

The surface quasi-geostrophic equation and its generalization

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HAPPY BIRTHDAY, PETER!

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Introduction

The surface quasi-geostrophic (SQG) equation assumes the form

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = \mathbf{0}, \quad (1)$$

where $\theta = \theta(\mathbf{x}, t)$ is a scalar function of $\mathbf{x} \in \mathbb{R}^2$ (or \mathbb{T}^2) and $t \geq 0$, and $0 < \alpha \leq 1$ and $\kappa \geq 0$ are parameters. The 2D velocity field \mathbf{u} is determined by θ through a stream function ψ ,

$$\mathbf{u} = \nabla^\perp \psi \equiv (-\partial_{x_2}, \partial_{x_1})\psi, \quad (-\Delta)^{\frac{1}{2}}\psi = \theta. \quad (2)$$

The fractional Laplacian, $(-\Delta)^\alpha$, can be defined through the Fourier transform, $\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi)$.

Writing Λ for $(-\Delta)^{\frac{1}{2}}$, then

$$\mathbf{u} = (-\partial_{x_2} \Lambda^{-1} \theta, \partial_{x_1} \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta),$$

where \mathcal{R}_j ($j = 1, 2$) are Riesz transforms.

When $\kappa > 0$, (1) is called the dissipative SQG equation. The case with $\alpha = \frac{1}{2}$ arises in geophysical studies of rapidly rotating fluids (see [J. Pedlosky](#) (Springer, 79), [Constantin, Majda and Tabak](#) (Nonlinearity, 94), [Majda and Tabak](#) (Physica D, 95) and [Constantin](#) (Springer, 06)).

When $\kappa = 0$, (1) is the inviscid SQG equation. It was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air. It is an important example of 2D active scalars and some of its distinctive features have made it an important testbed for turbulence theories.

The global regularity issue: given a sufficiently smooth data

$$\theta(x, 0) = \theta_0(x),$$

does the SQG have a global (in time) solution?

What's the difficulty? Try energy estimates to get global bounds!

It can be shown

$$\|\theta(\cdot, t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad 1 \leq p \leq \infty$$

using the pointwise inequality, $0 < \alpha \leq 2$,

$$2f(-\Delta)^\alpha f(x) \geq (-\Delta)^\alpha f^2(x) \quad (\text{Cordoba \& Cordoba, 2004})$$

$$\phi'(f)(-\Delta)^\alpha f(x) \geq (-\Delta)^\alpha \phi(f)(x) \quad \text{for convex } \phi,$$

The issue is the global H^1 -norm. $W = \nabla^\perp \theta$ satisfies

$$\partial_t W + u \cdot \nabla W + \kappa(-\Delta)^\alpha W = W \cdot \nabla u$$

which is in the same form as the 3D vorticity equation.

The SQG resembles, in many aspects, the 3D incompressible Navier-Stokes and Euler equations.

$$\nabla u = \mathcal{P}_3(\omega) \quad \text{for 3D fluid equations}$$

$$\nabla u = \mathcal{P}_2(\nabla^\perp \theta) \quad \text{for SQG}$$

where \mathcal{P}_3 and \mathcal{P}_2 are singular integral operators.

The SQG equation has recently attracted enormous attention. Many important results have been obtained. Here is a partial list of people: Abidi, Bae, Berselli, Caffarelli, Carrillo, Castro, Chae, Chamorro, C. Chan, Q. Chen, Z-M. Chen, Chuong, Constantin, A. Cordoba, D. Cordoba, Dabkowski, J. Deng, B.-Q. Dong, H. Dong, Dritschel, D. Du, Fefferman, Ferreira, Fontelos, Friedlander, Gancedo, Hmidi, T. Hou, Iyer, Ju, Keraani, Khouider, Kiselev, Lee, Lemarie-Rieusset, D. Li, P. Li, R. Li, Majda, Marchand, Mancho

May, Miao, Miura, Nazarov, Niche, Ohkitani, Pavlovic, Reinaud,
Resnick, Rodrigo, Rusin, Sadek, M. Schonbek, T. Schonbek, Z.
Shi, Silvstre, Stefanov, Tabak, Titi, Vasseur, Vicol, Volberg, H.
Wang, S. Wang, Wu, L. Xue, Yamada, H. Yu, B-Q. Yuan, J. Yuan,
Z. Zhang, Y. Zhou, and others.

There are two major cases:

- $\kappa = 0$, the inviscid SQG equation

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$$

The global regularity problem remains open. This is probably the simplest active scalar equation for which the global regularity is unknown. It shares parallel properties with the 3D Euler equations. For example, $W = \nabla^\perp \theta$ satisfies

$$\partial_t W + u \cdot \nabla W = W \cdot \nabla u.$$

Important results have been obtained.

Constantin, Majda and Tabak, 1994 Extensive numerical computations and geometric regularity criteria

Ohkitani and Yamada, 1997 Numerical computations

D. Cordoba, 1998, Nonexistence of simple hyperbolic blowup

D. Cordoba and C. Fefferman, 2001, 2002 Two level curves can not touch on a line segment

J. Deng, T. Y. Hou, and X. Yu, 2005

- D. Cordoba, Fontelos, A. Mancho and J. Rodrigo, 2005
- B. Khouider and E.S. Titi, 2008 (CPAM, inviscid regularization)
- D. Chae, 2008
- K. Ohkitani and Sakajo, 2010
- P. Constantin, M.-C. Lai, R. Sharma, Y.-H. Tseng and J. Wu, 2010
- T.Y. Hou and Z. Shi, 2010
- D. Chae, P. Constantin and J. Wu, Preprint.

- $\kappa > 0$, the dissipative SQG equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0$$

$\alpha = \frac{1}{2}$ appears to be critical. Consider the Fourier transform

$$\frac{d}{dt} \widehat{\theta}(\mathbf{l}, t) + \kappa |\mathbf{l}|^{2\alpha} \widehat{\theta}(\mathbf{l}, t) = - \sum_{\mathbf{j}+\mathbf{k}=\mathbf{l}} \frac{\mathbf{j}^\perp \cdot \mathbf{k}}{|\mathbf{j}|} \widehat{\theta}(\mathbf{j}, t) \widehat{\theta}(\mathbf{k}, t).$$

When $\alpha = \frac{1}{2}$, the nonlinear term is comparable to the dissipative term.

The study on the dissipative SQG is divided into three cases:

- 1) the sub-critical case ($\alpha > \frac{1}{2}$),
- 2) the critical case ($\alpha = \frac{1}{2}$), and
- 3) the super-critical case ($\alpha < \frac{1}{2}$).

This problem for the sub-critical case is more or less resolved (Constantin-Wu, Resnick). Important progress has been made on the critical and supercritical cases.

Critical SQG

Theorem. (Constantin-C  rdoba-Wu, *Indiana U. Math. J.*, 2001)

Let $\alpha = \frac{1}{2}$. Assume $\theta_0 \in \mathbf{H}^2$ and

$$\|\theta_0\|_{L^\infty} \leq C_\kappa \quad (\text{smallness})$$

Then there exists a unique global solution θ to the SQG equation satisfying

$$\|\theta(\cdot, t)\|_{H^2} \leq \|\theta_0\|_{H^2}.$$

Small data global existence results have been obtained in various functional settings, in $B_{2,1}^{2-2\alpha}$ by Chae-Lee (*Comm. Math. Phys.*, 2003), in Cordoba-Cordoba (CMP, 2004), in $\mathbf{B}_{p,q}^r$ by J. Wu (2004, 2006), in $\mathbf{H}^{2-2\alpha}$ by H. Miura (2006), N. Ju (2006), by Chen-Miao-Zhang (*Comm. Math. Phys.*, 2007), and by Hmidi and Keraani (Adv. Math., 2007).

Progress by Kiselev, Nazarov and Volberg, and by Caffarelli and Vasseur on the critical case $\alpha = \frac{1}{2}$.

Theorem (Kiselev, Nazarov and Volberg, *Invent. Math.* **167** (2007), 445-453.)

The critical SQG equation with **periodic** smooth initial data θ_0 has a unique global smooth solution. Moreover,

$$\|\nabla\theta\|_{L^\infty} \leq \mathbf{C}\|\nabla\theta_0\|_{L^\infty} \exp\{\mathbf{C}\|\theta_0\|_{L^\infty}\}.$$

Method of “Modulus of Continuity”:

$|\theta(x, t) - \theta(y, t)| \leq \omega(|x - y|)$ for all $t \geq 0$. Then

$$\|\nabla\theta(\cdot, t)\|_{L^\infty} \leq \omega'(0).$$

Theorem (Caffarelli and Vasseur, *Ann. of Math.* **171** (2010),
 1903-1930)

Consider a slightly more general form of the SQG equation

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^n, \quad t > 0 \quad (3)$$

with \mathbf{u} satisfying $\nabla \cdot \mathbf{u} = \mathbf{0}$ and each component determined by θ through a singular integral operator.

Let $\alpha = \frac{1}{2}$. Let $\theta_0 \in L^2(\mathbb{R}^n)$ and let θ be a Leray-Hopf weak solution, namely

$$\theta \in L^\infty([0, \infty); L^2(\mathbb{R}^n)) \cap L^2([0, \infty); \dot{H}^{\frac{1}{2}}(\mathbb{R}^n)).$$

Then θ is a classical solution for $t > 0$. In fact, for any $t_0 > 0$

$$\theta \in C^\infty(\mathbb{R}^n \times [t_0, \infty)).$$

The proof is to improve the regularity of θ successively: from L^2 to L^∞ , from L^∞ to Hölder and from Hölder to $C^{1,\beta}$.

1) From L^2 to L^∞ . For any $t > 0$,

$$\sup_{x \in \mathbb{R}^n} |\theta(x, t)| \leq C \frac{\|u_0\|_{L^2}}{t^{n/2}}, \quad \|u(\cdot, t)\|_{BMO} \leq C \frac{\|u_0\|_{L^2}}{t^{n/2}}.$$

- 2) From L^∞ to C^δ for some $\delta > 0$. De Giorgi iteration.
- 3) From C^δ to $C^{1,\beta}$ (classical solution). They write the critical QG equation as

$$\begin{aligned}\theta(x, t) &= P(\cdot, t) * \theta_0 - g(x, t) \\ g(x, t) &= \int_0^t \int P(x - y, t - \tau) \nabla \cdot (\mathbf{u} \theta)(y, \tau) dy d\tau.\end{aligned}$$

where

$$P(x, t) = C_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}.$$

They first show

$$g(\tilde{x} + h\mathbf{e}) - g(\tilde{x}) = O(h^{2\delta}), \quad \tilde{x} \equiv (x, t)$$

Two new proofs by Kiselev and Nazarov and by Constantin and Vicol.

Theorem (Kiselev and Nazarov, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 370 (2009))

Assume $(\theta, u) \in C^\infty(\mathbb{T}^d \times [0, T])$ satisfies the critical SQG equation. Assume that $\|u\|_{BMO} \leq B$ uniformly for $t \in [0, T]$. Then, there is $\beta = \beta(B, d)$ such that

$$\|\theta(\cdot, t)\|_{C^\beta} \leq C(\theta_0, B) \quad \text{for any } t \in [0, T].$$

A consequence is the global regularity of the critical SQG.

The idea is to characterize the Hölder space C^β as the dual of a local Hardy space. The key part is to show the transfer of evolution on the test function.

Theorem (Constantin and Vicol, arXiv: 1110.0179v1, 2 Oct 2011)

(Pointwise nonlinear lower bound) For any $f \in \mathcal{S}$, $\alpha \in (0, 2)$

$$\nabla f \cdot \Lambda^\alpha \nabla f \geq \frac{1}{2} \Lambda^\alpha |\nabla f|^2 + \frac{|\nabla f|^{2+\alpha}}{c \|f\|_{L^\infty}^\alpha}.$$

One consequence is the global regularity of the critical SQG with any data in Schwartz class. They define the so-called “Only Small Shocks” (denoted OSS_δ) meaning

$$\sup_{|x-y|< L} |\theta(x, t) - \theta(y, t)| \leq \delta.$$

They show that (1) $\theta_0 \in \text{OSS}_\delta$ implies $\theta \in \text{OSS}_{8\delta}$; and (2) $\theta \in \text{OSS}_\delta$ implies regularity.

Supercritical SQG: $\alpha < \frac{1}{2}$

There are many papers devoted to the supercritical case.

We summarize the major results in two recent papers

- Constantin and Wu, Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation, *Ann. Inst. H. Poincaré Anal. Non Lin.* **25**(2008), 1103-1110.
- Constantin and Wu, Hölder continuity of solutions of supercritical hydrodynamic transport equation, *Ann. Inst. H. Poincaré Anal. Non Lin.* **26**(2009), 159-180.

Theorem Let $\theta_0 \in \mathbf{L}^2(\mathbb{R}^n)$ and θ be a Leray-Hopf weak solution of (3) with $\alpha < \frac{1}{2}$. Then, for any $t > 0$,

$$\sup_{\mathbb{R}^n} |\theta(x, t)| \leq C \frac{\|\theta_0\|_{\mathbf{L}^2}}{t^{\frac{n}{4\alpha}}}, \quad \|\mathbf{u}(\cdot, t)\|_{\mathbf{BMO}(\mathbb{R}^n)} \leq C \frac{\|\theta_0\|_{\mathbf{L}^2}}{t^{\frac{n}{4\alpha}}}.$$

Theorem Let $0 < \alpha < \frac{1}{2}$. Let θ be a Leray-Hopf weak solution.

Assume, for $t_0 > 0$,

$$\theta \in L^\infty(\mathbb{R}^n \times [t_0, \infty))$$

and

$$u \in L^\infty([t_0, \infty); C^{1-2\alpha}(\mathbb{R}^n)).$$

Then, for some $\delta > 0$,

$$\theta \in C^\delta(\mathbb{R}^n \times [t_0, \infty)).$$

Open problem: Can the condition on u be removed?

Diego Chamorro attempted to remove the assumption

$$\mathbf{u} \in \mathbf{C}^{1-2\alpha}$$

but failed

arXiv: 1007.3919v1 [math.AP] 22 Jul 2010

arXiv: 1007.3919v1 [math.AP] 22 Oct 2010

arXiv: 1007.3919v4 [math.AP] 12 Apr 2011

Theorem Let $0 < \alpha < \frac{1}{2}$. Let θ be a Leray-Hopf weak solution. Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If

$$\theta \in L^\infty([t_0, t]; C^\delta(\mathbb{R}^2)),$$

then

$$\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).$$

Remark. This theorem also applies to the critical case.

Remark. Dong and Pavlovic (2010) generalized this result by replacing the Holder norm by more general Besov norms.

