Uniqueness of Some Weak Solutions for 2D Viscous Primitive Equations

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Abstract

First, a new sufficient condition for uniqueness of weak solutions is proved for the system of 2D viscous Primitive Equations. Second, global existence and uniqueness are established for several classes of weak solutions with partial initial regularity, including but not limited to those weak solutions with initial horizontal regularity, rather than vertical regularity. Our results and analyses for the problem with *physical* boundary conditions can be extended to those with other typical boundary conditions. Most of the results were not available before, even for the periodic case.

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Contents

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1 Introduction

Consider as in [21] the system of 2D viscous Primitive Equations (PE) for *three* dimensional Geophysical Fluid Dynamics in the *two* dimensional spacial domain:

$$D := \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; 0 < x < 1, -h < z < 0\right\},$$

where h is a positive constant.
Horizontal momentum equations:

$$\begin{aligned}\frac{\partial u}{\partial t} + (u,w)\cdot\nabla u &= -\frac{\partial p}{\partial x} + v + \Delta u,\\ \frac{\partial v}{\partial t} + (u,w)\cdot\nabla v &= -u + \Delta v.\end{aligned}$$

Hydrostatic balance:

$$\frac{\partial p}{\partial z} + \theta = 0.$$

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

Heat equation:

$$\frac{\partial \theta}{\partial t} + (u,w)\cdot\nabla\theta = \Delta\theta + Q.$$

In the above equations, gradient ∇ and Laplacian Δ are defined as following:

$$\nabla := (\partial_x, \partial_z) = (\partial_1, \partial_2), \quad \Delta := \partial_x^2 + \partial_z^2 = \partial_1^2 + \partial_2^2.$$

The unknowns in the above system of 2D viscous PEs are the fluid velocity $(u,v,w) \in \mathbb{R}^3$, the pressure p and the temperature θ. The heat source Q is given. For issues concerned in this article and for simplicity of presentation, Q is assumed to be independent of t. Upon minor modifications, all the results obtained in this article can be extended to the case for time-dependent Q under suitable assumptions for Q. Some of the coefficients in the above system are already simplified for conciseness of presentation. In particular, viscosity constant, diffusivity constant and Coriolis rotational frequency from β-plane approximation are set as 1. The effect of salinity is omitted for simplicity of presentation. All these simplifications lose no mathematical generality.

The boundary ∂D of D is partitioned into three parts $\Gamma_i \cup \Gamma_b \cup \Gamma_l$, where

$$\Gamma_i := \{(x,0) \in \overline{D}\}, \quad \Gamma_b := \{(x,-h) \in \overline{D}\}, \quad \Gamma_l := \{(x,z) \in \overline{D} \big| x = 0,1\}.$$

The following boundary conditions of the PEs are used:

$$\begin{aligned}
&\text{on } \Gamma_i: \quad u_z + \alpha_1 u = v_z + \alpha_2 v = w = \theta_z + \alpha_3 \theta = 0,\\
&\text{on } \Gamma_b: \quad u = v = w = \theta_z = 0,\\
&\text{on } \Gamma_l: \quad u = v = \theta_x = 0,
\end{aligned}$$

where $\alpha_i \geqslant 0$ for $i = 0, 1, 2$ and u_z, v_z, θ_x and θ_z are the corresponding partial derivatives of u, v and θ.

Integrating continuity equation and hydrostatic balance equation and using boundary condition $w(x,0,t) = 0$, one can express w and p as:

$$w(x,z,t) = \int_z^0 u_x(x,\zeta,t)d\zeta, \tag{1.1}$$

$$p(x,z,t) = q(x,t) + \int_z^0 \theta(x,\zeta,t)d\zeta. \tag{1.2}$$

where $q(x,t) = p(x,0,t)$ is the pressure on Γ_i. Eliminating w and p from the previous system of 2D viscous PE results in the following equivalent formulation:

$$\begin{aligned}
u_t - \Delta u + u u_x + \left(\int_z^0 u_x(x,\zeta,t)d\zeta\right) u_z &\\
+ q_x + \int_z^0 \theta_x(x,\zeta,t)d\zeta = v&, \qquad (1.3)
\end{aligned}$$

$$v_t - \Delta v + u v_x + \left(\int_z^0 u_x(x,\zeta,t)d\zeta\right) v_z = -u, \tag{1.4}$$

$$\theta_t - \Delta \theta + u \theta_x + \left(\int_z^0 u_x(x,\zeta,t)d\zeta\right) \theta_z = Q; \tag{1.5}$$

with the following boundary conditions:

$$u_z + \alpha_1 u\big|_{z=0} = u\big|_{z=-h} = u\big|_{x=0,1} = 0, \tag{1.6}$$

$$v_z + \alpha_2 v\big|_{z=0} = v\big|_{z=-h} = v\big|_{x=0,1} = 0, \tag{1.7}$$

$$\int_{-h}^{0} u_x(x,\zeta,t)d\zeta = 0, \tag{1.8}$$

$$\theta_z + \alpha_3 \theta\big|_{z=0} = \theta_z\big|_{z=-h} = \theta_x\big|_{x=0,1} = 0. \tag{1.9}$$

The above system of 2D viscous PE will be solved with suitable initial conditions:

$$u(x,z,0) = u_0(x,z), \quad v(x,z,0) = v_0(x,z), \quad \theta(x,z,0) = \theta_0(x,z). \tag{1.10}$$

Notice that the boundary condition $w(x,0,t) = 0$ is already embedded in the expression (1.1). The other boundary condition for w is given in (1.8). It follows from (1.6) and (1.8) that

$$\int_{-h}^{0} u(x,z,t)\ dz = 0. \tag{1.11}$$

The mathematical framework of the viscous primitive equations for large scale ocean flow in *three* dimensional spacial domain (3D viscous PE) was formulated in [19], where the notions of weak and strong solutions were defined and existence of weak solutions was proved. Existence of strong solutions *local in time* and their uniqueness were proved in [7] and [24]. Existence of strong solutions *global in time* was proved in [4] and [13] for the case when u and v satisfy *Neumann* boundary condition at the bottom. Existence of strong solutions *global in time* was proved in [15] for the case when u and v satisfy *physical* boundary conditions. See also the results in [8]. Uniform boundedness in H^1 of strong solutions global in time was proved in [9] and [16]. Uniform boundedness in H^2 of H^2 solutions global in time was proved in [10] and [12]. Global uniform boundedness in H^m $(m \geqslant 2)$ of H^m solutions was recently proved in [11].

One of the outstanding unresolved mathematical problems for 3D viscous PE is about uniqueness of weak solutions. Global existence and uniqueness of z-weak solutions to 3D viscous PE were proved in [22] for initial data in L^6. A "z-weak" solution is a weak solution (u, v, θ) such that

$$(u_z, v_z, \theta_z) \in L^\infty(0, T; (L^2)^3) \cap L^2(0, T; (H^1)^3).$$

Recently, global existence and uniqueness of z-weak solutions was proved in [10]. Uniqueness of weak solutions was proved in [14] for *continuous* initial data as well. See also [17] for a result on uniqueness of weak solutions for a class of discontinuous initial data.

Indeed, the problem of uniqueness of weak solution is still open even for 2D viscous PE. In [2], existence and uniqueness of z-weak solutions, in the name of "*weak vorticity solutions*", for 2D hydrostatic Navier-Stokes equations (2D hNSE) were proved for the case when u satisfies a *Robin* type *friction* boundary condition at the bottom of physical domain. The system of 2D hNSE is somewhat simplified from that of 2D PE (1.3)-(1.5). It includes only u and q as the unknown variables without v or θ. Existence and uniqueness of z-weak solution u for 2D hNSE were also proved in [3] for the case with *Dirichlet* boundary condition at the bottom. For the case when the physical domain is a square, existence and uniqueness of the weak solution u of the 2D hNSE were proved in [3] with even less demanding regularity:

$$u \in L^\infty(0, T; L^2_x H^{\frac{1}{2}}_z) \cap L^2(0, T; H^1_x H^{\frac{3}{2}}_z),$$

for both the cases of *Dirichlet* and *Neumann* boundary conditions at the bottom. See [3] for notational details. Finally, we mention that existence and uniqueness of z-weak solution (u, v, θ) for 2D viscous PE were proved in [20] for the case with *periodic* boundary conditions on (u, v, θ).

This paper will focus on the problem of uniqueness of weak solutions and uniform boundedness of norms of partial regularity for given initial partial regularity of weak solutions to the 2D viscous PE. It studies existence and uniqueness of weak solutions of the system (1.3)-(1.5) in D under *physical* boundary conditions (1.6)-(1.9).

First, a new sufficient condition for uniqueness of weak solutions is proved. Second, global existence of several classes of weak solutions with initial partial regularity is also proved. Finally, as an application of our new sufficient condition, uniqueness of these classes of weak solutions with initial *partial* regularity is also proved. These results are valid as well for other typical boundary conditions for 2D viscous PEs, since our proofs can be easily extended to those cases.

To present our analysis in complete details, we first give the definition of a weak solution carefully and then prove several important results about properties of weak solutions and strong solutions of 2D viscous PEs. These results are included in Theorems 3.1-3.4. Closely related important discussions are also presented in Remarks 3.1-3.3. This section, Section 3, shares some similarity with [17] in terms of strategy. However, the definition of a weak solution of the viscous PE used in this paper is somewhat *different* from those used in [2], [17] and [19] for viscous PE. Hence, many of our detailed arguments and ideas of the proofs are also *different* from those of [17]. Therefore, we choose to present the full proofs of all these results. These results will provide fundamental technical support for our analysis in the rest sections of this paper and some of them might also be new in the presented forms.

The main results for existence and uniqueness of weak solutions with initial partial regularity to be presented are Theorem 4.1, Theorem 5.1, Theorem 5.2 and Theorem 6.1. Except that the results for z-weak solutions were proved in [20] for periodic case, all of our main results were not previously known even for periodic case. Especially, global existence and uniqueness are proved for weak solutions with initial partial regularity in horizontal direction, rather than vertical direction. The same is proved for several other mixed cases as well. The main ideas of our analysis are careful manipulations of anisotrophic inequalities in Sobolev spaces.

The rest of this article is organized as follows:

In Section 2, we give the notations and some definitions, briefly review the background and recall some facts and known results which are important for later analysis. In Section 3, we first give the definition of weak solutions and strong solutions of 2D viscous PE with physical boundary conditions. Then, prove Theorem 3.1-3.4 about important properties of weak solutions and strong solutions of 2D viscous PE. These theorems will provide helpful technical support for our analysis in the sequel. In Section 4, we prove Theorem 4.1. It gives a new sufficient condition for uniqueness of weak solutions and generalizes an already known one. It is a crucial observation which will be used later in proving our new uniqueness results. In Section 5, we prove Theorem 5.1 and Theorem 5.2. These global existence results, for weak solutions in specific partial regularity classes, not only give corresponding global-in-time uniform bounds and absorbing sets, but also prepare our proof of uniqueness in next section. In Section 6, we prove Theorem 6.1 for uniqueness of the weak solutions with the specific initial partial regularity.

2 Preliminaries

In this paper, C denotes a positive absolute constant, the value of which might vary from line to line. Similarly, C_ε denotes a positive constant depending on $\varepsilon > 0$, the value of which may also vary at different occurrence. The following notations are used for real numbers A and B:

$$A \preccurlyeq B \quad \text{iff} \quad A \leqslant C \cdot B,$$

and

$$A \approx B \quad \text{iff} \quad c \cdot A \leqslant B \leqslant C \cdot A,$$

for some positive constants c and C independent of A and B.

Denote by $L^r(D)$, $L^r((0,1))$ and $L^r((-h,0))$ $(1 \leqslant r < +\infty)$ the classic Lebesgue L^r spaces with the norm:

$$\|\varphi\|_r = \begin{cases} \left(\int_D |\varphi(x,z)|^r \, dxdz\right)^{\frac{1}{r}}, & \forall \varphi \in L^r(D); \\ \left(\int_0^1 |\varphi(x)|^r \, dx\right)^{\frac{1}{r}}, & \forall \varphi \in L^r((0,1)); \\ \left(\int_{-h}^0 |\varphi(z)|^r \, dz\right)^{\frac{1}{r}}, & \forall \varphi \in L^r((-h,0)). \end{cases}$$

Standard modification is used when $r = \infty$. When there is no confusion, index $r = 2$ is omitted:

$$\|\varphi\| := \|\varphi\|_2.$$

Denote by $H^m(D)$ ($m \geqslant 1$) the classic Sobolev spaces for square-integrable functions on D with square-integrable weak derivatives up to order m. Domains of the functions spaces will be omitted from notations without confusion.

Some *anisotrophic* Lebesgue spaces and Sobolev spaces will be used. For example, for $r, s \in [1, \infty)$, $L^r_x(L^s_z)$ denotes the standard function space of (classes of) Lebesgue measurable functions on D such that

$$\begin{aligned}\|\varphi\|_{L^r_x(L^s_z)} &:= \left(\int_0^1 \|\varphi(x,\cdot)\|^r_{L^s_z} dx\right)^{\frac{1}{r}} \\ &:= \left[\int_0^1 \left(\int_{-h}^0 |\varphi(x,z)|^s dz\right)^{\frac{r}{s}} dx\right]^{\frac{1}{r}} < \infty,\end{aligned}$$

with stand modifications when r or s is ∞.

We will also use $C_B([\alpha, \infty))$ to denote the set of uniformly bounded functions in $C([\alpha, \infty))$, for an interval $[\alpha, \infty) \subset \mathbb{R}$.

