MTH 532 Homework 1

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Exercise 1

Let $k < l$ and let f be a smooth function on $\mathbb{R}^k$. Let F be a function on $\mathbb{R}^l$ such that $F(a_1, \cdots, a_k, 0, \cdots, 0) = f(a_1, \cdots, a_k)$. Consider $\pi\colon \mathbb{R}^k \to \mathbb{R}^k$ defined by $x \mapsto x$ and note that $F = f \circ \pi$. Hence, F is the composition of smooth functions and is therefore smooth itself.

On the other hand, suppose F is smooth on $\{(a_1, \cdots, a_k, 0, \cdots, 0)\}$. Let $i\colon \mathbb{R}^k \to \mathbb{R}^l$ be the smooth map defined by $(a_1, \cdots, a_k) \mapsto (a_1, \cdots, a_k, 0, \cdots, 0)$ and let f be a function on $\mathbb{R}^k$ such that $f(a_1, \cdots, a_k) = F(a_1, \cdots, a_k, 0, \cdots, 0)$. Then F is smooth, as it is the composition of the smooth maps f and i. $\square$

Exercise 2

Let f be smooth on $X \subset \mathbb{R}^N$. As f is smooth on X, for every $p \in X$, there is an open neighborhood $O(p) \subset \mathbb{R}^N$ such that $F\colon O(p) \to \mathbb{R}$ is smooth on $O(p)$ and $F(p) = f(p)$ for all $p \in O(p) \cap X$.

Now consider $Z \subset X$ and, for every $p \in Z$, take $O(p)$ and F as before. Then $F(p) = f(p)$ for all $p \in O(p) \cap Z$ and f is thus smooth on Z. $\square$

Exercise 4

a Let $f^{-1}\colon \mathbb{R}^k \to B_a$ be the smooth map defined by

$$f^{-1}(y) = \frac{ay}{\sqrt{a^2 + ||y||^2}}$$

Note that $f^{-1}(0) = 0$ and $\lim_{y\to\infty} f^{-1}(y) = a$. That is, f^{-1} maps $[0, \infty)^k$ to $[0, a)^k$, which makes intuitive sense as f did the opposite.

b Since X is a manifold, for every $x \in X$, there is a parameterization $p\colon U \to O(x)$ where $U \subset \mathbb{R}^k$ and $O(x)$ is an open neighborhood of x. But U can be the ball B_a of radius a, as there is always one small enough inside of U such that $x \in p(B_a) \subset V$. So consider $f^{-1} \circ p$ restricted to B_a, which is a parameterization of an open neighborhood of x with all of $\mathbb{R}^k$ as its domain. $\square$

Exercise 6

Let $h(x) = x^{1/3}$. Note that $f \circ h = h \circ f = id$ and that $h'(x) = \frac{1}{3}x^{-2/3}$. However $\lim_{x\to 0} h(x)$ does not exist, so h is not smooth and f is not a diffeomorphism. $\square$

Exercise 8

Let $a > 0$ and let H be the hyperboloid $\{(x, y, z) \mid x^2 + y^2 - z^2 = a\}$. Let B_a be the ball of radius a centered at the origin. The upper half of H can then be parameterized via $\phi\colon\ \mathbb{R}^2 - B_a \to \mathbb{R}^3$ defined by $(x, y) \mapsto (x, y, \sqrt{x^2 + y^2 - a})$. Similarly, the lower half of H can be parameterized by $(x, y) \mapsto (x, y, -\sqrt{x^2 + y^2 - a})$. Intuitively speaking, this involves lifting the plane minus B_a so that it "covers" the given half of H.

When $a = 0$, the point $(0, 0, 0)$ becomes a problem. Removing the origin from $\mathbb{R}^2$ leaves one component, whereas removing the origin from H leaves two components, so H is not a manifold. $\square$

Exercise 12

Let $N = (0, 0, 1)$ and let p be a point on S^2. The line through points N and p then has the equation

$$\begin{aligned} l(t) &= (0 + t(x - 0),\ 0 + t(y - 0),\ 1 + t(z - 1)) \\ &= (tx,\ ty,\ 1 + t(z - 1)) \end{aligned}$$

This line hits the xy-plane when $z = 0$, or when $1 + t(z - 1) = 0$, implying $t = \frac{1}{1-z}$. Hence,

$$\pi(x, y, z) = (\frac{x}{1 - z}, \frac{y}{1 - z})$$

To find π^{-1}, note that

$$\begin{aligned} \pi^{-1}(0, 0) &= -N \\ \pi^{-1}(1, 0) &= (1, 0, 0) \\ \pi^{-1}(0, 1) &= (0, 1, 0) \end{aligned}$$

and $||(x, y)|| = 1 \iff z = 0$. I couldn't think of a function $z = f(x, y)$ which satisfied these conditions, but google gave me $z = \frac{x^2+y^2-1}{x^2+y^2+1}$, which definitely works. This makes finding π^{-1} easy:

$$\pi^{-1}(x, y) = (\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2})$$

$\square$

Exercise 14

Let $(x, y) \in X \times Y$ and let $U \times V$ be an open neighborhood of (x, y) such that F is smooth on U, G is smooth on V, F restricted to $U \cap X$ equals f, and G restricted to $V \cap Y$ equals g. Note that $(U \times V) \cap (X \times Y) = (U \cap X) \times (V \cap Y)$. Hence, since $F \times G$ is smooth on $(U \cap X) \times (V \cap Y)$, it is also smooth on $(U \times V) \cap (X \times Y)$, so $f \times g$ is too. $\square$

Exercise 18

a From class, we had that $f^{(n)}(x) = P_n e^{-1/x^2}$, where P_n is a polynomial of order n or less. Thus $\lim_{x \to 0} f^{(n)}(x) = 0$ for all $n \in \mathbb{N}$, so f is smooth.

b Subtraction and $(x, y) \mapsto xy$ are smooth functions, so g, the composition of smooth functions, is too. Since g is smooth and positive function on (a, b), we have that $c = \int_{-\infty}^{\infty} g\ dx$ is nonzero. Hence, $h(x) = \frac{1}{c} \int_{-\infty}^{x} g\ dx = \frac{1}{c} G(x)$ by the Second Fundamental Theorem of Calculus. As $G' = g$ and g was smooth, h must be too, with $h^{(n)} = \frac{1}{c} g^{(n)}$ for all $n \in \mathbb{N}$.

c Consider the function $r(x) = 1 - h(||x||)$. $\square$