Remark. A weak solution at the regularity level \mathbf{C}^δ with

$\delta > 1 - 2\alpha$ or higher can not develop finite-time singularity.

Remark. Whether $\delta > 1 - 2\alpha$ can be removed remains open.

Eventual regularity for $\alpha < \frac{1}{2}$

L. Silvestre (2010) showed that a weak solution is eventually smooth if α is sufficiently close to $\frac{1}{2}$.

M. Dabkowski, arXiv: 1007.2970v1 [math.AP] 18 Jul 2010

For any $0 < \alpha < \frac{1}{2}$, there is $T = T(\alpha, \|\theta_0\|_{L^\infty})$ such that if the solution is smooth on $[0, T]$, then it is smooth for any $t > 0$.

A. Kiselev, arXiv: 1009.0542v1 [math.AP] 2 Sep 2010

Generalized modulus of continuity

C. Miao & L. Xue: arXiv:1011.6214 [math.AP] 29 Nov 2010

Generalized SQG: inviscid case

Consider the the generalized SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, & x \in \mathbb{R}^2, t > 0, \\ u = \nabla^\perp \psi \equiv (-\partial_{x_2} \psi, \partial_{x_1} \psi), & -\Lambda^2 \psi = P(\Lambda) \theta, \end{cases} \quad (4)$$

where $\theta = \theta(x, t)$ is a scalar function of $x \in \mathbb{R}^2$ and $t \geq 0$, u denotes the velocity field, ψ the stream function, and

$$\Lambda = (-\Delta)^{\frac{1}{2}}, \quad \widehat{P(\Lambda)\theta}(\xi) = P(|\xi|)\widehat{\theta}(\xi).$$

Special examples of (4) are

- When $P(\Lambda) = I$, (4) becomes

the 2D Euler vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega. \end{cases} \quad (5)$$

- When $P(\Lambda) = \Lambda$, (4) becomes

the inviscid SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad -\Lambda \psi = \theta. \end{cases} \quad (6)$$

The SQG is sometimes written as $u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$.

- When $P(\Lambda) = \Lambda^\beta$, (4) becomes
the generalized SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta \theta, \end{cases} \quad (7)$$

- When $P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma$, (4) becomes
the Log-log Euler equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 + \log(1 - \Delta)))^\gamma \theta. \end{cases} \quad (8)$$

Motivation The equations with $P(\Lambda) = \Lambda^\beta$ bridges the 2D Euler and the inviscid SQG. The 2D Euler has global regularity while the global regularity of the SQG appears to be extremely hard. Are there any equations in between have global regularity?

Numerical computations on two patches appear to indicate finite-time singularity. See D. Córdoba, M. Fontelos, A. Mancho and J. Rodrigo, Evidence of singularities for a family of contour dynamics equations, *Proc. Natl. Acad. Sci. USA* **102** (2005), 5949–5952.

In a recent paper

D. Chae, P. Constantin and J. Wu, Inviscid models generalizing the
2D Euler and the surface quasi-geostrophic equations, *Archive for
Rational Mechanics and Analysis* **202** (2011), 35-62.

we obtained the global regularity of an equation with velocity more
singular than the 2D Euler velocity

Global regularity for the Loglog-Euler equation

Let $P(\Lambda) = (\log(1 + \log(1 - \Delta)))^\gamma$

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 + \log(1 - \Delta)))^\gamma \theta, \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (9)$$

which we call the Loglog-Euler equation. If we think of $(\log(1 + \log(1 - \Delta)))^\gamma \theta$ as vorticity ω , we have unbounded vorticity which does not belong to the Yudovich class.

Theorem (Chae, Constantin and Wu: Archive for Rational Mechanics and Analysis, 2011)

Consider the initial-value problem (9) with γ and θ_0 satisfying

$$0 \leq \gamma \leq 1, \quad \theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B_{q,\infty}^s(\mathbb{R}^2) \quad (10)$$

where $2 < q \leq \infty$ and $s > 1$. Then the initial-value problem (9) has a unique global solution θ satisfying,

$$\theta \in L^\infty([0, \infty); B_{q,\infty}^s(\mathbb{R}^2)), \quad \nabla u \in L^\infty([0, \infty); B_{q,\infty}^{s_1}(\mathbb{R}^2)),$$

where $s_1 < s$.

Definition (Besov spaces)

For $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$, the inhomogeneous Besov space $B_{p,q}^s$ is defined by

$$B_{p,q}^s = \left\{ f \in \mathcal{S}' : \|f\|_{B_{p,q}^s} < \infty \right\},$$

where

$$\|f\|_{B_{p,q}^s} \equiv \begin{cases} \left(\sum_{j=-1}^{\infty} \left(2^{js} \|\Delta_j f\|_{L^p} \right)^q \right)^{1/q}, & \text{if } q < \infty, \\ \sup_{-1 \leq j < \infty} 2^{js} \|\Delta_j f\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

Remark

Because of the embedding relations

$$W_q^r \hookrightarrow B_{q,\infty}^r \hookrightarrow B_{q,\min\{2,q\}}^{r_1} \hookrightarrow W_q^{r_1}, \quad r > r_1,$$

we can conclude that any initial data in W_q^r with $2 < q \leq \infty$ and $r > 1$ would yield a global solution in $W_q^{r_1}$ for any $r_1 < r$.

To prove this theorem, we developed tools to handle operators $P(\Lambda)$ whose symbol $P(|\xi|)$ satisfying the following conditions

Assumption

The symbol $P = P(|\xi|)$ assumes the following properties:

- 1 P is continuous on \mathbb{R}^d and $P \in C^\infty(\mathbb{R}^d \setminus \{0\})$;
- 2 P is radially symmetric;
- 3 $P = P(|\xi|)$ is nondecreasing in $|\xi|$;
- 4 There exists two constants C and C_0 such that

$$\sup_{2^{-1} \leq |\eta| \leq 2} |(I - \Delta_\eta)^n P(2^j |\eta|)| \leq C P(C_0 2^j)$$

for any integer j and $n = 1, 2, \dots, 1 + [\frac{d}{2}]$.

(4) in Assumption is a very natural condition on symbols of Fourier multiplier operators in order for the operator to be bounded from L^p to L^p ($1 < p < \infty$) as in the Mihlin-Hörmander Multiplier Theorem. Some special examples of P are

$$P(\xi) = |\xi|^\beta \quad \text{with } \beta \geq 0,$$

$$P(\xi) = (\log(1 + |\xi|^2))^\gamma \quad \text{with } \gamma \geq 0,$$

$$P(\xi) = (\log(1 + \log(1 + |\xi|^2)))^\gamma \quad \text{with } \gamma \geq 0,$$

$$P(\xi) = (\log(1 + |\xi|^2))^\gamma |\xi|^\beta \quad \text{with } \gamma \geq 0 \text{ and } \beta \geq 0.$$

The first ingredient is the control for $\|\nabla u\|_{L^\infty}$.

Theorem

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by

$$(\nabla u)_{jk} = \mathcal{R}_I \mathcal{R}_m P(\Lambda) \theta,$$

where $1 \leq j, k, l, m \leq d$, $(\nabla u)_{jk}$ denotes the (j, k) -th entry of ∇u , \mathcal{R}_l denotes the Riesz transform, and P obeys Assumption. Then, for any integers $j \geq 0$ and $N \geq 0$,

$$\|S_N \nabla u\|_{L^p} \leq C_{p,d} P(2^N) \|S_N \theta\|_{L^p}, \quad 1 < p < \infty, \quad (11)$$

$$\|\Delta_j \nabla u\|_{L^q} \leq C_d P(2^j) \|\Delta_j \theta\|_{L^q}, \quad 1 \leq q \leq \infty, \quad (12)$$

$$\|S_N \nabla u\|_{L^\infty} \leq C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N P(2^N) \|S_{N+1} \theta\|_{L^\infty}, \quad (13)$$

where $C_{p,d}$ is a constant depending on p and d only and C_d 's depend on d only.



Another ingredient is the interpolation inequality.

Proposition

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field. Assume

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m (\log(I + \log(I - \Delta)))^\gamma \theta \quad (14)$$

Then, for any $1 \leq q \leq \infty$ and $s > d/q$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq \|\theta\|_{L^1 \cap L^\infty} + C \|\theta\|_{L^\infty} \log(1 + \|\theta\|_{B_{q,\infty}^s}) \\ &\times \left(\log \left(1 + \log(1 + \|\theta\|_{B_{q,\infty}^s}) \right) \right)^\gamma \end{aligned}$$

Global regularity for the Log-Euler is open

It is currently unknown if solutions of the Log-Euler equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 - \Delta))^\gamma \theta, \\ \theta(x, 0) = \theta_0(x). \end{cases} \quad (15)$$

are global in time.

$$P(\Lambda) = \Lambda^\beta$$

Consider the active scalar equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta \theta, \quad \Lambda = (-\Delta)^{1/2}, \quad \beta \geq 0. \end{cases} \quad (16)$$

When $\beta = 0$, this is the 2D Euler vorticity.

When $\beta > 0$, the global regularity issue is open.

We have the following regularity criterion.

As β increases, u becomes more singular and the global regularity problem becomes more difficult.

$$\begin{aligned}\|\nabla u\|_{L^\infty} &= \|\nabla \nabla^\perp \Delta^{-1} \Lambda^\beta \theta\|_{L^\infty} \\ &= \|\mathcal{R}_j \mathcal{R}_k \Lambda^\beta \theta\|_{L^\infty} \approx \|\Lambda^\beta \theta\|_{L^\infty} \log(1 + \|\theta\|_{H^3(\text{or } C^\sigma)})\end{aligned}$$

$\beta = 1$ corresponds to the SQG equation. When $\beta < 1$, the velocity is more regular. When $\beta > 1$, the velocity is more singular than the SQG velocity.

Theorem (Chae, Constantin and Wu: ARMA, 2011)

Consider (33) with $0 \leq \beta \leq 1$. Let θ be a solution of (33) corresponding to the data $\theta_0 \in C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $\sigma > 1$ and $q > 1$. Let $T > 0$. If θ satisfies

$$\int_0^T \|\theta(\cdot, t)\|_{C^\beta(\mathbb{R}^2)} dt < \infty,$$

then θ remains in $C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ on the time interval $[0, T]$.

In order to obtain global regularity, the standard idea is to bound

$$\begin{aligned} \|\nabla u\|_{L^\infty} &= \|\nabla \nabla^\perp \Delta^{-1} \Lambda^\beta \theta\|_{L^\infty} \\ &= \|\mathcal{R}_j \mathcal{R}_k \Lambda^\beta \theta\|_{L^\infty} \approx \|\Lambda^\beta \theta\|_{L^\infty} \log(1 + \|\theta\|_{H^3(\text{or } C^\sigma)}) \end{aligned}$$

The proof of this result involves two ingredients. The first one bounds the L^∞ -norm of S in terms of the logarithm of the Hölder-norm of θ .

Proposition

Let $0 \leq \beta \leq 1$. Assume that u and θ are related by

$$u = -\nabla^\perp \Lambda^{-2+\beta} \theta \quad (17)$$

If $\theta \in C^\sigma(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$ with $\sigma > \beta$ and $q > 1$,

$$\|S\|_{L^\infty} \leq C_1 \|\theta\|_{C^\beta} \ln(1 + \|\theta\|_{C^\sigma}) + C_2 \|\theta\|_{L^q}, \quad (18)$$

where C_1 and C_2 are constants depending on β , σ and q only.

Proposition

Let u be a velocity field and let $S = \frac{1}{2}(\nabla u + (\nabla u)^T)$. Let A be the back-to-labels map. Then,

$$\|\nabla_x A(\cdot, t)\|_{L^\infty} \leq \exp\left(\int_0^t \|S(\cdot, \tau)\|_{L^\infty} d\tau\right).$$

Proof of Theorem: For any $\sigma \leq 1$,

$$\|\theta(\cdot, t)\|_{C^\sigma} = \sup_{x \neq y} \frac{|\theta(x, t) - \theta(y, t)|}{|x - y|^\sigma} \leq \|\theta_0\|_{C^\sigma} \|\nabla_x A(\cdot, t)\|_{L^\infty}^\sigma.$$

$$\ln(1 + \|\theta(\cdot, t)\|_{C^\sigma}) \leq C \ln(1 + \|\theta_0\|_{C^\sigma} + \|\theta_0\|_{L^q}) \exp\left(C \int_0^t \|\theta(\cdot, \tau)\|_{C^\beta} d\tau\right).$$

Generalized SQG: dissipative case

Consider the generalized dissipative SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Lambda^2 \psi = P(\Lambda) \theta. \end{cases} \quad (19)$$

where $\kappa > 0$, $\alpha > 0$ and P satisfies the Assumption.

Special examples of (19) are

- When $P(\Lambda) = I$, (19) becomes

the 2D Navier-Stokes vorticity equation

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega + \kappa(-\Delta)^\alpha \omega = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \omega. \end{cases} \quad (20)$$

- When $P(\Lambda) = \Lambda$, (19) becomes

the dissipative SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad -\Lambda \psi = \theta. \end{cases} \quad (21)$$

The SQG is sometimes written as $u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$.

- When $P(\Lambda) = (\log(1 - \Delta))^\gamma$, (19) becomes the Log-Navier-Stokes equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 - \Delta))^\gamma \theta. \end{cases} \quad (22)$$

- When $P(\Lambda) = \Lambda^\beta$, (19) becomes the generalized dissipative SQG equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta \theta, \end{cases} \quad (23)$$

which was previously studied by Constantin, Iyer and Wu in 2008.

