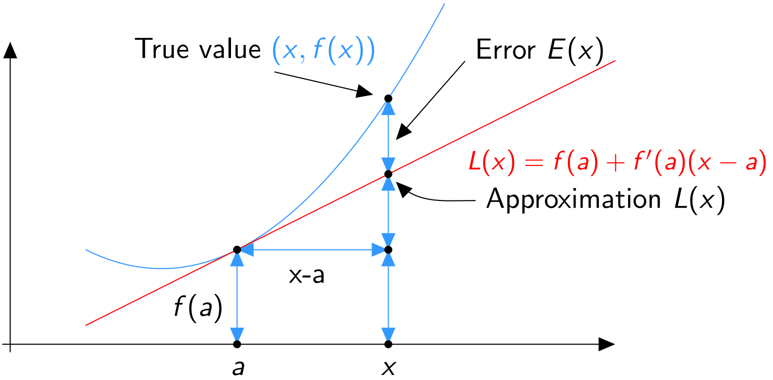
Calculus 1
Linear Approximation and the Derivative
Section 3.9

Visualization of the Tangent Line Approximation



The tangent line approximation and its error.

It can be shown that the tangent line approximation is the best linear approximation of f near a.

The Tangent Line Approximation

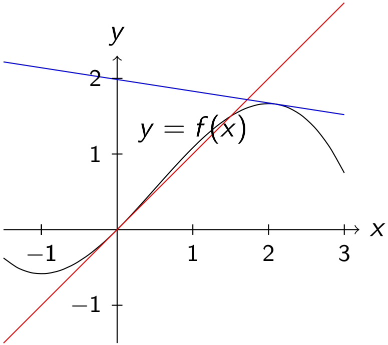
Suppose f is differentiable at a. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a)$$

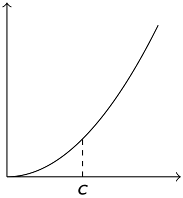
The expression $f(a) + f'(a)(x - a)$ is called the local linearization of f near $x = a$. We are thinking of a as fixed, so that $f(a)$ and $f'(a)$ are constant. The error $E(x)$, in the approximation is defined by

$$E(x) = f(x) - f(a) - f'(a)(x - a)$$

Look at the tangent line to $(0, 0)$. Now look at the tangent to $(2.1, 1.66)$. What conclusions could you make about the accuracy of any local linearizations at these two points?



In which of these graphs will using local linearity to approximate the value of the function near $x = c$ give the least error as Δx becomes larger?

(a)


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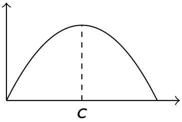
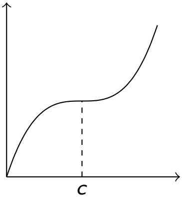
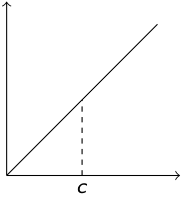


(c)


(d)


Answer

(d). Local linearity will give exact answers in graph (d) because the tangent line at $x = c$ is identical with the graph on the interval shown.

Example

What is the tangent line approximation for $f(x) = \sin(x)$ near $x = 0$?

Solution

The tangent line approximation of f near $x = 0$ is

$$f(x) \approx f(0) + f'(0)(x - 0)$$

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$ and $f'(0) = \cos(0) = 1$, and the approximation is

$$\sin(x) \approx x.$$

This means that, near $x = 0$, the function $f(x) = \sin(x)$ is well approximated by the function $y = x$. If we zoom in on the graphs of the functions $\sin(x)$ and x near the origin, we will not be able to tell them apart.

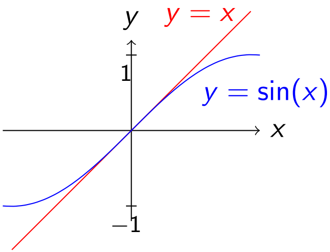


Figure : Tangent line approximation to $\sin(x)$ at $x = 0$.

Get your calculator out and find these sin values to 5 decimal places.

(a) $\sin(0.015)$

(b) $\sin(0.15)$

(c) $\sin(0.051)$

(d) $\sin(0.51)$

(e) $\sin(0.2)$

Get your calculator out and find these sin values to 5 decimal places.

(a) $\sin(0.015) \approx 0.01500$
(b) $\sin(0.15) \approx 0.14944$
(c) $\sin(0.051) \approx 0.04998$
(d) $\sin(0.51) \approx 0.47943$
(e) $\sin(0.2) \approx 0.90930$

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Now let's look at the error. Recall $E(x) = f(x) - L(x)$, but for us, $L(x) = x$, so the error is $\sin(x) - x$.

	$\sin(x)$to 5 decimals	$L(x) = x$	$E(x)$
(a)	0.01500	0.015	0
(b)	0.14944	0.15	-0.00056
(c)	0.04998	0.05	-0.00002
(d)	0.47943	0.5	-0.02057
(e)	0.90930	0.2	-1.0907

Theorem (Differentiability and Local Linearity)

Suppose f is differentiable at $x = a$, and $E(x)$ is the error in the tangent line approximation, that is,
$E(x) = f(x) - L(x) = f(x) - f(a) - f'(a)(x - a)$. Then,

$$\lim_{x \to a} \frac{E(x)}{x - a} = 0$$

Proof.

Using the definition of $E(x)$, we have

$$\frac{E(x)}{x-a} = \frac{f(x) - f(a) - f'(a)(x-a)}{x-a} = \frac{f(x) - f(a)}{x-a} - f'(a)$$

Taking the limit as $x \to a$, and using the definition of the derivative, we see that

$$\lim_{x \to a} \frac{E(x)}{x-a} = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x-a} - f'(a) \right) = f'(a) - f'(a) = 0.$$

An error estimate that will be developed later is

$$E(x) \approx \frac{f''(a)}{2}(x-a)^2.$$

This is part of the *Taylor Remainder Theorem* which allows for estimate is estimate of even more general approximations. It is one of the most important and useful theorems in science, engineering, and mathematics.

Example

Find a formula for the error, $E(x)$, in the tangent line approximation (linear approximation) to the function $f(x) = \sqrt{1+x}$ near $x = 0$. We could estimate the error using the suggested formula,

$$E(x) \approx \frac{f''(0)}{2}(x-2)^2$$

But this problem tells us to find the exact error. So we must find the linear approximation and subtract it from the function

$$L(x) = f(a) + f'(a)(x-a) = \sqrt{1+0} + \frac{1}{2\sqrt{1+0}}(x-0) = 1 + \frac{x}{2}$$

So,

$$E(x) = \sqrt{1+x} - 1 - \frac{x}{2}.$$

So now we find the value of the linear approximation, $L(x)$, and the error value of that approximation, $E(x)$, for the function $f(x) = \sqrt{1+x}$ at $x = 0.038$.

$$L(0.038) = 1 + 0.038/2 = 1.019$$

Also,

$$E(0.038) = f(0.038) - 1.019 = \sqrt{1.038} - 1.019 \approx -0.000177$$

Note: In real world problems, we usually can only estimate the error; we cannot compute it exactly.

Clicker Question

Find a formula for the error $E(x)$ in the tangent line approximation to the function $f(x) = x^2$ near $x = 3$.

(a) $E(x) = 9 - 9 - 6(x - 3)$

(b) $E(x) = 9 - 2x - 6(x - 3)$

(c) $E(x) = x^2 - 2x - 6(x - 3)$

(d) $E(x) = x^2 - 9 - 6(x - 3)$

(e) $E(x) = x^2 - 9 - 2x(x - 3)$