Hierarchy 1

"Bounding and Simulating Contextual Correlations in Quantum Theory"

Armin Tavakoli, Emmanuel Zambrini Cruzeiro, Roope Uola, Alastair A. Abbott

PRX Quantum 2, 020334 (2021); arXiv:2010.04751

Alastair A. Abbott

Inria Grenoble

SDP Hierarchies for Quantum Correlations

- Outer approximations of sets of quantum correlations using semidefinite programming (SDP)

- **Bell scenarios:** $p(a, b|x, y) = \langle\psi|E_a^x E_b^y|\psi\rangle$
  - Navascues-Pironio-Acin (NPA) hierarchy [NJP 2008]

- **Prepare-and-measure scenarios:**
  - Dimension bounded correlations [Navascues, Vertesi, PRL 2015]
  - Bounded overlap between states [Y. Wang *et al.*, npjQI 2019]

$$\left.\begin{array}{l} \text{Dimension bounded correlations} \\ \text{Bounded overlap between states} \end{array}\right\} \quad p(b|x, y) = \langle\psi_x|E_b^y|\psi_x\rangle = \mathrm{Tr}(E_b^y\,|\psi_x\rangle\langle\psi_x|)$$

SDP Hierarchies for Quantum Correlations

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- **Bell scenarios:** $p(a,b|x,y) = \langle\psi|E_a^x E_b^y|\psi\rangle$
  - Navascues-Pironio-Acin (NPA) hierarchy [NJP 2008]

- **Prepare-and-measure scenarios:**
  - Dimension bounded correlations [Navascues, Vertesi, PRL 2015]
  - Bounded overlap between states [Y. Wang *et al.*, npjQI 2019]
  - Contextuality?

$$\left.\begin{array}{l}\text{Dimension bounded correlations}\\ \text{Bounded overlap between states}\end{array}\right\} \quad p(b|x,y) = \langle\psi_x|E_b^y|\psi_x\rangle = \mathrm{Tr}(E_b^y\,|\psi_x\rangle\langle\psi_x|)$$

- **Difficulties:**
  - How to enforce operational equivalences?
  - Mixed states and POVMs? (Standard purification arguments don't go through)

Preparation Noncontextuality & Zero-Information Games

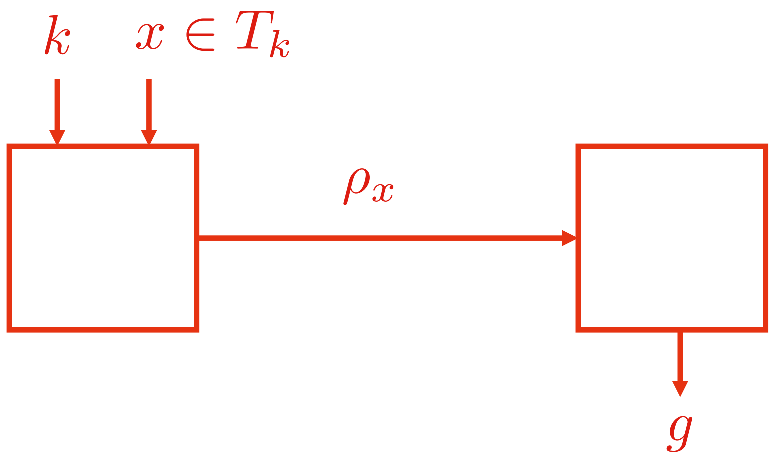
- Focus on preparation noncontextuality for simplicity

$$\frac{1}{2}(\rho_0 + \rho_1) = \frac{1}{2}(\rho_2 + \rho_3) = \frac{1}{2}(\rho_4 + \rho_5)$$

$$T_1 = \{\rho_0, \rho_1\}$$

$$T_2 = \{\rho_2, \rho_3\}$$

$$T_3 = \{\rho_4, \rho_5\}$$



operational equivalences satisfied $\iff \max p_{\text{guess}} = \max p(g = k) = \frac{1}{3}$