Constantin, Iyer and Wu, *Indiana U. Math. J.*, 2008

$$P(\Lambda) = \Lambda^{-(1-2\alpha)} \text{ or } \Lambda^{1-2\alpha} u = \nabla^\perp \nabla^{-1} \theta.$$

Kiselev, *Adv. Math.*, 2011

Nonlocal maximum principles for active scalars

Chae, Constantin and Wu, *Arch. Ration. Mech. Anal.*, 2011

Chae, Constantin and Wu, arXiv:1011.0171v1 [math.AP] 31 Oct 2010

Miao and Xue, arXiv:1011.6214 [math.AP] 29 Nov 2010

Chae, Constantin, Cordoba, Gancedo and Wu, arXiv:1101.3537v1
[math.AP] 18 Jan 2011

Dabkowski, Kiselev and Vicol, arXiv:1106.2137 [math.AP] 10 Jun
2011 slightly supercritical surface quasi-geostrophic equation

The magneto-geostrophic equation

Friedlander and Vicol, Ann. Inst. H. Poincar Anal. Non Linéaire,
2011

Friedlander and Vicol, arXiv:1105.1403 [math.AP] 13 May 2011

Friedlander, Rusin and Vicol, arXiv:1110.1129 [math.AP] 6 Oct
2011

The results presented below are from the joint work

D. Chae, P. Constantin and J. Wu, Dissipative models generalizing
the 2D Navier-Stokes and the SQG equations, arXiv:1011.0171v1
[math.AP] 31 Oct 2010

Definition (Extended Besov spaces)

Let $s \in \mathbb{R}$ and $1 \leq q, r \leq \infty$. Let $\{A_j\}_{j \geq -1}$ with $A_j \geq 0$ be a nondecreasing sequence. Let $f \in \mathcal{S}'$. We say $f \in B_{q,r}^{s,A}$ if f satisfies

$$\|f\|_{B_{q,r}^{s,A}} \equiv \left\| 2^{sA_j} \|\Delta_j f\|_{L^q} \right\|_{l^r} < \infty.$$

When $A_j = j + 1$, $B_{q,r}^{s,A}$ becomes the Besov space $B_{q,r}^s$. When

$$\frac{A_j}{j+1} \rightarrow 0, \quad j \rightarrow \infty$$

$B_{q,r}^{s,A}$ is called a sub-Besov space. When the limit tends to ∞ , $B_{q,r}^{s,A}$ is called a super-Besov space.

Global regularity for general $P(\Lambda)$

Theorem (Chae, Constantin and Wu: arXiv:1011.0171v1)

Consider the dissipative active scalar equation (19) with $\kappa > 0$, $\alpha > 0$ and $P(\xi)$ satisfying Condition 2.4. Let $s > 1$, $2 \leq q \leq \infty$ and $A = \{A_j\}_{j \geq -1}$ be a nondecreasing sequence with $A_j \geq 0$. Let $\theta_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{s,A}(\mathbb{R}^d)$. Assume either the velocity u is divergence-free or the solution θ is bounded in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for all time. If, there exists a constant C such that for all $j \geq -1$,

$$\sum_{k \geq j-1, k \geq -1} \frac{2^{sA_{j-2}} P(2^{k+1})}{2^{sA_k} P(2^{j+1})} < C \quad (24)$$

and

Theorem

$$\kappa^{-1} 2^{s(A_j - A_{j-2})} (j+2) P(2^{j+2}) 2^{-2\alpha j} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \quad (25)$$

then (19) has a unique global solution θ satisfying, for any $T > 0$

$$\theta \in L^\infty \left([0, T]; B_{q, \infty}^{s, A}(\mathbb{R}^d) \right).$$

Global regularity for $(\log(I - \Delta))^\gamma$

One special consequence is when $P(\Lambda) = (\log(I - \Delta))^\gamma$. Consider

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(I - \Delta))^\gamma \theta. \end{cases}$$

with $\kappa > 0$, $\alpha > 0$ and $\gamma \geq 0$. In this case

$$P(|\xi|) = (\log(I + |\xi|^2))^\gamma, \quad \gamma \geq 0 \quad \text{and} \quad A_j = (j+1)^b \quad \text{for some } b \leq 1, \tag{26}$$

(24) is trivially satisfied and the condition in (25) reduces to

$$2^{s((j+1)^b - j^b)} (j+2)^{1+\gamma} 2^{-2\alpha j} \rightarrow 0 \quad \text{as } j \rightarrow \infty, \tag{27}$$

which is obviously satisfied for any $\alpha > 0$.

Theorem

Consider the dissipative Log-Euler equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = (\log(1 - \Delta))^\gamma \theta \end{cases} \quad (28)$$

with $\kappa > 0$, $\alpha > 0$ and $\gamma \geq 0$. Assume that θ_0 satisfies

$$\theta_0 \in Y \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B_{q,\infty}^{s,A}(\mathbb{R}^2)$$

with $s > 1$, $2 \leq q \leq \infty$ and A given in (26). Then (28) has a unique global solution θ satisfying

$$\theta \in L^\infty([0, \infty); Y).$$

Theorem

Consider the active scalar equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta (\log(1 - \Delta))^\gamma \theta \end{cases} \quad (29)$$

with $\kappa > 0$, $\alpha > 0$, $0 \leq \beta < 2\alpha \leq 1$ and $\gamma \geq 0$. Assume the initial data $\theta_0 \in Y \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B_{q,\infty}^{s,A}(\mathbb{R}^2)$ with $s > 1$, $2 \leq q \leq \infty$ and $A_j = j + 1$. Then (29) has a unique global solution θ satisfying

$$\theta \in L^\infty([0, \infty); Y).$$

In conclusion, we developed a global regularity theory for the generalized SQG equations with very general Fourier multiplier operators. The extended Besov spaces proposed here include a range of spaces such as Sobolev spaces and Schwarz class.

In addition, the results presented here indicate that $\kappa(-\Delta)^\alpha$ for any $\alpha > 0$ does have a regularization effect.

Generalized SQG with singular velocities

We now turn to generalized SQG with singular velocities. These results are from a joint work

D. Chae, P. Constantin, D. Córdoba, F.Gancedo and J. Wu,
Generalized surface quasi-geostrophic equations with singular
velocities, arXiv:1101.3537v1 [math.AP] 18 Jan 2011.

Consider two models. The first one is given by

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi &= \Lambda^\beta \theta, \end{aligned} \tag{30}$$

where β is a real parameter satisfying $1 < \beta \leq 2$.

$\beta = 1$ corresponds to the SQG equation. When $\beta > 1$, u is more singular than the SQG equation since

$$u = \nabla^\perp \Delta^{-1} \Lambda^\beta \theta = \mathcal{R}^\perp \Lambda^{\beta-1} \theta$$

demands more regularity in θ .

When $\beta = 2$,

$$\psi = -\theta, \quad u = -\nabla^\perp \theta, \quad u \cdot \nabla \theta \equiv 0$$

and the equation is reduced to $\partial_t \theta = 0$, which has global solution.

Since the velocity in the case $1 < \beta < 2$ is more singular than the SQG, the issue is whether or not one can establish the local existence and uniqueness.

What is the difficulty? Let σ be a multi-index with $|\sigma| \leq m$.

$$\frac{d}{dt} \|D^\sigma \theta\|_{L^2}^2 = - \int D^\sigma (u \cdot \nabla \theta) D^\sigma \theta \, dx$$

The most singular part in the nonlinear term is

$$\int D^\sigma u \cdot \nabla \theta D^\sigma \theta \, dx$$

where $u = \Lambda^{\beta-1} \mathcal{R}^\perp \theta$ with $\beta > 1$. The idea is to rewrite the integral in terms of commutator. Realizing that for any skew-adjoint operator A , we have

$$\int A(f) fg \, dx = - \int f A(g) \, dx = -\frac{1}{2} \int f [A, g] f \, dx$$

$$\int D^\sigma u \cdot \nabla \theta D^\sigma \theta \, dx = -\frac{1}{2} \int D^\sigma \theta [\Lambda^{\beta-1} \mathcal{R}^\perp, \nabla \theta] D^\sigma \theta \, dx$$

Notice that $\mathcal{R}^\perp = \Lambda^{-1} \nabla^\perp$. We need a commutator estimate.

Proposition

For any $s \in R$,

$$\|[\Lambda^s \partial_x, g]f\|_{L^2} \leq C_\epsilon (\|\Lambda^s f\|_{L^2} \|g\|_{H^{2+\epsilon}} + \|f\|_{L^2} \|g\|_{H^{2+s+\epsilon}}).$$

Applying this proposition with $s = \beta - 2$, we get

$$\left| \int D^\sigma u \cdot \nabla \theta D^\sigma \theta \, dx \right| \leq C \|D^\sigma \theta\|_{L^2}^2 \|\nabla \theta\|_{H^{\beta+\epsilon}}.$$

Theorem (Local smooth solution)

Consider (30) with $1 < \beta \leq 2$. Assume that $\theta_0 \in H^m(\mathbb{R}^2)$ with $m \geq 4$. Then (30) has a unique local (in time) solution $\theta \in C([0, T]; H^m(\mathbb{R}^2))$ for some $T = T(\|\theta_0\|_{H^m}) > 0$

Theorem

Assume that $\theta_0 \in L^2(\mathbb{T}^2)$ has mean zero, namely

$$\int_{\mathbb{T}^2} \theta_0(x) dx = 0.$$

Then (30) has a global Leray-Hopf weak solution.

For $1 < \beta < 2$, the velocity is more singular and we need to write the nonlinear term as a commutator in terms of the stream function ψ .

The second equation is given by

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta &= 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi &= (\log(I - \Delta))^\mu \Lambda^\beta \theta, \end{aligned} \tag{31}$$

where $\kappa > 0$, $\alpha > 0$ and $\mu > 0$ are real parameters.

When $\beta = 2$ and $\mu = 0$, then $u = \nabla^\perp \theta$ and $u \cdot \nabla \theta = 0$. It becomes a linear equation and has global solution.

When $\beta = 2$ and $\mu > 0$, the velocity field $u = \nabla^\perp (\log(I - \Delta))^\mu \theta$ is logarithmically beyond the trivial case. Do we have a global solution? Do we even have a local solution?

K. Ohkitani was the first one who proposed this problem.

Theorem

Consider the active scalar equation (31) with $\kappa > 0$, $\alpha > 0$ and $\mu > 0$. Assume the initial data $\theta_0 \in H^4(\mathbb{R}^2)$. Then there exists $T > 0$ such that (31) has a unique solution $\theta \in C([0, T]; H^4(\mathbb{R}^2))$.

Again we need to deal with the nonlinear term

$$\int D^\sigma u \cdot \nabla \theta D^\sigma \theta \, dx$$

with $u = (\log(I - \Delta))^\mu \nabla^\perp \theta$ (one derivative higher plus logarithm)

A commutator estimate involving the logarithm of Laplace.

Proposition

Let $\mu \geq 0$. Let ∂_x denote a partial derivative, either ∂_{x_1} or ∂_{x_2} . Then, for any $\delta > 0$ and $\epsilon > 0$,

$$\| [(\ln(I - \Delta))^{\mu} \partial_x, g] f \|_{L^2} \leq C_{\mu, \epsilon, \delta} \left(1 + \left(\ln \left(1 + \frac{\|f\|_{\dot{H}^{\delta}}}{\|f\|_{L^2}} \right) \right)^{\mu} \right) \|f\|_{L^2} \|g\|_{H^{2+3\epsilon}},$$

where $C_{\mu, \epsilon, \delta}$ is a constant depending on μ , ϵ and δ only, \dot{H}^{δ} denotes the standard homogeneous Sobolev space and the brackets denote the commutator.

In addition, we also obtained local well-posedness of the contour dynamics under the first model. This result extends a previous work of F. Gancedo to $1 < \beta \leq 2$.

F. Gancedo, Existence for the a-patch model and the QG sharp front in Sobolev spaces, Adv. Math. 217 (2008), 2569-2598.

Logarithmically supercritical SQG

Work in progress

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta + \kappa \frac{\Lambda^{2\alpha}}{\log^{2\beta}(1 + \Lambda)} \theta &= 0, \\ u = \nabla^\perp \psi, \quad -\Lambda \psi &= \theta. \end{aligned} \tag{32}$$

Theorem

Let $\kappa > 0$, $\alpha > 0$ and $\beta \geq 0$. Let $\theta_0 \in L^2$ and let θ be a corresponding Leray-Hopf weak solution. Then, for any $t > 0$,

$$\|\theta(\cdot, t)\|_{L^\infty} \leq C \frac{\|\theta_0\|_{L^2}}{t^{\frac{1}{2\alpha_1}}}$$

where $0 < \alpha_1 < \alpha$ and C is a constant depending on α_1 and β .



Theorem

Let $\theta_0 \in L^2(\mathbb{R}^2)$ and let θ be a corresponding Leray-Hopf weak solution. Let $0 < t_1 < t_2 < \infty$. If $\theta \in L^\infty([t_1, t_2]; C^\delta)$ with $\delta > 1 - 2\alpha$, then

$$\theta \in C^\infty((t_1, t_2] \times \mathbb{R}^2).$$

We are also trying the method “Modulus of Continuity”.

Numerical results

Following the pioneering work of [Constantin, Majda and Tabak](#) and also the work of [Ohkitani and Yamada](#), we have done extensive numerical computations on the SQG and related equations.

How does β affect the regularity of the solutions to

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = \nabla^\perp \psi, \quad \Delta \psi = \Lambda^\beta \theta, \quad \Lambda = (-\Delta)^{1/2}, \quad \beta \geq 0, \\ \theta(x, 0) = \sin x_1 \sin x_2 + \cos x_2, \quad x \in [0, 2\pi] \times [0, 2\pi] \end{cases} \quad (33)$$

Some of the plots are from the paper

Peter Constantin, Ming-Chih Lai, Ramjee Sharma, Yu-Hou Tseng
and Jiahong Wu, New numerical results for the surface
quasi-geostrophic equation, Journal of Scientific Computing,
accepted for publication.

We thank T. Hou and K. Ohkitani for technical help.

- Parallel computing on total 128 machines with each being an Intel Pentium Xeon EM64T quad Core E5405 @ 2.0 GHz processors in Supercomputing Center at University of Oklahoma.
- Different uniform mesh sizes: 256×256 , 512×512 , 1024×1024 , 2048×2048 , and 4096×4096 .
- Parallel Fourier transforms were calculated by using mpi fftw (Fast Fourier Transform in the West) routines.
- The time integration was carried out by the fourth-order Runge-Kutta method. The time step $\Delta t = \Delta x / 10$.

$\beta = 1$ or the inviscid SQG equation

$$\theta_t + u \cdot \nabla \theta = 0, \quad \kappa > 0,$$

$$u = (-\partial_{x_2} \Lambda^{-1} \theta, \partial_{x_1} \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta)$$

$$\theta(x, 0) = \sin x_1 \sin x_2 + \cos x_2, \quad x \in [0, 2\pi] \times [0, 2\pi]$$

The L^2 -norm and helicity were monitored.

