# MTH 532 Homework 6

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## 1.7 Sard's Theorem and Morse Functions

### Exercise 4

Recall that the countable union of sets of measure zero has measure zero. $\mathbb{Q}$ is countable and each of its points has measure zero. Therefore, $\mathbb{Q}$ has measure zero. $\square$

### Exercise 6

Per the hint, let $f\colon S^1 \to S^k$ with $k > 1$. Then $p \in S^k$ is a regular value iff it is not in the image of $f$. But Sard's Theorem says the set of critical values of a smooth map has measure zero, so there must be a point $p_0 \notin f(S^1)$. Again, per the hint, recall that $S^k$ minus a single point is isomorphic to $\mathbb{R}^k$ via the stereographic projection. Hence, from problem 1.6.6, as $\mathbb{R}^k$ is contractible, we have that it is also simply connected. Thus $S^k$ is simply connected. $\square$

## 1.8 Embedding Manifolds in Euclidean Space

### Exercise 5

Let $p\colon T(X) \to X$ be the mentioned projection. Let $p = (x, v) \in T(X)$ and let $O(p)$ be an open neighborhood of $p$. Let $N$ be a neighborhood of $x$ such that $a\colon N \to \mathbb{R}^k$ is locally equivalent to the canonical submersion, i.e. $(x_1, x_2, \cdots, x_k) \mapsto (x_1, x_2, \cdots, x_l)$. Let $b$ be the same for the neighborhood $O(p)$. Let $p'$ be the restriction of $p$ to $O(p)$. Note that $a \circ p' \circ b^{-1}$ maps local coordinates to local coordinates, so $p$ is a local submersion at $p$, which was arbitrary. $\square$

### Exercise 6

Let $\vec{v}$ be a vector field on $X$, then, per the given definition, $\vec{v}(x)$ is tangent to $x$. Thus, we can define a smooth map $t\colon X \to T(X)$ defined by $x \mapsto (x, \vec{v}(x))$. Then $p \circ t$ is the identity map, so **(1)** implies **(2)**.

Conversely, assume there is a smooth map $t$ (as before) such that $p \circ t$ is the identity map. But then there is a function $\vec{v}$ such that $p(x, \vec{v}) = x$, so we may define $t$ as the map $x \mapsto (x, \vec{v}(x))$. By the definition of $T(X)$, we have that both $x$ and $\vec{v}(x)$ lie in $\mathbb{R}^N$. Thus, $\vec{v}$ is a vector field on $X$ and **(2)** implies **(1)**. $\square$

### Exercise 7

(Haha, I used this hint on my past algebra exam!) Let $x \in S^k$ and let $k$ be odd. Suppose $x = (x_1, x_2, \cdots, x_{k+1})$ and note that $x^\perp = (-x_2, x_1, \cdots, -x_{k+1}, x_k)$ is orthogonal to $x$. Let $\vec{v}$ be the map $x \mapsto x^\perp$. We wish only to show that $\vec{v}$ is nowhere vanishing, so note that $|\vec{v}(x)| = |x^\perp| = |x| = 1$. $\square$

### Exercise 8

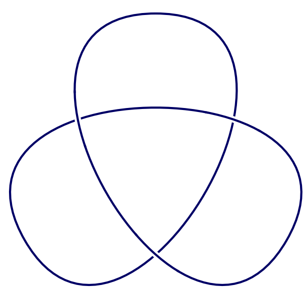
Note that from 1.6.7 (last week's homework) we have that $x \mapsto -x$ is homotopic to the identity iff $k$ is odd. But we just proved that $k$ is odd if $S^k$ has a nonvanishing vector field (and it was given that this does not occur for even $k$). $\square$

### Exercise 10

Let $X \subset \mathbb{R}^N$ be an immersion for $N > 2k$ (otherwise, it's not very interesting) and let $g\colon T(X) \to \mathbb{R}^N$ be the map $(x, v) \mapsto df_x(v)$. Then by Sard's Theorem we can pick a regular value $a$ which is not in the image of $g$. That means we can project $\mathbb{R}^{k+1}$ onto $\mathbb{R}^k$ via some map $\pi$, as there are $k$ dimensions orthogonal to $a$. We wish to show that $\pi \circ f$ is an immersion. This is true, as $d(\pi \circ f) = d\pi_{f(x)} \circ df_x = D$. So if $D(v)$ vanishes, then $df_x(v) = ta$ for some $t \in \mathbb{R}$ (we used this fact in class), which is impossible as $a$ is a regular value of $g$. Thus, we have dropped the dimension of our immersion from $N$ to $N-1$. Repeat until $N = 2k$. $\square$

## 2.2 One-Manifolds and Some Consequences

### Exercise 1



No, consider the 3-1 trefoil knot. $\square$