Quantum Circuits with Classical and Quantum Control of Causal Orders

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joint work with
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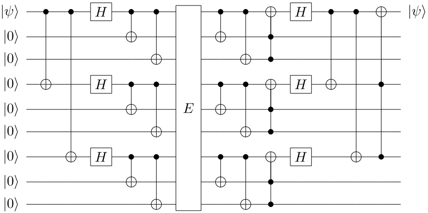
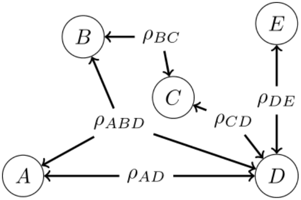
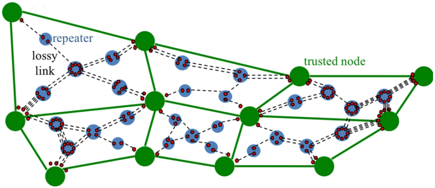
Causal Information Processing

In quantum information one usually assume a fixed, classical causal structure







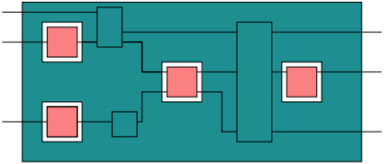
- Does this unnecessarily constrain quantum information processing?

Outline

Quantum Circuits

Quantum circuits, a.k.a. quantum combs, quantum networks

- [Chiribella, D'Ariano, Perinotti, PRL (2008) & PRA (2009)]



- Transform unknown operations $\mathcal{A}_1, \ldots, \mathcal{A}_N$ into $\mathcal{W}(\mathcal{A}_1, \ldots, \mathcal{A}_N)$
  - Formally they're supermaps
- Powerful framework for computing properties of unknown operations, distributed computation, tomography, metrology etc.

What maps can quantum circuits implement and how?

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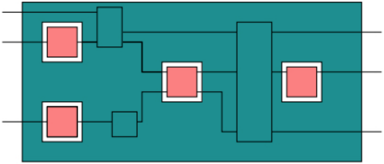

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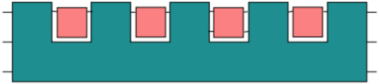

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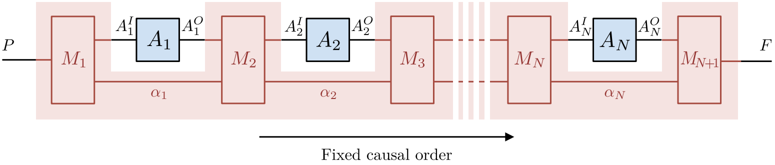
What maps can quantum circuits implement and how?

Quantum Circuits with Fixed Causal Order

Most general quantum circuit can be implemented as:



$\mathcal{M}_i$ CPTP maps and α_i ancillary systems:

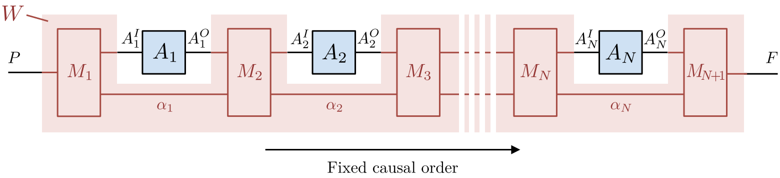
- e.g., $\mathcal{M}_{n+1} : A_n^O \alpha_n \to A_{n+1}^I \alpha_{n+1}$ for $1 \le n \le N-1$

Useful to represent operations as matrices in Choi picture:

- $M = \mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)$ with $|\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$
- $\mathcal{M}$ completely positive $\iff$ $M \ge 0$
- $\mathcal{M}$ trace-preserving $\iff$ $\operatorname{tr}_2 M = \mathbb{1}^1$

[Chiribella, D'Ariano, Perinotti, PRA (2009)]

Quantum Circuits with Fixed Causal Order



Inverse Choi isomorphism given by the link product:

- $\mathcal{M}(\rho) = M^{12} * \rho^1 := \mathrm{tr}_1[(\rho^1 \otimes \mathbb{1}^2) \cdot M^{T_1}]$
- Allows channels to be composed, etc.