Introduction
Generalized SQG: inviscid case
Generalized SQG: dissipative case
Generalized SQG with singular velocities
Logarithmically supercritical SQG

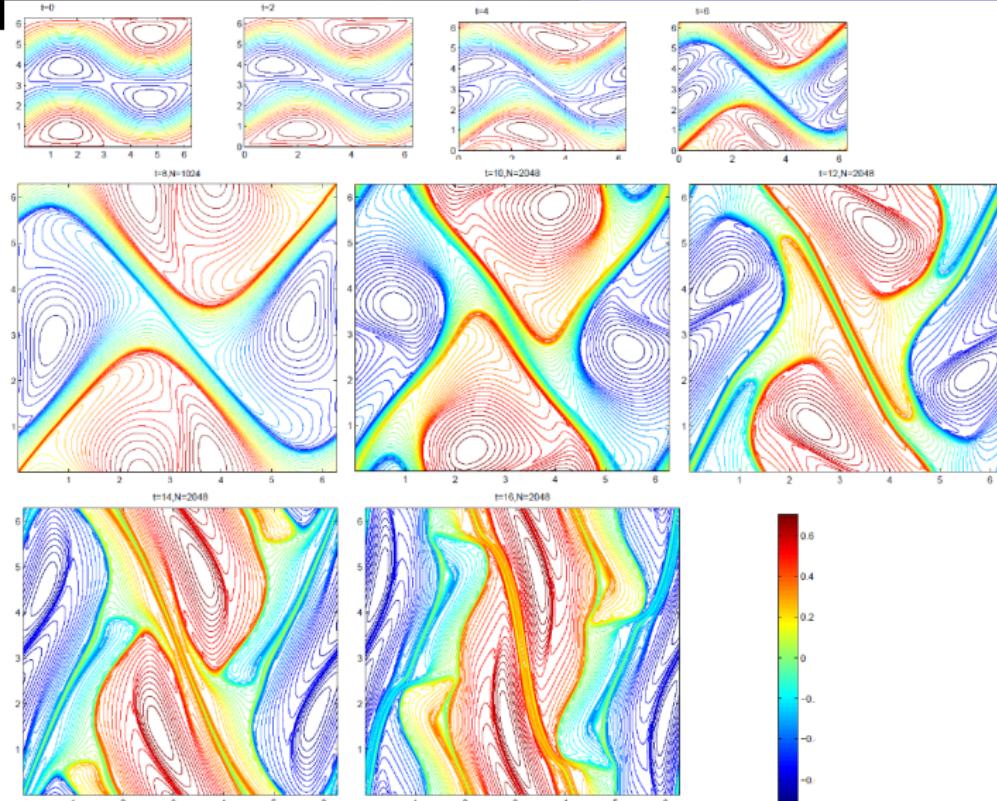


Figure: Contours of θ from $t=0$ to $t=16$

$t=16, [0,\pi] \times [0,\pi]$

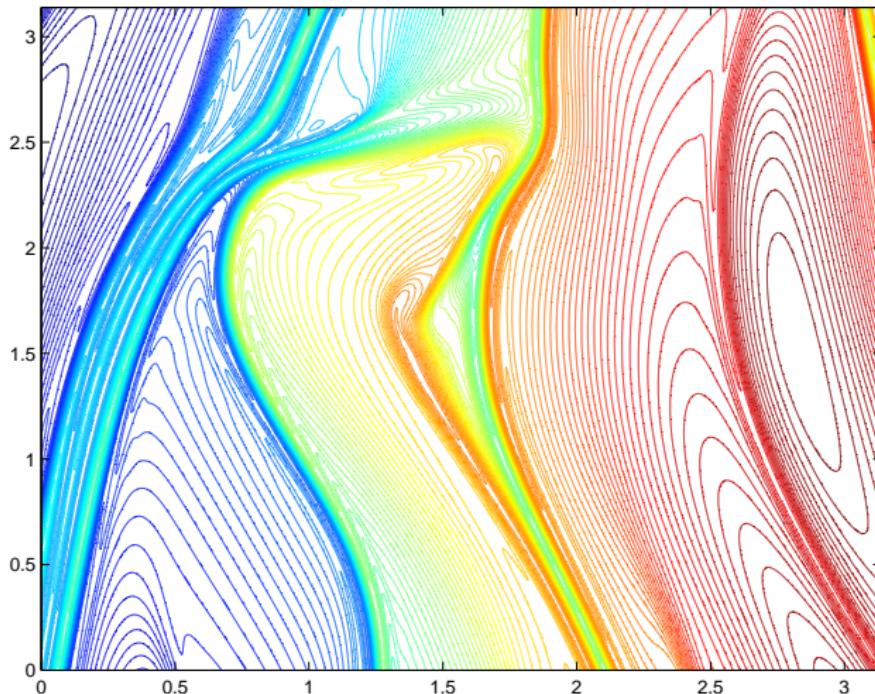


Figure: Contours of θ , $t = 16$, $N = 4096$, first quarter

$t=16, [\pi, 2\pi] \times [0, \pi]$

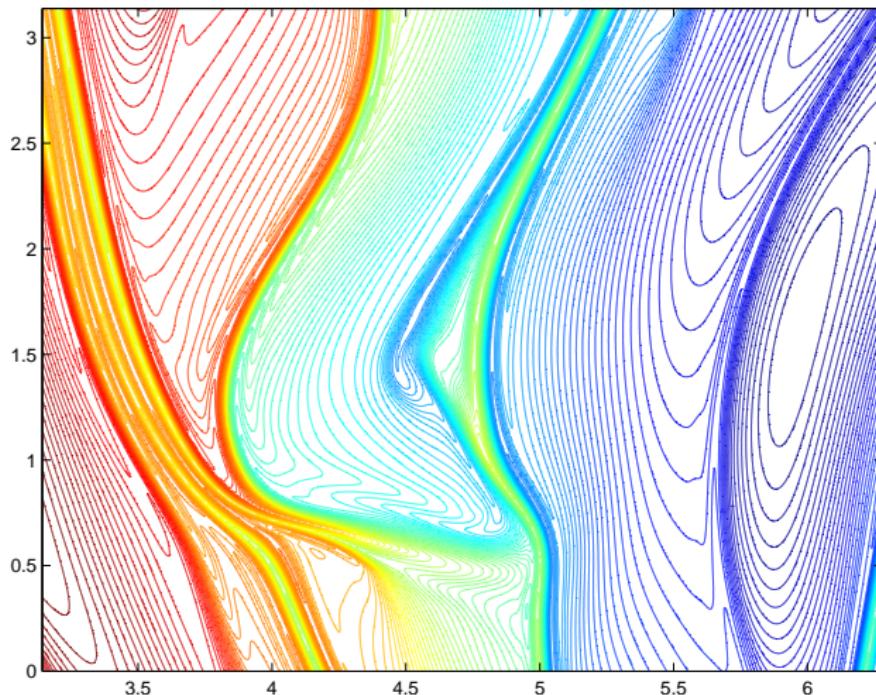


Figure: Contours of θ , $t = 16$, $N = 4096$, second quarter

$t=16, [0,\pi] \times [\pi, 2\pi]$

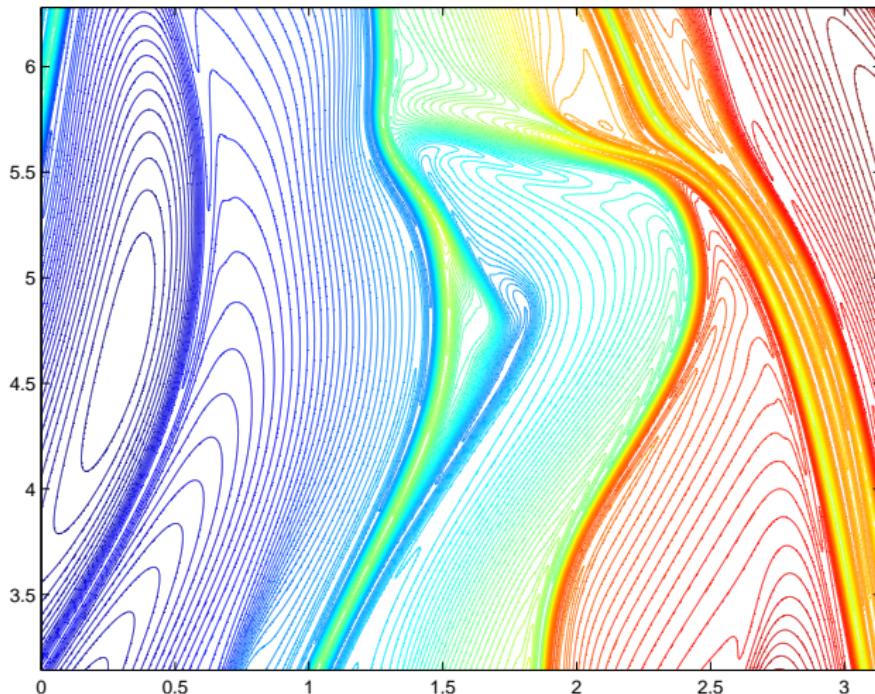


Figure: Contours of θ , $t = 16$, $N = 4096$, third quarter

$t=16, [\pi, 2\pi] \times [\pi, 2\pi]$

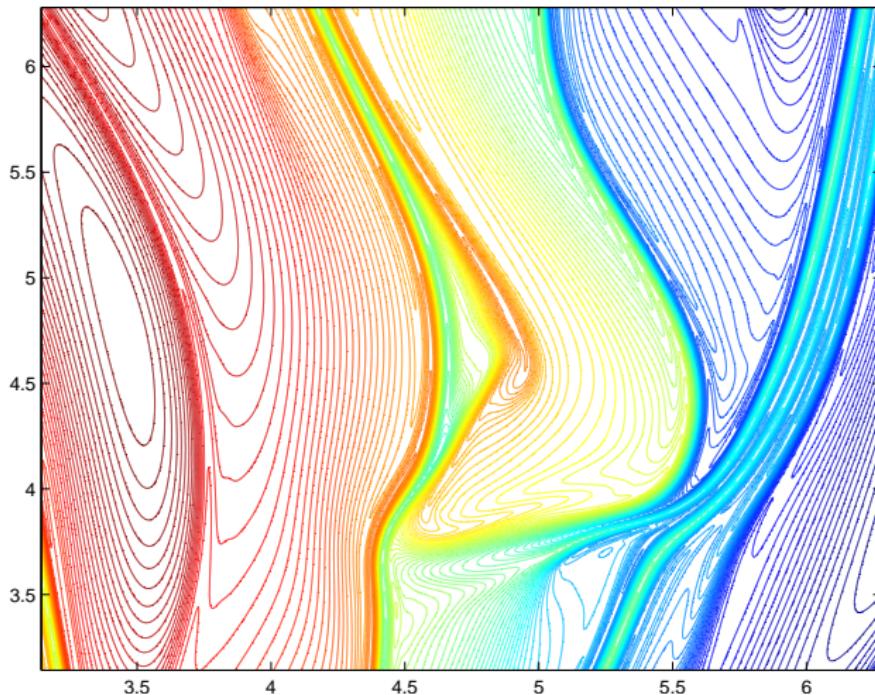


Figure: Contours of θ , $t = 16$, $N = 4096$, fourth quarter

$\beta = 0$ or the 2D Euler vorticity equation

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

$$u = (-\partial_{x_2} \Lambda^{-2} \theta, \partial_{x_1} \Lambda^{-2} \theta),$$

