Calculus 1
Exam 2 Review

Table of contents

Chain Rule, Power Rule

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b) Does $f(x)$ have any inflection points? If so, find the exact ordered pair (coordinates) for any such points.
Solution: Since $f''(1) = 0$ and $f(1) = 2$, we know that $(1, 2)$ is an inflection point.

Clicker

Which theorem would help you answer the following?
If f is differentiable on $[0, 1]$ and $f(0) < f(1)$, then there exists c in the interval $[0, 1]$ such that $f'(c) = \frac{f(1)-f(0)}{1-0}$

a) The Racetrack Principle

b) The Constant Function Theorem

c) The Mean Value Theorem

d) The Increasing Function Theorem

Local Linearization

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Solution:
Taking the derivative of the function and plugging in 0 yields:
$f'(0) = \frac{1}{5}\left(\frac{-1}{(2+(0))^2}\right) = \frac{-1}{20}$. This gives the slope of our approximation function.
Note $f(0) = \frac{1}{10}$. Then solve for b in $L(0) = \frac{1}{10} = \frac{-1}{20}(0) + b = b$, so $b = \frac{1}{10}$.

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Solution:

$L(1) = \frac{-1}{20} + \frac{1}{10} = \frac{1}{20}$.

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Local Linearization

Is $L(1)$ above or below the actual value of $f(x)$ at $x = 1$? *Be sure to explain your answer using complete sentences.* **Solution:**
$f''(x) = \frac{50}{(10+(x))^3} > 0$ for all $x \geq 0$, so the function is concave up on $[0, \infty)$.
This means we have an underestimate. I.e. $L(x) \leq f(x)$ for $x \in [0, \infty)$.
In particular, $L(1) \leq f(1)$.

Critical points

Given $f(x) = \frac{x^2 - x - 2}{x - 3}$

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a) Find the critical points of $f(x)$. **Solution:** $f'(x) = \frac{(2x-1)(x-3)-(x^2-x-2)}{(x-3)^2} = \frac{(x-5)(x-1)}{(x-3)^2}$. Hence the potential critical points are located at $x = 1, 3, 5$. However, $x = 3$ is not actually in the domain of the function so the critical points are located at $x = 1, 5$.

Critical points

Given $f(x) = \frac{x^2-x-2}{x-3}$

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b) Using your answers in part a, which critical points are local maximums and which are local minimums. show work to justify your answers.

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Solution:

$f'(x)$ $+$ $-$ $-$ $+$

x: 1, 3, 5

Hence, there is a maximum at $x = 1$, and a minimum at $x = 5$.

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Solution:

$$f(0) = \frac{2}{3} \qquad f(1) = 1$$
$$f(2) = 0$$

Therefore, the global maximum occurs at $(1, 1)$ and the global minimum occurs at $x = 0$.

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When does the equation $5 = x^3 + y^4$ have a horizontal tangent line?

a) $(0, \sqrt[4]{5})$

b) $(0, 0)$

c) $(\frac{-3}{4}, \frac{\sqrt[4]{347}}{\sqrt[4]{64}})$

d) $(\frac{-3}{4}, 0)$

Families of Functions

Find the formula for a function of the form $y = ax^3 - x + b$ with local minimum at $(4, 2)$.

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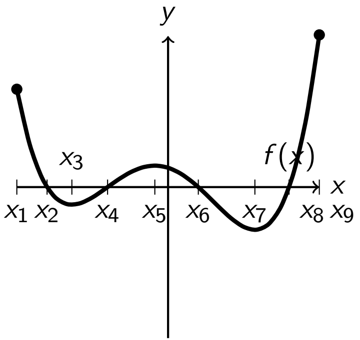
Find the formula for a function of the form $y = ax^3 - x + b$ with local minimum at $(4, 2)$.

Solution: First note: $y' = 3ax^2 - 1$ and $y'' = 6ax$. The second derivative implies that $a > 0$ to achieve a minimum. Now set the first derivative equal to 0 with the desired x-coordinate,
$0 = 3ax^2 - 1 \Rightarrow 1 = 3a(4)^2 \Rightarrow a = \frac{1}{48}$. Plug a into the original function as well as the desired coordinates:
$y = \frac{x^3}{48} - x + b \Rightarrow 2 = \frac{4}{3} - 4 + b \Rightarrow b = \frac{14}{3}$. Lastly conclude:
$y = \frac{x^3}{48} - x + 143.$

Interpreting Graphs



a) Which x-value(s) on the graph give an inflection point?
b) Which x-value(s) on the graph give an local minimum?
c) Which x-value(s) on the graph give an local maximum?
d) Which x-value(s) on the graph represent a negative first derivative?
e) Which x-value(s) on the graph represent a positive first derivative?
f) Which x-value(s) on the graph might have a second derivative equal to zero?