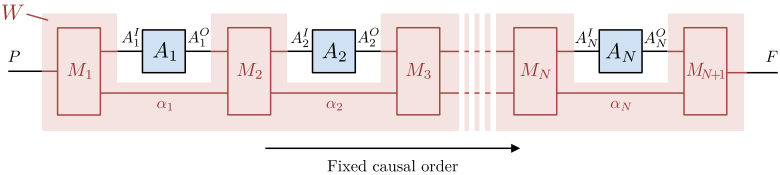
For input ρ output is

$$M_{N+1} * A_N * \cdots * M_2 * A_1 * M_1 * \rho = \underbrace{(M_1 * M_2 * \cdots * M_{N+1})}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

- The action of the circuit is characterised by W:

$$W = M_1 * M_2 * \cdots * M_{N+1} = \mathrm{Tr}_{\alpha_1 \cdots \alpha_N}\left[M_1 \cdot M_2^{T_{\alpha_1}} \cdots M_{N+1}^{T_{\alpha_N}}\right]$$

Quantum Circuits with Fixed Causal Order



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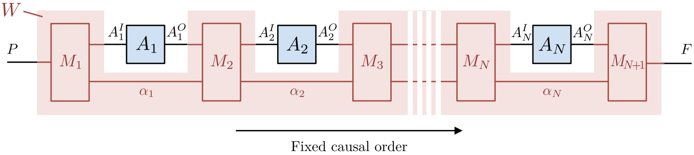
- W is the Choi representation of the supermap describing the circuit
- The $W^{A_1 \prec \cdots \prec A_N}$ that represent a circuit with causal order $A_1 \prec \cdots \prec A_N$ can be characterised via SDP constraints
    - From any W satisfying constraints one can extract the implementation effectively

Beyond Quantum Circuits?



Is the fixed causal structure of quantum circuits necessary?

- Convex mixture of circuits with different orders is physical, but still a classical causal structure
- More generally, can have classical "dynamical" causal orders

[Oreshkov & Giarmatzi, NJP 2016; Abbott *et al.*, PRA 2016.]

Why not a quantum superposition of causal orders?

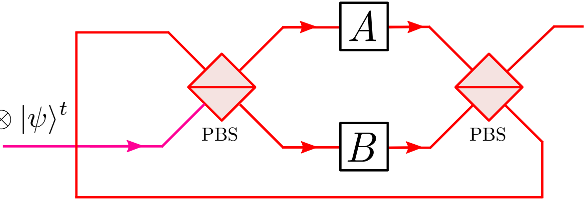
- What justification do we have for restricting the causal structure?
- In a quantum theory of gravity, spacetime itself would be quantum

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

Quantum Switch

The "Quantum Switch" uses a control system to apply two operations in a superposition of different orders

$\frac{1}{\sqrt{2}}(|H\rangle^c + |V\rangle^c) \otimes |\psi\rangle^t$



- A and B act on the target system and are each only applied once
- No well defined causal order between A and B: causally indefinite
- Implemented in several experiments
  - [Procopio *et al.*, Nat. Commun. 6 (2015)]; [Rubino *et al.*, Sci. Adv. 3 (2017)]; [Goswami *et al.*, PRL 121 (2018)]; [Pan et al., PRL (2019)]

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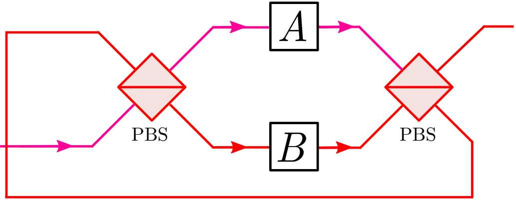

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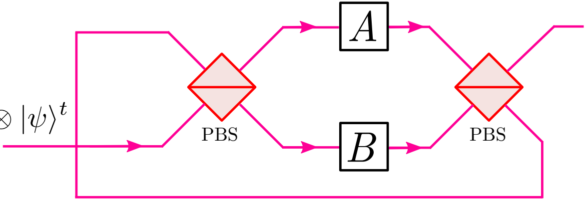