$$\omega(x, 0) = \sin x_1 \sin x_2 + \cos x_2.$$

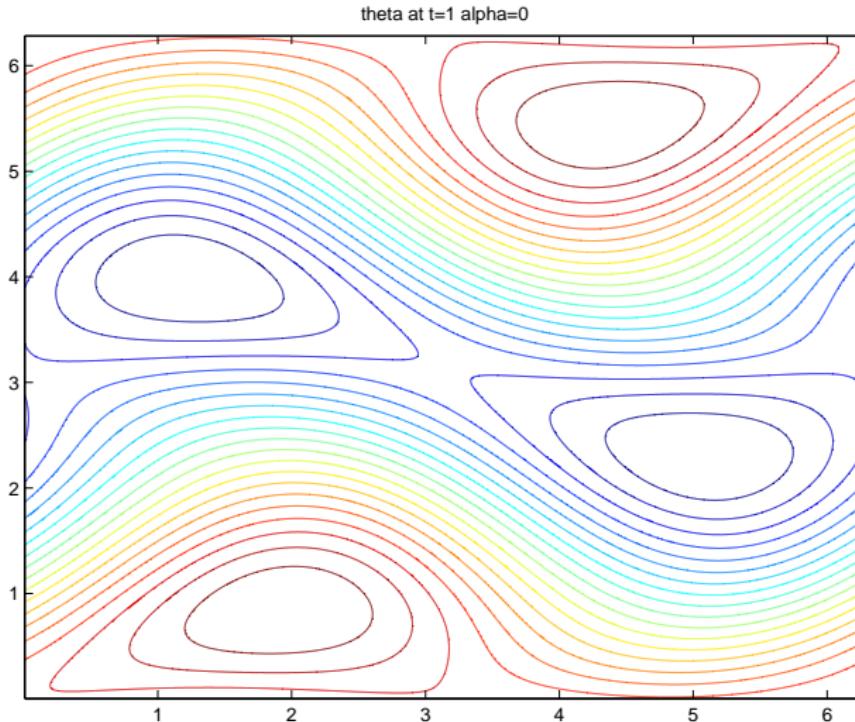


Figure: Contour of ω at $t = 1$

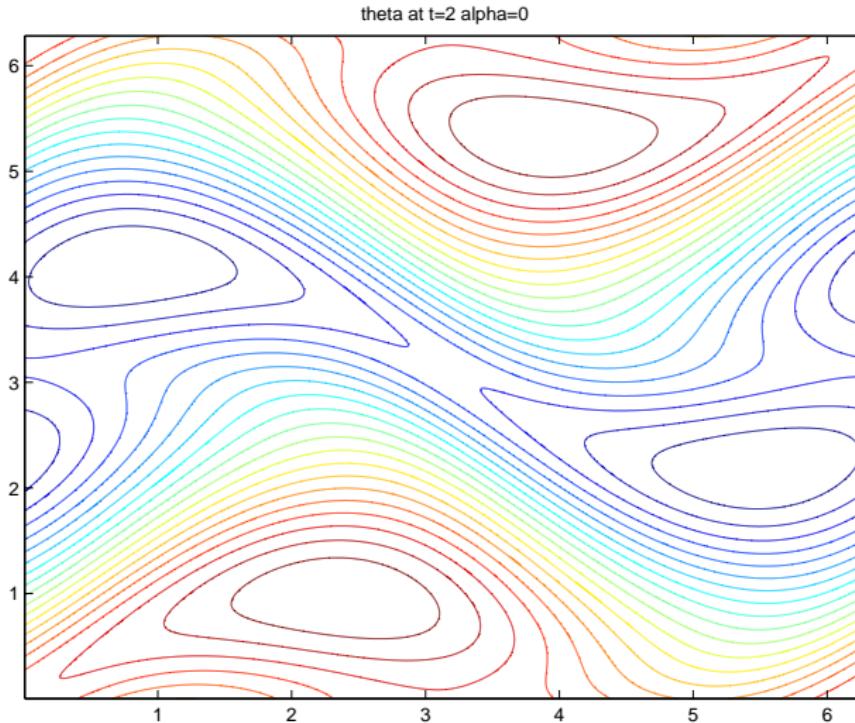


Figure: Contour of ω at $t = 2$

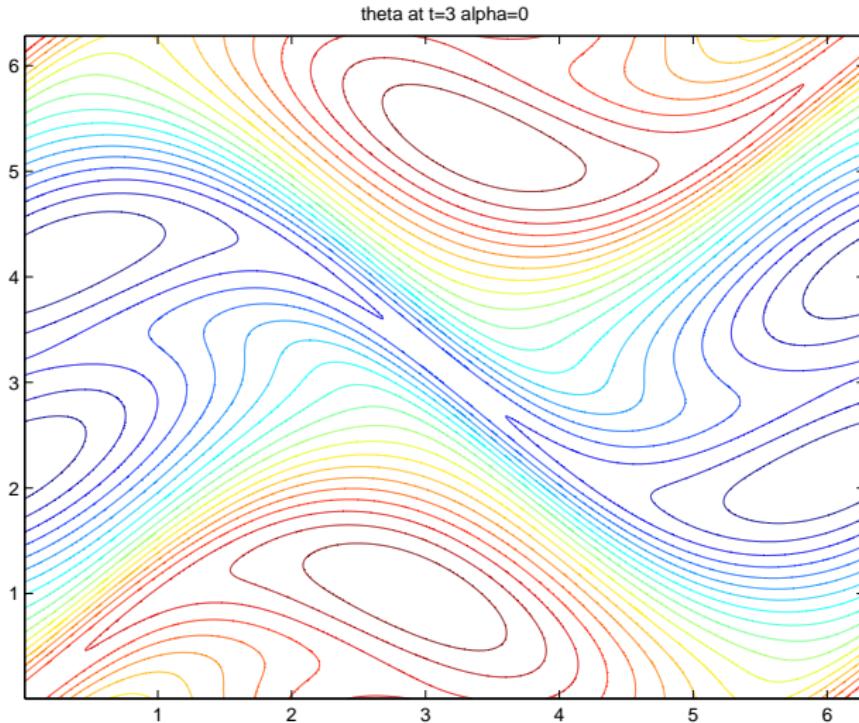


Figure: Contour of ω at $t = 3$

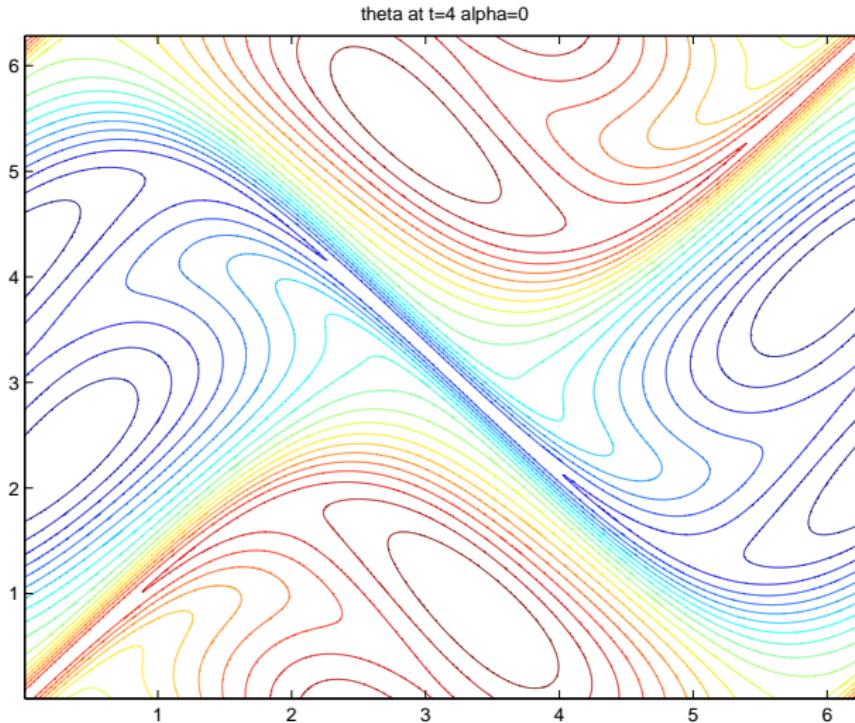


Figure: Contour of ω at $t = 4$

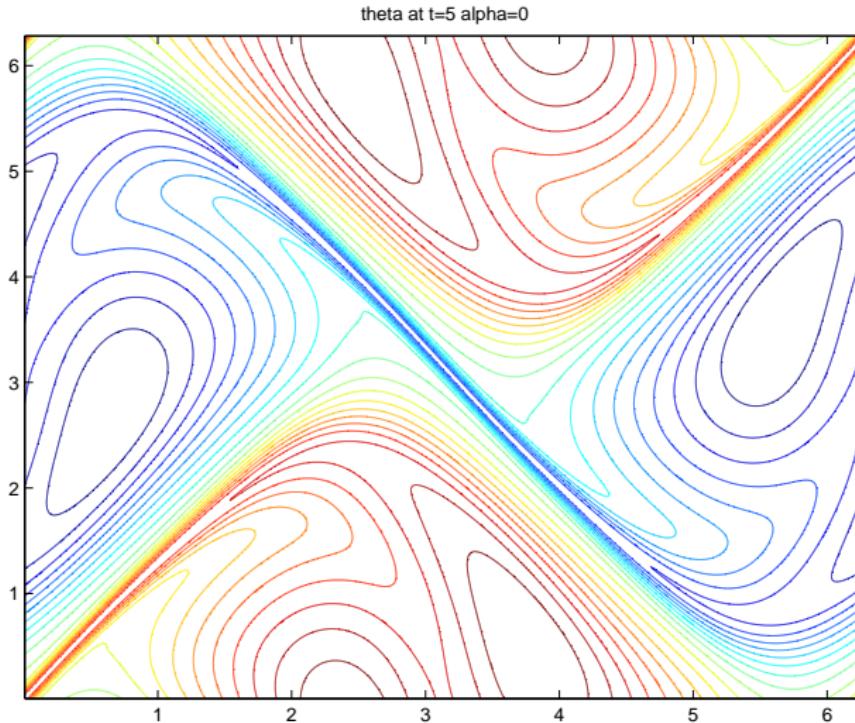


Figure: Contour of ω at $t = 5$

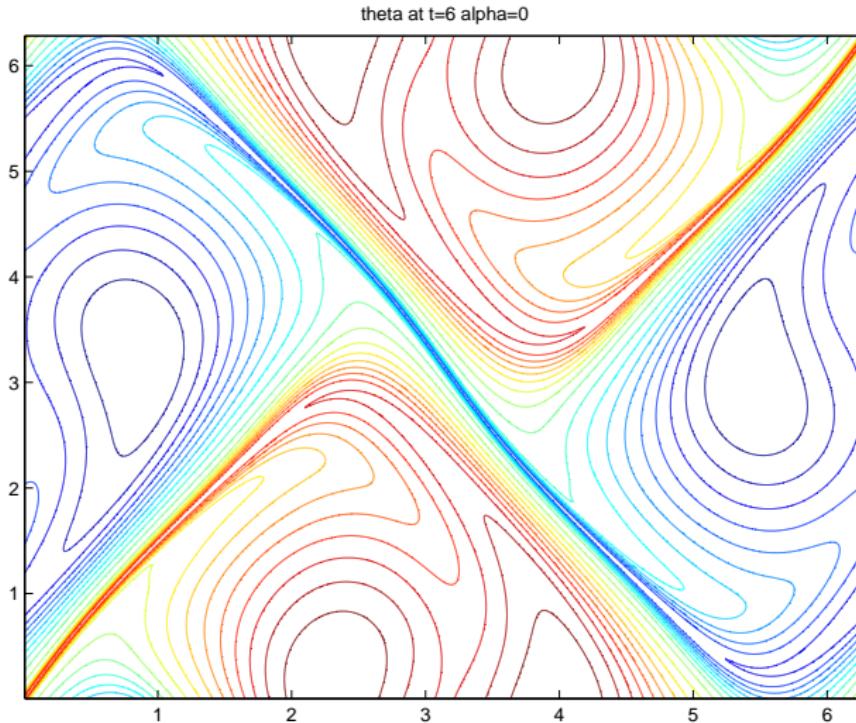


Figure: Contour of ω at $t = 6$

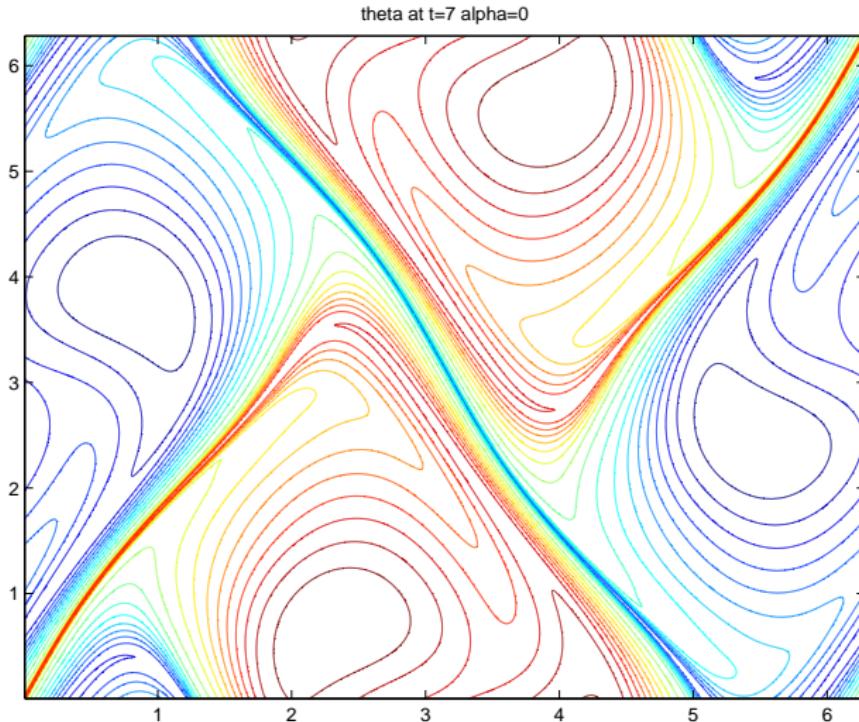


Figure: Contour of ω at $t = 7$

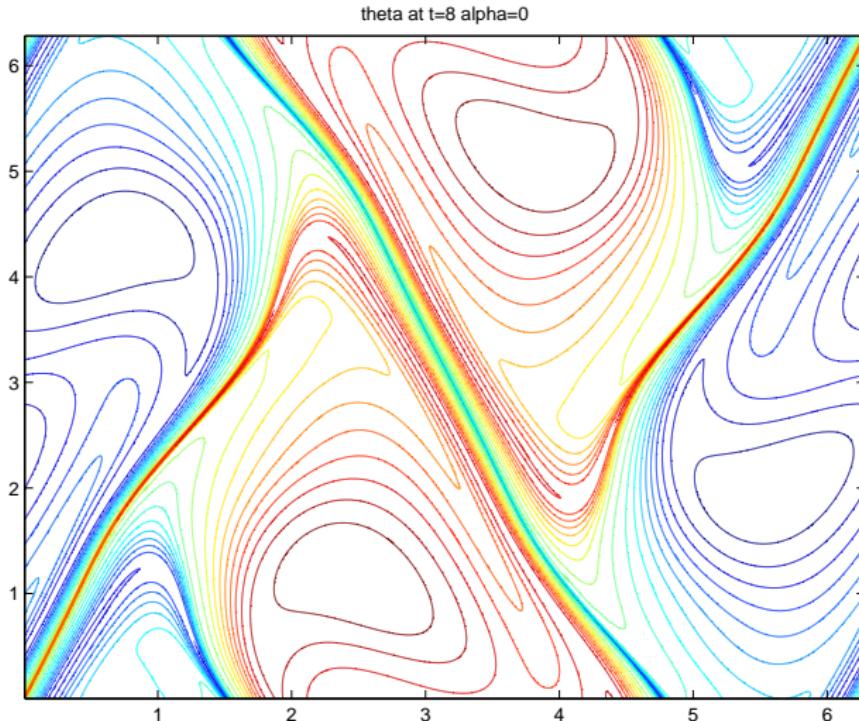


Figure: Contour of ω at $t = 8$

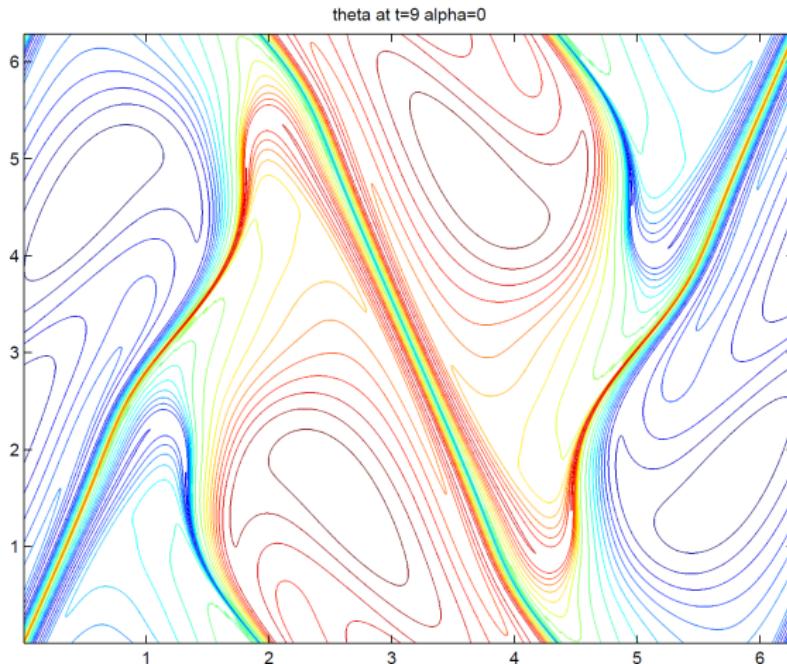


Figure: Contour of ω at $t = 9$

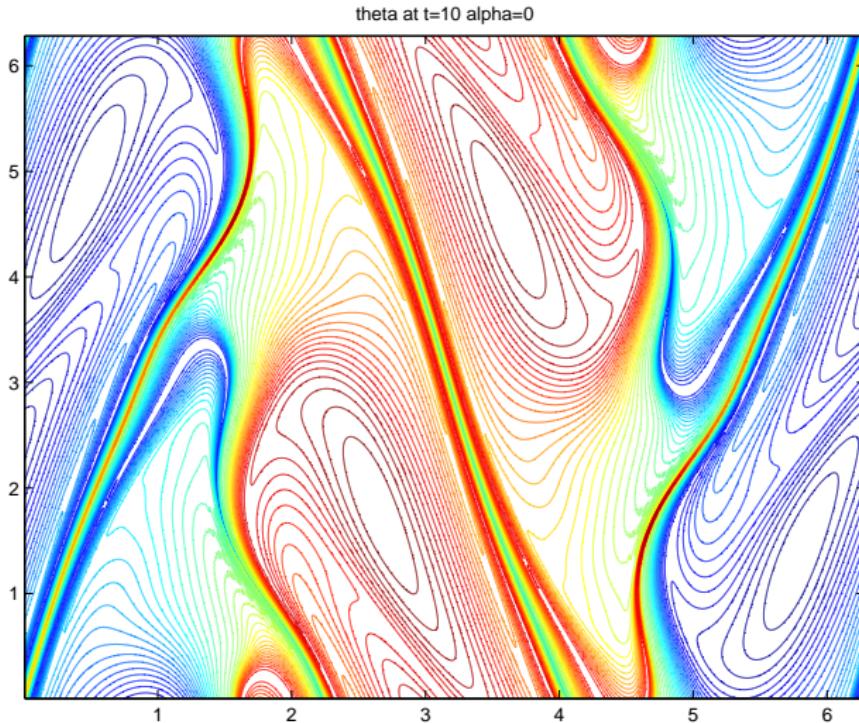


Figure: Contour of ω at $t = 10$

A side-by-side comparison between the SQG and the 2D Euler

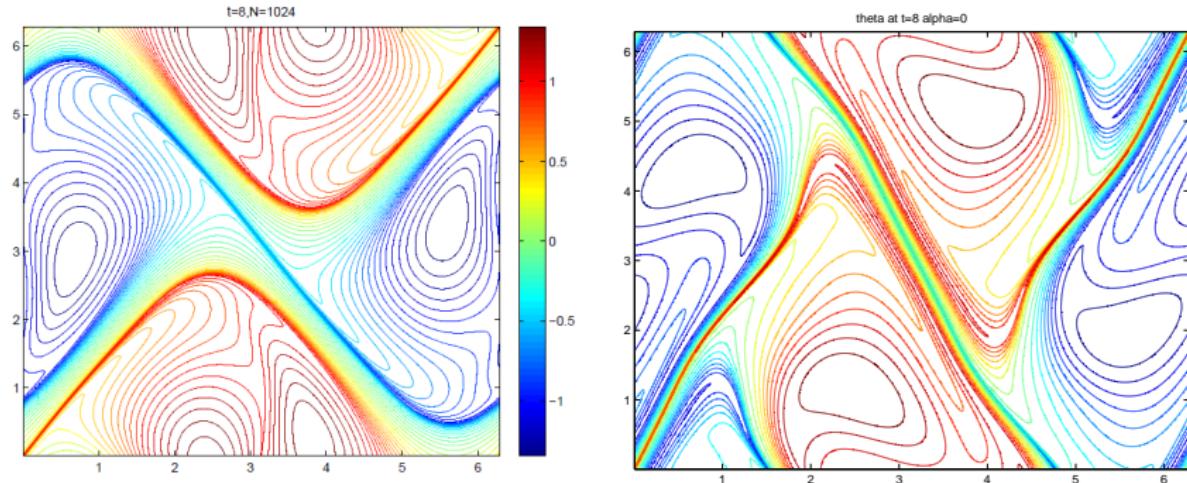


Figure: Left: contour of θ at $t = 8$; Right: contours of ω at $t = 8$

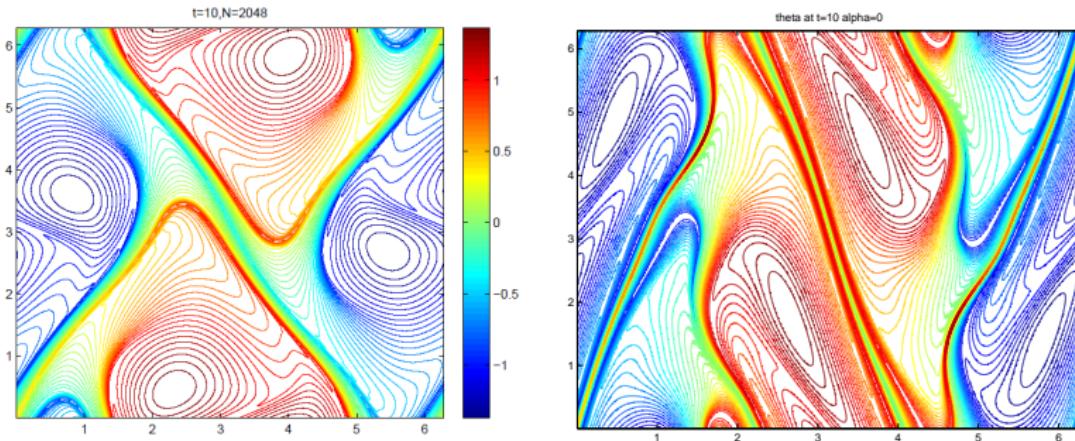


