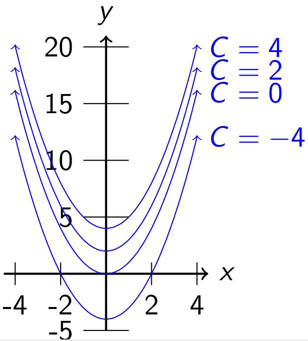
Calculus 1
Section 6.3
Differential Equations and Motion

Differential Equations

An equation of the form $\dfrac{dy}{dx} = f(x)$ is called a **differential equation.** Note it has a derivative in it. This will mean that the answer is a **function** (not a value such as $x = 5$). Finding the **general solution** to the differential equation means finding the general antiderivative $y = F(x) + C$ with $F'(x) = f(x)$.

Example

Find the general solution of the differential equation $\frac{dy}{dx} = 2x$. The antiderivative of $2x$ is $x^2 + C$. What is the antiderivative of $\frac{dy}{dx}$? It should be "y". So the general solution to $\frac{dy}{dx} = 2x$ is $y = x^2 + C$. This is a parabola opening up with its axis of symmetry on the y-axis but its vertex is $(0, C)$ for some constant C. The figure below shows several values of C.



Initial values

If a differential equation has an initial condition, say $y = 0$ when $x = 0$, then it is possible to get a specific solution to the problem. This is called an **initial value problem** and the given condition is called the **initial condition**.

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Example:

If $\frac{dy}{dx} = 2x$ and we have an initial condition $y(3) = 0$.

First we find the general solution, which we already have as $y = x^2 + C$.

Then we substitute the initial condition into the general solution to solve for C.

$0 = 3^2 + C$ or $C = -9$ and get the solution $y = x^2 - 9$.

Clicker Question 1

Find the solution of the initial value problem: $\dfrac{dy}{dx} = \sin(x)$, $y(\pi) = 2$.

(a) $y = 2x - \pi$

(b) $y = -\cos(x) + 3$

(c) $y = -\cos(x) + 1$

(d) $y = -\cos(x) + 2x - \pi$

(e) Unsure

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Solution: First find the general solution $y = -\cos(x) + C$. Then plug in the point $(\pi, 2)$, so $y(\pi) = -\cos(\pi) + C = 2$ thus $C = 1$ and we get $y = -\cos(x) + 1$.

Example

Ice is forming on a pond at a rate given by $\frac{dy}{dt} = k\sqrt{t}$ where y is the thickness of the ice, in cm, at time t, in hours since the ice started forming, and k is a positive constant. Find y as a function of t. You may assume there was no ice when the ice first began forming.

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Solution:

Solve the initial value problem $\frac{dy}{dt} = k\sqrt{t}$, $y(0) = 0$.

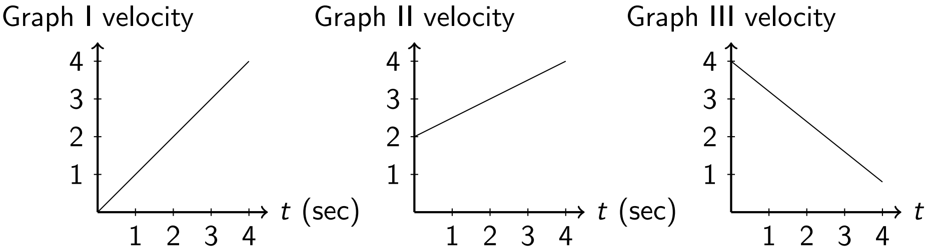
First find the general solution $y(t) = \frac{2}{3}kt^{3/2} + C$.

Then plug in the initial condition $y(0) = \frac{2}{3}k0^{3/2} + C = 0$ so $C = 0$, thus the solution is

$$y(t) = \frac{2}{3}kt^{3/2}.$$

Clicker Question 2

Graphs I-III below show velocity verse time for three different objects. Order graphs I-III in terms of the total distance traveled in four seconds, from greatest to least.