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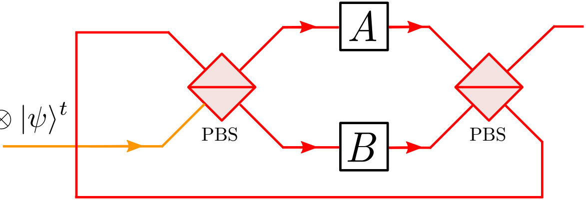

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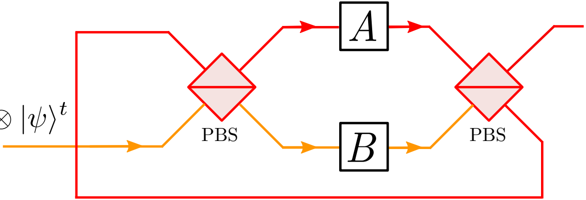

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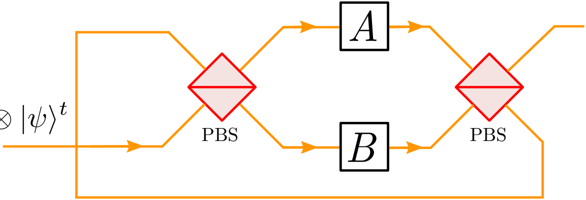

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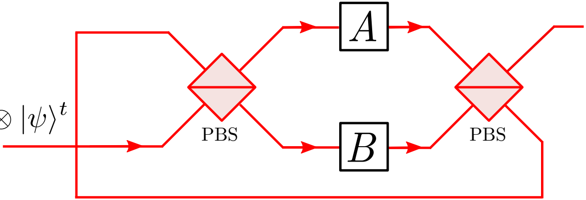
Causally Indefinite Information Processing

$\frac{1}{\sqrt{2}}(|H\rangle^c + |V\rangle^c) \otimes |\psi\rangle^t$



Is the quantum switch actually useful?

- If A and B are unitary transformations:

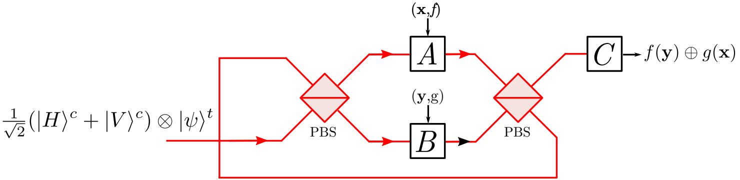
$$\frac{1}{\sqrt{2}}(|H\rangle^c + |V\rangle^c) \otimes |\psi\rangle^t \rightarrow \frac{1}{\sqrt{2}}(|H\rangle^c \otimes BA\,|\psi\rangle + |V\rangle^c \otimes AB\,|\psi\rangle)$$
$$= |+\rangle^c \otimes \tfrac{1}{2}\{A,B\}\,|\psi\rangle^t + |-\rangle^c \otimes \tfrac{1}{2}[A,B]\,|\psi\rangle^t$$

- Can then distinguish $\{A,B\} = 0$ and $[A,B] = 0$ by measuring the control
- Any quantum circuit needs to use A or B twice to do this
  - A simple query complexity advantage

Causally Indefinite Information Processing

Can generalise the quantum switch the N operations ($N!$ permutations)

- Query complexity: $O(N^2)$ advantage on a generalisation of the commutation/anticommutation problem [Araújo *et al.*, PRL 113 (2014)]
- Perfect channel discrimination [Chiribella, PRA 86 (2012)]
- Communication complexity: $O(2^N)$ advantage computing a distributed function with the quantum switch [Guérin *et al.*, PRL 117 (2016)]



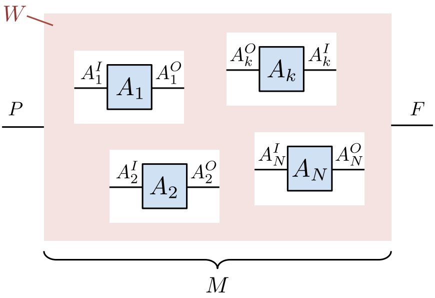
Causal indefiniteness is a resource for quantum information

- But how to generalise and study more rigorously?

