Calculus I - Lecture 1

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Today's lecture

- Congratulations to those of you who passed the CRA yesterday! If you did not pass yet, please reserve your spot to take it at the Learning Commons today!
- WebWork opens today; please work on it regularly in order not to miss any assignments. Don't forget to meet with your study group!
- Today: exponentials, logarithmic functions; inverse functions.

Proportionality

Directly Proportional

We say y is directly proportional to x if there is a nonzero constant k such that, $y = kx$. This k is called the constant of proportionality.

Inversely Proportional

We say that y is inversely proportional to x if y is proportional to the reciprocal of x, that is, $y = k/x$ for a nonzero constant k.

Clicker Question

Given the values in the table evaluate $f(g(1))$.

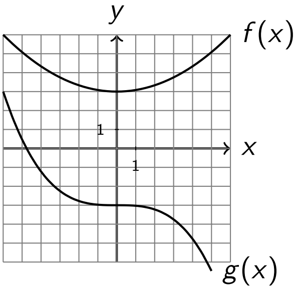
Table : Function Values

x	$f(x)$	$g(x)$
-2	1	-1
-1	2	1
0	-2	2
1	2	0
2	-1	-2

a) -2
b) -1
c) 0
d) 1
e) 2

Clicker Question

Given the values in the graph evaluate $g(f(0))$.



a) -3.6

b) -2.2

c) 0

d) 3.6

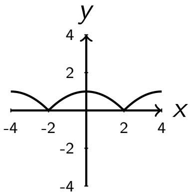
e) undefined

Shifts and Stretches

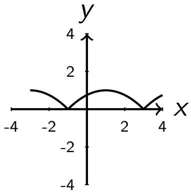
- Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$)
- A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.
- Replacing y by $(y k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x h)$ moves a graph to the right by h (to the left if h is negative).

Clicker Question

The graph given is that of $y = f(x)$. Which could be a graph of $cf(x)$?



a)



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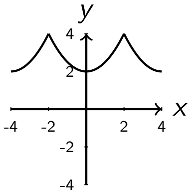

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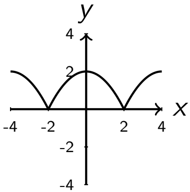
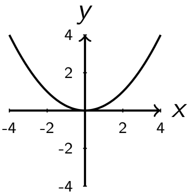
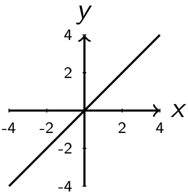


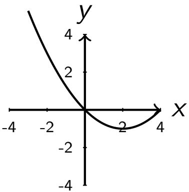
Clicker Question

Which of the following graphs might represent a function that is the sum of the functions represented in the graphs below?





a)



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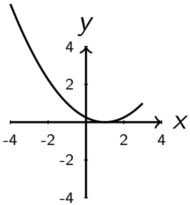
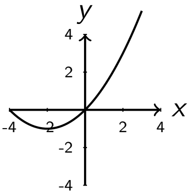
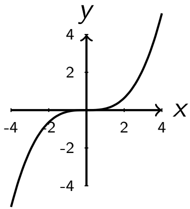


c)

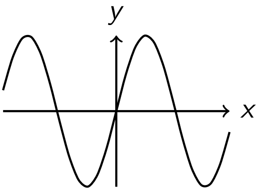


d)

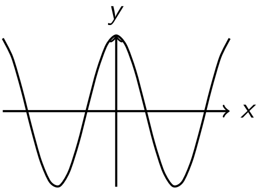


Clicker Question

Which of the graphs might represent the graph of $y = \sin(2x)$ given that the other functions represented are $y = \sin(x)$, $y = \sin(3x)$, $y = \cos(x)$?



a)



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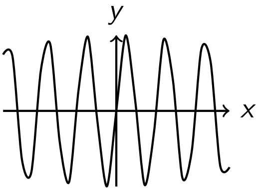

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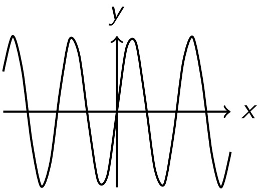


d)

Clicker Question

Which of the functions below is decreasing and concave down on $(0, \infty)$?

(A) $f(x) = e^{1-x}$

(B) $g(x) = \ln(1 + x)$

(C) $h(x) = 5 - (x - 2)^2$

(D) $j(x) = 5 - x^2$

(E) $k(x) = -5e^{-x}$

The exponential function

The equation

$$P(t) = P_0 a^t$$

gives an exponential function with base a. Then

$$\frac{P(t+1)}{P(t)} = \frac{P_0 a^{t+1}}{P_0 a^t} = a = \text{constant growth of growth/decay}$$

Growth: $a > 1$: Doubling time: the time it takes to double the initial amount

Decay $a < 1$: Half-life: the time it takes to decay to half of the initial amount

Example. If $P(t) = 5 \cdot 2^t$ what is the doubling time?

