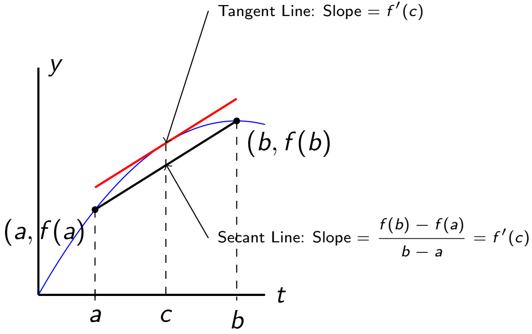
Calculus 1
Global Extrema

University of Nebraska-Lincoln

MVT - geometric interpretation

To understand this theorem geometrically, look at the figure. Join the points on the curve where $x = a$ and $x = b$ with a secant line and observe that the slope of secant line $= (f(b) - f(a))/(b - a)$.
Notice that there appears to be at least one point between a and b where the slope of the tangent line to the curve is precisely the same as the slope of the secant line.



Recall: The Mean Value Theorem was used to prove...

The Increasing Function Theorem

Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$.

- If $f'(x) > 0$ on $a < x < b$, then f is increasing on $a \leq x \leq b$.
- If $f'(x) \geq 0$ on $a < x < b$, then f is non-decreasing on $a \leq x \leq b$.

This theorem proves our geometric interpretation of relationship between the derivative's sign and the increasing/decreasing property of the function.

The Constant Function Theorem

Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. If $f'(x) = 0$ on $a < x < b$, then f is constant on $a \leq x \leq b$.

This theorem verifies what we know about a constant function ($f(x) = k$).

The Racetrack Principle

Suppose that g and h are continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, and that $g'(x) \leq h'(x)$ for $a < x < b$.

(1) "both at the beginning scenario": If $g(a) = h(a)$, then $g(x) \leq h(x)$ for $a \leq x \leq b$.

(2) "both at the end scenario": If $g(b) = h(b)$, then $g(x) \geq h(x)$ for $a \leq x \leq b$.

Example - Racetrack Principle

Use the Racetrack Principle to prove

$$e^x \geq 1 + x$$

for all $x \in \mathbb{R}$.

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Solution: Let $h(x) = e^x$ and $g(x) = 1 + x$. So $h(0) = g(0) = 1$, $h'(x) = e^x$ and $g'(x) = 1$.

Case 1. If $x \geq 0$ and we start at 0, then we are in the "both at the start" scenario ($a = 0$) with $e^x = h'(x) \geq g'(x) = 1$, so by the Racetrack Principle $h(x) \geq g(x)$ for $x \geq 0$.

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Use the Racetrack Principle to prove

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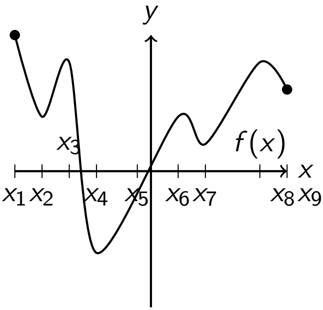
Case 2. If $x \leq 0$, then $1 = g'(x) \leq h'(x) = e^x$. We are now in the "both at the end" scenario ($b = 0$) so by the Racetrack Principle

$$g(x) \leq h(x), \quad x \leq 0.$$

Thus for all $x \in \mathbb{R}$ we have $e^x \geq 1 + x$.

Global Maxima and Minima

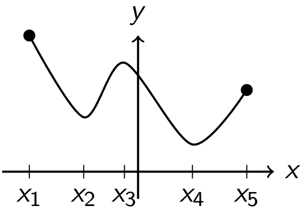


- local minima: x_2, x_4, x_7
- local maxima: x_3, x_6, x_8
- global maximum x_1
- global minimum x_4

- f has a global maximum at p if $f(p)$ is greater or equal than all values of f.
- f has a global minimum at p if $f(p)$ is less or equal than all values of f.

Clicker question

Where is the global maximum for the function below?



(A) x_1

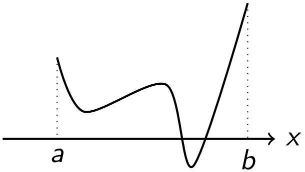
(B) x_3

(C) x_5

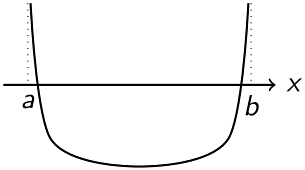
(D) there is no global maximum

The Extreme Value Theorem

Theorem. If f is continuous on $[a, b]$ then f has a global minimum and a global maximum on $[a, b]$.



Global extrema exist on a closed interval



Global extrema may not exist on an open interval

Clicker question

A global maximum is always a critical point.

(A) True

(B) False

Clicker question

A global maximum is always a critical point.

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A global maximum is always a critical point.

(A) True

(B) False

Answer: (B). If a global maximum is always an endpoint of the interval, it does not have to be a critical point. (The derivative need not be 0 or undefined at this point.)

Finding Global Extrema

Global Maxima and Minima on a Closed Interval: For a continuous function f on a closed interval $a \leq x \leq b$:

- Find the critical points of f in the interval.
- Evaluate the function at the critical points and at the endpoints, a and b. The largest value of the function is the global maximum; the smallest value is the global minimum.

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Global Maxima and Minima on an Open Interval or on $\mathbb{R}$: For a continuous function f

- find the value of f at all the critical points
- sketch a graph
- Look at values of f when x approaches the endpoints of the interval, or approaches $\pm\infty$, as appropriate. If there is only one critical point, look at the sign of f' on either side of the critical point.

Example on a closed interval

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Critical points are $x = 0, \quad x = \frac{3}{2}$. We compute values at endpoints and critical points:

$$f(-2) = -8e^4,\ f(0) = 0,\ f\left(\frac{3}{2}\right) = e^{-3}\frac{3^3}{2^3} \approx 0.16, \quad f(2) = 8e^{-4} \approx 0.14.$$

Hence, $x = \frac{3}{2}$ is a global maximum and $x = -2$ is a global minimum.

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Critical points are $x = 0, \quad x = \frac{3}{2}$. We compute limits at $\pm\infty$.

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = -\infty.$$

Hence, the function does not have global maxima and minima on $\mathbb{R}$.

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Solution. Compute $f'(x) = -\frac{1}{x}$.
There only critical point is at $x = 0$. However, the 0 is not in the domain of the function.
Do we need to check the endpoints?
Answer: No! They are not in the domain, and hence they cannot be points off local or global maximum or minimum.
Hence, the function does not have a global maximum on $(0,1)$.

Some announcements

- The deadline for passing the Gateway Exam is October 24.
- Solve the suggested problems and webwork from sections 4.1 and 4.2.
- For next time read section 4.3 (Optimization).