Process matrix formalism

The process matrix framework introduced by Oreshkov, Costa and Brukner to describe more general processes without assuming a global causal structure

- Characterises the W that represent general, consistent supermaps

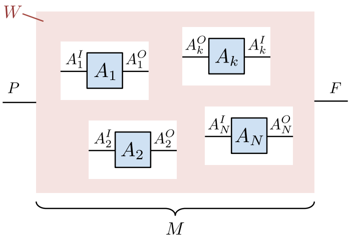


$$M = W * (A_1 \otimes \cdots \otimes A_N) = \mathrm{Tr}_{A_{\mathcal{N}}^{IO}} \left[W (A_1^T \otimes \cdots \otimes A_N^T \otimes \mathbb{1}^{PF}) \right]$$

- Circuits are a special case: fixed-order process

[Nat. Commun. 3, 1092 (2012)]

Causal Nonseparability

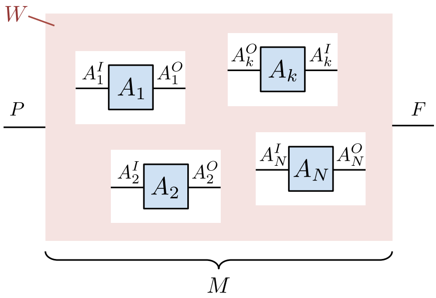


- Framework permits formal definition – and characterisation – of process matrices that are causally nonseparable
  - E.g., for two operations, causal separability means

$$W = q\, W^{A \prec B} + (1-q)\, W^{B \prec A}$$

  - General definition more subtle and only recently clarified and properly characterised [Wechs, AA, Branciard, NJP (2019)]
- Quantum switch can be written as a process matrix
  - Easily proven to be causally nonseparable

Beyond the Quantum Switch



"Top-down" definition means not clear if all W are physical

- Quantum switch is only noncausal process we know how to implement

Are there other classes of interesting noncausal processes?

Outline

Quantum Circuit Framework

Causally Indefinite Processes

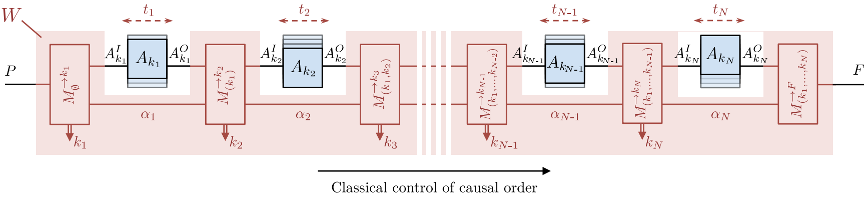
- Quantum switch
- Process matrix framework

Quantum Circuits with Classical and Quantum Controls of Causal Order

- Classically controlled circuits
- Coherently (quantum) controlled circuits

Classically Controlled Circuits

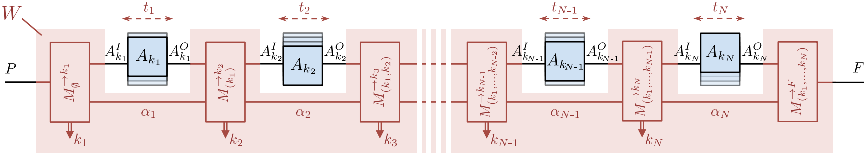
- First generalise circuits to allow for classical control of causal structure
  - Encompass mixed and dynamic causal orders
  - Oreshkov & Giarmatzi proposed quantum circuits with classical control of causal order (QC-CCs) as most general causal process



- At each time slot t_n exactly one operation A_{k_n} is applied
- $\{M_{(k_1,\dots,k_n)}^{\to k_{n+1}}\}_{k_{n+1}}$ are instruments: $\sum_{k_{n+1}} M_{(k_1,\dots,k_n)}^{\to k_{n+1}}$ is CPTP
- Crucial requirement: each operation applied once and only once, irrespective of the operations themselves

