RESEARCH STATEMENT

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1. Introduction

We know, from the work of Kahn and Marković [20], that a closed hyperbolic 3-manifold contains an abundance of essential, nearly Fuchsian surfaces. My research is concerned with studying the statistical properties of these surfaces and how that relates to the structure of the manifolds in which they live. Specifically, in my thesis, [3], I show that the weak-* limits of the area measures of a sequence of asymptotically Fuchsian minimal or pleated surfaces in a closed hyperbolic 3-manifold M are exactly the convex combinations of the area measures of the totally geodesic surfaces of M and the volume measure of M. (See Theorem 2.1.)

The work of Kahn and Marković answered a longstanding question of whether a given closed hyperbolic 3-manifold contains any essential surfaces. Their result had applications to other important problems about hyperbolic 3-manifolds of the past twenty years, such as the virtual Haken conjecture ([2]), solved by Agol, Wise and their collaborators.

In future investigations, I hope to refine my understanding of the limiting behaviors of such surfaces by determining the limiting measure of certain averages of *all* asymptotically Fuchsian surfaces in M. I would also like to understand what happens to the asymptotically Fuchsian surfaces of a manifold M as M itself is changing, say, as a sequence of Dehn fillings M_n approach a cusped manifold M in the geometric topology. More generally, I hope to obtain similar results in other settings, such as when M is a cusped finite-volume hyperbolic manifold, or the locally symmetric manifold associated to a lattice in a center-free semisimple complex Lie group. In a converse direction, I would also like to see whether it is possible to detect information about a manifold from its set of asymptotically Fuchsian surfaces. Finally, I would also like to investigate the distributional behaviors of quasifuchsian surfaces of M in the moduli spaces $\mathscr{M}_g$ of hyperbolic structures.

2. Limits of asymptotically Fuchsian surfaces in hyperbolic 3-manifolds

Let $M = \Gamma\backslash\mathbf{H}^3$ be a closed hyperbolic 3-manifold, where $\Gamma \leq \mathrm{PSL}_2\,\mathbf{C}$ is a lattice. We say a sequence of π_1-injective (essential) maps $f : S_i \to M$ of surfaces S_i is *asymptotically Fuchsian* if f_i is K_i-quasifuchsian with $K_i \to 1$ as $i \to \infty$. For an almost-everywhere differentiable map $f : S \to M$ of a surface into M, we let $\nu(f)$ and $\hat{\nu}(f)$ denote, respectively, the probability measures induced by f on the 2-plane Grassmann bundle $\mathrm{Gr}\,M$ of M and on the frame bundle $\mathrm{Fr}\,M$ of M. (When convenient we will write $\nu(f(S))$ instead of $\nu(f)$.)

We let $\mathscr{G}$ denote the set of immersed closed totally geodesic surfaces in M. For $T \in \mathscr{G}$, we let ν_T denote the area measure of T on $\operatorname{Gr} M$. We let $\nu_{\operatorname{Gr} M}$ denote the probability volume (Haar) measure of $\operatorname{Gr} M$. In a forthcoming article [3], we show that

Theorem 2.1. *The set of weak-* limits of $\nu(f_i)$, where $f_i : S_i \to M$ are asymptotically Fuchsian minimal or pleated maps of closed connected surfaces, consists of all measures of the form*

$$\nu = \alpha_M \nu_{\operatorname{Gr} M} + \sum_{T \in \mathscr{G}} \alpha_T \nu_T \tag{$\star$}$$

where $\alpha_M + \sum_{T\in\mathscr{G}} \alpha_T = 1$.

Theorem 2.1 is in contrast with the case in which the maps $f_i : S_i \to M$ are all Fuchsian and the S_i are all distinct. In that setting, the surfaces $f_i(S_i)$ equidistribute in M, namely

Theorem (Mozes–Shah)**.** *$\nu(f_i) \xrightarrow{\star} \nu_{\operatorname{Gr} M}$ as $i \to \infty$.*

This follows from a more general theorem of Mozes and Shah [39]. This is an article about unipotent dynamics, that builds on work of Dani, Margulis and Ratner. As shown more recently by Margulis–Mohammadi [34] and Bader–Fisher–Miller–Stover [6], if M contains infinitely many distinct totally geodesic surfaces (up to commensurability), then M is aritmhetic. Our setting is much more flexible – the Kahn–Marković surface subgroup theorem says that *any* closed hyperbolic 3-manifold M contains infinitely many commensurtability classes of K-quasifuchsian surfaces for any $K > 1$.

