Some Notes On Cobwebs

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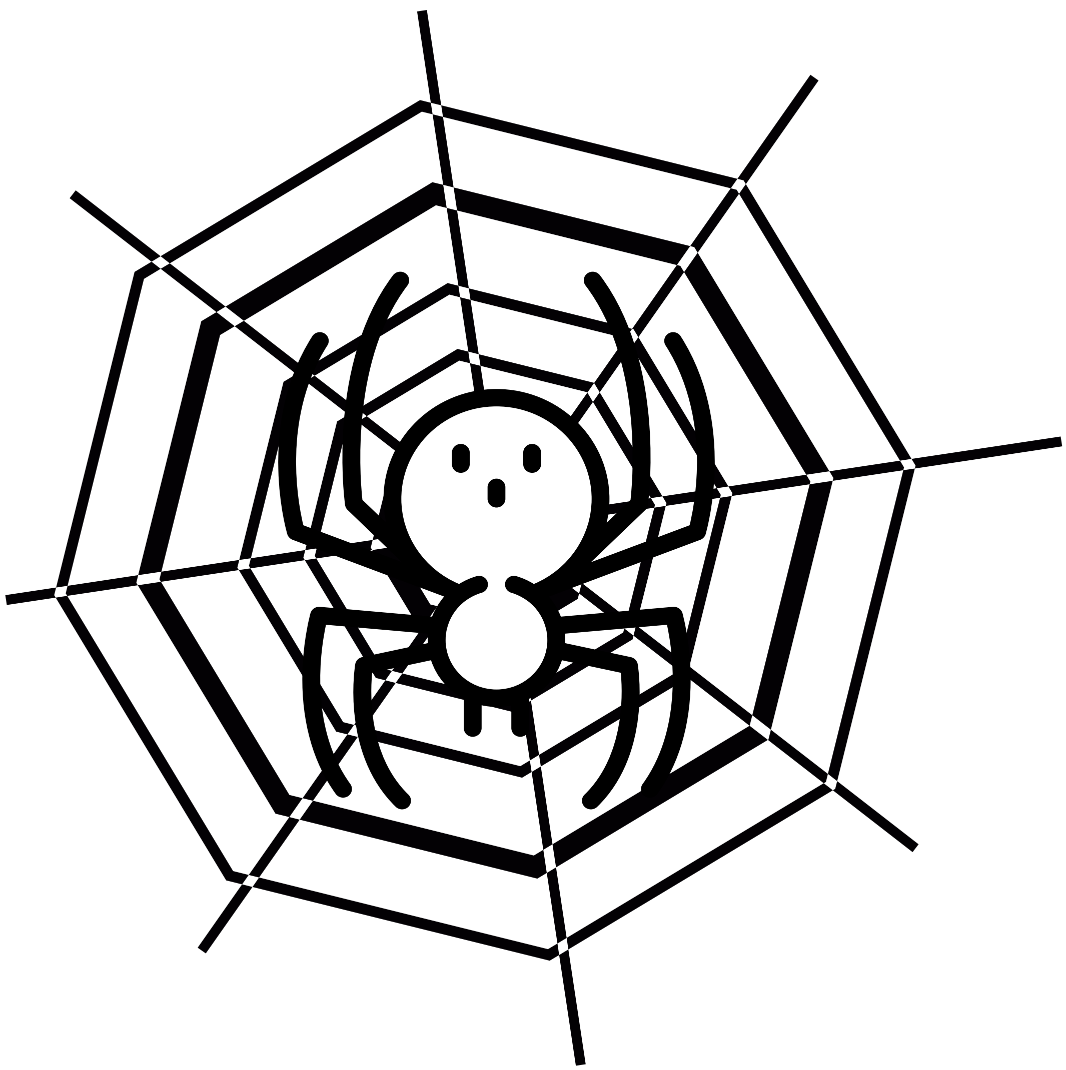
1 Introduction

So, I've had a couple of you get in touch and ask "Why do we use cobweb diagrams? What are they FOR?"
So, I've decided to write up some notes. You can read them, or not, as suits you. I'm going to explain cobwebs in multiple different ways, and you can decide which explanation suites you best. Then I'm going to go through a few examples.



