Introduction

Roughly 10 minutes maybe.

Questions

1 Multiple Choice, Section 1.1 Q.39

a. F (You can generalize for any $n \in \mathbb{N}$)
b. T (We already know we need different notations to denote them)
c. F (Affine Arithmetic)
d. F (Can draw a picture and then use vector addition)
e. T (Draw the same picture as before and use vector addition)
f. F (You can choose two parallel vectors)
g. T (You can probably prove this in the general sense)
h. F (Can see that span$\{(1,1,0),(1,0,0),(0,1,0)\} = \mathbb{R}^2 \neq \mathbb{R}^3$)
i. F (You can have more than two vectors spanning $\mathbb{R}^n$)
j. T (Sure, why not. You can't have two vectors spanning all of $\mathbb{R}^3$ and trivially can't have one)

2 Multiple Choice, Section 1.2 Q.40

a. T (You can get a general non-zero vector in $\mathbb{R}^n$ and verify this)
b. T (You can do the same as in a. but in reverse)
c. F (CE: Add $\mathbf{v}$ and $-\mathbf{v}$ and verify)
d. F (No, it has the unit vector and its additive inverse)
e. T (See above)
f. F (You can have an entire field of vectors perpendicular to one)
g. T (You can see that $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cdot \cos(\theta)$ and test for values of θ)
h. F (No, as $\mathbf{a} \cdot \mathbf{a} = ||\mathbf{a}||^2$)
i. F (You can have r being a negative number)
j. F (CE: just get v = (1,0), u = (0,-1) in $\mathbb{R}^2$. Then you can see that ||v-u|| $\neq$ 0)

3 Linear Combinations

Let $\mathbf{v} \in \mathbf{R}^2$ be arbitrary. WTS $\mathbf{v} = s[1,2] + t[-1,2]$ with scalar s, t.
Know that $\mathbf{v} = (v_1, v_2)$.
So we have the linear system:

$v_1 = s - t$
$v_2 = 2s + 2t$

Solving this in terms of v_1, v_2 gives us the following:
$s = \frac{v_2 + 2v_1}{4}$ and $s = \frac{v_2 - 2v_1}{4}$
Now need to show uniqueness.
General way of showing uniqueness - first assume there exists something else that satisfies the past condition.

So assume $\exists\, a, b$ so that $\mathbf{v} = a[1,2] + b[-1,2]$.
So $a[1,2] + b[-1,2] - (s[1,2] + t[-1,2]) = (a-s)[1,2] + (b-t)[-1,2] = 0$ to get another linear system.
$(a-s) - (b-t) = 0 \quad (1)$
$2(a-s) + 2(b-t) = 0 \quad (2)$
Multiply (1) by 2 and add to (2).
$(a-s) = 0 \implies a = s$, and substitute into either (1) or (2) to conclude that $b = t$ as well. So we can find a representation of any vector in $\mathbf{R}^2$ as a unique linear combination of the above vectors.

4 Linear Combinations

Definition of linear combination - let $a_1, ...a_k \in \mathbb{R}$ and $\mathbf{v}_1, ...\mathbf{v}_k \in \mathbb{R}^n$. Then the linear combination of these vectors $\mathbf{v}_1, ...\mathbf{v}_k$ with scalar coefficients is expressed as $a_1\mathbf{v}_1 + .. + a_k\mathbf{v}_k$.

To prove what we want, let $\mathbf{u} \in \mathbb{R}^n$ be arbitrary. WTS $\mathbf{u} = a_1\mathbf{e}_1 + 2a_2\mathbf{e}_2 + ... + na_n\mathbf{e}_n$ for scalars $a_1, ...a_k$.

(We can explain the following geometrically as well, maybe I'll do it in class)
Can see that for each vector $i\mathbf{e_i}$ that the following holds (with $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $u_i \in \mathbb{R}$):
$\mathbf{u} \cdot i \cdot \mathbf{e_i} = i \cdot u_i$, so we can pick $a_i = \frac{u_i}{i}$.
Then we can see that $\sum_{i=1}^n a_i\mathbf{v_i} = \sum_{i=1}^n (\frac{u_i}{i})(i \cdot e_i) = \sum_{i=1}^n u_i \cdot e_i = \mathbf{u}$ as wanted.
To show uniqueness, let $\mathbf{u} = \sum_{i=0}^k r_i\mathbf{v_i}$. Then we see that $r_i = \frac{1}{i} \cdot u_i$ as otherwise $\mathbf{u} \neq \sum_{i=0}^k r_i\mathbf{v_i}$.

