\nabla v\|_{L^2}^2 \|u\|_{L^4}^2 + \frac{1}{2\delta} \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2 \\
&\leq \frac{c_L \delta}{2} \|\nabla v\|_{L^2}^2 \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \frac{c_L}{2\delta} \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \\
&\leq \frac{c_L \delta}{2} \left(\frac{1}{2} \|\nabla v\|_{L^2}^2 \|u\|_{L^2}^2 + \frac{1}{2} \|\nabla u\|_{L^2}^2 \right) + \frac{1}{2} \frac{c_L^2}{4\epsilon \delta^2} \|\nabla v\|_{L^2}^2 \|w\|_{L^2}^2 + \frac{\epsilon}{2} \|\nabla w\|_{L^2}^2.
\end{aligned}$$

We obtain (26) by switching the roles of u and w after applying (5).

The proof of (27) requires us to estimate the components of the product differently. First, write

$$\left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dxdy \right| = \left| \int_{\Omega} \sum_{i,j=1}^2 u_i \partial_i v_j w_j \, dxdy \right| \leq \sum_{i,j=1}^2 \left| \int_{\Omega} u_i \partial_i v_j w_j \, dxdy \right|,$$

and then we estimate the terms of the sum separately.

(Case: $i = 1, j = 1$) For this case we proceed similarly as in the proof of (25), to obtain:

$$\begin{aligned}
\left| \int_{\Omega} u_1 \partial_1 v_1 w_1 \, dxdy \right| &\leq \|\nabla v_1\|_{L^2} \|u_1\|_{L^4} \|w_1\|_{L^4} \\
&\leq \frac{c_L}{2} \|\nabla v_1\|_{L^2}^2 \|u_1\|_{L^2}^2 \|\nabla u_1\|_{L^2}^2 + \frac{c_L}{2} \|\nabla v_1\|_{L^2}^2 \|w_1\|_{L^2}^2 \|\nabla w_1\|_{L^2}^2 \\
&\leq \frac{c_L \delta}{4} \|\nabla u_1\|_{L^2}^2 + \frac{c_L}{4\delta} \|\nabla v_1\|_{L^2}^2 \|u_1\|_{L^2}^2 + \frac{c_L \delta}{4} \|\nabla w_1\|_{L^2}^2 + \frac{c_L}{4\delta} \|\nabla v_1\|_{L^2}^2 \|w_1\|_{L^2}^2.
\end{aligned}$$

(Case: $i = 1, j = 2$) For this and the next case, we use (8):

$$\begin{aligned}
\left| \int_{\Omega} u_1 \partial_1 v_2 w_2 \, dxdy \right| &\leq c_B \|\nabla w_2\|_{L^2} \|\nabla v_2\|_{L^2} \|u_1\|_{L^2} \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right)^{1/2} \\
&\leq \frac{c_B \delta}{2} \|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v_2\|_{L^2}^2 \|u_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right) \\
&\leq \frac{c_B \delta}{2} \|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v_2\|_{L^2}^2 \|u_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right)
\end{aligned}$$

(Case: $i = 2, j = 1$) Similarly, we obtain:

$$\left| \int_{\Omega} u_2 \partial_2 v_1 w_1 \, dxdy \right| \leq \frac{c_B \delta}{2} \|\nabla u_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v_1\|_{L^2}^2 \|w_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla w_1\|_{L^2}}{2\pi \|w_1\|_{L^2}} \right) \right)$$

(Case: $i = 2, j = 2$) Now we use the divergence free conditions (i.e. $\partial_1 u_1 = -\partial_2 u_2$) and integrate by parts in order to obtain integrals in which the second components of u and w do not appear together:

$$\begin{aligned}
\int_{\Omega} u_2 \partial_2 v_2 w_2 \, dxdy &= - \int_{\Omega} \partial_2 u_2 v_2 w_2 \, dxdy - \int_{\Omega} u_2 v_2 \partial_2 w_2 \, dxdy = \int_{\Omega} \partial_1 u_1 v_2 w_2 \, dxdy + \int_{\Omega} u_2 v_2 \partial_1 w_1 \, dxdy \\
&= - \int_{\Omega} u_1 \partial_1 v_2 w_2 \, dxdy - \int_{\Omega} u_1 v_2 \partial_1 w_2 \, dxdy - \int_{\Omega} \partial_1 u_2 v_2 w_1 \, dxdy - \int_{\Omega} u_2 \partial_1 v_2 w_1 \, dxdy.
\end{aligned}$$

Now, each of these terms can be estimated similarly to the cases where $i \neq j$:

$$\begin{aligned} \left| \int_{\Omega} u_1 \partial_1 v_2 w_2 \, dx dy \right| &\leq \frac{c_B \delta}{2} \|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|u_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right) \\ \left| \int_{\Omega} u_1 v_2 \partial_1 w_2 \, dx dy \right| &\leq \frac{c_B \delta}{2} \|\nabla w_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|u_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right) \\ \left| \int_{\Omega} \partial_1 u_2 v_2 w_1 \, dx dy \right| &\leq \frac{c_B \delta}{2} \|\nabla u_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|w_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla w_1\|_{L^2}}{2\pi \|w_1\|_{L^2}} \right) \right) \\ \left| \int_{\Omega} u_2 \partial_1 v_2 w_1 \, dx dy \right| &\leq \frac{c_B \delta}{2} \|\nabla u_2\|_{L^2}^2 + \frac{c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|w_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla w_1\|_{L^2}}{2\pi \|w_1\|_{L^2}} \right) \right) \end{aligned}$$

Taking the sum of these 7 inequalities obtained from the 4 cases, we have:

$$\begin{aligned} \left| \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx dy \right| &\leq \frac{c_L \delta}{4} \|\nabla u_1\|_{L^2}^2 + \frac{3c_B \delta}{2} \|\nabla u_2\|_{L^2}^2 + \frac{c_L \delta}{4} \|\nabla w_1\|_{L^2}^2 + \frac{3c_B \delta}{2} \|\nabla w_2\|_{L^2}^2 \\ &\quad + \frac{c_L}{4\delta} \|\nabla v\|_{L^2}^2 \|u_1\|_{L^2}^2 + \frac{c_L}{4\delta} \|\nabla v\|_{L^2}^2 \|w_1\|_{L^2}^2 \\ &\quad + \frac{3c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|u_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla u_1\|_{L^2}}{2\pi \|u_1\|_{L^2}} \right) \right) + \frac{3c_B}{2\delta} \|\nabla v\|_{L^2}^2 \|w_1\|_{L^2}^2 \left(1 + \ln \left(\frac{\|\nabla w_1\|_{L^2}}{2\pi \|w_1\|_{L^2}} \right) \right). \end{aligned}$$

Setting $c = \max\{\frac{c_L}{4}, \frac{3c_B}{2}\}$ now yields (27). \square

The following lemma will be used in our analyses of the algorithms using measurements on only the first components of the reference solutions, where we will need to make use of (8), (9), or (27).

Lemma 4.3. *Let $\phi(r) = r - \gamma(1 + \ln(r))$, for some $\gamma > 0$. Then $\forall r \geq 1$,*

$$\phi(r) \geq -\gamma \ln(\gamma).$$

Proof. $\phi'(r) = 1 - \frac{\gamma}{r}$, so $\phi'(r) = 0 \iff r = \gamma$. Therefore,

$$\inf\{\phi(r) : r \geq 1\} = \min\{\phi(1), \phi(\gamma), \lim_{r \rightarrow \infty} \phi(r)\} = \min\{1 - \gamma, -\gamma \ln(\gamma), \infty\} = \min\{1 - \gamma, -\gamma \ln(\gamma)\}.$$

Define $\hat{\phi}(t) = 1 - t - (-t \ln(t)) = 1 + t(\ln(t) - 1)$. Then $\hat{\phi}'(t) = \ln(t) = 0 \iff t = 1$. Therefore

$$\inf\{\hat{\phi}(t) : t \geq 0\} = \min\left\{\lim_{t \rightarrow 0^+} \hat{\phi}(t), \hat{\phi}(1), \lim_{t \rightarrow \infty} \hat{\phi}(t)\right\} = \min\{1, 0, \infty\} = 0.$$

Hence, $1 - \gamma \geq -\gamma \ln(\gamma)$, so $\min\{\phi(r) : r \geq 1\} = -\gamma \ln(\gamma)$. \square

Proof of Theorem 3.1.

Let $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$.

Then η satisfies:

$$\partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (w \cdot \nabla) v - (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta).$$

Using the fact that $(w \cdot \nabla) v - (\tilde{w} \cdot \nabla) \tilde{v} = (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta$ we write:

$$\partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta = -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta).$$

Taking the inner product with η we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \alpha \|\nabla \eta\|_{L^2}^2 - \beta \langle \nabla \zeta, \nabla \eta \rangle + \langle (\zeta \cdot \nabla) v, \eta \rangle = -\langle \nabla(\pi - \tilde{\pi}), \eta \rangle - \mu \langle \mathbf{I}_h(\eta), \eta \rangle.$$

Now, by the divergence free condition,

$$-\langle \nabla(\pi - \tilde{\pi}), \eta \rangle := - \int_{\Omega} \nabla(\pi - \tilde{\pi}) \cdot \eta \, dxdy = \int_{\Omega} (\pi - \tilde{\pi}) \cdot (\nabla \cdot \eta) \, dxdy = 0.$$

By applying Cauchy-Schwarz inequality and (4),

$$|\beta \langle \nabla \zeta, \nabla \eta \rangle| \leq \frac{\beta}{2} \|\nabla \eta\|_{L^2}^2 + \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2,$$

and by rewriting $\langle I_h(\eta), \eta \rangle = \langle I_h(\eta) - \eta, \eta \rangle + \langle \eta, \eta \rangle$, we have:

$$-\mu \langle I_h(\eta), \eta \rangle = -\mu \langle I_h(\eta) - \eta, \eta \rangle - \mu \|\eta\|_{L^2}^2.$$

Thus, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \mu \|\eta\|_{L^2}^2 &\leq -\langle (\zeta \cdot \nabla) v, \eta \rangle - \mu \langle I_h(\eta) - \eta, \eta \rangle \\ &\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu |\langle I_h(\eta) - \eta, \eta \rangle| \\ &\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu \|I_h(\eta) - \eta\|_{L^2} \|\eta\|_{L^2} \\ &\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu c_1 h \|\nabla \eta\|_{L^2} \|\eta\|_{L^2} \\ &\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \frac{\mu c_1^2 h^2}{2} \|\nabla \eta\|_{L^2}^2 + \frac{\mu}{2} \|\eta\|_{L^2}^2, \end{aligned}$$

where in the last three lines we used Cauchy-Schwarz inequality, the definition of I_h , and Young's inequality. This leaves us with:

