Calculus 1
Limits

Dr. Adam Larios

30 August 2017

Idea of a Limit

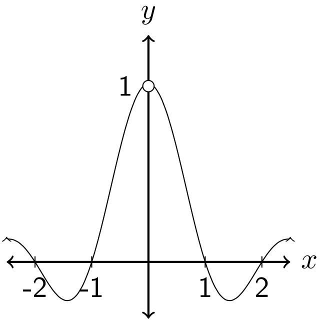
We write

$$\lim_{x \to c} f(x) = L$$

if the values of $f(x)$ approach L as x approaches c.

- **Example:**
  Use the graph to estimate

  $$\lim_{\theta \to 0} \frac{\sin \theta}{\theta}.$$

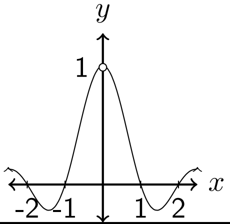


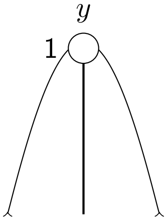
Definition of a Limit

Limit

A function f is defined on an interval around c, except perhaps at the point $x = c$. We define the limit of the function $f(x)$ as x approaches c to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write

$$\lim_{x \to c} f(x) = L$$





Question

Given the graph below, compute $\lim_{x\to 3} x^2 + 2$

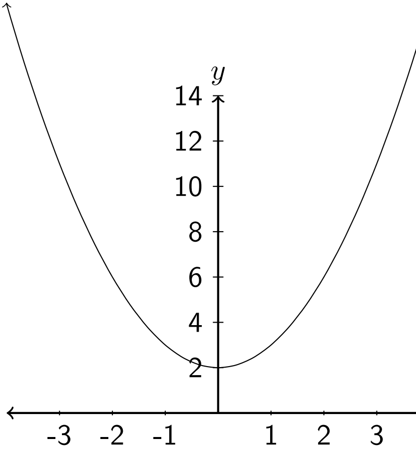
(a) $\lim_{x\to 3} x^2 + 2 = 10$

(b) $\lim_{x\to 3} x^2 + 2 = 11$

(c) $\lim_{x\to 3} x^2 + 2 = 12$

(d) $\lim_{x\to 3} x^2 + 2 = 13$

(e) Does not exist



Limit Properties

Properties of Limits

Assuming all the limits on the right-hand side exist:

1. If b is a constant, then $\lim\limits_{x\to c}(bf(x)) = b\left(\lim\limits_{x\to c} f(x)\right)$
2. $\lim\limits_{x\to c}(f(x)+g(x)) = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$
3. $\lim\limits_{x\to c}(f(x)\cdot g(x)) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$
4. $\lim\limits_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, provided $\lim\limits_{x\to c} g(x) \neq 0$
5. For any constant k, $\lim\limits_{x\to c} k = k$
6. $\lim\limits_{x\to c} x = c$

Clicker Question

Use algebra to compute $\lim_{x\to 1} x^2(x^3+2)$.

a) $\lim_{x\to 1} x^2(x^3+2) = 3$
b) $\lim_{x\to 1} x^2(x^3+2) = 2$
c) $\lim_{x\to 1} x^2(x^3+2) = 4$
d) Does not Exist

Clicker Question

Suppose that $\lim_{x \to 2} f(x) = 5$. Use algebra to compute $\lim_{x \to 2} x^2(f(x) + 2)$.

a) $\lim_{x \to 2} x^2(f(x) + 2) = 20$

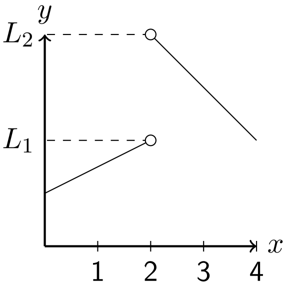
b) $\lim_{x \to 2} x^2(f(x) + 2) = 35$

c) $\lim_{x \to 2} x^2(f(x) + 2) = 100$

d) $\lim_{x \to 2} x^2(f(x) + 2) = 28$

e) Does not Exist

One and Two Sided Limits



For the function graphed we have that

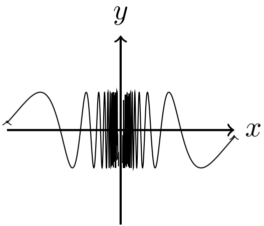
$$\lim_{x\to 2^-} f(x) = L_1$$

$$\lim_{x\to 2^+} f(x) = L_2$$

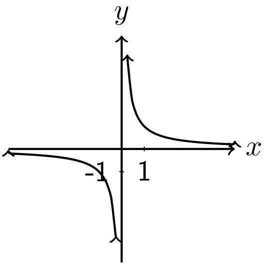
If the left and right hand limits were equal, that is, if $L_1 = L_2$, then we would say that $\lim\limits_{x\to 2} f(x)$ exists. However, since they are not equal, we say the limit does not exist.

