MAT 579 Functional Analysis II

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Spring 2012
revised April 6, 2012

Notes on: Group representation theory — Version 2

1. Introduction

These notes are intended to summarize our discussion of representations of locally compact groups. The style is quite informal, and no attempt is made to make the coverage complete. When a proof is missing, it means that I either regard the proof as routine, or that I might go back and fill in the proof later. Moreover, I'm jumping into the middle here, concentrating mostly on the latest material at the moment that I'm writing these notes; later I might go back and fill in earlier material (e.g., the basic properties of topological groups).

In this 2nd version I'm filling in a little bit more of the proofs. Also I'm taking this opportunity to fix any typos and other minor errors, clarify things here and there, or even make minor additions. Otherwise the content is the same as in the original version.

2. The group algebra

Let G be a locally compact (Hausdorff) group.

Definition 1. For each compact subset $K \subset G$ let

$$C_K(G) = \{f \in C_c(G) : \operatorname{supp} f \subset K\}.$$

Observation 2. With the sup norm, $C_K(G)$ is a Banach space, and can be identified with a closed subspace of $C(K)$. Also, if $K_1 \subset K_2$ then $C_{K_1}(G)$ is a closed subspace of $C_{K_2}(G)$.

Definition 3. The *inductive limit topology* on $C_c(G)$ is the strongest locally convex topology such that the inclusions $C_K(G) \hookrightarrow C_c(G)$ continuous, where $C_K(G)$ is given the sup norm topology.

Proposition 4. *Let X be a locally convex space, and let $T : C_c(G)$ to X be linear. Then T is continuous for the inductive limit topology if and only if $T|_{C_K(G)}$ is continuous for every compact $K \subset G$.*

Proof. This follows immediately from the definition of strongest locally convex topology. □

Here is another way to express the above continuity criterion:

Corollary 5. *A linear map $T : C_c(G) \to X$ (for locally convex X) is continuous if and only if $Tf_n \to 0$ whenever $\{f_n\}$ is a sequence in $C_c(G)$ such that*

1. *there exists a compact $K \subset G$ such that* $\operatorname{supp} f_n \subset K$ *for all n;*
2. *$f_n \to 0$ uniformly.*

Remark 6. In fact, it can be shown (although we will not need it) that a sequence $\{f_n\}$ in $C_c(G)$ converges to 0 in the inductive limit topology if and only if it satisfies (1)–(2) in the above corollary.

However, this does not generalize to arbitrary nets: if $\{f_i\}$ is a net in $C_c(G)$ that converges to 0, and if it satisfies (1) above, then it also satisfies (2). Slightly more generally, it suffices to have

(3) there exist $K \subset G$ and i_0 such that K is compact and $\operatorname{supp} f_i \subset K$ for all $i \geq i_0$.

But there are examples where $f_i \to 0$ in the inductive limit topology but doesn't satisfy (3).

Fortunately, though, for most (all?) practical purposes, we can restrict attention to nets in $C_c(G)$ satisfying (2)–(3). For example:

Lemma 7. *Let $\{f_i\}$ be a net in $C_c(G)$. If there exist $K \subset G$ and i_0 such that $\operatorname{supp} f_i \subset K$ for all $i \geq i_0$, and $f_i \to 0$ uniformly, then $f_i \to 0$ in the inductive limit topology.*

Proposition 8. *If μ is a Radon measure on G, then the linear functional*

$$f \mapsto \int_G f \, d\mu$$

on $C_c(G)$ is continuous for the inductive limit topology.

Proof. This follows immediately from the Dominated Convergence theorem. □

Definition 9. For $x \in G$, the *left-translation operator* L_x on $C_c(G)$ is given by

$$(L_x f)(y) = f(x^{-1}y) \qquad \text{for } f \in C_c(G), y \in G.$$

Lemma 10. *For $f \in C_c(G)$, the map*

$$x \mapsto L_x f : G \to L^1(G)$$

is continuous for the inductive limit topology.