$\iff I_{\text{accessible}} = \log_2 3 + \log_2 \max p_{\text{guess}} = 0$

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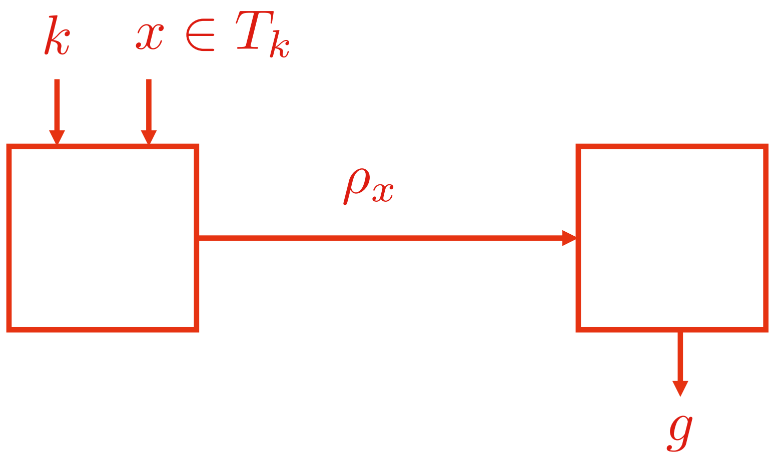

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- Adapt results from "informationally-restricted correlations" to handle OEs
  - [A. Tavakoli *et al.*, Quantum (2020)]
  - SDP hieararchy: [A. Tavakoli, E. Zambrini Cruzeiro, E. Woodhead, S. Pironio, arXiv:2007.16145]

SDP Hierarchy for Preparation Noncontextuality

Given a distribution $p(b|x,y)$ are there quantum states $\{\rho_x\}$ and POVMs $\{E_b^y\}$ reproducing the statistics $p(b|x,y) = \mathrm{Tr}(E_b^y \rho_x)$ and satisfying the operational equivalences?

Mixed states
Unbounded dimension

Projective measurements
(preparation NC)

SDP Hierarchy for Preparation Noncontextuality

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- Moment matrix $\Gamma : \Gamma_{ij} = \mathrm{Tr}(O_i^\dagger O_j)$ $\qquad O_i, O_j \in \{\mathbb{1}, \rho_x, E_b^y, \sigma, \rho_x \rho_{x'}, \rho_x E_b^y, \dots\}$

PSD operators used to enforce zero-information constraints (OEs)

	$\mathbb{1}$	ρ_0	ρ_1	E_0^0	E_0^1	σ	$\rho_0 E_0^0$	$\cdots$
$\mathbb{1}$								
ρ_0								
ρ_1								
E_0^0								
E_0^1								

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$$\begin{array}{c|cccccccc}
 & \mathbb{1} & \rho_0 & \rho_1 & E_0^0 & E_0^1 & \sigma & \rho_0 E_0^0 & \cdots \\
\hline
\mathbb{1} & \mathrm{Tr}(\mathbb{1}) & \mathrm{Tr}(\rho_0) & \mathrm{Tr}(\rho_1) & & & & & \\
\rho_0 & & & & & & & & \\
\rho_1 & & & & & & & & \\
E_0^0 & & & & & & & & \\
E_0^1 & & & & & & & &
\end{array}$$

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	$\mathbb{1}$	ρ_0	ρ_1	E_0^0	E_0^1	σ	$\rho_0 E_0^0$	$\cdots$
$\mathbb{1}$	$\mathrm{Tr}(\mathbb{1})$	1	1					
ρ_0								
ρ_1								
E_0^0								
E_0^1								

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$\mathbb{1}$	$\mathrm{Tr}(\mathbb{1})$	1	1				$\mathrm{Tr}(\rho_0 E_0^0)$	
ρ_0				$\mathrm{Tr}(\rho_0 E_0^0)$	$\mathrm{Tr}(\rho_0 E_0^1)$			
ρ_1				$\mathrm{Tr}(\rho_1 E_0^0)$	$\mathrm{Tr}(\rho_1 E_0^1)$			
E_0^0							$\mathrm{Tr}(E_0^0 \rho_0 E_0^0)$	
E_0^1								