Other Limits Which Do not Exist

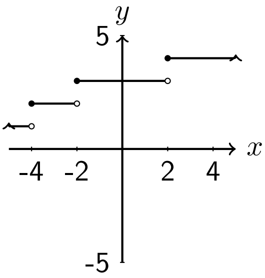
Here are three examples in which the limit fails to exist.



$f(x) = \sin(1/x)$



$f(x) = 1/x$



$$f(x) = \begin{cases} 1 & \text{if } x < -4 \\ 2 & \text{if } -4 \leq x < -2 \\ 3 & \text{if } -2 \leq x < 2 \\ 4 & \text{if } 2 \leq x \end{cases}$$

Limits at Infinity

Limits at Infinity

If $f(x)$ gets as close to a number L as we please when x gets sufficiently large, then we write

$$\lim_{x\to\infty} f(x) = L$$

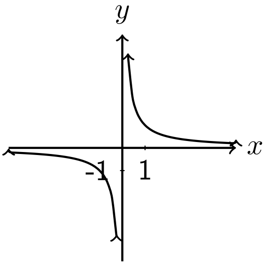
Similarly, if $f(x)$ approaches L when x is negative and has a sufficiently large absolute value, then we write

$$\lim_{x\to-\infty} f(x) = L$$

For example, consider the graph of $f(x) = 1/x$.

$$\lim_{x\to\infty} f(x) = 0$$

$$\lim_{x\to-\infty} f(x) = 0$$



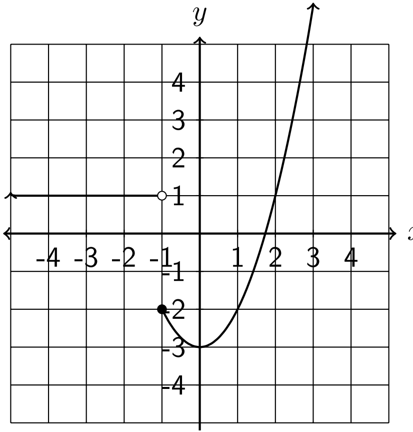
Where does the following limit fail to exist?

(a) $x = -3$

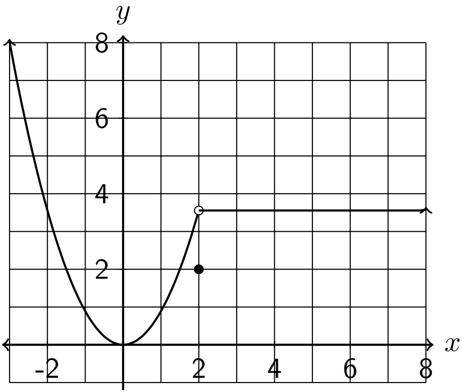
(b) $x = -1$

(c) $x = 2$

(d) It exists everywhere



How is this graph different?



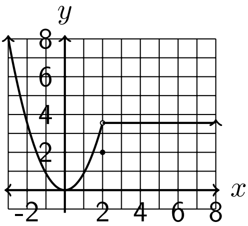
Definition of Continuity

Continuous

The function f is continuous at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c. The function is continuous on an interval $[a, b]$ if it is continuous at every point in the interval.



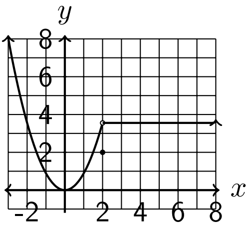
Definition of Continuity

Continuous

The function f is continuous at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c. The function is continuous on an interval $[a, b]$ if it is continuous at every point in the interval.



If the function can be made continuous then the limit should exist and should have the value needed to make the function continuous.

Some Theorems

Continuity of Sums, Products, and Quotients of Functions

Suppose that f and g are continuous on an interval and that b is a constant. Then, on that same interval, 1. $bf(x)$ is continuous.
2. $f(x) + g(x)$ is continuous.
3. $f(x)g(x)$ is continuous.
4. $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

Continuity of Composite Functions

If f and g are continuous, and if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.