Calculus 1
Integration by Substitution

The Guess-and-Check Method

Integration by substitution reverses the chain rule. Recall:

$$\frac{d}{dx}\left(f(g(x)\right) = f'(g(x)) \cdot g'(x).$$

Thus, any function which is the result of differentiating with the chain rule is the product of two factors, the "derivative of the outside" and the "derivative of the inside." If a function has this form, its antiderivative is $f(g(x))$.

For Example

Find

$$\int \left(3x^2 \cos(x^3)\right) dx$$

Solution: The function $3x^2 \cos(x^3)$ looks like the result of applying the chain rule: there is an "inside" function x^3 and its derivative $3x^2$ appears as a factor. Since the outside function is a cosine which has a sine as an antiderivative, we guess $\sin(x^3)$ for the antiderivative. Differentiating to check gives

$$\frac{d}{dx}(\sin(x^3)) = \cos(x^3) \cdot (3x^2).$$

Since this is what we began with, we know that

$$\int \left(3x^2 \cos(x^3)\right) dx = \sin(x^3) + C$$

The Method of Substitution

To Make a Substitution in an Integral

Let w be the "inside function" and $dq = w'(x)dx = \frac{dw}{dx}dx$. Then express the integrand in terms of w.

Find

$$\int \left(3x^2 \cos(x^3)\right) dx$$

Solution:
We look for an inside function whose derivative appears. In this case x^3. We let $w = x^3$. Then $dw = w'(x)dx = 3x^2dx$. The original integrand can now be completely rewritten in terms of the new variable w:

$$\begin{aligned}
\int 3x^2 \cos(x^3)dx &= \int \cos \underbrace{(x^3)}_{w} \cdot \underbrace{3x^2 dx}_{dw} \\
&= \int \cos w dw \\
&= \sin w + C = \sin(x^3) + C
\end{aligned}$$

Another Example

Find

$$\int xe^{x^2}\,dx$$

$$\begin{aligned}\int xe^{x^2}\,dx = \int \frac{2}{2}xe^{x^2}\,dx = \frac{1}{2}\int 2xe^{x^2}\,dx \\ = \frac{1}{2}\int e^w\,dw \\ = \frac{1}{2}e^w + C\end{aligned}$$

But we didn't start with w so we need to switch back to a function in terms of x. Therefore, the answer is $\frac{1}{2}e^{x^2} + C$ because $w = x^2$.

Another Example

Find

$$\int x^3\sqrt{x^4+5}dx$$

The inside function is x^4+5, with derivative $4x^3$. The integrand has a factor of x^3, and since the only thing missing is a constant factor we try $w = x^4+5$. Then $dw = w'(x)dx = 4x^3dx$, giving $\frac{1}{4}dw = x^3dx$. Thus,

$$\begin{aligned}
\int x^3\sqrt{x^4+5}dx &= \int \sqrt{w}\frac{1}{4}dw \\
&= \frac{1}{4}\int \sqrt{w}dw \\
&= \frac{1}{4}\frac{w^{3/2}}{\frac{3}{2}} + C \\
&= \frac{1}{6}\left(x^4+5\right)^{3/2} + C
\end{aligned}$$

Warning!!!

We saw in the preceding example that we can apply the substitution method when a constant factor is missing from the derivative of the inside function. However, we may not be able to use substitution if anything other than a constant factor is missing. For example, setting $w = x^4 + 5$ to find

$$\int x^2 \sqrt{x^4 + 5} dx$$

does us no good because $x^2 dx$ is not a constant multiple of $dw = 4x^3 dx$. Substitution works if the integrand contains the derivative of the inside function, to *within a constant factor.*

Note:

Integrals are, in some sense, harder than derivatives. With derivative in general, if you can write it down you can find a derivative, this is not true for integrals. Writing down something that appears to be simple may in fact be unsolvable!

