An Inviscid Regularization of the Velocity-Vorticity formulation of the 3D Navier-Stokes Equations

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Fayetteville, AR

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Outline

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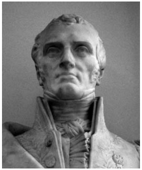
The Incompressible Navier-Stokes/Euler Equations



Claude L.M.H. Navier

George G. Stokes

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{Acceleration} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{Advection} = \underbrace{-\nabla p}_{\substack{Pressure\\ Gradient}} + \underbrace{\nu\triangle\vec{u}}_{\substack{Viscous\\ Diffusion}}$$

Incompressibility

$$\operatorname{div}\vec{u} = 0$$

Unknowns

$\vec{u}$:= Velocity (vector)

p := Pressure (scalar)

Parameter

ν := Kinematic Viscosity

Problem (J. Leray, 1933)

Can a singularity develop in the solutions?

- 2D case: No.
- 3D case: \$1,000,000 Clay Millennium Prize Problem
- 3D, $\nu = 0$ case: \$0 Pat on the back problem

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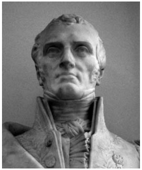

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Blow-up Criteria

Beale-Kato-Majda Criterion (1984)

$$\boldsymbol{\omega} := \nabla \times \boldsymbol{u} = \text{ Vorticity}$$
$$\int_0^T \|\boldsymbol{\omega}(t)\|_{L^\infty}\,dt < \infty \iff \text{Solution is regular on } [0,T].$$

Analytical Blow-up Criteria

- Beale, Kato, Majda, 1984
- Ponce, 1985
- Ferrari, 1993
- Constantin, Fefferman, 1993
- Constantin, Fefferman, Majda, 1996
- L., Titi, 2010
- Gibbon, Titi, 2013
- L., Titi, 2015

Computational Search for Blow-up

- Kerr, 1993, 2013
- Deng, Hou, Yu, 2005
- Hou, Li, 2008
- Hou, 2009
- Bustamante, Brachet, 2012
- Brachet, Bustamante, Krstulovic, Mininni, Pouquet, Rosenburg, 2013
- Lou, Hou, 2014
- L., Peterson, Titi, Wingate, 2017

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The Voigt α-Regularization

$$\begin{cases} -\alpha^2\partial_t\triangle\boldsymbol{u} + \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u} + \mathbf{f}, \\ \nabla\cdot\boldsymbol{u} = 0. \end{cases}$$

