Math 107-Lecture 22

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Comparison tests

The direct comparison test (DCT) Assume that we have $0 \leq a_n \leq b_n$ for all $n \geq 1$. Then

- If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges (because it is smaller)
- If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges (because it is larger)

The limit comparison test (LCT) Assume that $a_n, b_n > 0$ for all $n \geq 1$ and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0, \infty)$$

In other words, the two sequences have similar behavior at infinity. Then

$$\sum_{n=1}^{\infty} a_n (\mathrm{C}) \iff \sum_{n=1}^{\infty} b_n (\mathrm{C}).$$

Alternating series

Alternating Series Test (AST): Assume we have a sequence such that

$$c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0, \text{ and } \lim_{n\to\infty} c_n = 0.$$

Then the series $\displaystyle\sum_{n=0}^{\infty}(-1)^n c_n$ converges.

Example 1: The series

$$\sum_{n=0}^{\infty}(-1)^n \frac{3}{e^n}$$

is convergent by AST.

Example 2: However, for the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos n}{\sqrt{n}}$$

AST is inconclusive (Why?).

Absolute convergence

- If the series of absolute values $\displaystyle\sum_{n=n_0}^{\infty} |a_n|$ converges, then the original series $\displaystyle\sum_{n=n_0}^{\infty} a_n$ also converges and we say it converges absolutely.

  Example of absolute convergence: The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{\sqrt{n^3}}$$

  is convergent by AST; the series of absolute values $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^3}}$ also converges (Why?).

Conditional convergence

- If the series of absolute values $\displaystyle\sum_{n=n_0}^{\infty} |a_n|$ diverges, but the original series $\displaystyle\sum_{n=n_0}^{\infty} a_n$ converges, we say it converges conditionally.

  Example of conditional convergence: The series

$$\sum_{n=3}^{\infty}(-1)^n \frac{1}{\ln(n)}$$

  is convergent by AST, however the series of absolute values $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(n)}$ is divergent (Why?).

The ratio test

For situations when factorials are present, we have the following tool:
The ratio test. Let

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$$

- If $\rho < 1$ then the series converges absolutely.
- If $\rho > 1$ then the series diverges.
- If $\rho = 1$ then the test is *inconclusive*.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2n!}$

The root test

For situations when we have terms raised to power n, we employ The root test. Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

- If $\rho < 1$ then the series converges absolutely.
- If $\rho > 1$ then the series diverges.
- If $\rho = 1$ then the test is *inconclusive*.

Example: $\displaystyle\sum_{n=2}^{\infty} \left(\frac{n+1}{2n}\right)^n$

Clicker question #1

What can we say about the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n+1}{n^2+2n-1}$$

- It diverges by the divergence test
- It is absolutely convergent by the ratio test
- We can not conclude convergence because the alternating series test does not apply
- It is is conditionally convergent
- It diverges - by the comparison test

Quiz last week

Use comparison tests to study convergence of the series below:

1. $$\sum_{n=1}^{\infty} \frac{4 + 3^{1/n}}{3^n}$$

2. $$\sum_{n=1}^{\infty} \frac{n - \sin(n)}{n^{3/2} + 1}$$

More examples

- Use the root test to see if the series converges and, if it does, whether it converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2^n}}$$

  Is there an easier way?

- Use a comparison test to determine if the series below converges

$$\sum_{n=2}^{\infty} \frac{n-2}{4^n+2}$$

- Use the ratio test to study the series

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

Clicker question #2

Which test is **inconclusive** for the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{(\sqrt{5}-1)^n}$$

- The divergence test
- The ratio test
- The alternating series test
- The root test
- The comparison test

Peek at section 9.5 Power series

We will focus on the power series about $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $c_n \in \mathbb{R}$ is an arbitrary sequence.

- If all $c_n = 1$, $n \geq 0$ then we have
$$\sum_{n=1}^{\infty} (x-a)^n = \frac{1}{1-(x-a)}, \quad |x-a| < 1.$$
- The series depends on x; powers of $x - a$
- The convergence of the series will depend on values of x
- The sum of the series will depend on x.