[Oreshkov, Giarmatzi, NJP (2016)]

Process Matrix of a QC-CC



For input ρ, when operations applied in order $k_1, \dots, k_N$, output is

$$M^{\to F}_{(k_1,\dots,k_N)} * A_{k_N} * M^{\to k_N}_{(k_1,\dots,k_{N-1})} * \cdots * M^{\to k_3}_{(k_1,k_2)} * A_{k_2} * M^{\to k_2}_{(k_1)} * A_{k_1} * M_\emptyset^{\to k_1} * \rho$$

$$= \underbrace{M_\emptyset^{\to k_1} * M^{\to k_2}_{(k_1)} * M^{\to k_3}_{(k_1,k_2)} * \cdots * M^{\to k_N}_{(k_1,\dots,k_{N-1})} * M^{\to F}_{(k_1,\dots,k_N)}}_{\widetilde{W}_{(k_1,\dots,k_N,F)}} *(\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

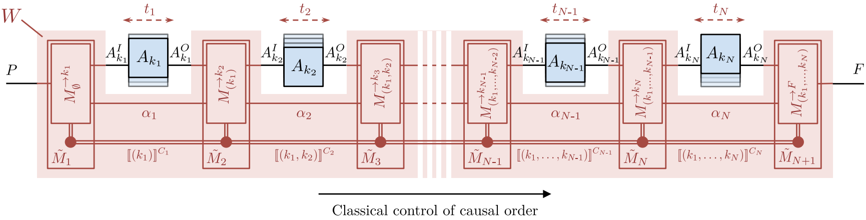
Process matrix of a QC-CC (Causally Separable)

$$W = \sum_{(k_1,\dots,k_N)} \widetilde{W}_{(k_1,\dots,k_N,F)}$$

- Can be characterised in terms of SDP constraints

Alternative Descriptions of QC-CCs

- Conditioning can be included in operations by introducing (classical) control system $[\![(k_1,\dots,k_n)]\!]^{C_n} := |(k_1,\dots,k_n)\rangle\!\langle(k_1,\dots,k_n)|^{C_n}$



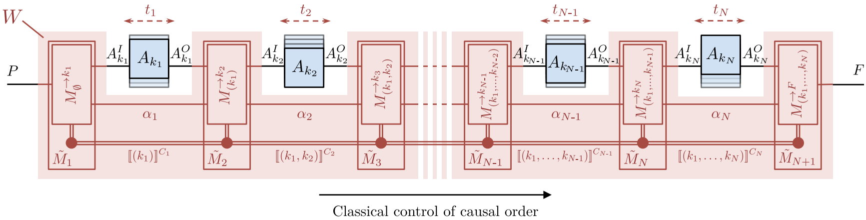
- Operations now given by the CPTP maps

$$\tilde{M}_1 := \sum_{k_1} \tilde{M}_\emptyset^{\to k_1} \otimes [\![(k_1)]\!]^{C_1},$$

$$\tilde{M}_{n+1} := \sum_{k_1,\dots,k_n,k_{n+1}} \tilde{M}_{(k_1,\dots,k_n)}^{\to k_{n+1}} \otimes [\![(k_1,\dots,k_n)]\!]^{C_n} \otimes [\![(k_1,\dots,k_n,k_{n+1})]\!]^{C_{n+1}}$$

$$M_{N+1} := \sum_{k_1,\dots,k_N} M_{(k_1,\dots,k_N)}^{\to F} \otimes [\![(k_1,\dots,k_N)]\!]^{C_N}$$

Alternative Descriptions of QC-CCs



- Defining global operations $\tilde{A}_n := \bigoplus_{k_n \in \mathcal{N}} A_{k_n}$ we have

$$\tilde{M}_{N+1} * \tilde{A}_N * \tilde{M}_N * \cdots * \tilde{A}_1 * \tilde{M}_1 * \rho = \underbrace{\sum_{k_1,\dots,k_N} W_{(k_1,\dots,k_N,F)}}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

