Calculus 1
The Derivative Function and Interpretations

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Example of a nondifferentiable function

Let $f(x) = |x+1|$. Then $f(x)$ is NOT differentiable at $x = -1$.

Hint: Use the limit definition for the piecewise defined function

$$|x+1| := \begin{cases} x+1, & x \geq -1 \\ -(x+1), & x < -1. \end{cases}$$

Computing the Derivative Algebraically

Recall that to find the derivative of $f(x) = 1/x$ at the point $x = 2$

Solution: The derivative is the limit of the difference quotient, so we look at

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$
$$= \lim_{h\to 0} \frac{2-(2+h)}{2h(2+h)} = \lim_{h\to 0} \frac{-h}{2h(2+h)}$$

Since the limit only examines values of h close to, but not equal to, zero, we can cancel h. We get

$$f'(2) = \lim_{h\to 0} \frac{-h}{2h(2+h)} = \frac{-1}{4}$$

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(i) Slope. The slope of the graph at a point is the same as the slope of the tangent line at that point, which is the value of the derivative.

$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h \to 0} \frac{(-1+h)^3 + 2(-1+h) - (-1)^3 - 2(-1)}{h} \\ &= \lim_{h \to 0} \frac{h^3 - 3h^2 + 5h}{h} = 5 \end{aligned}$$

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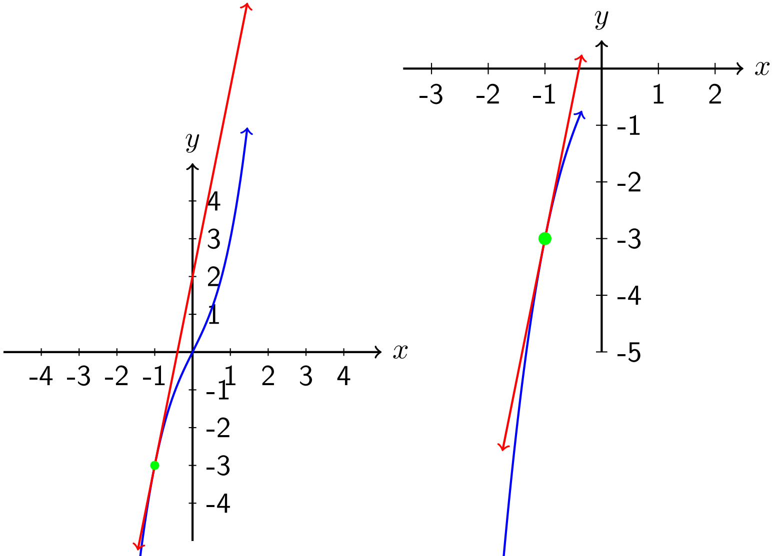
The eqn. of the tangent line is $\boxed{y - (-3) = 5(x - (-1))}$ or

Geometric viewpoint



Using the derivative for approximations

We have that the line $y = 5x + 2$ is tangent to $f(x) = x^3 + 2x$ at $x = -1$. Therefore

$$x^3 + 2x \approx 5x + 2 \quad \text{for } x \text{ close to } -1.$$

Thus

$$f(-1.01) = (-1.01)^3 + 2(-1.01) = -3.050301 \approx$$

$$\underbrace{5(-1.01) + 2}_{=y(-1.01)} = -3.05.$$

Clicker question

What is the slope of the graph of $f(x) = 5x^2 - 2x$ at $x = -2$?

A. 22
B. -22
C. -10
D. 10
E. it does not exist

Derivative as a function (Page 48 of your course packet)

The derivative function is defined for every x as the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. Here, we do not specify the point, it is a general x.

