Quantum Circuits with Classical and Quantum Control of Causal Orders

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joint work with
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[arXiv:1807.10557 & arXiv:19??:soon™]

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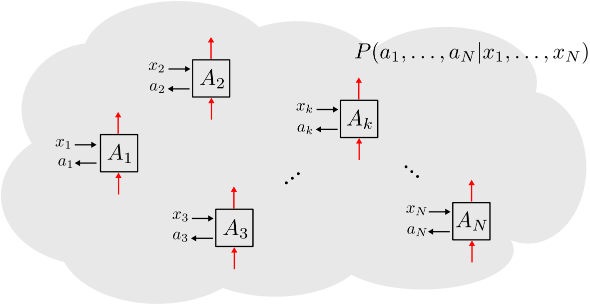
Outline

General Operational Framework



- What constraints does a global causal structure impose on:
  - The correlations $P(a_1, \ldots, a_N | x_1, \ldots, x_N)$?
  - The physical resource generating the correlations?
- Assume "local quantum mechanics":
  - Input/output Hilbert spaces $\mathcal{H}^{A_k^I}$ and $\mathcal{H}^{A_k^O}$
  - Parties perform completely positive maps $\mathcal{M}_{a|x}$

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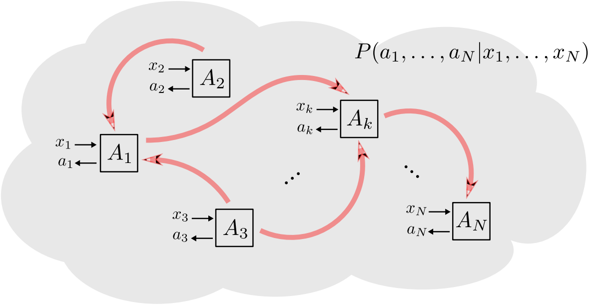

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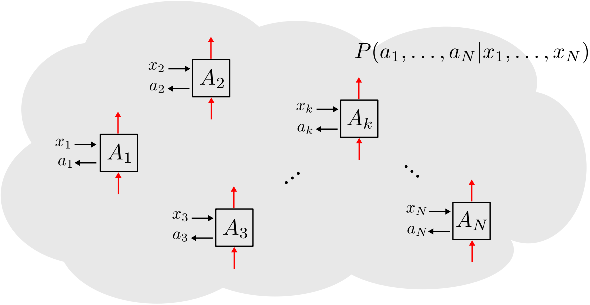

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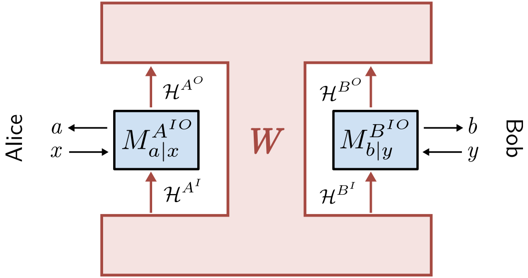
Bipartite Process Matrices



- Alice and Bob's operations: $\mathcal{M}_{a|x}$ and $\mathcal{M}_{b|y}$
  - Represent via CJ isomorphism as PSD matrices $M_{a|x}$ and $M_{b|y}$
  - $\sum_a \mathcal{M}_{a|x}$ is CPTP $\implies Tr_{A^O} \sum_a M_{a|x} = \mathbb{1}^{A^I}$

Correlations can be obtained via the generalised Born rule:

$$P(a,b|x,y) = \mathrm{Tr}\left[(M_{a|x}^T \otimes M_{b|y}^T) \cdot W\right]$$

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

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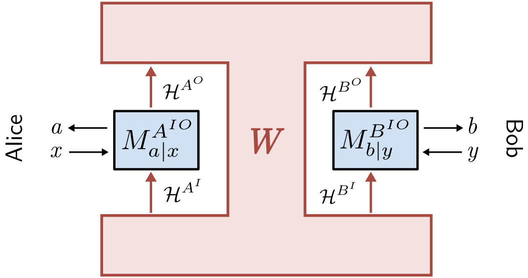


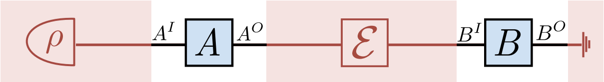
Requiring $P(a,b|x,y)$ to be a valid probability distribution, even when the parties share ancillary states ρ gives:

- Positivity: $W \geq 0$
- Normalisation: $W \in \mathcal{L}^{\{A,B\}}$ and $\mathrm{Tr}\, W = d_{A^O} d_{B^O}$
- $\mathcal{L}^{\{A,B\}}$ is linear subspace of “valid” process matrices

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

Fixed Order Process Matrices

- Some processes are compatible with a fixed causal order
  - Defined in terms of signalling constraints:
    - $A \prec B$ means B cannot signal to A
- E.g. channel: $W^{A\prec B} = \rho^{A^I} \otimes E^{A^O B^I} \otimes \mathbb{1}^{B^O}$



[Chiribella *et al.*, PRA 2009]; [Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

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- $\mathcal{L}^{A\prec B}$: subspace of valid processes compatible with $A \prec B$
- $W^{A\prec B} \in \mathcal{L}^{A\prec B}$ if:
  1. $W^{A\prec B} = \big(\mathrm{Tr}_{B^O}[W^{A\prec B}]\big) \otimes \mathbb{1}^{B^O}$
  2. $\underbrace{\mathrm{Tr}_{B^{IO}} W^{A\prec B}}_{\text{Reduced process for } A} = \big(\mathrm{Tr}_{A^O}[\mathrm{Tr}_{B^{IO}} W^{A\prec B}]\big) \otimes \mathbb{1}^{A^O}$

- Quantum circuit / channel with memory:

[Chiribella *et al.*, PRA 2009]; [Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

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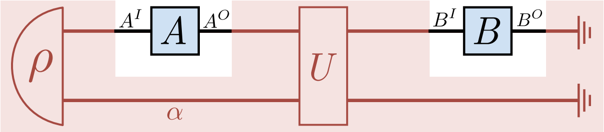

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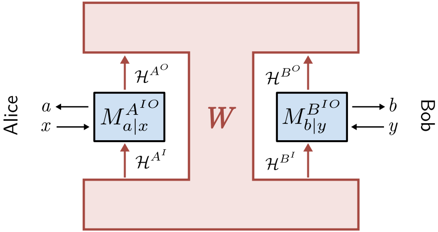
Causally Separable Process Matrices



Causally separable process matrix

$$W^{\text{sep}} = q\, W^{A \prec B} + (1-q)\, W^{B \prec A},$$

(with $W^{A \prec B} \in \mathcal{L}^{A \prec B}$, $W^{B \prec A} \in \mathcal{L}^{B \prec A}$)

- Well defined causal order in every experimental run
- Causally nonseparable process matrices exist
  - Local QM consistent with globally noncausal physics!
- Causal separability can be checked efficiently with SDPs

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

Defining Multipartite Causal Separability

- Process matrix formalism generalises easily to N parties $\mathcal{N} = \{A_1, \ldots, A_N\}$

- Restricted tripartite scenario where C has no outgoing system
  - Only relevant orders are $A \prec B \prec C$ and $B \prec A \prec C$

Restricted Tripartite Causal Separability [Araújo *et al.*, *NJP* 2015]

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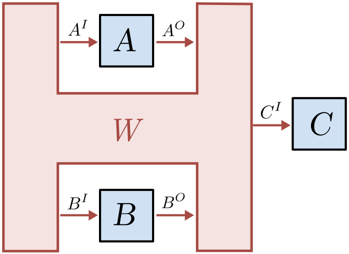

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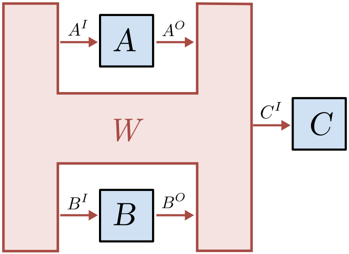
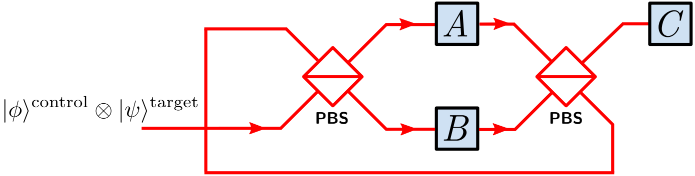


Example: Quantum Switch



- A "pure" 3-partite process matrix: $W^{\text{switch}} = |w\rangle\rangle\langle\langle w|$ with

$$|w\rangle\rangle = |\psi\rangle^{A_t^I} |\mathbb{1}\rangle\rangle^{A_t^O B_t^I} |\mathbb{1}\rangle\rangle^{B_t^O C_t^I} |0\rangle^{C_c^I} + |\psi\rangle^{B_t^I} |\mathbb{1}\rangle\rangle^{B_t^O A_t^I} |\mathbb{1}\rangle\rangle^{A_t^O C_t^I} |1\rangle^{C_c^I}$$

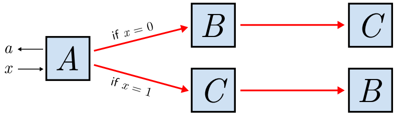
  - Causally non-separable
    - But cannot violate any "causal inequality"
  - Permits advantages: e.g., in query and communication complexities, communication through noisy channels
  - Physically realisable [Procopio *et al.*, Nat. Commun. 2015; Rubino *et al.*, Sci. Adv. 2017; Goswami *et al.*, PRL 2018.]

