# Math 107-Lecture 17

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# Proving convergence

- Use a known result for functions; E.g.

$$\lim_{x\to\infty} \frac{1}{x^2+1} = 0, \text{ hence, } \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

- Use the sandwich/squeeze theorem; E.g.

$$0 \le \frac{\sqrt{n}+1}{n^6+15n+3} \le \frac{1}{n} \to 0 \text{ as } n \to \infty$$

- Theorem 1 (old): The product between a bounded sequence and one that converges to zero is a sequence that converges to zero. E.g. $(-1)^n \dfrac{1}{n} \to 0$ as $(-1)^n$ is bounded and $\dfrac{1}{n} \to 0$.

## Proving convergence (cont)

- Theorem 2 (new): A bounded and monotone (i.e. it is always increasing, or always decreasing) sequence is convergent.

1. Example 1: $a_n := e^{1/n}$ is convergent because it is
   - bounded: $0 < e^{1/n} < e$
   - decreasing: $a_n = e^{1/n} > e^{1/(n+1)} = a_{n+1}$;

2. Example 2: $b_0 = 2,\ b_{n+1} := \dfrac{b_n}{2}$ is convergent because it is
   - decreasing: $b_{n+1} = \dfrac{b_n}{2} < b_n$
   - bounded: $0 < ... < b_{n+1} < b_n < ... < b_1 < b_0$

# Clicker question #1

What can you tell about the sequence

$$a_0 = 3, \quad a_{n+1} = \frac{2a_n}{3} \quad ?$$

A) unbounded and increasing, hence divergent
B) bounded and decreasing; we can't conclude convergence
C) bounded and increasing; we can't conclude convergence
D) bounded and decreasing, hence convergent
E) bounded and decreasing, hence divergent.

# Geometric Sums/Series

What is

$$S(x) = 1 + x + x^2 + x^3 + ...x^{n-1} + x^n \quad ?$$

Write

$$x \cdot S(x) = x + x^2 + x^3 + ... + x^n + x^{n+1}$$

and subtract the two equalities:

$$S(x) - xS(x) = S(x)(1 - x) = 1 - x^{n+1}.$$

Thus,

$$S(x) = 1 + x + x^2 + x^3 + ...x^{n-1} + x^n = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 1.$$

# What is so geometric about it?

A geometric sum is a sum of the form

$$S_n = a + ax + ax^2 + ... + ax^{n-1}$$

where $a$ is called the first term, $n$ is the number of terms, and $x$ is the ratio of the series. We have

$$S_n = a(1 + x + x^2 + ... + x^{n-1}) = a\,\frac{1 - x^n}{1 - x}, \quad x \neq 1.$$

Cool fact: It is called a geometric sum, because every term in the sequence is the geometric average of its neighbors.
A geometric series is an infinite sum of the form

$$S = \sum_{n=0}^{\infty} ax^n = \lim_{n \to \infty} S_n = \lim_{n \to \infty} a\frac{1 - x^n}{1 - x}, \quad x \neq 1.$$

# Clicker question #2

What is the limit

$$S = \lim_{n\to\infty} S_n = \sum_{n=0}^{\infty} ax^n$$

when $0 < x < 1$?

A) $\infty$
B) $0$
C) $ax$
D) $1$
E) $\dfrac{a}{1-x}$.

## The sum of the geometric series

Let

$$S = \sum_{n=0}^{\infty} ax^n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-x^n}{1-x}, \quad x \neq 1.$$

Then we have:

- If $|x| < 1$ then the series is convergent (i.e. the limit exists and is finite) and its sum is
$$S = \frac{a}{1-x}.$$
- If $|x| > 1$ then the series is divergent

# Applications

Geometric sums/series appear in **a lot** in applications.

Example 1. You invest \$100 in a savings account with 5% annual interest rate. How much do you have after 4 years?

After one year:

$$S_1 = 100 + \frac{5}{100}100 = 105$$

After two years:

$$S_2 = 105 + \frac{5}{100}105 = 110.25$$

After three years:

$$S_3 = 110.25 + \frac{5}{100}110.25 = 115.7625$$

After four years:

$$S_4 = 115.7625 + \frac{5}{100}115.7625 \approx 121.55$$

# A better way ...

Write

$$S_1 = S_0 \cdot (1.05)$$

then

$$S_2 = S_1 \cdot (1.05) = S_0 \cdot (1.05)^2$$

Similarly,

$$S_n = S_0 \cdot (1.05)^n.$$

New question: Assuming that we make \$100 deposits at the beginning of every year, how much would have at the end of 4 years?

$$S_4 + S_3 + S_2 + S_1 + S_0 = 100 \cdot [(1.05)^4 + (1.05)^3 + (1.05)^2(1.05) + 1]$$

# Bouncing balls

A ball is dropped from a height 10 ft and bounces. Each bounce is $3/4$ of the height of the bounce before.

1. Find an expression for the height to which the ball raises after it hits the floor for the $n$th time.
2. What's the **total** distance the ball has traveled when it hits the floor $n$-th time?
3. What's the total distance the ball travels if it keeps on bouncing?
4. How long will this process take? Use the fact that falling from the height of $h$ feet (or bouncing to this height) takes the ball about $\frac{1}{4}\sqrt{h}$ seconds.