Calculus 1
Implicit Differentiation
Section 3.7

What Is an Implicit Function?

In earlier chapters, most functions were written in the form

$$y = f(x)$$

Here, y is said to be an **explicit function** of x. An equation such as

$$x^2 + y^2 = 4$$

is said to give y as an **implicit function** of x. Its graph is the circle to the right. Since there are x-values which correspond to two y-values, y is not a function of x on the whole circle.

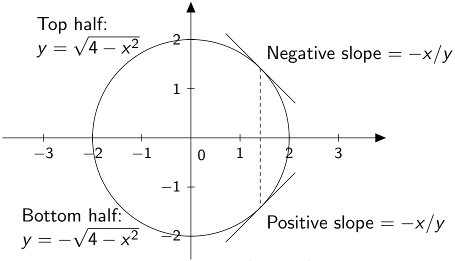


Figure : Graph of $x^2 + y^2 = 4$

Note that y is a function of x on the top half, and y is a *different* function of x on the bottom half.

Differentiating Implicitly

Let us consider the circle as a whole. The equation does represent a curve which has a tangent line at each point. The slope of this tangent can be found by differentiating the equation of the circle with respect to

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

If we think of y as a function of x and use the chain rule, we get

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = -\frac{x}{y}$. The derivative here depends on both x and y (instead of just on x). Differentiating the equation of the circle has given us the slope of the curve at all points except $(2, 0)$ and $(2, 0)$, where the tangent is vertical. In general, this process of **implicit differentiation** leads to a derivative whenever the expression for the derivative does not have a zero in the denominator.

So what does this mean? If an equation has both x's and y's in it, and y cannot be solved for explicitly, we need to find another way to compute $\frac{dy}{dx}$. The "other way" being implicit differentiation. Here's how it works.

Recall that $\frac{d(?)}{dx}$ is an operator that takes the derivative of "?" with respect to x. So if the "?" is y, the operator would take the derivative of y with respect to x:

$$\frac{d(y)}{dx} = \frac{dy}{dx}$$

If the "?" is x, then the operator would take the derivative of x with respect to x:

$$\frac{d(x)}{dx} = \frac{dx}{dx} = 1$$

This means that where ever there is a y in the equation, we will get a $\frac{dy}{dx}$.

Let's see the circle example again, a little more quickly:

$$x^2 + y^2 = 4$$

$$\Rightarrow$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}4 \text{ (Do the same thing to both sides.)}$$

$$\Rightarrow$$

$$2x + 2y\frac{dy}{dx} = 0 \text{ (Note the use of chain rule.)}$$

$$\Rightarrow$$

$$2y\frac{dy}{dx} = -2x$$

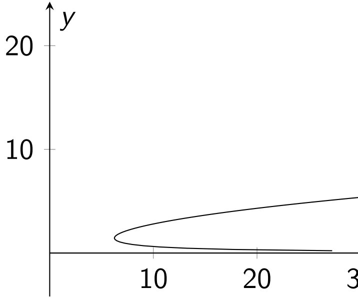
$$\Rightarrow$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

Find all points where the tangent line to $y^3 - xy = -6$ is either horizontal or vertical. **Solution** Differentiating implicitly,

$$3y^2\frac{dy}{dx} - y - x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{3y^2 - x}$$

The tangent is horizontal when the numerator of $\frac{dy}{dx}$ equals 0, so $y = 0$. Since we must also satisfy $y^3 - xy = -6$, we get $0 = -6$, which is impossible. We conclude that there are no points on the curve where the tangent line is horizontal. The tangent is vertical when the denominator of $\frac{dy}{dx}$ is 0, giving $3y^2 - x = 0$. Thus, $x = 3y^2$ at any point with a vertical tangent line. Again, we must also satisfy $y^3 - xy = -6$, so $y^3 = 3$. Solving for x, with this value of y, we conclude there is a vertical tangent at the point $(6.240, 1.442)$.



Clicker Question

When does the equation $3 = x^2 + y^3$ have a horizontal tangent line?

(a) $(0, 0)$

(b) $(\frac{1}{2}, \sqrt[3]{3})$

(c) $(\sqrt[3]{3}, 0)$

(d) $(0, \sqrt[3]{3})$

Example

Consider $2y = x^2 + \sin(y)$ (This can't be solved for y.) Find $\frac{dy}{dx}$.

First, apply $\frac{d}{dx}$:

$$\frac{d}{dx}2y = \frac{d}{dx}x^2 + \frac{d}{dx}\sin(y)$$

$$\Rightarrow \quad 2\frac{dy}{dx}y = 2x + \cos(y)\frac{dy}{dx}$$

$$\Rightarrow \quad 2\frac{dy}{dx}y - \cos(y)\frac{dy}{dx} = 2x$$

$$\Rightarrow \quad (2y - \cos(y))\frac{dy}{dx} = 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{2y - \cos(y)}$$

This technique is called **implicit differentiation**.

Implicit differentiation

1. Apply $\frac{d}{dx}$ to both sides.
2. Solve for $\frac{dy}{dx}$.

Find the equations of the tangent line to the curve at $(-1, 2)$, where

$$x^2 - xy + y^2 = 7$$

We need the slope. Since we can't solve for y, we use implicit differentiation to solve for $\frac{dy}{dx}$.

$$\frac{d}{dx}x^2 - \frac{d}{dx}(xy) + \frac{d}{dx}y^2 = \frac{d}{dx}7$$

$$\Rightarrow 2x - \left[x\frac{dy}{dx} + y\right] + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x - x\frac{dy}{dx} - y + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (2y - x)\frac{dy}{dx} = y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

So we see that $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$, so slope at $(-1, 2)$ is

$$m = \frac{2 - 2(-1)}{2(2) - (-1)} = \frac{2 + 2}{4 + 1} = \frac{4}{5}$$

Clicker Question

Find the equations of the tangent line to the curve at $(-1, 2)$, where

$$x^2 - xy + y^2 = 7$$

Finish the problem as a clicker question. (We found $m = \frac{4}{5}$)

(a) $y = \dfrac{4}{5}x + \dfrac{14}{5}$

(b) $y = \dfrac{5}{4}x + \dfrac{14}{5}$

(c) $y = \dfrac{4}{5}x + \dfrac{4}{5}$

(d) $y = \dfrac{5}{4}x + \dfrac{14}{5}$

Implicit differentiation gives us one of the most powerful tools in differentiation.

Differentiate $y = x^x$.

- Notice that the ordinary rules of differentation do not apply..
- So, what do you do?

Solving the problem

$$
\begin{aligned}
y &= x^x \\
\ln y &= \ln(x^x) && \text{(apply ln to both sides)} \\
\ln y &= x \ln(x) && \text{(Use a property of logs)} \\
\frac{1}{y}\frac{dy}{dx} &= x\frac{1}{x} + (1)\ln(x) && \text{(Differentiate both sides)} \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln(x) && \text{(Simplify)} \\
\frac{dy}{dx} &= y(1 + \ln(x)) && \text{(Simplify)} \\
\frac{dy}{dx} &= x^x(1 + \ln(x)) && \text{(Substitute back } y = x^x\text{)}
\end{aligned}
$$

Differentiate $y = x^{3x}$.

(a) $y' = x^{3x}(1 + \ln(3x))$
(b) $y' = 3 \ln x + \frac{3}{x^{3x}}$
(c) $y' = 3x \ln x$
(d) $y' = 3 + 3$
(e) $y' = x^{3x}(3 \ln x + 3)$