Calculus 1
Section 4.7 L'Hopital's Rule

L'Hopital's Rule

General form of L'Hopital's rule

If f and g are differentiable, and $f(a) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

provided the limit on the right exists.

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Example

Calculate: $\lim_{t \to 0} \dfrac{e^t - 1 - t}{t^2}$.

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Solution

Here $f(t) = e^t - 1 - t$ and $g(t) = t^2$. Then $f(0) = 0$ and $g(0) = 0$ so apply L'Hopital's rule:

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t\to 0} \frac{e^t - 1}{2t}.$$

Since $f'(0) = g'(0) = 0$ we can use L'Hopital's rule again:

$$\lim_{t\to 0} \frac{e^t - 1 - t}{t^2} = \lim_{t\to 0} \frac{e^t - 1}{2t} = \lim_{t\to 0} = \frac{e^t}{2} = \frac{e^0}{2} = \frac{1}{2}.$$

Clicker Question 1

Suppose $f(a) = g(a) = 0$. According to L'Hopital's rule $\dfrac{f(a)}{g(a)} = \dfrac{f'(a)}{g'(a)}$.

(a) True

(b) False

(c) Neither true or false

Clicker Question 1

Suppose $f(a) = g(a) = 0$. According to L'Hopital's rule $\dfrac{f(a)}{g(a)} = \dfrac{f'(a)}{g'(a)}$.

(a) True

(b) False

(c) Neither true or false

False. L'Hopital's rule concerns the equality of the **limit** not the equality of the functions.

Clicker Question 2

Suppose $f(a) = g(a) = 0$. Let $Q(t) = \dfrac{f(x)}{g(x)}$. Then $\lim_{x \to a} Q(x) = \lim_{x \to a} Q'(x)$.

(a) True

(b) False

(c) Neither true or false

Clicker Question 2

Suppose $f(a) = g(a) = 0$. Let $Q(t) = \dfrac{f(x)}{g(x)}$. Then $\lim_{x \to a} Q(x) = \lim_{x \to a} Q'(x)$.

(a) True

(b) False

(c) Neither true or false

False. L'Hopital's rule says apply the derivative to the numerator and denominator separately.

More examples

1. $\lim_{x\to 1} \dfrac{x^3+2x^2-2x-1}{x^3+4x^2-x-4}$
2. $\lim_{x\to 1} \dfrac{4x^3-6x+2}{e^{x-1}}$
3. $\lim_{x\to 1} \dfrac{\sin(\pi x)}{x^2-5x+4}$

Example 1

$$\lim_{x \to 1} \frac{x^3 + 2x^2 - 2x - 1}{x^3 + 4x^2 - x - 4}$$

Always first verify that both the numerator and denominator are 0, in this case by plugging in $x = 1$. Using L'Hopital's rule:

$$\lim_{x \to 1} \frac{x^3 + 2x^2 - 2x - 1}{x^3 + 4x^2 - x - 4} = \lim_{x \to 1} \frac{3x^2 + 4x - 2}{3x^2 + 8x - 1} = \frac{5}{10} = \frac{1}{2}$$

Example 2

$$\lim_{x \to 1} \frac{4x^3 - 6x + 2}{e^{x-1}}$$

Again first plug in $x = 1$, in this case $e^{1-1} = e^0 = 1$, the denominator is 1 thus:

$$\lim_{x \to 1} \frac{4x^3 - 6x + 2}{e^{x-1}} = \frac{0}{1} = 0$$

No need for L'Hopital's rule.

Example 3

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x^2 - 5x + 4}$$

First plug in $x = 1$ to both the numerator and denominator. Here we get 0 for both so use L'Hopital's rule:

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x^2 - 5x + 4} = \lim_{x \to 1} \frac{\cos(\pi x)\pi}{2x - 5} = \frac{-\pi}{-3} = \frac{\pi}{3}$$

Clicker Question 3

Evaluate:

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

(a) 0/0

(b) 0

(c) 2

(d) 1/2

(e) Not sure

L'Hopital's rule applies to limits involving infinity

provided that f and g are differentiable. For a any real number or $\pm\infty$,

- When $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ or
- When $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to\infty} g(x) = 0$.

it can be shown that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(a)}{g'(a)}$$

provided the limit on the right exists.

