MTH 532 Homework 5

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1.5 Transversality

Exercise 2

(a) Yes, this is one of the given examples.

(b) Yes, let v be a vector in the xy-plane such that v is not a multiple of $(3, 2, 0)$, then the three vectors v, $(3, 2, 0)$ and $(0, 4, -1)$ span $\mathbb{R}^3$.

(c) No, because the z-dimension of any combination will always be zero.

(d) Since $T_x\mathbb{R}^a = \mathbb{R}^a$, these two spaces are transversal iff $k + l \geq n$.

(e) Tricky. The two spaces span $\mathbb{R}^n$ iff $k \geq n$ or $l \geq n$, as one space is a subset of or equal to the other.

(f) Yes: $v = (a, b) \in V$ can be written as the sum of (b, b) and $(a - b, 0)$.

(g) Yes. Let $A \in \mathrm{M}(n)$ be a canonical basis element, i.e., only a single entry is nonzero and equal to 1. Then we can represent it via the sum of a symmetric (B) and skew-symmetric (C) matrix. Suppose $a_{ij} = 1$, then let $b_{ij} = b_{ji} = 1$, $c_{ij} = 1$, and $c_{ji} = -1$, then $A = \frac{1}{2}(A + B)$. $\square$

Exercise 4

Let $h \in T_yX \cap T_yZ$, then h is tangent to both X and Z, meaning it is tangent to $X \cap Z$. Hence, it is in the tangent space of $X \cap Z$, i.e., $h \in T_y(X \cap Z)$.

On the other hand, if X and Z are transversal, then $X \cap Z$ is a submanifold of both X and Z. Consider the natural inclusion maps $f\colon X \cap Z \to X$ and $g\colon X \cap Z \to Z$, then $df_p\colon T_p(X \cap Z) \to T_pX$ and $dg_p\colon T_p(X \cap Z) \to T_pZ$ imply that $T_p(X \cap Z) \subset T_pX$ and $T_p(X \cap Z) \subset T_pZ$. Therefore, $T_p(X \cap Z) \subset T_pX \cap T_pZ$.

Hence, $T_p(X \cap Z) = T_pX \cap T_pZ$. $\square$

Exercise 8

Since at any point, the hyperboloid and sphere both have tangent planes, in order to be transversal, these two planes must not overlap. Otherwise, the ambient space cannot be "filled."

Clearly, when $a = 1$ and $z = 0$, there is a problem, as both objects have tangent planes parallel to the z-axis. In fact, they are not transversal when $a = 1$. For $a > 1$, the two objects do not intersect, so transversality is uninterestingly true. For $a < 1$, the intersection is a horizontal circle; the tangent planes do not coincide, so the two objects are transversal. $\square$

Exercise 9

Recall that if an eigenvalue $\lambda = 1$, then there is a corresponding eigenvector v such that $Av = v$. Hence, if 1 is not an eigenvalue, then $Av \neq v$ for all $v \in V$. Thus $W \cap \Delta = \varnothing$, so transversality is vacuously true.

Conversely, suppose 1 is an eigenvalue of A, then $W \cap \Delta = \Delta$, so the tangent space of the intersection cannot span $V \times V$, meaning W is not transversal to Δ. Thus, 1 is not an eigenvalue of A. $\square$

1.6 Homotopy and Stability

Exercise 3

Let $x \sim y$ be an equivalence relation on X with $x \sim y$ iff there is a smooth curve $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Per the hint, we shall show $\sim$ is equivalent to homotopy.

Let $F\colon X \times [0,1] \to X$ be a homotopy with $F(c,0) = x$ and $F(c,1) = y$ for all $c \in X$. Then, if $x \sim y$ via $f\colon [0,1] \to X$, let $F(c,t)$ equal $f(t)$. Conversely, if F is a homotopy as defined above, then let $f(t) = F(c_0, t)$ for any $c \in X$, implying $x \sim y$. Hence, $\sim$ is an equivalence relation.

Suppose $X/{\sim}$ has more than one component, then it can be parameterized by disjoint open sets. However, then there are two points x and y which cannot be connected. This is a contradiction, as X is connected. Hence, $X/{\sim}$ has a single component and the equivalence classes under $\sim$ are open. $\square$

Exercise 4

Let X be contractible, $f\colon Y \to X$ be a smooth map of manifolds, $c(x)\colon X \to X$ be the constant map defined by $* \mapsto x$, and F be a homotopy between id, the identity map on X, and $c(x)$.

Next, construct a homotopy G between Y and X defined by $G(y,t) = F(f(y),t)$. Note that $G(y,0) = id \circ f(y) = f(y)$ and $G(y,1) = c(x)(1) = x$. Thus, any smooth map is homotopic to the constant map and, therefore, any two smooth maps of manifolds are homotopic. $\square$

Exercise 6

This follows from exercise **1.6.4**. Suppose $f\colon S^1 \to X$ is a smooth map and suppose X is contractible, then f is homotopic to a constant map on X. Let $x, y \in X$. Then there is a homotopy F_1 between x and id and a homotopy F_2 between id and y. Thus, there is a homotopy F_0 between x and y, so there is a smooth map connecting the two points. x and y were arbitrary, so X is connected. Hence, X is simply connected. $\square$

Exercise 7

Consider $S^{2k-1} \subset \mathbb{R}^{2k} \cong \mathbb{C}^k$. We know $z \mapsto ze^{i\pi}$ smoothly rotates a point $z \in \mathbb{C}$ by π. Consider S^{2k-1} as a subset of $\mathbb{C}^k$, i.e., $\{\, (z_1, \cdots, z_k) \mid z_1^2 + \cdots + z_k^2 = 1 \,\}$. Then there is a smooth homotopy $R\colon S^{2k-1} \times [0,1] \to S^{2k-1}$ defined by $(z_1, \cdots, z_k) \mapsto (z_1 e^{i\pi t}, \cdots, z_k e^{i\pi t})$ between the identity and antipodal maps. $\square$

Exercise 9

Consider $t > 0$ and note that, for $|tx| > 2$, one has $p(tx) = 0$. Therefore, f_t cannot have any of the properties **(a)** through **(f)**. For **(d)**, consider $Z = \{\, 0 \,\}$. Since, $f_t \equiv 0$ for $|x| > 2|t|^{-1}$, it is not transversal to Z. $\square$