MTH 532 Homework 7

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2.1 Manifolds with Boundary

Exercise 1

Let $U \subset \mathbb{R}^k$ and $V \subset H^k$ be open neighborhoods of 0. Note that $V - \{0\}$ that is simply connected but $U - \{0\}$ is not. Hence, there is no diffeomorphism between the two spaces. [Note: in a sense, problem 2.4.10 is another example of this and can also be generalized to any dimension.] $\square$

Exercise 2

Consider a diffeomorphism $f\colon X \to Y$ of manifolds with boundary. Pick $p \in \delta X$, then there is an open neighborhood $O(p)$ about p with the property that $f|_{O(p)}$ is a linear isomorphism. Furthermore, $O(p)$ can be parameterized by $U \subset H^k$ and $f|_{O(p)}$ is equivalent to $id\colon U \to U$. It follows from this parameterization that $x \in \delta X$ iff $\delta f(x) \in \delta Y$. $\square$

Exercise 4

Use the lemma from page 62. Consider the function $\pi\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $(x, y, z) \mapsto a - x^2 - y^2 + z^2$. Since $x^2 + y^2 - z^2 \leq a$, we have that $\pi(x) \geq 0$ for all $x \in H$, where H is the given solid hyperboloid. Thus, by the lemma, $\pi^{-1}([0, \infty]) = H$ is a manifold with boundary. $\square$

Exercise 6

Not sure how to "check," other than to draw a single line around the entire Mbius band without having to lift one's pen. Naturally, this doesn't work for the cylinder. $\square$?

If the strip is twisted n times before gluing, then the resulting object is diffeomorphic to the cylinder iff n is even and diffeomorphic to the Mbius band iff n is odd. $\square$

2.2 One-Manifolds and Some Consequences

Exercise 3

A map which rotates the torus about its center by π has no fixed points. The proof of the Brouwer theorem fails because the solid torus has a hole in its center, so the retraction to its boundary doesn't work. $\square$

Exercise 4

Per the hint, consider the map from the open ball $f\colon B(0, a) \to \mathbb{R}^k$ specified in problem 1.1.4. Next, consider a continuous function $g\colon \mathbb{R}^k \to \mathbb{R}^k$ with no fixed point, say $x \mapsto (x_1 + 1, x_2, \cdots, x_k)$. Note that f^{-1} exists. Thus, we have that $f^{-1} \circ g \circ f$ is a map from $B(0, 1)$ to itself with no fixed point. $\square$

2.4 Intersection Theory Mod 2

Exercise 7

Per the hint, note that $\deg_2(id) \equiv 1$. However, if S^1 were simply connected, then every map $f\colon S^1 \to S^1$ would be homotopic to a constant map. But $\deg_2(f) \equiv 0$ and homotopic maps must have the same degree modulo 2. This is a contradiction, so S^1 is not simply connected. $\square$

Exercise 10

Note that $T^2 = S^1 \times S^1$. Consider the simple, closed paths $S^1 \times \{a\}$ and $\{b\} \times S^1$, where $a, b \in S^1$. Note that they intersect at a single point: $(a, b) \in T^2$.

On the other hand, consider two simple, closed, nonintersecting paths on S^2. From problem 1.7.6, since S^2 is simply connected for $k > 1$, these paths are each contractible to a single point.

If there were a diffeomorphism between S^2 and T^2, this would be a contradiction, as they each have different intersection numbers modulo 2. $\square$