MTH 532 Homework 3

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Exercise 3

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a local diffeomorphism, then f is a local homeomorphism and, therefore, an open map. Note that f is continuous. Since $\mathbb{R}$ is connected, the image of f must be too. Hence, $f(\mathbb{R})$ is an open interval. Furthermore, df_x is everywhere nonzero, so f is injective. Thus, f maps diffeomorphically onto its image. $\square$

Exercise 4

Consider $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (e^x \cos y, e^x \sin y)$. Then df is

$$\begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

So det $df = e^{2x} \cos^2 y + e^{2x} \sin^y = e^{2x}$, which is everywhere nonzero. Hence, f is a local diffeomorphism. However, f is clearly not invertible, as it has period 2π, so it is not a diffeomorphism onto its image. $\square$

Exercise 6

(a) Let $f\colon A \to B$ and $g\colon C \to D$. If f and g are immersions, then the maps $df_a\colon T_aA \to T_bB$ and $dg_c\colon T_cC \to T_dD$ are injective. Note that $T_{(x,y)}(X \times Y) = T_xX \times T_yY$. Hence, $df_a \times dg_c\colon T_aA \times T_cC \to T_bB \times T_dD$ equals $d(f \times g)_{(a,c)}\colon T_{(a,c)}(A \times C) \to T_{(b,d)}(B \times D)$. Since df_a and dg_c are injective, so is $d(f \times g)_{(a,c)}$. Thus, $f \times g$ is an immersion.

(b) Let $f\colon A \to B$ and $g\colon B \to C$ be immersions. Note that $d(g \circ f)_x = dg_{f(x)} \circ df_x$. df_a and $dg_{f(a)}$ are injective, so $dg_{f(a)} \circ df_a$ is injective for all $a \in A$. Hence, $g \circ f$ is an immersion.

(c) Let Z be a submanifold of X, let $i\colon Z \hookrightarrow X$ be the inclusion map, and let $f\colon X \to Y$ be an immersion. Then $f|_Z\colon Z \to Y$ equals $f \circ i$. Since f is an immersion and i is the inclusion map, $f|_Z$ is also an immersion.

(d) If $f\colon X \to Y$ is an immersion at x and $y = f(x)$, then, by the Local Immersion Theorem, there are local coordinates about x and y such that $f(x_1, \cdots, x_k) = (x_1, \cdots, x_k, 0, \cdots, 0)$. However, if dim $X =$ dim Y, then $f(x_1, \cdots, x_k) = (x_1, \cdots, x_k)$, so f is indeed a local diffeomorphism. $\square$

Exercise 7

(a) Let $g\colon \mathbb{R}^1 \to S^1$ be defined by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$, then $dg_t = (-2\pi \sin 2\pi t, 2\pi \cos 2\pi t)$. Since $dg_t \neq (0, 0)$ for all t, by the Inverse Function Theorem, g is a local diffeomorphism.

(b) Let $G = g \times g \colon L \to S^1 \times S^1$ be defined by $(a, b) \mapsto (\cos 2\pi a, \sin 2\pi a, \cos 2\pi b, \sin 2\pi b)$, where $L \subset \mathbb{R}^2$ is a line of irrational slope. Without loss of generality, suppose L has no constant term, as that only serves to "shift" the image of L about the torus. That is, let $i \in \mathbb{R} - \mathbb{Q}$ and define $L = \{(x, y) \mid y = ix\}$.

Next, suppose $G(s, is) = G(t, it)$. Then $\cos 2\pi s = \cos 2\pi t$, so $s - t \in \mathbb{Z}$. Similarly, $\cos 2\pi is = \cos 2\pi it$, so $is - it = i(s - t) \in \mathbb{Z}$. But i is irrational, so $i(s - t)$ is in $\mathbb{Z}$ iff $s - t = 0$. Thus, G is injective. $\square$

Exercise 8

Let $h \colon \mathbb{R}^1 \to \mathbb{R}^2$ be defined by $t \mapsto \frac{1}{2}(e^t + e^{-t}, e^t - e^{-t})$. To show h is an embedding, we must show that it is an immersion, injective, and proper. Note that $dh_t = \frac{1}{2}\begin{bmatrix} e^t - e^{-t} \\ e^t + e^{-t} \end{bmatrix}$ is injective for all t, as e^{-t} is everywhere nonzero. Hence, h is an immersion.

Next, suppose h is not injective. Then there exist $a \neq b$ such that $h(a) = h(b)$. But then

$$\begin{aligned} e^a + e^{-b} &= e^b + e^{-a} \\ e^{a+b}(e^a + e^{-b}) &= (e^b + e^{-a})e^{a+b} \\ e^a(e^{a+b} + 1) &= (e^{b+a} + 1)e^b \\ e^a &= e^b \end{aligned}$$

Since $x \mapsto e^x$ is injective, $a = b$. A contradiction, so h is injective.

Since h is continuous, the preimage of a closed set is closed. We need only show the preimage of a bounded set is itself bounded. Suppose $B \subset \mathbb{R}^2$ is a bounded set. If $B \cap h(\mathbb{R}) = \varnothing$, we're done, as the empty set is trivially bounded. Otherwise, let $B_{x,y}(r)$ be the ball of radius r centered at (x, y) containing B. Let $x_0 = h^{-1}(x, h(x))$, then $h^{-1}(B)$ is bounded by the interval $(x_0 - r, x_0 + r)$.

Hence, h is proper and thus an embedding.

To show its image is one nappe of the hyperbola $x^2 - y^2 = 1$, consider

$$\begin{aligned} \frac{1}{4}(e^t + e^{-t})^2 - \frac{1}{4}(e^t - e^{-t})^2 &= 1 \\ (e^{2t} + 2 + e^{-2t}) - (e^{2t} - 2 + e^{-2t}) &= 4 \\ (e^{2t} - e^{2t}) + (e^{-2t} - e^{-2t}) + 4 &= 4 \end{aligned}$$

which clearly checks out. $\square$