Proof. If $x_i \to x$ in G, we can restrict to a tail of the net lying in a compact set, and it follows that the functions $L_{x_i} f$ have supports contained in a fixed compact set, and converge to $L_x f$ by uniform continuity of f. □

Definition 11. A Radon measure μ on G is *left-invariant* if

$$\int f \, d\mu = \int L_x f \, d\mu \qquad \text{for all } f \in C_c(G), x \in G.$$

Any nonzero left-invariant measure on G is called a *Haar measure*.

Remark 12. We could just as well define right-invariant measures, but we will always use left-invariant ones by default. Sometimes, for emphasis, one refers to *left Haar measure*, and the companion concept of *right Haar measure* can be developed as well.

We won't prove the following fundamental result concerning integration on G:

Theorem 13. *Every locally compact group has a Haar measure, which is essentially unique in the following sense: any two Haar measures are positive scalar multiples of each other* (*and of course every multiple of a Haar measure is a Haar measure*).

Example 14.

(1) For $\mathbb{R}$ we take Lebesgue measure.
(2) If G is compact, we typically normalize Haar measure to have total mass 1. For example, for $\mathbb{T}$ we take normalized arc-length measure. Equivalently, if we identify $\mathbb{T}$ with the interval $[0, 1)$ then Haar measure is Lebesgue measure.
(3) If G is discrete, for example $\mathbb{Z}$, we take counting measure.

Observation 15. For $x \in G$, the left-translation operator L_x is a linear isometry on $L^1(G)$.

The following lemma says that $x \mapsto L_x$ is strong-operator continuous from G to $B(L^1(G))$:

Lemma 16. *For $f \in L^1(G)$, the map*

$$x \mapsto L_x f : G \to L^1(G)$$

is continuous.

Proof. Since $C_c(G)$ is dense in $L^1(G)$, and the operators $\{L_x\}_{x \in G}$ are uniformly bounded, without loss of generality $f \in C_c(G)$. But now the statement is obvious, since the inductive limit topology is stronger than the 1-norm topology. □

Since Haar measure is left invariant, left translation doesn't change integrals; however, right translation does have an effect:

Lemma 17. *For all $y \in G$ there is a unique $\Delta(y) > 0$ such that*

$$\Delta(y) \int_G f(xy)\, dx = \int_G (f(x)\, dx \qquad \text{for } f \in C_c(G). \tag{1}$$

Moreover, $\Delta : G \to \mathbb{R}^+$ is a continuous homomorphism, and is independent of Haar measure.

Proof. Define a positive linear functional J on $C_c(G)$ by

$$J(f) = \int f(xy)\, dx.$$

A quick calculation shows that J is left-invariant, i.e.:

$$J(L_x f) = J(f) \qquad \text{for } x \in G, f \in C_c(G).$$

Thus by the properties of Haar measure there exists a unique $\Delta(y) > 0$ such that (1) holds.

To see that $\Delta : G \to \mathbb{R}^+$ is a homomorphism, we compute:

$$\begin{aligned} \Delta(y)\Delta(z) \int f(xyz)\, dx &= \Delta(z) \int f(xz)\, dx \\ &= \int f(x)\, dx \\ &= \Delta(yz) \int f(xyz)\, x. \end{aligned}$$

For the continuity, pick $f \in C_c(G)$ with $\int f \neq 0$, and choose a neighborhood V of e in G such that

$$\int f(xy)\, dx \neq 0 \qquad \text{for all } y \in V.$$

Then

$$\Delta(y) = \frac{\int f(x)\, dx}{\int f(xy)\, dx} \xrightarrow{y \to e} 1,$$

by inductive-limit continuity.

Finally, the uniqueness assertion follows from the uniqueness of Haar measure up to positive scalar multiples. □

Definition 18. Δ is the *modular function* on G.

Definition 19. G is *unimodular* if $\Delta \equiv 1$.

Example 20. G is unimodular if it is abelian or compact (because $\Delta(G)$ is then a compact subgroup of $\mathbb{R}^+$, which must therefore be $\{1\}$).