[Chiribella *et al.*, PRA 2013; Araújo *et al.*, PRL 2014]

Dynamical Causal Orders

In general, a causal process may have:

- Fixed causal orders: $A_{\sigma(1)} \prec \cdots \prec A_{\sigma(N)}$ (σ a permutation of $\{1, \ldots, N\}$)
- But also dynamical orders:



Recursive definition of causal correlations

Multipartite Causal Correlation [Oreshkov & Giarmatzi, NJP 2016; Abbott *et al.*, PRA 2016.]

1. Any single-partite distribution $P(a|x)$ is causal
2. For $N \geq 2$, P causal iff $P(\vec{a}|\vec{x}) = \sum_k q_k P_k(a_k|x_k) \underbrace{P_{k,x_k,a_k}(\vec{a}_{\mathcal{N}\setminus k}|\vec{x}_{\mathcal{N}\setminus k})}_{(N\text{-}1)\text{-partite causal correlation}}$

- Natural approach would be to generalise directly this for process matrices

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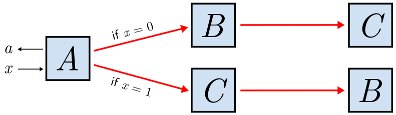

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Oreshkov & Giarmatzi's Definitions

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Idea: in any run, one party acts first and conditioned on their operation, the other parties also causally separable

- Need to define a notion of "conditional process": For a process W, party A_k and CP map M_k applied by A_k:

$$W_{|M_k} := W * M_k = \mathrm{Tr}_{A_k^{IO}}[(M_k^T \otimes \mathbb{1}^{\mathcal{N}\setminus k}) \cdot W]$$

Oreshkov & Giarmatzi's Causal Separability (OG-CS) [NJP 2016]

1. Any single-partite process W is causally separable
2. For $N \geq 2$, W causally separable iff $W = \sum_k q_k \underbrace{W_{(k)}}_{\substack{\text{Valid process compatible with } A_k \prec (\mathcal{N}\setminus A_k), \\ \text{s.t. } \forall M_k \text{ the } (N-1)\text{-partite conditional matrix} \\ W_{|M_k} \text{ is causally separable}}}$

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- Definition natural, but allows "activation of non-causality"
  - $W^{\text{act.}}$ causally separable but $W^{\text{act.}} \otimes \rho$ nonseparable
  - Process matrix framework constructed to allow for shared ancillary systems
  - Entanglement a different kind of resource and generally taken to be free in process matrix formulation
- Doesn't reduce to accepted definition for restricted tripartite scenario (i.e., when C always last)

Multipartite Causal Separability

A more robust definition is the following:

Multipartite Causal Separability [Wechs, AAA, Branciard, NJP 2019]

1. For $N = 1$, any W is causally separable
2. For $N \geq 2$ W is causally separable iff for all extensions $\rho \in A_{\mathcal{N}}^{I'}$

$$W \otimes \rho = \sum_{k \in \mathcal{N}} q_k W_{(k)}^{\rho}, \quad \text{where}$$

   (i) $W_{(k)}^{\rho}$ is a valid process compatible with $A_k \prec (\mathcal{N} \backslash A_k)$

   (ii) For any $M_k \in A_k^{II'O}$, $W_{|M_k}$ is causally separable

- This definition turns out to be equivalent to a notion of "extensible causal separability" defined by Oreshkov & Giarmatzi
  - In particular: the decomposition $\{W_{(k)}^{\rho}\}_k$ can be taken independent of ρ

Tripartite Causal Separability

- How to check if an N-partite W is causally separable?
- Recall bipartite characterisation: $W^{\text{sep}} = W^{A\prec B} + W^{B\prec A}$

Tripartite Causal Separability [equivalent to Oreshkov & Giarmatzi, NJP 2016]

Valid process compatible with A first (up to norm.)

$$\begin{aligned} W^{\text{sep}} &= \overbrace{\widetilde{W}_{(A,B,C)} + \widetilde{W}_{(A,C,B)}}^{W_{(A)}} + \overbrace{\widetilde{W}_{(B,A,C)} + \widetilde{W}_{(B,C,A)}}^{W_{(B)}} + \overbrace{\widetilde{W}_{(C,A,B)} + \widetilde{W}_{(C,B,A)}}^{W_{(C)}} \end{aligned}$$

Not necessarily a valid process

- All terms are positive semidefinite
- $\underbrace{\mathrm{Tr}_{B^{IO}C^{IO}} W_{(A)}}_{\text{Reduced process for } A} = \big(\mathrm{Tr}_{A^O}[\mathrm{Tr}_{B^{IO}C^{IO}} W_{(A)}]\big) \otimes \mathbb{1}^{A^O}$
- $\mathrm{Tr}_{C^{IO}} \widetilde{W}_{(A,B,C)} = \big(\mathrm{Tr}_{B^O}[\mathrm{Tr}_{C^{IO}} \widetilde{W}_{(A,B,C)}]\big) \otimes \mathbb{1}^{B^O}$
- $\widetilde{W}_{(A,B,C)} = (\mathrm{Tr}_{C^O} \widetilde{W}_{(A,B,C)}) \otimes \mathbb{1}^{C^O}$ (+ permutations of these)

- For any M_A, $(\widetilde{W}_{(A,B,C)})_{|M_A}$ *is* a valid process (compatible with $B \prec C$)
- Not a convex mixture of valid processes; allows for dynamical orders

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Example: Fourpartite Quantum Switch

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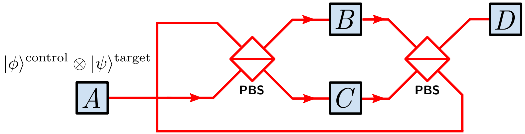

- A "pure" 4-partite process matrix: $W^{\text{switch}} = |w\rangle\rangle\langle\langle w|$ with

$$|w\rangle\rangle = |0\rangle^{A_c^O} |\mathbb{1}\rangle\rangle^{A_t^O B_t^I} |\mathbb{1}\rangle\rangle^{B_t^O C_t^I} |\mathbb{1}\rangle\rangle^{C_t^O D_t^I} |0\rangle^{D_c^I} + |1\rangle^{A_c^O} |\mathbb{1}\rangle\rangle^{A_t^O C_t^I} |\mathbb{1}\rangle\rangle^{C_t^O B_t^I} |\mathbb{1}\rangle\rangle^{B_t^O D_t^I} |1\rangle^{D_c^I}$$

  - Causally non-separable

- Tracing out D it becomes causally separable:

$$\underbrace{|0\rangle\langle 0|^{A_c^O} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{A_t^O B_t^I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B_t^O C_t^I} \otimes \mathbb{1}^{C_t^O}}_{\widetilde{W}_{(A,B,C)} \text{ – not a valid process}} + |1\rangle\langle 1|^{A_c^O} \otimes |\mathbb{1}\rangle\langle\mathbb{1}|^{A_t^O C_t^I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{C_t^O B_t^I} \otimes \mathbb{1}^{B_t^O}$$

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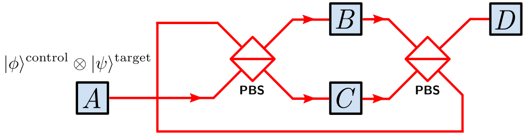

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N-partite Causal Separability

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\end{aligned}
$$

- Can generalise condition to 4 parties and beyond:

Sufficient Condition for Fourpartite Causal Separability

Valid process compatible with A first (up to norm.)