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ρ_0				$p(0\|0,0)$	$p(0\|0,1)$			
ρ_1				$p(0\|1,0)$	$p(0\|1,1)$			
E_0^0							$p(0\|0,0)$	
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$\mathbb{1}$	$\mathrm{Tr}(\mathbb{1})$	1	1	$\mathrm{Tr}(E_0^0)$	$\mathrm{Tr}(E_0^1)$	$\mathrm{Tr}(\sigma)$	$p(0|0,0)$	
ρ_0		$\mathrm{Tr}(\rho_0^2)$		$p(0|0,0)$	$p(0|0,1)$			
ρ_1			$\mathrm{Tr}(\rho_1^2)$	$p(0|1,0)$	$p(0|1,1)$			
E_0^0				$\mathrm{Tr}(E_0^0)$			$p(0|0,0)$	
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	$\mathbb{1}$	ρ_0	ρ_1	E_0^0	E_0^1	σ	$\rho_0 E_0^0$	$\cdots$
$\mathbb{1}$	$\mathrm{Tr}(\mathbb{1})$	1	1	$\mathrm{Tr}(E_0^0)$	$\mathrm{Tr}(E_0^1)$	$\mathrm{Tr}(\sigma)$	$p(0|0,0)$	
ρ_0		$\mathrm{Tr}(\rho_0^2)$		$p(0|0,0)$	$p(0|0,1)$			
ρ_1			$\mathrm{Tr}(\rho_1^2)$	$p(0|1,0)$	$p(0|1,1)$			
E_0^0				$\mathrm{Tr}(E_0^0)$			$p(0|0,0)$	
E_0^1					$\mathrm{Tr}(E_0^1)$			

Necessary Conditions

$$\Gamma \succeq 0$$
$$\Gamma_{\mathbb{1},\rho_x} = 1$$
$$\Gamma_{\rho_x, E_b^y} = p(b|x,y)$$
$$\vdots$$

- How to enforce the operational equivalences, positivity $(\rho_x \succeq 0)$?

Localising Matrices for Noncontextuality

- State positivity: $\Upsilon^x : (\Upsilon^x)_{ij} = \mathrm{Tr}(L_i^\dagger \rho_x L_j)$

$L_i, L_j \in \{\mathbb{1}, E_b^y, \rho_x, \dots\}$

$$
\begin{array}{c|cccc}
 & \mathbb{1} & E_0^0 & E_0^1 & \cdots \\
\hline
\mathbb{1} & 1 & p(0|x,0) & p(0|x,1) & \\
E_0^0 & & p(0|x,0) & \mathrm{Tr}(\rho_x E_0^1 E_0^0) & \\
E_0^1 & & & p(0|x,1) & \\
\vdots & & & &
\end{array}
\succeq 0
$$

$$
\begin{array}{c|cccccccc}
 & \mathbb{1} & \rho_0 & \rho_1 & E_0^0 & E_0^1 & \sigma & \rho_0 E_0^1 & \cdots \\
\hline
\mathbb{1} & \mathrm{Tr}(\mathbb{1}) & 1 & 1 & \mathrm{Tr}(E_0^0) & \mathrm{Tr}(E_0^1) & \mathrm{Tr}(\sigma) & p(0|0,1) & \\
\rho_0 & & \mathrm{Tr}(\rho_0^2) & & p(0|0,0) & p(0|0,1) & & & \\
\rho_1 & & & \mathrm{Tr}(\rho_1^2) & p(0|1,0) & p(0|1,1) & & & \\
E_0^0 & & & & \mathrm{Tr}(E_0^0) & & & \mathrm{Tr}(\rho_0 E_0^1 E_0^0) & \\
E_0^1 & & & & & \mathrm{Tr}(E_0^1) & & &
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E_0^0 & & p(0|x,0) & \mathrm{Tr}(\rho_x E_0^1 E_0^0) & \\
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\vdots & & & &
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\succeq 0
$$

$$
\begin{array}{c|cccccccc}
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\rho_0 & & \mathrm{Tr}(\rho_0^2) & & p(0|0,0) & p(0|0,1) & & & \\
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$$

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$$
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\vdots & & & &
\end{array}
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$$

$$
\begin{array}{c|cccccccc}
 & \mathbb{1} & \rho_0 & \rho_1 & E_0^0 & E_0^1 & \sigma & \rho_0 E_0^1 & \cdots \\
\hline
\mathbb{1} & \mathrm{Tr}(\mathbb{1}) & 1 & 1 & \mathrm{Tr}\left(E_0^0\right) & \mathrm{Tr}\left(E_0^1\right) & \frac{1}{3} & p(0|0,1) & \\
\rho_0 & & \mathrm{Tr}\left(\rho_0^2\right) & & p(0|0,0) & p(0|0,1) & & & \\
\rho_1 & & & \mathrm{Tr}\left(\rho_1^2\right) & p(0|1,0) & p(0|1,1) & & & \\
E_0^0 & & & & \mathrm{Tr}\left(E_0^0\right) & & & \mathrm{Tr}\left(\rho_0 E_0^1 E_0^0\right) & \\
E_0^1 & & & & & \mathrm{Tr}\left(E_0^1\right) & & &
\end{array}
$$

