# Math 107: Calculus II

Section 5.1: Distance Traveled

University of Nebraska-Lincoln

# Estimate the Distance a Car Travels

A car is moving with increasing velocity. The table below show the velocity every two seconds:

**Table 5.1** *Velocity of car every two seconds*

| Time (sec) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| Velocity (ft/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

We might estimate the distance traveled by assuming a constant velocity in each interval and use the formula

$$\text{distance} = \text{velocity} \times \text{time}$$

At least how far has the car traveled?

$$20 \cdot 2 + 30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 = 360 \text{ feet}$$

At most how far has the car traveled?

$$30 \cdot 2 + 38 \cdot 2 + 44 \cdot 2 + 48 \cdot 2 + 50 \cdot 2 = 420 \text{ feet.}$$

# How Do We Improve Our Estimate?

Velocity of car every second

| Time (sec) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Speed (ft/sec) | 20 | 26 | 30 | 34 | 38 | 41 | 44 | 46 | 48 | 49 | 50 |

$$\begin{aligned}\text{New lower estimate } &= 20\cdot 1 + 26\cdot 1 + 30\cdot 1 + 34\cdot 1 + 38\cdot 1\\ &\quad + 41\cdot 1 + 44\cdot 1 + 46\cdot 1 + 48\cdot 1 + 49\cdot 1\\ &= 376 \text{ feet} > 360 \text{ feet}\end{aligned}$$

$$\begin{aligned}\text{New upper estimate } &= 26\cdot 1 + 30\cdot 1 + 34\cdot 1 + 38\cdot 1 + 41\cdot 1\\ &\quad + 44\cdot 1 + 46\cdot 1 + 48\cdot 1 + 49\cdot 1 + 50\cdot 1\\ &= 406 \text{ feet} < 420 \text{ feet}\end{aligned}$$

The difference between upper and lower estimates is now 30 feet, half of what it was before. By halving the interval of measurement, we have halved the difference between the upper and lower estimates.

The table gives a car's velocity at time t. In each of the four fifteen-minute intervals, the car is either always speeding up or always slowing down. The car's route is a straight line with four towns on it. Town A is 60 miles from the starting point, town B is 70 miles from the starting point, town C is 73 miles from the starting point, and town D is 80 miles from the starting point. For each town, decide if the car has passed it, has not yet passed it, or may have passed it.

| $t$ (minutes) | 0 | 15 | 30 | 45 | 60 |
|---|---|---|---|---|---|
| $v$ (miles per hour) | 60 | 75 | 72 | 78 | 65 |

Which best describes the situation?

A. The car has definitely passed town A.

B. The car has possibly passed all four of the towns

Calculating an upper and lower estimate for the distance traveled, we discover that the car traveled at least 67.25 miles and at most 76.5 miles:

$$\begin{aligned}\text{Lower estimate } &= (60+72+72+65)0.25 = 67.25\\ \text{Upper estimate } &= (75+75+78+78)0.25 = 76.50.\end{aligned}$$

Thus, the car definitely passed town A, may have passed towns B and C, and definitely did not pass town D.

**Comment**

Note that the velocity of the car is not monotone. Therefore the upper and lower estimates for the distance traveled are not equal to the right and left sum estimates.

# Visualizing Distance on the Velocity Graph: Two-Second Data

To visualize the difference between the two estimates, look at the picture and imagine the light rectangles all pushed to the right and stacked on top of each other, giving a difference of $30 \times 2 = 60$.

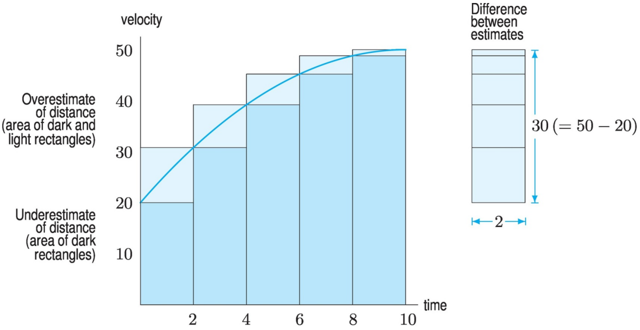
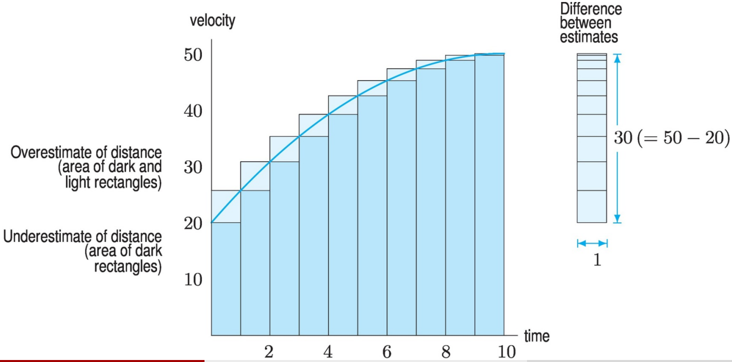



Figure: Velocity measured every 2 seconds.

