The Singularity's Tale

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CAMS Colloquium
University of Southern California
Los Angeles, CA

The Incompressible Navier-Stokes Equations

Claude L.M.H. Navier

George G. Stokes

Momentum Equation

$$\underbrace{\frac{\partial \vec{u}}{\partial t}}_{\textit{Acceleration}} + \underbrace{(\vec{u}\cdot\nabla)\vec{u}}_{\textit{Advection}} = \underbrace{-\nabla p}_{\substack{\textit{Pressure}\\ \textit{Gradient}}} + \underbrace{\nu\triangle\vec{u}}_{\substack{\textit{Viscous}\\ \textit{Diffusion}}}$$

Incompressibility

$$\text{div}\,\vec{u} = 0$$

Unknowns

$\vec{u}$:= Velocity (vector)

p := Pressure (scalar)

Parameter

ν := Kinematic Viscosity

Problem (J. Leray, 1933)

Can a singularity develop in the solutions?

- 2D case: No.
- 3D case: $1,000,000 Clay Millennium Prize Problem

Turbulence: A Wide Range of Scales

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Turbulence: A Wide Range of Scales $(10^{-2}m)$

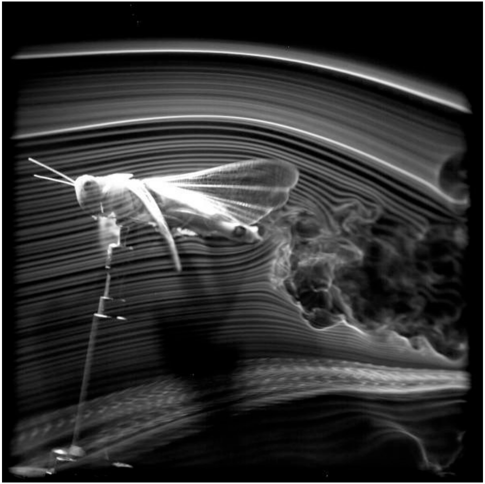
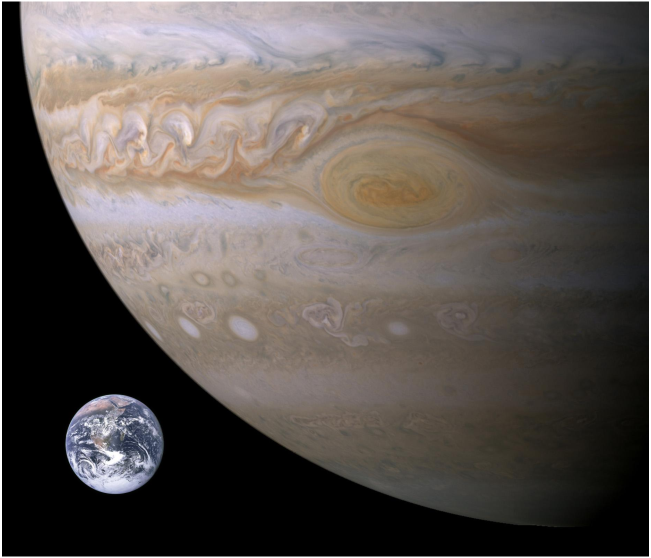
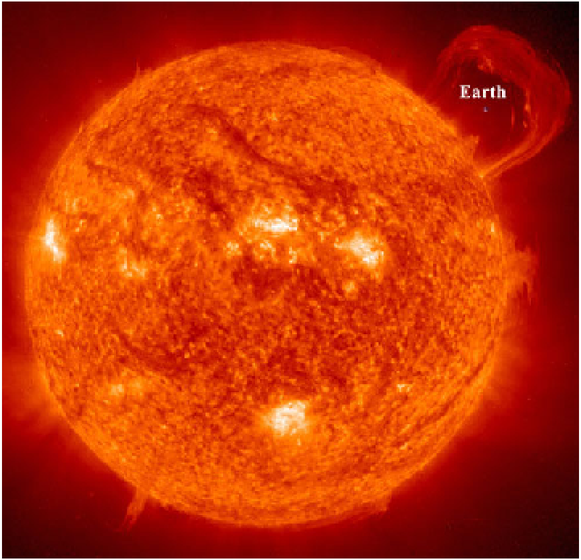


Image Credit: Animal Flight Group, Dept. of Zoology, Oxford University and Dr. John Young

