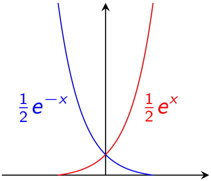
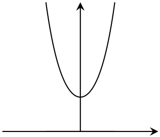
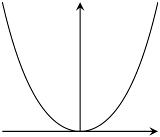
Calculus 1
Hyperbolic Functions
Section 3.8

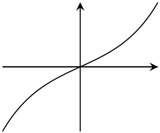
Clicker Question

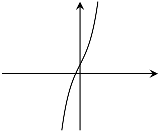
Consider the graphs of $f(x) = \frac{1}{2}e^x$ and $g(x) = \frac{1}{2}e^{-x}$ in the figure on the left. Which of the following functions represent $f(x) + g(x)$?



(a) 

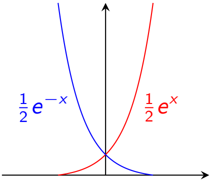
(b) 

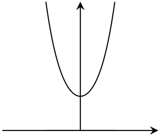
(c) 

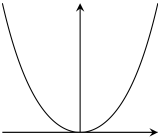
(d) 

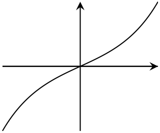
Clicker Question II

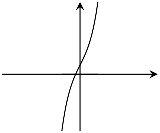
Consider the graphs of $f(x) = \frac{1}{2}e^{x}$ and $g(x) = \frac{1}{2}e^{-x}$ in the figure on the left. Which of the following functions represent $f(x) - g(x)$?



(a) 

(b) 

(c) 

(d) 

Hyperbolic Functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

Graphs of Hyperbolic Cosine and Sine

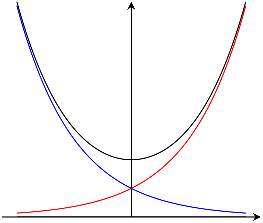


Figure : $\frac{1}{2}e^x$, $\frac{1}{2}e^{-x}$, $\cosh(x)$

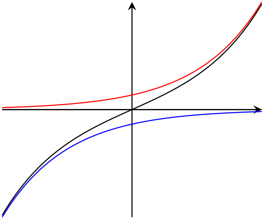


Figure : $\frac{1}{2}e^x$, $-\frac{1}{2}e^{-x}$, $\sinh(x)$

An interesting fact about the hyperbolic cosine: A cable hanging between two supports will form the shape of a hyperbolic cosine. In particular,

$$y = \frac{T}{w}\cosh\left(\frac{wx}{T}\right)$$

where T is the tension at its lowest points, and w is the weight of the cable per unit length. This curve is called the *catenary curve*.
A very famous (upside down) catenary curve is the Saint Louis Arch.

Properties of Hyperbolic Functions

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- $\sinh(0) = 0$
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Solution

From the figures above, it appears that as $x \to \infty$, the graph of $\cosh(x)$ resembles the graph of $\frac{1}{2}e^x$. Similarly, it appears that as $x \to -\infty$, the graph of $\cosh(x)$ resembles the graph of $\frac{1}{2}e^{-x}$. This behavior is explained by using the formulas for $\cosh(x)$ and $\sinh(x)$ and the facts that $e^{-x} \to 0$ as $x \to \infty$, and $e^x \to 0$ as $x \to -\infty$

Identities

Recall that the trig functions were defined on the unit circle and hence were called the *circular functions*. This is where we get the Pythagorean identity, setting $x = \cos\theta$ and $y = \sin\theta$

$$\cos^2\theta + \sin^2\theta = 1 \text{ because } x^2 + y^2 = 1$$

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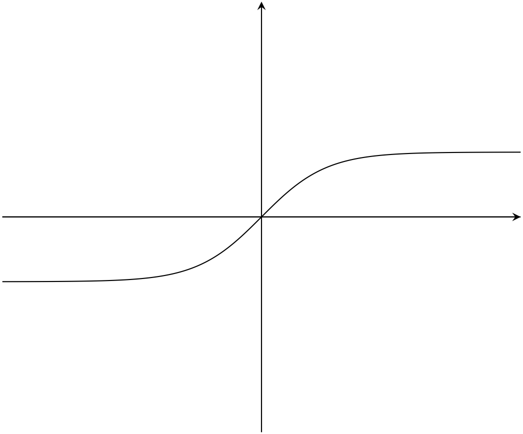
Identity involving cosh and sinh

$$\cosh^2\theta - \sinh^2\theta = 1$$

This identity shows us how the hyperbolic functions got their name. Suppose (x, y) is a point in the plane, and $x = \cosh\theta$ and $y = \sinh\theta$ for some θ. Then the point (x, y) lies on the hyperbola $x^2 - y^2 = 1$.

Now we wish to extend the analogy of the tangent trigonometric function to hyperbolic functions. We define the hyperbolic tangent as follows.

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



Derivatives of Hyperbolic Functions

We calculate the derivatives using the fact that $\frac{d}{dx}e^x = e^x$. The results are again reminiscent of the trigonometric functions.

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Compute the derivative of $\tanh x$.

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Example

Compute the derivative of $\tanh x$.

Solution

Using the Quotient Rule gives

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{(\cosh x)^2 - (\sinh x)^2}{(\cosh x)^2} = \frac{1}{(\cosh x)^2} = \operatorname{sech}^2 x.$$

Derivative Practice

Find the derivatives of the following functions.

- $f(x) = x^4 \sinh x + \cosh^2(3x)$
- $g(x) = \tanh(\cosh(2^x))$
- $h(x) = \ln(\tanh x)$
- $k(x) = \frac{\sinh(3x^2)}{\cosh(2x^3)}$