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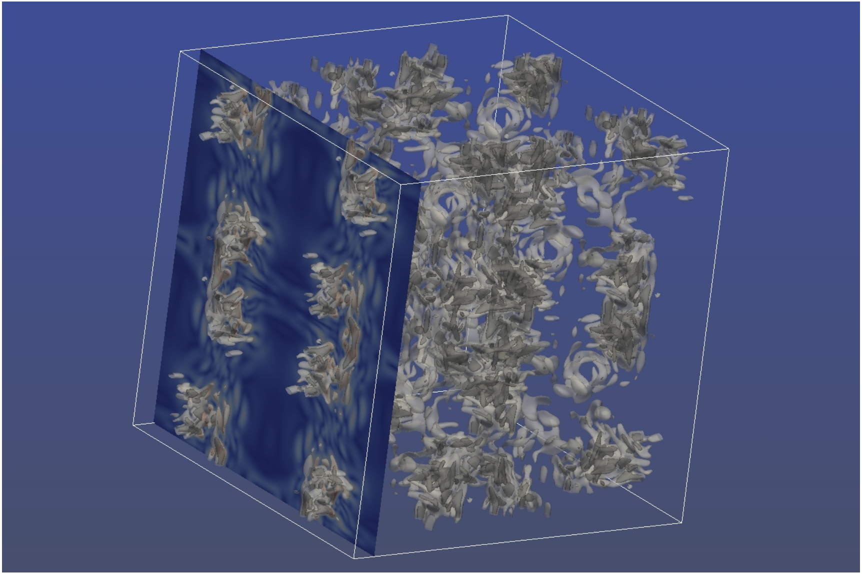
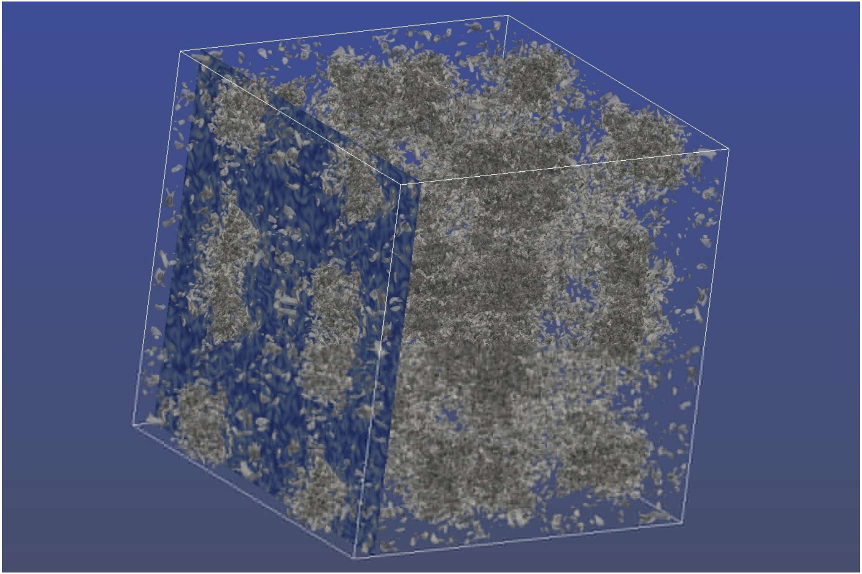
- Solutions do not experience instantaneous smoothing.
- Parabolic character of the equations is destroyed.
  (Recovered in the 'long term' behavior. (Kalantarov, Titi))



Vorticity magnitude, $T = 2$



Vorticity magnitude, $T = 6$

The Euler-Voigt Model

What makes the Voigt-regularization work?

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Modified Energy Equality (Cao, Lunasin, Titi, 2006)

$$\alpha^2\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^2}^2 = \alpha^2\|\nabla\boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{u}_0\|_{L^2}^2$$

Analytical Results: Regularity

$$\begin{cases} -\alpha^2\partial_t\triangle\boldsymbol{u} + \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\triangle\boldsymbol{u} \\ \nabla\cdot\boldsymbol{u} = 0 \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \end{cases} \tag{2.1}$$

Theorem (Global Existence)(Cao, Lunasin, Titi)

Let $\boldsymbol{u}_0 \in H^1$, $\nu \geq 0$. Then NSV has a unique solution in $C^1((-\infty,\infty),H^1)$ under either periodic or (if $\nu > 0$) homogeneous Dirichlet (no-slip) boundary conditions.

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Theorem (H^s Regularity and Analyticity)(L., Titi)

Let $\boldsymbol{u}_0 \in H^s$, $s \geq 0$, $\nu \geq 0$. Then NSV has a unique solution in $C^1((-\infty, \infty), V \cap H^s)$, under periodic boundary conditions. Furthermore, if $\boldsymbol{u}_0 \in V \cap C^\omega$, then $\boldsymbol{u} \in C^1((-\infty, \infty), V \cap C^\omega)$.

Analytical Results: Convergence

- Given initial data $\boldsymbol{u}_0 \in H^s$, $s \geq 3$.
- Let $\boldsymbol{u}$ be a solution to the Euler equations with initial data $\boldsymbol{u}_0$.
- Let $\boldsymbol{u}^\alpha$ be a solution of the Euler-Voigt equations with initial data $\boldsymbol{u}_0$.

Theorem (Convergence)(L., Titi)

Suppose $\boldsymbol{u} \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *for* $s \geq 3$. *Then* $\boldsymbol{u}^\alpha \to \boldsymbol{u}$ *in* $L^\infty([0,T], L^2)$.

