Genuinely Multipartite Noncausality

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joint work with
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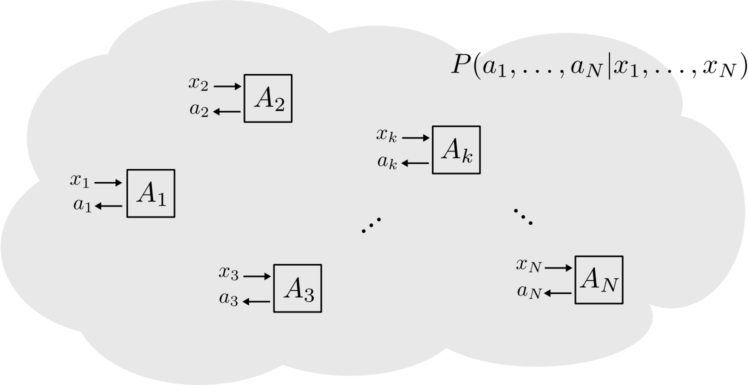
Operational Scenario



- What correlations $P(a_1, \ldots, a_N | x_1, \ldots, x_N)$ can be obtained if the parties interact in a global causal structure?

Oreshkov, Costa & Brukner, *Nat. Commun.* 2012.

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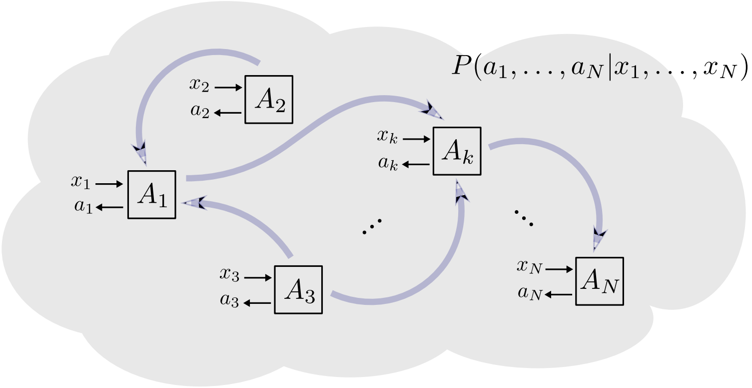

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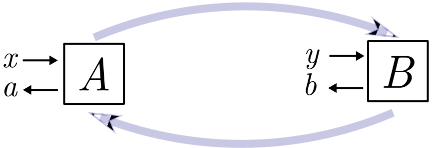
Bipartite Causal Correlations



If we have a fixed causal order, either:

- $A \prec B$: then B can't signal to A: $P^{A\prec B}(a|x,y) = P^{A\prec B}(a|x)$
    - $P^{A\prec B}(a,b|x,y) = P^{A\prec B}(a|x)P^{A\prec B}(b|y,x,a)$
- $B \prec A$: analogously, $P^{B\prec A}(b|x,y) = P^{B\prec A}(b|y)$
- Both $A \prec B$ and $B \prec A \implies$ no-signalling

Bipartite Causal Correlations

Compatible with *some* (perhaps probabilistic) causal order:

$$P^{\text{causal}}(a,b|x,y) = q\,P^{A\prec B}(a,b|x,y) + (1-q)\,P^{B\prec A}(a,b|x,y)$$

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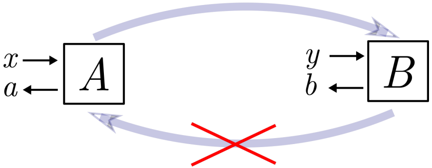

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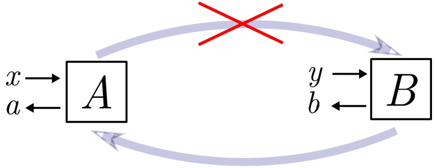

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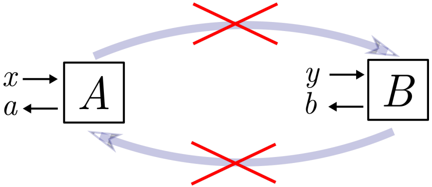

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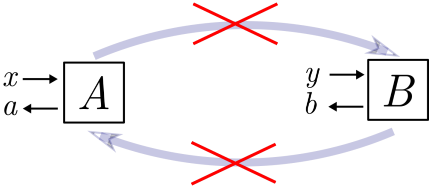

