Calculus I - Lecture 32
Monday, November 6, 2017

University of Nebraska-Lincoln

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Announcements

- Exam 3 will take place on Thursday, November 30, from 8:00pm to 9:30 pm. If you can not make it at this time for a university sanctioned reason, please fill out the form at:
  `https://www.math.unl.edu/alternate-exam-request`
  The form will close on Monday, November 20th at 5pm. The list of approvals will go out on Tuesday, November 21st before noon.

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- The Math Department has been working hard to improve student learning and engagement in calculus. As part of our efforts of ongoing improvement, we will be asking you to complete a brief survey in recitation tomorrow, to share with us your experiences in this calculus class this semester. Your responses do not get shared directly with your instructor. Your honest feedback will help us to continue to improve calculus for future semesters. Please bring your electronic devices to class.
  `https://ssp.qualtrics.com/jfe/form/SV_eeYZyi6wQStyXI1`

Clicker question #1

Which of the following is a TRUE statement?

(A) If a function f is continuous at a, then it has a limit at a

(B) If a function has a limit at a, then it is continuous at a

(C) The integral from a to b of a function gives the area under the graph of the function, hence it is always positive

(D) If a function takes a positive value and a negative value on an interval, then it must take the value zero on that interval as well.

(E) I have not read section 5.4, neither did I start the review for the final.

Answer: Only (A) is true (which contradicts (B)). C is false since an integral could be negative. (D) is only true for continuous functions (by the IVT).

Properties of the Definite Integral

Theorem (Properties of Limits of Integration). If a, b, and c are any numbers and f is a continuous function, then

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. Example:
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

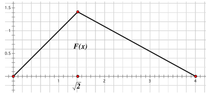
In words:

1. The integral from b to a is the negative of the integral from a to b.
2. The integral from a to c plus the integral from c to b is the integral from a to b. (This property holds for all numbers a, b, and c, not just for those satisfying $a < c < b$.)

1. For a partition of the interval $[a, b]$ we have $a = t_0 < t_1 < \ldots < t_n = b$ where $\Delta t = \frac{b-a}{n} = t_i - t_{i-1}$. Here Δt is always positive. But if we go from b to a (the opposite direction), then $\Delta t = t_i - t_{i-1}$ is negative. Therefore each Δt is negative and so if you switch the limits of integration you will have to put a negative out in front of the integral sign. An example: $\displaystyle\int_2^{-7} 2x\,dx = -\int_{-7}^{2} 2x\,dx$.

2. Let

$$F(x) = \begin{cases} x, & 0 \le x < \sqrt{2} \\ -(2\sqrt{2}+1)(x-4), & \sqrt{2} \le x < 4. \end{cases}$$



Clicker question #2

Suppose that $a < b < c$ and f is continuous. Only one is FALSE, which one?

(A) $\displaystyle\int_a^b f(x)\,dx = \int_c^b f(x)\,dx - \int_c^b f(x)\,dx$

(B) $\displaystyle\int_a^b f(x)\,dx = \int_c^b f(x)\,dx - \int_c^b f(x)\,dx$

(C) $\displaystyle\int_a^b f(x)\,dx = \int_c^b f(x)\,dx - \int_c^b f(x)\,dx$

(D) $\displaystyle\int_a^b f(x)\,dx = \int_c^b f(x)\,dx - \int_c^b f(x)\,dx$

Answer:

The property holds no matter what the order for a, b, c. Thus (B), (C) are true and (A) is false.

Also (D) is true, since $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Examples

1. Given that $\int_0^5 f(x)dx = 3$ and $\int_{-2}^5 f(x)dx = 2$ what is $\int_0^{-2} f(x)dx$?

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   Solution:

   $$\int_0^{-2} f(x)dx = -\int_{-2}^0 f(x)dx = -\left(\int_{-2}^5 f(x)dx - \int_0^5 f(x)dx\right) = 1$$

2. Find the value of a such that

$$\int_0^a 3x^2\,dx = \frac{1}{2}\int_0^2 3x^2\,dx.$$

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   Solution: By the Fundamental Theorem of Calculus
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   Solution: By the Fundamental Theorem of Calculus
   $$\int_0^2 3x^2\,dx = x^3|_0^2 = 8, \quad , \quad \int_0^a 3x^2\,dx = x^3|_0^a = a^3.$$
   Thus, to find a with the desired property we need to have $a^3 = \dfrac{8}{2} = 4$, hence $a = \sqrt[3]{4}$.

Properties of the Definite Integral

Theorem.

Properties of Sums and Constant Multiples of the Integrand Let f and g be continuous functions and let c be a constant. Then

- $\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
- $\int_a^b cf(x)\, dx = c \int_a^b f(x) dx$

In words:

1. The integral of the sum (or difference) of two functions is the sum (or difference) of their integrals.
2. The integral of a constant times a function is that constant times the integral of the function.

Clicker #3

Suppose that $\int_0^1 f(x)\,dx = 5$ and $\int_0^1 g(x)\,dx = -2$. Then $\int_0^1 3f(x) - 4g(x)\,dx = ?$

(A) 7

(B) -26

(C) -7

(D) 10

(E) 23

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(D) 10

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The answer is (D).

Area Between Curves

If the graph of $f(x)$ lies above the graph of $g(x)$ for $a \le x \le b$, then **the area between** f and g for $a \le x \le b$ is $\displaystyle\int_a^b (f(x) - g(x))dx$.