Break (10 minutes maybe)

5 Definition of a vector norm

Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 \ \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$

Know the definition of the norm of a vector $\mathbf{u} \in \mathbb{R}^n$ is $||\mathbf{u}|| = (\sum_{i=1}^n x_i^2)^{(\frac{1}{2})}$ with $\mathbf{u} = (x_1, x_2, ..., x_n)$. So from this observation we can see that $||\mathbf{u}||^2 = (\sum_{i=1}^n x_i^2) = \mathbf{u} \cdot \mathbf{u}$.

We know that vectors are closed under component wise addition in $\mathbb{R}^n$, so $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are both vectors in $\mathbb{R}^n$.

We know that the inner product of the sum and difference of two vectors looks like this:
$||\mathbf{u}+\mathbf{v}||^2 = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = ||\mathbf{u}|| + 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||$
$||\mathbf{u}-\mathbf{v}||^2 = (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = ||\mathbf{u}|| - 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||$

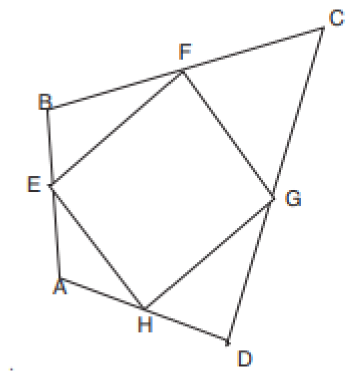
Due to distributivity of dot product over addition and the equivalent way of expressing the square of a vector norm. Also a good example for observing why the triangle inequality holds (Not a proof but maybe an intuitive thing to see).

Now plug in the values above to the right side of the given equation.

$\frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 - \frac{1}{4}||\mathbf{u}+\mathbf{v}||^2 = \frac{1}{4}(||\mathbf{u}|| + 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||)$ - $\frac{1}{4}(||\mathbf{u}|| - 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||)$
$= \frac{1}{2}\mathbf{u} \cdot \mathbf{v} + \frac{1}{2}\mathbf{u} \cdot \mathbf{v}$
$= \mathbf{u} \cdot \mathbf{v}$ as wanted.

6 Fun with vector addition

First we have to know how to equate vectors. $\mathbf{u} = \mathbf{v}$ iff $|\mathbf{u}| = |\mathbf{v}|$ and $u \cdot v = |u||v|$ (the two vectors are parallel). What I'm trying to say is that they don't necessary need to occupy the same space in $\mathbb{R}^n$ but simply need to be parallel and have the same magnitude.



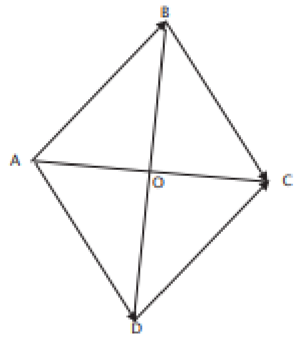
So WTS EF = HG and FG = EH
Start with:

EH = $\frac{1}{2}$BA + $\frac{1}{2}$AD
= $\frac{1}{2}$BD (We can pull $\frac{1}{2}$ out due to scalar multiplication being distributive over vector addition and the definition of vector addition)
= $\frac{1}{2}$(BC + CD) (Vector addition)
= $\frac{1}{2}$BC + $\frac{1}{2}$CD (Distributive property of vector addition)
= FC + CG (Bisection of the edges of the quadrilateral)
= FG (Vector addition)

So if we have the EH = FG, the second equality to prove is easy.
EF = EG - FG (Vector addition)
= EH + HG - FG
= EH + HG - EH (With FG = EH)
= HG as wanted.

7

Using vectors, show that the diagonals of a rhombus bisect each other. (A rhombus is a parallelogram with equal sides.)



With the above diagram, what do we have to show? And what information are we given? Well, we know that ||AB|| = ||BC|| = ||CD|| = ||DA|| and we need to show ||AO|| = ||OC|| and ||BO|| = ||OD||.

Let's think of a strategy to do this. First note that:
- AO = rAC, DO = kDB for $r, k \in [0, 1]$
- AC = AB + BC
- DB = AB - BC (As -BC = CB = DA)
- AB = AO + OB

So we want to show that $r = k = \frac{1}{2}$, and from there we can conclude that ||AO|| = ||OC|| and ||BO|| = ||OD|| as desired.

With our previous observations, we can see that:
AB = AO + OB
= rAC + kDB
= r(AB + BC) + k(AB - BC)
= $(r + k)$AB + $(r - k)$BC
$\implies$ 0 = -AB + $(r + k)$AB + $(r - k)$BC
$\implies$ 0 = (-1+$r + k$)AB + $(r - k)$BC

So now we have a system of equations:

$1 = r + k$
$0 = r - k$
$\implies r = k$ and $2k = 1 \implies k = r = \frac{1}{2}$ as wanted.