An important part of the proof of Theorem 2.1 is showing that the weak-* limits of convergent subsequences of $\nu(f_i)$ do not depend on whether f_i is minimal or pleated, or in particular on the choice of pleated map. This is despite the fact that, in the pleated case, the universal covers of $f_i(S_i)$ do not converge to a geodesic plane in the C^1 sense.

Theorem 2.2. *Suppose $f_i : S_i \to M$ are essential asymptotically Fuchsian maps of a closed connected surface. Let f_i^p and f_i^m be, respectively, pleated and minimal maps homotopic to f_i. Then, $\nu(f_{i_j}^p) \xrightarrow{\star} \nu$ as $j \to \infty$ if and only if $\nu(f_{i_j}^m) \xrightarrow{\star} \nu$ as $j \to \infty$.*

We show Theorem 2.2 by flowing the universal covers $\widetilde{f_i^m}(\widetilde{S_i})$ and $\widetilde{f_i^p}(\widetilde{S_i})$ normally into a component of the boundary of the convex core of $(f_i)_*(\pi_1 S_i)$. We argue that this process has uniformly small area distortion. In the pleated case, we need to argue away from the bending lamination of $\widetilde{f_i^p}(\widetilde{S_i})$ to avoid complicated wrinkles as in Figure 1. In the minimal case, we use the result of Seppi [38] that says that the principal curvatures of $\widetilde{f_i^m}(\widetilde{S_i})$ go uniformly to zero as the quasiconformal constant K_i tends to 1.

It was shown by Labourie [24] and Lowe [30] that the weak-* limits of $\hat{\nu}(f_i^m)$ are $\mathrm{PSL}_2\,\mathbf{R}$-invariant. Thus, from Theorem 2.2, so are the weak-* limits of $\hat{\nu}(f_i^p)$. From the Ratner measure classification theorem [36], one of the directions of Theorem 2.1 follows – any sequence of asymptotically Fuchsian minimal or pleated closed connected surfaces weak-* converges to a measure of the form ($\star$).

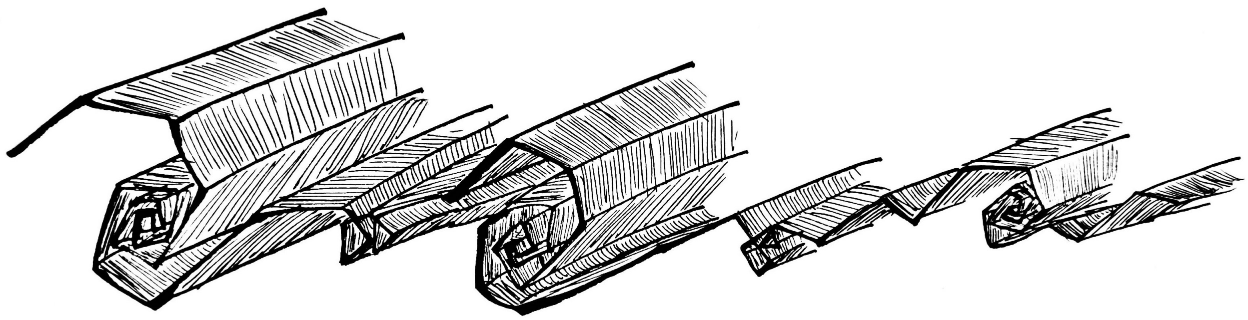

FIGURE 1. The universal covers of asymptotically Fuchsian pleated surfaces are not necessarily embedded in $\mathbf{H}^3$ and may develop wrinkles as above, so they are never C^1-close to a totally geodesic plane

To show the other direction of Theorem 2.1, namely, to construct a sequence of asymptotically Fuchsian minimal or pleated closed connected surfaces $f_i : S_i \to M$ so that $\nu(f_i)$ converges to a given measure of the form ($\star$), we rely crucially on the Kahn-Marković construction of quasifuchsian surfaces [20], which has a probabilistic flavor. The building blocks from which the nearly Fuchsian surfaces are assembled are the (ϵ, R)-good pants, which are the maps $f : P \to M$ from a pair of pants P taking the cuffs of P to (ϵ, R)-good curves in M – the closed geodesics with complex translation length 2ϵ-close to $2R$. Two (ϵ, R)-good pants f and g are equivalent if f is homotopic to $g \circ \phi$ for some orientation-preserving homeomorphism $\phi : P \to P$. Kahn and Marković show that a closed surface made by glueing (ϵ, R)-good pants in an (ϵ, R)-good way (namely with a certain shear parameter ϵ/R-close to 1) is $(1 + O(\epsilon))$-quasifuchsian.