The modular function also satisfies:

Lemma 21.
$$\int f(x)\,dx = \int f(x^{-1})\Delta(x^{-1})\,dx \qquad \text{for } f \in C_c(G).$$

Proof. Define a positive linear functional J on $C_c(G)$ by
$$J(f) = \int f(x^{-1})\Delta(x^{-1})\,dx.$$
Then J is left invariant, so there exists a unique $c > 0$ such that
$$J(f) = c\int f.$$

It remains to show that $c = 1$, choose $f \in C_c(G)$ such that $f \geq 0$ and $f \neq 0$, and define
$$g(x) = f(x) + \Delta(x^{-1})f(x^{-1}).$$
Then

- $\Delta(x^{-1})g(x^{-1}) = g(x)$;
- $\int g > 0$.

Thus
$$J(g) = c\int g = \int \Delta(x^{-1})g(x^{-1})\,dx = \int g,$$
so $c = 1$. □

Remark 22. Let μ be our (left) Haar measure on G. We get a right Haar measure ν on G by
$$\int f\,d\nu = \int f(x^{-1})\,d\mu(x) \qquad \text{for } f \in C_c(G).$$
It follows from the constructions that Δ agrees with the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.

Theorem 23. *$L^1(G)$ is a Banach $*$-algebra with multiplication given by* convolution*:*
$$(f * g)(x) = \int_G f(y)g(y^{-1}x)\,dx,$$
and involution
$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}),$$
*Moreover, $C_c(G)$ is a dense *-subalgebra of $L^1(G)$.*

Proof. All the required verifications can be done using the above properties, together with the Fubini and Tonelli theorems and changes of variables in integrals of the form $x \mapsto yx$ or xy (keeping in mind that the modular function must be used for the latter). □

Lemma 24. *For $f, g \in L^1(G), \psi \in L^\infty(G)$,*
$$\int f(x) \int L_x g(y)\psi(y)\,dy\,dx = \int f * g(y)\psi(y)\,dy.$$

Proof. Note that it follows from Lemma 16 that the inner integral on the left is a bounded continuous function of x. The lemma follows from a simple computation:

$$\begin{aligned}\int f(x)\int L_x g(y)\psi(y)\,dy\,dx &= \int\int f(x)g(x^{-1}y)\psi(y)\,dy\,dx\\ &= \int\int f(x)g(x^{-1}y)\,dx\,\psi(y)\,dy\\ &= \int f*g(y)\psi(y)\,dy.\end{aligned}$$
□

For future reference, we record the following definition and lemma:

Definition 25. For $x \in G$, the *right-translation operator* R_x on $C_c(G)$ is given by

$$(R_x f)(y) = f(yx^{-1})\Delta(x^{-1}).$$

Remark 26. We incorporate the factor $\Delta(x^{-1})$ in the above definition in order to make R_x an isometry for the 1-norm.

Lemma 27. *For $f, g \in C_c(g)$ and $x \in G$ we have*

$$g*(L_x f) = (R_x g)*f.$$

Proof. For $y \in G$ we have

$$\begin{aligned}\big(g*L_x f\big)(y) &= \int g(z)(L_x f)(z^{-1}y)\,dz\\ &= \int g(z)f(x^{-1}z^{-1}y)\,dz\\ &= \int g(zx^{-1})f(z^{-1}y)\Delta(x^{-1})\,dz \qquad (\text{after } z\mapsto zx^{-1})\\ &= \int (R_x g)(z)f(z^{-1}y)\,dz\\ &= \big(R_x g*f\big)(y).\end{aligned}$$
□

Definition 28. A *bounded approximate identity* for a Banach $*$-algebra A is a net $\{\phi_i\}_{i\in I}$ in A such that

(1) $\|\phi_i\| \le 1$ for all $i \in I$;
(2) $\phi_i^* = \phi_i$ for all $i \in I$;
(3) $\phi_i a \to a$ for all $a \in A$;
(4) $a\phi_i \to a$ for all $a \in A$.

Observation 29. Since ϕ_i is self-adjoint, each of (3) and (4) above implies the other.