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \frac{\mu}{2} \|\eta\|_{L^2}^2 \leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \frac{\mu c_1^2 h^2}{2} \|\nabla \eta\|_{L^2}^2.$$

Proceeding the same way for ζ , we have the following equations:

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \frac{\mu}{2} \|\eta\|_{L^2}^2 \leq \left| \int_{\Omega} (\zeta \cdot \nabla) v \cdot \eta \, dxdy \right|, \quad (28)$$

$$\frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} \right) \|\nabla \zeta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \eta\|_{L^2}^2 + \frac{\mu}{2} \|\zeta\|_{L^2}^2 \leq \left| \int_{\Omega} (\eta \cdot \nabla) w \cdot \zeta \, dxdy \right|. \quad (29)$$

We estimate the integrals in these equations using (25), with $\epsilon = \frac{\alpha - \beta}{2}$ and $\delta = \frac{\alpha - \beta}{c_L}$, and obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - \frac{\alpha - \beta}{4} \right) \|\nabla \eta\|_{L^2}^2 + \left(-\frac{\beta}{2} - \frac{\alpha - \beta}{4} \right) \|\nabla \zeta\|_{L^2}^2 \\ + \left(\frac{\mu}{2} - \frac{c_L^4}{4(\alpha - \beta)^3} \|\nabla v\|_{L^2}^2 \right) \|\eta\|_{L^2}^2 + \left(-\frac{\alpha - \beta}{4} \|\nabla v\|_{L^2}^2 \right) \|\zeta\|_{L^2}^2 \leq 0, \end{aligned} \quad (30)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - \frac{\alpha - \beta}{4} \right) \|\nabla \zeta\|_{L^2}^2 + \left(-\frac{\beta}{2} - \frac{\alpha - \beta}{4} \right) \|\nabla \eta\|_{L^2}^2 \\ + \left(\frac{\mu}{2} - \frac{c_L^4}{4(\alpha - \beta)^3} \|\nabla w\|_{L^2}^2 \right) \|\zeta\|_{L^2}^2 + \left(-\frac{\alpha - \beta}{4} \|\nabla w\|_{L^2}^2 \right) \|\eta\|_{L^2}^2 \leq 0. \end{aligned} \quad (31)$$

Then, adding (30) and (31), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\frac{\alpha - \beta}{2} - \frac{\mu c_1^2 h^2}{2} \right) (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) \\ + \left[\frac{\mu}{2} - \left(\frac{c_L^4}{4(\alpha - \beta)^3} + \frac{\alpha - \beta}{4} \right) (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \right] (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2) \leq 0. \end{aligned} \quad (32)$$

Thus, defining $Y(t) = \|\eta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2$ and $Z(t) = \|\nabla v(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2$, we have

$$\frac{d}{dt} Y + \psi Y \leq 0, \quad (33)$$

where $\psi(t) := \mu - \left(\frac{c_L^4 + (\alpha - \beta)^4}{2(\alpha - \beta)^3} \right) Z(t)$, provided that $\mu c_1^2 h^2 \leq \alpha - \beta$.

By Proposition 4.1 with $T = \frac{1}{\pi^2(\alpha - \beta)}$, ψ satisfies (10b) and if

$$\mu - \frac{c_L^4 + (\alpha - \beta)^4}{2T(\alpha - \beta)^3} (1 + T\pi^2(\alpha - \beta))(\alpha - \beta)G^2 > 0 \iff \mu > \frac{\pi^2(c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2,$$

then ψ also satisfies (10a), so we can apply Proposition 2.1 to Y and conclude that (\tilde{v}, \tilde{w}) converges exponentially in time to (v, w) .

The requirement on h is

$$h < \frac{\alpha - \beta}{\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1},$$

so $h \sim G^{-1}$.

□

Proof of Theorem 3.2.

Let $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$. Then η satisfies:

$$\partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (w \cdot \nabla) v - (\tilde{w} \cdot \nabla) \tilde{v} = -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta_1) e_1.$$

Using the fact that $(w \cdot \nabla) v - (\tilde{w} \cdot \nabla) \tilde{v} = (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta$ we write:

$$\partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta = -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta_1) e_1.$$

Taking the inner product with η we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \alpha \|\nabla \eta\|_{L^2}^2 - \beta \langle \nabla \zeta, \nabla \eta \rangle + \langle (\zeta \cdot \nabla) v, \eta \rangle = -\langle \nabla(\pi - \tilde{\pi}), \eta \rangle - \mu \langle \mathbf{I}_h(\eta_1), \eta_1 \rangle.$$

Now, by the divergence free condition, we have:

$$-\langle \nabla(\pi - \tilde{\pi}), \eta \rangle := -\int_{\Omega} \nabla(\pi - \tilde{\pi}) \cdot \eta \, dx dy = \int_{\Omega} (\pi - \tilde{\pi}) \cdot (\nabla \cdot \eta) \, dx dy = 0.$$

By applying Cauchy-Schwarz inequality and (4),

$$|\beta \langle \nabla \zeta, \nabla \eta \rangle| \leq \frac{\beta}{2} \|\nabla \eta\|_{L^2}^2 + \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2,$$

and by rewriting $\langle \mathbf{I}_h(\eta_1), \eta_1 \rangle = \langle \mathbf{I}_h(\eta_1) - \eta_1, \eta_1 \rangle + \langle \eta_1, \eta_1 \rangle$, we have:

$$-\mu \langle \mathbf{I}_h(\eta_1), \eta_1 \rangle = -\mu \langle \mathbf{I}_h(\eta_1) - \eta_1, \eta_1 \rangle - \mu \|\eta_1\|_{L^2}^2.$$

Thus, we obtain:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \mu \|\eta_1\|_{L^2}^2 &\leq -\langle (\zeta \cdot \nabla) v, \eta \rangle - \mu \langle \mathbf{I}_h(\eta_1) - \eta_1, \eta_1 \rangle \\
&\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu |\langle \mathbf{I}_h(\eta_1) - \eta_1, \eta_1 \rangle| \\
&\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu \|\mathbf{I}_h(\eta_1) - \eta_1\|_{L^2} \|\eta_1\|_{L^2} \\
&\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \mu c_1 h \|\nabla \eta_1\|_{L^2} \|\eta_1\|_{L^2} \\
&\leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \frac{\mu c_1^2 h^2}{2} \|\nabla \eta_1\|_{L^2}^2 + \frac{\mu}{2} \|\eta_1\|_{L^2}^2,
\end{aligned}$$

where in the last three lines we used the Cauchy-Schwarz inequality, the definition of \mathbf{I}_h , and Young's inequality. This leaves us with:

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \frac{\mu}{2} \|\eta_1\|_{L^2}^2 \leq |\langle (\zeta \cdot \nabla) v, \eta \rangle| + \frac{\mu c_1^2 h^2}{2} \|\nabla \eta_1\|_{L^2}^2,$$

or equivalently,

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \frac{\mu}{2} \|\eta_1\|_{L^2}^2 \leq \left| \int_{\Omega} (\zeta \cdot \nabla) v \cdot \eta \, dx dy \right|. \quad (34)$$

Now we apply Lemma 4.2 to estimate the nonlinear term with (27), yielding:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - c\delta \right) \|\nabla \eta\|_{L^2}^2 + \left(-\frac{\beta}{2} - c\delta \right) \|\nabla \zeta\|_{L^2}^2 \\
+ \left[\frac{\mu}{2} - \frac{c}{\delta} \|\nabla v\|_{L^2}^2 - \frac{c}{\delta} \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}}{2\pi \|\eta_1\|_{L^2}} \right) \right] \|\eta_1\|_{L^2}^2 \\
+ \left[-\frac{c}{\delta} \|\nabla v\|_{L^2}^2 - \frac{c}{\delta} \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla \zeta_1\|_{L^2}}{2\pi \|\zeta_1\|_{L^2}} \right) \right] \|\zeta_1\|_{L^2}^2 \leq 0. \quad (35)
\end{aligned}$$

Proceeding similarly with ζ we obtain:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - c\delta \right) \|\nabla \zeta\|_{L^2}^2 + \left(-\frac{\beta}{2} - c\delta \right) \|\nabla \eta\|_{L^2}^2 \\
+ \left[\frac{\mu}{2} - \frac{c}{\delta} \|\nabla w\|_{L^2}^2 - \frac{c}{\delta} \|\nabla w\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla \zeta_1\|_{L^2}}{2\pi \|\zeta_1\|_{L^2}} \right) \right] \|\zeta_1\|_{L^2}^2 \\
+ \left[-\frac{c}{\delta} \|\nabla w\|_{L^2}^2 - \frac{c}{\delta} \|\nabla w\|_{L^2}^2 \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}}{2\pi \|\eta_1\|_{L^2}} \right) \right] \|\eta_1\|_{L^2}^2 \leq 0. \quad (36)
\end{aligned}$$

Now, adding (35) and (36) and defining $Z(t) = \|\nabla v(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2$,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - 2c\delta \right) \|\nabla \eta\|_{L^2}^2 + \left(\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - 2c\delta \right) \|\nabla \zeta\|_{L^2}^2 \\
+ \left[\frac{\mu}{2} - \frac{c}{\delta} Z - \frac{c}{\delta} Z \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}}{2\pi \|\eta_1\|_{L^2}} \right) \right] \|\eta_1\|_{L^2}^2 \\
+ \left[\frac{\mu}{2} - \frac{c}{\delta} Z - \frac{c}{\delta} Z \left(1 + \ln \frac{\|\nabla \zeta_1\|_{L^2}}{2\pi \|\zeta_1\|_{L^2}} \right) \right] \|\zeta_1\|_{L^2}^2 \leq 0. \quad (37)
\end{aligned}$$

Since $\alpha > \beta$,

$$\gamma := (\alpha - \beta) - \frac{\mu c_1^2 h^2}{2} - 2c\delta \geq \frac{(\alpha - \beta)}{4} > 0,$$

provided that $h \leq (\alpha - \beta)^{\frac{1}{2}} c_1^{-1} \mu^{-\frac{1}{2}}$ and by choosing $\delta = \frac{(\alpha - \beta)}{8c}$.