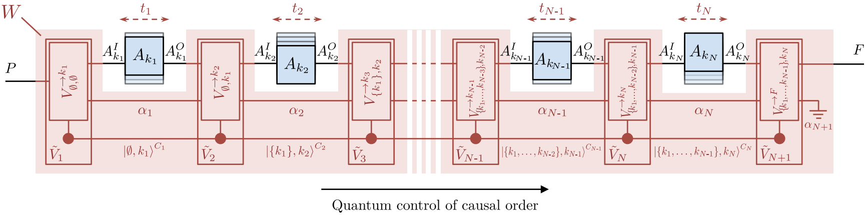
- Note that wlog we can take all operations to be purified isometries

$$M^{\to k_n}_{(k_1,\dots,k_{n-1})} = |V^{\to k_n}_{(k_1,\dots,k_{n-1})}\rangle\!\rangle\langle\!\langle V^{\to k_n}_{(k_1,\dots,k_{n-1})}|$$

- Suggests natural generalisation to quantum control of causal order

From Classical to Coherent Control

- Relax the control state to store only *which* operations have been performed, but not their order: $|\mathcal{K}_{n-1}, k_n\rangle^{C_n}$
  - Conditioning on $\mathcal{K} = \{k_1, \ldots, k_{n-1}\}$ allows different orders to interfere
  - Storing full history $|(k_1, \ldots, k_n)\rangle^{C_n}$ is more restrictive and included in this case by using ancillas



- Operations given by the isometries V_n with pure CJ representation

$$|\tilde{V}_{n+1}\rangle\!\rangle := \sum_{\substack{\mathcal{K}_{n-1} \\ k_n, k_{n+1}}} |\tilde{V}^{\to k_{n+1}}_{\mathcal{K}_{n-1}, k_n}\rangle\!\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C_n} \otimes |\mathcal{K}_n, k_{n+1}\rangle^{C_{n+1}},$$

$$|\tilde{V}_1\rangle\!\rangle := \sum_{k_1} |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle \otimes |\emptyset, k_1\rangle^{C_1}, \qquad |\tilde{V}_{N+1}\rangle\!\rangle := \sum_{k_N} |\tilde{V}^{\to F}_{\mathcal{N}\setminus\{k_N\}, k_N}\rangle\!\rangle \otimes |\mathcal{N}\setminus\{k_N\}, k_N\rangle^{C_N}$$

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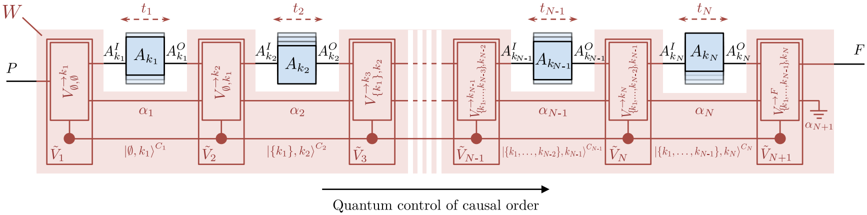

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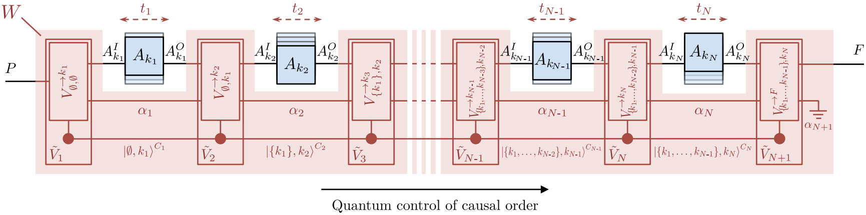
Coherently (Quantum) Controlled Circuits



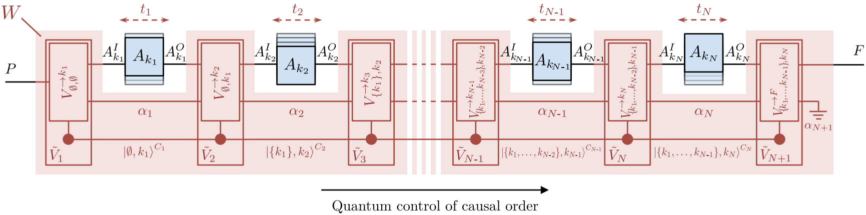
- Each $V^{\to k_{n+1}}_{\mathcal{K}_{n-1},k_n} : \mathcal{H}^{A^O_{k_n}\alpha_n} \to \mathcal{H}^{A^I_{k_{n+1}}\alpha_{n+1}}$ embedded in larger space
  - Control ensures that each party applied once and only once