Theorem 30. *$L^1(G)$ has a bounded approximate identity $\{\phi_i\}$, which can be chosen such that:*

- *$\phi_i \ge 0$ for all i;*
- *$\phi_i \in C_c(G)$ for all i;*
- *$\phi_i * f, f*\phi_i \to f$ in the inductive limit topology for all $f \in C_c(G)$;*
- *for every neighborhood U of e in G there exists i_0 such that $\operatorname{supp}\phi_i \subset U$ for all $i \ge i_0$;*
- *$\int_G \phi_i = 1$ for all i.*

Proof. Just make the obvious adjustments in the proof we gave in the abelian case. □

3. Group representations

Definition 31. Let H be a Hilbert space, and let $\mathcal{U}(H)$ denote the unitary group of H. A *representation of* G *on* H is a homomorphism $U : G \to \mathcal{U}(H)$ that is continuous for the weak operator topology on $\mathcal{U}(H)$.

Definition 32. Two representations $U : G \to \mathcal{U}(H)$ and $V : G \to \mathcal{U}(K)$ are *equivalent*, written $U \cong V$, if there is a unitary $W : H \to K$ such that

$$V = \operatorname{Ad} W \circ U,$$

where $\operatorname{Ad} W : B(H) \to B(K)$ is the isomorphism defined by

$$\operatorname{Ad} W(T) = WTW^* \qquad \text{for } T \in B(H).$$

Definition 33. An *invariant subspace* for a representation $U : G \to \mathcal{U}(H)$ is a closed subspace $M \subset H$ such that

$$U_x(M) \subset M \qquad \text{for } x \in G.$$

U is *irreducible* if it has no invariant subspaces other than $\{0\}$ and H.

Definition 34. The *subrepresentation* of a representation U of G *associated to* an invariant subspace M is the representation $U|_M : G \to \mathcal{U}(M)$ defined by

$$(U|_M)_x = U_x|_M \qquad \text{for } x \in G.$$

Proposition 35. *If M is an invariant subspace of a representation of G, then so is $M^\perp$.*

Definition 36. The *direct sum* of representations $U_i : G \to \mathcal{U}(H_i)$ $(i \in I)$ is the representation

$$U = \bigoplus_{i \in I} U_i : G \to \mathcal{U}\left(\bigoplus_{i \in I} H_i\right)$$

defined by

$$U_x(\xi_i) = \big((U_i)_x \xi_i\big) \qquad \text{for } (\xi)_i \in \bigoplus_i H_i.$$

Observation 37. If $U : G \to \mathcal{U}(H)$ is a representation and $\{M_i\}$ is a pairwise orthogonal family of invariant subspaces such that $H = \bigoplus_i M_i$, then

$$U \cong \bigoplus_i U|_{M_i},$$

and we blur the distinction between these. We refer to this as a *decomposition* of U.

U is irreducible precisely if it can't be decomposed in a nontrivial way.

4. Representations of Banach $*$-algebras

Let A be a Banach $*$-algebra.

Definition 38. A *representation* of A on a Hilbert space H is a $*$-homomorphism $\pi : A \to B(H)$.

Recall that, since $B(H)$ is a C^*-algebra, π is automatically bounded as a linear map, with norm at most 1.

Many of the basic concepts concerning representations of Banach $*$-algebras are similar to those of group representations: *equivalence*, *invariant subspace*, *irreducible representation*, *subrepresentation*, *direct sum*, and *decomposition*. In particular:

Proposition 39. *If M is an invariant subspace of a representation π of A, then so is $M^\perp$.*

Definition 40. A representation $\pi : A \to B(H)$ is *nondegenerate* if

$$\overline{\operatorname{span}}\{\pi(a)h : a \in A, h \in H\} = H.$$

It is not much of a restriction to only consider nondegenerate subspaces:

Proposition 41. *Let $\pi : A \to B(H)$ be a representation, and define a closed subspace M of H by*

$$M = \overline{\operatorname{span}}\{\pi(a)h : a \in A, h \in H\}.$$

Then M is invariant for π, $\pi|_M$ is nondegenerate, and $\pi_{M^\perp} = 0$.