Example. Find $f'(x)$ for $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} (2x + h) = 2x. \end{aligned}$$

Graphically

Based on the fact that

the slope of the graph=value of derivative

we see that

- If $f'(x) > 0$ on an open interval I then $f(x)$ is increasing on I.
- If $f'(x) < 0$ on an open interval I then $f(x)$ is decreasing on I.
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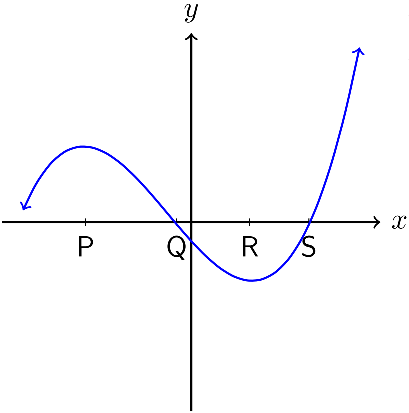
In the example above, for $f(x) = 2x^2 + 3x$ and $f'(x) = 4x + 3$ we have

- f is increasing when $f'(x) = 4x + 3 > 0$, i.e. $x \in (-\frac{3}{4}, \infty)$
- f is decreasing when $f'(x) = 4x + 3 < 0$, i.e. $x \in (-\infty, -\frac{3}{4})$.

Clicker question

What is the largest set on which the function graphed below is increasing?



- A $(-\infty, P) \cup (S, \infty)$
- B $(-\infty, Q) \cup (S, \infty)$
- C (R, ∞)
- D $(-\infty, P) \cup (S, \infty)$
- E $(-\infty, P) \cup (R, \infty)$

A few differentiation formulas

Using the definition of the derivative function we obtain:

- If $f(x) = k$ with k constant, then $f'(x) = 0$.
- If $f(x) = mx + b$, with m, b constant, then $f'(x) = m$
- (Power Rule:) If $n \neq 0$ and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

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- If $f(x) = mx + b$, with m, b constant, then $f'(x) = m$
- (Power Rule:) If $n \neq 0$ and $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Thus, if $f(x) = \frac{1}{x^{2/3}} = x^{-2/3}$ then $f'(x) = -\frac{2}{3}x^{-\frac{2}{3}-1} = -\frac{2}{3}x^{-\frac{5}{3}}$.

Interpretation of the derivative

If $C = f(w)$ is the cost (in dollars to dispose of w pounds of waste then

$$\frac{dC}{dw}\frac{\text{[dollars]}}{\text{[pounds]}} = f'(w)$$

has units of dollars/pound and gives us the rate of change for the cost with respect to the change in weight.

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If we have 100 pounds of waste, the disposal cost is $C = f(100)$. Suppose that we want to dispose of a little more than 100 pounds. About how much extra per pound (over 100 pounds) would we have to pay?

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$$\frac{dC}{dw}|_{w=100} = f'(100)\text{dollars/pound.}$$

Interpretation of the derivative(cont)

Suppose that $f(100) = 2,000$ dollars and $f'(100) = 4$ dollars per pound. About how much would it cost to dispose of 102 pounds of waste?

$$f(102) \approx \underbrace{f(100)}_{\text{cost of 100 pounds}} + \underbrace{f'(100)}_{\text{cost per additional pound}} \cdot \underbrace{(102 - 100)}_{\text{additional pounds over 1}}$$

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Hence

$$f(102) \approx 2000(\text{dollars}) + 4 \text{ dollars/pound} \cdot 2\text{pounds}$$

$$f(102) \approx 2008 \text{ dollars.}$$

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How much to dispose of about 95 pounds of waste?

$$f(95) \approx f(100)+f'(100)(95-100) = 2000+4\cdot(-5) = 1,980 \text{ dolla}$$

Clicker question

You invest \$1000 at an annual interest rate of r%, compounded continuously. At the end of 10 years, you have a balance of B dollars, where $B = g(r)$. What is the financial interpretation of $g'(5) = 165$?

- (A) The balance in your account after 5 years is \$165.
- (B) The balance grows at a rate of \$165 per % when $r = 5\%$.
- (C) If the interest rate increases from 5% to 6%, you would expect about \$165 more in your account after 10 yrs.