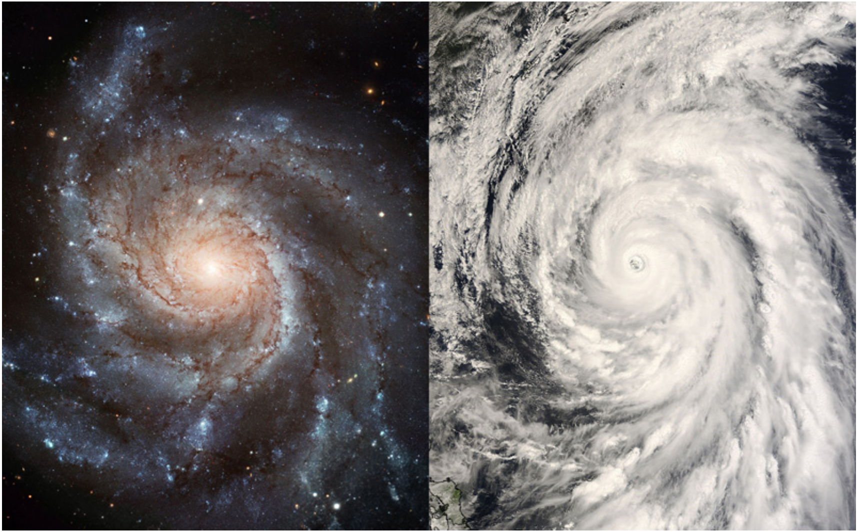
Turbulence: A Wide Range of Scales ($10^8 m$)



Turbulence: A Wide Range of Scales ($10^9 m$)



Turbulence: A Wide Range of Scales ($10^{21} m$)



Let's Understand The Problem

The Simplest PDE

$u = u(x,t) =$ some substance moved at velocity $c = \frac{dx}{dt}$.

$$\begin{aligned} 0 = \frac{du}{dt} &= \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \\ &= \frac{\partial u}{\partial t} + \quad c \,\frac{\partial u}{\partial x} \end{aligned}$$

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$$u_t + cu_x = 0$$

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Burgers Equation: Set $u = c$,

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Here, we will find a singularity.

Computer Time!

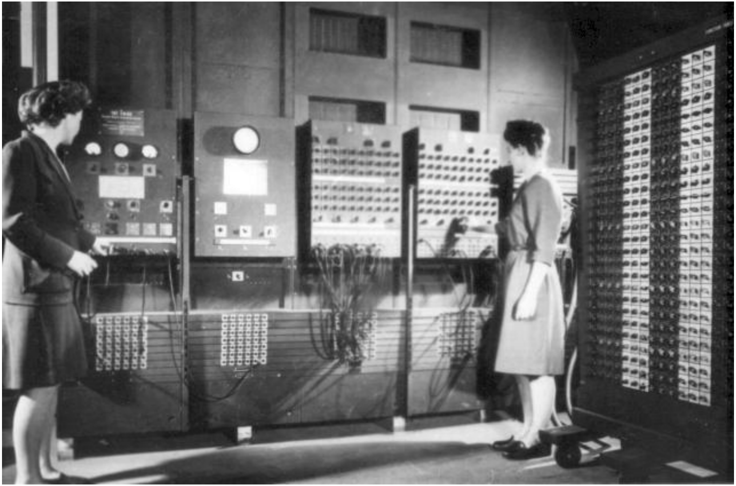

Figure : Programmers working on ENIAC, one of the first computers (c. 1946)

Burgers vs. Navier-Stokes

Burgers Equation

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0$$

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Take derivative $w = u_x$.

$$u_{xt} + u_x u_x + uu_{xx} = \nu u_{xxx}$$

$$w_t + w^2 + uw_x = \nu w_{xx}$$

If w is at a *maximum* ...

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So w $(= u_x)$ is decreasing at any maximum!

Burgers Equation: Higher dimensions

n-Dimensional Burgers

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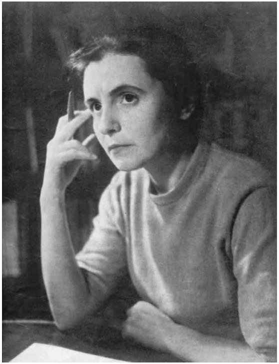
Olga Ladyzhenskaya: Maximum Principle for $\frac{1}{2}|\vec{u}|^2$.