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Specifically,

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla(\boldsymbol{u}(t) - \boldsymbol{u}^\alpha(t))\|_{L^2}^2 \leq C\alpha^2.$$

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Specifically,

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla(\boldsymbol{u}(t) - \boldsymbol{u}^\alpha(t))\|_{L^2}^2 \leq C\alpha^2(e^{Ct} - 1).$$

So that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}(t) - \boldsymbol{u}^\alpha(t)\|_{L^2} \sim \mathcal{O}(\alpha)$$

Blow-up Criterion 1

$$\|\boldsymbol{u}^\alpha\|_{L^2}^2 + \alpha^2\|\nabla\boldsymbol{u}^\alpha\|_{L^2}^2 = \|\boldsymbol{u}_0\|_{L^2}^2 + \alpha^2\|\nabla\boldsymbol{u}_0\|_{L^2}^2$$

$$\|\boldsymbol{u}\|_{L^2}^2 + \limsup_{\alpha\to 0^+} \alpha^2\|\nabla\boldsymbol{u}^\alpha\|_{L^2}^2 = \|\boldsymbol{u}_0\|_{L^2}^2$$

Theorem (Blow-up Criterion 1)(A.L., E.S. Titi, 2010, DCDS)

Suppose there exists a finite time $T_* > 0$ such that

$$\sup_{t\in[0,T_*)} \limsup_{\alpha\to 0^+} \alpha^2\|\nabla\boldsymbol{u}^\alpha(t)\|_{L^2}^2 > 0.$$

Then the 3D Euler equations develop a singularity on the interval $[0, T_*]$.

Note (vorticity!):

$$\|\nabla\boldsymbol{u}^\alpha\|_{L^2}^2 = \|\boldsymbol{\omega}^\alpha\|_{L^2}^2$$

- Similar Blow-up criteria exist for inviscid SQG (Khouider, Titi), inviscid Boussinesq (L., Lunasin, Titi).

Blow-up Criterion 2

Theorem (Blow-up Criterion 2)(L., Peterson, Titi, Wingate, 2016 Theor. Comp. Fluid Dyn.)

Suppose there exists a finite time $T_* > 0$ such that

$$\limsup_{\alpha\to 0^+}\left(\alpha \sup_{t\in[0,T^*]} \|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2}\right) > 0. \tag{3.1}$$

Then the 3D Euler equations develop a singularity on the interval $[0, T_*]$.

Moreover,

$$\limsup_{\alpha\to 0^+} \sup_{t\in[0,T]} \alpha^2\|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2}^2 \geq \sup_{t\in[0,T]} \limsup_{\alpha\to 0^+} \alpha^2\|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2}^2. \tag{3.2}$$

So the new criterion is stronger.

Remarks on Blow-up Criteria

Blow-up via limiting regularization

- Beale-Kato-Majda-type criteria track $\boldsymbol{\omega}(t)$, a quantity which comes from an equation that is **not known to be globally well-posed**! (Namely, 3D Euler.)
- Here, we track a quantity $\nabla \boldsymbol{u}^\alpha(t)$, coming from 3D Euler-Voigt equations, **which *are* globally well-posed.**

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- Here, we track a quantity $\nabla \boldsymbol{u}^\alpha(t)$, coming from 3D Euler-Voigt equations, **which *are* globally well-posed.**

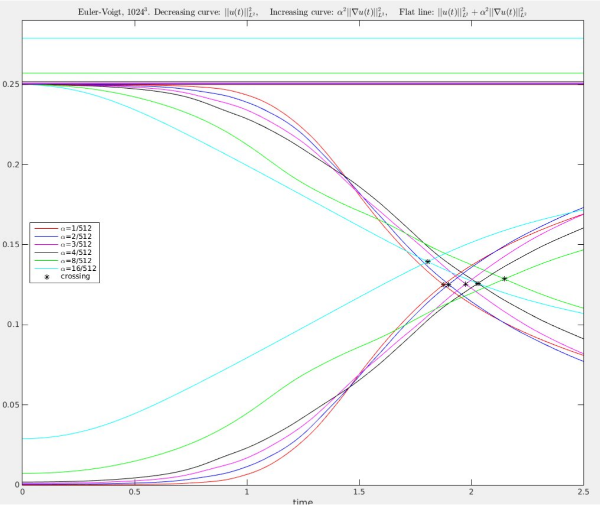
Comparison with Navier-Stokes ($\nu \to 0$)

