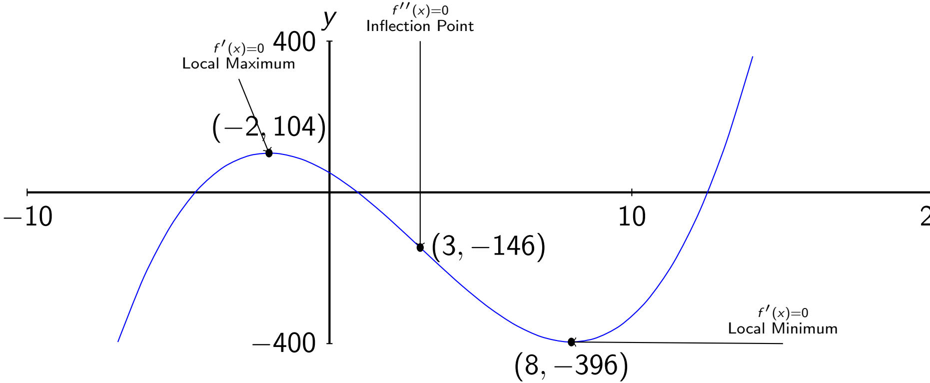
Calculus 1
Applications of the First and Second Derivative (page 102)

Do you recall these connections among a function and its derivatives?

	> 0	< 0
$f(x)$	the graph is above the x-axis	the graph is below the x-axis
$f'(x)$	f is increasing	f is decreasing
$f''(x)$	f is concave up	f is concave down

We now wish to expand our uses for derivatives to answer more problems.



$$f(x) = x^3 - 9x^2 - 48x + 52$$
$$f'(x) = 3x^2 - 18x - 48$$
$$f''(x) = 6x - 18$$

Local Maximum or Minimum

Definition

Let f be a function and p a point in the domain of f.

- f is said to have a *local minimum* at p if $f(p)$ is less than or equal to the values of f for points near p.
- f is said to have a *local maximum* at p if $f(p)$ is greater than or equal to the values of f for points near p.

Definition

Let f be a function and p a point in the domain of f. A *critical point* is a point p such that $f'(p)$ is either 0 or undefined. In addition, the point $(p, f(p))$ on the graph of f is also called a critical point. A *critical value* of f is the value, $f(p)$, at a critical point, p.

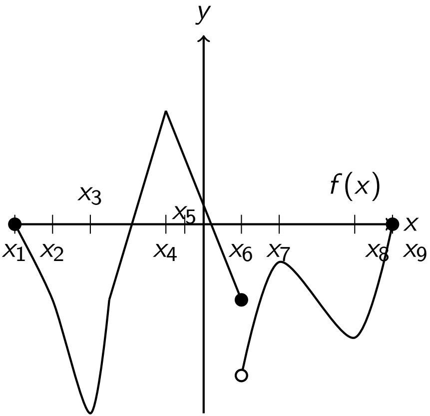
Local Maximum or Minimum

Local Extrema and Critical Points

Suppose f is defined on an interval and has a local maximum or minimum at the point $x = a$, which is not an endpoint of the interval. If f is differentiable at $x = a$, then $f'(a) = 0$. Thus, a is a critical point.

Local Maximum or Minimum

How many local minima does f have?



Clicker

Concerning the graph of the function in the figure, which of the following statements is true?

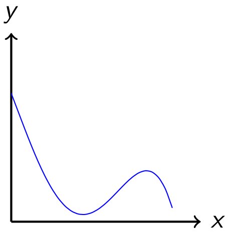
a) The derivative is zero at two values of x, both being local maxima.

b) The derivative is zero at two values of x, one is a local maximum while the other is a local minimum.

c) The derivative is zero at two values of x, one is a local maximum on the interval while the other is neither a local maximum nor a minimum.

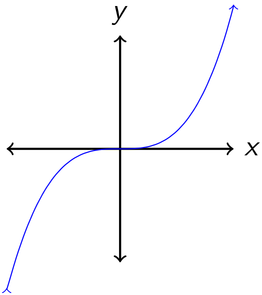
d) The derivative is zero at two values of x, one is a local minimum on the interval while the other s neither a local maximum nor a minimum.

e) The derivative is zero only at one value of x where it is a local minimum.



Warning!

Not every critical point is a local maximum or local minimum. Consider $f(x) = x^3$, which has a critical point at $x = 0$. The derivative, $f'(x) = 3x^2$, is positive on both sides of $x = 0$, so f increases on both sides of $x = 0$, and there is neither a local maximum nor a local minimum at $x = 0$.

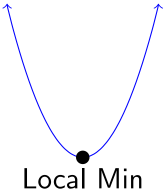


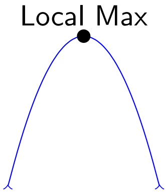
Local Maximum or Minimum

The First-Derivative Test for Local Maxima and Minima

Suppose p is a critical point of a continuous function f and $f'(p) = 0$.

- If f *if* changes from decreasing to increasing at p, then f has a local minimum at p.
- If f changes from increasing to decreasing at p, then f has a local maximum at p.