To argue it is indeed possible to form a closed surface by (ϵ, R)-good pants in an (ϵ, R)-good way, Kahn and Marković show that the good pants incident to a given good curve γ come in a well-distributed set of directions in the unit normal bundle $\mathrm{N}^1(\gamma)$. The version of this statement from the Kahn–Wright article [22] is particularly strong and lets you build a closed $(1 + O(\epsilon))$-quasifuchsian surface $S_{\epsilon,R}$ in M with *one* copy of each (ϵ, R)-good pants. I show that $\nu(S_{\epsilon,R(\epsilon)}) \xrightarrow{\star} \nu_{\mathrm{Gr}\,M}$ as $\epsilon \to 0$ and $R(\epsilon) \to \infty$. This uses the Kahn–Wright equidistribution of the incidence directions of good pants on a given good curve, as well as the fact, in a formulation due to Lalley [25], that the good curves themselves are asymptotically almost surely equidistributing in the unit tangent bundle $\mathrm{T}^1 M$.

There is no guarantee that the surfaces $S_{\epsilon,R}$ are *connected*. However, using ideas from the article of Liu and Marković [26], it is possible to show that if we take $N = N(\epsilon, R, M)$ copies of $S_{\epsilon,R}$, it is possible to perform surgeries around the good curves and reglue them in order to get connected surfaces $\hat{S}_{\epsilon,R}$ in M. These surfaces satisfy $\nu(\hat{S}_{\epsilon,R}) = \nu(S_{\epsilon,R})$, so in particular they equidistribute as $\epsilon \to 0$.

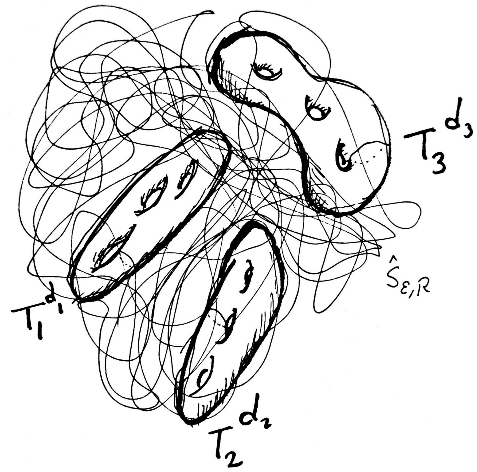

FIGURE 2. The family of surfaces $S_{\epsilon,R,\mathbf{d}}$, which can accumulate on the totally geodesic surfaces T by appropriately choosing the degrees d of their covers.

The surfaces $\hat{S}_{\epsilon,R}$ also form the bedrock for the construction of *nonequidistributing* asymptotically Fuchsian connected surfaces. Given an immersed totally geodesic surface $T \in \mathcal{G}$ in M, a theorem of Kahn and Marković [21] implies that T has a finite cover $\hat{T}$ built out of (ϵ, R)-good pants via (ϵ, R)-good glueings. (This result implies the Ehrenpreis conjecture.) We take a further cover T^d of $\hat{T}$ of degree $2d(T, \epsilon, R)$ that preserves the cuffs, and glue it back, in a $(3\epsilon, R)$-good way, in order to obtain a surface $S_{\epsilon,R,\mathbf{d}}$, where $\mathbf{d} = (d(T))_{T\in\mathcal{G}}$. This surface is closed, connected, and $(1 + O(\epsilon))$-quasifuchsian, and by making $d(T)$ grow in different rates as $\epsilon \to 0$, we can make $\nu(S_{\epsilon,R,\mathbf{d}})$ converge to any measure of the form $(\star)$.

A natural avenue for generalization is

Question 1. Does Theorem 2.1 also hold when M is a cusped hyperbolic 3-manifold of finite volume?

In [22], Kahn and Wright generalize the surface subgroup theorem to this setting, simplifying elements of the construction along the way. Now, the quasifuchsian surfaces are built mostly, but not exclusively, out of good pants – a new kind of component, the good hamster wheel, is introduced to veer the surfaces away from the cusps.