- Operational equivalences:

$$\frac{1}{2}(\rho_0 + \rho_1) = \frac{1}{2}(\rho_2 + \rho_3) = \frac{1}{2}(\rho_4 + \rho_5)$$

$$T_1 = \{\rho_0, \rho_1\},\ T_2 = \{\rho_2, \rho_3\},\ T_3 = \{\rho_4, \rho_5\}$$

$$
\exists \sigma : \begin{cases}
\mathrm{Tr}(\sigma) = \frac{1}{3}\ (= \max p_{\text{guess}}) \\
\sigma = \frac{1}{3}\left[\frac{1}{2}(\rho_0 + \rho_1)\right] \\
\sigma = \frac{1}{3}\left[\frac{1}{2}(\rho_2 + \rho_3)\right] \\
\sigma = \frac{1}{3}\left[\frac{1}{2}(\rho_4 + \rho_5)\right]
\end{cases}
$$

$$\Lambda^k : (\Lambda^k)_{ij} = \mathrm{Tr}\left(L_i^\dagger \left(\sigma - \frac{1}{3} \sum_{x \in T_k} \tfrac{1}{2} \rho_x\right) L_j\right)$$

$$\Lambda^k = 0$$

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$L_i, L_j \in \{\mathbb{1}, E_b^y, \rho_x, \ldots\}$

	$\mathbb{1}$	E_0^0	E_0^1	$\cdots$
$\mathbb{1}$	1	$p(0\|x,0)$	$p(0\|x,1)$	
E_0^0		$p(0\|x,0)$	$\mathrm{Tr}(\rho_x E_0^1 E_0^0)$	
E_0^1			$p(0\|x,1)$	
$\vdots$				

$\succeq 0$

	$\mathbb{1}$	ρ_0	ρ_1	E_0^0	E_0^1	σ	$\rho_0 E_0^1$	$\cdots$
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ρ_1			$\mathrm{Tr}(\rho_1^2)$	$p(0\|1,0)$	$p(0\|1,1)$			
E_0^0				$\mathrm{Tr}(E_0^0)$			$\mathrm{Tr}(\rho_0 E_0^1 E_0^0)$	
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$$\exists \sigma : \begin{cases} \mathrm{Tr}(\sigma) = \frac{1}{3} \ (= \max p_{\text{guess}}) \\ \sigma \succeq \frac{1}{3}\left[\frac{1}{2}(\rho_0 + \rho_1)\right] \\ \sigma \succeq \frac{1}{3}\left[\frac{1}{2}(\rho_2 + \rho_3)\right] \\ \sigma \succeq \frac{1}{3}\left[\frac{1}{2}(\rho_4 + \rho_5)\right] \end{cases}$$

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$$\Lambda^k \succeq 0$$

Necessary Conditions

$$\Gamma \succeq 0$$
$$\Upsilon^x \succeq 0$$
$$\Lambda^k \succeq 0$$
$$\Gamma_{\mathbb{1},\rho_x} = 1$$
$$\Gamma_{\rho_x, E_b^y} = p(b|x,y)$$
$$\Gamma_{\mathbb{1},\sigma} = \frac{1}{K}$$

