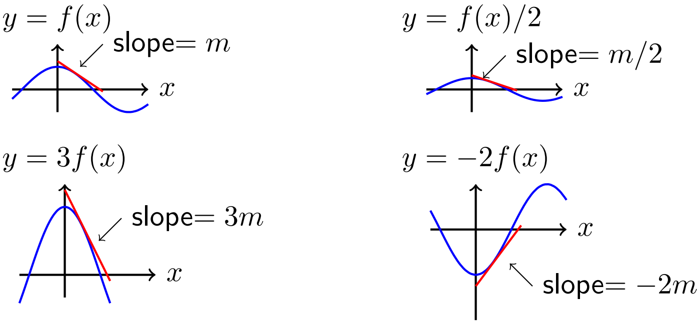
Calculus I
3.1 Powers and Polynomials
3.2 The Exponential Function

Constant Times a Function



Theorem 3.1: Derivative of a Constant Multiple

If f is differentiable and c is a constant, then

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Sum and Difference

Theorem 3.2: Derivative of Sum and Difference

If f and g are differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$$

Proof: Using the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)+g'(x).
\end{aligned}$$

Power Rule

The Power Rule

For any constant real number n,

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Examples: Use the power rule to differentiate the following functions.

(a) x^{25}

(b) $\dfrac{1}{x^3}$

(c) $\sqrt{x}$

(d) $\dfrac{1}{x^{1/3}}$

Example

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b) Recall that $\dfrac{1}{x^3} = x^{-3}$.

For $n = -3$: $\dfrac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \dfrac{-3}{x^4}$

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c) Recall that $\sqrt{x} = x^{1/2}$.

For $n = 1/2$: $\dfrac{d}{dx}(x^{1/2}) = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2} = \dfrac{1}{2x^{1/2}}$

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d) Rewrite $\frac{1}{x^{1/3}} = x^{-1/3}$. For $n = -1/3$:

$$\frac{d}{dx}(x^{-1/3}) = -1/3x^{-1/3-1} = -1/3x^{-4/3} = \frac{-1}{3x^{4/3}}$$

Clicker Question

For $f(x) = 2x^3 + x^2 + 3x + 7$ what is $f'(x)$?

a) $f'(x) = 3x^2 + 2x + 1$

b) $f'(x) = 6x^2 + x + 3$

c) $f'(x) = 6x^2 + 2x + 3$

d) $f'(x) = 6x^2 + 2x + 10$

Example

Find an equation of the line tangent to $f(x) = \frac{x^3}{2} - \frac{4}{3x}$ at $x = 2$.
First find the derivative of $f(x)$:

$$f'(x) = \frac{3x^2}{2} - \frac{4}{3}(-1x^{-2})$$

Recall that $f'(2)$ is the slope of the line tangent to $f(x)$ at $x = 2$.

$$f'(2) = \frac{3(2^2)}{2} + \frac{4}{3(2^2)} = \frac{19}{3}$$

The y-value at $x = 2$ is $f(2) = \frac{10}{3}$.
Then using point slope form the equation of a line tangent to $f(x)$ at $x = 2$ is:

$$(y - \frac{10}{3}) = \frac{19}{3}(x - 2)$$

A ball is tossed into the air from a bridge and its height y (in feet) above the ground t seconds after it is thrown is given by:

$$y = f(t) = -16t^2 + 50t + 36.$$

Find the velocity and acceleration of the ball as functions of time.

a) $v(t) = -16t + 50$ and $a(t) = -16$
b) $v(t) = -32t$ and $a(t) = -32$
c) $v(t) = -32t + 50$ and $a(t) = -32$
d) $v(t) = -32t + 50$ and $a(t) = -32 + 50$

Derivatives of Exponential Functions

We start by examining $f(x) = e^x$. Using the definition:

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x(e^h - 1)}{h} = e^x \lim_{h\to 0} \frac{e^h - 1}{h}.$$

Using a calculator we can approximate this limit as:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Thus

$$\frac{d}{dx}(e^x) = e^x.$$

Other bases

This limit is different for different bases. Consider $g(x) = a^x$. Using the definition:

$$g'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

Using a calculator we can approximate this limit for different values of a.

a	2	3	4	5	6	7
$\lim_{h \to 0} \frac{a^h - 1}{h}$	0.693	1.099	1.386	1.609	1.792	1.946
$\ln(a)$	0.693	1.099	1.386	1.609	1.792	1.946

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

Clicker Question

Given that $f(x) = (2)4^x + 3e^x$, what is $f'(x)$?

a) $f'(x) = (2)4^x + 3e^x$

b) $f'(x) = 2\ln(4)4^x + e^x$

c) $f'(x) = 2\ln(4)4^x + \ln(e)e^x$

d) $f(x) = 2\ln(4)4^x + 3e^x$

Clicker Question

Given that $g(x) = 3x^2 + \pi^x + \pi e^x + x^\pi$, what is $g'(x)$?

a) $g'(x) = 6x + \ln(\pi)\pi^x + \pi e^x + \pi x^{\pi - 1}$

b) $g'(x) = 6x + \pi^x + \pi e^x + \pi x^\pi$

c) $g'(x) = 6x + \ln(\pi)\pi^x + e^x + \pi x^{\pi - 1}$

d) $g'(x) = 6x + \pi^x + \pi e^x + x^{\pi - 1}$

Example

Some antique furniture increased very rapidly in price over the past decade. For example, the price of a particular rocking chair is well approximated by

$$V = 75(1.35)^t,$$

where V is in dollars and t is in years since 2000. Find the rate, in dollars per year, at which the price is increasing at time t.

Question

For what values of a is the function $f(x) = a^x$ increasing and for what values is it decreasing? Use the fact that, for $a > 0$,

$$\frac{d}{dx}(a^x) = \ln(a)a^x.$$