Some difficulties of trying to extend Theorem 2.1 to manifolds of finite volume are the following. To start, it is not clear that if $f_i : S_i \to M$ are asymptotically Fuchsian, then $\nu(f_i)$ has any subsequential weak-* limit at all. The surfaces $f_i(S_i)$ could a priori escape to the cusps. The good pants still come in a well-distributed set of directions around a good curve, provided the curve does not go too far into the cusp. It is also not obvious how to generalize to the cusped setting the ideas of Liu and Marković that allow us to build a connected surface going through every good curve.

3. Averaging all asymptotically Fuchsian surfaces in a hyperbolic 3-manifold

Theorem 2.1 tells us it is possible for connected asymptotically Fuchsian surfaces to accumulate in totally geodesic surfaces. That was achieved with a careful construction and it would be interesting to know whether this phenomenon is generic in any sense. In this section, we will describe a way of averaging all sequences of asymptotically Fuchsian surfaces and ask what that limiting measure looks like.

Let $\mathscr{S}(K, g, M)$ be the set of K-quasifuchsian surface subgroups of genus at most g, up to commensurability, in $\pi_1(M) \leq \mathrm{PSL}_2\,\mathbf{C}$. For each $\sigma \in \mathscr{S}(K, g, M)$, choose a minimal or pleated map $f_\sigma : S_\sigma \to M$ so that $(f_\sigma)_*(\pi_1 S) \in \sigma$.

Define

$$\nu_{K,g} = \frac{1}{\#\mathscr{S}(K,g,M)} \sum_{\sigma \in \mathscr{S}(K,g,M)} \nu(f_\sigma).$$

We can ask

Question 2. Does $\nu_{K,g}$ have a unique subsequential weak-* limit as $g \to \infty$?

Either way, let ν_K be such a subsequential limit. Again, we can ask

Question 3. Does ν_K have a unique subsequential weak-* limit as $K \to 1$?

From Theorem 2.1, we know that the weak-* limits of ν_K as $K \to 1$ are of the form

$$(\star) \qquad \nu = \alpha_M \nu_{\mathrm{Gr}\,M} + \sum_{T \in \mathscr{G}} \alpha_T \nu_T$$

where $\alpha_M + \sum_{T \in \mathscr{G}} \alpha_T = 1$.

Question 4. What are the possible coefficients α_M and α_T for ν?

This would be particularly interesting if it turned out that there is a unique limit $\lim_{K\to 1} \lim_{g\to\infty} \nu_{K,g}$; what would be the exact formula for it? The α_T not being all zero would imply that it is not so rare for asymptotically Fuchsian surfaces to accumulate on totally geodesic surfaces. One reason to expect that the α_T might not all vanish is that there is a large number of ways to produce non-equidistributing asymptotically Fuchsian surfaces as described in Section 2. Namely, there are many ways to glue the covers T^d of the totally geodesic surfaces T of M to the equidistributing connected surfaces $\hat{S}_{\epsilon,R}$.

It is relevant to these questions to understand the behavior of $\#\mathscr{S}(K, g, M)$ for large g as $K \to 1$. In [19], Kahn and Marković show that there are $c_1, c_2 > 0$ so that

$$(c_1 g)^{2g} \leq \#\mathscr{S}(g, M) \leq (c_2 g)^{2g},$$

where $\mathscr{S}(g, M)$ is the set of commensurability classes of quasifuchsian surface subgroups of genus up to g in M.

To obtain the lower bound, they construct a family of surfaces that is not nearly Fuchsian by gluing covers of two nearly Fuchsian surfaces at a nearly orthogonal angle. By counting such covers with a formula of Müller–Puchta [35], they show that there are at least $(c_1 g)^{2g}$ commensurability classes of the hybrid glued surfaces. It might be possible to construct as many nearly Fuchsian surfaces by glueing covers of nearly Fuchsian surfaces via good glueings. To do this, we might need to exploit what we know from the Liu–Marković article [26] – namely that there are connected nearly Fuchsian surfaces made with a fixed number $N(\epsilon, R, M)$ of good pants that go through every good cuff at a well-distributed set of directions. This would show that

$$\lim_{K \to 1} \#\mathscr{S}(K, g(K), M) \geq (c_1 g)^{2g}, \tag{$\circ$}$$

for some function $g(K)$ that goes to infinity as $K \to 1$ quickly enough.