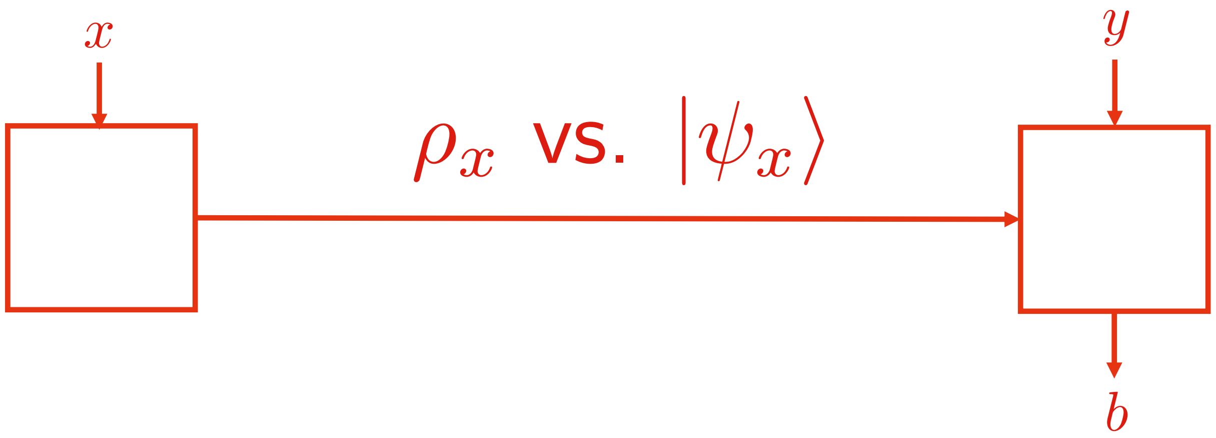
Applications

- Tight upper bounds for several noncontextuality inequalities
- Implementation freely available on github

See paper!

Are pure states as useful as mixed states in contextuality scenarios?

➢ Purifications may not satisfy preparation equivalences

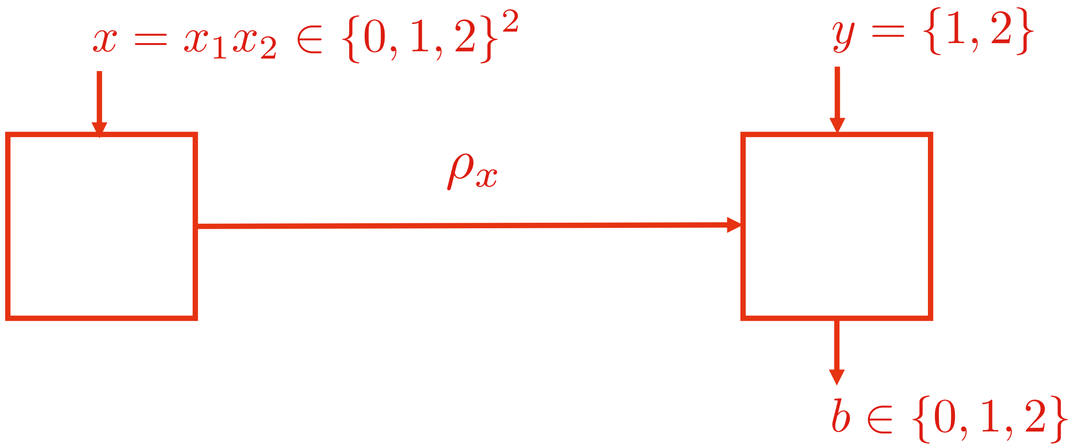


Pure hierarchy:

- Simplify: $(\rho_x)^2 = \rho_x$
- No need for positivity localising matrices (Υ^x): $(\rho_x)^2 = \rho_x \implies \rho_x \succeq 0$

Contextual Advantage from Mixed States

- Hameedi-Tavakoli-Marques-Bourennane inequality [PRL 2017]



RAC-based communication game:

$$p_{\text{win}} = \frac{1}{18} \sum_{x,y} p(b = x_y | x, y)$$

Operational equivalences:

$$\frac{1}{3}(\rho_{00} + \rho_{12} + \rho_{21}) = \frac{1}{3}(\rho_{01} + \rho_{10} + \rho_{22}) = \frac{1}{3}(\rho_{02} + \rho_{20} + \rho_{11})$$

$$\frac{1}{3}(\rho_{00} + \rho_{11} + \rho_{22}) = \frac{1}{3}(\rho_{02} + \rho_{10} + \rho_{21}) = \frac{1}{3}(\rho_{01} + \rho_{12} + \rho_{20})$$