Clicker

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int x \sin(x^2) dx$

b) $\int \frac{1}{x \ln x} dx$

c) $\int \frac{1}{\tan(x)} dx$

d) $\int x^2 (x^3 + 3)^6 dx$

Clicker

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int x \sin(x^2) dx$

b) $\int \frac{1}{x \ln x} dx$

c) $\int \frac{1}{\tan(x)} dx$

d) $\int x^2 (x^3 + 3)^6 dx$

(a) While $w = x^2$ is a useful substitution for (a), the result is not of the form $\int w^n dw$.

Clicker

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int \frac{4x^3+3}{\sqrt{x^4+3x}} dx$

b) $\int \frac{e^x - x^{-x}}{(e^x+e^{-x})^3} dx$

c) $\int \frac{2x}{x^2+1} dx$

d) $\int \frac{\sin(x)}{x} dx$

Clicker

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int \frac{4x^3+3}{\sqrt{x^4+3x}} dx$

b) $\int \frac{e^x - x^{-x}}{(e^x+e^{-x})^3} dx$

c) $\int \frac{2x}{x^2+1} dx$

d) $\int \frac{\sin(x)}{x} dx$

(d) Actually, there is no substitution that converts (d) into a function which has an elementary derivative.

Discussion

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int x^{16}(x^{17} + 16)^{16} dx$

b) $\int x^{16}(x^{17} + 16x)^{16} dx$

c) $\int \frac{18x}{1+6x^3} dx$

d) $\int \frac{e^x}{e^x+6} dx$

Discussion

Which of the integrals can not be converted to one of the form $\int w^n dw$ by a simple substitution where n is a constant?

a) $\int x^{16}(x^{17}+16)^{16}dx$

b) $\int x^{16}(x^{17}+16x)^{16}dx$

c) $\int \frac{18x}{1+6x^3}dx$

d) $\int \frac{e^x}{e^x+6}dx$

In (a), $w = x^{17}+16$ works and in (d) $w = e^x + 6$ will work.

Discussion

Which of the following integrals yields an arcsine or arctangent function upon integration after an appropriate substitution?

a) $\int \frac{x}{4-x^2}\,dx$

b) $\int \frac{4}{4+x^2}\,dx$

c) $\int \frac{x}{\sqrt{4-x^2}}\,dx$

d) $\int \frac{4}{\sqrt{4+x^2}}\,dx$

Definite Integrals by Substitution

To Use Substitution to Find Definite Integrals

Either

- Compute the indefinite integral, expressing an antiderivative in terms of the original variable, and then evaluate the result at the original limits,
- Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative back to the original variable.

Evaluate

$$\int_1^3 \frac{dx}{5-x}.$$

Definite Integrals by Substitution

To Use Substitution to Find Definite Integrals

Either

- Compute the indefinite integral, expressing an antiderivative in terms of the original variable, and then evaluate the result at the original limits,
- Convert the original limits to new limits in terms of the new variable and do not convert the antiderivative back to the original variable.

Evaluate

$$\int_1^3 \frac{dx}{5-x}.$$

Solution:

Let $w = 5 - x$, then $dw = -dx$. In particular, when $x = 1$, $w = 4$, and when $x = 3$, $w = 2$, therefore,

$$\int_1^3 \frac{dx}{5-x} = \int_4^2 \frac{-dw}{w} = -\ln|w|\Big|_4^2 = -(\ln 2 - \ln 4) = 0.693$$

More Complex Substitutions

Find

$$\int \sqrt{1+\sqrt{x}}dx$$

This time, the derivative of the inside function is nowhere to be seen. Nevertheless, we try $w = 1 + \sqrt{x}$. Then $w + 1 = \sqrt{x}$, so $(w+1)^2 = x$. Therefore, $2(w+1)dw = dx$. We have

$$\begin{aligned}
\int \sqrt{1+\sqrt{x}}dx &= \int \sqrt{w}2(w-1)dw = 2\int w^{1/2}(w-1)dw \\
&= 2\int (w^{3/2} - w^{1/2})dw = 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C \\
&= 2\left(\frac{2}{5}(1+\sqrt{x})^{5/2} - \frac{2}{3}(1+\sqrt{x})^{3/2}\right) + C
\end{aligned}$$