MTH 532 Homework 11

Roy Howie

April 13, 2017

4.4 Integration on Manifolds

Exercise 1

A 0-form f on a manifold X is the same as a real-valued function $f\colon X \to \mathbb{R}$; its integral on a single, positively (negatively) oriented point is $f(x)$ (respectively $-f(x)$). Hence, the integral of f, a 0-form, over Z, a finite collection of points, is the desired sum: $\sum_{z\in Z} \sigma(z)f(z)$. $\square$

Exercise 3

$$\int_a^b c^*\omega = \int_X \omega = \int_X df = \int_{\delta X} f = f(q) - f(p) \qquad \square$$

Exercise 4

Note that f preserves orientation, as it is smooth, and can be used to pull back from $[a, b]$ to $[a_1, b_1]$. Furthermore, $(g \circ f)^* = f^*g^*$. Hence,

$$\int_a^b c^*\omega = \int_{a_1}^{b_1} f^*c^*\omega = \int_{a_1}^{b_1} (c \circ f)^*\omega \qquad \square$$

Exercise 8

Define a 1-form on $\mathbb{R}^2 - \{ 0 \}$ by $\omega(x, y) = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy = udx + vdy$.

(a) Let C be a circle of radius r about the origin of the punctured plane. Let $\gamma\colon [0, 2\pi] \to C$ be an orientation preserving parameterization of C minus a single point defined by $t \mapsto (r\cos(t), r\sin(t))$.

$$\begin{aligned}
\int_C \omega &= \int_0^{2\pi} p^*\omega \\
&= \int_0^{2\pi} (ux' + vy')rdt \\
&= \int_0^{2\pi} \frac{r^2}{r^2}\left(\frac{\sin^2(t) + \cos^2(t)}{\cos^2(t) + \sin^2(t)}\right) rdt \\
&= 2\pi r
\end{aligned}$$

(b) Consider $\arctan(y/x)$, then

$$\begin{aligned}
\frac{\delta}{\delta x} \arctan\left(\frac{y}{x}\right) &= \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{\delta}{\delta x}\frac{y}{x}\right) = \frac{-\frac{y}{x^2}}{1 + \frac{y^2}{x^2}} = \frac{-y}{x^2 + y^2} \\
\frac{\delta}{\delta y} \arctan\left(\frac{y}{x}\right) &= \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{\delta}{\delta y}\frac{y}{x}\right) = \frac{\frac{1}{x}}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2 + y^2}
\end{aligned}$$

(c) Because the punctured plane is not simply connected. $\square$

Exercise 9

Only if: Let $h\colon \mathbb{R} \to S^1$ be defined by $t \mapsto (\cos(t), \sin(t))$. Restrict h to $[0, 2\pi] \subset \mathbb{R}$, then h is a parameterization, so we can consider its pullback. Hence, for any 1-form ω on S^1, we have $\int_{S^1} \omega = \int_0^{2\pi} h^*\omega$.

If: Define $g(t) = \int_0^t h^*\omega$ and suppose $\int_{S^1} \omega = 0$, then $g(t) = 0$, or that $t \in 2\pi\mathbb{Z}$. This follows from the fact that $g(t + 2\pi) = \int_0^{t+2\pi} h^*\omega = \int_0^t h^*\omega + \int_t^{t+2\pi} h^*\omega = g(t)$, as $\int_t^{t+2\pi} h^*\omega$ can be reparameterized by some orientation-preserving function $r\colon [0, 2\pi] \to [t, t + 2\pi]$, implying $\int_t^{t+2\pi} h^*\omega = \int_0^{2\pi} r^*h^*\omega = 0$.

Therefore, $g = f \circ h$ for some function f on S^1. $\square$

Exercise 10

Let $I_\omega = \int_{S^1} \omega$ and $I_\nu = \int_{S^1} \nu$ and denote $c = I_\omega / I_\nu$, which makes sense as $I_\nu \neq 0$. This implies $\int_{S^1} \omega - c\nu = 0$. Exercise **4.4.9** then implies that ω is the differential of some function f, i.e. $\omega = df$. $\square$