Noncontextuality inequality: $p_{\text{win}} \overset{\text{NC}}{\leq} 2/3$

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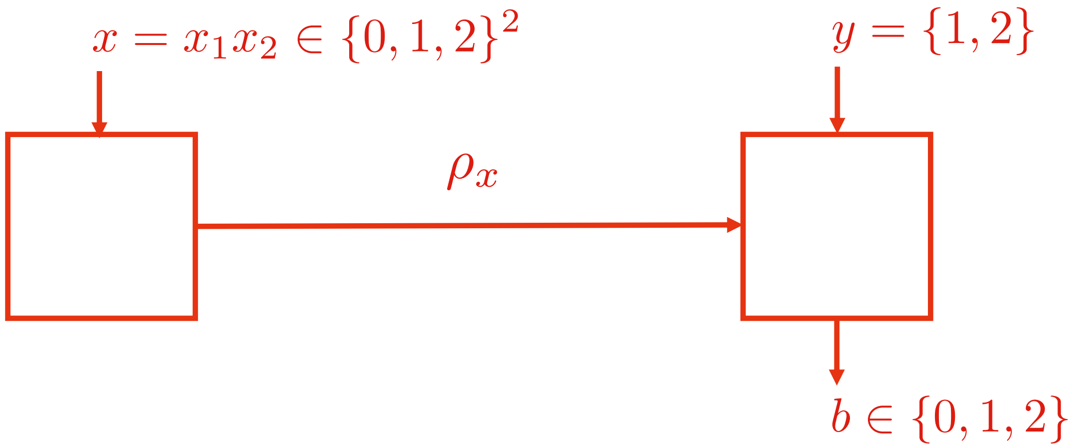

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Noncontextuality inequality: $p_{\text{win}} \overset{\text{NC}}{\leq} 2/3$

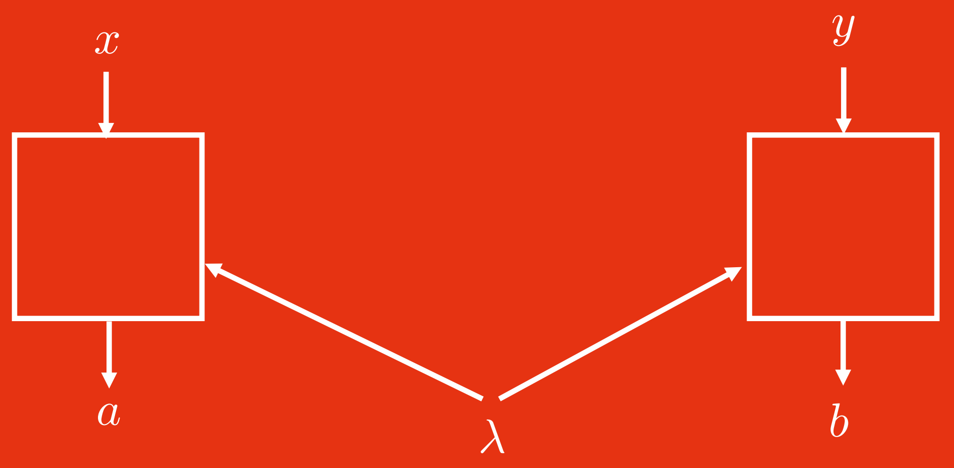
- SDP hierarchy: No contextual advantage from pure states! $p_{\text{win}} \overset{\Psi}{\lesssim} 2/3$
- Mixed states:
  - Achievable (d=6): $p_{\text{win}} \overset{Q}{\approx} 0.6979$
  - SDP hierarchy: $p_{\text{win}} \overset{Q}{\lesssim} 0.704$

Simulating Contextuality

What resources are needed to simulate contextual correlations (maybe post-quantum ones)

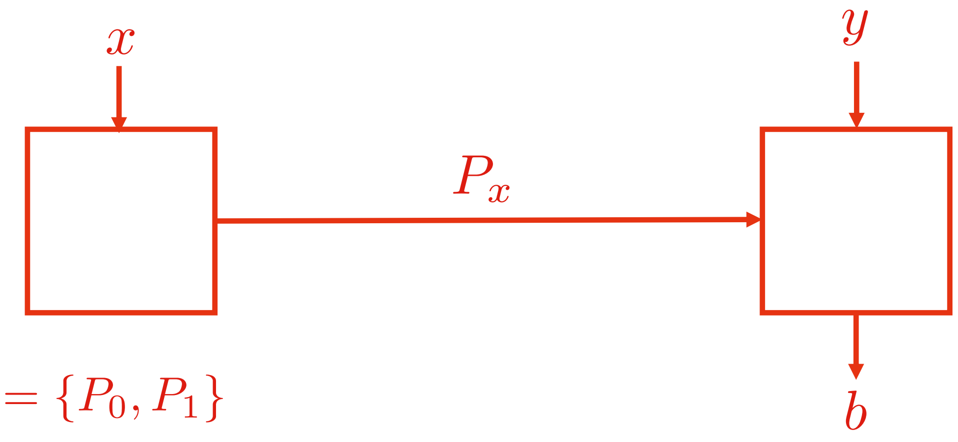
1. with a noncontextual, classical model?
2. with a quantum model?

