Calculus I - Optimization

University of Nebraska-Lincoln

Clicker question

How do we solve optimization problems as described in section 4.3 (reading material for today)?

(A) Trial and error.

(B) Draw graphs of the function to maximize and read off the information from the graph.

(C) Set up a function to optimize and use first and second derivatives.

(D) I have not read the section yet.

How to Solve It

How to Solve It is the title of a book by George Polya – the Father of Modern Heuristics. It has sold more copies than any other math book in history. It is a perennial best seller among math and science books. It was first published in 1945 and is still the best "How To Solve It" book available. We want to improve your problem solving ability by using Polya?s help. Polya based his ideas on a four-phase plan. Those phases are: 1) Understanding the problem; 2) Devising a plan; 3) Carrying out the plan; 4) Looking back.

Optimization – Example 1

On the same side of a straight river are two towns, and the townspeople want to build a pumping station, S (see Figure below). The pumping station is to be at the river's edge with pipes extending straight to the two towns. Which function must you minimize over the interval $0 \leq x \leq 4$ to find the location for the pumping station that minimizes the total length of the pipe?

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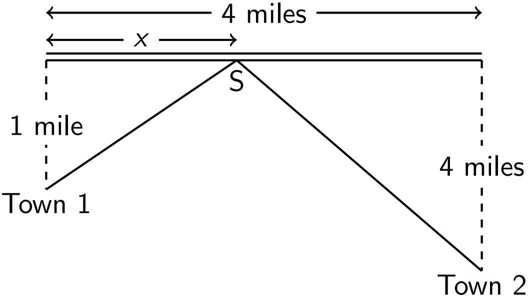


Follow-up Question

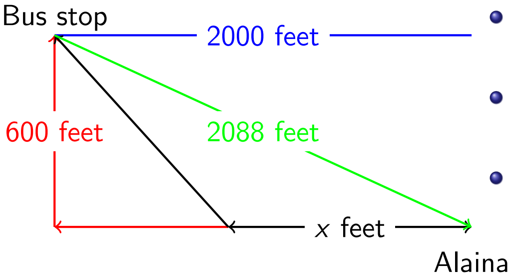
What function must be minimized if construction of the pipeline from Town 1 to the river is twice as expensive per foot as construction of the pipeline from Town 2 to the river and the goal is to minimize total construction cost?

Practical Tips for Modeling Optimization Problems

1) Make sure that you know what quantity or function is to be optimized.
2) If possible, make several sketches showing how the elements that vary are related. Label your sketches clearly by assigning variables to quantities which change.
3) Try to obtain a formula for the function to be optimized in terms of the variables that you identified in the previous step. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.
4) Find the critical points and evaluate the function at these points and the endpoints (if relevant) to find the global maxima and/or minima.

Example 2

Alaina wants to get to the bus stop as quickly as possible. The bus stop is across a grassy park, 2000 feet west and 600 feet north of her starting position. Alaina can walk west along the edge of the park on the sidewalk at a speed of 6 ft/sec. She can also travel through the grass in the park, but only at a rate of 4 ft/sec. What path gets her to the bus stop the fastest?



- all on grass time $= 2088/4 \approx 522$ seconds
- square corner time$= 2000/6 + 600/4 \approx 483$ seconds
- grass + sidewalk time $=f(x)$

Solution

Notation:

- $x =$ the distance that Alaina walks west along the sidewalk
- $y =$ the distance she walks through the grass

The total time is then

$$t = t_{\text{sidewalk}} + t_{\text{grass}} = \frac{x}{6} + \frac{y}{4}$$

since Time = Distance/Speed, and she can walk 6 ft/sec on the sidewalk and 4 ft/sec on the grass. By the Pythagorean Theorem $y = \sqrt{(2000-x)^2 + 600^2}$, hence

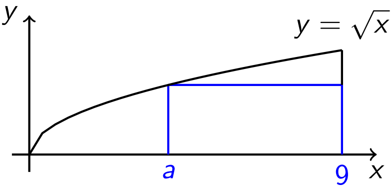
$$t = \frac{x}{6} + \frac{\sqrt{(2000-x)^2 + 600^2}}{4}, \quad 0 \le x \le 2000 =: f(x).$$

Computing the derivative of f and setting it equal to zero, we obtain a critical point $x_c \approx 1463$ ft, for a total time of ≈ 445 seconds.

Clicker question

The figure below shows the curves $y = \sqrt{x}, x = 9, y = 0$, and a rectangle with its sides parallel to the axes and its left end at $x = a$. What is the maximum perimeter of such a rectangle?

(A) 9.25
(B) 14
(C) 18
(D) 18.5
(E) 20.



Answer:

Consider the function (half the perimeter)

$$f(x) = \sqrt{x} + 9 - x, \quad 0 \leq x \leq 9.$$

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Answer:

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Then $f'(x) = -1 + \dfrac{1}{2\sqrt{x}}$ which is zero when $x = 1/4$. Note that $f''(x) = -\dfrac{1}{4}x^{-3/2} < 0$ for $0 < x < 9$, hence we have a maximum at the critical point.

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Then $f'(x) = -1 + \dfrac{1}{2\sqrt{x}}$ which is zero when $x = 1/4$. Note that $f''(x) = -\dfrac{1}{4}x^{-3/2} < 0$ for $0 < x < 9$, hence we have a maximum at the critical point. We have that $f\left(\dfrac{1}{4}\right) = 9.25$, which is half the perimeter, hence the full perimeter is 18.5.

Clicker question

You wish to maximize the volume of an open-topped rectangular box with a square base x cm by x cm and surface area 900 cm^2. Over what domain can the variable x vary?

(A) $0 < x < 900$

(B) $0 < x < 30$

(C) $0 < x < \sqrt{450}$

(D) $0 < x < \infty$.

More Practice Problems

- From each corner of a square piece of sheet metal, we remove a small square and turn up the edges to form an open box. What are the dimensions of the box with largest volume?
- A manufacturer needs to to produce a cylindrical container with a capacity of 1600 cm^3.The top and the bottom of the container are made from material that costs \$.05/cm^2, while the sides of the container are made from material costing \$0.03/cm^2. Find the dimensions that will minimize the company's cost of producing this container.
- The strength of a rectangular beam is proportional to the product of its width (smallest side) times the cube of its length. Find the dimensions of the strongest beam that can be cut from a log with circular cross sections of radius R.