Math 107-Lecture 7

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Announcements

- Gateway Exam: if you passed: Congratulations! If not, review the questions that you did not answer correctly and go today or tomorrow and take it again (you need to sign up to take it at the Love Library testing center).

Plan for today

1. Review numerical integration.
2. Improper integrals: motivation, definition, comparison tests.

Numerical Integration $\approx$ Approximation

Since ...

- Many functions do not have antiderivatives that can be written in terms of elementary functions; i.e. we can not get rid of the integration sign to write the function. Examples:

$$\int e^{x^2} dx, \quad \int e^{1/x} dx, \quad \int \sin(x^2) dx, \quad \int \ln(x)\cos(x) dx \ldots$$

- Many functions require **a lot of work** to find the exact answer, but we often just need a rough estimate.

... we approximate using the trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

Improper integrals. Going all the way to infinity.

- Since definite integrals can be computed on any intervals, would it make sense to have them on infinite intervals? In many situations we need to find cumulative behavior for very long intervals, so it is important to let the integration interval be infinite. But wouldn't that give us infinity always? We will see.
- We may also need to compute integrals on finite intervals but for functions that become infinite?

These two concepts will be analyzed through Type I and Type II integrals.

How do integrals come into play?

Assume that money is given continuously, so at time t the girl receives $\frac{1}{t^2}$ money. Then from time 1 to time N the girl would collect a total of

$$\int_1^N \frac{1}{t^2}\,dt.$$

Goal for improper integrals: Find out what is the total/cumulative impact of infinitely many small actions.

Clicker question #1

What is $M(N) = \int_1^N \frac{1}{x^2}\,dx$?

- $\frac{1}{N} - 1$
- $1 - \frac{1}{N}$
- $\ln(N^2)$
- $1 - \frac{1}{N} + C$
- we can't compute the integral.

Type I Integrals

We will let $N \to \infty$ (i.e. we will take a limit like we learned in Calculus I). We define

$$\int_a^\infty f(x)dx = \lim_{N\to\infty} \int_a^N f(x)dx$$

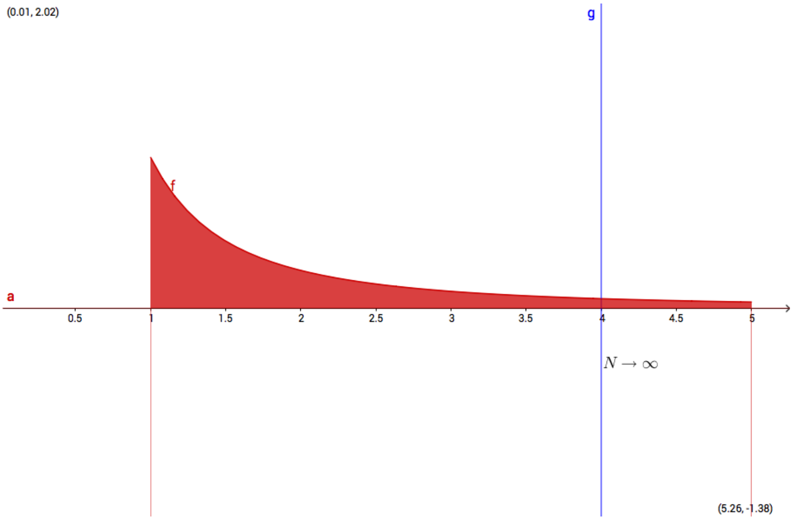
as a Type I improper integral. We say that

- the integral converges (C) if the limit exists and it is finite;
- the integral diverges (D) if the limit does not exist/approaches infinity.

Example:

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{N\to\infty}\left[\int_1^N \frac{1}{x^2}dx\right] = \lim_{N\to\infty}\left[1 - \frac{1}{N}\right] = 1 < \infty (C).$$

Stretching horizontally to infinity. Type I integrals



Type II Integrals

Consider now integrals on a finite interval, but when the integrand can become infinite. For a function f such that $\lim_{x\to a} f(x) = \pm\infty$ we define

$$\int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx$$

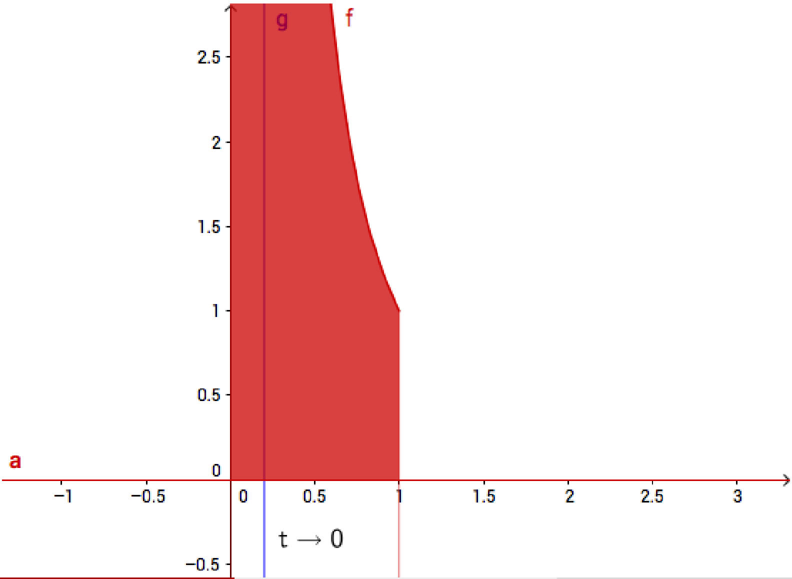
as a Type II improper integral. We say that

- the integral converges (C) if the limit exists and it is finite;
- the integral diverges (D) if the limit does not exist/approaches infinity.

Example:

$$\int_0^1 \frac{1}{x^2}dx = \lim_{t\to 0}\left[\int_t^1 \frac{1}{x^2}dx\right] = \lim_{t\to 0+}\left[-1+\frac{1}{t}\right] = \infty (D).$$

Stretching vertically to infinity. Type II integrals



More examples

Compute

- $\displaystyle\int_{-\infty}^{1} e^{3x}\,dx$ (Type I integral)
- $\displaystyle\int_{0}^{1} x^{-3/2}\,dx$ (Type II integral)

Mixed integrals or two "infinities"

Consider

$$\int_0^\infty x^{-3/2} dx \quad \text{Type I integral and Type II integral}$$

For integrals that have two issues (two "infinities") you must split the integration interval and analyze each "infinity" separately.

$$\int_0^\infty x^{-3/2} dx = \int_0^1 x^{-3/2} dx + \int_1^\infty x^{-3/2} dx$$

Both integrals must converge (i.e. be finite) for the integral to converge!

Clicker question #2

Is $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ convergent or divergent?

- convergent
- divergent (the limit is $-\infty$)
- divergent (the limit is $+\infty$)
- don't know how to do it
- we can't compute the integral.

Wrapping up:

- Today we covered Improper Integrals (7.6).
- For next time finish working all suggested problems from section 7.6.
- For next lecture read Section 7.7 – Comparison of Improper integrals.