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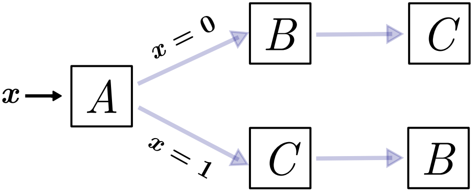
Multipartite Causal Correlations

With more than 2 parties, one can have:

- Fixed orders: $A_{\sigma(1)} \prec A_{\sigma(2)} \prec \cdots \prec A_{\sigma(N)}$ (σ a permutation of $\{1, \ldots, N\}$)
- But also dynamical orders:



- But one party always acts first; correlation *and even causal order* of the rest may depend on this party's input/output

Multipartite Causal Correlations

Parties $A_1, \ldots, A_N$, inputs $\vec{a} = (a_1, \ldots, a_N)$, outputs $\vec{x} = (x_1, \ldots, x_N)$

Recursive definition in multipartite case:

(Fully) Causal Correlations

1. For $N = 1$, any $P(a_1|x_1)$ is causal.
2. For $N \geq 2$, $P(\vec{a}|\vec{x})$ is causal iff it can be written

$$P(\vec{a}|\vec{x}) = \sum_k q_k P_k(a_k|x_k) \underbrace{P_{k,x_k,a_k}(\vec{a}_{\mathcal{N}\setminus k}|\vec{x}_{\mathcal{N}\setminus k})}_{\text{(N-1)-partite causal correlation}},$$

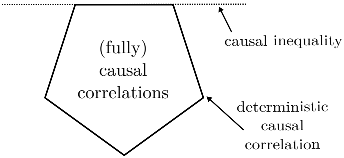
(with $\mathcal{N} = \{1, \ldots, N\}$, $q_k \geq 0$ and $\sum_k q_k = 1$).

Oreshkov & Giarmatzi, *NJP* 2016; Abbott, Giarmatzi, Costa & Branciard, *PRA* 2016.

Causal Polytope and Inequalities

For a given N-partite scenario, the set of causal correlations:

- Is a convex polytope
- Vertices are *deterministic* causal correlations
- Facets define causal inequalities



e.g. for $N = 2$, binary inputs and outputs, two families of inequalities:

$$\underbrace{P(a = y, b = x) \leq 1/2}_{\text{“guess your neighbour's input” game}} \quad \text{and} \quad \underbrace{P(x(a \oplus y) = y(b \oplus x) = 0) \leq 3/4}_{\text{“lazy” GYNI game}}$$

- Can both be violated by “process matrix correlations”

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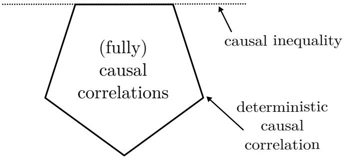

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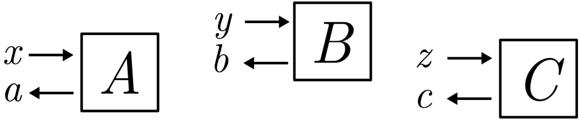
"Lazy" Tripartite Scenario

Simplest tripartite scenario:



- Binary inputs $x, y, z \in \{0, 1\}$
- Fixed outputs $a, b, c = 0$ when $x, y, z = 0$ and binary outputs $a, b, c \in \{0, 1\}$ when $x, y, z = 1$

Causal polytope is 19-dimensional (680 vertices, 13 074 facets)

- All 302 families of nontrivial causal inequalities can be violated by process matrix correlations
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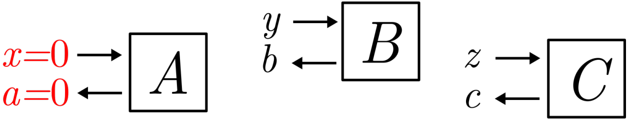

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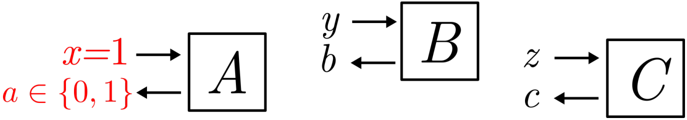

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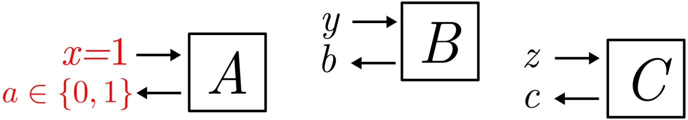