We want to apply Lemma 4.3 to the logarithmic terms in (37). To this end note that by (6), $\frac{\|\nabla \eta_1\|_{L^2}}{2\pi\|\eta_1\|_{L^2}} \geq 1$, so $\ln \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \geq \ln \frac{\|\nabla \eta_1\|_{L^2}^2}{2\pi\|\eta_1\|_{L^2}^2}$. Next, we write

$$\gamma \|\nabla \eta\|_{L^2}^2 \geq \frac{\gamma}{2} \|\nabla \eta\|_{L^2}^2 + \frac{4\pi^2\gamma}{2} \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \|\eta_1\|_{L^2}^2,$$

and consider

$$\begin{aligned} 2\pi^2\gamma \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \|\eta\|_{L^2}^2 - \frac{c}{\delta} Z \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \right) \|\eta_1\|_{L^2}^2 \\ = 2\pi^2\gamma \left(\frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} - \frac{c}{2\pi^2\gamma\delta} Z \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \right) \right) \|\eta_1\|_{L^2}^2. \end{aligned}$$

By Lemma 4.3,

$$\frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} - \frac{c}{2\pi^2\gamma\delta} Z \left(1 + \ln \frac{\|\nabla \eta_1\|_{L^2}^2}{4\pi^2\|\eta_1\|_{L^2}^2} \right) \geq -\frac{c}{2\pi^2\gamma\delta} Z \left(\ln \frac{c}{2\pi^2\gamma\delta} Z \right). \quad (38)$$

Hence, using (38) and defining $Y(t) = \|\eta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2$, we rewrite (37) as

$$\frac{1}{2} \frac{d}{dt} Y + \frac{\gamma}{2} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) + \left[\frac{\mu}{2} - \frac{c}{\delta} Z \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z \right) \right] (Y) \leq 0.$$

By (6),

$$\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2 \geq 4\pi^2 (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2) \geq 4\pi^2 (\|\eta_2\|_{L^2}^2 + \|\zeta_2\|_{L^2}^2),$$

and so

$$\frac{d}{dt} Y + \min \left\{ 4\pi^2\gamma, \mu - \frac{2c}{\delta} Z \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z \right) \right\} Y \leq 0. \quad (39)$$

Let

$$\psi(t) := \min \left\{ 4\pi^2\gamma, \mu - \frac{2c}{\delta} Z(t) \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z(t) \right) \right\},$$

and in order to apply Proposition 2.1 we only need to show that ψ satisfies (10a) and (10b). It is sufficient to show that for some $T, t_0 > 0$,

$$\mu - \limsup_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \frac{2c}{\delta} Z(s) \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z(s) \right) ds > 0, \quad (40)$$

and

$$\sup_{s > t_0} Z(s) \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z(s) \right) ds < \infty. \quad (41)$$

In fact, (41) follows directly from (17) with the t_0 given there.

To see (40), by Proposition 4.1 with $T = \frac{1}{\pi^2(\alpha-\beta)}$, we have:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \frac{2c}{\delta} Z(s) \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} Z(s) \right) ds \\ \leq \frac{2c}{\delta T} \left(1 + \ln \frac{c}{2\pi^2\gamma\delta} 10\pi^2(\alpha-\beta)^2 G^2 e^{CG^4} \right) \lim_{t \rightarrow \infty} \int_t^{t+T} Z(s) ds \\ \leq \frac{2c}{\delta T} (\tilde{c} + 2 \ln G + CG^4) (1 + T\pi^2(\alpha-\beta))(\alpha-\beta)G^2, \\ = 32\pi^2 c^2 (\alpha-\beta) (\tilde{c} + 2 \ln G + CG^4) G^2. \end{aligned}$$

Therefore, (40) holds by choosing $\mu > 32\pi^2 c^2 (\alpha-\beta) (\tilde{c} + 2 \ln G + CG^4) G^2$. In addition, the requirement $h \leq \frac{(\alpha-\beta)^{\frac{1}{2}}}{c_1} \mu^{-\frac{1}{2}}$ implies $h \sim G^{-3}$. □

Proof of Theorem 3.3.

Let $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$. Similarly to how we showed (34), the equation we obtain for η is

$$\frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} \right) \|\nabla \eta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \zeta\|_{L^2}^2 + \frac{\mu}{2} \|\eta\|_{L^2}^2 \leq \left| \int_{\Omega} (\zeta \cdot \nabla) v \cdot \eta \, dx dy \right|, \quad (42)$$

but now the equation for ζ is

$$\frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\nabla \zeta\|_{L^2}^2 - \frac{\beta}{2} \|\nabla \eta\|_{L^2}^2 \leq \left| \int_{\Omega} (\eta \cdot \nabla) w \cdot \zeta \, dx dy \right|. \quad (43)$$

We estimate the integral in (42) using (25), so (42) becomes:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{\mu c_1^2 h^2}{2} - \frac{\epsilon}{2} \right) \|\nabla \eta\|_{L^2}^2 + \left(-\frac{\beta}{2} - \frac{c_L \delta}{4} \right) \|\nabla \zeta\|_{L^2}^2 \\ + \left(\frac{\mu}{2} - \frac{c_L^2}{8\epsilon \delta^2} \|\nabla v\|_{L^2}^2 \right) \|\eta\|_{L^2}^2 + \left(-\frac{c_L \delta}{4} \|\nabla v\|_{L^2}^2 \right) \|\zeta\|_{L^2}^2 \leq 0, \end{aligned} \quad (44)$$

Similarly, we estimate the integral in (43) using (26), and get:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} - \frac{c_L \delta}{4} \right) \|\nabla \zeta\|_{L^2}^2 + \left(-\frac{\beta}{2} - \frac{\epsilon}{2} \right) \|\nabla \eta\|_{L^2}^2 \\ + \left(-\frac{c_L \delta}{4} \|\nabla w\|_{L^2}^2 \right) \|\zeta\|_{L^2}^2 + \left(-\frac{c_L^2}{8\epsilon \delta^2} \|\nabla w\|_{L^2}^2 \right) \|\eta\|_{L^2}^2 \leq 0. \end{aligned} \quad (45)$$

Adding (44) and (45),

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\eta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\zeta\|_{L^2}^2 + \left(\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - \epsilon \right) \|\nabla \eta\|_{L^2}^2 + \left(\alpha - \beta - \frac{c_L \delta}{2} \right) \|\nabla \zeta\|_{L^2}^2 \\ + \left(\frac{\mu}{2} - \frac{c_L^2}{8\epsilon \delta^2} (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \right) \|\eta\|_{L^2}^2 + \left(-\frac{c_L \delta}{4} (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \right) \|\zeta\|_{L^2}^2 \leq 0. \end{aligned}$$

Now, if we choose

$$h \leq \frac{(\alpha-\beta)^{1/2}}{c_1} \mu^{-1/2},$$

and $\epsilon = \frac{\alpha - \beta}{2}$, then $\alpha - \beta - \frac{\mu c_1^2 h^2}{2} - \epsilon \geq 0$.

Also, by choosing $\delta < \frac{\alpha - \beta}{c_L}$, we have

$$\gamma := \alpha - \beta - \frac{c_L \delta}{2} > \frac{\alpha - \beta}{2} > 0.$$

Then by applying (6) we obtain $\gamma \|\nabla \zeta\|_{L^2}^2 \geq \gamma 4\pi^2 \|\zeta\|_{L^2}^2$. Hence, defining $Y(t) = \|\eta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2$ and $Z(t) = \|\nabla v(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2$, we have:

$$\frac{d}{dt} Y + \psi Y \leq 0, \quad (46)$$

where $\psi(t) := \min \left\{ \mu - \frac{c_L^2}{4\epsilon\delta^2} Z(t), 8\pi^2\gamma - \frac{c_L\delta}{2} Z(t) \right\}$. Using Proposition 4.1 similarly as before, with $T = \frac{1}{\pi^2(\alpha - \beta)}$, ψ satisfies (10b) as well as (10a) provided that

$$\delta < \frac{\alpha - \beta}{c_L} \frac{4}{4 + (\alpha - \beta)^2 G^2} \implies 8\pi^2\gamma - \frac{c_L\delta}{2T} (1 + T\pi^2(\alpha - \beta))(\alpha - \beta)G^2 > 4(\alpha - \beta) > 0,$$

and

$$\mu > \frac{\pi^2 c_L^4 G^2 (4 + (\alpha - \beta)^2 G^2)^2}{16(\alpha - \beta)} \implies \mu - \frac{c_L^2}{4\epsilon\delta^2 T} (1 + T\pi^2(\alpha - \beta))(\alpha - \beta)G^2 > 0.$$

By choosing such a μ and δ , we can apply Proposition 2.1 to conclude that (\tilde{v}, \tilde{w}) converges exponentially in time to (v, w) .

Now the requirement we needed on h is

$$h < \frac{4(\alpha - \beta)}{\pi c_1 c_L^2 G (4 + (\alpha - \beta)^2 G^2)},$$

so $h \sim G^{-3}$.

□

4.2. Proof of H^1 Convergence Results with Type 1 Interpolants.

Proof of Theorem 3.4.