Bell Nonlocality



[Toner & Bacon, PRL 2003]

Contextuality



$$T_1 = \{P_0, P_1\}$$
$$T_2 = \{P_2, P_3\}$$
$$P_0 + P_1 \simeq P_2 + P_3$$

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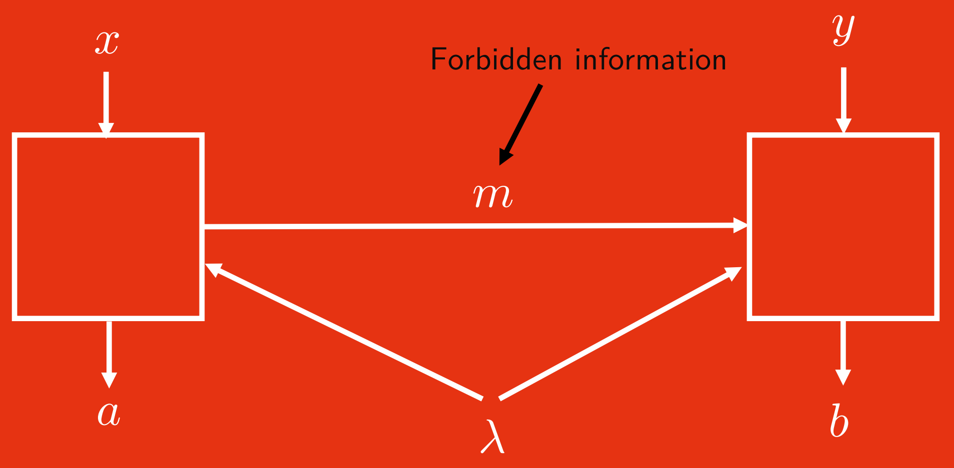
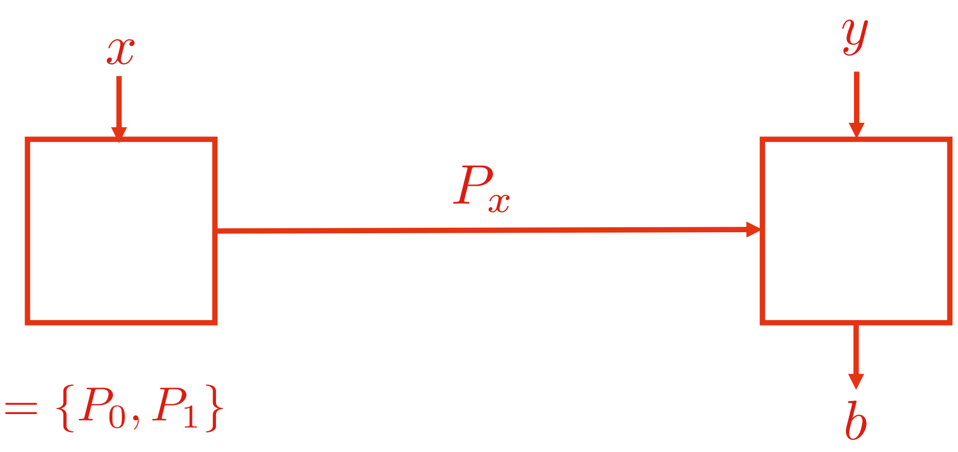
2. with a quantum model?

Bell Nonlocality



[Toner & Bacon, PRL 2003]

