Calculus I - Lecture 33
Wednesday, November 8, 2017

University of Nebraska-Lincoln

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The Family of Antiderivatives

If the derivative of F is f, we call F an antiderivative of f. For example, since the derivative of x^2 is $2x$, we say that x^2 is an antiderivative of $2x$. Notice that $2x$ has many antiderivatives, since $x^2 + 1, x^2 + 2$, and $x^2 + 3$, all have derivative $2x$. In fact, if C is any constant, we have

$$\frac{d}{dx}(x^2 + C) = 2x$$

so any function of the form $x^2 + C$ is an antiderivative of $2x$. The function $f(x) = 2x$ has a family of antiderivatives.

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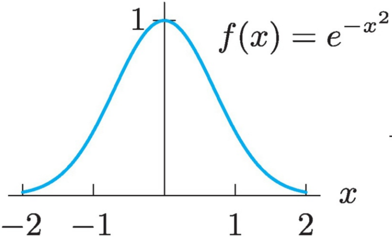
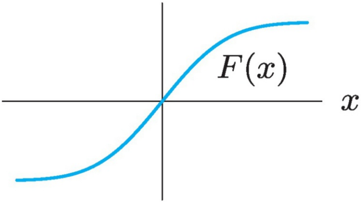
Example. If v is the velocity of a car and s is its position, then $v=ds/dt$ and s is an antiderivative of v. As before, $s+C$ is an antiderivative of v for any constant C. In terms of the car, adding C to s is equivalent to adding C to the odometer reading. Adding a constant to the odometer reading simply means measuring distance from a different point, which does not alter the car?s velocity.

Visualizing Antiderivatives Using Slopes. Example

Sketch a graph of the antiderivative F of $f(x) = e^{-x^2}$ with $F(0) = 0$.
Solution. The graph of $f(x) = e^{-x^2}$ is shown below. The slope of the antiderivative $F(x)$ is given by $f(x)$. Since $f(x)$ is always positive, the antiderivative $F(x)$ is always increasing. Since $f(x)$ is increasing for negative x, we know that $F(x)$ is concave up for negative x. Since $f(x)$ is decreasing for positive x, we know that $F(x)$ is concave down for positive x. Since $f(x) \to 0$ as $x \to -\infty$, the graph of $F(x)$ levels off at both ends.





Which of the following graphs (a)-(d) could represent an antiderivative of the function shown in Figure 6.1?

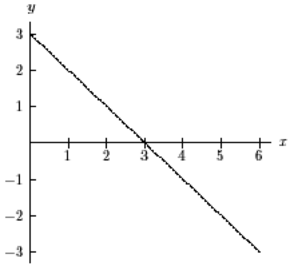
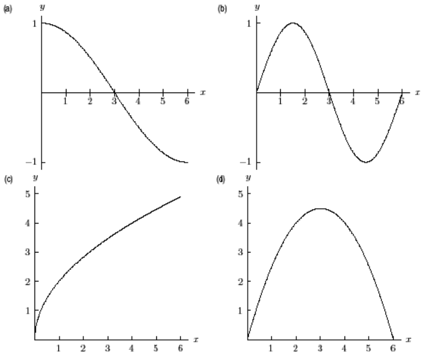


Figure 6.1



Answer

Because the graph of the function in Figure 6.1 is decreasing, the graph of the antiderivative must be concave down.
Since the function in Figure 6.1 is positive for $x < 3$, zero for $x = 3$, and negative for $x > 3$, the antiderivative has a local maximum at $x = 3$.

Clicker question #2

Which of the following graphs (a)-(d) could represent an antiderivative of the function shown in Figure 6.2?

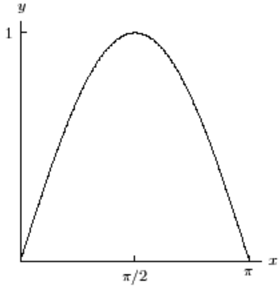
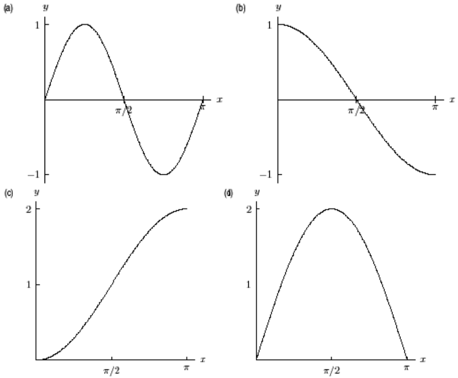


Figure 6.2

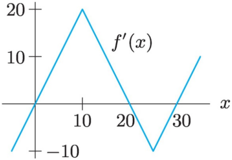


Answer to Clicker Question #2

(c). Because the graph in Figure 6.2 is always positive on this interval, the antiderivative must be increasing for this interval.

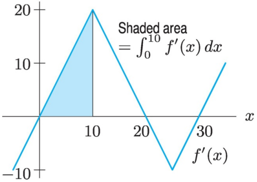
Computing Values of an Antiderivative Using Definite Integrals

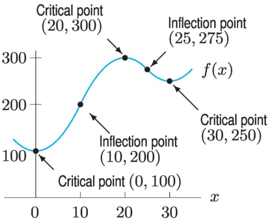
Figure 6.7 is the graph of the derivative $f'(x)$ of a function $f(x)$. It is given that $f(0) = 100$. Sketch the graph of $f(x)$, showing all critical points and inflection points of f and giving their coordinates.