This would not elucidate, however, how wide the gap between c_1 and c_2 is.

4. Asymptotically Fuchsian surfaces that minimize area in their genus

Let $\overline{g} = \overline{g}(K, g, M)$ be the largest genus that is realized by connected quasifuchsian surfaces in $\mathscr{S}(K, g, M)$. Let $\mathscr{A}(K, g, M)$ denote the connected quasifuchsian surfaces $\sigma \in \mathscr{S}(K, g, M)$ whose minimal representatives $f_\sigma : S_\sigma \to M$ have the least area among the $\sigma \in \mathscr{S}(K, g, M)$ of genus exactly $\overline{g}$. The set $\mathscr{A}(K, g, M)$ has at least one element, but it is not clear if it has exactly one element.

We define

$$\mu_{K,g} = \frac{1}{\#\mathscr{A}(K, g, M)} \sum_{\sigma \in \mathscr{A}(K,g,M)} \nu(f_\sigma).$$

As in the previous section, we can ask questions about the weak-* limits of $\mu_{K,g}$ as $g \to \infty$ and $K \to 1$. For example, we can fix a function $g(K)$ that goes to infinity quickly as $K \to 1$, as in ($\circ$) above.

Question 5. Does $\mu_{K,g(K)}$ have a unique weak-* limit as $K \to 1$? If so, is it equal to $\nu_{\mathrm{Gr}\, M}$?

This averaging process might miss the asympotically Fuchsian surfaces that accumulate on totally geodesic surfaces, which might then cause the $\mathrm{PSL}_2\,\mathbf{R}$-invariant limiting measures to always be the Haar measure of $\mathrm{Gr}\, M$.

Instead of averaging over $\mathscr{A}(K, g, M)$, we can study the behavior of a sequence $\sigma_K \in \mathscr{A}(K, g(K), M)$. We can ask, for example,

Question 6. Given a sequence $\sigma_K \in \mathscr{A}(K, g(K), M)$, what are the weak-* limits of $\nu(f_{\sigma_K})$ as $K \to 1$?

If these limits always turned out to be the volume measure $\nu_{\mathrm{Gr}\, M}$ of $\mathrm{Gr}\, M$, this would be a way of producing equidistributing increasingly Fuchsian connected surfaces in $\mathrm{Gr}\, M$ without relying on the Kahn–Marković construction of surfaces from glueing good pants.

It would also be interesting to investigate these questions for closed hyperbolizable 3-manifolds (M, h) endowed with a metric h of sectional curvature bounded above by

-1. An inspiration for this is the article of Calegari, Marques and Neves [8] that counts asymptotically Fuchsian minimal surfaces in (M, h) by area, and compares it with the count for M with the original hyperbolic metric.

Note that the limiting measures described in Sections 3 and 4 are analogues, for surfaces, of the Bowen–Margulis measure β on $\mathrm{T}^1 M$. The measure β is the weak-* limit, as $T \to \infty$, of the probability measures β_T supported on the closed geodesics of lengths up to T of M. It turns out that β is a measure of maximal entropy for the geodesic flow, and in our case, it coincides with the volume (Liouville) measure on $\mathrm{T}^1 M$.

5. Asymptotically Fuchsian surfaces in asymptotically cusped hyperbolic 3-manifolds

It would be interesting to know whether you can detect a cusp on a hyperbolic 3-manifold from the distributional behavior of its asymptotically Fuchsian surfaces. Suppose that $M_n \to M$ is a sequence of Dehn fillings of a cusped hyperbolic 3-manifold converging to M in the geometric topology. (Such sequences exist due to the Thurston hyperbolic Dehn filling theorem, see [41].)

For example, it would be interesting to investigate the following limit

$$\lim_{n\to\infty}\lim_{K\to 1}\lim_{g\to\infty}\frac{\#\mathscr{S}(K,g,M_n)}{\#\mathscr{S}(K,g,M)}.$$

If this limit exists and is different from 1, this would mean that there is a gap in the statistical behavior of asymptotically Fuchsian surfaces in closed versus cusped hyperbolic 3-manifolds.