Back to Navier-Stokes

$$\begin{cases} \vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} = \nu\triangle\vec{u} - \nabla p \\ \nabla\cdot\vec{u} = 0 \end{cases}$$

When do we have no singularity?

	$\nu > 0$	$\nu = 0$
Burgers		
Navier-Stokes 2D		
Navier-Stokes 3D		

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$$\omega = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \quad \text{(Vorticity)}$$

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Singularity? Open Problem. $((\vec{u} \cdot \nabla)\vec{u}, \vec{u}) \neq 0$

2D Kuramoto-Sivashinsky Equation

$$\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\triangle^2 \vec{u} - \triangle \vec{u}$$

Singularity? Open Problem.

Focus: 3D Euler Equations

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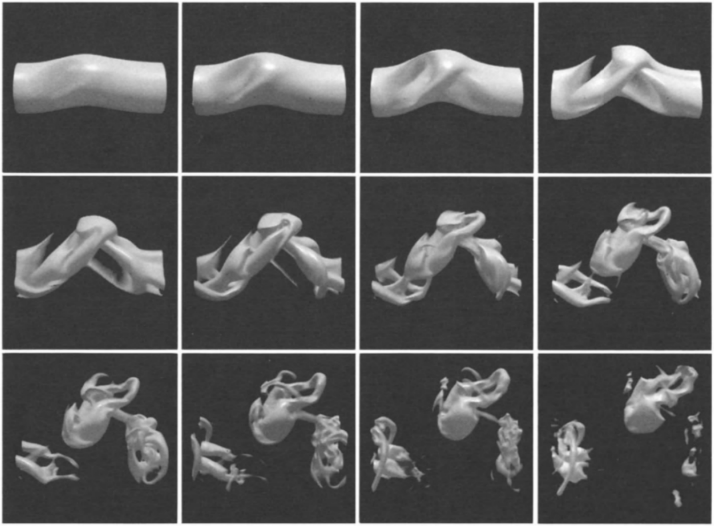

Image Credit: J. Bell, D. Marcus, *Comm. Math. Phys.* **147**, 371-394 (1992).

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Beale-Kato-Majda Criterion

$$\int_0^T \|\vec{\omega}(t)\|_{L^\infty}\, dt = \infty \iff \text{Singularity on } [0, T].$$

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Analytical Singularity Criteria

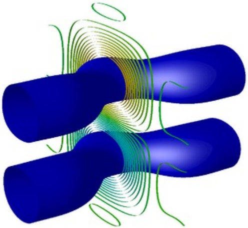
- Beale, Kato, Majda, 1984: $\|\vec{\omega}\|_{L^\infty}$
- Ponce, 1985
- Ferrari, 1993
- Constantin, Fefferman, 1993: $\omega/|\omega|$
- Constantin, Fefferman, Majda, 1996
- L., Titi, 2010: $\alpha\|\nabla\vec{u}^\alpha\|_{L^2}$
- Gibbon, Titi, 2013: $\nabla(\vec{\omega}\cdot\nabla\theta)\times\nabla\theta$
- L., Titi, 2015

Computational Search for singularity

- Brachet et al. '83,'84,'06,'12 ✓
- Pumir and Siggia ✓
- Grauer, Sideris 1991 ✓
- Kerr, 1993 ✓
- Deng, Hou, Yu, 2005 ✓
- Orlandi, Carnevale, 2007 ✓
- Hou, Li, 2008 ×
- Lou, Hou, 2014 ✓

Vortex Stretching

A geometric mechanism to prevent the singularity?