By denoting $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$ and subtracting the equations for (v, w) and (\tilde{v}, \tilde{w}) , we obtain the following equation for η and ζ

$$\begin{aligned} \partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta &= -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta), \\ \partial_t \zeta - \alpha \Delta \zeta + \beta \Delta \eta + (\eta \cdot \nabla) w + (\tilde{v} \cdot \nabla) \zeta &= -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\zeta). \end{aligned}$$

Taking the inner product with $-\Delta \eta$ and $-\Delta \zeta$, respectively, we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \alpha \|\Delta \eta\|_{L^2}^2 &= \beta \langle \Delta \zeta, \Delta \eta \rangle + \langle (\zeta \cdot \nabla) v, \Delta \eta \rangle + \langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle + \langle \nabla(\pi - \tilde{\pi}), \Delta \eta \rangle + \mu \langle \mathbf{I}_h(\eta), \Delta \eta \rangle, \\ \frac{1}{2} \frac{d}{dt} \|\nabla \zeta\|_{L^2}^2 + \alpha \|\Delta \zeta\|_{L^2}^2 &= \beta \langle \Delta \eta, \Delta \zeta \rangle + \langle (\eta \cdot \nabla) w, \Delta \zeta \rangle + \langle (\tilde{v} \cdot \nabla) \zeta, \Delta \zeta \rangle + \langle \nabla(\pi - \tilde{\pi}), \Delta \zeta \rangle + \mu \langle \mathbf{I}_h(\zeta), \Delta \zeta \rangle. \end{aligned}$$

Then, by the divergence-free condition,

$$\langle \nabla(\pi - \tilde{\pi}), \Delta \eta \rangle = - \int_{\Omega} (\pi - \tilde{\pi}) \cdot \Delta(\nabla \cdot \eta) dx dy = 0,$$

and similarly

$$\langle \nabla(\pi - \tilde{\pi}), \Delta \zeta \rangle = 0.$$

Also, by applying Cauchy-Schwarz inequality and (4), we have

$$\beta \langle \Delta \zeta, \Delta \eta \rangle \leq \frac{\beta}{2} \|\Delta \eta\|_{L^2}^2 + \frac{\beta}{2} \|\Delta \zeta\|_{L^2}^2.$$

Rewriting $\langle \mathbf{I}_h(\eta), -\Delta \eta \rangle = \langle \mathbf{I}_h(\eta) - \eta, -\Delta \eta \rangle + \langle \eta, \Delta \eta \rangle$, we have,

$$-\mu \langle \mathbf{I}_h(\eta), -\Delta \eta \rangle = -\mu \langle \mathbf{I}_h(\eta) - \eta, -\Delta \eta \rangle - \mu \|\nabla \eta\|_{L^2}^2,$$

and similarly,

$$-\mu \langle \mathbf{I}_h(\zeta), -\Delta \zeta \rangle = -\mu \langle \mathbf{I}_h(\zeta) - \zeta, -\Delta \zeta \rangle - \mu \|\nabla \zeta\|_{L^2}^2.$$

Adding up the equations for η and ζ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) + (\alpha - \beta) (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2) \\ & \leq |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| + |\langle (\eta \cdot \nabla) w, \Delta \zeta \rangle| + |\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle| + |\langle (\tilde{v} \cdot \nabla) \zeta, \Delta \zeta \rangle| \\ & \quad + \mu |\langle \mathbf{I}_h(\eta) - \eta, \Delta \eta \rangle| + \mu |\langle \mathbf{I}_h(\zeta) - \zeta, \Delta \zeta \rangle| - \mu (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2). \end{aligned}$$

Due to the properties of \mathbf{I}_h , we have

$$\begin{aligned} \mu |\langle \mathbf{I}_h(\eta) - \eta, \Delta \eta \rangle| & \leq \mu \|\mathbf{I}_h(\eta) - \eta\|_{L^2} \|\Delta \eta\|_{L^2} \leq \mu c_1 h \|\nabla \eta\|_{L^2} \|\Delta \eta\|_{L^2} \\ & \leq \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} \|\nabla \eta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2, \end{aligned}$$

and similarly, we obtain

$$\mu |\langle \mathbf{I}_h(\zeta) - \zeta, \Delta \zeta \rangle| \leq \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} \|\nabla \zeta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta \zeta\|_{L^2}^2.$$

Next, we estimate the nonlinear terms. First, by Hölder's and Sobolev inequalities, we obtain

$$\begin{aligned} |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| & \leq \int_{\Omega} |\zeta| |\nabla v| |\Delta \eta| dx dy \leq \|\zeta\|_{L^4} \|\nabla v\|_{L^4} \|\Delta \eta\|_{L^2} \\ & \leq \|\zeta\|_{L^2}^{1/2} \|\nabla \zeta\|_{L^2}^{1/2} \|\nabla v\|_{L^2}^{1/2} \|\Delta v\|_{L^2}^{1/2} \|\Delta \eta\|_{L^2} \\ & \leq \frac{4}{\alpha - \beta} \|\nabla v\|_{L^2} \|\Delta v\|_{L^2} \|\zeta\|_{L^2} \|\nabla \zeta\|_{L^2} + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\ & \leq \frac{4}{2\pi(\alpha - \beta)} \|\nabla v\|_{L^2} \|\Delta v\|_{L^2} \|\zeta\|_{L^2} \|\Delta \zeta\|_{L^2} + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\ & \leq \frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 \|\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{16} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2), \end{aligned}$$

where we used Poincaré's and Young's inequalities. The estimate for $\langle (\eta \cdot \nabla) w, \Delta \zeta \rangle$ is similarly, i.e., we have

$$|\langle (\eta \cdot \nabla) w, \Delta \zeta \rangle| \leq \frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla w\|_{L^2}^2 \|\Delta w\|_{L^2}^2 \|\eta\|_{L^2}^2 + \frac{\alpha - \beta}{16} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2).$$

Regarding $\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle$, we first rewrite it as

$$\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle = \langle (w \cdot \nabla) \eta, \Delta \eta \rangle - \langle (\zeta \cdot \nabla) \eta, \Delta \eta \rangle = \mathbf{(a)} + \mathbf{(b)}.$$

In order to estimate $\mathbf{(a)}$, we first observe that by the periodic boundary conditions, we have

$$\|\nabla \eta\|_{L^2}^2 = \int_{\Omega} \nabla \eta \cdot \nabla \eta dx dy = - \int_{\Omega} \eta \Delta \eta dx dy \leq \|\eta\|_{L^2} \|\Delta \eta\|_{L^2}. \quad (47)$$

Thus, we integrate by parts and proceed to estimate **(a)** as

$$\begin{aligned}
\langle (w \cdot \nabla) \eta, \Delta \eta \rangle &= \sum_{i,j,k=1}^2 \int_{\Omega} w_i \partial_i \eta_k \partial_{jj}^2 \eta_k \, dx dy = - \sum_{i,j,k=1}^2 \int_{\Omega} \partial_j w_i \partial_i \eta_k \partial_j \eta_k \, dx dy \\
&\leq \int_{\Omega} |\nabla w| |\nabla \eta|^2 \, dx dy \leq \|\nabla w\|_{L^2} \|\nabla \eta\|_{L^2} \|\Delta \eta\|_{L^2} \\
&\leq \frac{4}{\alpha - \beta} \|\nabla w\|_{L^2}^2 \|\nabla \eta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\
&\leq \frac{4}{\alpha - \beta} \|\nabla w\|_{L^2}^2 \|\eta\|_{L^2} \|\Delta \eta\|_{L^2} + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\
&\leq \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla w\|_{L^2}^4 \|\eta\|_{L^2}^2 + \frac{\alpha - \beta}{8} \|\Delta \eta\|_{L^2}^2.
\end{aligned}$$

By similar estimates and the analogy of (47) for ζ , i.e.,

$$\|\nabla \zeta\|_{L^2}^2 \leq \|\zeta\|_{L^2} \|\Delta \zeta\|_{L^2},$$

we estimate **(b)** as

$$\begin{aligned}
- \langle (\zeta \cdot \nabla) \eta, \Delta \eta \rangle &\leq \int_{\Omega} |\nabla \zeta| |\nabla \eta|^2 \, dx dy \leq \|\nabla \zeta\|_{L^2} \|\nabla \eta\|_{L^2} \|\Delta \eta\|_{L^2} \\
&\leq \frac{4}{\alpha - \beta} \|\nabla \eta\|_{L^2}^2 \|\nabla \zeta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\
&\leq \frac{4}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} \|\Delta \eta\|_{L^2} \|\Delta \zeta\|_{L^2} + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2 \\
&\leq \frac{2}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2) + \frac{\alpha - \beta}{16} \|\Delta \eta\|_{L^2}^2.
\end{aligned}$$

By a similar approach, we have

$$\langle (\tilde{v} \cdot \nabla) \zeta, \Delta \zeta \rangle = \langle (v \cdot \nabla) \zeta, \Delta \zeta \rangle - \langle (\eta \cdot \nabla) \zeta, \Delta \zeta \rangle = \mathbf{(c)} + \mathbf{(d)},$$

and **(c)** is bounded by

$$|\langle (v \cdot \nabla) \zeta, \Delta \zeta \rangle| \leq \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla v\|_{L^2}^4 \|\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{8} \|\Delta \zeta\|_{L^2}^2,$$

while we estimate **(d)** as

$$- \langle (\eta \cdot \nabla) \zeta, \Delta \zeta \rangle \leq \frac{2}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} (\|\Delta \zeta\|_{L^2}^2 + \|\Delta \eta\|_{L^2}^2) + \frac{\alpha - \beta}{16} \|\Delta \zeta\|_{L^2}^2.$$

Combining all the above estimates, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) + \underbrace{\left(\frac{\alpha - \beta}{2} - \frac{4}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} \right)}_{\text{(e)}} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2) \\
& \leq \underbrace{\left[\left(\frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 (\|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \|\Delta w\|_{L^2}^2) \right)}_{\text{(f)}} \right. \\
& \quad \left. + \underbrace{\left(\frac{4}{\alpha - \beta} \right)^3 (\|\nabla v\|_{L^2}^4 + \|\nabla w\|_{L^2}^4)}_{\text{(g)}} \right] (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2)}_{\text{(g)}} \\
& \quad + \underbrace{\left(\frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} - \mu \right)}_{\text{(i)}} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2). \tag{48}
\end{aligned}$$

Now choose h such that

$$\text{(i)} = \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} < \frac{\mu}{2}.$$

Thus, we have

$$h^2 < \frac{\alpha - \beta}{8\mu c_1^2}. \tag{49}$$

Moreover, by Theorem 3.1, we know that after a sufficiently large time T_1 , $\|\eta\|_{L^2}$ and $\|\zeta\|_{L^2}$ are small enough, so that we have

$$\|\eta\|_{L^2} \|\zeta\|_{L^2} \leq \frac{(\alpha - \beta)^2}{16},$$

which implies that

$$\frac{\alpha - \beta}{2} - \frac{4}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} \geq 0,$$

so we have:

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) + \frac{\mu}{2} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) \leq ((\text{f}) + (\text{g})) (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2).$$