6. Detecting a manifold from its asymptotically Fuchsian surfaces

This section is in keeping with the theme of trying to deduce something about M from its surfaces. Let $\rho : \pi_1 S_g \to \mathrm{PSL}_2\,\mathbf{C}$ be a discrete and faithful surface group representation, defined up to conjugacy in $\mathrm{PSL}_2\,\mathbf{C}$, with image $\Delta_\rho < \mathrm{PSL}_2\,\mathbf{C}$. Let $\ell_\rho(M)$ denote the number of Γ-conjugacy classes of $\Delta \leq \Gamma$ that are $\mathrm{PSL}_2\,\mathbf{C}$-conjugate to Δ_ρ. Define the *surface spectrum* of M and the *surface set* of M respectively as

$$\mathrm{Ssp}(M) = \{(\Delta_\rho, \ell_\rho(M)) : \ell_\rho(M) \neq 0\} \quad \text{and} \quad \mathrm{Sset}(M) = \{\Delta_\rho : \ell_\rho(M) \neq 0\}.$$

Let $\mathrm{Fset}(M)$ denote the elements of $\mathrm{Sset}(M)$ that are conjugate to virtually fibered subgroups of Γ. Note that $\mathrm{Fset}(M) \neq \emptyset$ due to the virtual fibering theorem [1].

In [32], McReynolds and Reid show that the number of closed orientable hyperbolic 3-manifolds N so that $\mathrm{Ssp}(M) = \mathrm{Ssp}(N)$ is finite, and moreover if $\mathrm{Ssp}(M) = \mathrm{Ssp}(N)$, then M and N are commensurable. In order to show that, they prove that the number of closed orientable hyperbolic 3-manifolds N so that $\mathrm{Fset}(M) = \mathrm{Fset}(N)$ is finite and that if $\mathrm{Fset}(M) = \mathrm{Fset}(N)$, then M and N are commensurable. This is sufficient because if $\mathrm{Ssp}(M) = \mathrm{Ssp}(N)$, then $\mathrm{Fset}(M) = \mathrm{Fset}(N)$, which in turn follows from the fact that Δ_ρ is conjugate to a virtually fibered subgroup of Γ exactly when ρ is geometrically infinite and finitely generated, a condition which does not depend on Γ. (This final fact follows from the tameness theorem of Bonahon [5] and the covering theorem of Thurston [41].)

Now let $\mathrm{Qset}^K(M)$ denote the elements of $\mathrm{Sset}(M)$ that are conjugate to K-quasifuchsian subgroups of Γ for some $K > 1$. We can ask

Question 7. Let $K > 1$. Is it true that the set of closed orientable hyperbolic 3-manifolds N so that $\mathrm{Qset}^K(M) = \mathrm{Qset}^K(N)$ is finite? Moreover, does $\mathrm{Qset}^K(M) = \mathrm{Qset}^K(N)$ imply M is commensurable to N?

Do the answers vary depending on whether K is close to one or not?

It would also be interesting to investigate these questions in the case where M is a cusped hyperbolic 3-manifold of finite volume. If the McReynolds–Reid result holds in this setting, it would be a way to detect whether a manifold has a cusp only using information about its surfaces.

More is known about the case when the manifold is arithmetic. Let $\mathrm{Tset}(M)$ denote the elements of $\mathrm{Sset}(M)$ that are Fuchsian (i.e., those corresponding to totally geodesic surfaces in M). Note that $\mathrm{Tset}(M) \subset \mathrm{Qset}^K(M)$ for every $K > 1$. For *arithmetic* complete orientable hyperbolic 3-manifolds M and N of finite volume that contain at least one totally geodesic surface, McReynolds and Reid show in [33] that if $\mathrm{Tset}(M) = \mathrm{Tset}(N)$, then M is commensurable to N.

7. Limits of asymptotically Fuchsian surfaces in locally symmetric manifolds

Another avenue of generalization is to study essential surfaces in higher-rank locally symmetric spaces. Kahn, Labourie and Mozes [18] prove a surface subgroup theorem for a semisimple center-free complex Lie group G. They show that if $\Gamma < G$ is cocompact, then Γ contains a surface subgroup. In fact, they show that Γ contains arbitrarily small deformations of Fuchsian surface subgroups of G, generalizing the notion of quasifuchsian subgroups of $\mathrm{PSL}_2\,\mathbf{C}$.