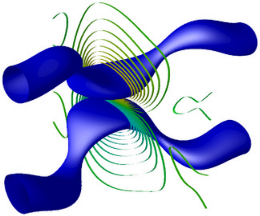


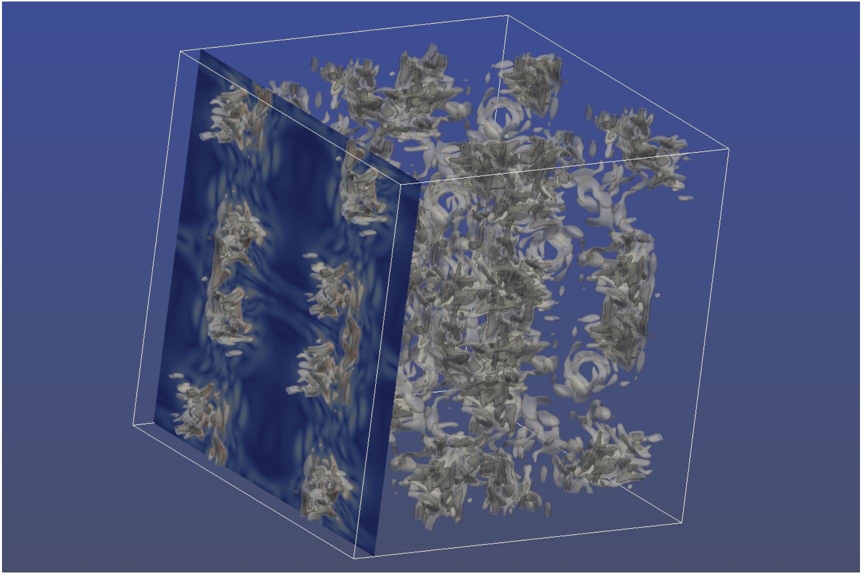
Image Credit: T.Y. Hou, R. Li, *J. Nonlinear Sci.* **16**(6), 639 (2006).

The Voigt α-Regularization

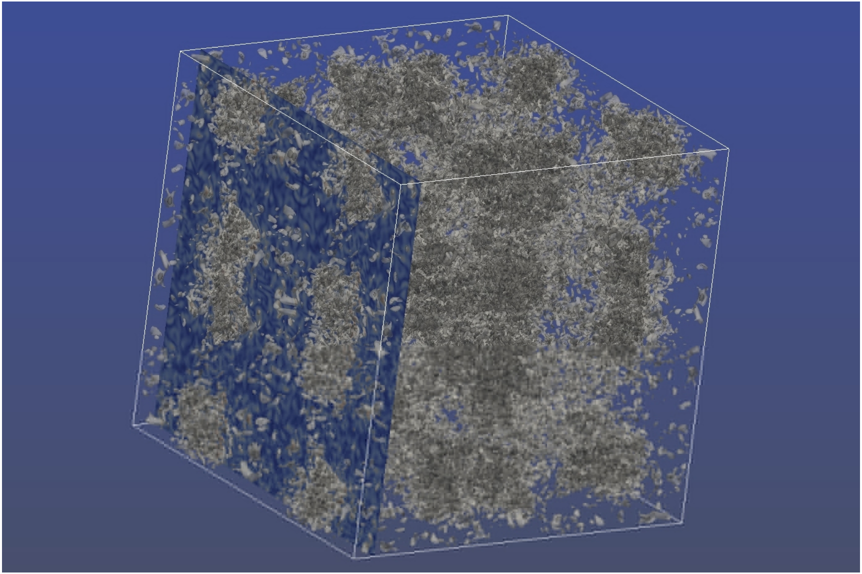
$$\begin{cases} -\alpha^2\partial_t\triangle\vec{u} + \partial_t\vec{u} + (\vec{u}\cdot\nabla)\vec{u} = -\nabla p + \nu\triangle\vec{u} + \mathbf{f}, \\ \nabla\cdot\vec{u} = 0. \end{cases}$$

Some Properties of the Voigt α-Regularization

- First studied by Oskolkov (1973) as a model for polymeric fluids.
- Same steady states as Navier-Stokes (or Euler) Equations.
- Global regularity in bounded domains, with $\nu > 0$ (Y. Cao, Lunasin, Titi).
- Global regularity in periodic case, with $\nu = 0$ (Y. Cao, Lunasin, Titi).
- Higher order regularity, Gevrey regularity, with $\nu \geq 0$ (L., Titi).
- Although the parabolic character of the equations is destroyed, global attractor is comprised of analytic functions (for analytic $\mathbf{f}$) (Kalantarov, Titi).
- Parabolic character of the equations is destroyed.
- Global attractor is comprised of analytic function (Kalantarov, Titi).
- Extensions to MHD (L., Titi), Boussinesq (L., Lunasin, Titi), Cahn-Hilliard (Gal, Tachim Medjo)



$$T = 2$$



$$T = 6$$

The Euler-Voigt Model

$$\begin{cases} -\alpha^2 \triangle \partial_t \vec{u} + \partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + \nabla p = 0 \\ \nabla \cdot \vec{u} = 0 \\ \vec{u}(0) = \vec{u}_0 \end{cases}$$

$$\frac{1}{2} \frac{d}{dt} \left(\alpha^2 \|\nabla \vec{u}\|_{L^2}^2 + \|\vec{u}\|_{L^2}^2 \right) = 0$$

