

# MTH 532 Homework 2

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## Exercise 1

Let  $f: U \rightarrow X$  and  $g: V \rightarrow Y$  be parameterizations, let  $X \subset Y$  be a submanifold, and let  $i: X \rightarrow Y$  be an inclusion map. This produces the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & Y \\ f \uparrow & & \uparrow g \\ U & \xrightarrow{\quad h \quad} & V \end{array}$$

Note that  $h = g^{-1} \circ i \circ f$ . However,  $h = g^{-1} \circ f$  too, as  $i$  is the inclusion map bringing  $X$  into  $Y$ . Without loss of generality and to simplify notation, let  $f(0) = g(0) = x \in X$ . Since  $g^{-1} \circ i \circ f = g^{-1} \circ f$ , taking the derivative on both sides yields  $d(g^{-1} \circ i \circ f) = d(g^{-1} \circ f)$ , or that  $dg_0^{-1} \circ di_x \circ df_0 = dg_0^{-1} \circ df_0$ . Moving terms gives  $di_x = (dg_0 \circ dg_0^{-1}) \circ (df_0 \circ df_0^{-1})$ . This is the identity map, so  $di_x$  is indeed an inclusion map.  $\square$

## Exercise 3

Let  $V$  be a vector subspace of  $\mathbb{R}^N$  with dimension  $n$ . Then there exists a parameterization  $f: \mathbb{R}^n \rightarrow V$ . But  $f$  is linear, so  $df = f$ . Thus,  $T_x(V) = df(\mathbb{R}^n) = f(\mathbb{R}^n) = V$ .  $\square$

## Exercise 4

Let  $f: X \rightarrow Y$  be a diffeomorphism, then there is a smooth map  $f^{-1}: Y \rightarrow X$  such that  $f \circ f^{-1} = Id$ . Note  $d Id = d(f \circ f^{-1})$ , implying that  $Id = df_x \circ df_y^{-1}$  for all  $x \in X$  with  $y = f(x)$ . Hence,  $df_x$  has an inverse in the form of  $df_y^{-1}$ , so it is indeed an isomorphism of tangent spaces.  $\square$

## Exercise 5

Let  $k \neq l$  and suppose  $f: \mathbb{R}^k \rightarrow \mathbb{R}^l$  is a diffeomorphism. Then  $df_x: T_x(\mathbb{R}^k) \rightarrow T_{f(x)}(\mathbb{R}^l)$  is an isomorphism of tangent spaces. Note  $T_x(\mathbb{R}^k) = \mathbb{R}^k$  and  $T_{f(x)}(\mathbb{R}^l) = \mathbb{R}^l$  for all  $x$ . But  $\dim \mathbb{R}^k = k \neq l = \dim \mathbb{R}^l$ , meaning  $df$  is not bijective. But  $df = f$ , so  $f$  is also not bijective. Contradiction! Hence,  $f$  is not a diffeomorphism.  $\square$

## Exercise 8

Let  $H = \{(x, y, z) \mid x^2 + y^2 - z^2 = a\}$  be a hyperboloid and let  $h: B_a(0) \rightarrow H \subset \mathbb{R}^3$  be a local parameterization of  $(\sqrt{a}, 0, 0)$ , where  $B_a(0)$  is the ball of radius  $a$  centered at the origin. Define  $h$  as

$(u, v) \mapsto (\sqrt{a + u^2 - v^2}, u, v)$ . (All credit to Wolfram for that parameterization.) Then  $dh$  is the matrix

$$\begin{bmatrix} u/x & -v/x \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where  $x = \sqrt{a + u^2 - v^2}$ . Therefore,  $dh_{(\sqrt{a}, 0, 0)}$  is the matrix  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$  and the tangent space is the span of  $dh_{(\sqrt{a}, 0, 0)}$ , or the  $yz$ -plane.  $\square$

## Exercise 9

- (1) Let  $a: U \rightarrow X$  and  $b: V \rightarrow Y$  be parameterizations of  $X$  and  $Y$ , respectively. Note that  $X \times Y = \text{Im}(a \times b)$ . Hence,  $T_{(x,y)}(X \times Y) = \text{Im } d(a \times b)_{(0,0)}$ , which can be rewritten as the image of  $\begin{bmatrix} da_0 & 0 \\ 0 & db_0 \end{bmatrix}$ , or  $T_x X \times T_y Y$ .
- (2) Let  $f: X \times Y \rightarrow X$  be the projection map and let  $a: U \rightarrow X$  and  $b: V \rightarrow Y$  be parameterizations. This forms a commutative diagram, meaning there is a map from  $U \times V$  to  $U$ ,  $h = a^{-1} \circ f \circ (a \times b)$ . For notation and without loss of generality, take  $(a \times b)(0, 0) = (x, y)$ . Shuffling terms yields  $f = a \circ h \circ (a \times b)^{-1}$ , implying  $df_{(x,y)} = da_0 \circ dh_{(0,0)} \circ d(a \times b)^{-1}_{(0,0)}$ , which is the desired projection.
- (3) Use  $a$  and  $b$  as before and let  $h: U \rightarrow U \times V$  defined by  $u \mapsto (u, 0)$ . Fix  $y = b(0)$  and let  $f: X \rightarrow X \times Y$  be the injective mapping  $x \mapsto (x, y)$ . Next, consider  $f = (a \times b) \circ h \circ a^{-1}$ . Note that  $dh = h$ , as  $h$  is linear. Therefore,  $df_x = (da_0 \times db_0) \circ h \circ da_0^{-1}$ . Thus,  $df_x(v) = (v, 0)$ .
- (4) Intuition: project each space and then smoothly map it to  $(x, 0)$  and  $(0, y)$  so that the direct sum “fills” the “containing” space.

Let  $\pi_x: X \times Y \rightarrow X$  and  $\pi_y: X \times Y \rightarrow Y$  be natural projections. Let  $i_x: X' \rightarrow \mathbb{R}^N$  and  $i_y: Y' \rightarrow \mathbb{R}^N$  be smooth inclusions defined by  $x \mapsto (x, 0)$  and  $y \mapsto (0, y)$ , respectively. ( $\mathbb{R}^N$  is assumed to “contain”  $X \times Y$ .) Then clearly  $f \times g = i_x \circ f \circ \pi_x + i_y \circ g \circ \pi_y$ . This implies  $d(f \times g)_{(x,y)} = df_x \times dg_y$ .  $\square$