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Which of the following is the equation of a tangent line to the curve $x^2 - y^2 + 2x = 7$ at the point $(2, 1)$?

a) $y = -x + 3$

b) $y = 3x - 5$

c) $y = x + 1$

d) $y = -3x + 7$

Implicit Differentiation

A function is defined implicitly by the equation $e^{2x} + \ln(y) = x^2 - xy^3$.

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a) Find the derivative $\dfrac{dy}{dx}$.

Solution:

$$2e^{2x} + \frac{1}{y}\frac{dy}{dx} = 2x - y^3 - x3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - y^3 - 2e^{2x}}{\frac{1}{y} + x3y^2}$$

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b) Find the equation of the line tangent to the graph at the point $(0, 1)$.

Solution: $m = \dfrac{dy}{dx}|_{(x,y)=(0,1)} = \frac{-3}{1} = -3$. Therefore, the equation of the tangent line is $y = -3x + 1$.

Optimization

A baker is trying to determine how many donuts to bake. The cost, in dollars of baking x donuts is given by the function $C(x) = 10 + 0.01x^2$. The amount of money brought in by selling x donuts is given by $N(x) = 2x$. Remember that profit is just revenue minus cost. You may assume that every donut baked is sold.

- Write an equation that models the total profit the baker makes as a function of the number of donuts baked.

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- Write an equation that models the total profit the baker makes as a function of the number of donuts baked.
  Solution:
  (profit) = (revenue) - (cost)
  $p(x) = 2x - (10 + 0.01x^2)$

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- How many donuts should the baker bake in order to maximize profit? How do you know this is a maximum and not a minimum? *Be sure to write your answer in a complete sentence using units.*

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  Solution:

  $p'(x) = 2 - .02x = 2 - \frac{2}{100}x = 2(1 - \frac{x}{100})$

  $p'(x) = 0$ when $x = 100$

  $x = 100$ is a global max since $p'(x)$ is positive for $x < 100$ and $p'(x)$ is negative for $x > 100$

  which implies $p(x)$ is increasing on $(-\infty, 1]$ and decreasing on $[1, \infty)$.

  The baker should bake 100 donuts in order to maximize profit.

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True of False, A global maximum is always a critical point.

a) True

b) False

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- What will the total profit be when the baker bakes and sells the number of donuts which maximizes profit? *Be sure to write your answer in a complete sentence using units.*

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  Solution:

  $p(100) = 2 \cdot 100 - (10 + \frac{1}{100}(100)^2) = 200 - (10 + 100) = 90$

  The total profit will be \$90 when the baker bakes and sells 100 donuts.

The Racetrack Principle

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Use the racetrack principle to show that $x \leq x^2$ for $x \geq 1$.
Solution: Let $f(x) := x$ and $g(x) := x^2$
$f(1) = 1$ and $g(1) = 1^2 = 1$ so $f(1) = g(1)$
$f'(x) = 1$ and $g'(x) = 2x \geq 2 > 1$ for all $x \geq 1$ so $g'(x) \geq f'(x)$ for all $x \geq 1$
Therefore, by the Racetrack Principle, $g(x) \geq f(x)$ for all $x \geq 1$.

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The maximum value of the function $f(x) = -x^2 + 3x + 2$ is what?

a) -2

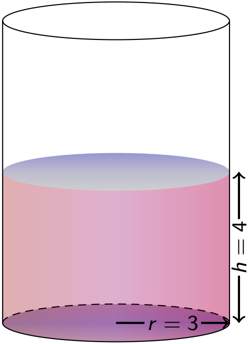
b) 0

c) 2

d) $\frac{17}{4}$

Related Rates

Gasoline is pouring into a cylindrical tank of radius 3 feet. When the height, h, of the gasoline is 4 feet, the height is increasing at 0.2 ft/sec. How fast is the volume of gasoline changing at that instant? *three decimal places.* Hint: Recall that the volume of a cylinder is given by $V = \pi r^2 h$.



Optimization

A square based rectangular box is to be constructed with a surface area of 384 cm^2. what dimensions will maximize the volume of this box?