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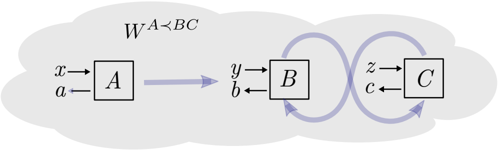
Many of the causal inequalities in the lazy tripartite scenario can be violated by W matrices compatible with, e.g., causal order $A \prec BC$



- Perhaps unsurprising given form of some causal inequalities
  - e.g. "conditional lazy GYNI":

$$P\big((1-x)y(b \oplus z) = (1-x)z(c \oplus y) = 0\big) \leq \frac{7}{8}$$

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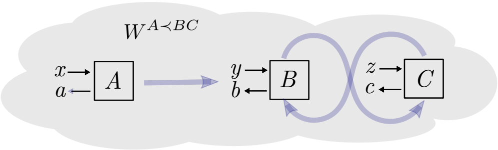

- Such correlations are of the form, e.g.

$$P^{A \prec BC}(abc|xyz) = P(a|x)\,\underbrace{P_{a,x}(bc|yz)}_{\text{noncausal}}$$

  - "partial", effectively bipartite, causal order between A and $\{B, C\}$
- Noncausality of $P^{A \prec BC}(abc|xyz)$ not a genuinely tripartite phenomenon

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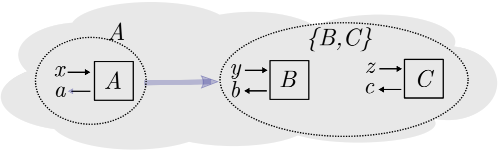

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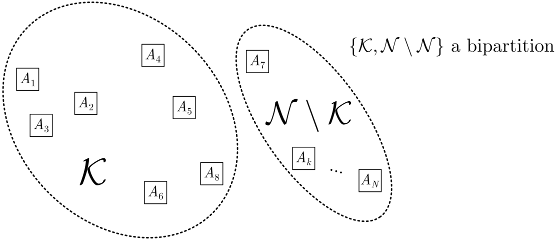
2-causal Correlations

"Genuinely N-partite noncausal correlations": no subset of parties should have a definite causal relation to any other subset

2-causal Correlations

$$P(\vec{a}|\vec{x}) = \sum_{\emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}} q_{\mathcal{K}} \underbrace{P_{\mathcal{K}}(\vec{a}_{\mathcal{K}}|\vec{x}_{\mathcal{K}})} \, \underbrace{P_{\vec{x}_{\mathcal{K}},\vec{a}_{\mathcal{K}}}(\vec{a}_{\mathcal{N}\setminus\mathcal{K}}|\vec{x}_{\mathcal{N}\setminus\mathcal{K}})}$$

any valid probability distributions



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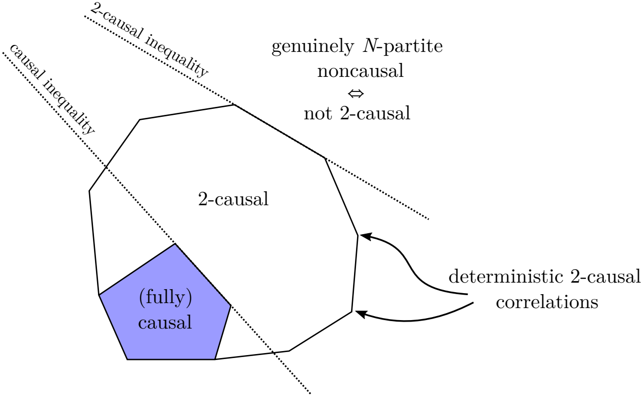
Genuinely N-partite noncausal correlations

$P(\vec{a}|\vec{x})$ is genuinely N-partite noncausal $\iff$ $P(\vec{a}|\vec{x})$ is not 2-causal

Characterising 2-causal correlations

Similar polytope structure to causal correlations

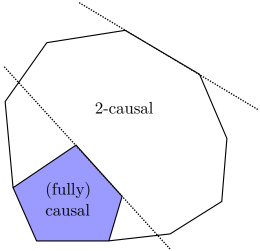
- Vertices are deterministic 2-causal correlations
- Facets give “2-causal” inequalities



Simplest Tripartite Scenario

Again we look at the lazy tripartite scenario:

- Dimension 19, 1 520 vertices, 21 154 facets
- 476 nonequivalent families of 2-causal inequalities (inc. 3 trivial ones)
- Only 2 nontrivial inequalities in common with the (fully) causal polytope
- All except 22 nontrivial inequalities can be saturated by fully causal correlations