For input $|\psi\rangle$, circuit applies transformation

$$|\tilde V_{N+1}\rangle\rangle * |\tilde A_N\rangle\rangle * |\tilde V_N\rangle\rangle * \cdots * |\tilde V_2\rangle\rangle * |\tilde A_1\rangle\rangle * |\tilde V_1\rangle\rangle * |\psi\rangle \in \mathcal{H}^{F\alpha_{n+1}}$$

with "pure link product" $|a\rangle^{\mathsf{A}} * |b\rangle^{\mathsf{B}} := \langle\langle \mathbb{1}|^{\mathsf{A}\cap\mathsf{B}} \big(\, |a\rangle \otimes |b\rangle \,\big) = \sum_i \langle i,i|^{(\mathsf{A}\cap\mathsf{B})^{\otimes 2}} \big(\, |a\rangle \otimes |b\rangle \,\big)$

QCs with Quantum Control of Causal Order



$$|\tilde{V}_{N+1}\rangle\rangle * |\tilde{A}_N\rangle\rangle * |\tilde{V}_N\rangle\rangle * \cdots * |\tilde{V}_2\rangle\rangle * |\tilde{A}_1\rangle\rangle * |\tilde{V}_1\rangle\rangle * |\psi\rangle$$

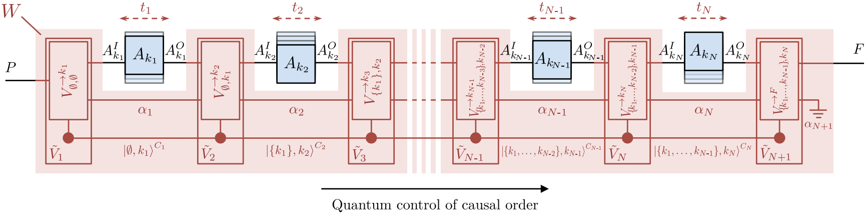
$$= \sum_{k_1,\ldots,k_N} \underbrace{|V^{\to k_1}_{\emptyset,\emptyset}\rangle\rangle * |V^{\to k_2}_{\emptyset,k_1}\rangle\rangle * |V^{\to k_3}_{\{k_1\},k_2}\rangle\rangle * \cdots * |V^{\to F}_{\{k_1,\ldots,k_{N-1}\},k_N}\rangle\rangle}_{|w_{(k_1,\ldots,k_N,F)}\rangle\rangle} * (|\psi\rangle \otimes |A_1\rangle\rangle \otimes \cdots \otimes |A_N\rangle\rangle)$$

Process matrix of a QC-QC

$$W = \mathrm{Tr}_{\alpha_{N+1}} |w\rangle\rangle\langle\langle w| , \quad \text{with} \quad |w\rangle\rangle := \sum_{k_1,\ldots,k_N} |w_{(k_1,\ldots,k_N,F)}\rangle\rangle$$

- Can again be characterised in terms of SDP constraints

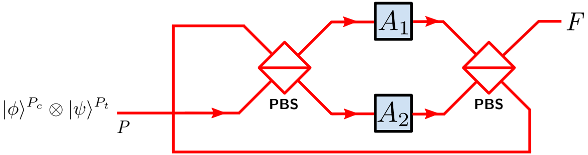
QC-QC Summary



- Classically controlled circuits recovered as a special case
  - But in general QC-QCs can be causally nonseparable
- QC-QCs are physically realisable, e.g., with a "quantum router"
- Realisation – in terms of the $\tilde{V}_n$ – can be effectively obtained from the any W satisfying the characterisation
- A framework within which to look for new examples and applications of noncausal processes