$$
\begin{aligned}
W &= W_{(A)} + W_{(B)} + W_{(C)} + W_{(D)} \\
&= \overbrace{\widetilde{W}_{(A,B)} + \widetilde{W}_{(A,C)} + \widetilde{W}_{(A,D)}}^{W_{(A)}} + \overbrace{\cdots}^{W_{(B)}} + \overbrace{\cdots}^{W_{(C)}} + \overbrace{\cdots}^{W_{(D)}} \\
&= \overbrace{\widetilde{W}_{(A,B,C,D)} + \widetilde{W}_{(A,B,D,C)}}^{\widetilde{W}_{(A,B)}} + \overbrace{\widetilde{W}_{(A,C,B,D)} + \widetilde{W}_{(A,C,D,B)}}^{\widetilde{W}_{(A,C)}} + \overbrace{\widetilde{W}_{(A,D,B,C)} + \widetilde{W}_{(A,D,C,B)}}^{\widetilde{W}_{(A,D)}} + \cdots + \cdots + \cdots
\end{aligned}
$$

For any CP map M_A, $(W_{(A,B)})_{|M_A}$ is valid, compatible with B first

For any CP maps M_A, M_B, $(W_{(A,B,C,D)})_{|M_A \otimes M_B}$ is valid, compatible with C first

- Conditions on terms given by linear constraints with same interpretation as before

N-partite Causal Separability

Tripartite Causal Separability

$$\begin{aligned} W^{\text{sep}} &= \overbrace{\widetilde{W}_{(A,B,C)} + \widetilde{W}_{(A,C,B)}}^{W_{(A)}} + \overbrace{\widetilde{W}_{(B,A,C)} + \widetilde{W}_{(B,C,A)}}^{W_{(B)}} + \overbrace{\widetilde{W}_{(C,A,B)} + \widetilde{W}_{(C,B,A)}}^{W_{(C)}} \end{aligned}$$

- Can generalise condition to 4 parties and beyond:

Sufficient Condition for N-partite Causal Separability

$$W^{\text{sep}} = \sum_{\pi \in \Pi} \widetilde{W}_\pi, \quad \text{with}$$

1. $\widetilde{W}_\pi \geq 0$ for each permutation π of $(1, \ldots, N) \equiv (A_1, \ldots, A_N)$
2. For every ordered subset $(k_1, \ldots, k_n)$ (with $1 \leq n \leq N$),

$$\widetilde{W}_{(k_1,\ldots,k_n)} := \sum_{\pi \in \Pi_{(k_1,\ldots,k_n)}} \widetilde{W}_\pi, \text{ satisfies}$$

$$\mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}} \widetilde{W}_{(k_1,\ldots,k_n)} = \left(\mathrm{Tr}_{A^O_{k_n}} \left[\mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}} \widetilde{W}_{(k_1,\ldots,k_n)}\right]\right) \otimes \mathbb{1}^{A^O_{k_n}}$$

- This constraint is sufficient for W^{sep} to be causally separable, is it necessary?

N-partite Causal Separability

Tripartite Causal Separability

$$
\begin{aligned}
W^{\text{sep}} &= \overbrace{\widetilde{W}_{(A,B,C)} + \widetilde{W}_{(A,C,B)}}^{W_{(A)}} + \overbrace{\widetilde{W}_{(B,A,C)} + \widetilde{W}_{(B,C,A)}}^{W_{(B)}} + \overbrace{\widetilde{W}_{(C,A,B)} + \widetilde{W}_{(C,B,A)}}^{W_{(C)}}
\end{aligned}
$$

- Can generalise condition to 4 parties and beyond:

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$$\widetilde{W}_{(k_1,\ldots,k_n)} := \mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}} \sum_{\pi \in \Pi_{(k_1,\ldots,k_n)}} \widetilde{W}_\pi, \ \text{satisfies}$$

$$\widetilde{W}_{(k_1,\ldots,k_n)} = \left(\mathrm{Tr}_{A^O_{k_n}} \widetilde{W}_{(k_1,\ldots,k_n)} \right) \otimes \mathbb{1}^{A^O_{k_n}}$$

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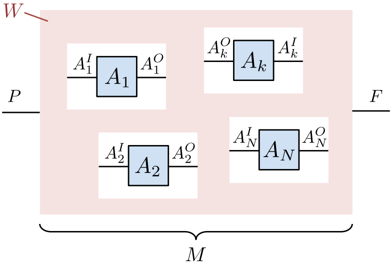
Outline

Process Matrices as Supermaps



- Useful to consider process matrices with global past P and future F
  - Interpret as parties with no input and output spaces, respectively
- Process matrices are supermaps, mapping $(A_1, \ldots, A_N) \to M$

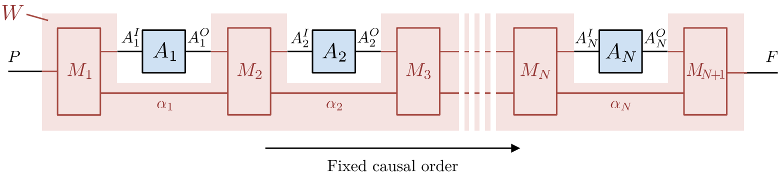
$$M = \mathrm{Tr}_{A_N^{IO}} \left[W (A_1^T \otimes \cdots \otimes A_N^T \otimes \mathbb{1}^{PF}) \right]$$
$$= W * (A_1 \otimes \cdots \otimes A_N) \quad \in \ PF,$$

- Output state given by $M * \rho = W * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$

[Araújo et al., Quantum 2017; Chiribella et al., EPL 2008.]

Quantum Circuits as Fixed-Order Processes

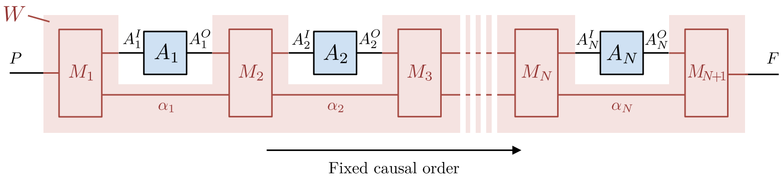
- Standard quantum circuits can be seen as fixed-order processes
  - Equivalently, quantum combs, quantum networks... [e.g. Chiribella *et al.*, PRA 2009]



Most general quantum circuit described by CPTP maps:

- $\mathcal{M}_1 : P \to A_1^I \alpha_1$, where α_1 is an ancillary system
- $\mathcal{M}_{n+1} : A_n^O \alpha_n \to A_{n+1}^I \alpha_{n+1}$ for $1 \leq n \leq N-1$
- $\mathcal{M}_{N+1} : A_N^O \alpha_N \to F$

Quantum Circuits with Fixed Causal Order



For input ρ output is

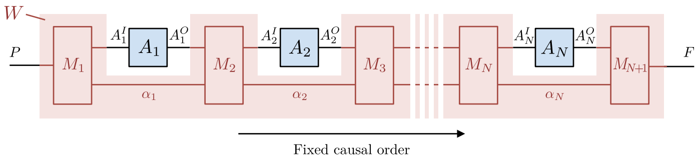
$$M_{N+1} * A_N * \cdots * M_2 * A_1 * M_1 * \rho = \underbrace{(M_1 * M_2 * \cdots * M_{N+1})}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

- W is defined uniquely by the maps M_n via the link product:

$$W = M_1 * M_2 * \cdots * M_{N+1} = \mathrm{Tr}_{\alpha_1 \cdots \alpha_N} \left[M_1 \otimes M_2^{T_{\alpha_1}} \otimes \cdots \otimes M_{N+1}^{T_{\alpha_N}} \right]$$

[Chiribella et al., PRL 2008.]

