MTH 532 Homework 4

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Exercise 7

Let $f\colon X \to Y$ with X compact and $\dim X = \dim Y$. Let y be a regular value of f. We wish to show $f^{-1}(y) = \{x_1, \cdots, x_n\}$ is finite and that there is a neighorbood U of y such that $f^{-1}(U)$ is equal to the disjoint union of V_1 through V_n, where V_i is a neighborhood of x_i and diffeomorphic to U.

First, apply the Inverse Function Theorem, which says each $x_i \in f^{-1}(y)$ has an open neighborhood $O(x_i)$ diffeomorphic to a neighborhood $O_i(y)$ of y. Since diffeomorphisms are bijective, we must have $O(x_i) \cap O(x_j) = \varnothing$ for all $i \neq j$. $f^{-1}(y)$ is finite as X is compact.

Next, apply the hint. Note that $O' = X - \bigcup O(x_i)$ is closed, so it is compact. Hence, $f(O')$ is compact and does not contain y. Let $U = Y - f(O')$ and let $\tilde{O}(x_i) = O(x_i) \cap f^{-1}(U)$, as desired. $\square$

Exercise 9

Consider the map $f\colon \mathrm{M}(2, \mathbb{R}) \to S(n)$, where $S(n)$ is the set of symmetric $n \times n$ matrices, defined by $A \mapsto AA^T$. Note that $O(n) = f^{-1}(I_n)$. Since f is continuous, $O(n)$ is closed.

Next, we take the hint by "flattening" $O(n)$ into $\mathbb{R}^{n^2}$. Note that, for $x \in \mathbb{R}^{n^2}$, we have $\|x\|^2 = \sum_{i=1}^{n^2} x_i^2$. Hence, for $A \in O(n)$, we have $\|A\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = \sum_{i=1}^{n} 1 = n$, so A is bounded. Since $O(n)$ is analogous to a subset of $\mathbb{R}^{n^2}$ and since it is closed and bounded, by Heine-Borel it is also compact. $\square$

Exercise 12

Consider the maps $f\colon \mathrm{M}(2, \mathbb{R}) \to \mathbb{R}^4$ and $\det\colon \mathbb{R}^4 \to \mathbb{R}$ defined by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a, b, c, d)$ and $(a, b, c, d) \mapsto ad - bc$, respectively. Let $A \in \mathrm{M}(2, \mathbb{R})$, then $d\det_{f(A)} = (d, -c, -b, a)$, which is surjective iff $f(A) \neq 0$. But a 2×2 matrix of rank 1 has at least one nonzero entry, so $d\det_{f(A)}$ is indeed onto, implying det is a submersion on any matrix of rank 1. Note that the only matrix of rank 0 is the zero matrix and that $\det^{-1}(0) = \{A \in \mathrm{M}(2, \mathbb{R}) \mid \operatorname{rank}(A) < 2\}$. Hence, det is a submersion on $\mathrm{M}(2, \mathbb{R}) - \{0\}$ and, therefore, a submanifold of $\mathbb{R}^4$. $\square$

Exercise A1

Note that $\mathrm{M}(n, \mathbb{C}) \cong \mathbb{C}^{n^2}$ and that $\det\colon \mathrm{M}(n, \mathbb{C}) \to \mathbb{C}$ is a smooth function. Since $\mathbb{C}$ is Hausdorff, $0 \in \mathbb{C}$ is closed. Likewise, because of the continuitity of det, $\det^{-1}(0)$ is closed too. Note that $\mathrm{GL}(n, \mathbb{C}) = M(n, \mathbb{C}) - \det^{-1}(0)$, so $\mathrm{GL}(n, \mathbb{C})$ is open and, therefore, a smooth manifold.

Furthermore, $\mathrm{GL}(n, \mathbb{C})$ is a group under matrix multiplication with identity I_n. Matrix multiplication is smooth, as it is a polynomial function in the entries of the product; matrix inversion is smooth because it is a rational function (Cramer's rule), as the determinant is non-vanishing in $\mathrm{GL}(n, \mathbb{C})$.

$\mathrm{GL}(n, \mathbb{C})$ is thus a Lie group of dimension $2n^2$, as its basis consists of matrices with one element in $\{1, i\}$. $\square$

Exercise A2

Consider the map $\det\colon \mathrm{M}(n,\mathbb{C}) \to \mathbb{C}$. Note that $\mathrm{SL}(n,\mathbb{C})$ is equal to the kernel of the determinant, i.e., $\mathrm{SL}(n,\mathbb{C}) = \det^{-1}(1)$. If 1 is a regular value of det, then $\dim \mathrm{SL}(n,\mathbb{C}) = \dim \mathrm{GL}(n,\mathbb{C}) - \dim \mathbb{C} = 2n^2 - 2$. Thus, we wish to show that 1 is a regular value. We do this by showing 0 is the only critical value:

$$\begin{aligned} d\det_0(h) &= \lim_{t\to 0} \frac{\det(0+th) - \det 0}{t} \\ &= \lim_{t\to 0} \frac{\det(th)}{t} \\ &= \lim_{t\to 0} t^{n-1}\det(h) \\ &= 0 \end{aligned}$$