Genuinely MP Causal Inequalities

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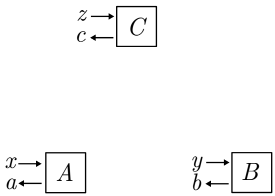
+&P_A(1|100) + P_C(1|001) - P_{AC}(11|101) \geq -1.
\end{aligned}
$$

- Each line is a conditional LGYNI inequality:

$$P_A(1|100)+P_B(1|010)-P_{AB}(11|110) \geq 0 \;\rightarrow\; P\big((1-z)x(a\oplus y) = (1-z)y(b\oplus x) = 0\big) \leq \frac{7}{8}$$

- Each pair plays the cond. LGYNI game and wins a point if they win it



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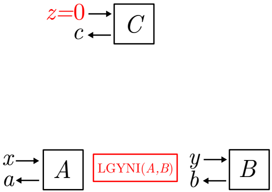

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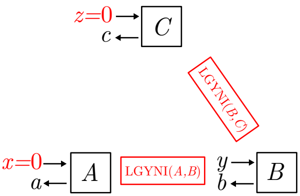

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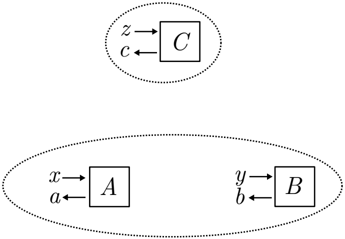

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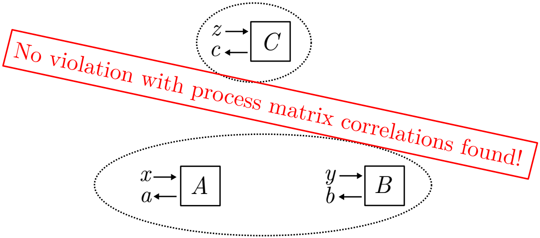

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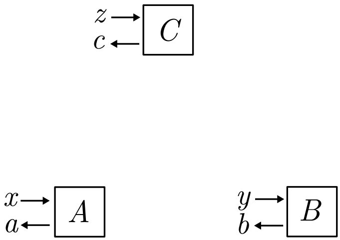

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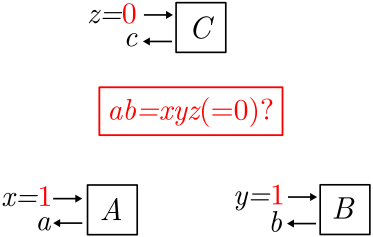

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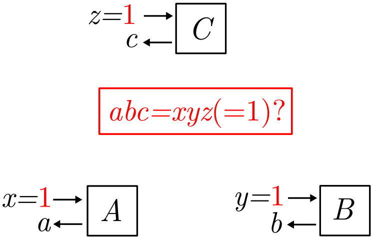
$$P\big(\tilde{a}\tilde{b}\tilde{c} = xyz\big) \;\leq\; 3/4, \quad \text{where } \tilde{a} = a \oplus x \oplus 1, \text{ etc.}$$

- Goal: product of *nontrivial* outputs $=$ product of inputs
- Outputting $(a, b, c) = (0, 0, 0)$ loses only if $x = y = z$



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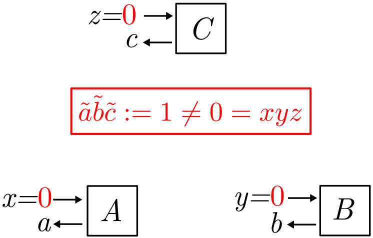

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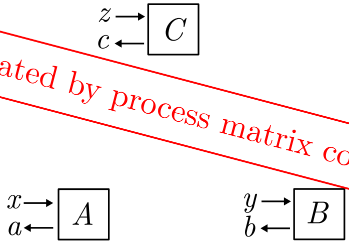


Can be violated by process matrix correlations!