Solution. The critical points of f occur at $x = 0$, $x = 20$, and $x = 30$, where $f'(x) = 0$. The inflection points of f occur at $x = 10$ and $x = 25$, where $f'(x)$ has a maximum or minimum. To find the coordinates of the critical points and inflection points of f, we evaluate $f(x)$ for $x = 0, 10, 20, 25, 30$. Using the Fundamental Theorem, we can express the values of $f(x)$ in terms of definite integrals. We evaluate the definite integrals using the areas of triangular regions under the graph of $f'(x)$, remembering that areas below the x-axis are subtracted.

Computing an Antiderivative Using Definite Integrals





Solution (cont.) Since $f(0) = 100$, the Fundamental Theorem gives us the following values of f, which are marked in Figure 6.9.

$$f(10) = f(0) + \int_0^{10} f'(x)dx = 100 + \text{shaded area} = 100 + \frac{1}{2} \cdot 10 \cdot 20 = 200$$

$$f(20) = f(10) + \int_{10}^{20} f'(x)dx = 200 + \frac{1}{2} \cdot 10 \cdot 20 = 300$$

$$f(25) = f(20) + \int_{20}^{25} f'(x)dx = 300 - \frac{1}{2} \cdot 5 \cdot 10 = 275$$

$$f(30) = f(25) + \int_{25}^{30} f'(x)dx = 275 - \frac{1}{2} \cdot 5 \cdot 10 = 250.$$

Consider the graph of $f'(x)$ in Figure 6.4. Which of the functions with values from Table 6.1 could represent $f(x)$?

Table 6.1

	x	0	2	4	6
(a)	$g(x)$	1	3	4	3
(b)	$h(x)$	5	7	8	7
(c)	$j(x)$	32	34	35	34
(d)	$k(x)$	-9	-7	-6	-7

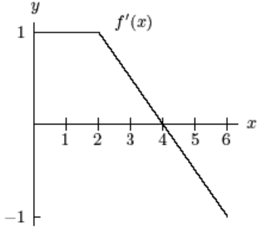


Figure 6.4

Consider the graph of $f'(x)$ in Figure 6.4. Which of the functions with values from Table 6.1 could represent $f(x)$?

Table 6.1

	x	0	2	4	6
(a)	$g(x)$	1	3	4	3
(b)	$h(x)$	5	7	8	7
(c)	$j(x)$	32	34	35	34
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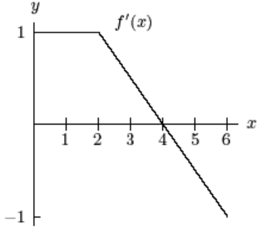
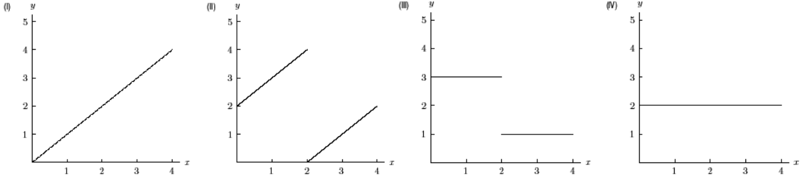


Figure 6.4

Answer. (a), (b), (c), (d). Note that only relative values of functions are important for this problem, not the actual values.

Clicker

Graphs of the derivatives of four functions are shown in (I) ? (IV). For the functions (not the derivative) list in increasing order which has the greatest change in value on the interval shown.



(A) (I), (IV), (III), (II)

(B) (I), (IV), (II), (III)

(C) (I) = (II), (IV), (III)

(D) (I) = (II), (III) = (IV)

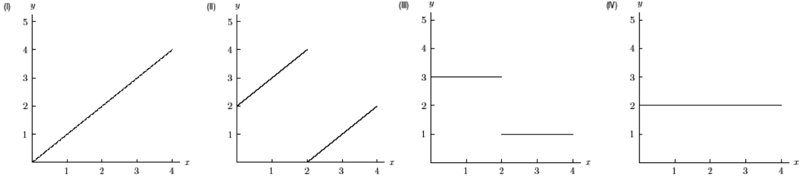
(E) (I) = (II) = (III) = (IV)

Answer

(E). The ordering will be given by the values of the area under the derivative curves. These areas are:
(I) $(1/2)(4)(4) = 8$, (II) $(1/2)(2)(4+2) + (1/2)(2)(2) = 8$, (III) $2(3) + 2(1) = 8$, and (IV) $4(2) = 8$.

Example

Graphs of the **derivatives** of four functions are shown in (I) ? (IV). For the **functions** (not the derivative) list in increasing order which has the greatest change in value on the interval shown.



(A) (I), (III), (IV), (II)

(B) (I) = (III), (IV), (II)

(C) (IV), (I) = (III), (II)

(D) (I) = (III) = (IV), (II)

(E) (I) = (II) = (III) = (IV)

Answer.

(D). The ordering will be given by the values of the area under these curves. These areas are (I) $4(2) + (1/2)(2)(2) = 10$, (II) $4(2) + 2(1/2)(2)(2) = 12$, (III) same as (I), 10, and (IV) $(1/2)(2)(2 + 4) + 2(2) = 10$.

Example

Suppose $F'(t) = t\cos t$ and $F(0) = 2$. Find $F(b)$ at the points $0, 0.1, 0.2, ..., 1.0.$

Solution. From FTC with $f(t) = F'(t) = t\cos t$ and $a = 0$ we get

$$F(b) - F(0) = \int_0^b t\cos t\,dt$$

so

$$F(b) = 2 + \int_0^b t\cos t\,dt.$$

We estimate the integral numerically, so we get the table of values below.

b	0	0.5	0.6	0.7	0.8	0.9	1
$F(b)$	2	2.117	2.164	2.216	2.271	2.327	2.382.