2 A metaphor to get us started:

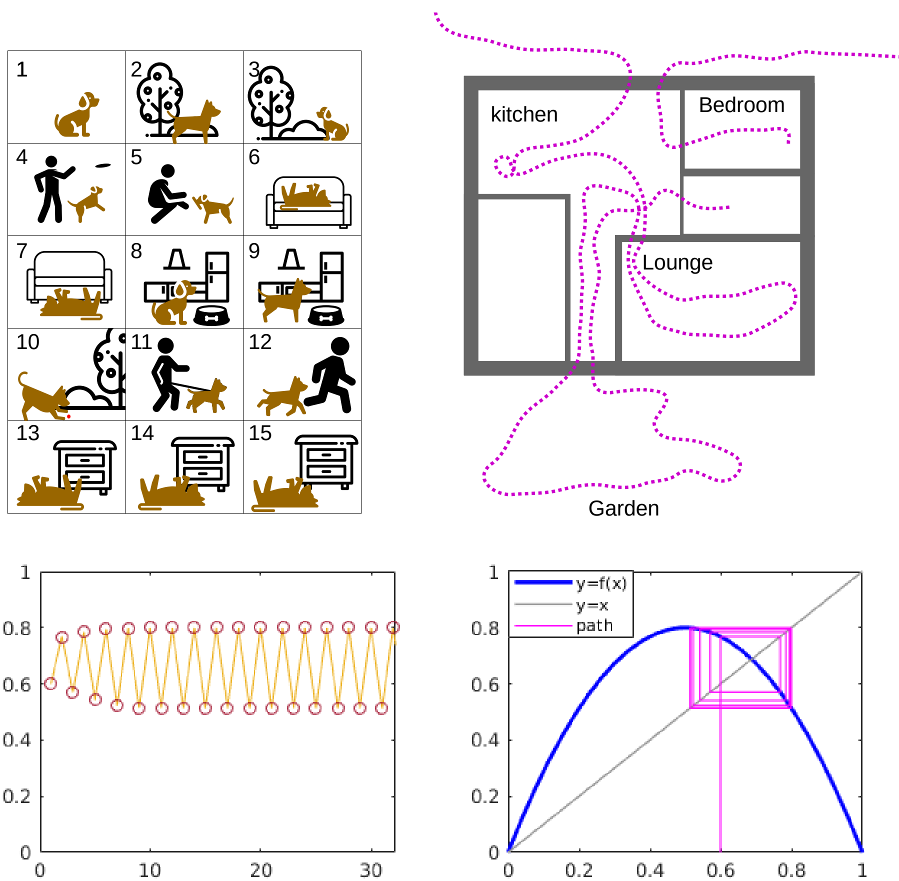
First off, lets start with an “intuitive” explanation for what cobwebs are, and what they are about. Imagine you have a dog. During the day, it runs around your house, sometimes sleeping in its bed, sometime hanging around the kitchen, sometimes going outside. Now, you want to know what the dog is up to during the day. One way of doing this is to take a photo of the dog once per hour, so that you have a collection of photos, showing where it goes and how it behaves. Each of these photos is timestamped.

This is good, in that it gives you a progression of time, and gives good detail at any given time.... but its also bad, because you don’t have any sense of *how* the dog got from one place to another, or why.

Now imagine you instead paint your dogs paws with fluorescent pink paint. Everywhere it goes, it leaves behind pink footprints. Now, at the end of the day, you get to see a path of everywhere your dog has gone. This is basically what a cobweb diagram is doing. Now... the catch is that the dogs path might criss-cross over itself many times. We get to see things like ‘after going outside, the dog ALWAYS comes back in to check its water bowl’, or ‘The pawprints leading to the bed come from many different locations; the dog can end up in its bed from many different places’

The catch with this plan (aside from making a mess of the house) is that we lose all sense of *time*. We know that the dog went from the kitchen to the garden, but we don’t know when this happened. We are gaining some information, and losing other information. We also end up with lines criss-crossing each other. This can (on occasion) be confusing.

Time course data and cobweb diagrams work on roughly the same way: timecourse data tells you where the system is at specific moments, and cobwebs show you how it got there.



