# MTH 532 Homework 2

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## Exercise 1

Let $f\colon U \to X$ and $g\colon V \to Y$ be parameterizations for, let $X \subset Y$ be a submanifold, and let $i\colon X \to Y$ be an inclusion map. This produces the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"'] & Y \\ U \arrow[u, "f"] \arrow[r, "h"] & V \arrow[u, "g"'] \end{tikzcd}$$

Note that $h = g^{-1} \circ i \circ f$. However, $h = g^{-1} \circ f$ too, as $i$ is the inclusion map bringing $X$ into $Y$. Without loss of generality and to simplify notation, let $f(0) = g(0) = x \in X$. Since $g^{-1} \circ i \circ f = g^{-1} \circ f$, taking the derivative on both sides yields $d(g^{-1} \circ i \circ f) = d(g^{-1} \circ f)$, or that $dg_0^{-1} \circ di_x \circ df_0 = dg_0^{-1} \circ df_0$. Moving terms gives $di_x = (dg_0 \circ dg_0^{-1}) \circ (df_0 \circ df_0^{-1})$. This is the identity map, so $di_x$ is indeed an inclusion map. $\square$

## Exercise 3

Let $V$ be a vector subspace of $\mathbb{R}^N$ with dimension $n$. Then there exists a parameterization $f\colon \mathbb{R}^n \to V$. But $f$ is linear, so $df = f$. Thus, $T_x(V) = df(\mathbb{R}^n) = f(\mathbb{R}^n) = V$. $\square$

## Exercise 4

Let $f\colon X \to Y$ be a diffeomorphism, then there is a smooth map $f^{-1}\colon Y \to X$ such that $f \circ f^{-1} = Id$. Note $d\, Id = d(f \circ f^{-1})$, implying that $Id = df_x \circ df_y^{-1}$ for all $x \in X$ with $y = f(x)$. Hence, $df_x$ has an inverse in the form of $df_y^{-1}$, so it is indeed an isomorphism of tangent spaces. $\square$

## Exercise 5

Let $k \neq l$ and suppose $f\colon \mathbb{R}^k \to \mathbb{R}^l$ is a diffeomorphism. Then $df_x\colon T_x(\mathbb{R}^k) \to T_{f(x)}(\mathbb{R}^l)$ is an isomorphism of tangent spaces. Note $T_x(\mathbb{R}^k) = \mathbb{R}^k$ and $T_{f(x)}(\mathbb{R}^l) = \mathbb{R}^l$ for all $x$. But $\dim \mathbb{R}^k = k \neq l = \dim \mathbb{R}^l$, meaning $df$ is not bijective. But $df = f$, so $f$ is also not bijective. Contradiction! Hence, $f$ is not a diffeomorphism. $\square$

## Exercise 8

Let $H = \{(x, y, z) \mid x^2 + y^2 - z^2 = a\}$ be a hyperboloid and let $h\colon B_a(0) \to H \subset \mathbb{R}^3$ be a local parameterization of $(\sqrt{a}, 0, 0)$, where $B_a(0)$ is the ball of radius $a$ centered at the origin. Define $h$ as

$(u,v) \mapsto (\sqrt{a+u^2-v^2}, u, v)$. (All credit to Wolfram for that parameterization.) Then $dh$ is the matrix

$$\begin{bmatrix} u/x & -v/x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $x = \sqrt{a+u^2-v^2}$. Therefore, $dh_{(\sqrt{a},0,0)}$ is the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and the tangent space is the span of $dh_{(\sqrt{a},0,0)}$, or the yz-plane. $\square$

## Exercise 9

(1) Let $a\colon U \to X$ and $b\colon V \to Y$ be parameterizations of $X$ and $Y$, respectively. Note that $X \times Y = \operatorname{Im}(a \times b)$. Hence, $T_{(x,y)}(X \times Y) = \operatorname{Im}\, d(a \times b)_{(0,0)}$, which can be rewritten as the image of $\left[\begin{smallmatrix} da_0 & 0 \\ 0 & db_0 \end{smallmatrix}\right]$, or $T_xX \times T_yY$.

(2) Let $f\colon X \times Y \to X$ be the projection map and let $a\colon U \to X$ and $b\colon V \to Y$ be parameterizations. This forms a commutative diagram, meaning there is a map from $U \times V$ to $U$, $h = a^{-1} \circ f \circ (a \times b)$. For notation and without loss of generality, take $(a \times b)(0,0) = (x,y)$. Shuffling terms yields $f = a \circ h \circ (a \times b)^{-1}$, implying $df_{(x,y)} = da_0 \circ dh_{(0,0)} \circ d(a \times b)^{-1}_{(0,0)}$, which is the desired projection.

(3) Use $a$ and $b$ as before and let $h\colon U \to U \times V$ defined by $u \mapsto (u,0)$. Fix $y = b(0)$ and let $f\colon X \to X \times Y$ be the injective mapping $x \mapsto (x,y)$. Next, consider $f = (a \times b) \circ h \circ a^{-1}$. Note that $dh = h$, as $h$ is linear. Therefore, $df_x = (da_0 \times db_0) \circ h \circ da_0^{-1}$. Thus, $df_x(v) = (v,0)$.

(4) Intuition: project each space and then smoothly map it to $(x,0)$ and $(0,y)$ so that the direct sum "fills" the "containing" space.

Let $\pi_x\colon X \times Y \to X$ and $\pi_y\colon X \times Y \to Y$ be natural projections. Let $i_x\colon X' \to \mathbb{R}^N$ and $i_y\colon Y' \to \mathbb{R}^N$ be smooth inclusions defined by $x \mapsto (x,0)$ and $y \mapsto (0,y)$, respectively. ($\mathbb{R}^N$ is assumed to "contain" $X \times Y$.) Then clearly $f \times g = i_x \circ f \circ \pi_x + i_y \circ g \circ \pi_y$. This implies $d(f \times g)_{(x,y)} = df_x \times dg_y$.

$\square$