No hope for a corresponding "well-posed-to-not-well-posed" blow-up criterion by viewing 3D Euler an invisicid limit of 3D Navier-Stokes, since 3D Navier-Stokes is not known to be well-posed.

Outline

Energy Balance at resolution 1024^3

$$\alpha^2\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\boldsymbol{u}\|_{L^2}^2 = \alpha^2\|\nabla \boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{u}_0\|_{L^2}^2$$



Computational Approach

Idea: Investigate the behavior of $f_t(\alpha) := \|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2}$

Implication of Blow-up Criterion 1

If $\alpha\|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2} \sim C\alpha^p$ as $\alpha \to 0$, then $p \leq 0$ implies blow-up.

Implication of Blow-up Criterion 2

If $\max_{t\in[0,T]} \|\nabla \boldsymbol{u}^\alpha(t)\|_{L^2} \sim C\alpha^p$ as $\alpha \to 0$. Then $p \leq -1$ implies blow-up.

Test: Benjamin-Bona-Mahony Equation

$$-\alpha^2 u_{txx} + u_t + u u_x = \nu u_{xx}, \qquad x \in \mathbb{T} = [-\pi, \pi], \quad \nu \geq 0$$
$$u(x,0) = \sin(x).$$

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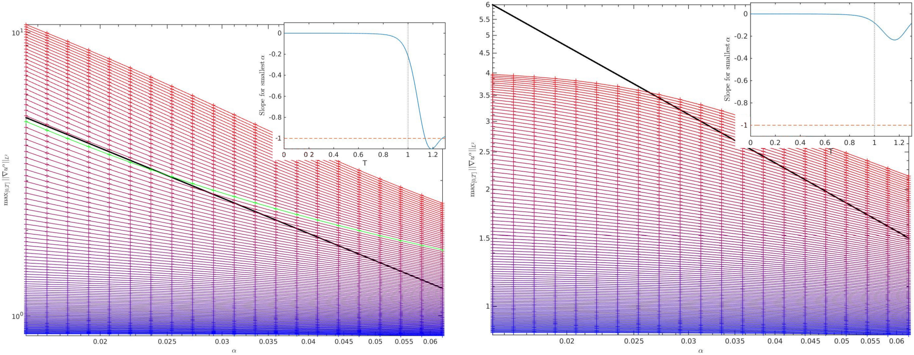


(a) α vs. $\|u_x^\alpha(t)\|_{L^2}$ for the BBM equations. $\nu = 0$ case.

(b) α vs. $\|u_x^\alpha(t)\|_{L^2}$ for the BBM equations. $\nu > 0$ case.

Figure: Simulations of the 1D BBM equations detecting the known singularity in the 1D Burgers equation at $T = 1$. Resolution: $8192 = 2^{13}$.

3D Euler Equations

Numerical Methods

Pseudo-spectral (i.e., Fourier) methods in space, RK-4 in time, $2/3$ dealiasing. Resolution 1024^3 spatial grid points. Adaptive Δt respecting advective CFL.