Following [21], the following function spaces are defined:

$$H := H_1 \times H_2 \times H_3, \quad V := V_1 \times V_2 \times V_3,$$

where

$$\begin{aligned} H_1 &:= \left\{\varphi \in L^2(D) \;\Big|\; \int_{-h}^0 \varphi(x,z)dz = 0, \varphi\big|_{\Gamma_l} = 0\right\}, \\ V_1 &:= \left\{\varphi \in H^1(D) \;\Big|\; \int_{-h}^0 \varphi(x,z)dz = 0,\ \varphi\big|_{\Gamma_l \cup \Gamma_b} = 0\right\}, \\ H_2 &:= L^2(D),\ V_2 := \left\{\varphi \in H^1(D) \;\Big|\; \varphi\big|_{\Gamma_l \cup \Gamma_b} = 0\right\}, \end{aligned}$$

$$H_3 := L^2(D), \quad V_3 := H^1(D), \quad \text{if } \alpha_3 > 0,$$

and

$$H_3 := \left\{\varphi \in L^2(D) \;\Big|\; \int_D \varphi = 0\right\},\ V_3 := H_3 \cap H^1(D), \quad \text{if } \alpha_3 = 0.$$

Define the bilinear form: $a_i : V_i \times V_i \to \mathbb{R}$ for $i = 1, 2, 3$, such that

$$a_i(\phi, \varphi) = \int_D \nabla\phi \cdot \nabla\varphi \; dxdz + \alpha_i \int_0^1 \phi(x,0)\varphi(x,0)\; dx,$$

and the corresponding linear operator $A_i : V_i \mapsto V_i'$, such that

$$\langle A_i v,\, \varphi\rangle = a_i(v,\varphi), \quad \forall v,\varphi \in V_i,$$

where V_i' is the dual space of V_i and $\langle\cdot,\,\cdot\rangle$ denotes scalar product between V_i' and V_i *and* the inner product in H_i. Define:

$$D(A_i) = \left\{\phi \in V_i \;\middle|\; A_i\phi \in H_i\right\}, \quad i = 1, 2, 3.$$

Define: $A : V \mapsto V'$ as $A(u, v, \theta) := (A_1 u, A_2 v, A_3\theta)$, for $(u, v, \theta) \in V$. Then,

$$D(A) = D(A_1) \times D(A_2) \times D(A_3).$$

Since A_i^{-1} is a self-adjoint compact operator in H_i, by the classic spectral theory, the power A_i^s can be defined for any $s \in \mathbb{R}$. Then

$$D(A_i)' = D(A_i^{-1})$$

is the dual space of $D(A_i)$ and

$$V_i = D(A_i^{\frac{1}{2}}), \quad V_i' = D(A_i^{-\frac{1}{2}}).$$

Moreover,

$$D(A_i) \subset V_i \subset H_i \subset V_i' \subset D(A_i)',$$

and the embeddings above are all continuous and compact and each space above is dense in the one following it. Define the norm of V_i as

$$\|\varphi\|_{V_i}^2 = a_i(\varphi,\varphi) = \langle A_i\varphi,\, \varphi\rangle = \langle A_i^{\frac{1}{2}}\varphi, A_i^{\frac{1}{2}}\varphi\rangle, \quad i = 1, 2, 3.$$

Then, for $\varphi \in V_i$ and $i = 1, 2, 3$,

$$\|\varphi\| \preccurlyeq \|\varphi\|_{V_i} \approx \|\varphi\|_{H^1}.$$

By the above discussion and elliptic regularity for linear 3D stationary PE (see e.g. [24]), we also have for $\varphi \in D(A_i)$ and $i = 1, 2, 3$,

$$\|\varphi\|_{V_i} \preccurlyeq \|A_i\varphi\| \approx \|\varphi\|_{H^2}.$$

Next, we introduce the following anisotrophy estimate which will very useful for our later discussion:

Lemma 2.1 *Let $\psi, \psi_x, \phi, \phi_z, \varphi \in L^2(D)$. Then,*

$$\left|\int_D \psi\phi\varphi\right| \leqslant \|\psi\|^{\frac{1}{2}}(\|\psi\| + \|\psi_x\|)^{\frac{1}{2}}(\|\phi\| + \|\phi_z\|)^{\frac{1}{2}}\|\varphi\|.$$

Proof:

$$\begin{aligned}
\left|\int_D \psi\phi\varphi\right| \leqslant & \int_0^1 \|\phi\|_{L_z^\infty}\|\psi\|_{L_z^2}\|\varphi\|_{L_z^2}\, dx \\
\preccurlyeq & \int_0^1 \|\phi\|_{L_z^2}^{\frac{1}{2}}(\|\phi\|_{L_z^2} + \|\phi_z\|_{L_z^2})^{\frac{1}{2}}\|\|\psi\|_{L_z^2}\|\varphi\|_{L_z^2} dx \\
\preccurlyeq & \|\psi\|_{L_x^\infty(L_z^2)} \int_0^1 \|\phi\|_{L_z^2}^{\frac{1}{2}}(\|\phi\|_{L_z^2} + \|\phi_z\|_{L_z^2})^{\frac{1}{2}}\|\varphi\|_{L_z^2}\, dx \\
\preccurlyeq & \|\psi\|_{L_z^2(L_x^\infty)}\|\phi\|^{\frac{1}{2}}(\|\phi\| + \|\phi_z\|)^{\frac{1}{2}}\|\varphi\| \\
\preccurlyeq & \|\psi\|^{\frac{1}{2}}(\|\psi\| + \|\psi_x\|)^{\frac{1}{2}}(\|\phi\| + \|\phi_z\|)^{\frac{1}{2}}\|\varphi\|.
\end{aligned}$$

□

An immediate application of Lemma 2.1 is the following result:

Lemma 2.2 *The following statements are valid:*

(a) *Suppose $(u, v, \theta) \in V$ and w is given by (1.1). Then, for $\varphi \in V_1$,*

$$|\langle uu_x,\, \varphi\rangle| \preccurlyeq \|u\|\|u_x\|^{\frac{1}{2}}\|u_z\|^{\frac{1}{2}}\|\varphi_x\|, \tag{2.1}$$

$$|\langle wu_z,\, \varphi\rangle| \preccurlyeq \|u\|\|u_x\|^{\frac{1}{2}}\|u_z\|^{\frac{1}{2}}\|\varphi_x\| + \|u\|^{\frac{1}{2}}\|u_x\|^{\frac{3}{2}}\|\varphi_z\|; \tag{2.2}$$

for $\varphi \in V_2$, with $i, j = 1, 2$, $i' = 3 - i$ and $j' = 3 - j$,

$$\begin{aligned}
|\langle uv_x,\, \varphi\rangle| \preccurlyeq\ & \|u_x\|\|v\|^{\frac{1}{2}}\|\partial_i v\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\partial_{i'}\varphi\|^{\frac{1}{2}} \\
& + \|u\|^{\frac{1}{2}}\|\partial_j u\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|\partial_{j'} v\|^{\frac{1}{2}}\|\varphi_x\|, \qquad (2.3) \\
|\langle wv_z,\, \varphi\rangle| \preccurlyeq\ & \|u_x\|\|v\|^{\frac{1}{2}}\|\partial_i v\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\partial_{i'}\varphi\|^{\frac{1}{2}} \\
& + \|u_x\|\|v\|^{\frac{1}{2}}\|v_x\|^{\frac{1}{2}}\|\varphi_z\|; \qquad (2.4)
\end{aligned}$$

for $\varphi \in V_3$, with $i, j = 1, 2$, $i' = 3 - i$ and $j' = 3 - j$,

$$\begin{aligned}
|\langle u\theta_x,\, \varphi\rangle| \preccurlyeq\ & \|u_x\|\|\theta\|^{\frac{1}{2}}\|\partial_i \theta\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\partial_{i'}\varphi\|^{\frac{1}{2}} \\
& + \|u\|^{\frac{1}{2}}\|\partial_j u\|^{\frac{1}{2}}\|\theta\|^{\frac{1}{2}}\|\partial_{j'} \theta\|^{\frac{1}{2}}\|\varphi_x\|, \qquad (2.5) \\
|\langle w\theta_z,\, \varphi\rangle| \preccurlyeq\ & \|u_x\|\|\theta\|^{\frac{1}{2}}\|\partial_i \theta\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\partial_{i'}\varphi\|^{\frac{1}{2}} \\
& + \|u_x\|\|\theta\|^{\frac{1}{2}}\|\theta_x\|^{\frac{1}{2}}\|\varphi_z\|. \qquad (2.6)
\end{aligned}$$

Therefore,

$$uu_x, wu_z \in V_1', \quad uv_x, wv_z \in V_2', \quad u\theta_x, w\theta_z \in V_3'. \tag{2.7}$$

(b) *Suppose* $(u, v, \theta) \in L^\infty(0,T;H) \cap L^2(0,T;V)$. *Then,*

$$\begin{gathered} uu_x \in L^2(0,T;V_1'), \quad wu_z \in L^{\frac{4}{3}}(0,T;V_1'), \\ uv_x, wv_z \in L^{\frac{4}{3}}(0,T;V_2'), \quad u\theta_x, w\theta_z \in L^{\frac{4}{3}}(0,T;V_3'). \end{gathered}$$

Proof: For $u, \varphi \in V_1$, Lemma 2.1 yields

$$|\langle uu_x, \varphi\rangle| \preccurlyeq \|u\|^{\frac{1}{2}}\|u_x\|^{\frac{3}{2}}\|\varphi\|^{\frac{1}{2}}\|\varphi_z\|^{\frac{1}{2}}, \quad \left|\left\langle u^2, \varphi_x\right\rangle\right| \preccurlyeq \|u\|\|u_x\|^{\frac{1}{2}}\|u_z\|^{\frac{1}{2}}\|\varphi_x\|.$$

Then, a density arguement using Lemma 2.1 again, along with the above two inequalities, proves

$$\langle uu_x, \varphi\rangle = -\frac{1}{2}\left\langle u^2, \varphi_x\right\rangle,$$

from which (2.1) follows. Similarly, Lemma 2.1 yields

$$|\langle wu_z, \varphi\rangle| \preccurlyeq \|w\|^{\frac{1}{2}}\|w_z\|^{\frac{1}{2}}\|\varphi\|^{\frac{1}{2}}\|\varphi_x\|^{\frac{1}{2}}\|u_z\| \leqslant \|u_x\|\|u_z\|\|\varphi\|^{\frac{1}{2}}\|\varphi_x\|^{\frac{1}{2}},$$

and

$$|\langle wu, \varphi_z\rangle| \preccurlyeq \|u\|^{\frac{1}{2}}\|u_x\|^{\frac{3}{2}}\|\varphi_z\|. \tag{2.8}$$

The above two inequalities plus (2.1), along with Lemma 2.1, then imply via a density arguement that

$$\langle wu_z, \varphi\rangle = \langle u_x u, \varphi\rangle - \langle wu, \varphi_z\rangle,$$

from which, we immediately prove (2.2) by (2.1) and (2.8).

Similarly, we can prove (2.3)-(2.6). Then, it is easy to prove (2.7) and part (b) using (2.1)-(2.6).

$\square$

3 Weak Solutions and Strong Solutions

In this section, some important properties about weak solutions and strong solutions will be discussed. These will provide important technical support in the proofs of the main results of this paper to be presented in the next few sections.

The following definitions of weak and strong solutions of the initial boundary value problem (1.3)-(1.10) for the 2D viscous PEs will be used in this paper:

Definition 3.1 Suppose $Q \in L^2(D)$, $(u_0, v_0, \theta_0) \in H$ and $T > 0$. The triple (u, v, θ) is called a *weak solution* of the viscous PEs (1.3)-(1.10) on the time interval $(0, T)$ if it satisfies (1.3)-(1.5) in weak sense, that is, if

$$(u, v, \theta) \in L^\infty(0, T; H) \cap L^2(0, T; V), \tag{3.1}$$

satisfies the follow equations in the sense of distribusion on $(0, T)$:

$$\langle u_t,\, \varphi\rangle + a_1(u, \varphi) + \left\langle (u, w)\cdot\nabla u - v + \int_z^0 \theta_x,\, \varphi\right\rangle = 0, \quad \forall \varphi \in V_1, \tag{3.2}$$

$$\langle v_t,\, \varphi\rangle + a_2(v, \varphi) + \langle (u, w)\cdot\nabla v - u,\, \varphi\rangle = 0, \quad \forall \varphi \in V_2, \tag{3.3}$$

$$\langle \theta_t,\, \varphi\rangle + a_3(\theta, \varphi) + \langle (u, w)\cdot\nabla \theta - Q,\, \varphi\rangle = 0, \quad \forall \varphi \in V_3, \tag{3.4}$$

where w is given by (1.1) in weak sense. Moreover,

$$\lim_{t\to 0^+} (u(t), v(t), \theta(t)) = (u_0, v_0, \theta_0), \tag{3.5}$$

in weak topology of H, and the following *energy inequalities* are satisfied for *almost every* $t_0 \in [0, T)$ and *almost every* $t \in (t_0, T)$:

$$\|u(t)\|^2 + 2\int_{t_0}^t \left(\|u(s)\|_{V_1}^2 + \left\langle -v + \int_z^0 \theta_x,\, u\right\rangle\right) ds \leqslant \|u(t_0)\|^2, \tag{3.6}$$

$$\|v(t)\|^2 + 2\int_{t_0}^t \left(\|v(s)\|_{V_2}^2 + \langle u,\, v\rangle\right) ds \leqslant \|v(t_0)\|^2, \tag{3.7}$$

$$\|\theta(t)\|^2 + 2\int_{t_0}^t \left(\|\theta(s)\|_{V_3}^2 - \langle Q,\, \theta(s)\rangle\right) ds \leqslant \|\theta(t_0)\|^2. \tag{3.8}$$

Further more, the above energy inequalities (3.6)-(3.8) are also satisfied for $t_0 = 0$ and for *almost every* $t \in (0, T)$.