Define $Y(t) = \|\nabla \eta(t)\|_{L^2}^2 + \|\nabla \zeta(t)\|_{L^2}^2$, and by appealing to Proposition 4.1, we can say that $(\text{f}) + (\text{g})$ is bounded by some number $\frac{M_G}{2}$. Also, by Theorem 3.1 we know that there exists constants $K, a > 0$ such that $\|\eta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2 \leq K e^{-at}$, $\forall t \geq T_1$. Putting all of this together, we have the following for all $t > T_1$:

$$\begin{aligned}
& \frac{d}{dt} Y(t) + \mu Y(t) \leq M_G K e^{-at}, \\
& \Rightarrow \frac{d}{dt} (e^{\mu t} Y(t)) \leq M_G K e^{(\mu - a)t}, \\
& \Rightarrow e^{\mu t} Y(t) - e^{\mu T_1} Y(T_1) \leq \frac{M_G K}{\mu - a} e^{(\mu - a)t} - \frac{M_G K}{\mu - a} e^{(\mu - a)T_1}, \\
& \Rightarrow Y(t) \leq Y(T_1) e^{-\mu(t - T_1)} + \frac{M_G K}{\mu - a} (e^{-at} - e^{-\mu(t - T_1) - aT_1}).
\end{aligned}$$

Therefore, $Y(t) = \|\nabla\eta(t)\|_{L^2}^2 + \|\nabla\zeta(t)\|_{L^2}^2 \rightarrow 0$ exponentially as $t \rightarrow \infty$ as long as μ and h satisfy the conditions of Theorem 3.1, as well as the new requirement (49). So, choosing

$$\mu > \frac{\pi^2(c_L^4 + (\alpha - \beta)^4)}{\alpha - \beta} G^2, \quad \text{and} \quad h < \frac{\alpha - \beta}{2\sqrt{2}\pi c_1 \sqrt{c_L^4 + (\alpha - \beta)^4}} G^{-1},$$

we have exponential convergence. \square

Next, we prove the H^1 decay estimates for the data assimilation scenario where measurement is only on v_1 and w_1 .

Proof of Theorem 3.5.

We still denote the difference of solutions to (11) and (14) by $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$. Similarly to the beginning of the proof of Theorem 3.4, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla\eta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2) + (\alpha - \beta) (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2) \\ & \leq |\langle (\zeta \cdot \nabla) v, \Delta\eta \rangle| + |\langle (\eta \cdot \nabla) w, \Delta\zeta \rangle| + |\langle (\tilde{w} \cdot \nabla) \eta, \Delta\eta \rangle| + |\langle (\tilde{v} \cdot \nabla) \zeta, \Delta\zeta \rangle| \\ & \quad + \mu |\langle \mathbb{I}_h(\eta_1) - \eta_1, \Delta\eta_1 \rangle| + \mu |\langle \mathbb{I}_h(\zeta_1) - \eta_1, \Delta\zeta_1 \rangle| - \mu \|\nabla\eta_1\|_{L^2}^2 - \mu \|\nabla\zeta_1\|_{L^2}^2, \end{aligned}$$

as well as

$$\mu |\langle \mathbb{I}_h(\eta_1) - \eta_1, \Delta\eta_1 \rangle| \leq \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} \|\nabla\eta_1\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta\eta_1\|_{L^2}^2,$$

and

$$\mu |\langle \mathbb{I}_h(\zeta_1) - \eta, \Delta\zeta_1 \rangle| \leq \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} \|\nabla\zeta_1\|_{L^2}^2 + \frac{\alpha - \beta}{16} \|\Delta\zeta_1\|_{L^2}^2.$$

The estimates for the nonlinear terms are also similar. Namely, we have

$$|\langle (\zeta \cdot \nabla) v, \Delta\eta \rangle| \leq \frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 \|\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{16} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2),$$

and

$$|\langle (\eta \cdot \nabla) w, \Delta\zeta \rangle| \leq \frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla w\|_{L^2}^2 \|\Delta w\|_{L^2}^2 \|\eta\|_{L^2}^2 + \frac{\alpha - \beta}{16} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2).$$

Also, by rewriting

$$\langle (\tilde{w} \cdot \nabla) \eta, \Delta\eta \rangle = \langle (w \cdot \nabla) \eta, \Delta\eta \rangle - \langle (\zeta \cdot \nabla) \eta, \Delta\eta \rangle$$

we obtain

$$\langle (w \cdot \nabla) \eta, \Delta\eta \rangle \leq \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla w\|_{L^2}^4 \|\eta\|_{L^2}^2 + \frac{\alpha - \beta}{8} \|\Delta\eta\|_{L^2}^2,$$

and

$$-\langle (\zeta \cdot \nabla) \eta, \Delta\eta \rangle \leq \frac{2}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2) + \frac{\alpha - \beta}{16} \|\Delta\eta\|_{L^2}^2.$$

Estimates for

$$\langle (\tilde{v} \cdot \nabla) \zeta, \Delta\zeta \rangle = \langle (v \cdot \nabla) \zeta, \Delta\zeta \rangle - \langle (\eta \cdot \nabla) \zeta, \Delta\zeta \rangle$$

also follow similarly, and we obtain

$$\langle (v \cdot \nabla) \zeta, \Delta\zeta \rangle \leq \left(\frac{4}{\alpha - \beta} \right)^3 \|\nabla v\|_{L^2}^4 \|\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{8} \|\Delta\zeta\|_{L^2}^2,$$

and

$$-\langle (\eta \cdot \nabla) \zeta, \Delta \zeta \rangle \leq \frac{2}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} (\|\Delta \zeta\|_{L^2}^2 + \|\Delta \eta\|_{L^2}^2) + \frac{\alpha - \beta}{16} \|\Delta \zeta\|_{L^2}^2.$$

Combining all the above estimates, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2) + \underbrace{\left(\frac{\alpha - \beta}{2} - \frac{4}{\alpha - \beta} \|\eta\|_{L^2} \|\zeta\|_{L^2} \right)}_{(e)} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2) \\ & \leq \underbrace{\left[\left(\frac{1}{4\pi^2} \left(\frac{4}{\alpha - \beta} \right)^3 (\|\nabla v\|_{L^2}^2 \|\Delta v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \|\Delta w\|_{L^2}^2) \right)}_{(j)} \right. \\ & \quad \left. + \underbrace{\left(\frac{4}{\alpha - \beta} \right)^3 (\|\nabla v\|_{L^2}^4 + \|\nabla w\|_{L^2}^4)}_{(k)} \right] (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2)}_{(l)} \\ & \quad + \underbrace{\left(\frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} - \mu \right)}_{(1)} (\|\nabla \eta_1\|_{L^2}^2 + \|\nabla \zeta_1\|_{L^2}^2). \end{aligned} \quad (50)$$

We choose h such that

$$(1) = \frac{4\mu^2 c_1^2 h^2}{\alpha - \beta} - \mu < 0. \quad (51)$$

In view of Theorem 3.2, after sufficiently large time $T_2 > 0$, $\|\eta\|_{L^2}$ and $\|\zeta\|_{L^2}$ are small enough so that

$$\|\eta\|_{L^2} \|\zeta\|_{L^2} < \frac{(\alpha - \beta)^2}{16}.$$

Thus, $(e) > \frac{1}{4}(\alpha - \beta) > 0$. Let us denote $Y(t) = \|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2$. Then, for all $t > T_2$, by applying Poincaré's inequality to the second term on the left-hand side of (50), it follows, due to (51), that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} Y(t) + \pi^2(\alpha - \beta) Y(t) & \leq M_G (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2) + ((1) - \mu) (\|\nabla \eta_1\|_{L^2}^2 + \|\nabla \zeta_1\|_{L^2}^2) \\ & \leq M_G (\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2) \\ & \leq K' M_G e^{-a't}, \end{aligned}$$

where $K' > 0$ and $a' > 0$ chosen so that is such that $\|\eta\|_{L^2}^2 + \|\zeta\|_{L^2}^2 \leq K' M_G e^{-a't}$ for all $t > T_2$ (this is permitted due to Theorem 3.2). This implies

$$\frac{d}{dt} \left(Y(t) e^{2\pi^2(\alpha - \beta)t} \right) \leq K' M_G e^{2\pi^2(\alpha - \beta)t} e^{-a't}.$$

Integrating, we arrive at

$$Y(t) \leq Y(T_2) e^{-2\pi^2(\alpha - \beta)(t - T_2)} + \frac{K' M_G}{2\pi^2(\alpha - \beta) - a'} \left(e^{-ta'} - e^{-2\pi^2(\alpha - \beta)(t - T_2) - a'T_2} \right).$$

(Note that, if necessary, one may choose a' slightly smaller so that $2\pi^2(\alpha - \beta) \neq a'$.) In particular, $Y(t) = \|\nabla \eta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2$ decays exponentially in time for all $t > T_2$, with h and μ chosen so that

$$h < (8\sqrt{2}\pi c_1 c)^{-1} (\tilde{c} + 2 \ln G + CG^4)^{-\frac{1}{2}} G^{-1}$$

and

$$\mu > 32\pi^2 c^2 (\alpha - \beta) (\tilde{c} + 2 \ln G + CG^4) G^2.$$

Thus, the proof of Theorem 3.5 is complete. \square

4.3. Proofs of the Results for Type 2 Interpolants.

Lemma 4.4. *Let $u, v, w \in H^2$ be divergence free. Then the following inequalities hold:*

$$\begin{aligned} (a) \quad \left| \int_{\Omega} (u \cdot \nabla) v \cdot \Delta w \, dx dy \right| &\leq 3c_T \|\nabla u_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2} \\ &\quad + (c_T + 4c_B) \|\Delta u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2} \\ &\quad + 2c_T \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2}, \end{aligned} \quad (52)$$

$$(b) \quad \left| \int_{\Omega} (u \cdot \nabla) v \cdot \Delta v \, dx dy \right| \leq (2c_B + 5c_T) \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}. \quad (53)$$

Proof. We start by writing

$$\begin{aligned} \int_{\Omega} (u \cdot \nabla) v \cdot \Delta w \, dx dy &= \int_{\Omega} u_1 \partial_x v_1 \Delta w_1 \, dx dy + \int_{\Omega} u_2 \partial_y v_1 \Delta w_1 \, dx dy \\ &\quad + \int_{\Omega} u_1 \partial_x v_2 \Delta w_2 \, dx dy + \int_{\Omega} u_2 \partial_y v_2 \Delta w_2 \, dx dy. \end{aligned}$$

Now we'll estimate each term individually.

