Calculus 1
The Fundamental Theorem and Interpretations

Motivation

Recall that we used Leibniz's derivative notation to understand the units of the derivative. But formally dy/dx meant the operator, d/dx operating on y. Integration can be thought of in that same way. Formally the the integral sign and the dx are one entity acting on f(x) (or y). That is the integral sign and the dx are the operator operating on f(x).
So to figure out the units of the integral we treat the integral symbol as telling us to sum the rectangles that are of height f(x) and width of dx. Thus if f(t) represents velocity and dt represents time, the units would be velocity times time or distance.

The Fundamental Theorem of Calculus

Notice that

$$\begin{aligned} F(b) - F(a) &= [\text{ Total change in } F(t) \text{ between } t = a \text{ and } t = b\,] \\ &= \int_a^b F'(t)\,dt \end{aligned}$$

In words, the definite integral of a rate of change gives the total change.

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Theorem (The Fundamental Theorem of Calculus)(FTC)

If f is continuous on the interval $[a, b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Since the terms being added up are products of the form "f(x) times a difference in x," the units of $\int_a^b f(t)\,dt$ is the product of the units of f(x) and the units of x.

Suppose $f(t) = F'(t)$. Which expression best represents the rate of change of F at $t = 3$?

(a) $f'(3)$
(b) $f(3) - f(0)$
(c) $F(3) - F(0)$
(d) $\int_0^3 f(t)\,dt$
(e) $f(3)$

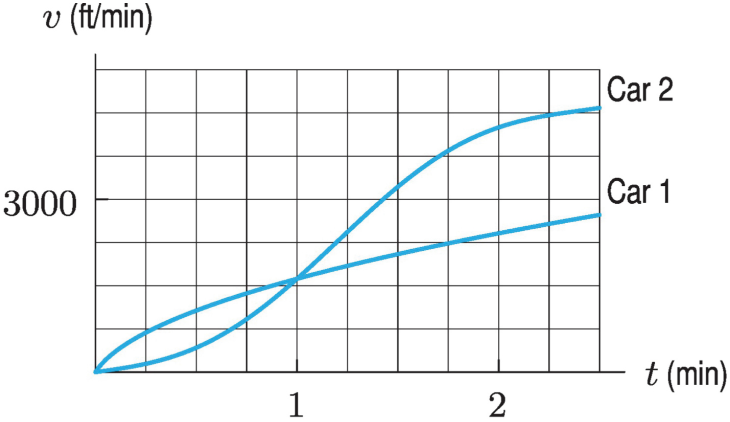
Answer

Expression (e).

Example: Distance Traveled

Two cars **start from rest** at a traffic light and then accelerate for several minutes. The figure below shows their velocities as a function of time.

(a) Which car is ahead after **one** minute?
(b) Which car is ahead after **two** minutes?



Example: Biology

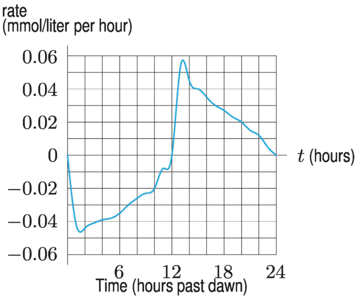
Biological activity in water is reflected in the rate at which carbon dioxide, CO_2, is added or removed. Plants take CO_2 out of the water during the day for photosynthesis and put CO_2 back into the water at night. Animals put CO_2 into the water all the time as they breathe. The figure below shows the rate of change of the CO_2 level in the pond. At dawn, there were 2.6000 mmol of CO_2 per liter of water.

(a) At what time was the CO_2 lowest? Highest?

(b) Estimate how much the CO_2 enters the pond from $t = 12$ to $t = 24$.

(c) Estimate how much the CO_2 enters the pond from $t = 0$ to $t = 12$.

(d) Does the CO_2 level appear to be approximately in



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Example: Biology (cont.)

Denote the total amount of CO_2 in the pond at time t by $F(t)$. The previous graph was then a graph of $F'(t)$.

(a) Since we are taking CO_2 out until $t = 12$, the level will be lowest at $t = 12$. After $t = 12$ we are putting back CO_2 and so the highest level will be at either dawn ($t = 0$) or $t = 24$.

(b) We want the total CO_2 in the pond, $F(24) - F(12)$. The FTC gives us $F(24) - F(12) = \int_{12}^{24} f(t)\,dt$, and we can estimate the integral by finding the area under the curve using the left- or right-hand rule.

(c) We want the total CO_2 in the pond, $F(12) - F(0)$. The FTC gives us $F(12) - F(0) = \int_0^{12} f(t)\,dt$, and again we can estimate the integral by finding the area under the curve using the left- or right-hand rule.

(d) The total amount in the beginning minus the total amount in the end is $F(24) - F(0)$. The FTC gives us $F(24) - F(0) = \int_0^{24} f(t)\,dt$, Thus, if we find the total (signed) area under the curve to be zero, this means the CO_2 level is in equilibrium.

The Definite Integral of a Rate of Change

Let $F(t)$ represent a bacteria population which is 5 million at time $t = 0$. After t hours, the population is growing at an instantaneous rate of 2^t million bacteria per hour. Estimate the **total increase** in the bacteria population during the first hour, and the population at $t = 1$.

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Solution

Since the rate at which the population is growing is $F'(t) = 2^t$, we have

$$\text{Change in population} =$$

We can approximate the area under the curve using a calculator and the left- or right-hand rule. It turns out $\int_0^1 2^t\,dt \approx 1.44$, so the total increase is about 1.44 million bacteria.

We can also find the total population after one hour, since $F(0) = 5$:

$$F(1) = F(0) + \int_0^1 2^t\,dt \approx 5 + 1.44 = 6.44 \text{ (million)}$$

Calculating Definite Integrals

Example.
Compute $\int_1^3 2x\,dx$ by two different methods.
Solution.
Using left- and right-hand sums, we can approximate this integral as accurately as we want. With $n = 100$, for example, the left-sum is 7.96 and the right sum is 8.04. Using $n = 500$ we learn

$$7.992 \leq \int_1^3 2s\,dx \leq 8.008.$$

The Fundamental Theorem, on the other hand, allows us to compute the integral exactly. We take $f(x) = 2x$. We know that if $F(x) = x^2$, then $F'(x) = 2x$. So we use $f(x) = 2x$ and $F(x) = x^2$ to obtain

$$\int_1^3 2x\,dx = F(3) - F(1) = 3^2 - 1^2 = 8.$$

Notice that to use the Fundamental Theorem to calculate a definite integral, we need to know the *anti-derivative* of F', namely F. Later we will discuss how anti-derivatives are computed.

A Third Method: Geometry

Since definite integrals are just areas under curves, if the curve is something simple like a line or a circle, we can sometimes calculate the integral using geometry. This is a shortcut that has limited use, but it can sometimes make certain problems easier.

$$\int_1^3 2x\,dx$$
$$=(\text{Area of big triangle})$$
$$-(\text{Area of big triangle})$$
$$=\tfrac{1}{2}(3)(6)-\tfrac{1}{2}(1)(2)=9-1=8.$$