Calculus I
Families of Functions

University of Nebraska-Lincoln

Clicker question #1

Suppose $f'(3) = 0$ and $f''(3) = 0$. Which of the following statements is true?

(A) f has a maximum at 3.

(B) f has an inflection point at 3.

(C) f has a minimum at 3.

(D) Not enough information given.

Problem # 1

A manufacturer needs to to produce a cylindrical container with a capacity of 1600 cm^3.The top and the bottom of the container are made from material that costs $\$.05/\text{cm}^2$, while the sides of the container are made from material costing $\$0.03/\text{cm}^2$. Find the dimensions that will minimize the company's cost of producing this container.

Clicker question #2

From each corner of a square piece of sheet metal of side 1, we remove a small square and turn up the edges to form an open box. What are the dimensions (length, width, height) of the box with largest volume?

(A) $1, 1, 0$ (there is no box that would maximize the volume)

(B) $\frac{2}{3}, \frac{2}{3}, \frac{1}{6}$

(C) $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$

(D) $\frac{3}{4}, \frac{3}{4}, \frac{1}{8}$.

Answer:

Consider the volume function

$$f(x) = (1 - 2x)^2 x$$

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$$f(x) = (1 - 2x)^2 x$$

which has the derivative $f'(x) = 12x^2 - 8x + 1$. Critical points are $x = 1/6$ and $x = 1/2$, so to find the maximum for this function we compute the second order derivative

$$f''(x) = -8 + 24x$$

which has to be negative at a maximum. Hence, we must have $x \leq \frac{1}{3}$, so we choose $x = 1/6$ as our solution. For $x = 1/6$ the dimensions of the box are $2/3, 2/3, 1/6$.

Families of functions: Motion Under Gravity

The position of an object moving vertically under the influence of gravity can be described by a function in the two-parameter family:

$$y = -4.9t^2 + v_0 t + y_0,$$

where t is time in seconds and y is the distance in meters above the ground. Why do we need the parameters v_0 and y_0?

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where t is time in seconds and y is the distance in meters above the ground. Why do we need the parameters v_0 and y_0? Notice that at time $t = 0$ we have $y = y_0$. Thus the parameter y_0 gives the height above ground of the object at time $t = 0$.

Since $\dfrac{dy}{dt} = -9.8t + v_0$, the parameter v_0 gives the velocity of the object at time $t = 0$. From this equation we see that $\dfrac{dy}{dt} = 0$ when $t = v_0/9.8$. This is the time when the object reaches its maximum height.

General families of functions

Any function that depends on a parameter (or more!) will generate a *family of functions*.

Examples:

- $f(x) = mx + n$
- $g(x) = ax^2 + bx + c$
- $h(x) = A\sin(Bx)$
- $j(x) = C\cos(Dx)$

Clicker question #3

Let $f(x) = ax + \dfrac{b}{x}$. If $a = 9$ and $b = 4$, what are the critical points of $f(x)$?

(A) $-4/9$

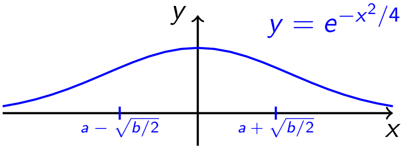
(B) 0

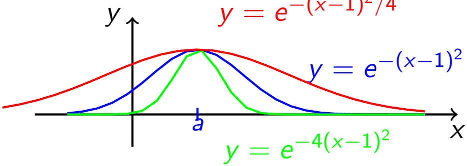
(C) $\pm 4/9$

(D) No critical points

(E) $\pm 2/3$

The Bell-Shaped Curve $y = e^{-(x-a)^2/b}$

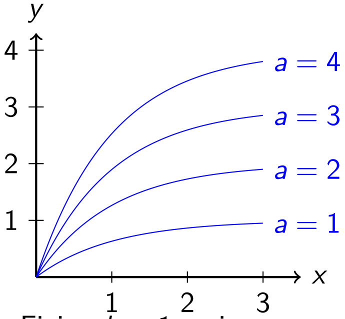




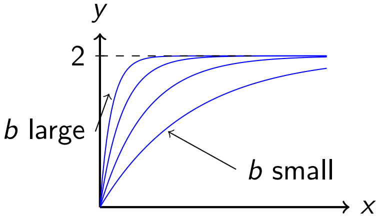
- Symmetry about a and a maximum at $x = a$.
- Inflection points at $x = a \pm \sqrt{b/2}$. At the inflection points, $y = e^{-1/2} \approx 0.6$.
- The parameter a determines the location of the center of the bell and b determines how narrow or wide the bell is. If b is small, then the inflection points are close to a and the bell is sharply peaked near a; if b is large, the inflection points are farther away from a and the bell is spread out.