# Visualizing Distance on the Velocity Graph: One-Second Data

To visualize the difference between the two estimates, look the picture below. This difference can be calculated by stacking the light rectangles vertically, giving a rectangle of the same height as before but of half the width. Its area is therefore half what it was before. Again, the height of this stack is 30, but its width is now 1, giving a difference 30.



If the velocity is positive, the total distance traveled is the area under the velocity curve.

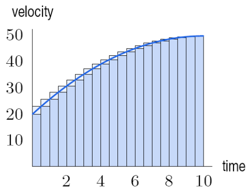



Figure 5.3: Velocity measured every 1/2 second

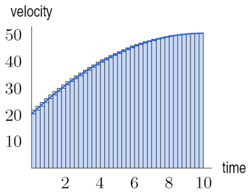



Figure 5.4: Velocity measured every 1/4 second

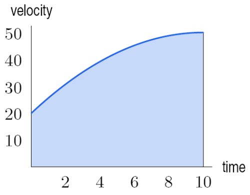
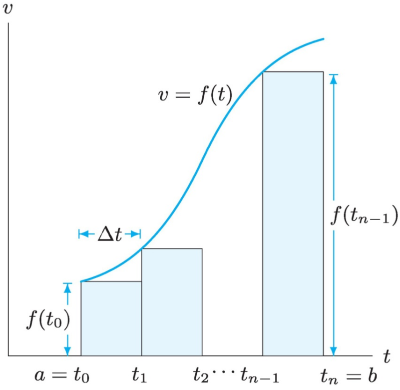
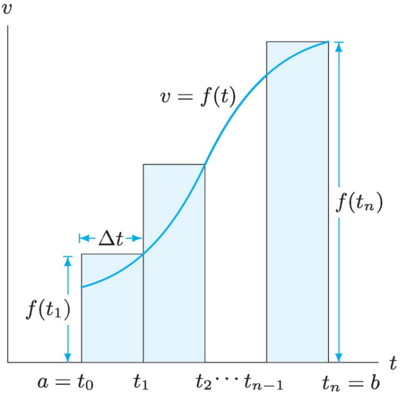



Figure 5.5: Distance traveled is area under curve

# Left- and Right-Hand Sums

If f is an increasing function, as in Figures 5.8 and 5.9, the left-hand sum is an underestimate and the right-hand sum is an overestimate of the total distance traveled. If f is decreasing, as in Figure 5.10 (next slide), then the roles of the two sums are reversed.



Consider the graph below. Give an interval on which the left-hand sum approximation of the area under the curve on that interval is an **underestimate**.

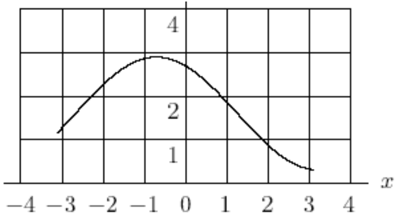



Figure 5.1

- A $[-3, -1]$
- B $[0, 1]$
- C $[0, 2]$
- D $[0, 3]$

# Answer

(a) $[-3, -1]$ because the function is increasing there.

**Comment**

Follow-up Question. What about the right-hand sum? Answer. On the interval $[0, 3]$ the right-hand sum is an underestimate because the function is decreasing there. On the interval $[-3, -1]$ the right-hand sum is an overestimate because the function is increasing there.

# More About Left- and Right-Hand Sums

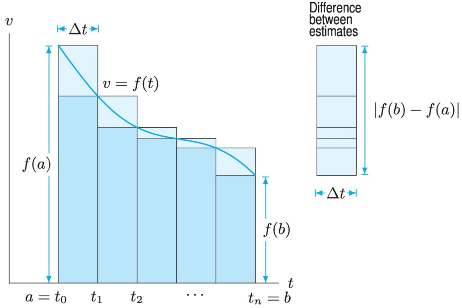



Figure: Figure 5.10: Left and right sums if f is decreasing.

For either increasing or decreasing velocity functions, the exact value of the distance traveled lies somewhere between the two estimates. Thus, the accuracy of our estimate depends on how close these two sums are. For a function which is increasing throughout or decreasing throughout the interval $[a, b]$: Difference between upper and lower estimates $= |f(b) - f(a)|\Delta t$.

## Clicker Question

The following expressions give estimates for the distance a rocket travels, where $v = f(t)$ is an increasing function that gives the rocket's velocity at time t (in seconds). Which expression best matches the given description?
*An underestimate of the distance the rocket travels during the first 2 seconds.*

- A $3f(3)$
- B $f(1) + f(2)$
- C $f(2) + f(3)$
- D $2f(1)$
- E $f(0) + f(1)$

# Answer

Expression (E).

## Clicker Question

The following expressions give estimates for the distance a rocket travels, where $v = f(t)$ is an increasing function that gives the rocket's velocity at time t (in seconds). Which expression best matches the given description?

*An overestimate of the distance the rocket travels from time $t = 1$ to time $t = 3$.*

- A. $3f(3)$
- B. $f(1) + f(2)$
- C. $f(2) + f(3)$
- D. $2f(1)$
- E. $f(0) + f(1)$

# Answer

Expression (C).