MTH 532 Homework 12

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4.5 Exterior Derivative

Exercise 1

(a) Let $\omega = z^2\, dx \wedge dy + (z^2 + 2y)\, dx \wedge dz$, then $d\omega = (2z - 2)\, dx \wedge dy \wedge dz$.

(b) Let $\omega = 13x\, dx + y^2\, dy + xyz\, dz$, then $d\omega = xz\, dy \wedge dz - yz\, dz \wedge dx$.

(c) If f, g are functions, then $d(fdg) = dfdg + f(ddg)$, but $d(dg) = 0$, so $d(fdg) = dfdg = \nabla f \nabla g$.

(d) Let $\omega = (x + 2y^3)(\, dz \wedge dx + \frac{1}{2}\, dy \wedge dx)$, then $d\omega = 6y^2\, dx \wedge dy \wedge dz$. $\square$

Exercise 2

Let $F = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$, then $\operatorname{curl} F = \nabla \times F = \left(0, 0, \frac{\delta}{\delta x}\frac{x}{x^2+y^2} - \frac{\delta}{\delta y}\frac{-y}{x^2+y^2}\right) = \left(0, 0, \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right) = \vec{0}$. It cannot be written as the gradient of any function because derivatives are unique and the gradient of $z = \arctan(y/x)$ restricted to x and y produces the given vector field, as found in exercise **4.4.8b**.

4.7 Stokes Theorem

Exercise 6

Let D be a compact region in $\mathbb{R}^3$ with smooth boundary S. Let $D' = D - B_0$, where B_0 is a ball of infinitesmal radius centered at the origin. Let $\vec{F} = q\vec{r}/r^3 = q(x, y, z)(x^2 + y^2 + z^2)^{-3/2}$. Note that $\operatorname{div}(\vec{F}) = 0$ for all $r \neq 0$. Denote $\delta B_0 = S_R$, a sphere of radius r. Then,

$$\begin{aligned}
\int_S \vec{F} \cdot \vec{n}\, dA &= \int_{D'} \operatorname{div}(\vec{F})\, dV \\
&= \int_D 0\, dV - \int_{B_0} \operatorname{div}(\vec{F})\, dV \\
&= \int_{S_R} \vec{F} \cdot \vec{n}\, dA \\
&= \int_{S_R} (q\vec{r}/r^3) \cdot (\vec{r}/r)\, dA \\
&= q/r^2 \int_{S_R} dA = 4\pi q
\end{aligned}$$

$\square$

Exercise 8

Let $X = \delta W$ with W compact and let $f \colon X \to Y$ be a smooth map. Let ω be a closed k-form on Y with $k = \dim X$. Suppose f extends to all of W. Note that W has dimension $l = k + 1$, so ω is a $l - 1$ form.

Apply the generalized Stokes theorem:

$$\int_X f^*\omega = \int_W d(f^*\omega) = \int_W f^*(d\omega) = 0$$

This follows from the fact that ω is closed on Y. $\square$

Exercise 9

Let $f_0, f_1 \colon X \to Y$ be smooth homotopic maps and let X be a smooth, boundaryless manifold of dimension k. Suppose ω is a closed k-form on Y. Let $W = X \times [0,1]$ and consider the homotopy $F \colon W \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Note W is a manifold with boundary, so, by exercise **4.7.8**, we have that $\int_{\delta W} F^*\omega = 0$. On the other hand, we have

$$\int_{\delta W} F^*\omega = \int_{X\times[0,1]} F^*\omega = \int_{X\times\{0\}} F^*\omega + \int_{-X\times\{1\}} F^*\omega = \int_X f_0^*\omega - \int_X f_1^*\omega$$

Thus, $\int_X f_0^*\omega = \int_X f_1^*\omega$. $\square$

Exercise 10

Let X be a simply connected manifold, ω be a closed 1-form on X, and γ be a closed curve in X. Note that γ can contracted to a single point, as X is simply connected. Fix $c \in X$ and let $f_c \colon S^1 \to X$ be defined by $* \mapsto c$. Consider the homotopy between γ and f_c, then, $\int_{S^1} \gamma^*\omega = \int_{S^1} f_c^*\omega$ by exercise **4.7.9**. Hence,

$$\oint_\gamma \omega = \int_{S^1} \gamma^*\omega = \int_{S^1} f_c^*\omega = 0$$

This follows from **4.7.8**, as f_c extends to all of D^2 and $\delta D^2 = S^1$, implying $\int_{\delta D^2} f_c^*\omega = 0$. $\square$