Exponential Model with a Limit $y = a(1 - e^{-bx})$

We consider $a, b > 0$. The y-values in the graphs represent quantities that are increasing by leveling off at the value of a as $x \to \infty$.



Fixing $b = 1$ varing a

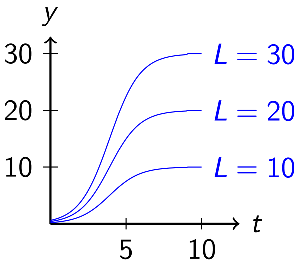


Fixing $a = 2$ varying b

The Logistic Model

For positive constants L, A and k, a **logistic function** has the form

$$y = f(t) = \frac{L}{1 + Ae^{-kt}}$$

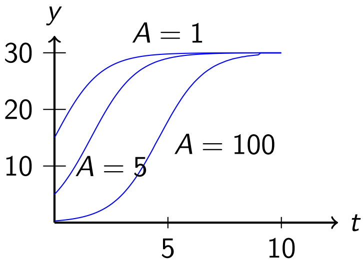


Fixing $A = 50$, $k = 1$, and varying L

The values of y level off as $t \to \infty$ to L because $Ae^{-kt} \to 0$ as $t \to \infty$. Thus $y = L$ is called the limit value or carrying capacity, and represents the maximum sustainable population.

The Logistic Model (continued)

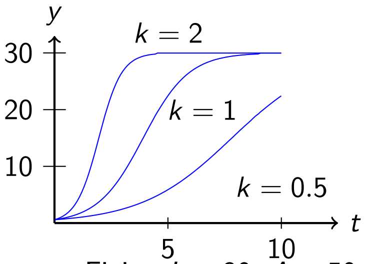
$$y = f(t) = \frac{L}{1 + Ae^{-kt}}$$



Fixing $L = 30$, $k = 1$, and varying A

We investigate the effect of parameter A with k and L fixed. The parameter A alters the point at which the curve intercepts the y-axis. At $t = 0$ we have $y = \frac{L}{1+A}$.

The Logistic Model (continued)



Fixing $L = 30$, $A = 50$, and varing k

We investigate the effect of parameter k with A and L fixed. The parameter k affects the rate at which the function approaches the limiting value L. If k is small, the graph rises slowly; if k is large the graph rises steeply. At $t = 0$ we have

$$\frac{dy}{dt} = \frac{LAk}{(1 + A)^2}$$

so the initial slope of a logistic curve is proportional to k.

Clicker question #4

The figure below shows two members of the logistic family $f(t) = \frac{L}{1+Ae^{-kt}}$. Which parameters are the same for the two graphs?

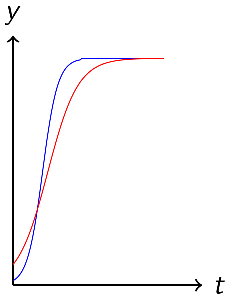
(a) None

(b) L

(c) k

(d) A

(e) both L and k.



Wrapping up

- Today we finished optimization: setting up the problem (to maximize/minimize) and solving it. What are the steps of an optimization problem?

Wrapping up

- Today we finished optimization: setting up the problem (to maximize/minimize) and solving it. What are the steps of an optimization problem?
- We also covered families of functions (section 4.4).
- Clicker question #5: How comfortable are you with the material on maxima/minima (sections 4.1–4.3)?
  (A) Very; (B) Quite comfortable; (C) Average; (D) Not quite;
  (E) It's not good – I need to get some help.

Final announcements

- The deadline for passing the Gateway Exam is October 24.
- Solve the suggested problems and webwork from sections 4.2 and 4.3.
- For next time read section 4.6 (Related rates). The lecture will start with a clicker question from your reading assignment!

Enjoy the midsemester break! (No classes on Monday and Tuesday 10/17-10/18)