Examples

1. $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}}$
2. $\lim_{x \to \infty} (x + e^{2x})e^{-2x}$
3. $\lim_{x \to \infty} \frac{(\ln(x))^3}{x^2}$

Example 1

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$$

Here both the numerator and the denominator are going to ∞ so apply L'Hopital's rule:

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/(2x^{1/2})} = \lim_{x\to\infty} \frac{2x^{1/2}}{x} = \lim_{x\to\infty} \frac{2}{x^{1/2}} = 0$$

Example 2

$$\lim_{x\to\infty}(x+e^{2x})e^{-2x}$$

Here we initially do not get ∞/∞ instead we get $\infty \times 0$ but we we can rewrite $e^{-2x} = 1/e^{2x}$ and get both numerator and denominator going to ∞, then we can apply L'Hopital's Rule:

$$\lim_{x\to\infty}(x+e^{2x})e^{-2x} = \lim_{x\to\infty}\frac{(x+e^{2x})}{e^{2x}} = \lim_{x\to\infty}\frac{(1+2e^{2x})}{2e^{2x}}$$

Still get ∞/∞ so apply L'Hopital's rule again:

$$\lim_{x\to\infty}\frac{(1+2e^{2x})}{2e^{2x}} = \lim_{x\to\infty}\frac{4e^{2x}}{4e^{2x}} = 1$$

Example 3 $\lim_{x\to\infty} \frac{(\ln(x))^3}{x^2}$

Here both the numerator and denominator are going to ∞ so apply L'Hopital's rule:

$$\lim_{x\to\infty} \frac{(\ln(x))^3}{x^2} = \lim_{x\to\infty} \frac{2(\ln(x))^2(1/x)}{2x} = \lim_{x\to\infty} \frac{(\ln(x))^2}{x^2}$$

The numerator and denominator still both going to ∞ so apply L'Hopital's rule again:

$$\lim_{x\to\infty} \frac{(\ln(x))^2}{x^2} = \lim_{x\to\infty} \frac{2\ln(x)(1/x)}{2x} = \lim_{x\to\infty} \frac{\ln(x)}{x^2}$$

The numerator and denominator still both going to ∞ so apply L'Hopital's rule again:

$$\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

Dominance: Powers, Polynomials, Exponentials, and Logarithms'

Check that any exponential function of the form e^{kx} with $k > 0$ dominates any power function Ax^p with A and p positive as $x \to \infty$.

Solution

We apply L'Hopital's rule repeatedly to the fraction $\dfrac{Ax^p}{e^{kx}}$:

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = \lim x \to \infty \frac{Apx^{p-1}}{ke^{kx}} = \lim_{x\to\infty} \frac{Ap(p-1)x^{p-2}}{k^2e^{kx}} = \ldots$$

Keep applying L'Hopital's rule until the power of x is no longer positive, the denominator is always going to ∞ thus

$$\lim_{x\to\infty} \frac{Ax^p}{e^{kx}} = 0$$

so e^{kx} dominates Ax^p.

Clicker Question 4

Which is dominate $\sqrt{x}$ or $\ln(x)$?

(a) $\ln(x)$

(b) $\sqrt{x}$

(c) neither

(d) Not sure how to start

Defintion of e

Evaluate

$$\lim_{x\to\infty}(1+\frac{1}{x})^x$$

This does not initially look like a use for L'Hopital's rule, to use the rule we are going to use the properties of ln. Write $y=(1+\frac{1}{x})^x$ and find the limit of $\ln(y)$:

$$\lim_{x\to\infty}\ln(y)=\lim_{x\to\infty}\ln((1+\frac{1}{x})^x)=\lim_{x\to\infty}x\ln(1+\frac{1}{x})$$

Why do we rewrite this way? So we can bring the power down, still not in the form 0/0 so again rewrite:

$$\lim_{x\to\infty}x\ln(1+\frac{1}{x})=\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x}$$

Now we get 0/0 and apply L'Hopital's rule:

$$=\lim_{x\to\infty}\frac{\frac{1}{(1+1/x)}(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}\frac{1}{1+1/x}=1$$

Since $\lim_{x\to\infty}\ln(y)=1$ we have $\lim_{x\to\infty}y=e^1=e$ so

$$\lim_{x\to\infty}(1+\frac{1}{x})^x=e$$

Example

Arrange in order by dominance as $x \to \infty$, from least to most dominant.

1. x^{100}
2. $x(\ln(x))^2$
3. $\dfrac{e^{2x}}{x}$
4. x^2
5. $e^x \ln(x)$

Example

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1. x^{100}
2. $x(\ln(x))^2$
3. $\dfrac{e^{2x}}{x}$
4. x^2
5. $e^x \ln(x)$

In order from least to most dominant

$$x(\ln(x))^2 < x^2 < x^{100} < e^x \ln(x) < \frac{e^{2x}}{x}$$