By (9) we have:

$$\begin{aligned} \left| \int_{\Omega} u_1 \partial_x v_1 \Delta w_1 \, dx dy \right| &\leq c_T \|\nabla u_1\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2} \\ &\leq c_T \|\nabla u_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2}, \end{aligned} \quad (54)$$

and

$$\begin{aligned} \left| \int_{\Omega} u_1 \partial_x v_2 \Delta w_2 \, dx dy \right| &\leq c_T \|\nabla u_1\|_{L^2} \|\nabla v_2\|_{L^2} \|\Delta w_2\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2} \\ &\leq c_T \|\nabla u_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2}. \end{aligned} \quad (55)$$

Using integration by parts and the divergence free condition, we have:

$$\begin{aligned} \int_{\Omega} u_2 \partial_y v_1 \Delta w_1 \, dx dy &= - \int_{\Omega} \partial_x u_2 \partial_y v_1 \partial_x w_1 \, dx dy - \int_{\Omega} \partial_y u_2 \partial_y v_1 \partial_y w_1 \, dx dy \\ &\quad + \int_{\Omega} u_2 \partial_{yy} v_2 \partial_x w_1 \, dx dy - \int_{\Omega} u_2 \partial_{yy} v_1 \partial_y w_1 \, dx dy, \end{aligned}$$

so applying (8) to the first two integrals and (9) to the second two, we obtain:

$$\begin{aligned} \left| \int_{\Omega} u_2 \partial_y v_1 \Delta w_1 \, dx dy \right| &\leq c_B \|\Delta u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2} \\ &\quad + c_T \|\nabla u\|_{L^2} \|\nabla w_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2}. \end{aligned} \quad (56)$$

Again by integrating by parts and using the divergence free condition, we obtain

$$\begin{aligned} \int_{\Omega} u_2 \partial_y v_2 \Delta w_2 \, dx dy &= \int_{\Omega} \partial_x u_1 v_2 \Delta w_2 \, dx dy \\ &\quad + \int_{\Omega} \Delta u_2 v_2 \partial_x w_1 \, dx dy + \int_{\Omega} u_2 \Delta v_2 \partial_x w_1 \, dx dy \\ &\quad + 2 \int_{\Omega} \partial_x u_2 \partial_x v_2 \partial_x w_1 \, dx dy + 2 \int_{\Omega} \partial_y u_2 \partial_y v_2 \partial_x w_1 \, dx dy. \end{aligned}$$

Now, estimating with (8) and (9) we have:

$$\begin{aligned} \left| \int_{\Omega} u_2 \partial_y v_2 \Delta w_2 \, dx dy \right| &\leq c_T \|\nabla u_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2} \\ &\quad + c_T \|\Delta u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2} \\ &\quad + c_T \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2} \\ &\quad + 4c_B \|\Delta u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2}. \end{aligned} \quad (57)$$

Combining (54), (55), (56), and (57), we obtain:

$$\begin{aligned} \left| \int_{\Omega} (u \cdot \nabla) v \cdot \Delta w \, dx dy \right| &\leq 3c_T \|\nabla u_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta w\|_{L^2} \left(1 + \ln \frac{\|\Delta u_1\|_{L^2}}{2\pi \|\nabla u_1\|_{L^2}} \right)^{1/2} \\ &\quad + (c_T + 4c_B) \|\Delta u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2} \\ &\quad + 2c_T \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} \|\nabla w_1\|_{L^2} \left(1 + \ln \frac{\|\Delta w_1\|_{L^2}}{2\pi \|\nabla w_1\|_{L^2}} \right)^{1/2}, \end{aligned}$$

so (a) is proven.

In order to prove (b), we first write

$$\begin{aligned} \int_{\Omega} (u \cdot \nabla) v \cdot \Delta v \, dx dy &= \int_{\Omega} u_1 \partial_x v_1 \Delta v_1 \, dx dy + \int_{\Omega} u_2 \partial_y v_1 \Delta v_1 \, dx dy \\ &\quad + \int_{\Omega} u_1 \partial_x v_2 \Delta v_2 \, dx dy + \int_{\Omega} u_2 \partial_y v_2 \Delta v_2 \, dx dy. \end{aligned}$$

Similar to the proof of (a), we proceed to estimate each term individually by appealing to (8) or (9), by integrating by parts and using the divergence free conditions.

By applying (9), we have:

$$\left| \int_{\Omega} u_1 \partial_x v_1 \Delta v_1 \, dx dy \right| \leq c_T \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}, \quad (58)$$

and

$$\left| \int_{\Omega} u_2 \partial_y v_1 \Delta v_1 \, dx dy \right| \leq c_T \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}, \quad (59)$$

and using the divergence free condition, we obtain

$$\begin{aligned} \left| \int_{\Omega} u_2 \partial_y v_2 \Delta v_2 \, dx dy \right| &= \left| - \int_{\Omega} u_2 \partial_x v_1 \Delta v_2 \, dx dy \right| \\ &\leq c_T \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}. \end{aligned} \quad (60)$$

To estimate the remaining integral, we write:

$$\int_{\Omega} u_1 \partial_x v_2 \Delta v_2 \, dx dy = \int_{\Omega} u_1 \partial_x v_2 \partial_{xx} v_2 \, dx dy + \int_{\Omega} u_1 \partial_x v_2 \partial_{yy} v_2 \, dx dy.$$

Now,

$$\begin{aligned} \int_{\Omega} u_1 \partial_x v_2 \partial_{yy} v_2 \, dx dy &= - \int_{\Omega} u_1 \partial_x v_2 \partial_y \partial_x v_1 \, dx dy \\ &= \int_{\Omega} \partial_x u_1 \partial_x v_2 \partial_y v_1 \, dx dy + \int_{\Omega} u_1 \partial_{xx} v_2 \partial_y v_1 \, dx dy, \end{aligned}$$

so

$$\left| \int_{\Omega} u_1 \partial_x v_2 \partial_{yy} v_2 \, dx dy \right| \leq (c_B + c_T) \|\nabla u\|_{L^2} \|\Delta v\|_{L^2} \|\nabla v_1\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}. \quad (61a)$$

For the other term, we have

$$\int_{\Omega} u_1 \partial_x v_2 \partial_{xx} v_2 \, dx dy = - \int_{\Omega} \partial_x u_1 \partial_x v_2 \partial_x v_2 \, dx dy - \int_{\Omega} u_1 \partial_{xx} v_2 \partial_x v_2 \, dx dy,$$

so,

$$\int_{\Omega} u_1 \partial_x v_2 \partial_{xx} v_2 \, dx dy = -\frac{1}{2} \int_{\Omega} \partial_x u_1 \partial_x v_2 \partial_x v_2 \, dx dy.$$

Next,

$$\begin{aligned} -\frac{1}{2} \int_{\Omega} \partial_x u_1 \partial_x v_2 \partial_x v_2 \, dx dy &= \frac{1}{2} \int_{\Omega} \partial_y u_2 \partial_x v_2 \partial_x v_2 \, dx dy \\ &= - \int_{\Omega} u_2 \partial_x \partial_y v_2 \partial_x v_2 \, dx dy = \int_{\Omega} u_2 \partial_{xx} v_1 \partial_x v_2 \, dx dy \\ &= - \int_{\Omega} \partial_x u_2 \partial_x v_1 \partial_x v_2 \, dx dy - \int_{\Omega} u_2 \partial_x v_1 \partial_{xx} v_2 \, dx dy. \end{aligned}$$

Therefore,

$$\left| \int_{\Omega} u_1 \partial_x v_2 \partial_{xx} v_2 \, dx dy \right| \leq (c_B + c_T) \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2}. \quad (61b)$$

Hence, by combining (58), (59), (60), (61a), and (61b), we obtain:

$$\left| \int_{\Omega} (u \cdot \nabla) v \cdot \Delta v \, dx dy \right| \leq (2c_B + 5c_T) \|\nabla u\|_{L^2} \|\nabla v_1\|_{L^2} \|\Delta v\|_{L^2} \left(1 + \ln \frac{\|\Delta v_1\|_{L^2}}{2\pi \|\nabla v_1\|_{L^2}} \right)^{1/2},$$

as claimed. \square

Proof of Theorem 3.6.

Let (\tilde{v}, \tilde{w}) be a strong solution corresponding to (v, w) , which is known to exist on some short time interval $[0, T_0)$. Assume T_0 is the largest time such that

$$\sup_{t \in [0, T_0)} (\|\nabla \tilde{v}(t)\|_{L^2}^2 + \|\nabla \tilde{w}(t)\|_{L^2}^2) \leq 50\pi^2(\alpha - \beta)^2 G^2 e^{CG^4},$$

and suppose that $T_0 < \infty$.

Then we know that

$$\limsup_{t \rightarrow T_0^-} (\|\nabla \tilde{v}(t)\|_{L^2}^2 + \|\nabla \tilde{w}(t)\|_{L^2}^2) = \sup_{t \in [0, T_0)} (\|\nabla \tilde{v}(t)\|_{L^2}^2 + \|\nabla \tilde{w}(t)\|_{L^2}^2) = 50\pi^2(\alpha - \beta)^2 G^2 e^{CG^4}. \quad (62)$$

Let $\eta = v - \tilde{v}$ and $\zeta = w - \tilde{w}$. Then we have the following equation for η :

$$\partial_t \eta - \alpha \Delta \eta + \beta \Delta \zeta + (\zeta \cdot \nabla) v + (\tilde{w} \cdot \nabla) \eta = -\nabla(\pi - \tilde{\pi}) - \mu \mathbf{I}_h(\eta_1) e_1.$$

Taking the inner product with $-\Delta \eta$, we obtain:

$$\frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \alpha \|\Delta \eta\|_{L^2}^2 - \beta \langle \Delta \zeta, \Delta \eta \rangle - \langle (\zeta \cdot \nabla) v, \Delta \eta \rangle - \langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle = \langle \nabla(\pi - \tilde{\pi}), \Delta \eta \rangle - \mu \langle \mathbf{I}_h(\eta_1), -\Delta \eta_1 \rangle$$

Now, by the divergence free condition, we have:

$$\langle \nabla(\pi - \tilde{\pi}), \Delta \eta \rangle = - \int_{\Omega} (\pi - \tilde{\pi}) \cdot \Delta(\nabla \cdot \eta) \, dx dy = 0,$$

and by applying Cauchy-Schwarz inequality and (4),

$$|\beta \langle \Delta \zeta, \Delta \eta \rangle| \leq \frac{\beta}{2} \|\Delta \eta\|_{L^2}^2 + \frac{\beta}{2} \|\Delta \zeta\|_{L^2}^2.$$

