# Convergence/divergence tests

- **DCT:** $0 \le a_n \le b_n$; $\sum_{n=1}^{\infty} b_n$ (C)$\Rightarrow \sum_{n=1}^{\infty} a_n$ (C); $\sum_{n=1}^{\infty} a_n$ (D)$\Rightarrow \sum_{n=1}^{\infty} b_n$ (D)
- **LCT:** $\lim_{n\to\infty} \frac{a_n}{b_n} = L \in (0,\infty)$ then $\sum_{n=1}^{\infty} a_n(\mathrm{C}/D) \iff \sum_{n=1}^{\infty} b_n(\mathrm{C}/D)$.
- **AST:** If $c_1 \ge c_2 \ge c_3 \ge \ldots \ge 0$, $\lim_{n\to\infty} c_n = 0$ then $\sum_{n=0}^{\infty}(-1)^n c_n$ (C)
- **Ratio/Root** Let $\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ (Ratio) $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (Root).
  $\rho < 1$ (AC); $\rho > 1$ (D); $\rho = 1$ *inconclusive*.

# More examples

- Test whether the following series converges absolutely or conditionally.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2^n}}$$

Is there an easier way?

- Use a comparison test to determine if the series below converges

$$\sum_{n=2}^{\infty} \frac{n-2}{4^n+2}$$

- Use the ratio test to study the series

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

## Clicker question #2 - from last time

Which test is **inconclusive** for the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{(\sqrt{5}-1)^n}$$

- The divergence test
- The ratio test
- The alternating series test
- (The root test)
- The comparison test

# 9.5 Power series

In this section we focus on the power series about $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $c_n \in \mathbb{R}$ is an arbitrary sequence.

- If all $c_n = 1$, $n \geq 0$ then we have
  $$\sum_{n=1}^{\infty} (x-a)^n = \frac{1}{1-(x-a)}, \quad |x-a| < 1.$$
- The series depends on $x$; powers of $x - a$
- The convergence of the series will depend on values of $x$
- The sum of the series will depend on $x$.

# Theorem

If a power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = \alpha$, then it converges **absolutely** for $|x| < |\alpha|$. If it diverges for $x = |\beta|$ then it diverges for $|x| > |\beta|$.

- A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a **radius of convergence** $R$;
- It converges absolutely when $|x-a| < R$ and diverges when $|x-a| > R$; Hence, to find $R$ it is enough to work with absolute values;
- The collection of all $x$ for which the series converges is called the **interval of convergence**. The convergence at the end-points must be checked separately.

# Why does the interval of convergence matter?

- If $R = \infty$, then the series converges absolutely for all $x$.
- If $R = 0$ then the series converges only when $x = a$.
- We can add, subtract, multiply, term-by term differentiate, integrate series **within the interval of convergence**. The result converges at every interior point of the interval of convergence of the original.

# How do we find the radius of convergence?

- **Ratio (and root) test** are most useful to determine the radius of convergence. **Apply it to absolute value of the series**
- For the end-points use the usual analysis for series.

# Example 1

Specify $a$ and $c_n$ for the following series. Find the radii and intervals of convergence. Determine for which $x$ the series converge absolutely, for which $x$ it diverges, and for which $x$, if any, it converges conditionally.

$$\sum_{n=0}^{\infty}(2x-6)^n$$

- $a =$
- $c_n =$
- Apply the **Ratio Test** (need the limit of the ratio $< 1$):
- $R =$
- Interval of convergence
- Check each end-point of the interval of convergence

# Clicker question #1

For the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$$

determine $a$ and $c_n$

- $a = 2, \quad c_n = (-1)^n$
- $a = 2^n, \quad c_n = \dfrac{(-1)^n}{n}$
- $a = 2, \quad c_n = \dfrac{(-1)^n}{n}$
- $a = -\dfrac{2}{n}, \quad c_n = (-1)^n$
- $a = -2, \quad c_n = \dfrac{(-1)^n}{n}$

# Clicker question #2

For the same power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$$

determine the interval of convergence:

- $(-\infty, \infty)$
- $(-3, -1)$
- $[-3, -1]$
- $(-3, -1]$
- $[-3, -1)$