The exponential function in the natural base

We will (naturally!) consider exponentials (and later, logarithms) in base

$$e \approx 2.718281828459\ldots \text{ (irrational number)}$$

Example. Write the general exponential function in the natural base

$$P(t) = P_0 a^t = P_0 (e^k) t = P_0 e^{kt}, \quad a = e^k.$$

Thus, we will have

- exponential growth if $a > 1$, which gives $k > 0$
- exponential decay if $a < 1$, which gives $k < 0$

The number k is called **the continuous rate of growth/decay.** To find k we will need the logarithmic function.

Clicker Question

During 1988, Nicaragua's inflation rate averaged 1.3% a day. Which formula below represents the above statement? (Assume t is measured in days.)

(A) $I(t) == I_0 e^{0.013t}$

(B) $I(t) = I_0(1.013)^t$

(C) $I(t) = I_0(1.013)t$

(D) $I(t) = I_0(1.3)^t$

(E) $I(t) = I_0 e^{1.3t}$

Inverses

Invertible

A function has an inverse if (and only if) its graph intersects any horizontal line at most once. If a function has an inverse, we say it is invertible.

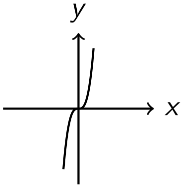
Inverse Function

If the function f is invertible, its inverse is defined as follows $f^{-1}(y) = x$ means $y = f(x)$.

Clicker Question

Which of the following could be graphs of functions that have inverses?

a)



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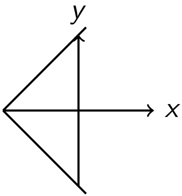

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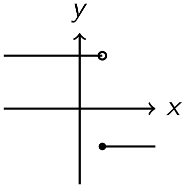

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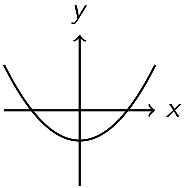

The inverse function. The logarithmic function

Inverse functions. If for each y in the range of f there exists exactly one value of x such that $f(x) = y$, then f has an inverse at y denoted by f^{-1} such that

$$f(x) = y \quad \iff \quad f^{-1}(y) = x$$

Hence

$$f(f^{-1}(y)) = y \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The **inverse** of the exponential function $f(x) = e^x$ is the natural **logarithmic** function $f^{-1}(x) = \ln x$ so we have

$$\ln x = c \iff e^c = x$$

We also have the useful identities

$$\ln e^c = c \quad e^{\ln x} = x.$$

Page 30 in the textbook has some of the properties of the logarithmic function. Be sure to review them carefully!

Example of population growth problem

Suppose a population initially has 20 members and it increases by 30% each year. What is its rate of growth?

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Suppose a population initially has 20 members and it increases by 30% each year. What is its rate of growth?
Solution. We would like to use the natural base model so

$$P(t) = P(0)e^{kt} \implies P(0) = P_0 = 20.$$

The growth rate can be computed from

$$P(1) = P_0 e^k \implies \frac{P(1)}{P(0)} = e^k = 1 + 0.3 = 1.3$$

Hence, $k = \ln(1.3) \approx 0.2624$, so

$$P(t) = 2e^{.2624t}.$$

Example of half-life problem

The quantity of a substance decreases by 35% every 100 days. What is its half life?

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The quantity of a substance decreases by 35% every 100 days. What is its half life?

Solution. We start with $P(t) = P(0)e^{kt}, \quad k < 0$. Let us take the unit of time to be 100 days. Since P decreases by 35% every time unit, we have

$$P(t+1) = P(t)(1-0.35) = 0.65P(t) \implies \frac{P(t+1)}{P(t)} = 0.65$$

From the model equation we have

$$\frac{P(t+1)}{P(t)} = \frac{P(0)e^{k(t+1)}}{P(0)e^{kt}} = e^k = 0.65 \implies P(t) = P_0(0.65)^t.$$

Example of half-life problem (solution continued)

We need to find t_h (half-life) such that $P(t_h) = \dfrac{P_0}{2}$ hence

$$P_0(0.65)^{t_h} = \frac{P_0}{2} \iff (0.65)^{t_h} = \frac{1}{2} \iff \ln(0.65^{t_h}) = \ln\frac{1}{2}$$

By using the properties of the logarithms we have

$$t\ln(0.65) = \ln\frac{1}{2} \implies t = \frac{\ln(1/2)}{\ln(0.65)} \approx 1.609.$$

The half life would be about 160.9 days, in which we would have half of the initial amount.

Wrapping up

- We reviewed exponential and logarithmic functions.
- Work on the suggested problems from sections 1.1, 1.3. 1.2, 1.4 by Friday (from syllabus and webwork).
- Read section 1.5 before lecture on Friday.
- Register your clickers, and bring them to lecture on Friday!