Math 107-Lecture 24

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Convergence/divergence tests

- **DCT:** $0 \leq a_n \leq b_n$; $\sum_{n=1}^{\infty} b_n$ (C)$\Rightarrow \sum_{n=1}^{\infty} a_n$ (C); $\sum_{n=1}^{\infty} a_n$ (D)$\Rightarrow \sum_{n=1}^{\infty} b_n$ (D)
- **LCT:** $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty)$ then $\sum_{n=1}^{\infty} a_n(C/D) \iff \sum_{n=1}^{\infty} b_n(C/D)$.
- **AST:** If $c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0$, $\lim_{n\to\infty} c_n = 0$ then $\sum_{n=0}^{\infty}(-1)^n c_n$ (C)
- **Ratio/Root** Let $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ (Ratio) $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (Root).
  $\rho < 1$ (AC); $\rho > 1$ (D); $\rho = 1$ *inconclusive*.

Clicker question #1

For the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n(2x+3)^{2n}}{n^2}$$

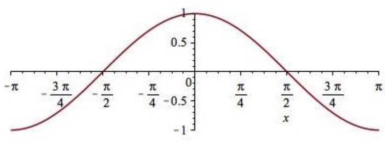
determine the interval of convergence:

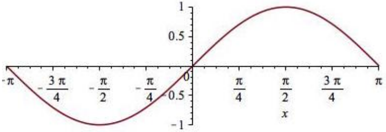
- $(-\frac{3}{2}, \frac{3}{2})$
- $[-2, -1)$
- $[-\frac{3}{2}, \frac{3}{2}]$
- $(-2, -1]$
- $[-2, -1]$

Linearizations

Write down the linearizations of $\cos(x)$ and $\sin(x)$ near $a = 0$.

$$f(x) \approx L(x) := f(a) + f'(a)(x-a), \qquad \text{for } x \text{ near } a$$





Linearization for $\cos x$

x	$L(x) = 1$	$1 - \frac{x^2}{2}$	$\cos x$
0.1	1	0.995	0.99500416...
0.4	1	0.92	0.92106099...
1	1	0.5	0.54030230...

Linearization for $\sin x$

x	$L(x) = x$	$x - \frac{x^3}{6}$	$\sin x$
0.1	0.1	$0.09983333\ldots$	$0.099833416\ldots$
0.4	0.4	$0.38941834\ldots$	$0.38933333\ldots$
1	1	$0.83333333\ldots$	$0.84147098\ldots$

Remark: Polynomials $x - x^3/6$ and $1 - x^2/2$ can be used as reasonably good approximations of $\sin(x)$ and $\cos(x)$ near the origin. These approximations are better than the corresponding linear approximations.

How do we find these higher order polynomial approximations?

We look for

$$f(x) \approx P_n(x) := a_0 + a_1x + a_2x^2 + \dots a_nx^n, \quad \text{near} x = 0.$$

Then we would like the function and its n derivatives to coincide at $x = 0$.

$$f(0) = P_n(0) = a_0$$

For the derivatives we have

$$f'(0) = P_n'(0) = a_1; \quad f''(0) = P_n''(0) = 2a_2; \quad f'''(0) = P_n'''(0) = 2 \cdot 3a_3 \dots$$

Hence:

$$a_0 = f(0); \quad a_1 = f'(0); \quad a_2 = \frac{f''(0)}{2}; \dots \quad a_n := \frac{f^{(n)}(0)}{n!}.$$

Taylor Polynomials

The Taylor polynomial of degree n for the function $f(x)$ near $x = a$ is the n-th degree polynomial

$$p_n(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!}$$

In fact, we have

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots + f^{(n)}(a)\frac{(x-a)^n}{n!} \pm \int_a^x \frac{(s-x)^n}{n!} f^{(n+1)}(s)\,ds$$

Example 1. Find the 3rd-degree Taylor polynomial of $\sin(x)$ near $a = 0$.

Taylor polynomials

"Taylor polynomial tango":
`https://www.youtube.com/watch?v=msFbHyw043g`)
Taylor polynomial of degree n for f near a:

$$p_n(x) = f(a) \quad + \quad f'(a)\frac{(x-a)}{1!} \quad + \quad f''(a)\frac{(x-a)^2}{2!} \quad + \cdots f^{(n)}(a)\frac{(x-a)^n}{n!}$$

Clicker question #2

What is the coefficient of x^3 in the 3rd-degree Taylor polynomial of e^{2x} near $x = 1$?

- $\frac{2^3 e^2}{3!}$
- $\frac{1}{3!}$
- $\frac{2^3}{3!}$
- $\frac{e^2}{3!}$
- $\frac{2e}{3}$