Rewriting $\langle \mathbf{I}_h(\eta_1), -\Delta \eta_1 \rangle = \langle \mathbf{I}_h(\eta_1) - \eta_1, -\Delta \eta_1 \rangle + \langle \eta_1, \Delta \eta_1 \rangle$, we have,

$$-\mu \langle \mathbf{I}_h(\eta_1), -\Delta \eta_1 \rangle = -\mu \langle \mathbf{I}_h(\eta_1) - \eta_1, -\Delta \eta_1 \rangle - \mu \|\nabla \eta_1\|_{L^2}^2,$$

so we obtain:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \left(\alpha - \frac{\beta}{2} \right) \|\Delta \eta\|_{L^2}^2 - \frac{\beta}{2} \|\Delta \zeta\|_{L^2}^2 + \mu \|\nabla \eta_1\|_{L^2}^2 \\ \leq |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| + |\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle| + \mu |\langle \mathbf{I}_h(\eta_1) - \eta_1, \Delta \eta_1 \rangle|. \end{aligned} \quad (63)$$

By the properties of \mathbf{I}_h , we have

$$\begin{aligned} \mu |\langle \mathbf{I}_h(\eta_1) - \eta_1, \Delta \eta_1 \rangle| &\leq \mu \|\mathbf{I}_h(\eta_1) - \eta_1\|_{L^2} \|\Delta \eta_1\|_{L^2} \\ &\leq \mu (c_2 h \|\nabla \eta_1\|_{L^2} + c_3 h^2 \|\Delta \eta_1\|_{L^2}) \|\Delta \eta_1\|_{L^2} \\ &\leq \frac{\mu^2}{2(\alpha - \beta)} (c_2 h \|\nabla \eta_1\|_{L^2} + c_3 h^2 \|\Delta \eta_1\|_{L^2})^2 + \frac{\alpha - \beta}{2} \|\Delta \eta_1\|_{L^2}^2 \\ &\leq \frac{\mu^2 c_2^2 h^2}{\alpha - \beta} \|\nabla \eta_1\|_{L^2}^2 + \frac{\mu^2 c_3^2 h^4}{\alpha - \beta} \|\Delta \eta_1\|_{L^2}^2 + \frac{\alpha - \beta}{2} \|\Delta \eta_1\|_{L^2}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \left(\frac{\alpha}{2} - \frac{\mu^2 c_3^2 h^4}{\alpha - \beta} \right) \|\Delta \eta\|_{L^2}^2 - \frac{\beta}{2} \|\Delta \zeta\|_{L^2}^2 + \mu \left(1 - \frac{\mu c_2^2 h^2}{\alpha - \beta} \right) \|\nabla \eta_1\|_{L^2}^2 \\ \leq |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| + |\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle|. \end{aligned} \quad (64)$$

Note that $1 - \frac{\mu c_2^2 h^2}{\alpha - \beta} > \frac{1}{2}$, and $\frac{\mu^2 c_3^2 h^4}{\alpha - \beta} < \frac{\alpha - \beta}{4}$ as long as

$$h^2 < \frac{\alpha - \beta}{2\mu \max\{c_2^2, c_3\}}. \quad (65)$$

Now we estimate the nonlinear terms using Lemma 4.4. By (52), we obtain

$$\begin{aligned} |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| &\leq 3c_T \|\nabla \zeta_1\|_{L^2} \|\nabla v\|_{L^2} \|\Delta \eta\|_{L^2} \left(1 + \ln \frac{\|\Delta \zeta_1\|_{L^2}}{2\pi \|\nabla \zeta_1\|_{L^2}} \right)^{1/2} \\ &\quad + (c_T + 4c_B) \|\Delta \zeta\|_{L^2} \|\nabla v\|_{L^2} \|\nabla \eta_1\|_{L^2} \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right)^{1/2} \\ &\quad + 2c_T \|\nabla \zeta\|_{L^2} \|\Delta v\|_{L^2} \|\nabla \eta_1\|_{L^2} \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right)^{1/2}, \end{aligned}$$

so by applying (4), we obtain

$$\begin{aligned} |\langle (\zeta \cdot \nabla) v, \Delta \eta \rangle| &\leq \frac{\alpha - \beta}{32} (\|\Delta \eta\|_{L^2}^2 + \|\Delta \zeta\|_{L^2}^2 + 4\pi^2 \|\nabla \zeta\|_{L^2}^2) \\ &\quad + \frac{72c_T^2}{(\alpha - \beta)} \|\nabla \zeta_1\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\Delta \zeta_1\|_{L^2}}{2\pi \|\nabla \zeta_1\|_{L^2}} \right) \\ &\quad + \frac{64(1 + 4\pi^2)(c_T^2 + c_B^2)}{4\pi^2(\alpha - \beta)} (\|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) \|\nabla \eta_1\|_{L^2}^2 \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right). \end{aligned}$$

Also, we use (6) to write $4\pi^2 \|\nabla \zeta\|_{L^2}^2 \leq \|\Delta \zeta\|_{L^2}^2$.

For the other term, we first apply (53), and obtain

$$|\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle| \leq (2c_B + 5c_T) \|\nabla \tilde{w}\|_{L^2} \|\nabla \eta_1\|_{L^2} \|\Delta \eta\|_{L^2} \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right)^{1/2}.$$

Then, by (4), we have

$$|\langle (\tilde{w} \cdot \nabla) \eta, \Delta \eta \rangle| \leq \frac{\alpha - \beta}{32} \|\Delta \eta\|_{L^2}^2 + \frac{200(c_B + c_T)^2}{\alpha - \beta} \|\nabla \tilde{w}\|_{L^2}^2 \|\nabla \eta_1\|_{L^2}^2 \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right).$$

Combining these estimates with (64), we have:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \eta\|_{L^2}^2 + \left(\frac{\alpha}{2} - \frac{5(\alpha - \beta)}{16} \right) \|\Delta \eta\|_{L^2}^2 - \left(\frac{\beta}{2} + \frac{\alpha - \beta}{16} \right) \|\Delta \zeta\|_{L^2}^2 \\ + \left[\frac{\mu}{2} - \gamma_0 (\|\nabla \tilde{w}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) \left(1 + \ln \frac{\|\Delta \eta_1\|_{L^2}}{2\pi \|\nabla \eta_1\|_{L^2}} \right) \right] \|\nabla \eta_1\|_{L^2}^2 \\ - \gamma_0 \|\nabla v\|_{L^2}^2 \left(1 + \ln \frac{\|\Delta \zeta_1\|_{L^2}}{2\pi \|\nabla \zeta_1\|_{L^2}} \right) \|\nabla \zeta_1\|_{L^2}^2 \leq 0, \end{aligned} \quad (66)$$

where

$$\gamma_0 := \frac{200(c_B + c_T)^2}{\alpha - \beta} = \max \left\{ \frac{72c_T^2}{(\alpha - \beta)}, \frac{64(1 + 4\pi^2)(c_T^2 + c_B^2)}{4\pi^2(\alpha - \beta)}, \frac{200(c_B + c_T)^2}{\alpha - \beta} \right\}.$$

Adding (66) with the corresponding inequality for $\frac{d}{dt}\|\nabla\zeta\|_{L^2}^2$, we obtain:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla\eta\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{8} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2) \\ & + \left[\frac{\mu}{2} - \gamma_0 (\|\nabla\tilde{w}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\Delta v\|_{L^2}^2) \right] \left(1 + \ln \frac{\|\Delta\eta_1\|_{L^2}^2}{2\pi\|\nabla\eta_1\|_{L^2}^2} \right) \|\nabla\eta_1\|_{L^2}^2 \\ & + \left[\frac{\mu}{2} - \gamma_0 (\|\nabla\tilde{v}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\Delta w\|_{L^2}^2) \right] \left(1 + \ln \frac{\|\Delta\zeta_1\|_{L^2}^2}{2\pi\|\nabla\zeta_1\|_{L^2}^2} \right) \|\nabla\zeta_1\|_{L^2}^2 \leq 0. \end{aligned} \quad (67)$$

Using (6), we have

$$\frac{\alpha - \beta}{8} \|\Delta\eta\|_{L^2}^2 \geq \frac{\alpha - \beta}{16} \|\Delta\eta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \frac{\|\Delta\eta_1\|_{L^2}^2}{4\pi^2\|\nabla\eta_1\|_{L^2}^2} \|\nabla\eta\|_{L^2}^2$$

and

$$\frac{\alpha - \beta}{8} \|\Delta\zeta\|_{L^2}^2 \geq \frac{\alpha - \beta}{16} \|\Delta\zeta\|_{L^2}^2 + \frac{\alpha - \beta}{16} \frac{\|\Delta\zeta_1\|_{L^2}^2}{4\pi^2\|\nabla\zeta_1\|_{L^2}^2} \|\nabla\zeta\|_{L^2}^2.$$

Then, by defining

$$r(u) = \frac{\|\Delta u_1\|_{L^2}^2}{4\pi^2\|\nabla u_1\|_{L^2}^2}$$

and

$$\gamma = \gamma_0 (\|\nabla\tilde{v}\|_{L^2}^2 + \|\nabla\tilde{w}\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2),$$

we rewrite (66) as:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla\eta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2) + \frac{\alpha - \beta}{16} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2) \\ & + \left[\frac{\mu}{2} + r(\eta) - \gamma(1 + \ln r(\eta)) \right] \|\nabla\eta_1\|_{L^2}^2 + \left[\frac{\mu}{2} + r(\zeta) - \gamma(1 + \ln r(\zeta)) \right] \|\nabla\zeta_1\|_{L^2}^2 \leq 0. \end{aligned}$$

Now we apply Lemma 4.3 and conclude that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\nabla\eta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2) + \frac{\alpha - \beta}{16} (\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2) \\ & + \left[\frac{\mu}{2} - \gamma \ln(\gamma) \right] \|\nabla\eta_1\|_{L^2}^2 + \left[\frac{\mu}{2} - \gamma \ln(\gamma) \right] \|\nabla\zeta_1\|_{L^2}^2 \leq 0. \end{aligned}$$

