Math 107-Lecture 4

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Announcements

- Recall: Practice Problems for the Gateway Exam are now posted. The grades do NOT count towards your grade (only the homework grades in webwork will be imported into Canvas). Soon, practice exams will be posted (all problems included in one exam).
- Study Stops are now open; see schedule and updates at `http://success.unl.edu/current/study-stop-schedule`.

Plan for today

1. Integration by parts formula review
2. Integration with partial fractions:
   - motivation
   - decomposition of rational functions in partial fractions
   - integration of rational functions.

Integration by Parts

This method is the equivalent of the product rule for differentiation

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

For definite integrals it is given by

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

For definite integrals we have

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx$$

Partial Fractions

Motivation: Polynomials are **straight-forward** to integrate, how about rational functions? Rational functions are quotients of two polynomials, e.g.

$$\frac{x+2}{x^2-3}, \quad \frac{x+2}{x^2-3}, \quad \frac{x+2}{x^2-3}.$$

But NOT

$$\frac{\sqrt{x}}{x^2-3}, \quad \frac{x+2}{x^{3/2}-3}.$$

Idea: Write the rational functions in terms of very simple fractions (called partial fractions) which we know how to integrate.

Simple Example:

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

And we know how to integrate both terms on the right hand side.

Integrals of basic rational functions:

1. $$\int \frac{1}{x} dx = \ln|x| + C$$
2. $$\int \frac{1}{x^7} dx = -\frac{x^{-6}}{6} + C$$
3. $$\int \frac{dx}{1+x^2} = \arctan(x) + C$$
4. $$\int \frac{x}{1+x^2} dx =$$
5. $$\int \frac{1}{2x+1} dx =$$
6. $$\int \frac{1}{(3x-2)^7} dx =$$
7. $$\int \frac{dx}{1+\left(x/\sqrt{3}\right)^2} =$$
8. $$\int \frac{x-3}{1+(x-3)^2} dx =$$

What are partial fractions and how do we find them?

1. with powers of linear terms in the denominator; anything that is of the form:

$$\frac{A}{(Bx+C)^k}, \quad A, B, C \in \mathbb{R}, k = \text{ a positive integer; e.g. } \frac{-3}{(2x+\pi)^4}.$$

2. with powers of irreducible quadratic terms in the denominator, of the form

$$\frac{Ax+B}{(Cx^2+Dx+E)^k}, \quad A, B, C, D, E \in \mathbb{R}, k = \text{ a positive integer}$$

E.g.

$$\frac{-x+1}{(x^2+\pi)^3}$$

Decomposition in partial fractions

Theorem. All rational functions can be decomposed as a polynomial and a sum of partial fractions.

Question. How do we find this decomposition?

Case 1: if the degree of the numerator is larger than the degree of the denominator, perform a long division to obtain a polynomial and a fraction which has the degree of the numerator is smaller than the degree of the denominator; then apply Case 2 to the fraction.

Case 2: if the degree of the numerator is smaller than the degree of the denominator, factor the denominator in irreducible terms then write a decomposition in simple fractions.

Case 2

- For a factor of the form $(Bx+C)^k$ in the denominator write a decomposition of the form

$$\frac{A_1}{(Bx+C)}+\frac{A_2}{(Bx+C)^2}+\cdots+\frac{A_k}{(Bx+C)^k}$$

  Find $A_1, A_2, \cdots A_k$.

- For an irreducible factor of the form $(Cx^2+Dx+E)^k$ in the denominator write a decomposition of the form

$$\frac{A_1x+B_1}{(Cx^2+Dx+E)}+\frac{A_2x+B_2}{(Cx^2+Dx+E)^2}+\cdots+\frac{A_kx+B_k}{(Cx^2+Dx+E)^k}$$

  Find $A_1, B_1, A_2, B_2, \cdots A_k, B_k$.

Examples of decompositions in partial fractions

Example 1.

$$\frac{x+5}{x^2-2x-8} = \frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

Two methods for finding A and B:
1. Multiply the equality by $x^2 - 2x - 8 = (x-4)(x+2)$ and equate coefficients of similar terms
2. Give specific values to x (here, 2 are needed as we need to find 2 constants).
We find $A = 3/2$, $B = -1/2$.

Clicker Question #1

The decomposition in partial fractions of

$$\frac{2x^3+7}{(x-2)^2(x^2+2x+1)(x^2+1)}$$

is given by

- $\dfrac{A}{(x-2)^2}+\dfrac{B}{x^2+2x+1}+\dfrac{C}{x^2+1}$
- $\dfrac{A_1}{(x-2)}+\dfrac{A_2}{(x-2)^2}+\dfrac{B}{x^2+2x+1}+\dfrac{C}{x^2+1}$
- $\dfrac{A_1}{(x-2)}+\dfrac{A_2}{(x-2)^2}+\dfrac{Bx+C}{x^2+2x+1}+\dfrac{D}{x^2+1}$
- $\dfrac{A_1}{(x-2)}+\dfrac{A_2}{(x-2)^2}+\dfrac{B}{x^2+2x+1}+\dfrac{Cx+D}{x^2+1}$
- $\dfrac{A_1}{(x-2)}+\dfrac{A_2}{(x-2)^2}+\dfrac{B_1}{x+1}+\dfrac{B_2}{(x+1)^2}+\dfrac{Cx+D}{x^2+1}$

Integrating the partial fractions

1. $\dfrac{A}{(Bx+C)^k}dx$ – use the substitution $u = Bx + C$, then integrate u^{-k} [careful if $k = 1$ which integrated gives a log function!]
2. If you encounter an irreducible quadratic factor in the denominator $Ax^2 + Bx + C$
   - complete the square to get it in the form $A(x + D)^2 + E$, where $E > 0$.
   - use the substitution $u = x + D$, then integrate $\dfrac{1}{u^2+1}$ [we will not cover the case $\dfrac{1}{(u^2+1)^k}$, $k > 1$]

Clicker Question #2

Completing the square in

$$x^2 + 3x + 3$$

gives

- $(x + \frac{3}{2})^2 + \frac{3}{4}$
- $(x + 3)^2 - 3x - 9$
- $x(x + 3) + 3$
- $x = 0$, $x = -3$
- $(x + 3)^2$

(Full) Example.

Integrate

$$\int \frac{x^4 - 3x^2 + 2}{x^3 + x} dx$$

- Long division
- Factor the denominator
- Integrate the polynomial
- Integrate the partial fractions.

Wrapping up:

- Today we covered integration with partial fractions from 7.4.
- For next time finish working the first set of suggested problems from section 7.4.
- For next lecture, read Trigonometric substitution (section 7.4).
- Purchase your worksheet packets from the bookstore!