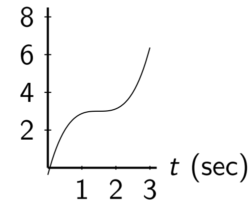


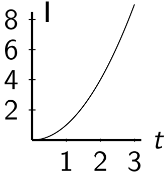
1. I, II, III
2. III, II, I
3. II, III,I
4. II=III,I
5. I=II=III

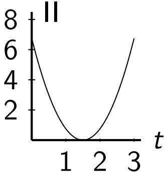
Solution:

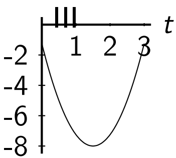
The distance traveled in this situation is the area under the graph. The order from most area to least is: Graph II, Graph III, Graph I or (C).

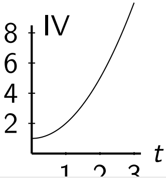
Consider the following velocity graph. Which of the following could be associated graph of position versus time?











(a) I

(b) II

(c) III

(d) IV

(e) I and IV

Solution

(e) The velocity function is positive and increasing so the position function is increasing and concave up. Both I and IV match that, the difference between I and IV is the initial condition.

Equations of Motion

Example

An object is thrown vertically upward with a speed of 10m/sec from a height of 2 meters above the ground. Find the highest point it reaches and the time it hits the ground if you know that the acceleration due to gravity is -9.8m/sec^2.

Solution

We must find the position function $s(t)$. First find the velocity function by solving the initial value problem $\frac{dv}{dt} = -9.8$ with $v(0) = 10$.

The general solution is $v(t) = -9.8t + C$, we know $v = 10$ when $t = 0$ so $C = 10$.

To find the position function we have a second initial value problem:
$\frac{ds}{dt} = -9.8t + 10$ with $s(0) = 2$

The general solution is $s(t) = -4.9t^2 + 10t + C$ with $s(0) = 2$ so $C = 2$.
Thus we have found the position function $s(t) = -4.9t^2 + 10t + 2$

The time that it reaches its highest point is a local max, so find when $s'(t) = 0$ or $v = -9.8t + 10 = 0$, at $t = 10/9.8 \approx 1.02$ sec. The highest point is then $s(1.02) = -4.9(1.02)^2 + 10(1.02) + 2 \approx 7.10$ meters.

The time the object reaches the ground is when $s = 0$, so solve $0 = -4.9t^2 + 10t + 2$.

Using the quadratic formula we get $t \approx -.18$ and2.22sec. Since the time must be positive the time the object reaches the ground again is $t \approx 2.22$.

History of the Equations of Motion

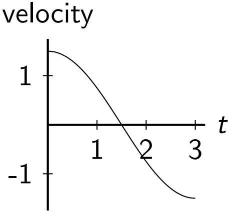
Our derivation of the formulas for the velocity and the position of the body hides an almost 2000-year struggle to understand the mechanics of falling bodies, from Aristotle's Physics to Galileo's Dialogues Concerning Two New Sciences.

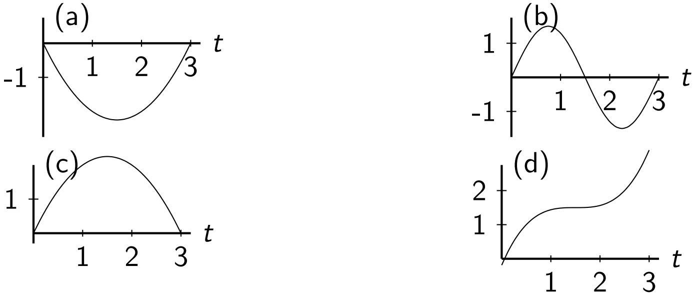
Galileo demonstrated that a body falling under the influence of gravity does so with constant acceleration. Assuming we can neglect air resistance, this constant acceleration is independent of the mass of the body. This last fact was the outcome of Galileo's famous observation around 1600 that a heavy ball and a light ball dropped off the Leaning Tower of Pisa hit the ground at the same time.

Nearly a hundred years after Galileo's experiment, Newton formulated his laws of motion and gravity, which gave a theoretical explanation of Galileo's experimental observation that the acceleration due to gravity is independent of the mass of the body. According to Newton, acceleration is caused by force, and in the case of falling bodies, that force is the force of gravity.

Clicker Question 3

The figure to the right is a graph of velocity vs. time. Which of the figures (a)–(d) could be a graph of position vs. time?





Solution

(c). Because the velocity is zero at $x \approx 1.5$ and because the velocity goes from a postive value to a negative value through this point, there will be a local maximum there.