Here's a perhaps unexpected benefit of nondegeneracy:

Lemma 42. *Let $\pi : A \to B(H)$ be a nondegenerate representation, and let $h \in H$. Then $h = 0$ if and only if $\pi(a)h = 0$ for all $a \in A$.*

Proof. For the nontrivial direction, assume that $\pi(a)h = 0$ for all $a \in A$. Then for all $a \in A, k \in H$,

$$\begin{aligned} 0 &= \langle \pi(a)h, k\rangle \\ &= \langle h, \pi(a)^*k\rangle \\ &= \langle h, \pi(a^*)k\rangle. \end{aligned}$$

Since the vectors $\pi(a^*)k$ for $a \in A, k \in H$ span a dense subset of H, we have $h = 0$. $\square$

We will need the following gadget in the next section:

Definition 43. Let $\pi : A \to B(H)$ be a representation, and let $h, k \in H$. Define $\psi_{h,k} \in A^*$ by

$$\psi_{h,k}(a) = \langle \pi(a)h, k\rangle \qquad \text{for } a \in A.$$

Observation 44. If $A = L^1(G)$, then $A^* = L^\infty(G)$, and for a representation $\pi : L^1(G) \to B(H)$ and $h, k \in H$ the element $\psi_{h,k} \in L^\infty$ is characterized by

$$\int_G f(x)\psi_{h,k}(x)\,dx = \langle \pi(f)h, k\rangle \qquad \text{for } f \in L^1(G).$$

5. Correspondence between representations of G and $L^1(G)$

Theorem 45. *Let $U : G \to \mathcal{U}(H)$ be a representation. Then there is a unique representation $\pi_U : L^1(G) \to B(H)$ such that*

$$\langle \pi_U(f)h, k\rangle = \int_G f(x)\langle U_x h, k\rangle\,dx \qquad \text{for } f \in C_c(G), h, k \in H. \tag{2}$$

Moreover, π_U is nondegenerate.

Before we start the proof, note that the above integral is only used for $f \in C_c(G)$.

Proof. First of all, note that the integral in (2) is well-defined since U is weak-operator continuous. We need to know that there is a unique operator $\pi_U(f) \in B(H)$ satisfying (2). For this, note that the right-hand side of (2) is a sesquilinear form on H, which is bounded because

$$\left|\int f(x)\langle U_x h, k\rangle\,dx\right| \le \int |f(x)|\,|\langle U_x h, k\rangle|\,dx$$

$$
\begin{aligned}
&\le \int |f(x)|\,\|U_x h\|\|k\|\,dx\\
&= \int |f(x)|\,dx\,\|h\|\|k\| \qquad \text{because } U_x \text{ is unitary}\\
&= \|f\|_1\|h\|\|k\|.
\end{aligned}
$$

Thus by the theory of bounded linear operators on Hilbert space, there is a unique operator $\pi_U(f)$ satisfying (2).

Routine computations, sometimes using Fubini's theorem and changing variables of integration using translation, show that $\pi_U : C_c(G) \to B(H)$ is linear, bounded, multiplicative, and involutive, and therefore extends uniquely to a representation $\pi_U : L^1(G) \to B(H)$. For example, here is why it is involutive:

$$
\begin{aligned}
\langle h, \pi_U(f)^* k\rangle &= \langle \pi_U(f)h, k\rangle\\
&= \int f(x)\langle U_x h, k\rangle\,dx\\
&= \int f(x)\langle h, U_x^* k\rangle\,dx\\
&= \int f(x)\langle h, U_{x^{-1}} k\rangle\,dx\\
&= \int f(x^{-1})\Delta(x^{-1})\langle h, U_x k\rangle\,dx \qquad (\text{after } x \mapsto x^{-1})
\end{aligned}
$$