If $(u_0, v_0, \theta_0) \in V$, then (u, v, θ) is called a *strong solution* of (1.3)-(1.10) on the time interval $[0, T)$ if it satisfies (3.2)-(3.5) and

$$(u, v, \theta) \in L^\infty(0, T; V) \cap L^2(0, T; D(A)). \tag{3.9}$$

If $T > 0$ in the above can be arbitrarily large, then the corresponding weak or strong solution is *global*. □

Remark 3.1 There are somewhat different ways to define weak solutions of the PE. For examples, see [1],[2], [17], [19], [21] and [24]). Especially, to define a weak solution of the 3D PE with physical boundary conditions, the domain of φ in (3.2)-(3.4) was chosen as $D(A_i)$ in [19], [21] and [24], for $i = 1, 2, 3$ respectively instead of V_i. Definition 3.1 is formally more restrictive than the one given in [19], [21] and [24]. However, $D(A_i)$ is dense in V_i and, by Lemma 2.2, the nonlinear terms of the 2D PE are in V_i' for $(u, v, \theta) \in V$ and in $L^{\frac{4}{3}}(0, T; V')$ for $(u, v, \theta) \in L^\infty(0, T; H) \cap L^2(0, T; V)$. *Thus, for 2D case, a weak solution defined in [19], [21] and [24], if satisfying* (3.6)-(3.8)*, is also a weak solution in the sense of Definition 3.1.* □

We first state and prove the following theorem on some basic properties satisfied by every weak solution.

Theorem 3.1 *There exists at least one global weak solution of* (1.3)-(1.10) *in the sense of Definition 3.1. If (u, v, θ) is a weak solution on $(0, T)$, then*[1]

$$\langle u, \varphi_1 \rangle, \langle v, \varphi_2 \rangle, \langle \theta, \varphi_3 \rangle \in C([0, T]), \quad \forall\ (\varphi_1, \varphi_2, \varphi_3) \in H. \tag{3.10}$$

Moreover, there exists a zero measure set $E \subset (0, \infty)$, such that

$$\lim_{E^c \ni t \to 0^+} \|(u(t), v(t), \theta(t)) - (u_0, v_0, \theta_0)\|_H = 0, \tag{3.11}$$

where $E^c := (0, \infty) \setminus E$.

[1]If $T = \infty$, the space $C([0, T])$ in (3.10) is replaced by $C_B([0, \infty))$.

Proof: It is proved in [19] that there exists at least one global weak solution (in their sense) for 3D PE and it satisfies energy inequalities (3.6)-(3.8). Moreover, any weak solution as defined in [19] is weakly continuous from $[0, T]$ into H if T is finite and weakly continuous from $[0, T)$ into H if $T = \infty$. By Remark 3.1, these results imply existence of at least one global weak solution (u, v, θ) of (1.3)-(1.10) in the sense of Definition 3.1 and that (3.10) is satisfied by *any* weak solution (u, v, θ) in the sense of Definition 3.1.

By Lemma 2.1 and Lemma 2.2, we can also prove existence of at least one global weak solutions (u, v, θ) of the 2D problem (1.3)-(1.10) using Definition 3.1 directly, by following the standard approach of [23] for Navier-Stokes Equations. Moreover, we can prove that any weak solution (u, v, θ) satisfies (3.10).

Finally, due to the fact that a weak solution satisfies the energy inequalities (3.6)-(3.8) for $t_0 = 0$ and for almost every $t \in [0, T]$ by Definition 3.1, there exists a zero measure set $E \subset (0, \infty)$ such that, for all $t \in E^c$,

$$\|u(t)\|^2 + 2\int_0^t \left(\|u(s)\|_{V_1}^2 + \left\langle -v + \int_z^0 \theta_x,\, u \right\rangle\right) ds \leqslant \|u_0\|^2, \tag{3.12}$$

$$\|v(t)\|^2 + 2\int_0^t \left(\|v(s)\|_{V_2}^2 + \langle u,\, v\rangle\right) ds \leqslant \|v_0\|^2, \tag{3.13}$$

$$\|\theta(t)\|^2 + 2\int_0^t \left(\|\theta(s)\|_{V_3}^2 - \langle Q,\, \theta(s)\rangle\right) ds \leqslant \|\theta_0\|^2. \tag{3.14}$$

Notice that, by definition, $(u, v, \theta) \in L^2(0, T; V)$. Therefore, taking $\limsup$ on both sides of (3.12) for $t(\in E^c) \to 0^+$, we have

$$\limsup_{E^c \ni t \to 0^+} \|u(t)\|^2 \leqslant \|u_0\|^2.$$

By (3.10) and weak lower semicontinuity, we also have

$$\|u_0\|^2 \leqslant \liminf_{E^c \ni t \to 0^+} \|u(t)\|^2.$$

Thus,

$$\lim_{E^c \ni t \to 0^+} \|u(t)\|^2 = \|u_0\|^2.$$

Hence,

$$\begin{aligned}\lim_{E^c \ni t \to 0^+} \|u(t) - u_0\|^2 &= \lim_{E^c \ni t \to 0^+} (\|u(t)\|^2 - 2\,\langle u(t),\, u_0\rangle + \|u_0\|^2) \\ &= \lim_{E^c \ni t \to 0^+} (\|u(t)\|^2 - 2\|u_0\|^2) = 0.\end{aligned}$$

The second equality above is due to weak continuity (3.10). This weak continuity argument was also used in [17] to prove (3.11).

□

Different proofs of existence of global weak solutions of 3D PE can also be found in [1] and [24]. A different set of boundary conditions and a somewhat different definition of weak solution were used in [1], [2] and [17].

Remark 3.2 Existence and uniqueness of global strong solution for 3D PE with *Neumann boundary condition* for (u, v) at botton was proved in [4]. See also [13] for a different proof of existence of global strong solution with the same boundary conditions when initial data is in H^2. Existence and uniqueness of global strong solution was proved in [15] and [16] for 3D viscous PE with *physical boundary condition*. The strong solutions are uniformly bounded in V and a bounded absorbing set for the solutions exists in V. These results apply to the 2D case as well. See also [21] for a direct proof of global existence of the strong solution of the 2D viscous PE (1.3)-(1.10). Moreover, following the argument of [9] for the case of 3D PE with Neumann boundary conditions, we can prove (see [5]) for the 3D PE with physical boundary conditions that, if $(u_0, v_0, \theta_0) \in V$, then the strong solution (u, v, θ) satisfies

$$(u_t, v_t, \theta_t) \in L^2(0, T; H), \qquad \forall T > 0, \tag{3.15}$$

and

$$(u(t), v(t), \theta(t)) \in C_B([0, \infty); V). \tag{3.16}$$

□

The next theorem shows that energy *equalities* are satisfied by every strong solution. Therefore, the strong solution is also a weak solution.

Theorem 3.2 *Let (u, v, θ) be the unique global strong solution of (1.3)-(1.10) with $(u_0, v_0, \theta_0) \in V$. Then, for every $t_0 \in [0, \infty)$ and $t \in (t_0, \infty)$,*

$$\|u(t)\|^2 + 2\int_{t_0}^t \left(\|u(s)\|_{V_1}^2 + \left\langle -v(s) + \int_z^0 \theta_x(s),\, u(s)\right\rangle\right) ds = \|u(t_0)\|^2, \tag{3.17}$$

$$\|v(t)\|^2 + 2\int_{t_0}^t \left(\|v(s)\|_{V_2}^2 + \langle u(s),\, v(s)\rangle\right) ds = \|v(t_0)\|^2, \tag{3.18}$$

$$\|\theta(t)\|^2 + \int_{t_0}^t \left(\|\theta(s)\|_{V_3}^2 - \langle Q,\, \theta(s)\rangle\right) ds = \|\theta(t_0)\|^2. \tag{3.19}$$

Therefore, the strong solution is also a weak solution.

Proof: By (3.9), (3.15) and a lemma of Lions and Magenes (see [18] and [23]), we have in the sense of distribution on $(0, \infty)$

$$\frac{d}{dt}\|u\|^2 = 2\langle u_t,\, u\rangle\,, \ \frac{d}{dt}\|v\|^2 = 2\langle v_t,\, v\rangle\,, \ \frac{d}{dt}\|\theta\|^2 = 2\langle \theta_t,\, \theta\rangle\,. \tag{3.20}$$

Therefore, by (3.16), (3.20) and Definition 3.1, we have in *classic* sense on $[0, \infty)$,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|u\|_{V_1}^2 + \left\langle \int_z^0 \theta_x,\, u\right\rangle = \langle v,\, u\rangle\,, \tag{3.21}$$

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|v\|_{V_2}^2 + \langle u,\, v\rangle = 0, \tag{3.22}$$

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \|\theta\|_{V_3}^2 = \langle Q,\, \theta\rangle\,. \tag{3.23}$$

In the above derivation, we have also used the cancellation property for $(u, v, \theta) \in V$:

$$\langle (u, w)\cdot\nabla u,\, u\rangle = \langle (u, w)\cdot\nabla v,\, v\rangle = \langle (u, w)\cdot\nabla\theta,\, \theta\rangle = 0, \tag{3.24}$$

which can be justified by Lemma 2.1, as shown in the proof of Lemma 2.2. Integrating (3.21)-(3.23) finishes the proof.

□

Weak-strong uniqueness was proved in [17]. That is, a weak solution with initial data in the space of strong solutions V is the strong solution with the same initial data, and is thus the unique weak (and strong) solution. The definition of weak solution used in [17] is somewhat different from Definition 3.1. In the following, we give a completely different proof of weak-strong uniqueness result. Our prove is a direct proof using Definition 3.1 and properties of weak and strong solutions. The argument of our proof is also different from that of [23] for a related uniqueness result.

Theorem 3.3 *Let (u, v, θ) be a weak solution of (1.3)-(1.10) on $(0, T)$ in the sense of Definition 3.1 with $(u_0, v_0, \theta_0) \in V$. Then, (u, v, θ) is the strong solution of (1.3)-(1.10). Thus, (u, v, θ) is the unique weak (and strong) solution.*

Proof: Let (u_1, v_1, θ_1) be the global strong solution and (u_2, v_2, θ_2) be a weak solution on $(0, T)$ for some $T > 0$, with the same initial value $(u_0, v_0, \theta_0) \in V$. For convenience of presentation, assume T is finite. The case of $T = \infty$ is then an easy consequence.

Use w_i and q_i, for $i = 1, 2$, to denote corresponding vertical velocity and surface pressure. Denote:

$$(\tilde{u}, \tilde{v}, \tilde{\theta}, \tilde{w}, \tilde{q}) := (u_1 - u_2, v_1 - v_2, \theta_1 - \theta_2, w_1 - w_2, q_1 - q_2).$$

Notice that, by (3.15) and (3.16), for the strong solution,

$$u_1 \in C([0, T]; V_1) \cap L^2(0, T; D(A_1)), \quad u_{1,t} \in L^2(0, T; H_1).$$

For a weak solution u_2,

$$u_2 \in L^\infty(0, T; H_1) \cap L^2(0, T; V_1), \quad u_{2,t} \in L^1(0, T; V_1').$$

Therefore, by standard regularization approximation on $(0, T)$, we can obtain sequences of functions $\{u_{1,m}\}_{m=1}^\infty$ and $\{u_{2,m}\}_{m=1}^\infty$ such that

$$u_{1,m} \in C^\infty([0, T]; D(A_1)), u_{2,m} \in C^\infty([0, T]; V_1), \quad \forall\, m \geqslant 1,$$

and as $m \to \infty$,

$$\begin{aligned} u_{1,m} &\to u_1 \quad \text{in } L^2_{loc}(0, T; D(A_1)) \text{ and } C([0, T]; V_1), \\ (u_{1,m})_t &\to u_{1,t} \quad \text{in } L^2_{loc}(0, T; H_1), \\ u_{2,m} &\to u_2 \quad \text{in } L^2_{loc}(0, T; V_1) \\ (u_{2,m})_t &\to u_{2,t} \quad \text{in } L^1_{loc}(0, T; V_1'), \end{aligned} \tag{3.25}$$

It is obvious that, for any $m \geqslant 1$,

$$\frac{d}{dt}\left\langle u_{1,m},\, u_{2,m}\right\rangle = \left\langle (u_{1,m})_t,\, u_{2,m}\right\rangle + \left\langle (u_{2,m})_t,\, u_{1,m}\right\rangle. \tag{3.26}$$

As $m \to \infty$, we have by (3.25) that, in $L^1_{loc}(0,T)$,

$$\begin{aligned}
\left\langle u_{1,m},\, u_{2,m}\right\rangle &\to \left\langle u_1,\, u_2\right\rangle, \\
\left\langle (u_{1,m})_t,\, u_{2,m}\right\rangle &\to \left\langle u_{1,t},\, u_2\right\rangle, \\
\left\langle (u_{2,m})_t,\, u_{1,m}\right\rangle &\to \left\langle u_{2,t},\, u_1\right\rangle.
\end{aligned} \tag{3.27}$$

These convergences are also valid in the distribution sense. Therefore, we can take the limit $m \to \infty$ in (3.26) in the sense of distribution to obtain

$$\frac{d}{dt}\left\langle u_1,\, u_2\right\rangle = \left\langle u_{1,t},\, u_2\right\rangle + \left\langle u_{2,t},\, u_1\right\rangle, \tag{3.28}$$

in the sense of distribution. Notice that

$$\left\langle u_{1,t},\, u_2\right\rangle \in L^2(0,T), \quad \left\langle u_{2,t},\, u_1\right\rangle \in L^1(0,T).$$

Therefore, $\langle u_1,\, u_2\rangle \in W^{1,1}(0,T)$. Thus, it is absolutely continuous in t and for any $t_0 \in [0,T)$ and $t \in (t_0,T]$,

$$\left\langle u_1(t),\, u_2(t)\right\rangle = \left\langle u_1(t_0),\, u_2(t_0)\right\rangle + \int_{t_0}^{t} \left(\left\langle u_{1,t}(s),\, u_2(s)\right\rangle + \left\langle u_{2,t}(s),\, u_1(s)\right\rangle\right) ds. \tag{3.29}$$

By the regularity of (u_i, v_i, θ_i), $i = 1, 2$, we have as well

$$u_{i,t} = -A_1 u_i - (u_i, w_i)\cdot\nabla u_i + v_i - \int_z^0 \theta_{i,x}, \quad i = 1, 2. \tag{3.30}$$