Indeed, 0 is a critical value. Next, consider $A \in \mathrm{GL}(n,\mathbb{C})$:

$$\begin{aligned} d\det_A(A) &= \lim_{t\to 0} \frac{\det(A+tA) - \det(A)}{t} \\ &= \lim_{t\to 0} \frac{(1+t)^n\det(A) - \det(A)}{t} \\ &= \det(A)\lim_{t\to 0} \frac{(1+t)^n - 1}{t} \\ &= \det(A)n \end{aligned}$$

Which is nonzero, as $\det(A) \neq 0$. Hence, 1 is a regular value of det. □

Exercise A3

Consider the map $f\colon \mathrm{M}(n,\mathbb{C}) \to \mathrm{H}(n)$ defined by $A \mapsto AA^*$, where $\mathrm{H}(n)$ is the set of all $n \times n$ Hermitian matrices and A^* is the conjugate transpose. Note that $f^{-1}(I_n)$, where I_n is the identity matrix, equals $U(n)$.

Next, denote AA^* as C, then $C^* = (AA^*)^* = A^*(A^*)^* = A^*A = C$, so C is Hermitian. Then, observe that Hermitian matrices are of the form

$$\begin{bmatrix} a & c & \cdots & x \\ \bar{c} & b & \cdots & y \\ \vdots & \vdots & \ddots & \vdots \\ \bar{x} & \bar{y} & \cdots & z \end{bmatrix}$$

Note that there are n real-valued elements along the diagonal. The elements in the upper half of the matrix are dependent on those in the lower half, hence the rest of the matrix has $1+2+\cdots+(n-1)$ free complex variables. Thus, $H(n)$ has dimension $2*[1+2+\cdots+(n-1)]+n = n^2$.

Next, we wish to show that I_n is a regular value of f:

$$\begin{aligned} df_A(h) &= \lim_{t\to 0} \frac{f(A+th) - f(A)}{t} \\ &= \lim_{t\to 0} \frac{(A+th)(A+th)^* - AA^*}{t} \\ &= \lim_{t\to 0} \frac{AA^* + thA^* + tAh^* + t^2hh^* - AA^*}{t} \\ &= \lim_{t\to 0} hA^* + Ah^* + thh^* \\ &= hA^* + Ah^* \end{aligned}$$

I_n is then a regular value iff df_A is surjective for all $A \in U(n)$. Let $C \in U(n)$, then $C = \frac{1}{2}C + \frac{1}{2}C^*$. Then, solving for h in $hA^* = \frac{1}{2}C$ yields $h = \frac{1}{2}CA$. This checks out, as

$$\begin{aligned} hA^* + Ah^* &= (\frac{1}{2}CA)A^* + A(\frac{1}{2}CA)^* \\ &= \frac{1}{2}C(AA^*) + \frac{1}{2}A(A^*C^*) \\ &= \frac{1}{2}CI_n + \frac{1}{2}I_nC \\ &= C \end{aligned}$$

So df_A is indeed surjective for all $A \in U(n)$. $\square$

Exercise A4

Let $U(1)$ be the unitary group and let S^1 be the circle group, then $f\colon U(1) \to S^1$ defined by $(a+ib) \mapsto (a, b)$ is a diffeomorphism. Suppose $x \in U(1)$, then $x\bar{x} = 1$. Recall, for complex z, $\|z\|^2 = z\bar{z}$. Therefore, $\|x\| = 1$, which implies $x \in S^1$. The reverse direction is no different.

$SU(2)$ is the set $\{A \in U(2) \mid \det A = 1\}$. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SU(2)$, then we have that $A^{-1} = \bar{A}^T$, or that $\left(\begin{smallmatrix} d & \text{-}b \\ \text{-}c & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{smallmatrix}\right)$. Thus, A must be of the form $\left(\begin{smallmatrix} a & \text{-}\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ for $a, b \in \mathbb{C}$. Since $\det A = 1$, note that $a\bar{a} + b\bar{b} = 1$.

Next, consider the smooth maps $\sigma\colon \mathbb{R}^4 \to \mathbb{C}^2$ and $\phi\colon \mathbb{C}^2 \to M_2\mathbb{C}$ defined by $(x, y, z, w) \mapsto (x + iy, z + iw)$ and $(a, b) \mapsto \left(\begin{smallmatrix} a & \text{-}\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$, respectively. Let $m = \phi \circ \sigma$.

Suppose $s = (x, y, z, w) \in S^3$, then $x^2 + y^2 + z^2 + w^2 = 1$. Denote $\sigma(s)$ as $(a, b) = (x + iy, z + iw)$. Recall, for $z = x + iy$, one has $z\bar{z} = x^2 + y^2$. Hence, $a\bar{a} + b\bar{b} = 1$, so $m(s) \in SU(2)$. Since σ and ϕ are both smooth and invertible, S^3 is diffeomorphic to $SU(2)$ via m. $\square$