It remains to show that π_U is nondegenerate. Since

$$
\|p\| = \sup\bigl\{|\langle p, k\rangle| : \|k\| \le 1\bigr\} \qquad \text{for all } p \in H,
$$

it suffices to show that for fixed $h, k \in H$ we have $\langle \pi_U(\phi_i)h, k\rangle \to \langle h, k\rangle$, where $\{\phi_i\}$ is an approximate identity with all the above properties. Let $\epsilon > 0$. Since U is weak-operator continuous, we can find a neighborhood U of e such that

$$
\bigl|\langle U_x h, k\rangle - \langle h, k\rangle\bigr| \le \epsilon \qquad \text{for all } x \in U.
$$

Then we can choose i_0 such that for all $i \ge i_0$ we have $\operatorname{supp}\phi_i \subset U$, and hence

$$
\begin{aligned}
\bigl|\langle \pi_U(\phi_i)h, k\rangle - \langle h, k\rangle\bigr| &= \left|\int_G \phi_i(x)\langle U_x h, k\rangle\,dx - \int_G \phi_i(x)\langle h, k\rangle\,dx\right|\\
&= \left|\int_G \phi_i(x)\bigl(\langle U_x h, k\rangle - \langle h, k\rangle\bigr)\,dx\right|\\
&\le \int_U \phi_i(x)\bigl|\langle U_x h, k\rangle - \langle h, k\rangle\bigr|\,dx\\
&\le \epsilon \int_U \phi_i(x)\,dx\\
&= \epsilon.
\end{aligned}
$$

Now uniqueness follows from (2) and density of $C_c(G)$ in $L^1(G)$. □

Definition 46. With the above notation π_U is the *integrated form* if U.

Remark 47. Formally (and this can be made precise, although we probably will not do so), we can write

$$\pi_U(f) = \int_G f(x) U_x \, dx.$$

In the other direction:

Theorem 48. *Let $\pi : L^1(G) \to B(H)$ be a nondegenerate representation. Then there exists a representation $U : G \to \mathcal{U}(H)$ such that $\pi = \pi_U$. Moreover, U is determined by*

$$U_x \pi(f) h = \pi(L_x f) h \qquad \text{for } x \in G, f \in C_c(G), h \in H. \tag{3}$$

Proof. Define a dense subspace H_0 of H by

$$H_0 = \operatorname{span}\{\pi(f)h : f \in C_c(g), h \in H\}.$$

For $x \in G$, we want to define U_x on H_0 by

$$U_x \sum_1^n \pi(f_i) h_i = \sum_1^n \pi(L_x f_i) h_i$$

To see that U_x is well-defined, it suffices to show that if $\sum_1^n \pi(f_i)h_i = 0$ then $\sum_1^n \pi(L_x f_i) h_i = 0$. For $g \in C_c(G)$,

$$\begin{aligned}
\pi(g) \sum_1^n \pi(L_x f_i) h_i &= \sum_1^n \pi(g)\pi(L_x f_i) h_i \\
&= \sum_1^n \pi(g * L_x f_i) h_i \\
&= \sum_1^n \pi(R_x g * f_i) h_i \\
&= \sum_1^n \pi(R_x g)\pi(f_i) h_i \\
&= \pi(R_x g) \sum_1^n \pi(f_i) h_i \\
&= 0.
\end{aligned}$$

It follows that $\sum_1^n \pi(L_x f_i) h_i = 0$. Therefore U_x is well-defined on H_0, and is obviously linear. Also, it follows right from the construction that $x \mapsto U_x$ is a homomorphism from G into the group of invertible linear operators on the vector space H_0, with $U_x^{-1} = U_{x^{-1}}$.