From (3.29)-(3.30), we obtain, for any $t_0 \in [0,T)$ and $t \in (t_0,T]$,

$$\begin{aligned}
\left\langle u_1(t),\, u_2(t)\right\rangle &+ 2\int_{t_0}^{t} a_1(u_1(s), u_2(s))ds \\
&+ \int_{t_0}^{t}\left(\left\langle (u_1, w_1)\cdot\nabla u_1 - v_1 + \int_z^0 \theta_{1,x},\, u_2\right\rangle\right) ds \\
&+ \int_{t_0}^{t}\left(\left\langle (u_2, w_2)\cdot\nabla u_2 - v_2 + \int_z^0 \theta_{1,x},\, u_1\right\rangle\right) ds \\
&= \left\langle u_1(t_0),\, u_2(t_0)\right\rangle.
\end{aligned} \tag{3.31}$$

Similarly, we can also obtain, for any $t_0 \in [0,T)$ and $t \in (t_0,T]$,

$$\begin{aligned} \langle v_1(t),\, v_2(t)\rangle &+ 2\int_{t_0}^t a_2(v_1(s), v_2(s))ds \\ &+ \int_{t_0}^t \left(\langle (u_1,w_1)\cdot\nabla v_1 + u_1,\, v_2\rangle\right) ds \\ &+ \int_{t_0}^t \left(\langle (u_2,w_2)\cdot\nabla v_2 + u_2,\, v_1\rangle\right) ds \\ &= \langle v_1(t_0),\, v_2(t_0)\rangle\,, \end{aligned} \tag{3.32}$$

and for any $t_0 \in [0,T)$ and $t \in (t_0,T]$,

$$\begin{aligned} \langle \theta_1(t),\, \theta_2(t)\rangle &+ 2\int_{t_0}^t a_3(\theta_1(s), \theta_2(s))ds \\ &+ \int_{t_0}^t \langle (u_1,w_1)\cdot\nabla \theta_1,\, \theta_2\rangle\, ds \\ &+ \int_{t_0}^t \langle (u_2,w_2)\cdot\nabla \theta_2,\, \theta_1\rangle\, ds \\ &= \langle \theta_1(t_0),\, \theta_2(t_0)\rangle + \int_{t_0}^t \langle Q,\, \theta_1+\theta_2\rangle\, ds. \end{aligned} \tag{3.33}$$

By Defintion 3.1, (u_2, v_2, θ_2) satifies the energy inequalities (3.6)-(3.8) for *almost every* $t \in (0,T]$. By Theorem 3.2, (u_1, v_1, θ_1) satifies the energy equalities (3.17)-(3.19) for any $t_0 \in [0,T)$ and $t \in (t_0,T]$. Combining these with (3.31)-(3.33), we obtain, for *almost every* $t \in (0,T]$,

$$\|\tilde{u}(t)\|^2 + 2\int_0^t \|\tilde{u}\|_{V_1}^2 ds \leqslant -2\int_0^t \left\langle \tilde{u}u_{1,x} + \tilde{w}u_{1,z} + \int_z^0 \tilde{\theta}_x d\zeta + \tilde{v},\, \tilde{u}\right\rangle ds, \tag{3.34}$$

$$\|\tilde{v}(t)\|^2 + 2\int_0^t \|\tilde{v}\|_{V_2}^2 ds \leqslant -2\int_0^t \langle \tilde{u}v_{1,x} + \tilde{w}v_{1,z} - \tilde{u},\, \tilde{v}\rangle\, ds, \tag{3.35}$$

$$\|\tilde{\theta}(t)\|^2 + 2\int_0^t \|\tilde{\theta}\|_{V_3}^2 ds \leqslant -2\int_0^t \left\langle \tilde{u}\theta_{1,x} + \tilde{w}\theta_{1,z},\, \tilde{\theta}\right\rangle ds. \tag{3.36}$$

Notice that some cancellations are used in the derivation of the above inequalities, which can be justified using Lemma 2.1. We omit justification of these cancellations here, since we have done similar justifications before.

Now, we estimate the terms on the right-hand side of equations (3.34)-(3.36). By Lemma 2.1, we have

$$\begin{aligned}
|\langle \tilde{u}u_{1,x},\, \tilde{u}\rangle| &\preccurlyeq \|u_{1,x}\|\|\tilde{u}\|\|\tilde{u}\|_{V_1},\\
|\langle \tilde{w}u_{1,z},\, \tilde{u}\rangle| &\preccurlyeq \|u_{1,z}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}\|_{V_1}^{\frac{3}{2}},\\
|\langle \tilde{u}v_{1,x},\, \tilde{v}\rangle| &\preccurlyeq \|v_{1,x}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}\|_{V_1}^{\frac{1}{2}}\|\tilde{v}\|^{\frac{1}{2}}\|\tilde{v}\|_{V_2}^{\frac{1}{2}},\\
|\langle \tilde{w}v_{1,z},\, \tilde{v}\rangle| &\preccurlyeq \|v_{1,z}\|\|\tilde{u}\|_{V_1}\|\tilde{v}\|^{\frac{1}{2}}\|\tilde{v}\|_{V_2}^{\frac{1}{2}},\\
\left|\left\langle \tilde{u}\theta_{1,x},\, \tilde{\theta}\right\rangle\right| &\preccurlyeq \|\theta_{1,x}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}\|_{V_1}^{\frac{1}{2}}\|\tilde{\theta}\|^{\frac{1}{2}}\|\tilde{\theta}\|_{V_2}^{\frac{1}{2}},\\
\left|\left\langle \tilde{w}\theta_{1,z},\, \tilde{\theta}\right\rangle\right| &\preccurlyeq \|\theta_{1,z}\|\|\tilde{u}\|_{V_1}\|\tilde{\theta}\|^{\frac{1}{2}}\|\tilde{\theta}\|_{V_2}^{\frac{1}{2}}.
\end{aligned} \tag{3.37}$$

Summing up (3.34)-(3.36), applying the estimates in (3.37) and using Cauchy-Schwartz inequality, we get for *almost every* $t \in (0, T]$,

$$\begin{aligned}
\|(\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t))\|_H^2 + \int_0^t \|(\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t))\|_V^2 \\
\preccurlyeq \int_0^t (1 + \|(u_1, v_1, \theta_1)\|_V^4)\|(\tilde{u}, \tilde{v}, \tilde{\theta})\|_H^2 ds.
\end{aligned}$$

Applying a generalized version of Gronwall lemma to the above inequality yields

$$\|(\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t))\|_H = 0, \qquad \text{for a.e. } t \in [0, T].$$

Therefore, $(u_1, v_1, \theta_1) = (u_2, v_2, \theta_2)$, for almost every $t \in [0, T]$.

□

Remark 3.3 We have in fact proved, for general $(\tilde{u}(0), \tilde{v}(0), \tilde{\theta}(0)) \in H$ and $(u_1(0), v_1(0), \theta_1(0)) \in V$, the following Lipschitz continuity property for every[2] $t \in [0, T]$,

$$\begin{aligned}
&\|(\tilde{u}(t), \tilde{v}(t), \tilde{\theta}(t))\|_H^2 \\
\preccurlyeq &\|(\tilde{u}(0), \tilde{v}(0), \tilde{\theta}(0))\|_H^2 \exp\left\{\int_0^t (1 + \|(u_1, v_1, \theta_1)\|_V^4) ds\right\}.
\end{aligned} \tag{3.38}$$

□

[2]See (3.42).

The following theorem gives a much deeper discreption of a weak solution than Definition 3.1 and Theorem 3.1 combined.

Theorem 3.4 *Let $(0,T)$ be the largest interval of existenece for a weak solution (u,v,θ) of the problem (1.3)-(1.10) with $(u_0,v_0,\theta_0)\in H$. Then, $T=\infty$. Moreover,*

$$(u,v,\theta)\in C((0,\infty),V), \tag{3.39}$$

$$(u(t),v(t),\theta(t))\in D(A), \quad \textit{for a.e. } t>0, \tag{3.40}$$

and, for any $t_0\in[0,\infty)$, $t\in(t_0,\infty)$, the energy equalities (3.17)-(3.19) are valid. Moreover,

$$\lim_{t\to 0^+}\|(u(t),v(t),\theta(t))-(u_0,v_0,\theta_0)\|_H=0. \tag{3.41}$$

Therefore,

$$(u,v,\theta)\in C_B([0,\infty),H). \tag{3.42}$$

Proof: Let $(0,T)$ be the largest interval of existence for a weak solution (u,v,θ) of (1.3)-(1.10). Since $(u,v,\theta)\in L^2(0,T;V)$, we have

$$(u(t),v(t),\theta(t))\in V, \quad \text{for a.e. } t\in(0,T].$$

Choose $\tau\in(0,T)$ such that $(u(\tau),v(\tau),\theta(\tau))\in V$ and that (3.6)-(3.8) are satisified with $t_0=\tau$. Then, by Remark 3.2, there is a strong solution (u_1,v_1,θ_1) of (1.3)-(1.10) on $[\tau,\infty)$ such that

$$(u_1(\tau),v_1(\tau),\theta_1(\tau))=(u(\tau),v(\tau),\theta(\tau)).$$

By Definition 3.1, (u,v,θ) is a weak solution on $[\tau,T]$. Therefore, by Theorem 3.3, $(u_1,v_1,\theta_1)=(u,v,\theta)$ on $[\tau,T]$. By Theorem 3.2, (u_1,v_1,θ_1) is also a weak solution on $[\tau,\infty)$. By Theorem 3.3 again, (u_1,v_1,θ_1) is the unique weak solution on $[\tau,\infty)$. This proves $T=\infty$.

Notice that the above τ can be chosen arbitrarily small. Therefore, (3.39) follows from continuity property (3.16) for a strong solution, (3.40) follows from the definition of a strong solution, and by Theorem 3.2, for any $t_0>0$ and all $t\in(t_0,\infty)$, the energy equalities (3.17)-(3.19) are satisfied.

Next, we prove validity of (3.17)-(3.19) for $t_0 = 0$ and all $t > 0$. By Theorem 3.1, there exists a set $E \subset (0, \infty)$ such that (3.11) is valid. So, we can choose a sequence

$$\{t_n\}_{n=1}^{\infty} \subset (0, \infty) \setminus E,$$

which is monotonically decreasing to 0 as $n \to \infty$ and

$$\lim_{n\to\infty} \|(u(t_n), v(t_n), \theta(t_n)) - (u_0, v_0, \theta_0)\|_H = 0. \tag{3.43}$$

Since $t_n > 0$ for every, we have just proved, for any $t > t_n$,

$$\|u(t)\|^2 + 2\int_{t_n}^t \left(\|u(s)\|_{V_1}^2 + \left\langle -v(s) + \int_z^0 \theta_x(s),\, u(s) \right\rangle \right) ds = \|u(t_n)\|^2, \tag{3.44}$$

$$\|v(t)\|^2 + 2\int_{t_n}^t \left(\|v(s)\|_{V_2}^2 + \langle u(s),\, v(s)\rangle \right) ds = \|v(t_n)\|^2, \tag{3.45}$$

$$\|\theta(t)\|^2 + \int_{t_n}^t \left(\|\theta(s)\|_{V_3}^2 - \langle Q,\, \theta(s)\rangle \right) ds = \|\theta(t_n)\|^2. \tag{3.46}$$

Now, take the limit $n \to \infty$ in (3.44)-(3.46) and using the continuity property (3.43) and the fact that $(u, v, \theta) \in L^2(0, \infty; V)$, we have, for any $t > 0$,

$$\|u(t)\|^2 + 2\int_0^t \left(\|u(s)\|_{V_1}^2 + \left\langle -v(s) + \int_z^0 \theta_x(s),\, u(s) \right\rangle \right) ds = \|u_0\|^2, \tag{3.47}$$

$$\|v(t)\|^2 + 2\int_0^t \left(\|v(s)\|_{V_2}^2 + \langle u(s),\, v(s)\rangle \right) ds = \|v_0\|^2, \tag{3.48}$$

$$\|\theta(t)\|^2 + \int_0^t \left(\|\theta(s)\|_{V_3}^2 - \langle Q,\, \theta(s)\rangle \right) ds = \|\theta_0\|^2. \tag{3.49}$$

These are (3.17)-(3.19) for $t_0 = 0$ and all $t > 0$.

Moreover, by (3.47) and that $(u, v, \theta) \in L^2(0, \infty; V)$, we obtain

$$\limsup_{t\to 0^+} \|u(t)\|^2 \leqslant \|u_0\|^2.$$

By (3.10), we also have

$$\|u_0\|^2 \leqslant \liminf_{t\to 0^+} \|u(t)\|^2.$$

Thus,

$$\lim_{t\to 0^+} \|u(t)\|^2 = \|u_0\|^2.$$

Using (3.10) again, we obtain

$$\lim_{t\to 0^+} \|u(t) - u_0\|^2 = 0.$$

Similarly, we have

$$\lim_{t\to 0^+} \|v(t) - v_0\|^2 = \lim_{t\to 0^+} \|\theta(t) - \theta_0\|^2 = 0.$$

This proves (3.41).

Finally, by definition, as a weak solution,

$$(u, v, \theta) \in L^\infty(0, T_0; H), \quad \forall T_0 > 0.$$

As a strong solution, the uniform boundedness in V is valid:

$$(u, v, \theta) \in L^\infty(T_0, \infty; V).$$

Therefore,

$$(u, v, \theta) \in L^\infty(0, \infty; H).$$

Then, (3.42) follows form the above uniform boundedness in H, (3.39) and (3.41).