QC-FO Characterisation



The constraint that the M_n are CPTP maps and thus satisfy $\mathrm{Tr}_{A_{n+1}^I \alpha_{n+1}} M_{n+1} = \mathbb{1}^{A_n^O \alpha_n}$ allow the W of QC-FOs to be characterised

- They are precisely process matrices compatible with $P \prec A_1 \prec \cdots \prec A_N \prec F$

QC-FOs compatible with order $P \prec A_1 \prec \cdots \prec A_N \prec F$

$$\mathrm{Tr}_F\, W = W_{(N)} \otimes \mathbb{1}^{A_N^O},$$

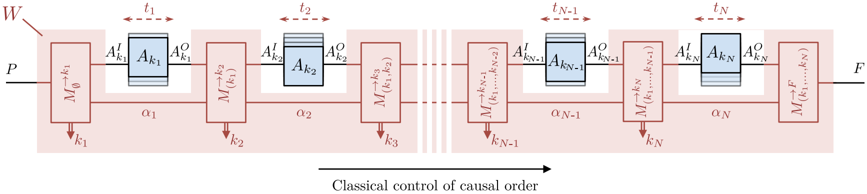
$$\mathrm{Tr}_{A_{n+1}^I}\, W_{(n+1)} = W_{(n)} \otimes \mathbb{1}^{A_n^O} \quad \forall\, n = 1, \ldots, N-1,$$

and $\quad \mathrm{Tr}_{A_1^I}\, W_{(1)} = \mathbb{1}^P.$

where $W_{(n)} := \frac{1}{d_n^O d_{n+1}^O \cdots d_N^O} \mathrm{Tr}_{A_n^O A_{\{n+1,\ldots,N\}}^{IO} F}\, W$ are reduced process matrices

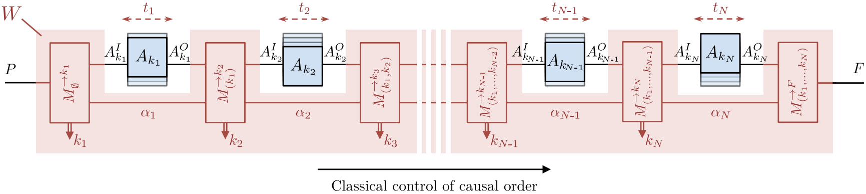
Classically Controlled Circuits

- Causal separability allows for mixed or dynamical causal orders
  - Goes beyond QC-FOs; how can such causal processes be realised?
  - Oreshkov & Giarmatzi [NJP, 2016] suggested causal separability corresponds to quantum circuits with classical control of causal order (QC-CCs): "classically controlled quantum circuits"

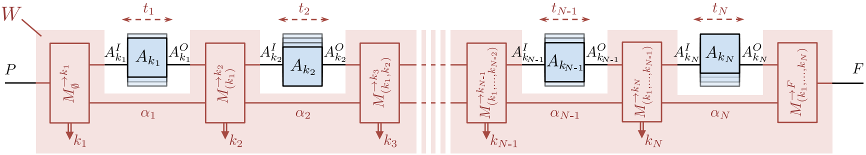


- At each time slot t_n exactly one operation A_{k_n} is applied
- Crucial requirement: each operation applied once and only once, irrespective of the operations themselves
  - Needed to ensure W gives a valid supermap

Classically Controlled Circuits



- Outcome of instrument $\{M^{\to k_{n+1}}_{(k_1,\ldots,k_n)}\}_{k_{n+1}}$ determines the $(n+1)$th party
  - Can depend on previous parties and operations $\to$ allows dynamical causal order
- Technicality: the $M^{\to k_{n+1}}_{(k_1,\ldots,k_n)} \in A^O_{k_n}\alpha_n A^I_{k_{n+1}}\alpha_{n+1}$ belong to different spaces
  - Can solve by embedding in common direct-sum output space



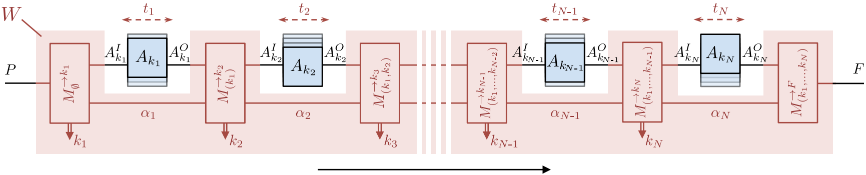
For input ρ, when operations applied in order $k_1, \ldots, k_N$, output is

$$M^{\to F}_{(k_1,\ldots,k_N)} * A_{k_N} * M^{\to k_N}_{(k_1,\ldots,k_{N-1})} * \cdots * M^{\to k_3}_{(k_1,k_2)} * A_{k_2} * M^{\to k_2}_{(k_1)} * A_{k_1} * M^{\to k_1}_{\emptyset} * \rho$$
$$= \underbrace{M^{\to k_1}_{\emptyset} * M^{\to k_2}_{(k_1)} * M^{\to k_3}_{(k_1,k_2)} * \cdots * M^{\to k_N}_{(k_1,\ldots,k_{N-1})} * M^{\to F}_{(k_1,\ldots,k_N)}}_{\widetilde{W}_{(k_1,\ldots,k_N,F)}} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

Process matrix of a QC-CC

$$W = \sum_{(k_1,\ldots,k_N)} \widetilde{W}_{(k_1,\ldots,k_N,F)}$$

QC-CC Characterisation



Classical control of causal order

- Intuitively clear that QC-CCs are causally separable
  - What about the conjectured converse claim?

Characterisation of circuits with classically controlled order

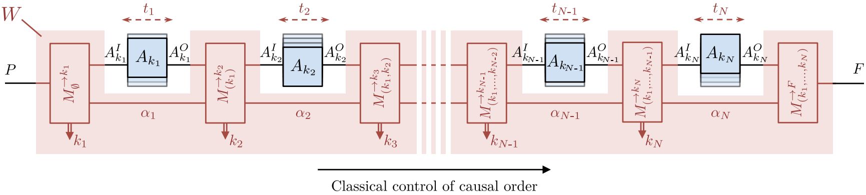
A process W represents a classically controlled circuit iff it satisfies our sufficient conditions for causal separability.

- Can constructively give the QC-CC components from any such W^{sep}

Recall characterisation: $W^{\text{sep}} = \sum_{\pi\in\Pi} \widetilde{W}_\pi$; $\widetilde{W}_{(k_1,\dots,k_n)} = \left(\mathrm{Tr}_{A_O^{k_n}} \widetilde{W}_{(k_1,\dots,k_n)}\right) \otimes \mathbb{1}^{A_O^{k_n}}$,

where $\widetilde{W}_{(k_1,\dots,k_n)} := \mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\dots,k_n\}}} \sum_{\pi\in\Pi_{(k_1,\dots,k_n)}} \widetilde{W}_\pi$

QC-CC Characterisation



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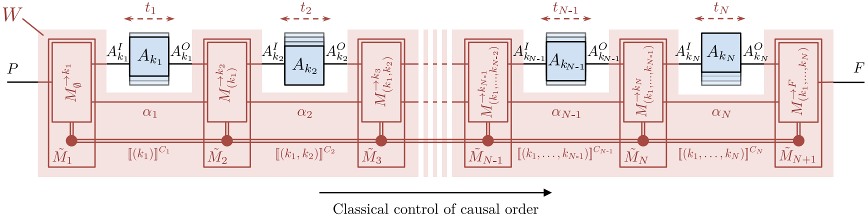
Alternative Descriptions of QC-CCs

- Conditioning can be included in operations by introducing (classical) control system $[\![(k_1,\dots,k_n)]\!]^{C_n} := |(k_1,\dots,k_n)\rangle\langle(k_1,\dots,k_n)|^{C_n}$

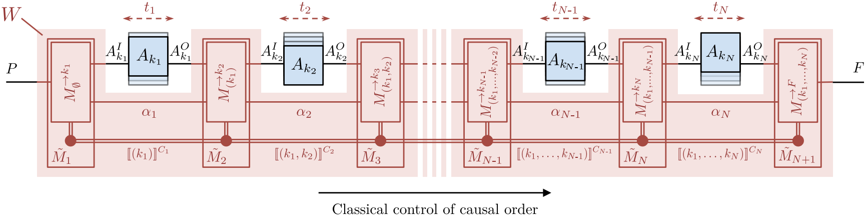


- Operations now given by the CPTP maps

$$\tilde M_{n+1} := \sum_{k_1,\dots,k_n,k_{n+1}} \tilde M_{(k_1,\dots,k_n)}^{\to k_{n+1}} \otimes [\![(k_1,\dots,k_n)]\!]^{C_n} \otimes [\![(k_1,\dots,k_n,k_{n+1})]\!]^{C_{n+1}},$$

$$\tilde M_1 := \sum_{k_1} \tilde M_\emptyset^{\to k_1} \otimes [\![(k_1)]\!]^{C_1}, \qquad M_{N+1} := \sum_{k_1,\dots,k_N} M_{(k_1,\dots,k_N)}^{\to F} \otimes [\![(k_1,\dots,k_N)]\!]^{C_N}$$