3 The Algebra:

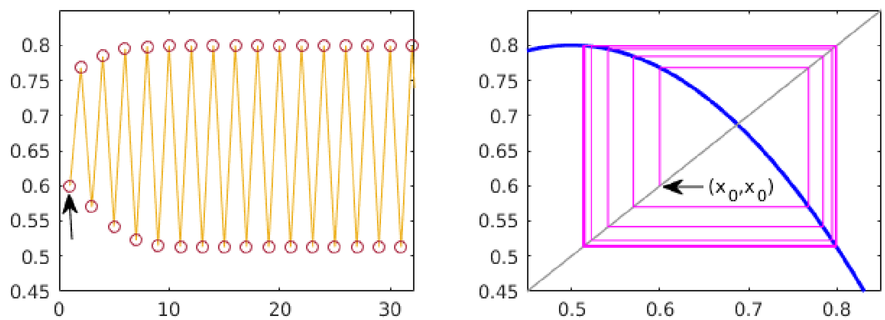
Difference equations work by taking an number and applying a function to it to find a new value. We use this value as our new number in the next round, and repeat the process over and over again.

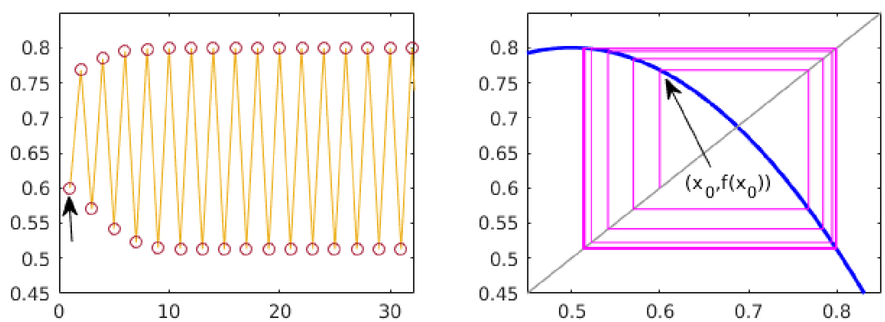
We might write this as follows:

$$x_0 \to f(x_0) = x_1 \to f(x_1) = x_2 \to f(x_2) = x_3 \to f(x_3) = x_4 \to f(x_4) = x_5 \to f(x_5) = x_6 \to ...$$

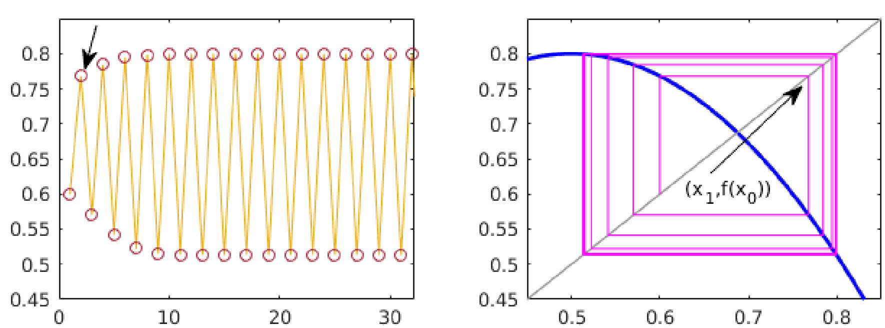
There are two processes here, two types of relationship. There is the connection $x_n \to f(x_n)$, where the arrow means 'x_n is used to calculate $f(x_n)$'. There is also $f(x_n) = x_{n+1}$, where here we are saying "x_{n+1} is EQUAL to $f(x_n)$".

Lets say we start are some point on the cobweb diagram (x_0, x_0). We want to represent the relationship between x_0 and $f(x_0)$, so we move vertically to the point $(x_0, f(x_0))$. This point will be on the line $y = f(x)$. The location of this point is a visual representation of the relationship $x_0 \to f(x_0)$; by going here we are saying "x_0 and $f(x_0)$ are related to one another", but instead of saying it with words we we are saying it with a diagram.





The next relationship in our series of equations is $f(x_0) = x_1$. To represent this by moving to the point $(x_1, f(x_0))$. Because we are keeping the y coordinate the same, we move horizontally. Because $x_1 = f(x_0)$ this point will be on the line $y = x$. This point represents the relationship $x_1 = f(x_0)$; once again, by going to this point, we are saying "x_1 and $f(x_0)$ are related to one another", in a visual way.