□

4 A Sufficient Condition for Uniqueness

In this section, we present a new sufficient condition for uniqueness of weak solutions of 2D viscous PE (1.3)-(1.10). First, we mention the following result for a sufficient condition for uniqueness of weak solutions of (1.3)-(1.10):

Proposition 4.1 *Let (u_i, v_i, θ_i), for $i = 1, 2$, be weak solutions of (1.3)-(1.10). Suppose $(u_0, v_0, \theta_0) \in H$ and for some $T > 0$,*

$$(u_{1,z}, v_{1,z}, \theta_{1,z}) \in L^4(0, T; [L^2(D)]^3).$$

Then, $(u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)$ for $t \in [0, T]$.

Proposition 4.1 was proved in [7] for the 2D hydrostatic Navier-Stokes Equations, where v, θ are neglected. In [7], the definition of weak solution is somewhat different from Definition 3.1 and the boundary condtion is also somewhat different.

As the first main result of this section, the following Theorem 4.1 generalizes Proposition 4.1 and allows one to find new classes of weak solutions of the system of (1.3)-(1.10), within which the weak solutions are unique. Especially, it is crucial for proving our main uniqueness result in Section 6.

Theorem 4.1 *Let (u_i, v_i, θ_i), for $i = 1, 2$, be weak solutions of (1.3)-(1.10). Suppose $(u_0, v_0, \theta_0) \in H$ and for some $T > 0$,*

$$(u_{1,z}, v_{1,z}, \theta_{1,z}) \in \left[L^4(0,T; L^2(D)) \cup L^2(0,T; L^\infty_x(L^2_z))\right]^3. \tag{4.1}$$

Then, $(u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)$ for all $t \in [0, T]$.

Proof:

We will prove Lipschitz continuity of the weak solutions with respect to initial data in L^2, assuming (u_1, v_1, θ_1) satisfies the regularity condition (4.1).

Denote:

$$\widetilde{u} = u_1 - u_2, \quad \widetilde{v} = v_1 - v_2, \quad \widetilde{\theta} = \theta_1 - \theta_2,$$

$$\widetilde{u}_0 = u_{1,0} - u_{2,0}, \quad \widetilde{v}_0 = v_{1,0} - v_{2,0}, \quad \widetilde{\theta}_0 = \theta_{1,0} - \theta_{2,0},$$

and

$$\widetilde{w} = w_1 - w_2, \quad \widetilde{q} = q_1 - q_2.$$

Let $t_0 \in (0, T)$. Then (u_1, v_1, θ_1) and (u_2, v_2, θ_2) are both strong solutions on $[t_0, T]$. Therefore, we can follow the proof of Theorem 3.3 and apply it to (u_1, v_1, θ_1) and (u_2, v_2, θ_2) on $[t_0, T]$ to obtain, for *every* $t \in [t_0, T]$,

$$\begin{aligned} &\|\tilde{u}(t)\|^2 + 2\int_{t_0}^t \|\tilde{u}\|^2_{V_1} ds \\ &\quad = \|\tilde{u}(t_0)\|^2 - 2\int_{t_0}^t \left\langle \tilde{u}u_{1,x} + \tilde{w}u_{1,z} + \int_z^0 \tilde{\theta}_x d\zeta + \tilde{v},\ \tilde{u} \right\rangle ds, \end{aligned} \tag{4.2}$$

$$\|\tilde{v}(t)\|^2 + 2\int_{t_0}^{t} \|\tilde{v}\|_{V_2}^2 ds = \|\tilde{v}(t_0)\|^2 - 2\int_{t_0}^{t} \langle \tilde{u}v_{1,x} + \tilde{w}v_{1,z} - \tilde{u},\, \tilde{v}\rangle\, ds, \quad (4.3)$$

$$\|\tilde{\theta}(t)\|^2 + 2\int_{t_0}^{t} \|\tilde{\theta}\|_{V_3}^2 ds = \|\tilde{\theta}(t_0)\|^2 - 2\int_{t_0}^{t} \left\langle \tilde{u}\theta_{1,x} + \tilde{w}\theta_{1,z},\, \tilde{\theta}\right\rangle ds. \quad (4.4)$$

Notice that *equalities* (4.2)-(4.4) are obtained similarly to *inequailies* (3.34)-(3.36). Since (u_1, v_1, θ_1) and (u_2, v_2, θ_2) are both strong solutions on $[t_0, T]$, we indeed have above *equalities* and they are valid for *every* $t \in [t_0, T]$.

By Theorem 3.4, $(u_i, v_i, \theta_i) \in C([t_0, T], V)$ for $i = 1, 2$. Thus, (4.2)-(4.4) along with Lemma 2.1 imply $\|\tilde{u}\|, \|\tilde{v}\|, \|\tilde{\theta}\| \in C^1([t_0, T])$. Therefore, we have in *classic sense for every* $t \in [t_0, T]$ that

$$\frac{1}{2}\frac{d}{dt}\|\tilde{u}\|^2 + \|\tilde{u}\|_{V_1}^2 = -\left\langle \tilde{u}u_{1,x} + \tilde{w}u_{1,z} + \int_z^0 \tilde{\theta}_x d\zeta + \tilde{v},\, \tilde{u}\right\rangle, \quad (4.5)$$

$$\frac{1}{2}\frac{d}{dt}\|\tilde{v}\|^2 + \|\tilde{v}\|_{V_2}^2 = -\left\langle v_{1,x}\tilde{u} + \tilde{w}v_{1,z} - \tilde{u},\, \tilde{v}\right\rangle, \quad (4.6)$$

$$\frac{1}{2}\frac{d}{dt}\|\tilde{\theta}\|^2 + \|\tilde{\theta}\|_{V_3}^2 = -\left\langle \tilde{u}\theta_{1,x} + \tilde{w}\theta_{1,z},\, \tilde{\theta}\right\rangle. \quad (4.7)$$

Summing up (4.5)-(4.7) yields, for every $t \in [t_0, T]$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|^2 + \|\tilde{v}\|^2 + \|\tilde{\theta}\|^2\right) + \|\tilde{u}\|_{V_1}^2 + \|\tilde{v}\|_{V_2}^2 + \|\tilde{\theta}\|_{V_3}^2 \\ &= -\left\langle \tilde{u}u_{1,x} + \tilde{w}u_{1,z} + \int_z^0 \tilde{\theta}_x d\zeta,\, \tilde{u}\right\rangle \\ &\quad - \left\langle v_{1,x}\tilde{u} + \tilde{w}v_{1,z},\, \tilde{v}\right\rangle - \left\langle \tilde{u}\theta_{1,x} + \tilde{w}\theta_{1,z},\, \tilde{\theta}\right\rangle \end{aligned} \quad (4.8)$$

Now, we estimates all the terms on the right side of (4.8). The bilinear term is easily estimated by Cauchy-Schwartz inequality:

$$\left|\left\langle \int_z^0 \tilde{\theta}_x d\zeta,\, \tilde{u}\right\rangle\right| \leqslant \|\tilde{\theta}_x\|\|\tilde{u}\| \leqslant C_\varepsilon\|\tilde{u}\|^2 + \varepsilon\|\tilde{\theta}_x\|^2. \quad (4.9)$$

Next, by Agmon's inequality, we have

$$\begin{aligned} |\langle \tilde{u}u_{1,x},\, \tilde{u}\rangle| &\leqslant \int_0^1 \|\tilde{u}\|_{L_z^\infty}\left(\int_{-h}^0 |u_{1,x}\tilde{u}|dz\right)dx \\ &\leqslant \int_0^1 \|\tilde{u}\|_{L_z^2}^{\frac{1}{2}}\|\tilde{u}_z\|_{L_z^2}^{\frac{1}{2}}\|u_{1,x}\|_{L_z^2}\|\tilde{u}\|_{L_z^2}dx \\ &\leqslant \|\tilde{u}\|_{L_x^\infty(L_z^2)}\int_0^1 \|\tilde{u}\|_{L_z^2}^{\frac{1}{2}}\|\tilde{u}_z\|_{L_z^2}^{\frac{1}{2}}\|u_{1,x}\|_{L_z^2}dx \\ &\leqslant \|\tilde{u}\|_{L_x^\infty(L_z^2)}\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_z\|^{\frac{1}{2}}\|u_{1,x}\|, \end{aligned}$$

where Hölder's inequality is applied in the last step. Then, by Minkowski's inequality:

$$\|\tilde{u}\|_{L^\infty_x(L^2_z)} \leqslant \|\tilde{u}\|_{L^2_z(L^\infty_x)},$$

we get

$$\begin{aligned}
|\langle \tilde{u}u_{1,x},\, \tilde{u}\rangle| \leqslant& \|\tilde{u}\|_{L^2_z(L^\infty_x)}\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_z\|^{\frac{1}{2}}\|u_{1,x}\| \\
\leqslant& \text{ (Agmon's inequality)} \\
\leqslant& \|u_{1,x}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_x\|^{\frac{1}{2}}\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_z\|^{\frac{1}{2}} \\
=& \|u_{1,x}\|\|\tilde{u}\|\|\tilde{u}_x\|^{\frac{1}{2}}\|\tilde{u}_z\|^{\frac{1}{2}} \\
\leqslant& C_\varepsilon\|u_{1,x}\|^2\|\tilde{u}\|^2 + \varepsilon\|\nabla\tilde{u}\|^2.
\end{aligned} \tag{4.10}$$

It follows similarly that

$$\begin{aligned}
|\langle \tilde{u}v_{1,x},\, \tilde{v}\rangle| \leqslant& \|v_{1,x}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_x\|^{\frac{1}{2}}\|\tilde{v}\|^{\frac{1}{2}}\|\tilde{v}_z\|^{\frac{1}{2}} \\
\leqslant& C_\varepsilon\|v_{1,x}\|^2(\|\tilde{u}\|^2 + \|\tilde{v}\|^2) + \varepsilon(\|\tilde{u}_x\|^2 + \|\tilde{v}_z\|^2),
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\left|\left\langle \tilde{u}\theta_{1,x},\, \tilde{\theta}\right\rangle\right| \leqslant& \|\theta_{1,x}\|\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_x\|^{\frac{1}{2}}\|\tilde{\theta}\|^{\frac{1}{2}}\|\tilde{\theta}_z\|^{\frac{1}{2}} \\
\leqslant& C_\varepsilon\|\theta_{1,x}\|^2(\|\tilde{u}\|^2 + \|\tilde{\theta}\|^2) + \varepsilon(\|\tilde{u}_x\|^2 + \|\tilde{\theta}_z\|^2).
\end{aligned} \tag{4.12}$$

Finally, similar to above estimates, we have

$$\begin{aligned}
|\langle \tilde{w}u_{1,z},\, \tilde{u}\rangle| \leqslant& \int_0^1 \|\tilde{w}\|_{L^\infty_z}\left(\int_{-h}^0 |u_{1,z}\tilde{u}|dz\right)dx \\
\leqslant& \int_0^1 \|\tilde{w}\|^{\frac{1}{2}}_{L^2_z}\|\tilde{w}_z\|^{\frac{1}{2}}_{L^2_z}\|u_{1,z}\|_{L^2_z}\|\tilde{u}\|_{L^2_z}dx.
\end{aligned}$$

The right-hand side of the above inequality can be further estimated in two different ways:

$$\begin{aligned}
|\langle \tilde{w}u_{1,z},\, \tilde{u}\rangle| \leqslant& C\|\tilde{u}\|_{L^\infty_x(L^2_z)}\int_0^1 \|\tilde{u}_x\|_{L^2_z}\|u_{1,z}\|_{L^2_z}dx \\
\leqslant& C\|\tilde{u}\|_{L^2_z(L^\infty_x)}\|u_{1,z}\|\|\tilde{u}_x\| \\
\leqslant& C\|\tilde{u}\|^{\frac{1}{2}}\|\tilde{u}_x\|^{\frac{1}{2}}\|u_{1,z}\|\|\tilde{u}_x\| \\
\leqslant& C_\varepsilon\|u_{1,z}\|^4\|\tilde{u}\|^2 + \varepsilon\|\tilde{u}_x\|^2
\end{aligned} \tag{4.13}$$

and

$$
\begin{aligned}
|\langle \tilde{w} u_{1,z},\ \tilde{u}\rangle| \leqslant & C\|u_{1,z}\|_{L_x^\infty(L_z^2)} \int_0^1 \|\tilde{u}_x\|_{L_z^2}\|\tilde{u}\|_{L_z^2} dx \\
\leqslant & C\|u_{1,z}\|_{L_x^\infty(L_z^2)}\|\tilde{u}_x\|\|\tilde{u}\| \\
\leqslant & C_\varepsilon\|u_{1,z}\|^2_{L_x^\infty(L_z^2)}\|\tilde{u}\|^2 + \varepsilon\|\tilde{u}_x\|^2.
\end{aligned} \tag{4.14}
$$

In the above estimates (4.13) and (4.14), we have used ideas similar to those used in (4.10). It now follows, similar to (4.13) and (4.14), that

$$
\begin{aligned}
|\langle \tilde{w} v_{1,z},\ \tilde{v}\rangle| \leqslant & C\|\tilde{v}\|_{L_x^\infty(L_z^2)} \int_0^1 \|\tilde{u}_x\|_{L_z^2}\|v_{1,z}\|_{L_z^2} dx \\
\leqslant & C\|\tilde{v}\|_{L_z^2(L_x^\infty)}\|v_{1,z}\|\|\tilde{u}_x\| \\
\leqslant & C\|\tilde{v}\|^{\frac{1}{2}}\|\tilde{v}_x\|^{\frac{1}{2}}\|v_{1,z}\|\|\tilde{u}_x\| \\
\leqslant & C_\varepsilon\|v_{1,z}\|^4\|\tilde{v}\|^2 + \varepsilon(\|\tilde{u}_x\|^2 + \|\tilde{v}_x\|^2),
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
|\langle \tilde{w} v_{1,z},\ \tilde{v}\rangle| \leqslant & C\|v_{1,z}\|_{L_x^\infty(L_z^2)} \int_0^1 \|\tilde{u}_x\|_{L_z^2}\|\tilde{v}\|_{L_z^2} dx \\
\leqslant & C\|v_{1,z}\|_{L_x^\infty(L_z^2)}\|\tilde{u}_x\|\|\tilde{v}\| \\
\leqslant & C_\varepsilon\|v_{1,z}\|^2_{L_x^\infty(L_z^2)}\|\tilde{v}\|^2 + \varepsilon\|\tilde{u}_x\|^2,
\end{aligned} \tag{4.16}
$$

and

$$
\left|\left\langle \tilde{w}\theta_{1,z},\ \tilde{\theta}\right\rangle\right| \leqslant C_\varepsilon\|\tilde{\theta}_{1,z}\|^4\|\tilde{u}\|^2 + \varepsilon(\|\tilde{u}_x\|^2 + \|\tilde{\theta}_x\|^2), \tag{4.17}
$$

$$
\left|\left\langle \tilde{w}\theta_{1,z},\ \tilde{\theta}\right\rangle\right| \leqslant C_\varepsilon\|\theta_{1,z}\|^2_{L_x^\infty(L_z^2)}\|\tilde{\theta}\|^2 + \varepsilon\|\tilde{u}_x\|^2. \tag{4.18}
$$