Summary of Violations

- Same see-saw approach as for violating fully causal inequalities

	Fully causal	2-causal
Can be violated classically	284	284
No classical violation found	18	187
No violation found	0	2
	302	473

Violation of many inequalities with classical processes still possible

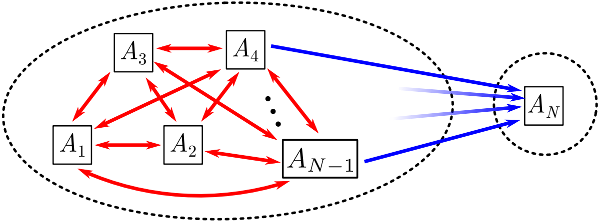
- Algebraic maximum violation only ever obtained by classical processes
- Same instruments give best violations for both sets of inequalities

$\mathcal{N}$-partite inequalities

Both inequalities can be generalised to N parties:

$$\sum_{\{i,j\}\subset\mathcal{N}} L_N(i,j) \geq -\binom{N-1}{2}, \quad \text{where} \quad \begin{aligned} L_N(i,j) =& P\big(a_i = 1 | x_i x_j = 10, \vec{x}_{\mathcal{N}\setminus\{i,j\}} = \vec{0}\big) \\ &+ P\big(a_j = 1 | x_i x_j = 01, \vec{x}_{\mathcal{N}\setminus\{i,j\}} = \vec{0}\big) \\ &- P\big(a_i a_j = 11 | x_i x_j = 11, \vec{x}_{\mathcal{N}\setminus\{i,j\}} = \vec{0}\big). \end{aligned}$$

- Cond. LGYNI between every pair of parties: $L_N(i,j) \geq 0$
- *Not* a facet for $N = 4$



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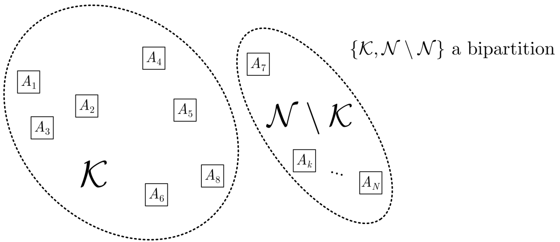
$$P\big(\tilde{a}\tilde{b}\tilde{c} = xyz\big) \ \leq \ 3/4 \quad \longrightarrow \quad P(\Pi_k \tilde{a}_k = \Pi_k x_k) \leq 1 - 2^{-N+1}$$

- Facet for $N = 4$, conjectured for all N

Refining the Definition



$\mathcal{P}$-causal Correlations

Given a partition $\mathcal{P}$, $P(\vec{a}|\vec{x})$ is $\mathcal{P}$-*causal* iff P is causal when considered as an effective $|\mathcal{P}|$-partite correlation, where each $\mathcal{A}_\ell \in \mathcal{P}$ defines an effective party, i.e.

$$P(\vec{a}|\vec{x}) = \sum_{\mathcal{A}_\ell\in\mathcal{P}} q_{\mathcal{A}_\ell}\, P_{\mathcal{A}_\ell}(\vec{a}_{\mathcal{A}_\ell}|\vec{x}_{\mathcal{A}_\ell})\, \underbrace{P_{\vec{x}_{\mathcal{A}_\ell},\vec{a}_{\mathcal{A}_\ell}}(\vec{a}_{\mathcal{N}\setminus\mathcal{A}_\ell}|\vec{x}_{\mathcal{N}\setminus\mathcal{A}_\ell})}_{(\mathcal{P}\setminus\mathcal{A}_\ell)\text{-causal}}$$

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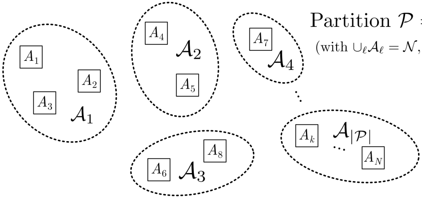

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M-causal Correlations

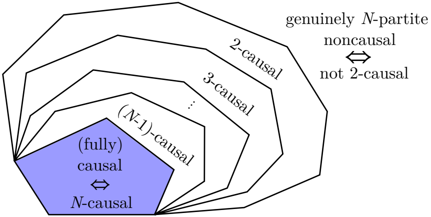
How many groups of parties are needed to reproduce a given $P(\vec{a}|\vec{x})$?

M-causal Correlations

$P(\vec{a}|\vec{x})$ is M-*causal* iff it is a convex mixture of $\mathcal{P}$-causal correlations for various partitions $\mathcal{P}$ with $|\mathcal{P}| \geq M$ subsets, i.e.

$$P(\vec{a}|\vec{x}) = \sum_{\mathcal{P}:\,|\mathcal{P}|\geq M} q_{\mathcal{P}}\, P_{\mathcal{P}}(\vec{a}|\vec{x})$$

- Defines a hierarchy of correlations with M-causal $\implies$ M'-causal if $M \geq M'$



Size-S-causal Correlations

What size groups are needed to reproduce a given $P(\vec{a}|\vec{x})$?