Alternative Descriptions of QC-CCs



- Defining global operations $\tilde{A}_n := \bigoplus_{k_n \in \mathcal{N}} A_{k_n}$ we have

$$\tilde{M}_{N+1} * \tilde{A}_N * \tilde{M}_N * \cdots * \tilde{A}_1 * \tilde{M}_1 * \rho = \underbrace{\sum_{k_1,\ldots,k_N} W_{(k_1,\ldots,k_N,F)}}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

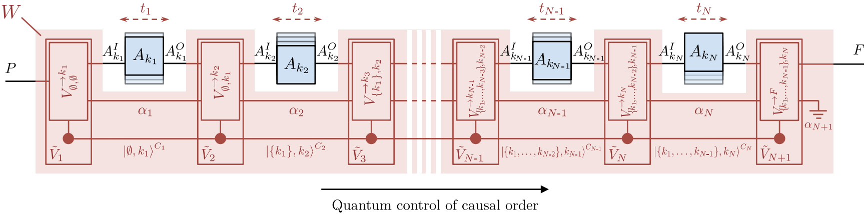
- Note that wlog we can take all operations to be purified isometries

$$M^{\to k_n}_{(k_1,\ldots,k_{n-1})} = |V^{\to k_n}_{(k_1,\ldots,k_{n-1})}\rangle\!\rangle\langle\!\langle V^{\to k_n}_{(k_1,\ldots,k_{n-1})}|$$

- Suggests natural generalisation to quantum control of causal order

From Classical to Coherent Control

- Relax the control state to store only *which* operations have been performed, but not their order: $|\mathcal{K}_{n-1}, k_n\rangle^{C_n}$
  - Conditioning on $\mathcal{K} = \{k_1, \dots, k_{n-1}\}$ allows different orders to interfere
  - Storing full history $|(k_1, \dots, k_n)\rangle^{C_n}$ is more restrictive and included in this case by using ancillas



- Operations given by the isometries V_n with pure CJ representation

$$|\tilde{V}_{n+1}\rangle\rangle := \sum_{\substack{\mathcal{K}_{n-1} \\ k_n, k_{n+1}}} |\tilde{V}^{\to k_{n+1}}_{\mathcal{K}_{n-1}, k_n}\rangle\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C_n} \otimes |\mathcal{K}_n, k_{n+1}\rangle^{C_{n+1}},$$

$$|\tilde{V}_1\rangle\rangle := \sum_{k_1} |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\rangle \otimes |\emptyset, k_1\rangle^{C_1}, \qquad |\tilde{V}_{N+1}\rangle\rangle := \sum_{k_N} |\tilde{V}^{\to F}_{\mathcal{N}\setminus\{k_N\}, k_N}\rangle\rangle \otimes |\mathcal{N}\setminus\{k_N\}, k_N\rangle^{C_N}$$

From Classical to Coherent Control

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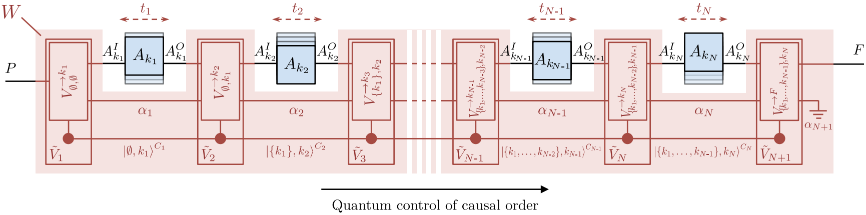

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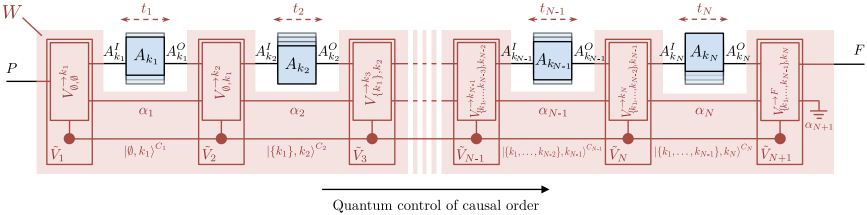
$$|\tilde{V}_1\rangle\!\rangle := \sum_{k_1} |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle \otimes |\emptyset, k_1\rangle^{C_1}, \qquad |\tilde{V}_{N+1}\rangle\!\rangle := \sum_{k_N} |\tilde{V}^{\to F}_{\mathcal{N}\setminus\{k_N\}, k_N}\rangle\!\rangle \otimes |\mathcal{N}\setminus\{k_N\}, k_N\rangle^{C_N}$$

Coherently (Quantum) Controlled Circuits



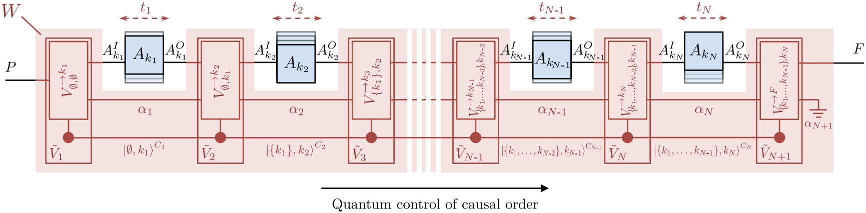
- Each $V^{\to k_{n+1}}_{\mathcal{K}_{n-1},k_n} : \mathcal{H}^{A^O_{k_n}\alpha_n} \to \mathcal{H}^{A^I_{k_{n+1}}\alpha_{n+1}}$ embedded in larger space
  - Control ensures that each party applied once and only once

For input $|\psi\rangle$, circuit applies transformation

$$|\tilde{V}_{N+1}\rangle\rangle * |\tilde{A}_N\rangle\rangle * |\tilde{V}_N\rangle\rangle * \cdots * |\tilde{V}_2\rangle\rangle * |\tilde{A}_1\rangle\rangle * |\tilde{V}_1\rangle\rangle * |\psi\rangle \in \mathcal{H}^{F\alpha_{n+1}}$$

with "pure link product" $|a\rangle^{\mathsf{A}} * |b\rangle^{\mathsf{B}} := \langle\langle \mathbb{1}|^{\mathsf{A}\cap\mathsf{B}} \left(|a\rangle \otimes |b\rangle\right) = \sum_i \langle i,i|^{(\mathsf{A}\cap\mathsf{B})^{\otimes 2}} \left(|a\rangle \otimes |b\rangle\right)$

QCs with Quantum Control of Causal Order



- To identify the process matrix, note that for input $|\psi\rangle$

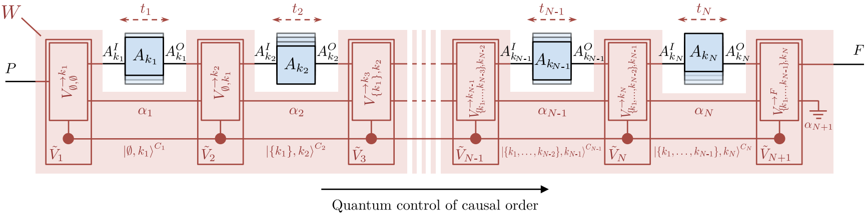
$$|\tilde{V}_{N+1}\rangle\!\rangle * |\tilde{A}_N\rangle\!\rangle * |\tilde{V}_N\rangle\!\rangle * \cdots * |\tilde{V}_2\rangle\!\rangle * |\tilde{A}_1\rangle\!\rangle * |\tilde{V}_1\rangle\!\rangle * |\psi\rangle$$

$$= \sum_{k_1,\ldots,k_N} \underbrace{|V^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle * |V^{\to k_2}_{\emptyset,k_1}\rangle\!\rangle * |V^{\to k_3}_{\{k_1\},k_2}\rangle\!\rangle * \cdots * |V^{\to F}_{\{k_1,\ldots,k_{N-1}\},k_N}\rangle\!\rangle}_{|w_{(k_1,\ldots,k_N,F)}\rangle\!\rangle} * (|\psi\rangle \otimes |A_1\rangle\!\rangle \otimes \cdots \otimes |A_N\rangle\!\rangle)$$

Process matrix of a QC-QC

$$W = \mathrm{Tr}_{\alpha_{N+1}} |w\rangle\!\rangle\langle\!\langle w| \,, \quad \text{with} \quad |w\rangle\!\rangle := \sum_{k_1,\ldots,k_N} |w_{(k_1,\ldots,k_N,F)}\rangle\!\rangle$$