Plug the estimates (4.9)-(4.12), (4.13) or (4.14), (4.15) or (4.16), and (4.17) or (4.18) into (4.8) and choose sufficiently small $\varepsilon > 0$ to collect the dissipation terms. Then, assume that (u_1, v_1, θ_1) satisfies (4.1) to apply Gronwall lemma to finish the proof. Notice that global regularity result of weak solutions of (1.3)-(1.10) is also used in justifying applicability of Gronwall lemma. As an example to demonstrate the datails, we now finish the proof for a special case of (4.1) when (u_1, v_1, θ_1) satisfies:

$$
(u_1, v_1, \theta_1)_z \in \left[L^2(0,T; L_x^\infty(L_z^2))\right]^3. \tag{4.19}
$$

Plug (4.9)-(4.12), (4.14), (4.16) and (4.18) into (4.8) and choose sufficiently small $\varepsilon > 0$, we obtain, for $t \in [t_0, T]$,

$$\begin{aligned}
&\frac{d}{dt}\|(\tilde{u},\tilde{v},\tilde{\theta})\|_H^2 + \|(\tilde{u},\tilde{v},\tilde{\theta})\|_V^2 \\
&\preccurlyeq \left[1 + \|(u_1,v_1,\theta_1)_x\|_H^2 + \|(u_1,v_1,\theta_1)_z\|^2_{(L^\infty_x(L^2_z))^3}\right] \|(\tilde{u},\tilde{v},\tilde{\theta})\|_H^2.
\end{aligned} \tag{4.20}$$

Noticing (4.19), we can use Gronwall inequality to obtain, for $t \in [t_0, T]$,

$$\begin{aligned}
&\|(\tilde{u}(t),\tilde{v}(t),\tilde{\theta}(t))\|_H^2 \\
\preccurlyeq&\|(\tilde{u}(t_0),\tilde{v}(t_0),\tilde{\theta}(t_0))\|_H^2 \\
&\times \exp\left\{\int_0^t \left[1 + \|(u_1,v_1,\theta_1)_x\|_H^2 + \|(u_1,v_1,\theta_1)_z\|^2_{(L^\infty_x(L^2_z))^3}\right] ds\right\}.
\end{aligned}$$

Now, take the limit $t_0 \to 0+$ and use Theorem 3.4, we get

$$\begin{aligned}
&\|(\tilde{u}(t),\tilde{v}(t),\tilde{\theta}(t))\|_H^2 \\
\preccurlyeq&\|(\tilde{u}_0,\tilde{v}_0,\tilde{\theta}_0)\|_H^2 \\
&\times \exp\left\{\int_0^t \left[1 + \|(u_1,v_1,\theta_1)_x\|_H^2 + \|(u_1,v_1,\theta_1)_z\|^2_{(L^\infty_x(L^2_z))^3}\right] ds\right\}.
\end{aligned}$$

Since the above inequality is independent of t_0 and t_0 can be chosen arbitrarily small, it is valid for all $t \in (0, T]$. This proves that $(u_1, v_1, \theta_1) \equiv (u_2, v_2, \theta_2)$, if $(\tilde{u}_0, \tilde{v}_0, \tilde{\theta}_0) = (0, 0, 0)$. The other cases covered by (4.1) can be similarly proved.

□

5 Global Existence

In this section, we prove global in time *uniform boundedness* of the norms of some partial derivatives of the weak solutions. These results are also important for proving our uniqueness result in Section 6.

Let us mention first that it is not immediately clear whether or not the global uniform $L^2_x H^{\frac{1}{2}}_z$ boundedness for the solutions of the 2D hydrostatic Navier-Stokes equations as obtained in [3] can be extended to the problem of (1.3)-(1.10). This is due to the fact that the boundary conditions (1.6) and (1.7) for (u, v) are different from the boundary condition (1.9) for θ and the possibility that α_1 and α_2 may be different. This problem will be studied elsewhere.

We begin with a theorem for global in time uniform boundedness of (u_z, v_z, θ_z) in $[L^2(D)]^3$ for (1.3)-(1.10). Recall that global existence and uniqueness of z-weak solutions were proved in [20] for 2D viscous PE in the case of periodic boundary conditions; and in [10] for 3D viscous PE in case of Neumann boundary condition for horizontal velocity at bottom of the physical domain. However, these analyses do not apply to the system (1.3)-(1.10) due to different boundary conditions. A possible approach might be a proper modification of that of [3] in obtaining boundedness for z-weak solutions of the simplified 2D hydrostatic Navier-Stokes equations, where v and θ were omitted. Nevertheless, new issues will come up again due to boundary conditions. Instead, we will take advantage of a result of [21] directly in our proof of the following Theorem 5.1.

Theorem 5.1 *Suppose $Q \in L^2(D)$, $(u_0, v_0, \theta_0) \in H$ and (u, v, θ) is a weak solution of (1.3)-(1.10). The following statements are valid:*

(a) *If $\partial_z u_0 \in L^2(D)$, then there exists a weak solution (u, v, θ) of (1.3)-(1.10), such that*

$$u_z \in L^\infty(0, \infty; L^2(D)) \cap L^2(0, \infty; H^1(D)).$$

Moreover, there exists a bounded absorbing set for u_z in $L^2(D)$.

(b) *If $(\partial_z u_0, \partial_z v_0) \in (L^2(D))^2$, then there exists a weak solution (u, v, θ) of (1.3)-(1.10), such that*

$$(u_z, v_z) \in L^\infty(0, \infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_z, v_z) in $[L^2(D)]^2$.

(c) *If $(\partial_z u_0, \partial_z \theta_0) \in (L^2(D))^2$, then there exists a weak solution (u, v, θ) of (1.3)-(1.10), such that*

$$(u_z, \theta_z)) \in L^\infty(0, \infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_z, θ_z) in $[L^2(D)]^2$.

Remark: Quite unexpectedly, it seems to be a non-trivial problem whether or not global in time uniform boundedness of $\|(u_z, v_x)\|_{[L^2(D)]^2}$ or $\|(u_z, \theta_x)\|_{[L^2(D)]^2}$ is still valid when $(u_{0,z}, v_{0,x}) \in [L^2(D)]^2$ or $(u_{0,z}, \theta_{0,x}) \in [L^2(D)]^2$.

$\square$

Proof:

Step 1. Proof of part (a) of Theorem 5.1.

By Theorem 3.4, we can choose a monotonically decreasing sequence

$$\{t_n\}_{n=1}^{\infty} \subset (0, \infty), \text{ such that } \lim_{n\to\infty} t_n = 0,$$

and

$$(u, v, \theta) \in C([t_n, \infty), V) \cap L^2(t_n, \infty; D(A)), \ \forall\, n \geqslant 1. \tag{5.1}$$

Moreover, there exists an absorbing set for (u, v, θ) in V, when the time interval $[t_1, \infty)$ is considered. Therefore, *what is still needed to be proved is just the following:*

$$u_z \in L^{\infty}(0, t_1; L^2(D)) \cap L^2(0, t_1; H^1(D)). \tag{5.2}$$

By the estimate of $\|u_z\|$ in §3.3 of [21] for a strong solution (u, v, θ) on $[t_n, \infty)$ with initial data $(u(t_n), v(t_n), \theta(t_n) \in V$ and by Theorem 3.3, we have for almost every $t \in [t_n, \infty)$,

$$\begin{aligned} &\frac{d}{dt}\left(\|u_z(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2\right) + \|\nabla u_z\|^2 + \alpha_1 \|u_x|_{z=0}\|^2 \\ &\quad \preccurlyeq \|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2. \end{aligned} \tag{5.3}$$

Notice that (5.1) is used in deriving (5.3). Therefore, we have for $t \in [t_n, t_1]$ with $n > 1$,

$$\begin{aligned} &\|u_z(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \int_{t_n}^{t} \left(\|\nabla u_z\|^2 + \alpha_1 \|u_x|_{z=0}\|^2\right) ds \\ \leqslant &\|u_z(t_n)\|^2 + \alpha_1 \|u(t_n)|_{z=0}\|^2 + C \int_{t_n}^{t} \left[\|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2\right] ds. \end{aligned} \tag{5.4}$$

Since $u(t_n) \in V_1$, we have

$$\|u(t_n)|_{z=0}\| = \left\| \int_{-h}^{0} u_z(t_n) dz \right\| \preccurlyeq \|u_z(t_n)\|. \tag{5.5}$$

Due to the fact that (u, v, θ) is a weak solution on $(0, \infty)$, we also have, for $t \in [t_n, t_1]$,

$$\int_{t_n}^{t} (\|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2) ds \leqslant \int_0^{t_1} (\|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2) ds, \tag{5.6}$$

the upper bound of which depends only on $\|(u_0, v_0, \theta_0)\|_H$, $\|Q\|$ and t_1. Combining (5.4)-(5.6), we have, for $t \in [t_n, t_1]$,

$$\begin{aligned} &\|u_z(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \int_{t_n}^{t} \left(\|\nabla u_z\|^2 + \alpha_1 \|u_x|_{z=0}\|^2\right) ds \\ \preccurlyeq &\|u_z(t_n)\|^2 + \int_0^{t_1} \left[\|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2\right] ds. \end{aligned} \tag{5.7}$$

Now, choose any $\phi \in \mathscr{C}_0^\infty(D)$. Then, $\phi_z \in V_1$. Thus, by weak continuity of (u, v, θ) on $[0, \infty)$ (see Theorem 3.1), we have

$$\lim_{n\to\infty} \langle u_z(t_n) - \partial_z u_0,\, \phi\rangle = -\lim_{n\to\infty} \langle u(t_n) - u_0,\, \phi_z\rangle = 0.$$

Since $\mathscr{C}_0^\infty(D)$ is dense in $L^2(D)$, we have weak convergence:

$$u_z(t_n) \rightharpoonup \partial_z u_0, \quad \text{in} \quad L^2(D).$$

Therefore, $\{u_z(t_n)\}_{n=1}^\infty$ is bounded in $L^2(D)$. Now, taking the limit $t_n \to 0$ in (5.7), we have, for all $t \in (0, t_1)$,

$$\begin{aligned} &\|u_z(t)\|^2 + \alpha_1 \|u(t)|_{z=0}\|^2 + \int_0^{t} \left(\|\nabla u_z\|^2 + \alpha_1 \|u_x|_{z=0}\|^2\right) ds \\ \preccurlyeq &\sup_{n\geqslant 1} \|u_z(t_n)\|^2 + \int_0^{t_1} \left[\|\nabla u\|^2 + \|v\|^2 + \|\theta_x\|^2\right] ds. \end{aligned} \tag{5.8}$$

This proves (5.2) and thus finishes **Step 1**.

Step 2. Proof of Theorem 5.1 part (b).

For simplicity of presentation, in the proof of part (b) of Theorem 5.1, we will only provide the key estimate of $\|v_z\|$ under the assumption that (u, v, θ) is a *strong* solution on $[0, \infty)$. The justification that this estimate is sufficient for a rigorous proof of part (b) of Theorem 5.1 is almost the same as the one we provided in **Step 1** for our proof of part (a) of Theorem 5.1. Thus, it is omitted for conciseness.