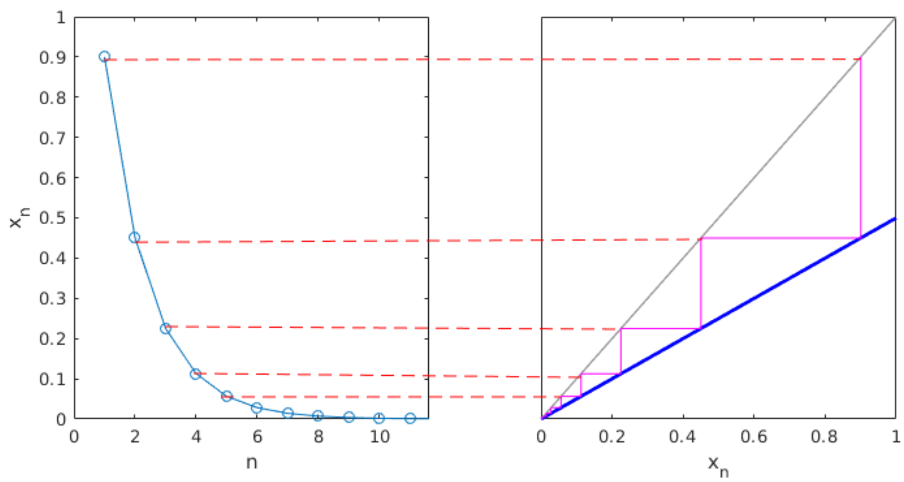
Next, we want to represent the relationship $x_1 \to f(x_1)$. We move to the point $(x_1, f(x_1))$. Because x_1 stays the same, only the y coordinate changes, and this is a vertical movement (either up or down). Once again, we are going to be on the line $y = f(x)$.

The cobweb continues in this manner forever- moving vertically to the line $y = f(x)$ and horizontally to the line $y = x$ over and over again. We do two different things because we are trying to represent two different types of relationship. The PURPOSE of the cobweb diagram is to represent all the relationships in the equation

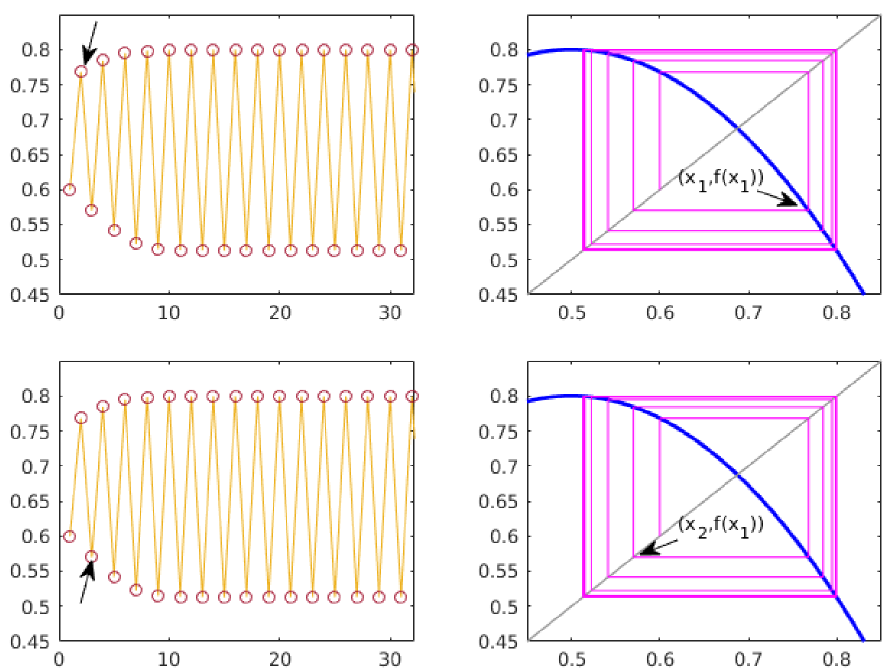
$$x_0 \to f(x_0) = x_1 \to f(x_1) = x_2 \to f(x_2) = x_3 \to f(x_3) = x_4 \to f(x_4) = x_5 \to f(x_5) = x_6 \to ...$$

in one tidy diagram.