A bit more precisely, choose an embedding $\mathfrak{s} : \mathfrak{sl}_2\mathbf{R} \to \mathfrak{g}$. This defines a flag variety $\mathbf{F} = \mathbf{F}(\mathfrak{s})$, which is identified with G/P for a parabolic subgroup $P(\mathfrak{s})$. A *circle* in $\mathbf{F}$ is a map $\eta : P^1(\mathbf{R}) \to \mathbf{F}$ that is equivariant under a representation $\mathrm{PSL}_2\,\mathbf{R} \to G$ conjugate to the one defined by $\mathfrak{s}$. A map $\xi : P^1(\mathbf{R}) \to \mathbf{F}$ is ζ-*Sullivan* for some $\zeta > 0$ if it is, in a quantitative sense, close to a circle. Given $\zeta > 0$, Kahn, Labourie and Mozes show that a cocompact lattice $\Gamma < G$ contains a surface subgroup $\rho(\Sigma)$, where $\Sigma < \mathrm{PSL}_2\,\mathbf{R}$ is a Fuchsian surface group and $\rho : \Sigma \to G$ is an embedding so there is a ζ-Sullivan ρ-equivariant map $\xi : P^1(\mathbf{R}) \to \mathbf{F}$.

For $\zeta > 0$ small enough, the map $\xi : P^1(\mathbf{R}) \to \mathbf{F}$ turns out to be $\mathbf{F}$-Anosov as well, so this connects to a well-developed part of higher Teichmüller–Thurston theory.

The locally symmetric manifold $M = \Gamma\backslash G/K$, where $K < G$ is a maximal compact subgroup, is a classifying space for Γ. Thus, an embedding $\rho : \Sigma \to \Gamma$ induces a π_1-injective continuous map $h : S \to M$, where S is the topological surface so that $\pi_1 S = \Sigma$.

From the work of Sacks–Uhlenbeck and Schoen–Yau, it follows that h is homotopic to a possibly branched area-minimizing minimal map $f : S \to M$.

In the case that there is a ρ-equivariant ζ-Sullivan map $P^1(\mathbf{R}) \to \mathbf{F}$, we can ask questions about $f : S \to M$ such as

Question 8. Is f unique?

Question 9. Is there a bound on the principal curvatures of $f(S)$ in terms of ζ?

The answer to Question 9 is affirmative when $G = \mathrm{PSL}_2\,\mathbf{C}$, due to the work of Seppi [38], who shows that the principal curvatures of $f(S)$ go to zero uniformly as $\zeta \to 0$. Uhlenbeck [42] shows that quasifuchsian minimal surfaces with principal curvatures in $(-1, 1)$ are unique in their homotopy classes, so in particular for $G = \mathrm{PSL}_2\,\mathbf{C}$, the answer to Question 8 is yes when their quasiconformal quality is sufficiently close to 1. (Otherwise the answer can be no, see [16] for example.) Similar results have been obtained for rank-one Lie groups by Jiang [17].

As before, we define a sequence of π_1-injective maps $f_i : S_i \to M$ of closed surfaces to be asymptotically Fuchsian if $(f_i)_* : \pi_1 S_i \to \Gamma$ is ζ_i-Sullivan for $\zeta_i \to 0$ as $i \to \infty$. (This definition is with respect to a choice of embedding $\mathfrak{s} : \mathfrak{sl}_2\mathbf{R} \to \mathfrak{g}$.) Again, for an almost-everywhere differentiable $f : S \to M$, we let $\hat{\nu}(f)$ denote the probability measure induced by f on $\Gamma\backslash G$. We can ask

Question 10. Let $f_i : S_i \to M$ be a sequence of asymptotically Fuchsian minimal maps of closed surfaces S_i. Are the weak-* limits of $\hat{\nu}(f_i)$ invariant under the right action of $\mathrm{PSL}_2\,\mathbf{R}$?

If the answer to the above question is yes, we can apply the Ratner measure classification and conclude that these weak-* limits are convex combinations of the volume measures of the closed orbits of the closed Lie subgroups $L \leq G$ containing the image of the representation $\mathrm{PSL}_2\,\mathbf{R} \to G$ corresponding to the embedding $\mathfrak{s} : \mathfrak{sl}_2\mathbf{R} \to \mathfrak{g}$. It would be interesting to know whether all of those measures can occur, as in Theorem 2.1, or if there are restrictions.