Using (6) again, we have

$$\|\Delta\eta\|_{L^2}^2 + \|\Delta\zeta\|_{L^2}^2 \geq 4\pi^2 (\|\nabla\eta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2),$$

so by defining

$$Y = \|\nabla\eta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2,$$

and

$$\psi = \min \left\{ \frac{\pi^2(\alpha - \beta)}{2}, \mu - 2\gamma \ln(\gamma) \right\}$$

we obtain:

$$\frac{d}{dt} Y + \psi Y \leq 0. \quad (68)$$

Thus, as long as we choose $\mu > \frac{\pi^2(\alpha-\beta)}{2} + 2\gamma \ln(\gamma)$, we conclude by Gronwall's inequality that

$$Y(t) \leq Y(0)e^{-\pi^2(\alpha-\beta)t/2}, \quad \forall t \in [0, T_0).$$

By (62), (17), and (18),

$$\gamma \leq \gamma_0 \left(60\pi^2(\alpha-\beta)^2 G^2 e^{CG^4} + c_M(\alpha-\beta)^2 G^2 \left[1 + \left(1 + G^2 e^{CG^4} \right) \left(1 + e^{CG^4} + G^4 e^{CG^4} \right) \right] \right) < \infty,$$

so on the time interval $[0, T_0)$, such a μ is available. Specifically, it is sufficient to choose

$$\mu \geq 1200(c_B + c_T)^2(60\pi^2 + c_M)(\alpha-\beta)G^2(1+G^2)^3 e^{2CG^4} (\tilde{c} + 2\ln(G) + 3\ln(1+G^2) + 2CG^4), \quad (69)$$

where $\tilde{c} := \ln(600(c_B + c_T)^2(60\pi^2 + c_M)(\alpha-\beta))$, so

$$\mu \sim (\alpha-\beta)G^{12}e^{2CG^4}. \quad (70)$$

Therefore, for all $t \in [0, T_0)$, we obtain

$$Y(t) \leq Y(0) \leq 2\|\nabla v_0\|_{L^2}^2 + 2\|\nabla \tilde{v}_0\|_{L^2}^2 + 2\|\nabla w_0\|_{L^2}^2 + 2\|\nabla \tilde{w}_0\|_{L^2}^2 \leq 20\pi^2(\alpha-\beta)^2 G^2 e^{CG^4}.$$

This implies that, in fact,

$$\sup_{t \in [0, T_0)} (\|\nabla \tilde{v}(t)\|_{L^2}^2 + \|\nabla \tilde{w}(t)\|_{L^2}^2) \leq 40\pi^2(\alpha-\beta)^2 G^2 e^{CG^4},$$

which is a contradiction to (62).

Hence we have $T_0 = \infty$, and $(\tilde{v}(t), \tilde{w}(t))$ converges exponentially in time to $(v(t), w(t))$ in the H^1 norm, and we have established the estimate:

$$\sup_{t \in [0, \infty)} (\|\nabla \tilde{v}(t)\|_{L^2}^2 + \|\nabla \tilde{w}(t)\|_{L^2}^2) \leq 50\pi^2(\alpha-\beta)^2 G^2 e^{CG^4}.$$

Also, our restriction on μ (70) is in fact sufficient to guarantee convergence on $[0, \infty)$, with our restriction (65) on h , which we see now means we can choose

$$h \sim G^{-6}e^{-CG^4}.$$

□

4.4. Determining Interpolants.

Proof of Theorem 3.8.

The proof proceeds exactly as that of Theorem 3.1, where $\delta^{(1)} \equiv \delta^{(2)} \equiv \epsilon^{(1)} \equiv \epsilon^{(2)} \equiv 0$, with a few differences. As before, we let $\eta = v - \tilde{v}$ and then we obtain a differential inequality for $\|\eta\|_{L^2}$. We get the same inequality as before but with two extra terms.

After subtracting the equations for v and \tilde{v} , we have $f - (f + \delta^{(1)}) = -\delta^{(1)}$ for the forcing term, and after taking the inner product with η we have

$$\left| \int_{\Omega} \delta^{(1)} \cdot \eta \, dx dy \right| \leq \|\delta^{(1)}\|_{L^2} \|\eta\|_{L^2} \leq \frac{1}{\mu} \|\delta^{(1)}\|_{L^2}^2 + \frac{\mu}{4} \|\eta\|_{L^2}^2.$$

Also, we have $\mu \mathbf{I}_h(u + \epsilon^{(1)} - \tilde{u}) = \mu \mathbf{I}_h(u - \tilde{u}) + \mu \mathbf{I}_h(\epsilon^{(1)})$, and after taking the inner product with η , we obtain

$$\left| \mu \int_{\Omega} \mathbf{I}_h(\epsilon^{(1)}) \cdot \eta \, dx dy \right| \leq \mu \|\mathbf{I}_h(\epsilon^{(1)})\|_{L^2} \|\eta\|_{L^2} \leq \mu \|\mathbf{I}_h(\epsilon^{(1)})\|_{L^2}^2 + \frac{\mu}{4} \|\eta\|_{L^2}^2.$$

We have similar additions for the inequality we derive for $\zeta := w - \tilde{w}$.

Thus, letting $Y(t) = \|\eta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2$ and proceeding as before, we eventually get:

$$\frac{d}{dt} Y + \psi Y \leq \phi,$$

where

$$\psi(t) := \frac{\mu}{2} - \left(\frac{c_L^4 + (\alpha - \beta)^4}{2(\alpha - \beta)^3} \right) (\|\nabla v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2),$$

and

$$\phi(t) := \frac{1}{\mu} \left(\|\delta^{(1)}\|_{L^2}^2 + \|\delta^{(2)}\|_{L^2}^2 \right) + \mu \left(\|\mathbf{I}_h(\epsilon^{(1)})\|_{L^2}^2 + \|\mathbf{I}_h(\epsilon^{(2)})\|_{L^2}^2 \right).$$

Since $\|\delta^{(1)}\|_{L^2}, \|\delta^{(2)}\|_{L^2} \rightarrow 0$ and $\|\mathbf{I}_h(\epsilon^{(1)})\|_{L^2}, \|\mathbf{I}_h(\epsilon^{(2)})\|_{L^2} \rightarrow 0$, we have $\|\phi\|_{L^2} \rightarrow 0$. Therefore, by Proposition 2.1, $\|v - \tilde{v}\|_{L^2}, \|w - \tilde{w}\|_{L^2} \rightarrow 0$ as $t \rightarrow \infty$. \square

Proof of Theorem 3.10.

Let μ be large enough to satisfy the requirements of Theorem 3.1 with $(v^{(1)}, w^{(1)})$ as the reference solution, and note that our h and \mathbf{I}_h satisfy Theorem 3.1. Let (\tilde{v}, \tilde{w}) be the corresponding solution.

Then $\|v^{(1)}(t) - \tilde{v}(t)\|_{L^2} \rightarrow 0$ and $\|w^{(1)}(t) - \tilde{w}(t)\|_{L^2} \rightarrow 0$, and for some π , \tilde{v} and \tilde{w} satisfy the following equations:

$$\begin{aligned} \partial_t \tilde{v} - \alpha \Delta \tilde{v} + \beta \Delta \tilde{w} + (\tilde{w} \cdot \nabla) \tilde{v} + \nabla \pi &= f^{(1)} + \mu \mathbf{I}_h \left(v^{(1)} - \tilde{v} \right) \\ &= f^{(2)} + (f^{(1)} - f^{(2)}) + \mu \mathbf{I}_h \left(v^{(2)} + (v^{(1)} - v^{(2)}) - \tilde{v} \right), \\ \partial_t \tilde{w} - \alpha \Delta \tilde{w} + \beta \Delta \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{w} + \nabla \pi &= g^{(1)} + \mu \mathbf{I}_h \left(w^{(1)} - \tilde{w} \right) \\ &= g^{(2)} + (g^{(1)} - g^{(2)}) + \mu \mathbf{I}_h \left(w^{(2)} + (w^{(1)} - w^{(2)}) - \tilde{w} \right). \end{aligned}$$

Therefore, setting $\delta^{(1)} := f^{(1)} - f^{(2)}$ and $\delta^{(2)} := g^{(1)} - g^{(2)}$, and $\epsilon^{(1)} := v^{(1)} - v^{(2)}$ and $\epsilon^{(2)} := w^{(1)} - w^{(2)}$, we see that (\tilde{v}, \tilde{w}) must be the unique solution guaranteed by Theorem 3.8, with $(v^{(2)}, w^{(2)})$ as the reference solution.

Therefore $\|v^{(2)}(t) - \tilde{v}(t)\|_{L^2} \rightarrow 0$ and $\|w^{(2)}(t) - \tilde{w}(t)\|_{L^2} \rightarrow 0$.

Thus,

$$\|v^{(1)}(t) - v^{(2)}(t)\|_{L^2} \leq \|v^{(1)}(t) - \tilde{v}(t)\|_{L^2} + \|\tilde{v}(t) - v^{(2)}(t)\|_{L^2} \rightarrow 0,$$

and

$$\|w^{(1)}(t) - w^{(2)}(t)\|_{L^2} \leq \|w^{(1)}(t) - \tilde{w}(t)\|_{L^2} + \|\tilde{w}(t) - w^{(2)}(t)\|_{L^2} \rightarrow 0.$$

\square

5. CONCLUDING REMARKS

We have shown that, in the language of the reformulated equations, solutions (\tilde{v}, \tilde{w}) of the data assimilation equations will converge to the corresponding true values (v, w) in L^2 , even if measurements are only taken for only one of v and w . This equates to having to take measurements on either $u + b$ or $u - b$. Could one prove that it is sufficient to collect data on just u or just b and still get convergence, similar to the result for the reformulated variables?

If one were to consider collecting data only on the magnetic field, b , then the problem is evident when we take $b(t) \equiv \tilde{b}(t) \equiv g \equiv 0$ for all $t \geq 0$, because we then have u and \tilde{u} satisfying the Navier-Stokes equations with different initial conditions and no data assimilation. Hence, there is an asymmetry between the original system and the reformulated system.

The answer to the question for collecting data on the velocity field, u , is open. However, since we've demonstrated that the algorithm works with knowledge of only the sum of measurements on u and b , it may be that the knowledge of the velocity field is what makes this work, and so a u -measurement only algorithm

is hopeful. However, since it seems we shouldn't be able to prove the convergence of a b -measurement only algorithm, and the Elsässer variable formulation does not distinguish u and b , a proof of a u -measurement only algorithm would have to be in terms of the original variables.

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