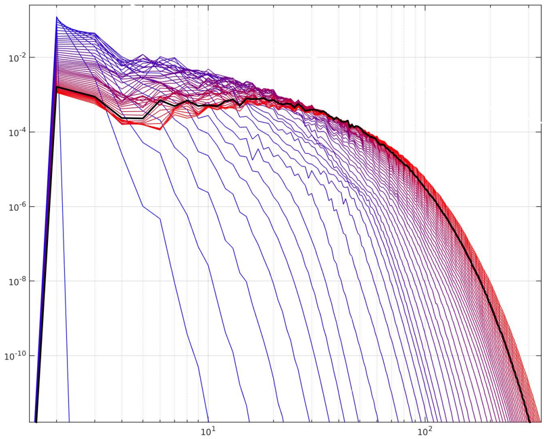
Initial data: Taylor Green Vortex

$$\begin{cases} u_1 = \quad \sin(2\pi x)\cos(2\pi y)\cos(2\pi z), \\ u_2 = -\cos(2\pi x)\sin(2\pi y)\cos(2\pi z), \\ u_3 = 0. \end{cases}$$

First use of Taylor-Green initial data

Brachet, Meiron, Orszag, Nickel, Morf, Frisch (1983, JFM; 1984, JSP)
Blow-up time ≈ 4.2.

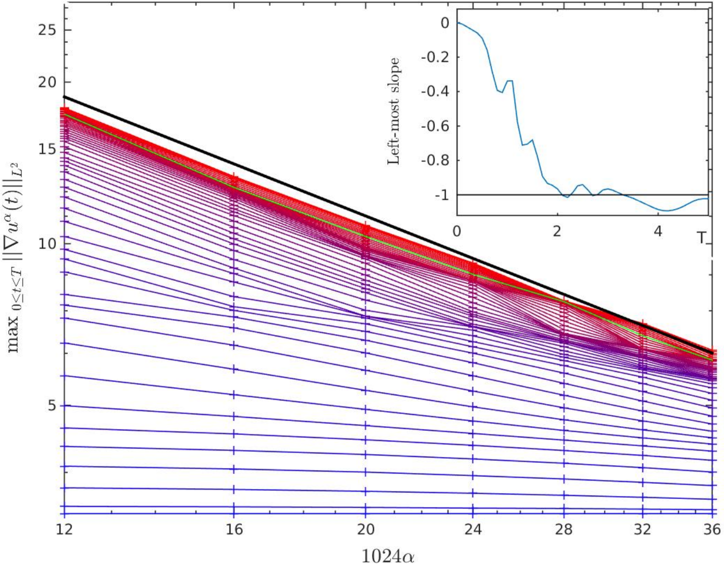
Spectrum $3D$ Euler-Voigt



Energy spectrum vs. wave number at times $t = 0.0, 0.1, \ldots, 4.9, 5.0$.
Black curve: $t = \mathbf{4.2}$, where critical slope is observed. $(\alpha = 12/1024.)$
Resolution: 1024^3 ($>$ 1 billion grid points. Processor time ≈ 10 years.)

Blow-up Test for 3D Euler

Result: Near $\alpha \approx 16/1024$, $t \approx 4.2$ we observe $\|\nabla \boldsymbol{u}^\alpha(t)\| \sim C\alpha^{-1.1}$.



Outline

Velocity-Vorticity Formulation

Navier-Stokes

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\triangle\boldsymbol{u} + \nabla\times\mathbf{f} \\ \nabla\cdot\boldsymbol{u} = 0 \end{cases}$$

Vorticity: $\boldsymbol{\omega} := \nabla\times\boldsymbol{u}$. (Biot-Savart: $\boldsymbol{u} = -\triangle^{-1}\nabla\times\boldsymbol{\omega}$).

$$\partial_t\boldsymbol{\omega} + (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} = (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} + \nu\triangle\boldsymbol{\omega}$$

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Vector identity: $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \boldsymbol{\omega}\times\boldsymbol{u} + \frac{1}{2}\nabla|\boldsymbol{u}|^2$.

Velocity-Vorticity Formulation

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Vorticity: $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$. (Biot-Savart: $\boldsymbol{u} = -\triangle^{-1} \nabla \times \boldsymbol{\omega}$).

$$\partial_t \boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + \nu \triangle \boldsymbol{\omega}$$

Vector identity: $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{u} + \frac{1}{2}\nabla |\boldsymbol{u}|^2$.