Example. Find the area of the shaded region between two parabolas in the figure below.

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Using Symmetry to Evaluate Integrals

- If f is even, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ (left graph)
- If g is odd, then $\int_{-a}^{a} g(x)dx = 0$ (right graph)

Example

For which of the following functions $f(x)$ is $\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$ true?

(A) e^{-x^2}

(B) $x \sin x$

(C) $x \cos x$

(D) $x + \sin x$

(E) $x + \cos x$

(F) $x^2 + \sin x$

(G) $x^2 + \cos x$

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Solution. This is true for even functions, so it is true for (A), (B), and (G). For odd functions, $\int_{-1}^{1} f(x)dx = 0$ so it is false for (C) and (D). For (E) and (F), which are neither odd nor even, it is false.

Example

Figure below contains the graph of $F(x)$, while the graphs in (a)-(d) are those of $F'(x)$. Which shaded region represents $F(b) - F(a)$?

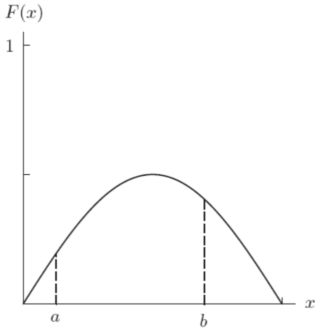
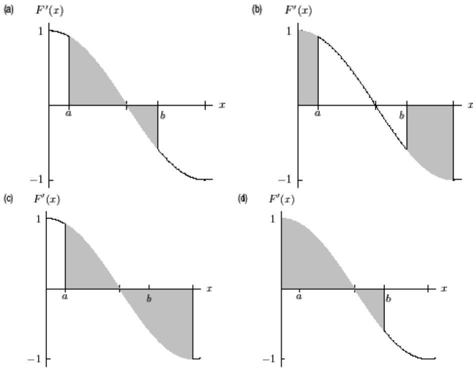


Figure 5.6



Solution

The correct answer is (a). Because $F(b) - F(a) = \int_a^b F'(x)dx$ the left-hand and right - hand limits of the integral must be a and b, respectively.

Follow-up Question. What feature of the graph of $F'(x)$ tells you that $F(b) > F(a)$?

Solution

The correct answer is (a). Because $F(b) - F(a) = \int_a^b F'(x)dx$ the left-hand and right - hand limits of the integral must be a and b, respectively.

Follow-up Question. What feature of the graph of $F'(x)$ tells you that $F(b) > F(a)$?

Answer. More area is shaded above the $x-$axis than below. Thus, $F(b) - F(a) > 0$ so $F(b) > F(a)$.

Comparison of Definite Integrals

Let f and g be continuous functions.

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
  $$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$
- If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

The Definite Integral as an Average

The average value of f from a to b is:

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

How to Visualize the Average on a Graph: The definition of the average value tells us that:

$$(\text{The average value of } f \text{ from } a \text{ to } b)\cdot(b-a)=\int_a^b f(x)\,dx.$$

The Definite Integral as an Average (cont)

Example:

Suppose that $C(t)$ represents the daily cost of heating your house, measured in dollars per day, where t is time measured in days and $t = 0$ corresponds to January 1, 2008. Interpret

$$\int_0^{90} C(t)\,dt \text{ and } \frac{1}{90-0}\int_0^{90} C(t)\,dt.$$

Solution. The units for the first expression are (dollars/day) (days) = dollars. The integral represents the total cost in dollars to heat your house for the first 90 days of 2008, namely the months of January, February, and March. The second expression is measured in (1/days)(dollars) or dollars per day, the same units as $C(t)$. It represents the average cost per day to heat your house during the first 90 days of 2008.

Example for average value

Find the average value of $g(t) = 1 + t$ over $[0, 2]$.

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Solution. The average value is given by

$$\frac{1}{2-0}\int_0^2 g(t)\,dt = \frac{1}{2-0}\left[t + \frac{t^2}{2}\right]_0^2 = \frac{1}{2-0}\left(2 + \frac{2^2}{2} - (0 + \frac{0^2}{2})\right) = 2.$$

Clicker question #4

For which interval is the average value of $\cos x$ over the interval smallest?

(A) $0 \leq x \leq \pi$
(B) $\pi/2 \leq x \leq 3\pi/2$
(C) $3.14 \leq x \leq 3.5$
(D) $4.71 \leq x \leq 4.72$

Answer

From smallest to largest, (D), (B), (A), (C).

On $0 \le x \le \pi$, the average value of $\cos x$ is 0; on $\pi/2 \le x \le 3\pi/2$, the average value is negative. Since $3\pi/2$ is closer to the left-end of $4.71 \le x \le 4.72$, and because $\cos x$ changes from negative to positive, and the graph of $\cos x$ is symmetric about $x = 3\pi/2$, the average value of $\cos x$ on this interval is small and positive.

On $3.14 \le x \le 3.15$, the average value is almost -1.

Review for final exam

So far: Functions: linear, exponential, logarithmic, trigonometric functions, limits, and continuity.

For next time: Speed and derivatives; interpretation

- How do you compute the derivative of a function using the definition? (try $f(x) = x^2 + 3x - 5$)
- On what intervals is the function $x + \cos x$ increasing?

Solve the suggested problems in the syllabus from sections 2.1, 2.2, 2.3, and 2.4.

Wrapping up

- Work out the problems from section 5.4.
- For next time read 6.1. The lecture will start with a clicker question from your reading assignment and the review material!