Each time the cobweb hits $y = x$, the corresponding x value is one of our x_n. This can be seen on the cobweb diagram by noticing that each '$y = x$ corner' corresponds with the *height* of a point on the timecourse.



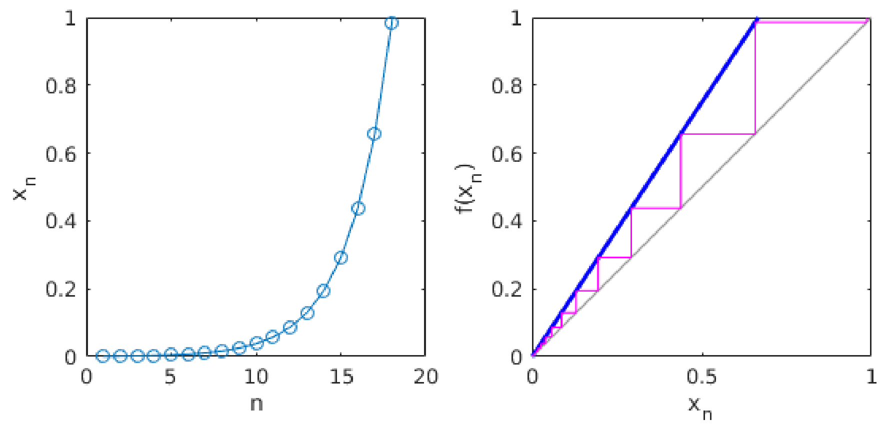
By drawing the lines $y = f(x)$ and $y = x$ on our diagram, it is easy to see WHEN the pink line is going to hit a corner.



4 Examples

Let's talk about some example cobweb diagrams. The two most common patterns that you are going to see on a cobweb diagram are spirals and staircases. We'll talk about staircases first.

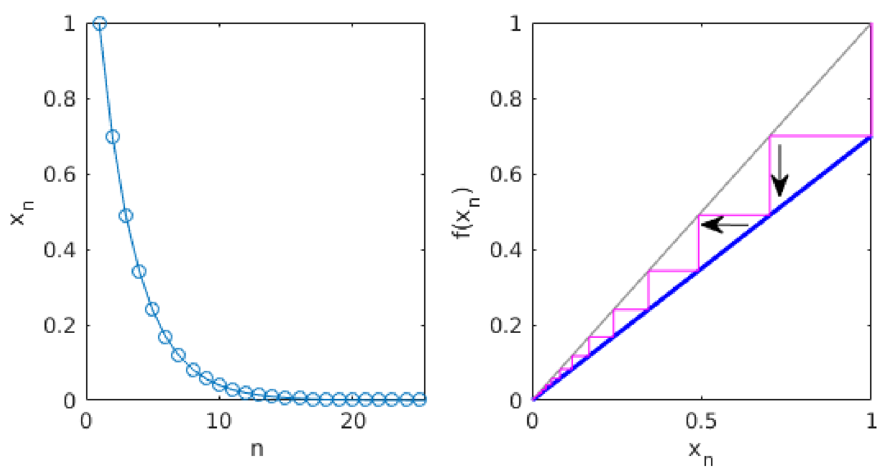
4.1 Staircases



Staircases occur when we are near a steady state and $f'(x) > 0$. THIS cobweb diagram represents exponential growth away from a steady state.

Each 'step' on the staircase corresponds with one x_n value on the timecourse diagram; the flat part of the step has the same y value as the corresponding x_n. As we move further and further from the steady state, the 'steps' on our cobweb diagram get larger and larger. They travel in only one direction (up and to the right), and this matches with the fact that x_n is only getting larger as n increases.

While our first example showed exponential growth away from a steady state, staircases also show up when we have exponential decay towards a steady state:



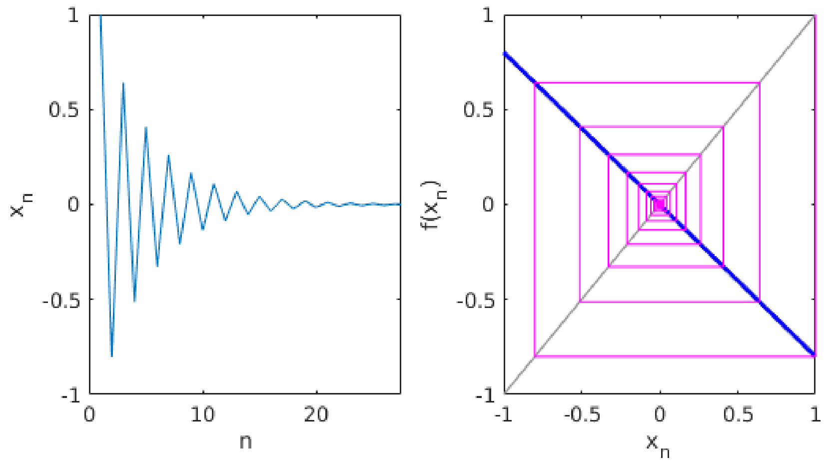
Here, the 'steps' get smaller and smaller as we draw close to the point where the two lines cross. Correspondingly, on the timecourse, $x_{n+1} - x_n$ gets smaller and smaller as we draw close to the steady state.

It can sometimes be difficult to tell if a cobweb diagram is moving 'up' or 'down'. In order to tell the difference, remember that we move *vertically* to the function and *horizontally* to the line $y = x$. When the blue function line is *above* $y = x$ then x_n will increase. When it is below, x_n will decrease.

When $0 < f'(x) < 1$ we expect to see exponential decay (a staircase that is moving in towards the steady state). When $f'(x) > 1$ we see exponential growth (a staircase moving out away from the steady state.

4.2 Spirals

Another common feature to see in a Cobweb diagram is a spiral:



When we see a spiral on a cobweb diagram, we know that our system has some central point, and x_n is moving down below the point... and back up above it... and down below it... and back up above it... and so on.

What this means on our timecourse diagram is zig-zags. Both x_n and the cobweb diagram are constantly switching which direction they are travelling in.

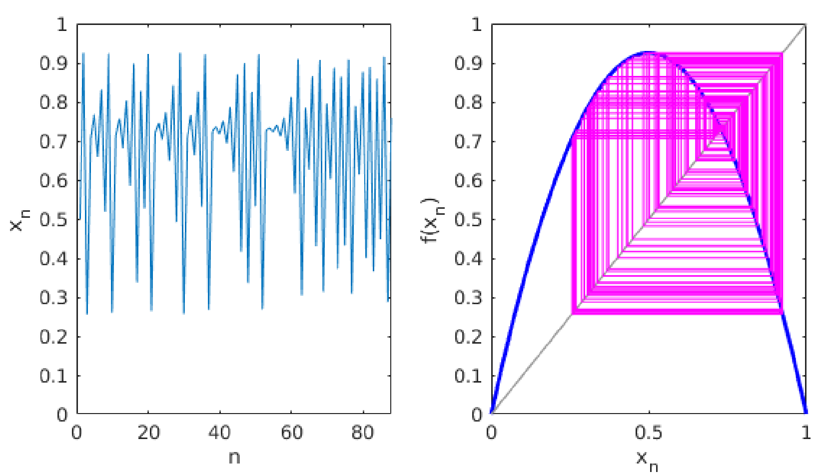
When dealing with spirals, we can either spiral inwards towards our focal point, or spiral outwards, further and further away. On the timecourse diagram, this corresponds to zig-zags that decay towards nothing (as above), or zig-zags which become more dramatic over time.

Decaying zig-zags occur when we have a steady state with $-1 < f'(x) < 0$. This means we have a stable steady state $-1 < f'(x) < 1$, but because $f'(x) < 0$ the 'error' keeps switching sign.

Growing zig-zags most often occur when $f'(x) < -1$ at some steady state.

