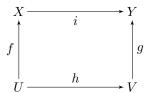
MTH 532 Homework 2

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Exercise 1

Let $f: U \to X$ and $g: V \to Y$ be parameterizations, let $X \subset Y$ be a submanifold, and let $i: X \to Y$ be an inclusion map. This produces the following commutative diagram:



Note that $h = g^{-1} \circ i \circ f$. However, $h = g^{-1} \circ f$ too, as i is the inclusion map bringing X into Y. Without loss of generality and to simplify notation, let $f(0) = g(0) = x \in X$. Since $g^{-1} \circ i \circ f = g^{-1} \circ f$, taking the derivative on both sides yields $d(g^{-1} \circ i \circ f) = d(g^{-1} \circ f)$, or that $dg_0^{-1} \circ di_x \circ df_0 = dg_0^{-1} \circ df_0$. Moving terms gives $di_x = (dg_0 \circ dg_0^{-1}) \circ (df_0 \circ df_0^{-1})$. This is the identity map, so di_x is indeed an inclusion map.

Exercise 3

Let V be a vector subspace of \mathbb{R}^N with dimension n. Then there exists a parameterization $f \colon \mathbb{R}^n \to V$. But f is linear, so df = f. Thus, $T_x(V) = df(\mathbb{R}^n) = f(\mathbb{R}^n) = V$.

Exercise 4

Let $f: X \to Y$ be a diffeomorphism, then there is a smooth map $f^{-1}: Y \to X$ such that $f \circ f^{-1} = Id$. Note $d \ Id = d(f \circ f^{-1})$, implying that $Id = df_x \circ df_y^{-1}$ for all $x \in X$ with y = f(x). Hence, df_x has an inverse in the form of df_y^{-1} , so it is indeed an isomorphism of tangent spaces.

Exercise 5

Let $k \neq l$ and suppose $f: \mathbb{R}^k \to \mathbb{R}^l$ is a diffeomorphism. Then $df_x: T_x(\mathbb{R}^k) \to T_{f(x)}(\mathbb{R}^l)$ is an isomorphism of tangent spaces. Note $T_x(\mathbb{R}^k) = \mathbb{R}^k$ and $T_{f(x)}(\mathbb{R}^l) = \mathbb{R}^l$ for all x. But dim $\mathbb{R}^k = k \neq l = \dim \mathbb{R}^l$, meaning df is not bijective. But df = f, so f is also not bijective. Contradiction! Hence, f is not a diffeomorphism. \square

Exercise 8

Let $H = \{(x, y, z) \mid x^2 + y^2 - z^2 = a\}$ be a hyperboloid and let $h: B_a(0) \to H \subset \mathbb{R}^3$ be a local parameterization of $(\sqrt{a}, 0, 0)$, where $B_a(0)$ is the ball of radius a centered at the origin. Define h as

 $(u,v)\mapsto (\sqrt{a+u^2-v^2},u,v)$. (All credit to Wolfram for that parameterization.) Then dh is the matrix

$$\begin{bmatrix} u/x & -v/x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $x = \sqrt{a + u^2 - v^2}$. Therefore, $dh_{(\sqrt{a},0,0)}$ is the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the tangent space is the span of $dh_{(\sqrt{a},0,0)}$, or the yz-plane.

Exercise 9

- (1) Let $a: U \to X$ and $b: V \to Y$ be parameterizations of X and Y, respectively. Note that $X \times Y = \operatorname{Im}(a \times b)$. Hence, $T_{(x,y)}(X \times Y) = \operatorname{Im} d(a \times b)_{(0,0)}$, which can be rewritten as the image of $\begin{bmatrix} da_0 & 0 \\ 0 & db_0 \end{bmatrix}$, or $T_x X \times T_y Y$.
- (2) Let $f: X \times Y \to X$ be the projection map and let $a: U \to X$ and $b: V \to Y$ be parameterizations. This forms a commutative diagram, meaning there is a map from $U \times V$ to U, $h = a^{-1} \circ f \circ (a \times b)$. For notation and without loss of generality, take $(a \times b)(0,0) = (x,y)$. Shuffling terms yields $f = a \circ h \circ (a \times b)^{-1}$, implying $df_{(x,y)} = da_0 \circ dh_{(0,0)} \circ d(a \times b)_{(0,0)}^{-1}$, which is the desired projection.
- (3) Use a and b as before and let $h: U \to U \times V$ defined by $u \mapsto (u,0)$. Fix y = b(0) and let $f: X \to X \times Y$ be the injective mapping $x \mapsto (x,y)$. Next, consider $f = (a \times b) \circ h \circ a^{-1}$. Note that dh = h, as h is linear. Therefore, $df_x = (da_0 \times db_0) \circ h \circ da_0^{-1}$. Thus, $df_x(v) = (v,0)$.
- (4) Intuition: project each space and then smoothly map it to (x,0) and (0,y) so that the direct sum "fills" the "containing" space.

Let $\pi_x \colon X \times Y \to X$ and $\pi_y \colon X \times Y \to Y$ be natural projections. Let $i_x \colon X' \to \mathbb{R}^N$ and $i_y \colon Y' \to \mathbb{R}^N$ be smooth inclusions defined by $x \mapsto (x,0)$ and $y \mapsto (0,y)$, respectively. (\mathbb{R}^N is assumed to "contain" $X \times Y$.) Then clearly $f \times g = i_x \circ f \circ \pi_x + i_y \circ g \circ \pi_y$. This implies $d(f \times g)_{(x,y)} = df_x \times dg_y$.