MTH 532 Homework 8

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2.4 Intersection Theory Mod 2

Exercise 1

Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $z \mapsto z^7 + \cos(|z|^2)(1 + 93z^4)$. To show there is a $z \in \mathbb{C}$ such that $f(z) = 0$, note that $f(-10) < 0 < f(10)$. As f is continuous, by the Intermediate Value Theorem, there is an $z \in \mathbb{C}$ such that $f(z) = 0$. (I suspect the book wanted us to construct a homotopy between z^7 and f, then consider $\deg_2(f)$ over a sufficiently large disk.) $\square$

Exercise 11

Suppose $f\colon X \to Y$ is not surjective, there there is a point y outside the image of f. Note y is transversal to $f(X)$, as there is no intersection, impying $I_2(f, \{ y \}) = 0$. But Y is connected, so there is a homotopy between y and some other point y_0 contained in the image of f. As homotopic maps have the same degree mod 2, $\deg_2(f) = 0$. Thus, for surjective maps, $\deg_2(f) = 1$. $\square$

Exercise 12

Use exercise **2.4.11**. In this case, a map $f\colon X \to Y$ cannot be surjective, so it must be that $\deg_2(f) = 0$. $\square$

3.2 Orientation

Exercise 2

(a) Note in this case the linear isomorphism A from $V \to V$ which sends β to $\beta' = \{ v_1, \cdots, cv_i, \cdots, cv_k \}$ is the identity matrix with the entry at row and column i replaced by c. Hence, $\det A = c$, so β' is equivalently oriented iff $c > 0$ and β' is oppositely oriented iff $c < 0$.

(b) Let $E_{i,j}$ be the elementary row operator matrix which switches row i with row j. Note $\det E_{i,j} = -1$ and that $E_{i,j}$ performs the mentioned change of ordered basis. Hence, transposing two elements produces an oppositely oriented basis.

(c) Again, as in **(b)**, this is another elementary row operation matrix, but with determinant one, thus thus producing an equivalently oriented basis. $\square$

Exercise 4

Use exercise **3.2.2b**. Let $\dim V_1 = a$ and $\dim V_2 = b$, with ordered bases $\beta_1 = \{ x_1, \cdots, x_a \}$ and $\beta_2 = \{ y_1, \cdots, y_b \}$, respectively. Then $V_1 \oplus V_2$ has an ordered basis $B = \{ x_1, \cdots, x_a, y_1, \cdots, y_b \}$. Similarly, $V_2 \oplus V_1$ has an ordered basis $B' = \{ y_1, \cdots, y_b, x_1, \cdots, x_a \}$. Hence, we can construct a linear map which is the composition of the elementary row operator which switches v_i with v_{i+1}. To go from B to B', this would have to be applied b times for each element in V_1, i.e. ab times. [Start with x_a and move it "to the end," then do x_{a-1}, etc.]

Thus, the determinant of this linear map is $(-1)^{ab} = (-1)^{(\dim V_1)(\dim V_2)}$. $\square$

Exercise 6

Let $\beta = \{ v_1, \cdots, v_{k-1} \}$ be an ordered basis for δH^k and let n be the outward facing normal. Then $\beta' = \{ n, \beta \}$ has the same sign as β. Note that β' is oppositely oriented compared to the standard ordered basis on $\mathbb{R}^k$. We can then "move" n to the end of the basis, which changes the sign by $(-1)^{k-1}$. We can then change n to $-n$. Thus, for even k, the sign does not change, as desired, for we found the opposite orientation of $\mathbb{R}^{k-1}$. $\square$