We'll show that the operator U_x is isometric on H_0, and consequently it extends uniquely to an isometric linear operator on H, and moreover this extension is unitary because its range contains the dense subspace H_0. For $f, g \in C_c(G), h, k \in H$ we have

$$\begin{aligned}
\langle U_x \pi(f) h, U_x \pi(g) k \rangle &= \langle \pi(L_x f) h, \pi(L_x g) k \rangle \\
&= \langle \pi(L_x f)^* \pi(L_x g) h, k \rangle \\
&= \Big\langle \pi\big((L_x f)^* * L_x g\big) h, k \Big\rangle \\
&= \int \big((L_x f)^* * L_x g\big)(y) \langle U_y h, k \rangle \, dy
\end{aligned}$$

$$
\begin{aligned}
&= \int \int (L_x f)^*(z) L_x g(z^{-1}y)\, dz \langle U_y h, k\rangle\, dy \\
&= \int \int \overline{L_x f(z^{-1})} \Delta(z^{-1}) g(x^{-1}z^{-1}y) \langle U_y h, k\rangle\, dz\, dy \\
&= \int \int \overline{f(x^{-1}z^{-1})} \Delta(z^{-1}) g(x^{-1}z^{-1}y) \langle U_y h, k\rangle\, dz\, dy \\
&= \int \int \overline{f(z^{-1})} \Delta(xz^{-1}) g(z^{-1}y) \Delta(x^{-1}) \langle U_y h, k\rangle\, dz\, dy \\
&\qquad\qquad\qquad \text{(after } z \mapsto zx^{-1}\text{)} \\
&= \int \int f^*(z) g(z^{-1}y) \langle U_y h, k\rangle\, dz\, dy \\
&\qquad\qquad\qquad \text{(cancelling } \Delta(x)\Delta(x^{-1})\text{)} \\
&= \int f^* * g(y) \langle U_y h, k\rangle\, dy \\
&= \cdots \\
&= \langle \pi(f)h, \pi(g)k\rangle,
\end{aligned}
$$

and it follows fhat for a typical vector $\sum_1^n \pi(f_i)h_i \in H_0$ we have

$$
\begin{aligned}
\left\| U_x \sum_1^n \pi(f_i)h_i \right\|^2 &= \sum_{i,j} \langle U_x \pi(f_i)h_i, U_x \pi(f_j)h_j\rangle \\
&= \sum_{i,j} \langle \pi(f_i)h_i, \pi(f_j)h_j\rangle \\
&= \left\| \sum_1^n \pi(f_i)h_i \right\|^2 .
\end{aligned}
$$

We use the same notation U_x for the extension to a unitary on H. Then by continuity it quickly follows that the map $U : G \to \mathcal{U}(H)$ is a homomorphism.

To show that U is weak-operator continuous, since the operators U_x are uniformly bounded in the operator norm, it suffice to check vectors in the dense subspace H_0: and by linearity it suffices to observe that, for $f, g \in C_c(G), h, k \in H$, if $x_i \to x$ in G then

$$
\begin{aligned}
\langle U_{x_i} \pi(f)h, \pi(g)k\rangle &= \langle \pi(L_{x_i} f)h, \pi(g)k\rangle \\
&\to \langle \pi(L_x f)h, \pi(g)k\rangle \qquad \text{(because } L_{x_i} f \to L_x f \text{ in } L^1(G)\text{)} \\
&= \langle U_x \pi(f)h, \pi(g)k\rangle,
\end{aligned}
$$

Thus U is a representation of G on H.

To see that $\pi = \pi_U$, let $f \in C_c(G)$. Then the following computation, with $g \in C_c(G), h, k \in H$, suffices:

$$
\begin{aligned}
\langle \pi(f)\pi(g)h, k\rangle &= \langle \pi(f * g)h, k\rangle \\
&= \int f * g(y) \psi_{h,k}(y)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int f(x) \int L_x g(y) \psi_{h,k}(y)\, dy\, dx \qquad \text{(by Lemma 24)} \\
&= \int f(x) < \pi(L_x g)h, k\rangle\, dx \\
&= \int f(x) \langle U_x \pi(g)h, k\rangle\, dx \\
&= \langle \pi_U(f)\pi(g)h, k\rangle.
\end{aligned}
$$

Finally, the last statement of the theorem is now obvious. □

Remark 49. Taken together, the above two theorems give a 1-1 correspondence $U \mapsto \pi_U$ from the class of representations of G to the class of nondegenerate representations of $L^1(G)$.