QC-QC Characterisation



- As for QC-CCs, can characterise such W with SDP constraints

Characterisation of circuits with quantum control of causal order

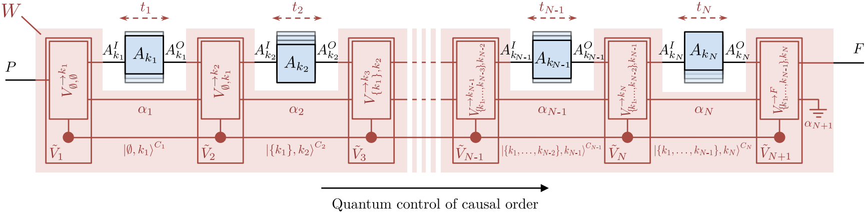
W is the process matrix of a QC-QC iff $\exists$ PSD matrices $W_{(\mathcal{K},\ell)} \in PA^{IO}_{\mathcal{K}} A^I_\ell$ $\forall \mathcal{K} \subsetneq \mathcal{N}, \ell \in \mathcal{N} \setminus \mathcal{K}$ satisfying

$$\mathrm{Tr}_F\, W = \sum_{k\in\mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A^O_k},$$

$$\forall \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}, \ \sum_{\ell\in\mathcal{N}\setminus\mathcal{K}} \mathrm{Tr}_{A^I_\ell} \widetilde{W}_{(\mathcal{K},\ell)} = \sum_{k\in\mathcal{K}} \widetilde{W}_{(\mathcal{K}\setminus\{k\},k)} \otimes \mathbb{1}^{A^O_k},$$

and $$\sum_{\ell\in\mathcal{N}} \mathrm{Tr}_{A^I_\ell} W_{(\emptyset,\ell)} = \mathbb{1}^P.$$

QC-QC Characterisation

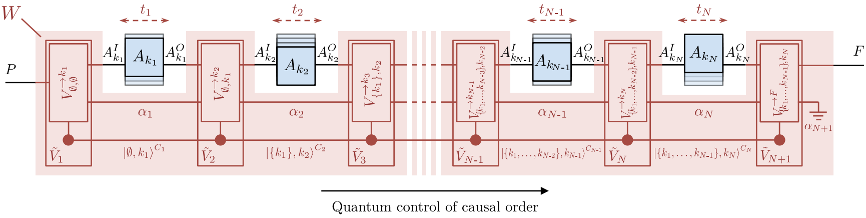


- As for QC-CCs, can characterise such W with SDP constraints

Three-operation QC-QC Characterisation

- $\mathrm{Tr}_F\, W = \widetilde{W}_{(\{B,C\},A)} \otimes \mathbb{1}^{A^O} + \widetilde{W}_{(\{A,C\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{A,B\},C)} \otimes \mathbb{1}^{C^O}$
- $\mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A,B\},C)} = \widetilde{W}_{(\{A\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{B\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{B^I}\, \widetilde{W}_{(\{A\},B)} + \mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A\},C)} = W_{(\{\emptyset\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{A^I}\, W_{(\{\emptyset\},A)} + \mathrm{Tr}_{B^I}\, W_{(\{\emptyset\},B)} + \mathrm{Tr}_{C^I}\, W_{(\{\emptyset\},C)} = \mathbb{1}^P$

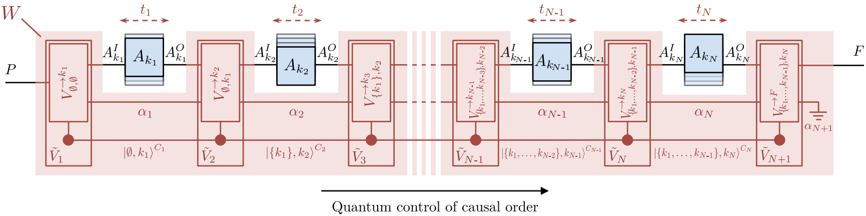
QC-QC Summary



- QC-QCs are physically realisable, e.g., with a "quantum router"
- Realisation – in terms of the $\tilde{V}_n$ – can be effectively obtained from the any W satisfying the characterisation
  - Can be checked and obtained via SDPs, or witnesses obtained

- Classically controlled circuits are recovered as a special case
  - But QC-QCs can be causally nonseparable in general

QC-QC Summary



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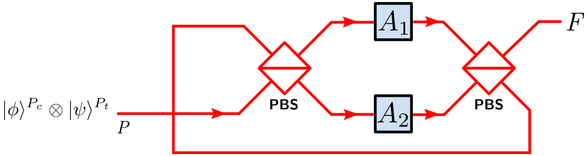
  - But QC-QCs can be causally nonseparable in general

Example: Quantum Switch

$N = 2$, d-dimensional target system and 2-dimensional "control":
$\mathcal{H}^P = \mathcal{H}^{P_t} \otimes \mathcal{H}^{P_c}$ and $\mathcal{H}^F = \mathcal{H}^{F_t} \otimes \mathcal{H}^{F_c}$



The controlled operations are

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_\mathrm{c}} |\mathbb{1}\rangle\rangle^{P_\mathrm{t} A_{k_1}^I}, \quad |V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = |\mathbb{1}\rangle\rangle^{A_{k_1}^O A_{k_2}^I}, \quad |V_{\{k_1\},k_2}^{\to F}\rangle\rangle = |k_1\rangle^{F_\mathrm{c}} |\mathbb{1}\rangle\rangle^{A_{k_2}^O F_\mathrm{t}}$$

Process vector is then

$$\begin{aligned} |w_\mathsf{s}\rangle\rangle &:= |w_{(P,A_1,A_2,F)}\rangle\rangle + |w_{(P,A_2,A_1,F)}\rangle\rangle \\ &= |V_{\emptyset,\emptyset}^{\to A_1}\rangle\rangle * |V_{\emptyset,A_1}^{\to A_2}\rangle\rangle * |V_{\{A_1\},A_2}^{\to F}\rangle\rangle + |V_{\emptyset,\emptyset}^{\to A_2}\rangle\rangle * |V_{\emptyset,A_2}^{\to A_1}\rangle\rangle * |V_{\{A_2\},A_1}^{\to F}\rangle\rangle \\ &= |1\rangle^{P_\mathrm{c}} |\mathbb{1}\rangle\rangle^{P_\mathrm{t} A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O F_\mathrm{t}} |1\rangle^{F_\mathrm{c}} + |2\rangle^{P_\mathrm{c}} |\mathbb{1}\rangle\rangle^{P_\mathrm{t} A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O F_\mathrm{t}} |2\rangle^{F_\mathrm{c}} \end{aligned}$$

Standard four-partite switch recovered as $W_\mathsf{switch} = |w_\mathsf{s}\rangle\rangle\langle\langle w_\mathsf{s}|$

Beyond the Quantum Switch?

- N-partite generalisation of the quantum switch is a QC-QC
  - Essentially the extent of known "interesting" causally nonseparable processes
- Do QC-QCs offer something new, or are they all "equivalent" to the switch?
- Need a better understanding of causally nonseparable resources and free operations
  - Taddei, Nery and Aolita [arXiv:1903.06180]: local operations and controlled non-signalling operations ofr bipartite processes
  - Composition: Possible compositions severely restricted [Guérin et al., NJP 2019], but can, e.g., concatenate switches, or insert them inside other switches

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Beyond the Quantum Switch?

However, recall characterisation of QC-QCs

$$\operatorname{Tr}_F W = \sum_{k\in\mathcal{N}} \widetilde{W}_{(\mathcal{N}\backslash\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

$$\forall \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}, \sum_{\ell\in\mathcal{N}\backslash\mathcal{K}} \operatorname{Tr}_{A_\ell^I} \widetilde{W}_{(\mathcal{K},\ell)} = \sum_{k\in\mathcal{K}} \widetilde{W}_{(\mathcal{K}\backslash\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

and $\displaystyle\sum_{\ell\in\mathcal{N}} \operatorname{Tr}_{A_\ell^I} W_{(\emptyset,\ell)} = \mathbb{1}^P.$

- The $\widetilde{W}_{(\mathcal{N}\backslash\{k\},k)}$ need not, *a priori*, be valid process matrices
  - $\operatorname{Tr}_F W$ not necessarily a mixture of valid process matrices compatible with fixed last parties, i.e.

$$\operatorname{Tr}_F W \neq q_1 W^{\{A_1,A_2\}\prec A_3} + q_2 W^{\{A_1,A_3\}\prec A_2} + q_3 W^{\{A_2,A_3\}\prec A_1}$$

- Seems like such W can't be obtained by composing switches
  - Numerically, such processes seem to exist: further study needed to find (and interpret) nice examples
  - In particular, can one obtain new types of advantages with QC-QCs – which, by construction, are realisable

QC-QCs and Causal Correlations

Can quantum circuits with quantum control of causal order violate causal inequalities?