Taking inner product of (1.4) with $-v_{zz}$ yields:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|v_z\|^2+\alpha_2\|v(z=0)\|^2\right)+&\|\nabla v_z\|^2+\alpha_2\|v_x(z=0)\|^2 \\ &=\langle uv_x+wv_z-u,\, v_{zz}\rangle\,. \end{aligned} \tag{5.9}$$

The following computations are used in deriving (5.9):

$$\begin{aligned} -\int v_t v_{zz} =&-\int_0^1\left(v_tv_z\Big|_{z=-h}^0-\int_{-h}^0 v_{zt}v_z\right)dx \\ =&\frac{1}{2}\frac{d}{dt}\|v_z\|^2+\int_0^1 v_t\alpha_2 v\Big|_{z=0} \\ =&\frac{1}{2}\frac{d}{dt}\|v_z\|^2+\frac{\alpha_2}{2}\frac{d}{dt}\|v(z=0)\|^2, \end{aligned}$$

$$\begin{aligned} \int_D v_{zz}v_{xx} =&\int_{-h}^0\left(v_{zz}v_x\Big|_{x=0}^1-\int_0^1 v_{xzz}v_xdx\right)dz \\ =&-\int_D v_{xzz}v_xdxdz \\ =&-\int_0^1\left(v_{xz}v_x\Big|_{z=-h}^0-\int_{-h}^0 v_{xz}^2dz\right)dx \\ =&\|v_{xz}\|^2+\alpha_2\|v_x(z=0)\|^2. \end{aligned}$$

The two trilinear terms on the right-hand side of (5.9) will be estimated in the following.
First, we have

$$\begin{aligned} \int_D wv_zv_{zz} =&\frac{1}{2}\int_D w\partial_z(v_z^2) \\ =&\frac{1}{2}\int_0^1\left(wv_z^2\Big|_{-h}^0-\int_{-h}^0 w_zv_z^2\right)dx \\ =&\frac{1}{2}\int_D u_xv_z^2 \\ \leqslant& C\|u_x\|\|v_z\|\|v_{zx}\|^{\frac{1}{2}}\|v_{zz}\|^{\frac{1}{2}} \\ \leqslant& C_\varepsilon\|u_x\|^2\|v_z\|^2+\varepsilon\|\nabla v_z\|^2. \end{aligned} \tag{5.10}$$

Contrary to the common intuition from experience, the other trilinear term is more complicated to deal with. Integrating by parts and applying boundary conditions, one has

$$
\begin{aligned}
\int_D uv_x v_{zz} =& \int_0^1 \left[uv_x v_z \Big|_{z=-h}^{0} - \int_{-h}^{0} (u_z v_x v_z + uv_{xz} v_z) dz \right] dx \\
=& - \alpha_2 \int_0^1 uv_x v \Big|_{z=0} dx - \int_D u_z v_x v_z dxdz - \int_D uv_{xz} v_z dxdz \\
=:& I_0 + I_1 + I_2.
\end{aligned}
$$

In the following, we estimate I_0, I_1 and I_2 respectively. First, we have

$$
\begin{aligned}
|I_1| \leqslant& \|u_z\| \|v_x\|^{\frac{1}{2}} \|v_{xz}\|^{\frac{1}{2}} \|v_z\|^{\frac{1}{2}} \|v_{zx}\|^{\frac{1}{2}} \\
\leqslant& \|u_z\| \|v_x\|^{\frac{1}{2}} \|v_z\|^{\frac{1}{2}} \|v_{xz}\| \\
\leqslant& C_\varepsilon (\|u_z\|^4 + \|v_x\|^2 \|v_z\|^2) + \frac{\varepsilon}{2} \|v_{xz}\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| \leqslant& \|v_{xz}\| \|u\|^{\frac{1}{2}} \|u_z\|^{\frac{1}{2}} \|v_z\|^{\frac{1}{2}} \|v_{xz}\|^{\frac{1}{2}} \\
=& \|u\|^{\frac{1}{2}} \|u_z\|^{\frac{1}{2}} \|v_z\|^{\frac{1}{2}} \|v_{xz}\|^{\frac{3}{2}} \\
\leqslant& C_\varepsilon \|u\|^2 \|u_z\|^2 \|v_z\|^2 + \frac{\varepsilon}{2} \|v_{xz}\|^2.
\end{aligned}
$$

Noticing that $u(0, z, t) = 0$, we have

$$
u^2(x, 0, t) = 2 \int_0^x u(\xi, 0, t) u_x(\xi, 0, t) d\xi.
$$

Thus,

$$
\|u(z=0)\|_\infty^2 \leqslant 2 \|u(z=0)\| \|u_x(z=0)\|.
$$

So, we can estimate I_0 as following:

$$
\begin{aligned}
|I_0| \leqslant& \alpha_2 \|u(z=0)\|_\infty \|v(z=0)\| \|v_x(z=0)\| \\
\leqslant& \frac{\alpha_2}{2} \|u(z=0)\|_\infty^2 \|v(z=0)\|^2 + \frac{\alpha_2}{2} \|v_x(z=0)\|^2 \\
\leqslant& \alpha_2 \|u(z=0)\| \|u_x(z=0)\| \|v(z=0)\|^2 + \frac{\alpha_2}{2} \|v_x(z=0)\|^2.
\end{aligned}
\tag{5.11}
$$

Due to the boundary condition $u(z=-h) = 0$, it holds that,

$$
u(z=0) = \int_{-h}^{0} u_z(x, \zeta, t) d\zeta,
$$

and thus

$$
\begin{aligned}
\|u(z=0)\| = & \left[\int_0^1 \left(\int_{-h}^0 u_z d\zeta\right)^2 dx\right]^{\frac{1}{2}} \\
\leqslant & \left[\int_0^1 \left(\int_{-h}^0 |u_z| d\zeta\right)^2 dx\right]^{\frac{1}{2}} \\
\leqslant & \text{ (Minkowski inequality)} \\
\leqslant & \int_{-h}^0 \left(\int_0^1 u_z^2 dx\right)^{\frac{1}{2}} dz \\
\leqslant & h^{\frac{1}{2}} \|u_z\|.
\end{aligned}
\tag{5.12}
$$

Similar to (5.12), we also have $\|u_x(z=0)\| \leqslant h^{\frac{1}{2}}\|u_{xz}\|$. Therefore,

$$
I_0 \leqslant \alpha_2 h \|u_z\| \|u_{xz}\| \|v(z=0)\|^2 + \frac{\alpha_2}{2}\|v_x(z=0)\|^2.
$$

Combining the above estimates of I_1, I_2 and I_0, we have

$$
\begin{aligned}
\left|\int_D u v_x v_{zz} dx dz\right| \leqslant & C_\varepsilon (\|u\|^2 \|u_z\|^2 + \|v_x\|^2) \|v_z\|^2 \\
& + \alpha_2 h \|u_z\| \|u_{xz}\| \|v(z=0)\|^2 + C_\varepsilon \|u_z\|^4 \\
& + \varepsilon \|v_{xz}\|^2 + \frac{\alpha_2}{2} \|v_x(z=0)\|^2.
\end{aligned}
\tag{5.13}
$$

Finally, it follows from (5.9), (5.10) and (5.13) that, for $\varepsilon > 0$ chosen sufficiently small,

$$
\begin{aligned}
& \frac{d}{dt}\left(\|v_z\|^2 + \alpha_2 \|v(z=0)\|^2\right) + \|\nabla v_z\|^2 + \alpha_2 \|v_x(z=0)\|^2 \\
\leqslant & C(\|u_x\|^2 + \|v_x\|^2 + \|u\|^2\|u_z\|^2)\|v_z\|^2 \\
& + C\left(\alpha_2 \|u_z\| \|u_{xz}\| \|v(z=0)\|^2 + \|u_z\|^4 + \|u\|^2\right) \\
\leqslant & C(\|u_x\|^2 + \|v_x\|^2 + \|u\|^2\|u_z\|^2 + \|u_z\|\|u_{xz}\|) \\
& \times (\|v_z\|^2 + \alpha_2 \|v(z=0)\|^2) + C(\|u_z\|^4 + \|u\|^2).
\end{aligned}
\tag{5.14}
$$

Notice that, by (5.12), for $v_{0,z} \in L^2$,

$$
\|v_0(z=0)\| \leqslant h^{\frac{1}{2}} \|v_{0,z}\| < \infty.
$$

By, Theorem 5.1 (a), (5.14) and Gronwall lemma, we have local in time boundedness of $\|v_z\|$ on some interval $[0, t_0]$, with $t_0 > 0$. This yields global uniform boundedness and an absorbing set for $\|v_z\|$, since we have uniform boundedness and an absorbing set for (u, v, θ) in V when considered in $[t_0, \infty)$ as a strong solution.

Finally, with (5.14) we can jusify as in our proof of part (a), that Theorem 5.1 (b) is valid for a weak solution (u, v, θ) when $\partial_z u_0, \partial_z v_0 \in L^2(D)$.

Proof of Theorem 5.1 (c) is similar to that for (b).

□

Notice that vertical regularity $(u_z, v_z, \theta_z) \in L^2$ of weak solutions of 2D and 3D viscous PE played a very prominent or even crucial role in almost all previous analytic works in dealing with solution regularity and uniqueness properties, for example, as shown in (4.1). However, to the contrary of this intuitive impression, we see next that horizontal regularity might actually force weak solutions to behave somewhat better, at least for the 2D problem. This is manifested in the following Theorem 5.2, which is our second main result of this section.

Theorem 5.2 *Suppose $Q \in L^2(D)$, $(u_0, v_0, \theta_0) \in H$ and (u, v, θ) is a weak solution of (1.3)-(1.10). The following statements are valid:*

(a) *If $\partial_x u_0 \in L^2(D)$, then*

$$u_x \in L^\infty(0, +\infty; L^2(D)) \cap L^2(0, \infty; H^1(D)).$$

Moreover, there exists a bounded absorbing set for u_x in $L^2(D)$.

(b) *If $(\partial_x u_0, \partial_x v_0) \in [L^2(D)]^2$, then*

$$(u_x, v_x) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_x, v_x) in $[L^2(D)]^2$.

(c) *If $(\partial_x u_0, \partial_x \theta_0) \in [L^2(D)]^2$, then*

$$(u_x, \theta_x) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_x, θ_x) in $[L^2(D)]^2$.

(d) *If* $(\partial_x u_0, \partial_z v_0) \in [L^2(D)]^2$*, then*

$$(u_x, v_z) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_x, v_z) *in* $[L^2(D)]^2$*.*

(e) *If* $(\partial_x u_0, \partial_z \theta_0) \in [L^2(D)]^2$*, then*

$$(u_x, \theta_z) \in L^\infty(0, +\infty; [L^2(D)]^2) \cap L^2(0, \infty; [H^1(D)]^2).$$

Moreover, there exists a bounded absorbing set for (u_x, θ_z) *in* $[L^2(D)]^2$*.*

Proof:

Again, for simplicity of presentation, in the proof of Theorem 5.2, we will only provide the key estimates of $\|u_x\|$, $\|v_x\|$, $\|\theta_x\|$, $\|v_z\|$ and $\|\theta_z\|$ under the assumption that (u, v, θ) is a *strong* solution on $[0, \infty)$. The justification that these estimates are sufficient for a rigorous proof of Theorem 5.2 is almost the same as the one we provided in our proof of Theorem 5.1 (a). Thus, it is omitted for conciseness.

We provide these key estimates in three steps.

Step 1. Estimate for $\|u_x\|_2$.

Taking inner product of (1.3) with $-u_{xx}$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2 + &\|\nabla u_x\|^2 + \alpha_1 \|u_x(z=0)\|^2 \\ &= \left\langle u u_x + w u_z + v + q_x + \int_z^0 \theta_x(x, \zeta, t) d\zeta,\ u_{xx} \right\rangle.\end{aligned} \tag{5.15}$$

The following computations, along with the boundary conditions (1.6)-(1.9) and the constraint (1.11), are used in the derivation of (5.15):

$$-\int_D u_t u_{xx} dx dz = -\int_{-h}^0 \left(u_t u_x \Big|_{x=0}^1 - \int_0^1 u_{xt} u_x \right) = \frac{1}{2}\frac{d}{dt}\|u_x\|^2,$$

$$\begin{aligned}\int_D u_{xx} u_{zz} dx dz &= \int_{-h}^0 \left(u_x u_{zz} \Big|_{x=0}^1 - \int_0^1 u_x u_{xzz} dx \right) dz \\ &= -\int_D u_x u_{xzz} dx dz \\ &= -\int_0^1 \left(u_x u_{xz} \Big|_{z=-h}^0 - \int_{-h}^0 |u_{xz}|^2 dz \right) dx \\ &= \alpha_1 \|u_x(z=0)\|^2 + \|u_{xz}\|^2,\end{aligned}$$

$$\int_D q_x u_{xx} dxdz = \int_0^1 q_x \left(\int_{-h}^0 u_{xx} dz \right) dx = \int_0^1 q_x \partial_x \left(\int_{-h}^0 u_x dz \right) dx = 0.$$

Now, we estimate the two tri-linear terms on the right-hand side of (5.15) as following:

$$\begin{aligned}\langle uu_x,\, u_{xx}\rangle =& \int_{-h}^0 \int_0^1 u\partial_x \left(\frac{u_x^2}{2}\right) dxdz = -\frac{1}{2}\int_D u_x^3 \\ \leqslant& C\|u_x\|^2 \|u_{xx}\|^{\frac{1}{2}} \|u_{xz}\|^{\frac{1}{2}} \\ \leqslant& C_\varepsilon \|u_x\|^4 + \varepsilon \|\nabla u_x\|^2.\end{aligned}$$

$$\begin{aligned}\langle wu_z,\, u_{xx}\rangle \leqslant& C\|w\|^{\frac{1}{2}} \|w_z\|^{\frac{1}{2}} \|u_z\|^{\frac{1}{2}} \|u_{zx}\|^{\frac{1}{2}} \|u_{xx}\| \\ \leqslant& C\|u_x\| \|u_z\|^{\frac{1}{2}} \|u_{zx}\|^{\frac{1}{2}} \|u_{xx}\| \\ \leqslant& C_\varepsilon \|u_z\|^2 \|u_x\|^4 + \varepsilon \|\nabla u_x\|^2.\end{aligned}$$

Therefore, it follows from (5.15) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2 +& \|\nabla u_x\|^2 + \alpha_1 \|u_x(z=0)\|^2 \\ \leqslant& 2\varepsilon \|\nabla u_x\|^2 + C_\varepsilon (1 + \|u_z\|^2)\|u_x\|^4 + (\|v\| + \|\theta_x\|)\|u_{xx}\| \\ \leqslant& 3\varepsilon \|\nabla u_x\|^2 + C_\varepsilon (1 + \|u_z\|^2)\|u_x\|^4 + C_\varepsilon (\|v\|^2 + \|\theta_x\|^2)\end{aligned}$$

Thus, if $\varepsilon > 0$ is chosen sufficiently small, then

$$\begin{aligned}\frac{d}{dt}\|u_x\|^2 +& \|\nabla u_x\|^2 + \alpha_1 \|u_x(z=0)\|^2 \\ \leqslant& C(1 + \|u_z\|^2)\|u_x\|^4 + C(\|v\|^2 + \|\theta_x\|^2).\end{aligned} \tag{5.16}$$

A local in time upper bound of $\|u_x\|$ can be obtained from (5.16) as follows. By (5.16),

$$\frac{d}{dt}\|u_x\|^2 \leqslant C(1 + \|u_z\|^2 + \|v\|^2 + \|\theta_x\|^2)(\|u_x\|^2 + 1)^2.$$

Denote:

$$y(t) := \|u_x\|^2 + 1, \quad g(t) := C(1 + \|u_z\|^2 + \|v\|^2 + \|\theta_x\|^2).$$

Then

$$y'(t) \leqslant g(t) y^2(t).$$

Notice that $y(t) \geqslant 1$, $g(t) \geqslant C(>0)$ and $g \in L^1(0,+\infty)$. Therefore,

$$y(t) \leqslant \frac{y(0)}{1 - y(0)\int_0^t g(s)ds},$$

for $t \in (0, t_*)$ where t_* is decided by the following equation:

$$y(0)\int_0^{t_*} g(s)ds = 1.$$

Thus, for $t \in (0, t_*)$

$$\|u_x\|^2 + 1 \leqslant \frac{\|u_{0,x}\|^2 + 1}{1 - (\|u_{0,x}\|^2 + 1)\int_0^t g(s)ds}.$$

This finishes the proof of local in time boundedness of $\|u_x\|$.
One the other hand, there exists $t_0 \in (0, t_*)$, such that

$$(u(t_0), v(t_0), \theta(t_0)) \in V.$$

Therefore, by the result of uniform boundedness of strong solutions (see §3.3 of [21] and [16]) and its uniqueness on $[t_0, +\infty)$, there exists a bounded absorbing set for (u, v, θ) in V for $t \in [t_0, \infty)$, thus a bounded absorbing set for u_x in L^2 for $t \in [0, \infty)$. It also proves the uniform boundedness:

$$u_x \in L^\infty(0, +\infty; L^2).$$

Then, integrating (5.16) for t from 0 to ∞ proves

$$\nabla u_x \in L^2(0, +\infty; L^2).$$

This finishes proof of Theorem 5.2 (a).

