Clicker Survey

How do you feel about the first test?

(a) Great
(b) Good
(c) Average
(d) No so good
(e) Terrible

Clicker Survey

Do you think you need to:

(a) Study about the same for the next test
(b) Study harder for the next test
(c) Study less for the next test
(d) Study Sooner for the next test

We have already used inverse functions to solve things such as $\tan(\arcsin(3/5)) = ?$
We now wish to take advantage of the fact that

$$f(f^{-1}(x)) = x$$

to find some derivatives of new functions.
Let's practice the idea on something we already know:

$$\text{If } f(x) = \sqrt{x}, \text{ then } (f(x))^2 = x.$$

Take the derivative of both sides:

$$2(f(x))\frac{df}{dx} = 1$$

$$\text{Therefore,} \qquad \frac{df}{dx} = \frac{1}{2f(x)} = \frac{1}{2\sqrt{x}}$$

as expected.

The Derivative of $\ln(x)$

We use the chain rule to differentiate an identity involving $\ln x$.
Since $e^{\ln x} = x$, we can differentiate both sides. On the one hand we have:

$$\frac{d}{dx} e^{\ln x} = \frac{d}{dx} x = 1.$$

Also, by the chain rule,

$$\frac{d}{dx} e^{\ln x} = e^{\ln x} \frac{d}{dx} \ln(x) = x \frac{d}{dx} \ln(x)$$

Thus, dividing by x,

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Clicker Question

Find the derivative of $\ln(x^2 + 2)$.

(a) $f'(x) = \frac{1}{ln(x^2+2)}$

(b) $f'(x) = \frac{1}{2x}$

(c) $f'(x) = \frac{1}{x^2+2}$

(d) $f'(x) = \frac{1}{x^2+2}2x$

Derivative of a^x Revisited

In Section 3.2, we saw that the derivative of a^x is proportional to a^x. Now we see another way of calculating the constant of proportionality. We use the identity

$$\ln(a^x) = x \ln a$$

Differentiating both side, using the chain rule, and remembering that $\ln a$ is a constant, we obtain:

$$\frac{d}{dx}(\ln a^x) = \frac{1}{a^x}\frac{1}{a^x}\frac{d}{dx}a^x = \ln a$$

So, we have the result from Section 3.2 that

$$\frac{d}{dx}a^x = a^x \ln a$$

Derivatives of Inverse Trigonometric Functions

To find $\frac{d}{dx}(\arctan x)$, we use the identity $\tan(\arctan x) = x$. Differentiating both sides gives

$$\frac{d}{dx}(\tan(\arctan x)) = \frac{d}{dx}x = 1$$

But also, from the chain rule,

$$\frac{d}{dx}(\tan(\arctan x)) = \frac{1}{\cos^2(\arctan x)}\frac{d}{dx}(\arctan x)$$

So,

$$\frac{d}{dx}(\arctan x) = \cos^2(\arctan x)$$

Now, using the identity $1 + \tan^2\theta = 1/\cos^2(\theta)$, and setting $\theta = \arctan x$, we find

$$\cos^2(\arctan x) = \frac{1}{1+\tan^2\theta} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}$$

Derivative of the Arcsine and Examples

By a similar argument, we obtain the result

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

Examples

(a) Differentiate $f(t) = \arctan(t^2)$.

(b) Differentiate $g(\theta) = \arcsin(\tan\theta)$.

Solution: Use the chain rule.

$$\frac{d}{dt}\arctan(t^2) = \frac{1}{1+(t^2)^2}\frac{d}{dt}(t^2) = \frac{2t}{1+t^4} \tag{1}$$

Also,

$$\frac{d}{d\theta}\arcsin(\tan\theta) = \frac{1}{\sqrt{1-\tan^2\theta}}\frac{d}{d\theta}(\tan\theta) = \frac{1}{\sqrt{1-\tan^2\theta}}\frac{1}{\cos^2\theta}$$

Derivative of a General Inverse Function

In general, if a function f has a differentiable inverse, f^{-1}, we find its derivative by differentiating $f(f^{-1}(x)) = x$ by using the chain rule, yielding the following result:

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Exercise

Use the table and the fact that $f(x)$ is invertible and differentiable everywhere to find $(f^{-1})'(3)$

x	f(x)	f'(x)
3	1	7
6	2	10
9	3	5

Solution Note that $f^{-1}(3) = 9$, since $f(9) = 3$. Then $(f^{-1})'(3) = 1/f'(9) = 1/5$.

If $f(x)$ is given by the followin table, what is $\frac{d}{dx}(f^{-1}(x))$ evaluated at $x = 1$?

x	-2	-1	0	1	2
f(x)	3	1	4	2	0
f'(x)	1	2	5	3	4

Challenge Problem

$$f(x) = \arctan\left(\frac{\sin^2((\ln(x) + 3))}{3}\right)$$

Find $f'(x)$.