8. Asymptotically flat surfaces in $\mathrm{SL}_d\,\mathbf{Z}\backslash \mathrm{SL}_d\,\mathbf{R}$ for $d = 3, 4$

Keeping with the higher-rank theme, articles of Long–Reid–Thistlethwaite [27] and Long–Thistlethwaite [28] describe infinite families $\Sigma_{n,g} < \mathrm{SL}_d\,\mathbf{Z}$ of Zariski-dense surface subgroups of a *fixed* genus $g \geq 2$ for $d = 3, 4$. As described in the section above, this gives rise to an infinite family of minimal maps $f_{n,g} : S_g \to M$, where S_g is the closed surface of genus g and $M = \mathrm{SL}_d\,\mathbf{Z}\backslash \mathrm{SL}_d\,\mathbf{R}/\,\mathrm{SO}_d$. Since the minimal surfaces $f_{n,g}(S_g)$ are all distinct, their area goes to infinity as $n \to \infty$. From the Gauss–Bonnet formula, it follows that increasingly large portions of $f_{n,g}(S_g)$ have Gaussian curvatures close to zero for a fixed g and $n \to \infty$. As before, let $\nu(f_{n,g})$ denote the probability area measure of $f_{n,g}(S_g)$ on the Grassmann bundle $\mathrm{Gr}\,M$.

Question 11. For a fixed $g \geq 2$, are the weak-* limits of $\nu(f_{n,g})$ as $n \to \infty$ supported in the flats of M?

9. Asymptotic distribution of quasifuchsian surfaces in moduli spaces

The main theme in many of the sections above is studying how the quasifuchsian surfaces $f : S \to M$ of a hyperbolic 3-manifold M are distributed in its Grassmann bundle $\mathrm{Gr}\, M$. It would also be interesting to learn how the hyperbolic surfaces S endowed with the pullback metric from f are distributed in their moduli spaces.

Kahn and Marković [19] show that, for large g, there are at least $(c_1 g)^{2g}$ and at most $(c_2 g)^{2g}$ conjugacy classes of quasifuchsian surface subgroups of genus at most g in $\pi_1(M)$, where $0 < c_1 \le c_2$ are universal constants. Fletcher, Kahn and Marković [13] show that, for large g, a number between $(c_1' g)^{2g}$ and $(c_2' g)^{2g}$ of Teichmüller balls of radius 1 cover the ϵ-thick part $\mathscr{M}_g^{\geq \epsilon}$ of the moduli space $\mathscr{M}_g$ of hyperbolic surfaces of genus g, where $0 < c_1' \le c_2'$ are universal constants. Thus, the following question is a way of asking whether the surface subgroups of genus g in a given $\pi_1(M)$ are well distributed in $\mathscr{M}_g$:

Question 12. Is there a universal constant $C > 0$ so that, for large enough g, a Teichmüller ball of radius 1 in $\mathscr{M}_g$ has no more than C^g surface subgroups of genus g in $\pi_1(M)$?

A version of this question that might be easier to answer is

Question 13. Is there a universal constant $C > 0$ so that, for large enough g, a Teichmüller ball of radius 1 in $\mathscr{M}_g$ has no more than C^g surface subgroups of genus g in $\pi_1(M)$ that are built out ot (ϵ, R)-good glueings of (ϵ, R)-good pants for some (ϵ, R)?

From the work of Liu–Marković [26], it is known that given $\epsilon > 0$, there is $R \gtrsim_\epsilon 1$ so that a finite cover of every nearly Fuchsian surface in M may be built out of (ϵ, R)-good glueings of (ϵ, R)-good pants. Still, it is not clear that an affirmative answer to Question 13 would imply an affirmative answer for Question 12.

These questions were inspired by the article of Vasuvedan [43], which shows that, in a similar sense, random triangulated surfaces are well-distributed in moduli spaces of large genus.

It would be also interesting to learn about how the *R-perfect surfaces* are distributed in their moduli spaces. These are the hyperbolic surfaces that admit a pants decomposition with cuffs of size exactly R that are glued with a shear parameter of size exactly 1. (In the language of the Kahn–Marković surface subgroup construction, they are made by glueing $(R, 0)$-good pants via $(R, 0)$-good glueings.)

Let $P_{R,g} \subset \mathscr{M}_g$ be the set of R-perfect surfaces in $\mathscr{M}_g$. We may ask

Question 14. What is the asymptotic behavior of $\#P_{R,g}$ in R and g?

Question 15. Let $\nu_{R,g}$ be the uniform probability measure supported in $P_{R,g}$. What are the weak-* limits, if they exist, of $\nu_{R,g}$, as $R \to \infty$ for a large g? For example, are they equal to the Weil–Petersson, or Masur–Veech measure on $\mathscr{M}_g$?

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