QC-QC correlations are causal

Let W be a QC-QC with trivial spaces $\mathcal{H}^P$ and $\mathcal{H}^F$. Then the correlations

$$P(a_1, \ldots, a_N | x_1, \ldots, x_N) = \mathrm{Tr}[W \cdot (M_{a_1|x_1}^T \otimes \cdots \otimes M_{a_N|x_N}^T)]$$

are causal for any instruments $\{M_{a_i|x_i}\}_{x_i}$.

- Can noncausal correlations be realised in nature?
  - Would require going beyond this type of generic, coherently controlled circuit
- QC-QCs nevertheless have potential for new advantages arising from indefinite causal order
  - New classes of physically realisable, causally nonseparable processes
  - Use as "quantum super-instruments", generalising quantum testers, for transformation tasks

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Summary & Outlook

- Definition of multipartite causal (non)separability
- Characterisation of causally separable process matrices
  - Separate necessary and sufficient conditions
  - Coincide for $N = 2, 3$; in general?
  - Necessary condition allows construction of witnesses of causal nonseparability
- Quantum circuits with classical control of causal order
  - Coincide with sufficient condition for causal separability
- Quantum circuits with quantum control of causal order
  - Potential new realisable, causally nonseparable, circuits beyond the quantum switch?
  - Do QC-QCs provide new information theoretical advantages?
  - Need for resource theoretical treatment for such processes
  - Are there other classes of physically realisable processes?

[arXiv:1807.10557 + new paper soon]

Choi Isomorphism and Link Product

- $|\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$ is the "pure Choi isomorphism" of an identity channel
- Pure Choi isomorphism: for an operator A, $|A\rangle\rangle = \mathbb{1} \otimes A\,|\mathbb{1}\rangle\rangle$
- Mixed Choi isomorphism: for a CP map $\mathcal{M}$, $M = \mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)$
- Inverse Choi isomorphism given by the link product: $\mathcal{M}(\rho) = M * \rho$; $A\,|\psi\rangle = |A\rangle\rangle * |\psi\rangle$

Constraints for Process Matrix Validity

Recall the notation:

$$ {}_X W := (\mathrm{Tr}_X W) \otimes \frac{\mathbb{1}^X}{d_X}, \quad {}_1 W := W, \quad {}_{[\sum_X \alpha_X X]} W := \sum_X \alpha_X \, {}_X W, $$

Space of valid process matrices

$$ W \in \mathcal{L}^{\mathcal{N}} \;\Leftrightarrow\; \forall\, \chi \subseteq \mathcal{N}, \chi \neq \emptyset, \; {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash\chi}} W = 0, $$

Space of valid process compatible with A first

$$ W \in \mathcal{L}^{A_k \prec (\mathcal{N}\backslash A_k)} $$

$$ \Leftrightarrow \; {}_{[1-A_O^k]A_{IO}^{\mathcal{N}\backslash k}} W = 0 \quad \text{and} \quad \forall\, \chi \subseteq \mathcal{N}\backslash k, \chi \neq 0, \, {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash k\backslash\chi}} W = 0, $$

Causal Separability: Necessary Conditions

- Explicit necessary conditions can be obtained by choosing specific CP maps and ancillas at each level of the recursive definition
- Ognyan and Giarmatzi showed how such a choice proves sufficient conditions also necessary in tripartite case
  1. ρ: maximally entangled state for each pair of parties
  2. M_{A_k}: $|\Phi^+\rangle\!\langle\Phi^+|$ – M.E.S. between A^{IO} and half of ancilla between A_k and some $A_{k'}$
- "Teleports" A_k's system on A_k^{IO} to $A_{k'}^{I'}$

$$\underbrace{W_{(k)}^{\rho}}_{N\text{-partite, } A_k \text{ first}} \longrightarrow \underbrace{W_{(k)}^{A_k^{IO}\to A_{k'}^{I'}}}_{(N-1)\text{-partite, formally equivalent to } W_{(k)}} \otimes \rho' := (W_{(k)}^{\rho})_{|M_{A_k}}$$

- Any constraints obeyed by $W_{(k)}^{A_k^{IO}\to A_{k'}^{I'}}$ must be obeyed by $W_{(k)}$ once Hilbert spaces relabelled
  - Can repeat for each $k' \neq k$

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Necessary Condition for Causal Separability

Necessary condition for N-partite causal separability

An N-partite $W^{\text{sep}} \in \mathcal{W}^{\text{sep}}$ must have a decomposition $W = \sum_{k\in\mathcal{N}} W_{(k)}$ where:

1. $W_{(k)}$ is a valid process compatible with $A_k \prec (\mathcal{N}\backslash A_k)$
2. For each $k' \neq k$, $W_{(k)}^{A_k^{IO} \to A_{k'}^{I'}}$ is an $(N-1)$-partite causally separable process
   - $\implies$ obeys the necessary conditions for $(N-1)$-partite processes

- Coincides with separable condition for $N = 3$ [Oreshkov & Giarmatzi, NJP 2016]
- Also reduced 4-partite scenario (no output for D, c.f. quantum switch)
- Note that decomposition may differ for each k'
  - Satisfying these conditions with a unique decomposition would imply the sufficient conditions
- No known counterexample, but unclear whether the conditions coincide in general

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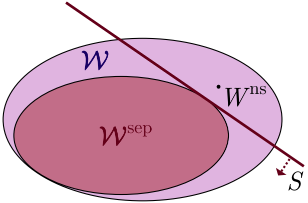
Witnesses of Causal Nonseparability

Causally separable process matrix

$$W^{\mathsf{sep}} = q\, W^{A \prec B} + (1-q)\, W^{B \prec A},$$

- Convex cone of (non-normalised) causally separable processes:

$$\mathcal{W}^{\mathsf{sep}} = (\mathcal{P} \cap \mathcal{L}^{A \prec B}) + (\mathcal{P} \cap \mathcal{L}^{B \prec A})$$



Witness of causal nonseparability

$\forall\, W^{\mathsf{ns}} \notin \mathcal{W}^{\mathsf{sep}},\ \exists S :$

$\mathrm{Tr}[S^T \cdot W^{\mathsf{ns}}] < 0$, and

$Tr[S^T \cdot W^{\mathsf{sep}}] \geq 0 \quad \forall\, W^{\mathsf{sep}} \in \mathcal{W}^{\mathsf{sep}}$

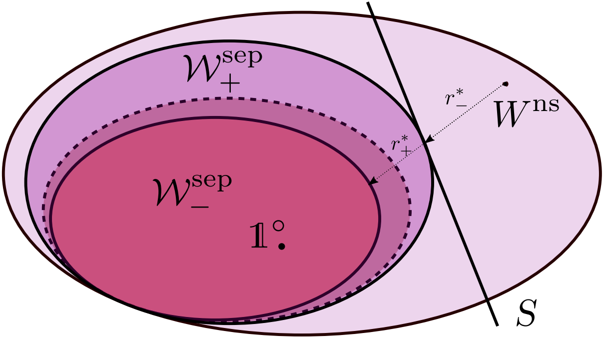
[Araújo *et al.*, NJP 2015; Branciard, Sci. Rep. 2016]

- Witnesses can be efficiently constructed by semidefinite programming (SDP)
- Witnesses can be measured experimentally

[Rubino *et al.*, Sci. Adv. 2017; Goswami *et al.*, PRL 2018]

Witnessing Causal Nonseparability

- Both necessary and sufficient conditions define convex cones $\mathcal{W}_+^{\mathsf{sep}}$, $\mathcal{W}_-^{\mathsf{sep}}$ of (non-normalised) process matrices



- Membership can be tested with SDP
- Dual SDP from necessary condition gives causal witnesses
- So far no numerical evidence that $\mathcal{W}_-^{\mathsf{sep}} \neq \mathcal{W}_+^{\mathsf{sep}}$, but...