Figure: Left: contour of θ at $t = 10$; Right: contours of ω at $t = 10$

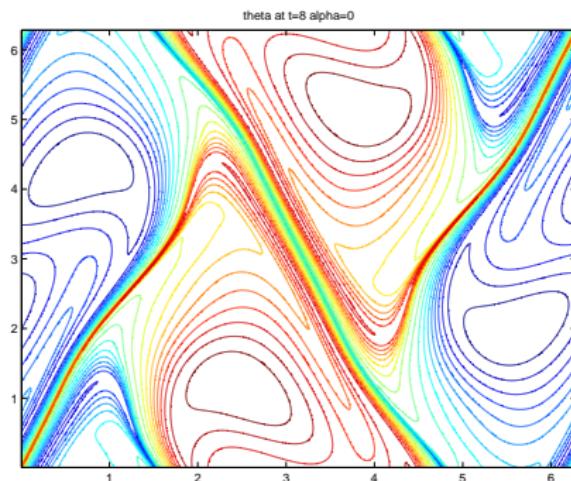
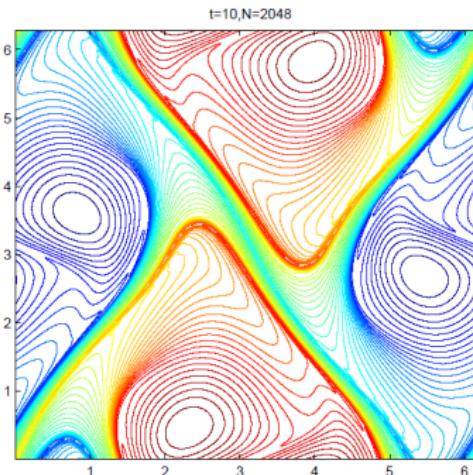
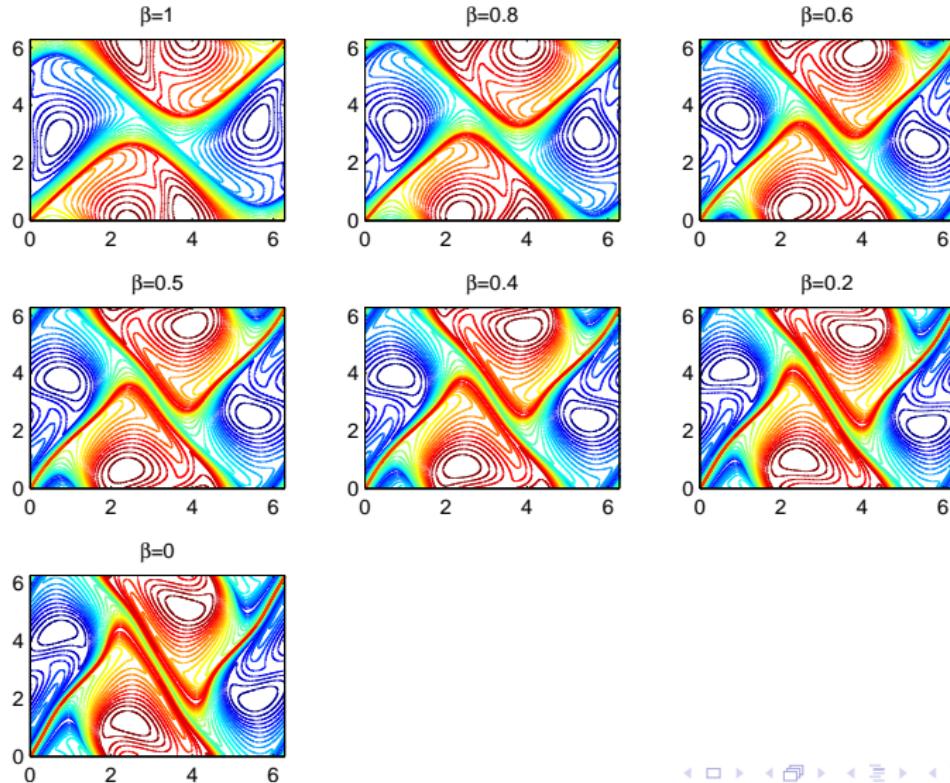


Figure: Left: contour of θ at $t = 10$; Right: contours of ω at $t = 8$



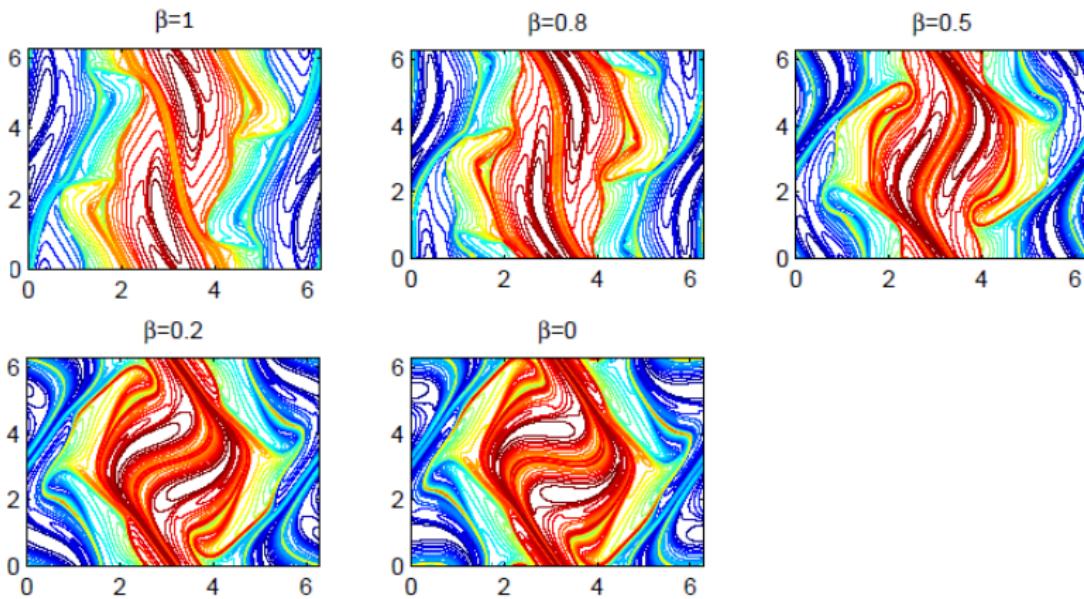


Figure: Comparison of contours for various β at $t = 16$

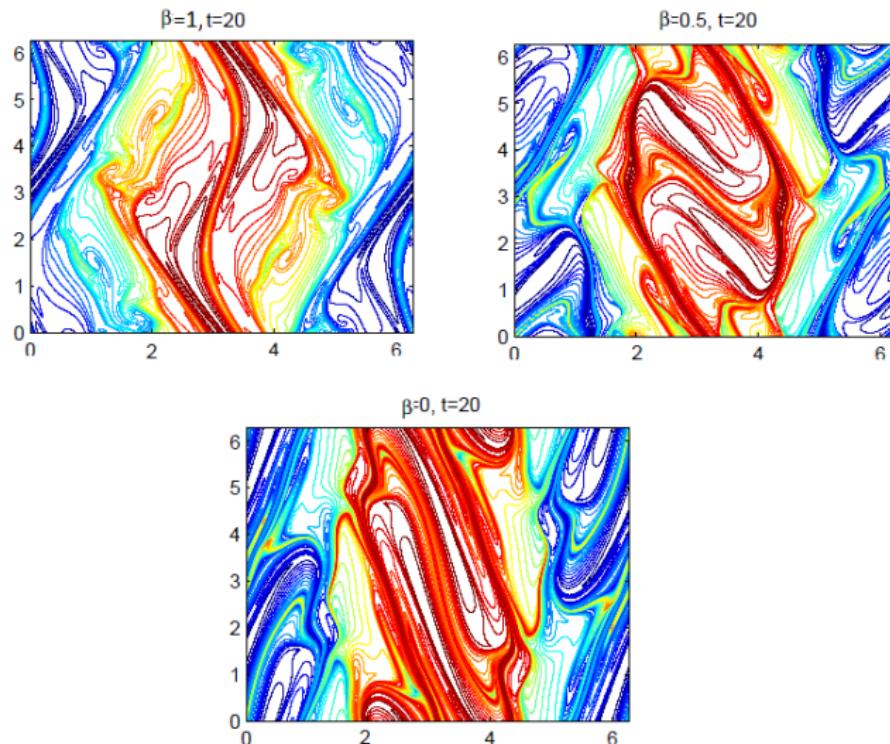


Figure: Comparison of contours for various β at $t = 20$.

In conclusion, we have covered the following topics

- A summary of results for the critical and supercritical SQG
- Recent work for the generalized inviscid SQG
- Recent work for the generalized dissipative SQG
- Generalized SQG with singular velocities
- Logarithmically supercritical SQG equation
- Numerical results

Some Proofs

The rest of slides detail the proofs of the major theorems. They are divided into three subsections: bounds for ∇u , proofs for the theorem in the inviscid case and proofs for the theorem in the viscous case.

Bounds for ∇u

First we need to control $\|\nabla u\|_{L^\infty}$.