Velocity-Vorticity Formulation

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla \left(p + \frac{1}{2}|\boldsymbol{u}|^2\right) = \nu \triangle \boldsymbol{u} + \mathbf{f}, \\ \partial_t \boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = \nu \triangle \boldsymbol{\omega} + \nabla \times \mathbf{f}, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{\omega} = 0. \end{cases}$$

Idea: View $\boldsymbol{\omega}$ as decoupled from $\boldsymbol{u}$ (No Biot-Savart Law).

Velocity-Vorticity (VV) Formulation

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u} + \nabla \pi & = \nu \triangle \boldsymbol{u} + \qquad \mathbf{f}, \\ \partial_t \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{w} & = \nu \triangle \boldsymbol{w} + \nabla \times \mathbf{f}, \\ \nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{w} & = 0, \\ \boldsymbol{u}(0) & = \boldsymbol{u}_0, \\ \boldsymbol{w}(0) & = \boldsymbol{w}_0 = \nabla \times \boldsymbol{u}_0, \end{cases}$$

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Incomplete History (mostly numerical work):

- Guevremont, Habashi, Hafez, 1990
- Gatski, 1991
- Wu, Wu, Wu, 1995
- Meitz, Fasel, 2000
- Wong, Baker, 2002
- Lo, Young, Murugesan, 2006
- Heister, Olshanskii, Rebholz, 2017
- Olshanskii, Rebholz, Salgodo 2018

Velocity-Vorticity-Voigt

Idea

- Global well-posedness in 3D for VV is still (obviously) open.

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Velocity-Vorticity-Voigt System (L., Pei, Rebholz, 2018, JDE)

$$\begin{cases} (I-\alpha^2\triangle)\partial_t \boldsymbol{u} + \boldsymbol{w}\times\boldsymbol{u} + \nabla\pi &= \nu\triangle\boldsymbol{u} + \mathbf{f}, \\ \partial_t\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} &= \nu\triangle\boldsymbol{w} + \nabla\times\mathbf{f}, \\ \nabla\cdot\boldsymbol{u} = \nabla\cdot\boldsymbol{w} &= 0. \end{cases}$$

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Boundary conditions (Olshanskii, Rebholz, Salgodo, Galvin, 2015, CMAME)

$$\begin{cases} \boldsymbol{u}\big|_{\partial\Omega} = \mathbf{0}, \\ \boldsymbol{w}\cdot\mathbf{n}\big|_{\partial\Omega} = \mathbf{0}, \\ (\nu(\nabla\times\boldsymbol{w})\times\mathbf{n} - (\mathbf{f}-\nabla\pi)\times\mathbf{n})\big|_{\partial\Omega} = \mathbf{0}. \end{cases}$$

VVV: Weak and Strong Solutions

Definition of weak solution

Let $T > 0$ be arbitrary. Suppose $\boldsymbol{u}_0 \in V$, $\boldsymbol{w}_0 \in H$, and $f \in L^2(0,T;H)$. We call the pair $(\boldsymbol{u},\boldsymbol{w})$ a *weak solution* on the time interval $[0,T]$ to the VVV system, if $\boldsymbol{u} \in C(0,T;V)$, $\boldsymbol{u}_t \in L^2(0,T;V)$, $\boldsymbol{w} \in C_w(0,T;H) \cap L^2(0,T;V)$, $\boldsymbol{w}_t \in L^2(0,T;H^{-1})$, and moreover, $(\boldsymbol{u},\boldsymbol{w})$ satisfies

$$\begin{cases} \alpha^2((\boldsymbol{u}_t,\psi)) + (\boldsymbol{u}_t,\psi) + ((\boldsymbol{u},\psi)) + \langle \boldsymbol{w}\times\boldsymbol{u},\psi\rangle = (\mathbf{f},\psi)\,, \\ \langle \boldsymbol{w}_t,\psi\rangle + ((\boldsymbol{w},\psi)) - \langle B(\boldsymbol{u},\psi),\boldsymbol{w}\rangle - \left\langle \widetilde{B}(\boldsymbol{w},\boldsymbol{u}),\psi\right\rangle = -(\mathbf{f},\nabla\times\psi)\,, \end{cases}$$

holds for any $\psi \in L^2(0,T;V)$.