Cones $\mathcal{W}^{\text{sep}}$ and $\mathcal{S}$ for tripartite case

Adopt the notation $\mathcal{L}_X = \{W|{}_XW = 0\}$.

$$\begin{aligned}\mathcal{W}^{\text{sep}} = \mathcal{L}_{[1-A_O]B_{IO}C_{IO}} \cap \big(&\mathcal{P} \cap \mathcal{L}_{[1-B_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]B_{IO}} \cap \mathcal{L}_{[1-B_O]}\big) \\ + \mathcal{L}_{[1-B_O]A_{IO}C_{IO}} \cap \big(&\mathcal{P} \cap \mathcal{L}_{[1-A_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]A_{IO}} \cap \mathcal{L}_{[1-A_O]}\big) \\ + \mathcal{L}_{[1-C_O]A_{IO}B_{IO}} \cap \big(&\mathcal{P} \cap \mathcal{L}_{[1-A_O]B_{IO}} \cap \mathcal{L}_{[1-B_O]} + \mathcal{P} \cap \mathcal{L}_{[1-B_O]A_{IO}} \cap \mathcal{L}_{[1-A_O]}\big),\end{aligned}$$

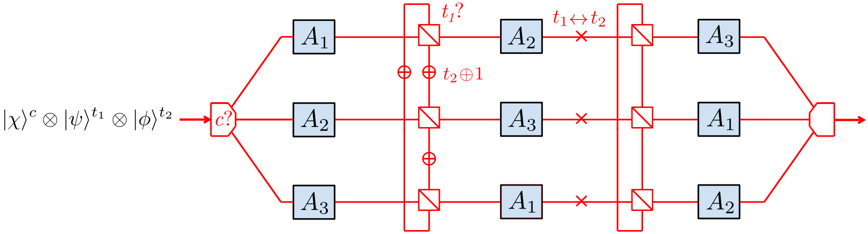
$$\begin{aligned}\mathcal{S} = &\Big(\mathcal{L}^{\perp}_{[1-A_O]B_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-B_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]B_{IO}} + \mathcal{L}^{\perp}_{[1-B_O]})\Big) \\ \cap &\Big(\mathcal{L}^{\perp}_{[1-B_O]A_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-A_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]A_{IO}} + \mathcal{L}^{\perp}_{[1-A_O]})\Big) \\ \cap &\Big(\mathcal{L}^{\perp}_{[1-C_O]A_{IO}B_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-A_O]B_{IO}} + \mathcal{L}^{\perp}_{[1-B_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-B_O]A_{IO}} + \mathcal{L}^{\perp}_{[1-A_O]})\Big).\end{aligned}$$

Example: New type of 3-operation QC-QC

$N = 3$, two qubit "targets" and 3-dimensional "control"

- Initial control state $|k_1\rangle^{P_c}$ determines first party
- First party acts on first target qubit $|\psi\rangle^{P_{t_1}}$
- Output of A_{k_1} determines (dynamically, coherently) k_2 and conditions a flip on second target $|\phi\rangle^{P_{t_2}}$, which is swapped to become "active" target *after* A_{k_2}
- A_{k_3} then acts on this second target qubit

Can be represented in an "unravelled" form as:



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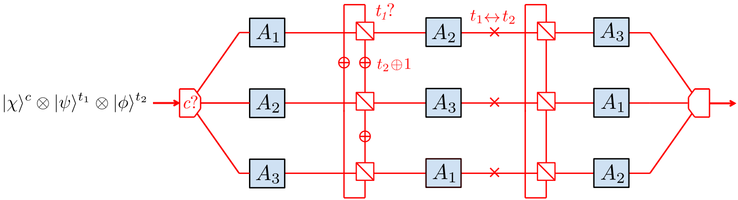

Controlled operations can be written

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_c}\, |\mathbb{1}\rangle\rangle^{P_{t_1} A_{k_1}^I}\, |\mathbb{1}\rangle\rangle^{P_{t_2}\alpha_1}$$

$$|V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = |0\rangle^{A_{k_1}^O}\, |0\rangle^{A_{k_2}^I}\, |\mathbb{1}\rangle\rangle^{\alpha_1\alpha_2}, \text{ and } |V_{\emptyset,k_1}^{\to k_2'}\rangle\rangle = |1\rangle^{A_{k_1}^O}\, |1\rangle^{A_{k_2'}^I}\, |X\rangle\rangle^{\alpha_1\alpha_2} \text{ for } k_2 \neq k_2'$$

$$|V_{\{k_1\},k_2}^{\to k_3}\rangle\rangle = |\mathbb{1}\rangle\rangle^{A_{k_2}^O \alpha_3}\, |\mathbb{1}\rangle\rangle^{\alpha_2 A_{k_3}^I}$$

$$|V_{\{k_1,k_2\},k_3}^{\to F}\rangle\rangle = |k_3\rangle^{F_c}\, |\mathbb{1}\rangle\rangle^{A_{k_3}^O F_{t_2}}\, |\mathbb{1}\rangle\rangle^{\alpha_3 F_{t_1}}$$

giving (with cyclic permutations for $k_1 = 2, 3$)

$$|w_{(P,A_1,A_2,A_3,F)}\rangle\rangle = |1\rangle^{P_c}\, |\mathbb{1}\rangle\rangle^{P_{t_1} A_1^I}\, |00\rangle^{A_1^O A_2^I}\, |\mathbb{1}\rangle\rangle^{A_2^O F_{t_1}}\, |\mathbb{1}\rangle\rangle^{P_{t_2} A_3^I}\, |\mathbb{1}\rangle\rangle^{A_3^O F_{t_2}}\, |3\rangle^{F_c}$$

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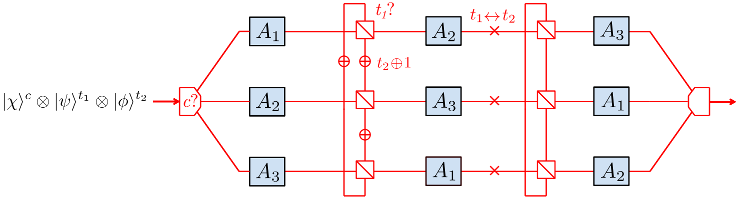

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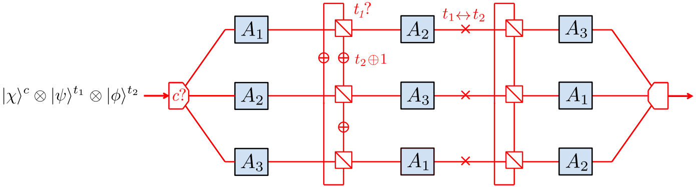

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- W is easily seen to be causally nonseparable
  - Pure process, not compatible with any of A, B or C being first (after P)
- Appears qualitatively different to the quantum switch
  - Two target qubits, one of which is also used to control the causal order
- But how to prove W is fundamentally inequivalent to quantum switch?
  - Could imagine composing switches, using control of one as target for another, etc.
  - Need a more complete resource theory, e.g. generalising Taddei, Nery and Aolita's proposal for bipartite processes [arXiv:1903.06180]

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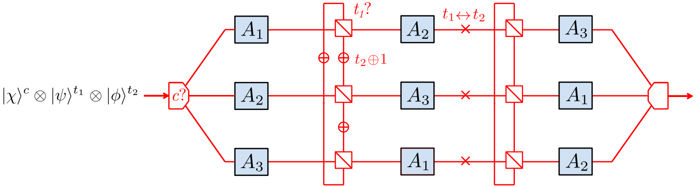

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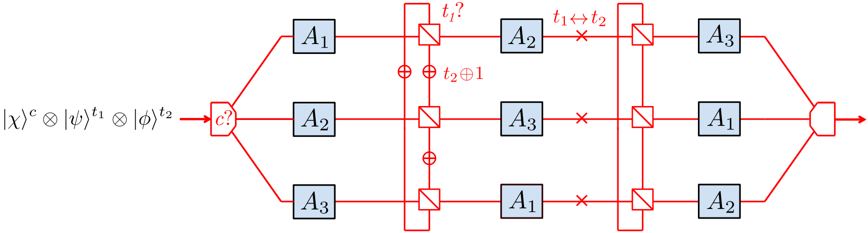

- Crucial difference: $\mathrm{Tr}_F\, W$ is causally nonseparable and cannot be written as a mixture of valid process matrices with fixed last parties, i.e.

$$\mathrm{Tr}_F\, W \neq q_1 W^{\{A_1,A_2\}\prec A_3} + q_2 W^{\{A_1,A_3\}\prec A_2} + q_3 W^{\{A_2,A_3\}\prec A_1}$$

- Recall characterisation: $\mathrm{Tr}_F\, W = \sum_{k\in\mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O}$
  - the $\widetilde{W}_{(\mathcal{N}\setminus\{k\},k)}$ need not be valid process matrices
- Seems like no composition of quantum switches could give this property!