Contextuality



$T_1 = \{P_0, P_1\}$

$T_2 = \{P_2, P_3\}$

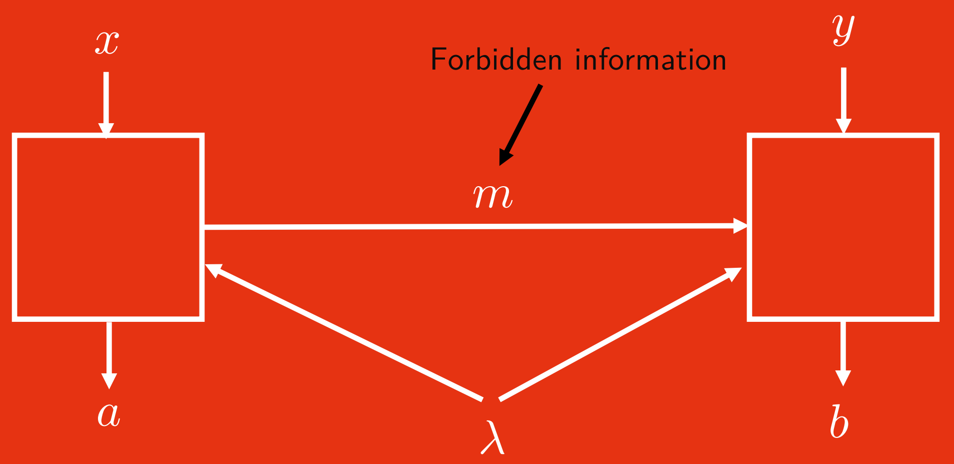
$P_0 + P_1 \simeq P_2 + P_3$

Simulating Contextuality

What resources are needed to simulate contextual correlations (maybe post-quantum ones)

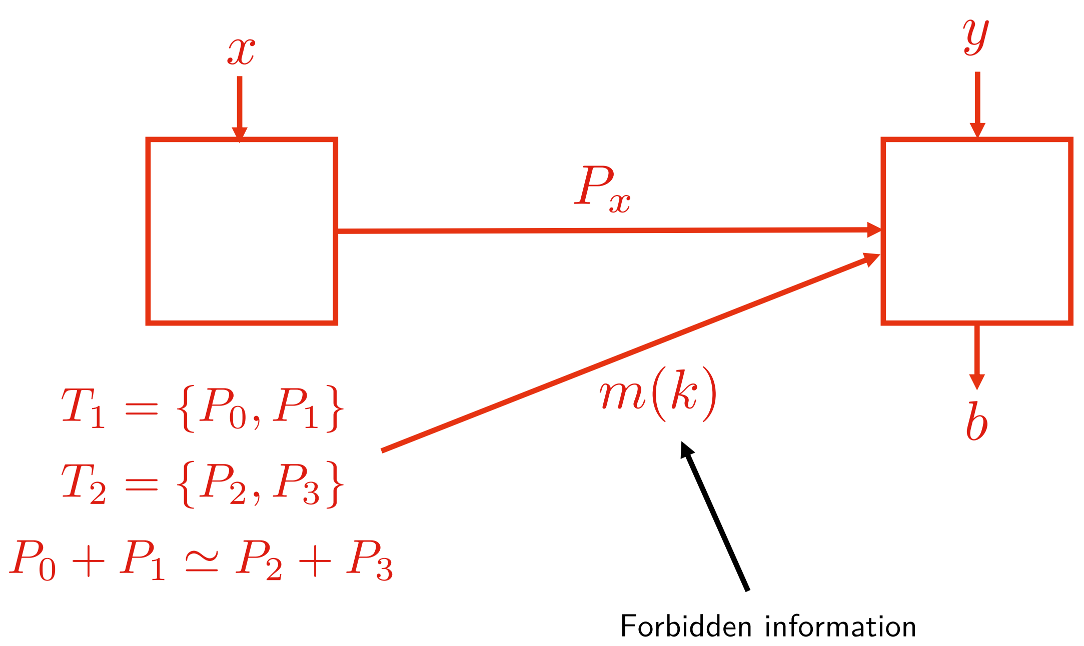
1. with a noncontextual, classical model?
2. with a quantum model?

Bell Nonlocality



[Toner & Bacon, PRL 2003]

Contextuality



Simulation Cost of Preparation Contextuality

Simulation cost: quantifies information about preparation equivalences available to Bob

$$C(p) = \min \mathcal{I}_{\text{accesible}} = \log_2 K + \log_2(\max p_{\text{guess}})$$

min over all models (classical or quantum)
reproducing $p(b|x,y)$

Can lower bound $C(p)$ using SDP hierarchy:

- Rather than imposing $\Gamma_{\mathbb{1},\sigma} = \frac{1}{K}$ we instead minimise $\Gamma_{\mathbb{1},\sigma} = \mathrm{Tr}(\sigma)$ and use

$$p_{\text{guess}} \leq \mathrm{Tr}(\sigma)$$

- Classical models $\approx$ commuting operators

Parity Oblivious Multiplexing