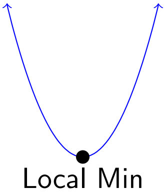


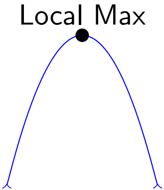


The Second-Derivative Test for Local Maxima and Minima

Suppose p is a critical point of a continuous function f and $f'(p) = 0$.

- If f is concave up at p, then f has a local minimum at p.
- If f is concave down at p, then f has a local maximum at p.





Concavity and Inflection Points

A point, p, at which the graph of a continuous function, f, changes concavity is called an inflection point of f.

Suppose f'' is defined on both sides of a point p:

- If f'' is zero or undefined at p, then p is a possible inflection point.
- To test whether p is an inflection point, check whether f'' changes sign at p.

Example

For $x \geq 0$, find the local maxima and minima and inflection points for $g(x) = xe^{-x}$ and sketch the graph of g.

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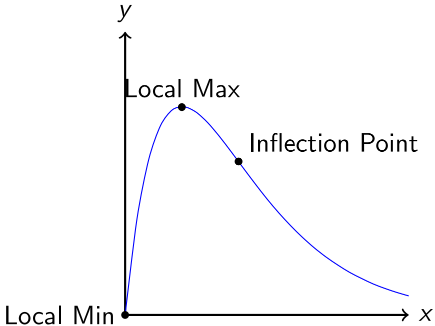
Solution

Taking derivatives and simplifying we have

$$g'(x) = (1-x)e^{-x}$$

$$g''(x) = (x-2)e^{-x}$$

So $x = 1$ is a critical point, and $g' > 0$ for $x < 1$ and $g' < 0$ for $x > 1$. Hence g increases to a local maximum at $x = 1$ and then decreases. Since $g(0) = 0$ and $g(x) > 0$ for $x > 0$, there is a local minimum at $x = 0$. Also, $g(x) \to 0$ as $x \to \infty$. There is an inflection point at $x = 2$ since $g'' < 0$ for $x < 2$ and $g'' > 0$ for $x > 2$.



Inflection Points and Local Maxima and Minima of the Derivative

Suppose a function f has a continuous derivative. If f'' changes sign at p, then f has an inflection point at p, and f' has a local minimum or a local maximum at p.

Practice

Find the inflection points of $f(x) = 4x^4 + 39x^3 - 15x^2 + 8$.

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SOLUTION
We find the x-coordinates of the inflection points by determining where $f''(x)$ changes sign. $f'(x) = 4 \cdot 4x^3 + 39 \cdot 3x^2 - 15 \cdot 2x$, so $f''(x) = 4 \cdot 4 \cdot 3x^2 + 39 \cdot 3 \cdot 2x - 30 = 6(8x - 1)(x + 5)$. So we see that $f''(x)$ changes sign at $x = 1/8$ and $x = -5$.

Practice

Find and classify the critical points of $f(x) = 2x^4(6-x)^5$ as local maxima and minima.

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SOLUTION
Differentiating using the product rule gives
$f'(x) = 2\left(4x^3(6-x)^5 - 5x^4(6-x)^4\right)$. Critical points are where this is zero, or, where $2x^3(6-x)^4\left(4(6-x)-5x\right) = 2x^3(6-x)^4\left(24-9x\right) = 0$. So critical points are $x = 0$, $x = 6$ and $x = 24/9$. Because 4, the multiplicity of the root $x = 6$, is even, the derivative does not change sign at $x = 6$. Checking points on either side of the other two critical points shows that $x = 0$ is a local minimum (f' changes from negative to positive) and $x = 24/9$ is a local maximum.

Practice

Find constants a and b in the function $f(x) = axe^{bx}$ such that $f(\frac{1}{6}) = 1$ and the function has a local maximum at $x = \frac{1}{6}$.

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SOLUTION

Using the product rule on the function $f(x) = axe^{bx}$, we have $f'(x) = ae^{bx} + abxe^{bx} = ae^{bx}(1 + bx)$. We want $f(\frac{1}{6}) = 1$, and since this is to be a maximum, we require $f'(\frac{1}{6}) = 0$. These conditions give

$$f(1/6) = \frac{a}{6}e^{b/6} = 1$$

and

$$f'(1/6) = ae^{b/6}(1 + b/6) = 0.$$

Since $ae^{b/6}$ is non-zero, we can divide both sides of the second equation by $ae^{b/6}$ to obtain $1 + \frac{b}{6} = 0$. This implies $b = -6$. Plugging $b = -6$ into the first equation gives us $a(\frac{1}{6})e^{-1} = 1$, or $a = 6e$. How do we know we have a maximum at $x = \frac{1}{6}$ and not a minimum? Since $f'(x) = ae^{bx}(1 + bx) = ()e^{-6x}(1 - 6x)$, and $(6e)e^{-6x}$ is always positive, it follows that $f'(x) > 0$ when $x < \frac{1}{6}$ and $f'(x) < 0$ when $x > \frac{1}{6}$. Since f' is positive to the left of $x = \frac{1}{6}$ and negative to the right of $x = \frac{1}{6}$, $f(\frac{1}{6})$ is a local maximum.