Modified Energy Equality (Cao, Lunasin, Titi, 2006)

$$\alpha^2 \|\nabla \vec{u}(t)\|_{L^2}^2 + \|\vec{u}(t)\|_{L^2}^2 = \alpha^2 \|\nabla \vec{u}_0\|_{L^2}^2 + \|\vec{u}_0\|_{L^2}^2$$

Convergence

- Given initial data $\vec{u}_0 \in H^s$, $s \geq 3$.
- Let $\vec{u}$ be a solution to the Euler equations with initial data $\vec{u}_0$.
- Let $\vec{u}^\alpha$ be a solution of the Euler-Voigt equations with initial data $\vec{u}_0$.

Theorem (Convergence)(A.L., E.S. Titi, 2010, DCDS)

Suppose $\vec{u} \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ *for* $s \geq 3$.
Then $\vec{u}^\alpha \to \vec{u}$ *in* $L^\infty([0,T], L^2)$.

Specifically,

$$\|\vec{u}(t) - \vec{u}^\alpha(t)\|_{L^2}^2 + \alpha^2 \|\nabla(\vec{u}(t) - \vec{u}^\alpha(t))\|_{L^2}^2 \leq C\alpha^2(e^{Ct} - 1).$$

So that

$$\sup_{t \in [0,T]} \|\vec{u}(t) - \vec{u}^\alpha(t)\|_{L^2} \sim \mathcal{O}(\alpha)$$

Singularity Criterion 1

$$\|\vec{u}^\alpha\|_{L^2}^2 + \alpha^2\|\nabla\vec{u}^\alpha\|_{L^2}^2 = \|\vec{u}_0\|_{L^2}^2 + \alpha^2\|\nabla\vec{u}_0\|_{L^2}^2$$
$$\|\vec{u}\|_{L^2}^2 + \limsup_{\alpha\to 0^+} \alpha^2\|\nabla\vec{u}^\alpha\|_{L^2}^2 = \|\vec{u}_0\|_{L^2}^2$$

Theorem (Singularity Criterion 1)(A.L., E.S. Titi, 2010, DCDS)

Suppose there exists a finite time $T_* > 0$ such that

$$\sup_{t\in[0,T_*)} \limsup_{\alpha\to 0^+} \alpha^2\|\nabla\vec{u}^\alpha(t)\|_{L^2}^2 > 0.$$

Then the 3D Euler equations develop a singularity on the interval $[0, T_*]$.

Note (vorticity!):

$$\|\nabla\vec{u}^\alpha\|_{L^2}^2 = \|\boldsymbol{\omega}^\alpha\|_{L^2}^2$$

- Similar singularity criteria exist for inviscid SQG (Khouider, Titi), inviscid Boussinesq (L., Lunasin, Titi).

Singularity Criterion 2

Theorem (Singularity Criterion 2)(A.L., E.S. Titi, submitted)

Suppose there exists a finite time $T_* > 0$ *such that*

$$\limsup_{\alpha\to0^+}\left(\alpha \sup_{t\in[0,T^*]}\|\nabla\vec{u}^\alpha(t)\|_{L^2}\right) > 0. \tag{1.1}$$

Then the 3D Euler equations develop a singularity on the interval $[0, T_*]$.

Moreover,

$$\limsup_{\alpha\to0^+}\sup_{t\in[0,T]}\alpha^2\|\nabla\vec{u}^\alpha(t)\|_{L^2}^2 \geq \sup_{t\in[0,T]}\limsup_{\alpha\to0^+}\alpha^2\|\nabla\vec{u}^\alpha(t)\|_{L^2}^2. \tag{1.2}$$

So the new criterion is stronger.

Computational Approach

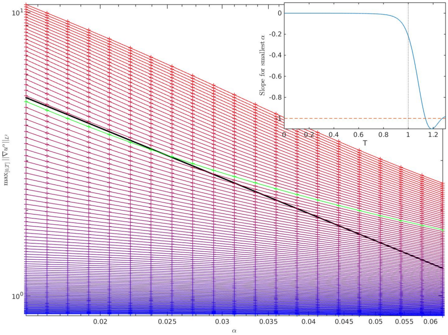
<u>Idea:</u> Investigate the behavior of $f(t,\alpha) := \|\nabla\vec{u}^\alpha(t)\|_{L^2}$

