MTH 532 Homework 9

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3.3 Oriented Intersection Number

Exercise 2

(a) From class, we have that the degree of the antipodal map $S^k \to S^k$ is $(-1)^{k+1}$, as it consists of $k+1$ maps each sending x to $-x$. Therefore, orientation is preserved for k odd and reversed for k even.

(b) Homotopic maps of connected spaces have the same degree. The identity map has a degree of 1. Thus, the antipodal map is homotopic to the identity iff k is odd.

(c) From exercises **1.8.7** and **1.8.8**, we have that S^k has a nonvanishing vector field when its antipodal map is homotopic to the identity map. From **(b)**, this occurs iff k is odd.

(d) No, as $-1 \equiv 1 \pmod 2$, so whether orientation is preserved is lost. $\square$

Exercise 7

Per the hint, note that $z \mapsto \bar{z}$ is orientation reversing, i.e. the map which sends a complex number to its complex conjugate has degree -1. Then, use exercise **3.3.10**: for $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\deg(g \circ f) = \deg f * \deg f$.

As $z \mapsto \bar{z}^m$ is the composition of $z \mapsto \bar{z}$ and $z \mapsto z^m$, which have degrees of -1 and m, respectively, we have that $z \mapsto \bar{z}^m$ has degree $-m$. $\square$

Exercise 10

Consider $X \xrightarrow{f} Y \xrightarrow{g} Z$ and let $c \in Z$ be a regular value of $g \circ f$. Note that c is also a regular value of g and that $g^{-1}(c)$ are regular values of f. Furthermore, recall that, for a regular value q and $f^{-1}(q) = \{ p_1, \cdots, p_k \}$, the degree of f is defined as $\deg f = \sum_{i=1}^{k} \varepsilon_i(f)$, where $\varepsilon_i(f) = \operatorname{sign} \det df_{p_i} \in \{ -1, 1 \}$.

Let $g^{-1}(c) = \{ y_1, \cdots, y_m \}$ and let $f^{-1}(y_i) = \{ x_{i1}, \cdots, x_{in} \}$.

$$\begin{aligned}
\deg(g \circ f) &= \sum_{i=1}^{mn} \varepsilon_i(g \circ f) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_i(g) \varepsilon_{ij}(f) \\
&= \sum_{i=1}^{m} \varepsilon_i(g) \sum_{j=1}^{n} \varepsilon_{ij}(f) \\
&= \deg(g) * \deg(f)
\end{aligned}$$

$\square$