Theorem

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field. Assume that u is related to a scalar θ by

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m P(\Lambda) \theta,$$

where $1 \leq j, k, l, m \leq d$, $(\nabla u)_{jk}$ denotes the (j, k) -th entry of ∇u , \mathcal{R}_l denotes the Riesz transform, and P obeys Assumption. Then, for any integers $j \geq 0$ and $N \geq 0$,

$$\|S_N \nabla u\|_{L^p} \leq C_{p,d} P(2^N) \|S_N \theta\|_{L^p}, \quad 1 < p < \infty, \quad (34)$$

$$\|\Delta_j \nabla u\|_{L^q} \leq C_d P(2^j) \|\Delta_j \theta\|_{L^q}, \quad 1 \leq q \leq \infty, \quad (35)$$

$$\|S_N \nabla u\|_{L^\infty} \leq C_d \|\theta\|_{L^1 \cap L^\infty} + C_d N P(2^N) \|S_{N+1} \theta\|_{L^\infty}, \quad (36)$$



We make use of Mihlin and Hörmander Multiplier Theorem (see [Stein, p.96]) to prove (34).

Theorem

Suppose that $Q(\xi)$ is of class C^k in the complement of the origin of \mathbb{R}^d , where $k > \frac{d}{2}$ is an integer. Assume also that

$$|D^\alpha Q(\xi)| \leq B |\xi|^{-|\alpha|}, \quad \text{whenever } |\alpha| \leq k. \quad (37)$$

Then $Q \in \mathcal{M}_q$, $1 < q < \infty$. That is, $\|T_Q f\|_{L^q} \leq C_q \|f\|_{L^q}$, where T_Q is defined by

$$\widehat{T_Q f}(\xi) = Q(\xi) \widehat{f}(\xi).$$

Write $(\widehat{S_N \nabla u})_{jk}(\xi) = Q(\xi) P(C_0 2^N) \widehat{S_N \theta}(\xi)$ and verify that Q satisfies the condition of this theorem.

The Proof of (35):

$$(\Delta_N \nabla u)_{jk} = \mathcal{R}_I \mathcal{R}_m P(\Lambda) \Delta_N \theta$$

and

$$(\widehat{\Delta_N \nabla u})_{jk}(\xi) = -\frac{\xi_I \xi_m}{|\xi|^2} P(|\xi|) \widehat{\Delta_N \theta}(\xi).$$

Then

$$(\widehat{\Delta_N \nabla u})_{jk}(\xi) = -\frac{\xi_I \xi_m}{|\xi|^2} P(|\xi|) \tilde{\phi}_N(\xi) \widehat{\Delta_N \theta}(\xi)$$

or

$$(\Delta_N \nabla u)_{jk} = g * \Delta_N \theta,$$

The rest is to verify that $g \in L^1$.

Proof for the inviscid case

We recall the theorem here

Theorem

Consider the initial-value problem (15) with γ and θ_0 satisfying

$$0 \leq \gamma \leq 1, \quad \theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B_{q,\infty}^s(\mathbb{R}^2)$$

where $2 < q \leq \infty$ and $s > 1$. Then the initial-value problem (15) has a unique global solution θ satisfying,

$$\theta \in L^\infty([0, \infty); B_{q,\infty}^s(\mathbb{R}^2)), \quad u \in L^\infty([0, \infty); B_{q,\infty}^{1+s_1}(\mathbb{R}^2)),$$

where $s_1 < s$.

In order to prove the global regularity result for the inviscid model, we need the interpolation inequality.

Proposition

Let $u : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a vector field. Assume

$$(\nabla u)_{jk} = \mathcal{R}_l \mathcal{R}_m (\log(I + \log(I - \Delta)))^\gamma \theta \quad (38)$$

Then, for any $1 \leq q \leq \infty$ and $s > d/q$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq \|\theta\|_{L^1 \cap L^\infty} + C \|\theta\|_{L^\infty} \log(1 + \|\theta\|_{B_{q,\infty}^s}) \\ &\quad \times \left(\log \left(1 + \log(1 + \|\theta\|_{B_{q,\infty}^s}) \right) \right)^\gamma \end{aligned}$$

The proof consists of two major components. The first component derives a global *a priori* bound while the second constructs a unique local in time solution through the method of successive approximation.

We start with the part on the global *a priori* bound. This part is further divided into two steps. The first step shows that for any $d/q < \sigma < 1$ and any $T > 0$ and $t \leq T$

$$\|\theta(t)\|_{B_{q,\infty}^\sigma} \leq C(T, \|\theta_0\|_X), \quad X = L^1 \cap L^\infty \cap B_{q,\infty}^\sigma$$

and the second step establishes the global bound in $B_{q,\infty}^{\sigma_1}$ for some $\sigma_1 > 1$. A finite number of iterations then yields the global bound

First of all, $\theta_0 \in L^1 \cap L^\infty$ implies that the corresponding solution θ of (15) satisfies the *a priori* bound

$$\|\theta(\cdot, t)\|_{L^1 \cap L^\infty} \leq \|\theta_0\|_{L^1 \cap L^\infty}, \quad t \geq 0. \quad (39)$$

In the process of establishing the *a priori* bound, we can avoid using the divergence-free condition on u .

Let $j \geq -1$ be an integer. Applying Δ_j to (15) and following a standard decomposition, we have

$$\partial_t \Delta_j \theta = J_1 + J_2 + J_3 + J_4 + J_5 \quad (40)$$

where

$$\begin{aligned}
 J_1 &= - \sum_{|j-k| \leq 2} [\Delta_j, S_{k-1}(u) \cdot \nabla] \Delta_k \theta, \\
 J_2 &= - \sum_{|j-k| \leq 2} (S_{k-1}(u) - S_j(u)) \cdot \nabla \Delta_j \Delta_k \theta, \\
 J_3 &= -S_j(u) \cdot \nabla \Delta_j \theta, \\
 J_4 &= - \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1}(\theta)), \\
 J_5 &= - \sum_{k \geq j-1} \Delta_j (\Delta_k u \cdot \nabla \tilde{\Delta}_k \theta)
 \end{aligned}$$

with $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$.

Multiplying (47) by $\Delta_j \theta |\Delta_j \theta|^{q-2}$, integrating in space and applying Hölder's inequality, we have

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^q} \leq \|J_1\|_{L^q} + \|J_2\|_{L^q} + \|J_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}. \quad (41)$$

By a standard commutator estimate,

$$\|J_1\|_{L^q} \leq C \sum_{|j-k| \leq 2} \|\nabla S_{k-1} u\|_{L^\infty} \|\Delta_k \theta\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leq C \|\nabla \tilde{\Delta}_j u\|_{L^\infty} \|\Delta_j \theta\|_{L^q}.$$

By integration by parts,

$$\|J_3\|_{L^q} \leq C \|\nabla \cdot S_j u\|_{L^\infty} \|\Delta_j \theta\|_{L^q}.$$

For J_4 and J_5 , we have

$$\begin{aligned}
 \|J_4\|_{L^q} &\leq \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^\infty} \|\nabla S_{k-1} \theta\|_{L^q} \\
 &\leq C \sum_{|j-k|\leq 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m\leq k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q}, \\
 \|J_5\|_{L^q} &\leq C \sum_{k\geq j-1} \|\Delta_k u\|_{L^\infty} \|\tilde{\Delta}_k \nabla \theta\|_{L^q} \\
 &\leq C \sum_{k\geq j-1} \|\nabla \Delta_k u\|_{L^\infty} \|\tilde{\Delta}_k \theta\|_{L^q}.
 \end{aligned}$$

By Proposition 6.4, for any $\sigma \in \mathbb{R}$,

$$\|J_1\|_{L^q} \leq C \sum_{|j-k|\leq 2} \|\nabla u\|_{L^\infty} 2^{-\sigma(k+1)} 2^{\sigma(k+1)} \|\Delta_k \theta\|_{L^q} \quad (42)$$

$$\leq C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty} \sum_{k\geq j-1} 2^{\sigma(j-k)} \quad (43)$$

$$\leq C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty}, \quad (44)$$

where C is a constant depending on σ only. It is clear that $\|J_2\|_{L^q}$ and $\|J_3\|_{L^q}$ obey the same bound. For any $\sigma < 1$, we have

$$\begin{aligned}
 \|J_4\|_{L^q} &\leq C \|\nabla u\|_{L^\infty} \sum_{|j-k| \leq 2} \sum_{m < k-1} 2^{m-k} 2^{-\sigma(m+1)} 2^{\sigma(m+1)} \|\Delta_m \theta\|_{L^q} \\
 &\leq C \|\nabla u\|_{L^\infty} \|\theta\|_{B_{q,\infty}^\sigma} \sum_{|j-k| \leq 2} \sum_{m < k-1} 2^{m-k} 2^{-\sigma(m+1)} \\
 &= C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty} \sum_{|j-k| \leq 2} 2^{\sigma(j-k)} \sum_{m < k-1} 2^{(m-k)(1-\sigma)} \\
 &\leq C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty}.
 \end{aligned}$$

where C is a constant depending on σ only and the condition $\sigma < 1$ is used to guarantee that $(m - k)(1 - \sigma) < 0$. For any $\sigma > 0$,

$$\begin{aligned}\|J_5\|_{L^q} &\leq C \|\nabla u\|_{L^\infty} 2^{-\sigma(j+1)} \sum_{k \geq j-1} 2^{\sigma(j-k)} 2^{\sigma(k+1)} \|\tilde{\Delta}_k \theta\|_{L^q} \\ &\leq C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty}.\end{aligned}$$

Collecting these estimates, we obtain, for any $0 < \sigma < 1$,

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^q} \leq C 2^{-\sigma(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\nabla u\|_{L^\infty}.$$

Integrating in time yields

$$\|\theta(t)\|_{B_{q,\infty}^\sigma} \leq \|\theta_0\|_{B_{q,\infty}^\sigma} + C \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^\sigma} \|\nabla u(\tau)\|_{L^\infty} d\tau.$$

Invoking the interpolation inequality in Proposition 6.4, we obtain,

for $d/q < \sigma < 1$,

$$\begin{aligned}\|\theta(t)\|_{B_{q,\infty}^\sigma} &\leq \|\theta_0\|_{B_{q,\infty}^\sigma} + C \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^\sigma} \left[\|\theta\|_{L^1 \cap L^\infty} + (1 + \|\theta\|_{L^\infty}) \right. \\ &\quad \times \log(1 + \|\theta\|_{B_{q,\infty}^\sigma}) \left. \left(\log \left(1 + \log(1 + \|\theta\|_{B_{q,\infty}^\sigma}) \right) \right) \right]^\gamma\end{aligned}$$

It then follows from Gronwall's inequality that, for any $T > 0$,

$$\|\theta(t)\|_{B_{q,\infty}^\sigma} \leq C(T, \|\theta_0\|_x), \quad t \leq T.$$

We now continue with the second step. Since $d < q \leq \infty$, we can choose σ satisfying

$$\frac{d}{q} < \sigma < 1, \quad \sigma + 1 - \frac{d}{q} > 1$$

and then set σ_1 satisfying

$$1 < \sigma_1 < \sigma + 1 - \frac{d}{q}.$$

This step establishes the global bound for $\|\theta\|_{B_{q,\infty}^{\sigma_1}}$. J_1 , J_2 and J_3 and J_5 can be bounded the same way as before, namely

$$\|J_1\|_{L^q}, \|J_2\|_{L^q}, \|J_3\|_{L^q}, \|J_5\|_{L^q} \leq C 2^{-\sigma_1(j+1)} \|\theta\|_{B_{q,\infty}^{\sigma_1}} \|\nabla u\|_{L^\infty}.$$

$\|J_4\|_{L^q}$ is estimated differently and bounded by the global bound in the first step. We start with the bound

$$\|J_4\|_{L^q} \leq C \sum_{|j-k| \leq 2} \|\nabla \Delta_k u\|_{L^\infty} \sum_{m < k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q}.$$

By Bernstein's inequality, we have

$$\begin{aligned} \|\nabla \Delta_k u\|_{L^\infty} &\leq 2^{\frac{dk}{q}} \|\nabla \Delta_k u\|_{L^q} \\ &\leq 2^{\frac{dk}{q}} (\log(2+k))^\gamma \|\Delta_k \theta\|_{L^q}. \end{aligned}$$

Clearly,

$$\begin{aligned} \sum_{m < k-1} 2^{m-k} \|\Delta_m \theta\|_{L^q} &= 2^{-\sigma k} \sum_{m < k-1} 2^{(m-k)(1-\sigma)} 2^{\sigma m} \|\Delta_m \theta\|_{L^q} \\ &\leq C 2^{-\sigma k} \|\theta\|_{B_{q,\infty}^\sigma}. \end{aligned}$$

$$\begin{aligned}
 \|J_4\|_{L^q} &\leq C \sum_{|j-k|\leq 2} 2^{\frac{dk}{q}} (\log(2+k))^\gamma \|\Delta_k \theta\|_{L^q} 2^{-\sigma k} \|\theta\|_{B_{q,\infty}^\sigma} \\
 &= C 2^{-\sigma_1(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \sum_{|j-k|\leq 2} 2^{\sigma_1(j-k)} (\log(2+k))^\gamma 2^{(\sigma_1 + \frac{d}{q} - \sigma)k} \|\Delta_k \theta\|_{L^q} \\
 &= C 2^{-\sigma_1(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\theta\|_{B_{q,\infty}^{\sigma_2}} \sum_{|j-k|\leq 2} 2^{\sigma_1(j-k)} (\log(2+k))^\gamma 2^{(\sigma_1 + \frac{d}{q} - \sigma - \sigma_2)k}
 \end{aligned}$$

where $\sigma_2 < 1$ is chosen very close to 1 and satisfies

$$\sigma_1 + \frac{2}{q} - \sigma - \sigma_2 < 0.$$

Then, by the global bound in the first step,

$$\|J_4\|_{L^q} \leq C 2^{-\sigma_1(j+1)} \|\theta\|_{B_{q,\infty}^\sigma} \|\theta\|_{B_{q,\infty}^{\sigma_2}} \leq C(T, \|\theta_0\|_X) 2^{-\sigma_1(j+1)}.$$

Collecting the estimates in this step, we have

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^q} \leq C 2^{-\sigma_1(j+1)} \|\theta\|_{B_{q,\infty}^{\sigma_1}} \|\nabla u\|_{L^\infty} + C(T, \|\theta_0\|_X) 2^{-\sigma_1(j+1)}.$$

By Proposition 6.4, for any $d/q < \sigma < 1$,

$$\begin{aligned} \|\nabla u\|_{L^\infty} &\leq \|\theta\|_{L^1 \cap L^\infty} + (1 + \|\theta\|_{L^\infty}) \\ &\quad \times \log(1 + \|\theta\|_{B_{q,\infty}^\sigma}) \left(\log \left(1 + \log(1 + \|\theta\|_{B_{q,\infty}^\sigma}) \right) \right)^\gamma \\ &\leq C(T, \|\theta_0\|_X). \end{aligned}$$

Therefore,

$$\|\theta(t)\|_{B_{q,\infty}^{\sigma_1}} \leq \|\theta_0\|_{B_{q,\infty}^{\sigma_1}} + C(T, \|\theta_0\|_X) \left(1 + \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^{\sigma_1}} d\tau \right).$$

Proof for the dissipative model

The proof is divided into two main parts. The first part establishes the global (in time) *a priori* bound on solutions of (??) while the second part briefly describes the construction of a unique local (in time) solution.