Implication of Singularity Criterion 2

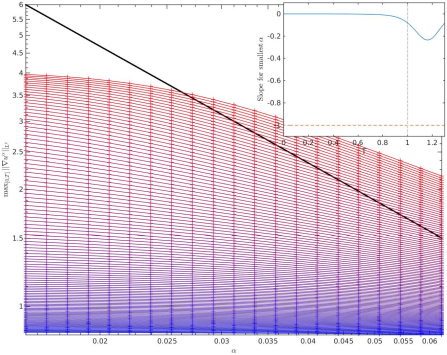
If $\max_{t\in[0,T]}\|\nabla\vec{u}^\alpha(t)\|_{L^2} \sim C\alpha^p$ as $\alpha\to 0$. Then $p\leq -1$ implies singularity formation.

Benjamin-Bona-Mahony Equation

$$-\alpha^2 u_{txx} + u_t + uu_x = \nu u_{xx}, \qquad x \in \mathbb{T} = [-\pi, \pi]$$
$$u(x,0) = \sin(x).$$



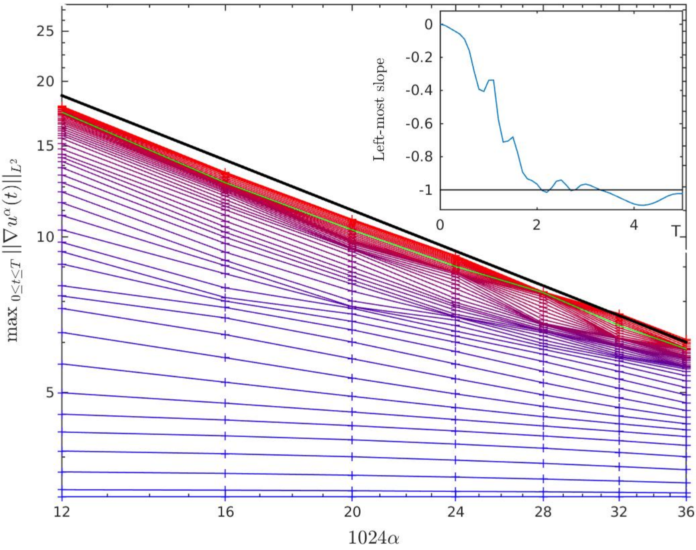
(a) $\max_{t\in[0,T]} \|\nabla u^\alpha(t)\|$ vs. α for the BBM equation, $\nu = 0$.



(b) $\max_{t\in[0,T]} \|\nabla u^\alpha(t)\|$ vs. α for the BBM equation, $\nu > 0$.

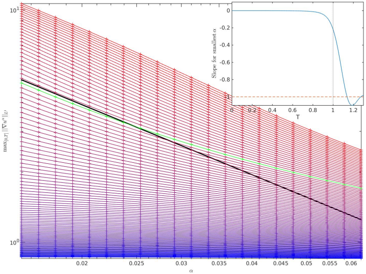
Figure : Simulations of the 1D BBM equation, detecting the known singularity in the 1D Burgers equation.

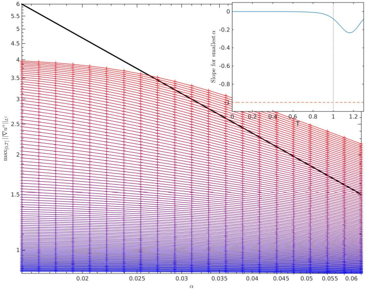
Second Singularity Criterion (Resolution: 1024^3)



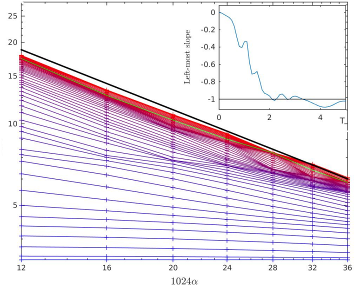
Summing Up

- Is there a singularity? Both cases are
- Evidence of blow-up of 3D Euler by simulating *well-posed problems*.
- Similar singularity criteria exist for many other equations.

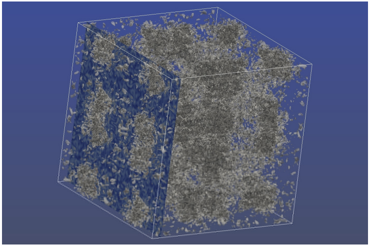




BBM



Euler-Voigt



Thank you!