Size-S-causal Correlations

$P(\vec{a}|\vec{x})$ is *size-S-causal* iff it is a convex mixture of $\mathcal{P}$-causal correlations for partitions $\mathcal{P}$ whose subsets are no larger than S, i.e.

$$P(\vec{a}|\vec{x}) = \sum_{\mathcal{P}:\, \max_{\mathcal{A}\in\mathcal{P}}|\mathcal{A}|\leq S} q_{\mathcal{P}}\, P_{\mathcal{P}}(\vec{a}|\vec{x})$$

- Both M- and size-S-causal hierarchies strict
- Both give (fully)-causal and 2-causal correlations as extreme cases
- Incomparable hierarchies: give distinct quantifications of how genuinely multipartite a noncausal correlation is

Summary & Outlook

- Definition of genuinely multipartite noncausal correlations as negation of 2-causality
  - characterised by 2-causal inequalities, most of which can be violated by W correlations
  - some inequalities have natural generalisations to N parties
- Robustness of definition? Recall issues with definitions of genuine multipartite nonlocality[1]
  - Svetlichny's definition permits "activation" of nonlocality
  - Need to understand noncausality as a resource
- Refinements of 2-causality: how genuinely multipartite a noncausal resource is needed to reproduce a correlation
  - M- and size-S-causality give different, incomparable measures: which is more relevant and when?
- Genuinely multipartite causal nonseparability?

[1] Svetlichny, *PRD* 1987; Gallego *et al.*, *PRL* 2012; Bancel *et al.*, *PRA* 2013.

Thank you!

[paper on arXiv soon™]

Incomparable Hierarchies

In general the polytope of M-causal correlations does not contain, nor is contained in, the polytope of size-S-causal correlations (for fixed M, S):

- All inclusions follow from

$$|\mathcal{P}| - 1 + \left(\max_{\mathcal{A}\in\mathcal{P}} |\mathcal{A}|\right) \leq N \leq |\mathcal{P}| \cdot \left(\max_{\mathcal{A}\in\mathcal{P}} |\mathcal{A}|\right)$$

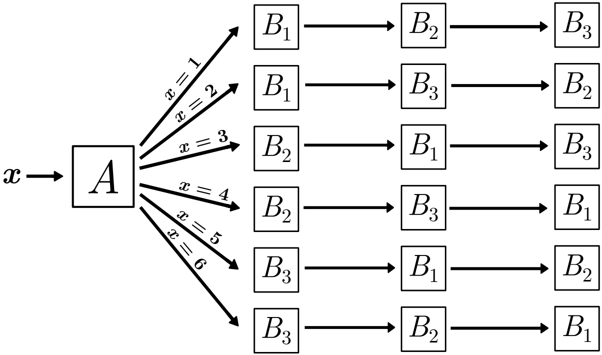
Inclusion Relations

- If a correlation is M-causal, then it is size-S-causal for all $S \geq N-M+1$;
- If a correlation is size-S-causal, then it is M-causal for all $M \leq \lceil \frac{N}{S} \rceil$.

Noninclusions of $\mathcal{P}$-causal Correlations

The following example is

- fully causal (i.e., 4-causal)
- Not $\mathcal{P}$-causal for any $|\mathcal{P}| = 3$
  - For any pair $\{A, B_i\}$ or $\{B_i, B_j\}$ there is a value of x for which another party comes between them



Process Matrix Violations

- Given all but one of W, $\{M_{a|x}\}_a$, $\{M_{b|y}\}_b$, $\{M_{c|z}\}_c$, finding the last one giving maximal violation of a causal inequality is an SDP problem
- Start with random instruments and use iterative 'see-saw' approach
- Best violation always found with the instruments:

$$\left\{M_{0|0} = |\mathbb{1}\rangle\!\rangle\langle\!\langle\mathbb{1}|\right\}, \left\{M_{0|1} = |0\rangle\langle 0| \otimes |1\rangle\langle 1|,\ M_{1|1} = |1\rangle\langle 1| \otimes |0\rangle\langle 0|\right\}$$

- For classical violations, best violation uses instead

$$\left\{M'_{0|0} = \frac{1}{2}(\mathbb{1} + ZZ) = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\right\}$$