# MTH 532 Homework 1

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### Exercise 1

Let k < l and let f be a smooth function on  $\mathbb{R}^k$ . Let F be a function on  $\mathbb{R}^l$  such that  $F(a_1, \dots, a_k, 0, \dots, 0) = f(a_1, \dots, a_k)$ . Consider  $\pi \colon \mathbb{R}^k \to \mathbb{R}^k$  defined by  $x \mapsto x$  and note that  $F = f \circ \pi$ . Hence, F is the composition of smooth functions and is therefore smooth itself.

On the other hand, suppose F is smooth on  $\{(a_1, \dots, a_k, 0, \dots, 0)\}$ . Let  $i : \mathbb{R}^k \to \mathbb{R}^l$  be the smooth map defined by  $(a_1, \dots, a_k) \mapsto (a_1, \dots, a_k, 0, \dots, 0)$  and let f be a function on  $\mathbb{R}^k$  such that  $f(a_1, \dots, a_k) = F(a_1, \dots, a_k, 0, \dots, 0)$ . Then F is smooth, as it is the composition of the smooth maps f and i.

### Exercise 2

Let f be smooth on  $X \subset \mathbb{R}^N$ . As f is smooth on X, for every  $p \in X$ , there is an open neighborhood  $O(p) \subset \mathbb{R}^N$  such that  $F: O(p) \to \mathbb{R}$  is smooth on O(p) and F(p) = f(p) for all  $p \in O(p) \cap X$ .

Now consider  $Z \subset X$  and, for every  $p \in Z$ , take O(p) and F as before. Then F(p) = f(p) for all  $p \in O(p) \cap Z$  and f is thus smooth on Z.

#### Exercise 4

**a** Let  $f^{-1}$ :  $\mathbb{R}^k \to B_a$  be the smooth map defined by

$$f^{-1}(y) = \frac{ay}{\sqrt{a^2 + ||y||^2}}$$

Note that  $f^{-1}(0) = 0$  and  $\lim_{y\to\infty} f^{-1}(y) = a$ . That is,  $f^{-1}$  maps  $[0,\infty)^k$  to  $[0,a)^k$ , which makes intuitive sense as f did the opposite.

**b** Since X is a manifold, for every  $x \in X$ , there is a parameterization  $p \colon U \to O(x)$  where  $U \subset \mathbb{R}^k$  and O(x) is an open neighborhood of x. But U can be the ball  $B_a$  of radius a, as there is always one small enough inside of U such that  $x \in p(B_a) \subset V$ . So consider  $f^{-1} \circ p$  restricted to  $B_a$ , which is a parameterization of an open neighborhood of x with all of  $\mathbb{R}^k$  as its domain.

### Exercise 6

Let  $h(x) = x^{1/3}$ . Note that  $f \circ h = h \circ f = id$  and that  $h'(x) = \frac{1}{3}x^{-2/3}$ . However  $\lim_{x\to 0} h(x)$  does not exist, so h is not smooth and f is not a diffeomorphism.

### Exercise 8

Let a>0 and let H be the hyperboloid  $\{(x,y,z)\mid x^2+y^2-z^2=a\}$ . Let  $B_a$  be the ball of radius a centered at the origin. The upper half of H can then be parameterized via  $\phi\colon \mathbb{R}^2-B_a\to\mathbb{R}^3$  defined by  $(x,y)\mapsto (x,y,\sqrt{x^2+y^2-a})$ . Similarly, the lower half of H can be parameterized by  $(x,y)\mapsto (x,y,-\sqrt{x^2+y^2-a})$ . Intuitively speaking, this involves lifting the plane minus  $B_a$  so that it "covers" the given half of H.

When a = 0, the point (0, 0, 0) becomes a problem. Removing the origin from  $\mathbb{R}^2$  leaves one component, whereas removing the origin from H leaves two components, so H is not a manifold.

### Exercise 12

Let N = (0,0,1) and let p be a point on  $S^2$ . The line through points N and p then has the equation

$$l(t) = (0 + t(x - 0), \ 0 + t(y - 0), \ 1 + t(z - 1))$$
  
=  $(tx, \ ty, \ 1 + t(z - 1))$ 

This line hits the xy-plane when z=0, or when 1+t(z-1)=0, implying  $t=\frac{1}{1-z}$ . Hence,

$$\pi(x, y, z) = (\frac{x}{1-z}, \frac{y}{1-z})$$

To find  $\pi^{-1}$ , note that

$$\pi^{-1}(0,0) = -N$$
  

$$\pi^{-1}(1,0) = (1,0,0)$$
  

$$\pi^{-1}(0,1) = (0,1,0)$$

and  $||(x,y)||=1 \iff z=0$ . I couldn't think of a function z=f(x,y) which satisfied these conditions, but google gave me  $z=\frac{x^2+y^2-1}{x^2+y^2+1}$ , which definitely works. This makes finding  $\pi^{-1}$  easy:

$$\pi^{-1}(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2})$$

### Exercise 14

Let  $(x,y) \in X \times Y$  and let  $U \times V$  be an open neighborhood of (x,y) such that F is smooth on U, G is smooth on V, F restricted to  $U \cap X$  equals f, and G restricted to  $V \cap Y$  equals g. Note that  $(U \times V) \cap (X \times Y) = (U \cap X) \times (V \cap Y)$ . Hence, since  $F \times G$  is smooth on  $(U \cap X) \times (V \cap Y)$ , it is also smooth on  $(U \times V) \cap (X \times Y)$ , so  $f \times g$  is too.

#### Exercise 18

- **a** From class, we had that  $f^{(n)}(x) = P_n e^{-1/x^2}$ , where  $P_n$  is a polynomial of order n or less. Thus  $\lim_{x\to 0} f^{(n)}(x) = 0$  for all  $n \in \mathbb{N}$ , so f is smooth.
- **b** Subtraction and  $(x,y) \mapsto xy$  are smooth functions, so g, the composition of smooth functions, is too. Since g is smooth and positive function on (a,b), we have that  $c=\int_{-\infty}^{\infty}g\ dx$  is nonzero. Hence,  $h(x)=\frac{1}{c}\int_{-\infty}^{x}g\ dx=\frac{1}{c}G(x)$  by the Second Fundamental Theorem of Calculus. As G'=g and g was smooth, h must be too, with  $h^{(n)}=\frac{1}{c}G^{(n)}$  for all  $n\in\mathbb{N}$ .
- **c** Consider the function r(x) = 1 h(||x||).