Definition of strong solution

Let $T > 0$ be an arbitrarily given time. Suppose $\boldsymbol{u}_0 \in V$, $\boldsymbol{w}_0 \in V$, and $\mathbf{f} \in L^2(0,T;H_{\text{curl}})$. We call the pair $(\boldsymbol{u},\boldsymbol{w})$ a *strong solution* on the time interval $[0,T]$ to the VVV system, if it is a weak solution and satisfies additionally $w \in C([0,T];V) \cap L^2(0,T;D(A))$, and $w_t \in L^2(0,T;H)$.

VVV: GWP Theorems

(L., Pei, Rebholz, 2018, JDE)

Suppose $\boldsymbol{u}_0 \in V$, $\boldsymbol{w}_0 \in H$, and $\mathbf{f} \in L^2(0,T;H)$. Then, the VVV sytem possesses a unique global weak solution (u,w) that satisfies $\nabla \cdot w = 0$. Moreover, the following energy equality holds for a.e. $t \in [0,T]$.

$$\begin{aligned}&\alpha^2\|\nabla \boldsymbol{u}(t)\|_{L^2}^2 + \|\boldsymbol{u}(t)\|_{L^2}^2 + 2\int_0^t \|\nabla \boldsymbol{u}(s)\|_{L^2}^2\,ds\\ =&\alpha^2\|\nabla \boldsymbol{u}_0\|_{L^2}^2 + \|\boldsymbol{u}_0\|_{L^2}^2 + 2\int_0^t (\boldsymbol{u}(s),\mathbf{f}(s))\,ds\end{aligned}$$

(L., Pei, Rebholz, 2018, JDE)

For the initial data $\boldsymbol{u}_0 \in V$, $\boldsymbol{w}_0 \in V$, and $\mathbf{f} \in L^2(0,T;H_{\text{curl}})$, there exists a unique strong solution $(\boldsymbol{u},\boldsymbol{w})$. Moreover, if we further assume that the initial data $\boldsymbol{u}_0 \in H^s \cap V$, $\boldsymbol{w}_0 \in H^s \cap V$, and $\mathbf{f} \in L^2(0,T;H^{s-1}_{\text{curl}})$ for $s \geq 2$, $s \in \mathbb{N}$, then, the solution $\boldsymbol{u} \in C_w(0,T;H^s \cap V)$ and $\boldsymbol{w} \in C_w(0,T;H^s \cap V) \cap L^2(0,T;H^{s+1} \cap V)$.

VVV: Convergence of vorticity

Convergence of vorticity (L., Pei, Rebholz, 2018, JDE)

Denote by $\boldsymbol{\omega} := \nabla \times \boldsymbol{u}$ the vorticity of the flow and let $\boldsymbol{u}_0 \in H^4 \cap V$, $\mathbf{f} \in H^2$. Then, we have

$$\begin{aligned}&\|\boldsymbol{\omega}(t) - w(t)\|_{L^2}^2 + \alpha^2\|\nabla\boldsymbol{\omega}(t) - \nabla\boldsymbol{w}(t)\|_{L^2}^2 + \int_0^t \|\nabla\boldsymbol{\omega}(s) - \nabla\boldsymbol{w}(s)\|_{L^2}^2\, ds \\ \leq& C_0 e^{Ct} + \frac{\tilde{K}\alpha^2}{C}(e^{Ct} - 1),\end{aligned}$$

where C_0 depends on the initial data and $\tilde{K}$ is explained in the proof. If we further assume $\boldsymbol{w}_0 = \nabla \times \boldsymbol{u}_0$, then,

$$\begin{aligned}&\|\boldsymbol{\omega}(t) - \boldsymbol{w}(t)\|_{L^2}^2 + \alpha^2\|\nabla\boldsymbol{w}(t) - \nabla\boldsymbol{\omega}(t)\|_{L^2}^2 + \int_0^t \|\nabla\boldsymbol{\omega}(s) - \nabla\boldsymbol{w}(s)\|_{L^2}^2\, ds \\ \leq& K\alpha^2(e^{Ct} - 1),\end{aligned}$$