For notational convenience, we write

$Y = L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap B_{q,\infty}^{s,A}(\mathbb{R}^d)$. The first part derives the global bound, for any $T > 0$,

$$\|\theta(\cdot, t)\|_{B_{q,\infty}^{s,A}} \leq C(T, \|\theta_0\|_Y) \quad \text{for } t \leq T \quad (45)$$

and we distinguish between two cases: $q < \infty$ and $q = \infty$. The dissipative term is handled differently in these two cases.

We start with the case when $q < \infty$. When the velocity field u is divergence-free, $\theta_0 \in L^1 \cap L^\infty$ implies the corresponding solution θ of (??) satisfies the *a priori* bound

$$\|\theta(\cdot, t)\|_{L^1 \cap L^\infty} \leq \|\theta_0\|_{L^1 \cap L^\infty}, \quad t \geq 0. \quad (46)$$

When u is not divergence-free, (46) is assumed. The divergence-free condition is not used in the rest of the proof.

Let $j \geq -1$ be an integer. Applying Δ_j to (??) and following a standard decomposition, we have

$$\partial_t \Delta_j \theta + \kappa(-\Delta)^\alpha \Delta_j \theta = J_1 + J_2 + J_3 + J_4 + J_5, \quad (47)$$

where

$$J_1 = - \sum_{|j-k| \leq 2} [\Delta_j, S_{k-1}(u) \cdot \nabla] \Delta_k \theta, \quad (48)$$

$$J_2 = - \sum_{|j-k| \leq 2} (S_{k-1}(u) - S_j(u)) \cdot \nabla \Delta_j \Delta_k \theta, \quad (49)$$

$$J_3 = -S_j(u) \cdot \nabla \Delta_j \theta, \quad (50)$$

$$J_4 = - \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1}(\theta)), \quad (51)$$

$$J_5 = - \sum_{k \geq j-1} \Delta_j (\tilde{\Delta}_k u \cdot \nabla \Delta_k \theta) \quad (52)$$

with $\tilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1}$. We multiply (47) by $\Delta_j \theta |\Delta_j \theta|^{q-2}$ and integrate in space. Integrating by parts in the term associated with J_3 , we obtain

$$\begin{aligned} - \int_{\mathbb{R}^d} (S_j(u) \cdot \nabla \Delta_j \theta) \Delta_j \theta |\Delta_j \theta|^{q-2} dx &= \frac{1}{q} \int_{\mathbb{R}^d} (\nabla \cdot S_j u) |\Delta_j \theta|^q dx \\ &= \int_{\mathbb{R}^d} \tilde{J}_3 |\Delta_j \theta|^{q-1} dx, \end{aligned}$$

where \tilde{J}_3 is given by

$$\tilde{J}_3 = \frac{1}{q} (\nabla \cdot S_j u) |\Delta_j \theta|.$$

Applying Hölder's inequality, we have

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \|\Delta_j \theta\|_{L^q}^q + \kappa \int \Delta_j \theta |\Delta_j \theta|^{q-2} (-\Delta)^\alpha \Delta_j \theta \, dx \\ \leq \left(\|J_1\|_{L^q} + \|J_2\|_{L^q} + \|\tilde{J}_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q} \right) \|\Delta_j \theta\|_{L^q}^q \end{aligned}$$

For $j \geq 0$, we have the lower bound (see [?] and [?])

$$\int \Delta_j \theta |\Delta_j \theta|^{q-2} (-\Delta)^\alpha \Delta_j \theta \geq C 2^{2\alpha j} \|\Delta_j \theta\|_{L^q}^q. \quad (54)$$

For $j = -1$, this lower bound is invalid. Still we have

$$\int \Delta_j \theta |\Delta_j \theta|^{q-2} (-\Delta)^\alpha \Delta_j \theta \geq 0. \quad (55)$$

Attention is paid to the case $j \geq 0$ first. Inserting (54) in (53) leads to

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^q} + \kappa 2^{2\alpha j} \|\Delta_j \theta\|_{L^q} \leq \|J_1\|_{L^q} + \|J_2\|_{L^q} + \|\tilde{J}_3\|_{L^q} + \|J_4\|_{L^q} + \|J_5\|_{L^q}.$$

By a standard commutator estimate,

$$\|J_1\|_{L^q} \leq C \sum_{|j-k| \leq 2} \|\nabla S_{k-1} u\|_{L^\infty} \|\Delta_k \theta\|_{L^q}.$$

By Hölder's and Bernstein's inequalities,

$$\|J_2\|_{L^q} \leq C \|\nabla \tilde{\Delta}_j u\|_{L^\infty} \|\Delta_j \theta\|_{L^q}.$$

Clearly,

$$\|\tilde{J}_3\|_{L^q} \leq C \|\nabla \cdot S_j u\|_{L^\infty} \|\Delta_j \theta\|_{L^q}.$$

For J_4 and J_5 , we have

$$\begin{aligned} \|J_4\|_{L^q} &\leq \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^\infty} \|\nabla S_{k-1} \theta\|_{L^q}, \\ \|J_5\|_{L^q} &\leq \sum_{k\geq j-1} \|\tilde{\Delta}_k u\|_{L^\infty} \|\Delta_k \nabla \theta\|_{L^q} \\ &\leq C \sum_{k\geq j-1} \|\nabla \tilde{\Delta}_k u\|_{L^\infty} \|\Delta_k \theta\|_{L^q}. \end{aligned}$$

These terms can be further bounded as follows. By Theorem 6.1,

$$\begin{aligned}
 \|\nabla S_k u\|_{L^\infty} &\leq \|\theta_0\|_{L^1 \cap L^\infty} + Ck P(2^{k+1}) \|S_{k+1} \theta\|_{L^\infty} \\
 &\leq \|\theta_0\|_{L^1 \cap L^\infty} + Ck P(2^{k+1}) \|\theta_0\|_{L^\infty}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \|J_1\|_{L^q} &\leq C \|\theta_0\|_{L^1 \cap L^\infty} \sum_{|j-k| \leq 2} (1 + Ck P(2^{k+1})) 2^{-sA_k} 2^{sA_k} \|\Delta_k \theta\|_{L^q} \\
 &\leq C 2^{-sA_j} \|\theta_0\|_{L^1 \cap L^\infty} \|\theta\|_{B_{q,\infty}^{s,A}} \sum_{|j-k| \leq 2} (1 + Ck P(2^{k+1})) 2^{s(A_j - A_k)}.
 \end{aligned}$$

Since A_j is a nondecreasing function of j ,

$$2^{s(A_j - A_k)} \leq 2^{s(A_j - A_{j-2})} \quad \text{for } |k - j| \leq 2, \quad (56)$$

where we have adopted the convention that $A_l \equiv 0$ for $l < -1$.

Consequently,

$$\|J_1\|_{L^q} \leq C 2^{-sA_{j-2}} \|\theta_0\|_{L^1 \cap L^\infty} \|\theta\|_{B_{q,\infty}^{s,A}} (1 + (j+2)P(2^{j+2})) .$$

Clearly, $\|J_2\|_{L^q}$ and $\|J_3\|_{L^q}$ admits the same bound as $\|J_1\|_{L^q}$. By Bernstein's inequality and Theorem 6.1,

$$\begin{aligned}\|J_4\|_{L^q} &\leq C \sum_{|j-k|\leq 2} \|\nabla \Delta_k u\|_{L^q} \|S_{k-1} \theta\|_{L^\infty} \\ &\leq C \|\theta\|_{L^\infty} \sum_{|j-k|\leq 2} P(2^{k+1}) \|\Delta_k \theta\|_{L^q}.\end{aligned}$$

By (56), we have

$$\|J_4\|_{L^q} \leq C 2^{-sA_{j-2}} \|\theta_0\|_{L^\infty} \|\theta\|_{B_{q,\infty}^{s,A}} P(2^{j+2}).$$

By Theorem 6.1,

$$\begin{aligned}
 \|J_5\|_{L^q} &\leq C \sum_{k \geq j-1} P(2^{k+1}) \|\tilde{\Delta}_k \theta\|_{L^\infty} \|\Delta_k \theta\|_{L^q} \\
 &\leq C \|\theta_0\|_{L^\infty} \sum_{k \geq j-1} P(2^{k+1}) \|\Delta_k \theta\|_{L^q} \\
 &\leq C \|\theta_0\|_{L^\infty} 2^{-sA_{j-2}} P(2^{j+1}) \|\theta\|_{B_{q,\infty}^{s,A}} \sum_{k \geq j-1} \frac{2^{sA_{j-2}}}{P(2^{j+1})} \frac{P(2^{k+1})}{2^{sA_k}}
 \end{aligned}$$

By (24),

$$\|J_5\|_{L^q} \leq C \|\theta_0\|_{L^\infty} 2^{-sA_{j-2}} P(2^{j+1}) \|\theta\|_{B_{q,\infty}^{s,A}}.$$

Collecting all the estimates, we have, for $j \geq 0$,

$$\begin{aligned} \frac{d}{dt} \|\Delta_j \theta\|_{L^q} + \kappa 2^{2\alpha j} \|\Delta_j \theta\|_{L^q} &\leq C 2^{-sA_{j-2}} \|\theta_0\|_{L^1 \cap L^\infty} \\ &\quad \times \|\theta\|_{B_{q,\infty}^{s,A}} (1 + (j+2)P(2^{j+2})) . \end{aligned}$$

That is,

$$\frac{d}{dt} \left(e^{\kappa 2^{2\alpha j} t} \|\Delta_j \theta\|_{L^q} \right) \leq C e^{\kappa 2^{2\alpha j} t} 2^{-sA_{j-2}} \|\theta_0\|_{L^1 \cap L^\infty} \|\theta\|_{B_{q,\infty}^{s,A}} (1 + (j+2)P$$

Integrating in time and multiplying by $2^{sA_j} \cdot e^{-\kappa 2^{2\alpha j} t}$, we obtain, for

$j \geq 0$,

$$2^{sA_j} \|\Delta_j \theta\|_{L^q} \leq 2^{sA_j} e^{-\kappa 2^{2\alpha j} t} \|\Delta_j \theta_0\|_{L^q} + K_j, \quad (57)$$

where

$$K_j = C \|\theta_0\|_{L^1 \cap L^\infty} (1 + (j+2)P(2^{j+2})) 2^{s(A_j - A_{j-2})} \int_0^t e^{-\kappa 2^{2\alpha j}(t-\tau)} \|\theta(\tau)\|$$

To further the estimates, we fix $t_0 \leq T$ and let $t \leq t_0$. It is easy to see that K_j admits the upper bound

$$\begin{aligned} K_j &\leq C \|\theta_0\|_{L^1 \cap L^\infty} (1 + (j+2)P(2^{j+2})) 2^{s(A_j - A_{j-2})} \\ &\quad \times \frac{1}{\kappa 2^{2\alpha j}} \left(1 - e^{-\kappa 2^{2\alpha j} t}\right) \sup_{0 \leq \tau \leq t_0} \|\theta(\tau)\|_{B_{q,\infty}^{s,A}}. \end{aligned}$$

According to (25), there exists an integer j_0 such that, for $j \geq j_0$,

$$K_j \leq \frac{1}{2} \sup_{0 \leq \tau \leq t_0} \|\theta(\tau)\|_{B_{q,\infty}^{s,A}}. \quad (58)$$

For $0 \leq j \leq j_0$,

$$K_j \leq C \|\theta_0\|_{L^1 \cap L^\infty} (1 + (j_0 + 2)P(2^{j_0+2})) \max_{0 \leq j \leq j_0} 2^{s(A_j - A_{j-2})} \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} d\tau. \quad (59)$$

We now turn to the case when $j = -1$. By combining (47) and (55) and estimating $\|J_1\|_{L^q}$ through $\|J_5\|_{L^q}$ in a similar fashion as for the case $j \geq 0$, we obtain

$$\|\Delta_{-1}\theta(t)\|_{L^q} \leq \|\Delta_{-1}\theta(0)\|_{L^q} + C \|\theta_0\|_{L^1 \cap L^\infty} \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} d\tau. \quad (60)$$

Putting (57) and (60) together, we find, for any $j \geq -1$,

$$2^{sA_j} \|\Delta_j \theta\|_{L^q} \leq \|\theta_0\|_{B_{q,\infty}^{s,A}} + K_j, \quad (61)$$

where K_j obeys the bound (58) for $j \geq j_0$ and the bound in (59) for $-1 \leq j < j_0$. Applying $\sup_{j \geq -1}$ to (61) and using the simple fact that

$$\sup_{j \geq -1} K_j \leq \sup_{j \geq j_0} K_j + \sup_{-1 \leq j < j_0} K_j,$$

we obtain

$$\|\theta(t)\|_{B_{q,\infty}^{s,A}} \leq \|\theta_0\|_{B_{q,\infty}^{s,A}} + \frac{1}{2} \sup_{0 \leq \tau \leq t_0} \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} + C(\theta_0, j_0) \int_0^t \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} d\tau$$

where

$$C(\theta_0, j_0) = C \|\theta_0\|_{L^1 \cap L^\infty} (1 + (j_0 + 2)P(2^{j_0+2})) \max_{0 \leq j \leq j_0} 2^{s(A_j - A_{j-2})}.$$

Now taking supremum over $t \in [0, t_0]$, we obtain

$$\sup_{0 \leq \tau \leq t_0} \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} \leq 2 \|\theta_0\|_{B_{q,\infty}^{s,A}} + C(\theta_0, j_0) \int_0^{t_0} \|\theta(\tau)\|_{B_{q,\infty}^{s,A}} d\tau,$$

Gronwall's inequality then implies (45) for any $t \leq t_0 \leq T$. This finishes the case when $q < \infty$.

We now turn to the case when $q = \infty$. For $j \geq 0$, applying Δ_j yields

$$\partial_t \Delta_j \theta + S_j u \cdot \nabla (\Delta_j \theta) + \kappa (-\Delta)^\alpha \Delta_j \theta = J_1 + J_2 + J_4 + J_5$$

where J_1 , J_2 , J_4 and J_5 are as defined in (48), (49), (51) and (52), respectively. According to Lemma ?? below, we have

Happy Birthday, Peter!