Step 2. Estimate of $\|v_x\|_2$ and $\|\theta_x\|$.
Similar to **Step 1**, taking inner product of (1.4) with $-v_{xx}$ yields:

$$\frac{1}{2}\frac{d}{dt}\|v_x\|^2 + \|\nabla v_x\|^2 + \alpha_2\|v_x(z=0)\|^2 = \langle uv_x + wv_z - u,\, v_{xx}\rangle\,. \tag{5.17}$$

The tri-linear terms on the right-hand side of (5.17) can be estimated as following:

$$
\begin{aligned}
\int_D u v_x v_{xx} =& \frac{1}{2}\int_D u\partial_x(v_x^2)dxdz \\
=& -\frac{1}{2}\int_D u_x v_x^2 dxdz \\
\leqslant& C\|u_x\|\|v_x\|\|v_{xx}\|^{\frac{1}{2}}\|v_{xz}\|^{\frac{1}{2}} \\
\leqslant& C_\varepsilon\|u_x\|^2\|v_x\|^2 + \varepsilon\|\nabla v_x\|^2.
\end{aligned}
$$

$$
\begin{aligned}
\int_D w v_z v_{xx} \leqslant& C\|w\|^{\frac{1}{2}}\|w_z\|^{\frac{1}{2}}\|v_z\|^{\frac{1}{2}}\|v_{xz}\|^{\frac{1}{2}}\|v_{xx}\| \\
\leqslant& C\|u_x\|\|v_z\|^{\frac{1}{2}}\|v_{xz}\|^{\frac{1}{2}}\|v_{xx}\| \\
\leqslant& C_\varepsilon\|u_x\|^4\|v_z\|^2 + \varepsilon\|\nabla v_x\|^2.
\end{aligned}
$$

Thus, if $\varepsilon > 0$ is chosen sufficiently small, then

$$
\begin{aligned}
\frac{d}{dt}\|v_x\|^2 +& \|\nabla v_x\|^2 + \alpha_2\|v_x(z=0)\|^2 \\
\leqslant& C(\|u_x\|^2\|v_x\|^2 + \|u_x\|^4\|v_z\|^2 + \|u\|^2).
\end{aligned} \tag{5.18}
$$

Similarly, we have

$$
\begin{aligned}
\frac{d}{dt}\|\theta_x\|^2 +& \|\nabla \theta_x\|^2 + \alpha_0\|\theta_x(z=0)\|^2 \\
\leqslant& C(\|u_x\|^2\|\theta_x\|^2 + \|u_x\|^4\|\theta_z\|^2 + \|Q\|^2).
\end{aligned} \tag{5.19}
$$

Notice the fact that $v_z, \theta_z \in L^2(0,\infty; L^2)$. Thus, as argued in **Step 1**, Theorem 5.2 (b) follows from Theorem 5.2 (a) and (5.18); Theorem 5.2 (c) follows from Theorem 5.2 (a) and (5.19).

Step 3. Estimate of $\|v_z\|_2$ and $\|\theta_z\|$.

Recall (5.9). The two tri-linear terms on the right-hand side of (5.9) can be estimated in the following.

First, we have

$$
\begin{aligned}
\int_D u v_x v_{zz} \leqslant& C\|u\|^{\frac{1}{2}}\|u_x\|^{\frac{1}{2}}\|v_x\|^{\frac{1}{2}}\|v_{xz}\|^{\frac{1}{2}}\|v_{zz}\| \\
\leqslant& C_\varepsilon\|u\|^2\|u_x\|^2\|v_x\|^2 + \varepsilon\|\nabla v_z\|^2
\end{aligned} \tag{5.20}
$$

Next, the estimate (5.10) will be used again. Thus, if $\varepsilon > 0$ is chosen sufficiently small, then

$$\frac{d}{dt}\left(\|v_z\|^2 + \alpha_2\|v(z=0)\|^2\right) + \|\nabla v_z\|^2 + \alpha_2\|v_x(z=0)\|^2 \leqslant C(\|u\|^2\|u_x\|^2\|v_x\|^2 + \|u_x\|^2\|v_z\|^2 + \|u\|^2). \tag{5.21}$$

Similarly, we have

$$\frac{d}{dt}\left(\|\theta_z\|^2 + \alpha_0\|\theta(z=0)\|^2\right) + \|\nabla \theta_z\|^2 + \alpha_2\|\theta_x(z=0)\|^2 \leqslant C(\|u\|^2\|u_x\|^2\|\theta_x\|^2 + \|u_x\|^2\|\theta_z\|^2 + \|Q\|^2). \tag{5.22}$$

Now, Theorem 5.2 (d) and (e) follow from (5.21) and (5.22) respectively as in **Step 1** and **Step 2**.

□

6 Uniqueness

In this section, we state and prove our last main result, Theorem 6.1, on uniqueness of weak solutions of 2D viscous PE (1.3)-(1.10) when some initial partial regularity is assumed.

Theorem 6.1 *Suppose $Q \in L^2(D)$, $(u_0, v_0, \theta_0) \in H$ and (u, v, θ) is a weak solution of (1.3)-(1.10). Suppose further that one of the following initial regularity is valid:*

$$\begin{aligned} &(\partial_x u_0, \partial_x v_0, \partial_x \theta_0) \in (L^2(D))^3, \ \textit{or} \ (\partial_x u_0, \partial_x v_0, \partial_z \theta_0) \in (L^2(D))^3, \\ \textit{or} \ &(\partial_x u_0, \partial_z v_0, \partial_x \theta_0) \in (L^2(D))^3, \ \textit{or} \ (\partial_x u_0, \partial_z v_0, \partial_z \theta_0) \in (L^2(D))^3, \\ \textit{or} \ &(\partial_z u_0, \partial_z v_0, \partial_z \theta_0) \in (L^2(D))^3. \end{aligned}$$

Then, the following are valid:

(a) *The norm for the corresponding solution regularity is uniformed bounded for all time $t \geqslant 0$ and an absorbing set exists for the norm of the corresponding solution regularity.*

(b) *The weak solution is unique.*

Proof:

The claim (a) for solution regularity is an immediate consequence of Theorem 5.1 and Theorem 5.2. Moreover, assuming any one of the above five initial conditions, we have

$$(u_{xz}, v_{xz}, \theta_{xz}) \in \left[L^2(0,\infty; L^2(D))\right]^3. \tag{6.1}$$

Notice that

$$\begin{aligned}
\|u_z\|_{L^\infty_x(L^2_z)} =& \left\| \left(\int_{-h}^0 |u_z|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty_x} \\
&\leqslant \text{(Minkowski inequality)} \\
&\leqslant \left(\int_{-h}^0 \|u_z\|^2_{L^\infty_x} \right)^{\frac{1}{2}} \\
&\leqslant \text{(Agmon's inequality)} \\
&\leqslant C \left(\int_{-h}^0 \|u_z\|_{L^2_x} \|u_{xz}\|_{L^2_x} \right)^{\frac{1}{2}} \\
&\leqslant C \|u_z\|^{\frac{1}{2}} \|u_{xz}\|^{\frac{1}{2}}.
\end{aligned}$$

Therefore, for any $T > 0$,

$$\begin{aligned}
\int_0^T \|u_z\|^2_{L^\infty_x(L^2_z)} dt \leqslant& C \int_0^T \|u_z\| \|u_{xz}\| dt \\
\leqslant& C \left(\int_0^T \|u_z\|^2 \, dt \right)^{\frac{1}{2}} \left(\int_0^T \|u_{xz}\|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned} \tag{6.2}$$

Notice that for any weak solution (u, v, θ),

$$(u_z, v_z, \theta_z) \in L^2(0, T; L^2(D)).$$

Thus, by (6.1) and (6.2), we have

$$u_z \in L^2(0, T; L^\infty_x(L^2_z)).$$

Similarly,

$$(u_z, v_z, \theta_z) \in \left[L^2(0, T; L^\infty_x(L^2_z))\right]^3.$$

Thus, Theorem 6.1 (b) is proved by Theorem 4.1.

□

References

[1] D. Bresch, F. Guillén-Gonzáez, N. Masmoudi and M.A. Rodríguez-Bellido, *Asymptotic derivation of a Navier condition for the primitive equations*, Asymp. Anal. **33**(2003), no. 1, 237-259.

[2] D. Bresch, F. Guillén-Gonzáez, N. Masmoudi and M.A. Rodríguez-Bellido, *On the uniqueness of weak solutions of the two-dimensional primitive equations*, Diff. Integral Eq. **16**(2003), no. 1, 77-94.

[3] D. Bresch, A. Kazhikhov and J. Lemoine, *On the two-dimensional hydrostatic Navier-Stokes equations*, SIAM J. Math. Anal. 36 (2004/05), no. 3, 796-814.

[4] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. (2) **166** (2007), no. 1, 245-267.

[5] L. Evans and R. Gastler, *Some results for the primitive equations with physical boundary conditions*. Z. Angew. Math. Phys. **64** (2013), no. 6, 17291744.

[6] G. P. Galdi, *An Introduction to the Navier-Stokes Initial-Boundary Value Problem*, Fundamental directions in mathematical fluid mechanics, 1-70, Adv. Math. Fluid Mech., Birkhäuser, Basel, 2000.

[7] F. Guillén-Gonzáez, N. Masmoudi and M.A. Rodríguez-Bellido, *Anisotropic estimates and strong solutions of the Primitive Equations*, Diff. Integral Eq. **14**(2001), no. 1, 1381-1408.

[8] M. Hieber and T. Kashiwabara, *Global well-posedness of the three-dimensional primitive equations in Lp-space*. Arch. Ration. Mech. Anal. 221, 1077-1115 (2016).

[9] N. Ju, *The global attractor for the solutions to the 3D viscous primitive equations*, Discrete and Continuous Dynamical Systems, 17 (2007), no. 1, 159-179.

[10] N. Ju, *On H^2 solutions and z-weak solutions of the 3D primitive equations*, Indiana University Mathematics Journal, **66**, (2017), no. 3.

[11] N. Ju, *Global Uniform Boundedness of Solutions to viscous 3D Primitive Equations with Physical Boundary Conditions*, 2017, arXiv:1710.04622.

[12] N. Ju and R. Temam, *Finite Dimensions of the Global Attractor for 3D Primitive Equations with Viscosity*, J. Nonlinear Sci. 25 (2015), no. 1, 131-155.

[13] G. Kobelkov, *Existence of a solution 'in the large' for ocean dynamics equations*, J. Math. Fluid Mech. **9** (2007), no. 4 588-610.

[14] I. Kukavica, Y. Pei, W. Rusin and M. Ziane, *Primitive equations with continuous initial data*. Nonlinearity 27 (2014), no. 6, 1135-1155.

[15] I. Kukavica and M. Ziane, *On the regularity of the primitive equations of the ocean*, Nonlinearity, **20**, (2007), no. 12, 2739-2753.

[16] I. Kukavica and M. Ziane, *Uniform gradient bounds for the primitive equations of the ocean*, Differential Integral Equations **21**, (2008), no. 9-10, 837-849.

[17] J. Li and E. Titi, *Existeence and Uniqueness of weak solutions to viscous primitive equations for a certain class of discontinuous initial data.* SIAM. J. Math. Anal., **49**, (2017), no. 1, 1-28.

[18] J. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer, Berlin, 1972.

[19] J. Lions, R. Temam and S. Wang, *On the equations of the large scale Ocean*, Nonlinearity **5**, (1992), 1007-1053.

[20] M. Petcu, *On the backward uniqueness of primitive equations*, J. Math. Pures Appl., **87**, (2007), 275-289.

[21] M. Petcu, R. Temam and M. Ziane, *Some Mathematical Problems in Geophysical Fluid Dynamics*, Handbook of Numerical Analysis, Vol. XIV. Special Volume : computational methods for the atmosphere and the oceans, 577-750, Handb. Numer. Anal., 14, Elsevier/North-Holland, Amsterdam, 2009.

[22] T. Tachim Medjo, *On the uniqueness of z-weak solutions of the three-dimensional primitive equations of the ocean*, Nonlinear Anal. Real World Appl. **11** (2010), no. 3, 1413-1421.

[23] R. Temam, *Navier-Stokes Equations, Theorey and Numerical Analysis*, reprinted by American Mathematical Society, 2001.

[24] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, vol 3, S. Friedlander and D. Serre Editors, Elsevier, 2004, 535-658.