for a.e. $t > 0$, i.e., $\|\boldsymbol{w} - \boldsymbol{\omega}\|_{L^\infty(0,T;L^2)} \sim \mathcal{O}(\alpha)$ and $\|\boldsymbol{w} - \boldsymbol{\omega}\|_{L^2(0,T;V)} \sim \mathcal{O}(\alpha)$. In particular, $\|\boldsymbol{w} - \boldsymbol{\omega}\|_{L^\infty(0,T;L^2)} \to 0$ and $\|\boldsymbol{w} - \boldsymbol{\omega}\|_{L^2(0,T;V)} \to 0$ as $\alpha \to 0$.

VVV: Convergence of velocity

Convergence of velocity (L., Pei, Rebholz, 2018, JDE)

Denote by $\widetilde{\boldsymbol{\omega}} := \nabla \times \widetilde{\boldsymbol{u}}$ the vorticity of $\widetilde{\boldsymbol{u}}$ in NS, and let $\boldsymbol{u}_0$, $\mathbf{f}$, and $T > 0$ be the same as in prev. Theorem, and set $\boldsymbol{w}_0 = \nabla \times \boldsymbol{u}_0$ and $\widetilde{\boldsymbol{u}}_0 = \boldsymbol{u}_0$. Then, for any $\alpha \in (0, 1]$,

$$
\begin{aligned}
&\|\boldsymbol{\omega}(t) - \widetilde{\boldsymbol{\omega}}(t)\|_{L^2}^2 + \|\boldsymbol{u}(t) - \widetilde{\boldsymbol{u}}(t)\|_{L^2}^2 + \alpha^2\|\nabla\boldsymbol{u}(t) - \nabla\widetilde{\boldsymbol{u}}(t)\|_{L^2}^2 \\
&\quad + \int_0^t \|\nabla\boldsymbol{u}(s) - \nabla\widetilde{\boldsymbol{u}}(s)\|_{L^2}^2\,ds \\
\leq & C\alpha^2
\end{aligned}
$$

for a.e. $t > 0$ in the interval of existence of the solution to NSE, say, up to $T > 0$ and the constant C depends on $\|\widetilde{\boldsymbol{u}}\|_{H^3}$, $\|\boldsymbol{u}\|_{H^3}$, as well as $\|\mathbf{f}\|_{H^2_{\text{curl}}}$.
In particular, we have $\|\boldsymbol{\omega} - \widetilde{\boldsymbol{\omega}}\|_{L^\infty(0,T;H)} \to 0$, $\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}\|_{L^\infty(0,T;H)} \to 0$, and $\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}\|_{L^2(0,T;V)} \to 0$ as $\alpha \to 0$.

VVV: Comparison with MHD

Remarks

- Analysis of 3D VVV is distinct from analysis of “3D Voigt-MHD” with Voigt-regularization on momentum equation (Larios, Titi, 2011).

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- Key idea is to notice that one may first obtain an energy estimate purely in terms of u, and then use this bound to obtain a bound on w.
- Higher-order estimates on u are not w-independent, but can be obtained using a bootstrapping technique, going back and forth between the two equations.

VVV: Invsicid Case?

Remarks

- Do analogous results hold in the inviscid (Euler-Voigt) case?
- Vorticity stretching term $(w \cdot \nabla)u$ can no longer be controlled in the same way, as higher-order derivatives cannot be absorbed into the viscosity, so higher-order estimates are needed, but as with other α-models, this is not currently understood.
- In the proof of convergence as $\alpha \to 0$, the estimates depend crucially on the fact that $\int_0^T \|\nabla u(t)\|_{L^2}^2 \, dt$ is bounded *independently* of $\alpha \in (0, 1]$, which is a property that one does not have in the Euler-Voigt equations.

Thank you!