4.3 Chaos

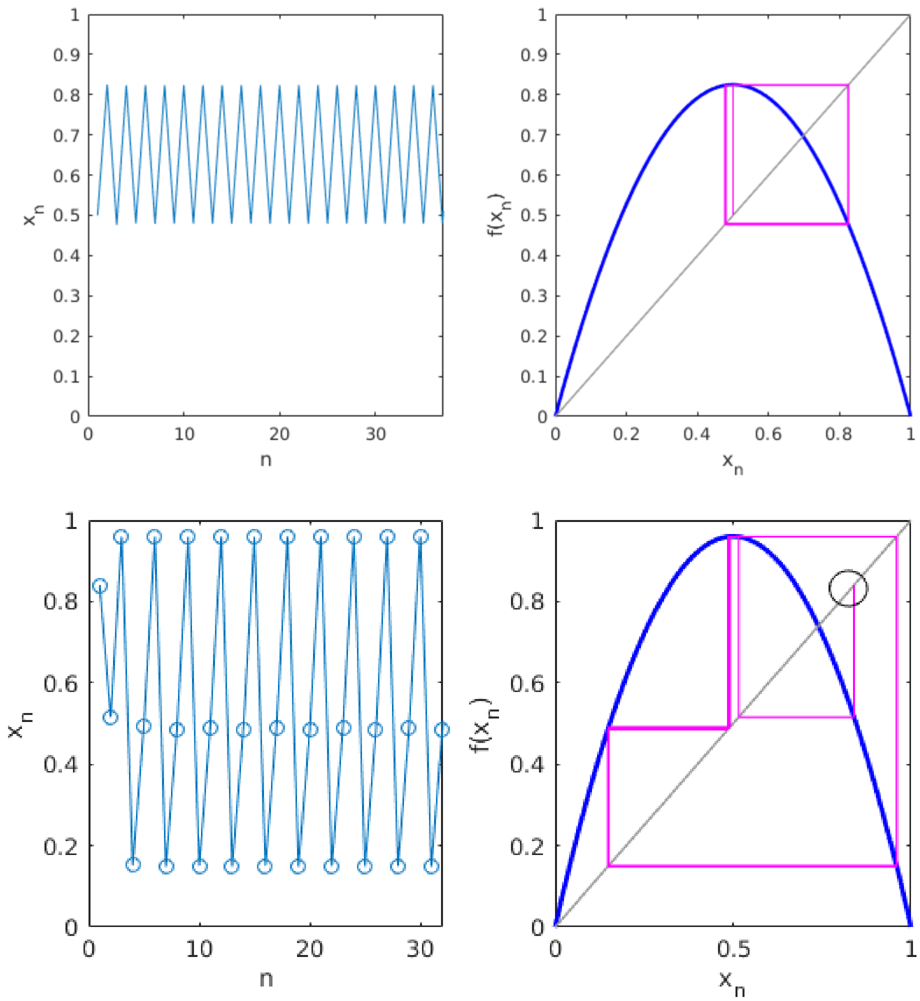
If your Cobweb diagram is super messy, with your line criss-crossing over itself over and over again, then in all likelihood you have chaos:



On cobweb diagrams, chaos is often identified by the trajectory 'filling out' a particular region of space- tangling over itself until there is no white left on the diagram. This indicates that x_n will visit *many different* points within the region during its wild wanderings.

4.4 Shapes

Sometimes on your Cobweb diagram you will observe a solid 'shape' such as a rectangle, L shape or similar



These shapes represent cycles: a collection of points that the cobweb diagram will visit over and over again.

Often when we plot cobweb diagrams of this sort there will be a brief 'tail' starting at the initial condition as the systems settles in on the repeating cycle. Once the system is near the stable cycle it will get closer and closer to the 'true' cycle. The cobweb trajectory will get closer and closer to the rectangle, L, or more complicated shape associated with the cycle. Often it can be useful to plot the cobweb diagram *without* this extra line caused by the initial conditions. You can either select initial conditions that are already ON your cobweb diagram, or instruct matlab not to include the first hundred or so points of the cobweb:

```
plot(X(100:end),Y(100:end));
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Once you have a clear shape on your cobweb diagram the most important question you want to ask is 'how long is the cycle this represents?'

2-cycles are represented by rectangles. 3 cycles are represented by 'L' shapes. A cycle with k points in it will correspond to a shape with $2k$ corners; k corners on the $y = f(x)$ line and k points on the $y = x$ line.

As always in cobweb diagrams, the horizontal lines in the above cobwebs above line up perfectly with the points in the timecourse.

5 Exercises:

To test your understanding of cobweb diagrams, look over the following cobwebs and decide what you expect to happen on the corresponding timecourse diagram. Try to describe what is happening in as much detail as possible; for example, if you expect a cycle, what is the length of the cycle. If there is exponential decay, what are we decaying to. I don't plan to post answers online, but you can check your answers by talking to your classmates, posting your guesses on Pizza, or emailing me.