MTH 532 Homework 10

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4.2 Exterior Algebra

Exercise 1

Suppose $T \in \Lambda^p(V^*)$ and $v_1, \cdots, v_p \in V$ are linearly dependent, then there are constants $\{ a^1, \cdots, a^p \}$, not all zero, such that $a^i v_i = 0$. Without loss of generality, suppose $v_1 = a^2 v_2 + \cdots + a^p v_p$. Therefore,

$$\begin{aligned} T(v_1, \cdots, v_p) &= T(a^2 v_2 + \cdots + a^p v_p, v_2, \cdots, v_p) \\ &= a^2 T(v_2, v_2, \cdots, v_p) + \cdots + a^p T(v_p, v_2, \cdots, v_p) \\ &= 0 + \cdots + 0 \\ &= 0 \end{aligned}$$

□

Exercise 2

Suppose $\phi_1, \cdots, \phi_p \in V^*$ are linearly dependent, then there are constants $a^1, \cdots, a^p$, not all zero, such that $a^i \phi_i = 0$. That is, without loss of generality, ϕ_1 can be written as the sum $c^2\phi_2 + \cdots + c^p \phi_p$. Recall $\phi_i \wedge \phi_i = 0$, $c \wedge \phi_i = c\phi_i$ for constant c, and that the wedge product distributes over addition. Therefore,

$$\begin{aligned} \phi_1 \wedge \cdots \wedge \phi_p &= (c^2\phi_2 + \cdots + c^p\phi_p) \wedge \phi_2 \wedge \cdots \wedge \phi_p \\ &= c^2\phi_2 \wedge \phi_2 \wedge \cdots \wedge \phi_p + \cdots + c^p\phi_p \wedge \phi_2 \wedge \cdots \wedge \phi_p \\ &= 0 + \cdots + 0 \\ &= 0 \end{aligned}$$

□

Exercise 6

(a) Let $A\colon V \to V$ be a linear isomorphism which sends the basis $B = \{ v_1, \cdots, v_k \}$ to $B' = \{ v_1', \cdots, v_k' \}$.

If B and B' are equivalently oriented, then $\det A$ is positive. Next, for $T \in \Lambda^k(V)$, note that $A^*T = \det(A)T$. However, A^*T was also defined to be $T(Av_1, \cdots, Av_k) = T(v_1', \cdots, v_k')$. Thus, $T(B)$ and $T(B')$ have the same sign.

Conversely, suppose $T(B)$ and $T(B')$ have the same sign. Note that $T(B') = A^*T = \det(A)T(B)$, implying that $\det A$ is positive. Therefore, B and B' are equivalently oriented.

(b) Let $T, S \in \Lambda^k(V^*)$ be two nonzero elements. Suppose $B = \{ v_1, \cdots, v_k \}$ is a positively oriented ordered basis. Without loss of generality, suppose $T(B)$ is positive and $S(B)$ is negative. Let $A\colon V \to V$ be a linear isomorphism, then $T(B)$ and $S(AB)$ should have the same sign iff $\det A$ is negative. From **4.2.6a**, this is clearly the case, as $S(AB = S(Av_1, \cdots, Av_k) = A^*S(B) = \det(A)S(B)$. This produces a natural orientation, as we have defined two equivalence classes: positive and negative.

(c) Let $T \in \Lambda^k(V^*)$ be nonzero. Let B and B' be two ordered bases. We define an orientation on V by saying B and B' are equivalently oriented iff $T(B)$ and $T(B')$ have the same sign. Let $A\colon V \to V$ be a linear isomorphism such that $A(B) = B'$.

Do the opposite of **4.2.6b**. Note $T(B') = A^*T(B) = \det(A)T(B)$. Thus, $T(B')$ and $T(B)$ have the same sign iff $\det A$ is positive, implying that B and B' are equivalently oriented. □