Example: Quantum Switch

Quantum switch is a QC-QC: $\mathcal{H}^P = \mathcal{H}^{P_t} \otimes \mathcal{H}^{P_c}$ and $\mathcal{H}^F = \mathcal{H}^{F_t} \otimes \mathcal{H}^{F_c}$



The controlled operations are

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_\text{c}} |\mathbb{1}\rangle\rangle^{P_\text{t} A_{k_1}^I}, \quad |V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = |\mathbb{1}\rangle\rangle^{A_{k_1}^O A_{k_2}^I}, \quad |V_{\{k_1\},k_2}^{\to F}\rangle\rangle = |k_1\rangle^{F_\text{c}} |\mathbb{1}\rangle\rangle^{A_{k_2}^O F_\text{t}}$$

Process vector is then

$$\begin{aligned} |w_\text{s}\rangle\rangle &:= |w_{(P,A_1,A_2,F)}\rangle\rangle + |w_{(P,A_2,A_1,F)}\rangle\rangle \\ &= |V_{\emptyset,\emptyset}^{\to A_1}\rangle\rangle * |V_{\emptyset,A_1}^{\to A_2}\rangle\rangle * |V_{\{A_1\},A_2}^{\to F}\rangle\rangle + |V_{\emptyset,\emptyset}^{\to A_2}\rangle\rangle * |V_{\emptyset,A_2}^{\to A_1}\rangle\rangle * |V_{\{A_2\},A_1}^{\to F}\rangle\rangle \\ &= |1\rangle^{P_\text{c}} |\mathbb{1}\rangle\rangle^{P_\text{t} A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O F_\text{t}} |1\rangle^{F_\text{c}} + |2\rangle^{P_\text{c}} |\mathbb{1}\rangle\rangle^{P_\text{t} A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O F_\text{t}} |2\rangle^{F_\text{c}} \end{aligned}$$

Standard process matrix of switch recovered as $W_\text{switch} = |w_\text{s}\rangle\rangle\langle\langle w_\text{s}|$

$$\text{Tr}_F W_\text{switch} = \underbrace{|1\rangle\langle 1|^{P_c} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{P_t A_1^I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{A_1^O A_2^I} \otimes \mathbb{1}^{A_2^O}}_{W^{A_1 \prec A_2}} + \underbrace{|2\rangle\langle 2|^{P_c} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{P_t A_2^I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{A_2^O A_1^I} \otimes \mathbb{1}^{A_1^O}}_{W^{A_2 \prec A_1}}$$

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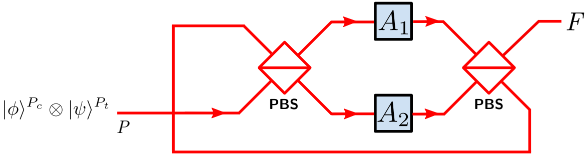

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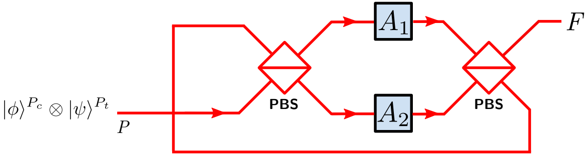

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Beyond the Quantum Switch?

- N-partite generalisation of the quantum switch is also a QC-QC
- Do QC-QCs offer something new, or are they all "equivalent" to the switch?
- Need a better understanding of causally nonseparable resources and free operations
  - Taddei, Nery and Aolita [arXiv:1903.06180]: local operations and controlled non-signalling operations ofr bipartite processes
  - Composition: Possible compositions severely restricted [Guérin et al., NJP 2019], but can, e.g., concatenate switches, or insert them inside other switches

Can find QC-QCs such that $\mathrm{Tr}_F W$ not a mixture of valid process compatible with fixed last parties, i.e.

$$\mathrm{Tr}_F W \neq q_1 W^{\{A_1,A_2\}\prec A_3} + q_2 W^{\{A_1,A_3\}\prec A_2} + q_3 W^{\{A_2,A_3\}\prec A_1}$$

- Qualitatively different to the switch

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Summary & Outlook

- Quantum mechanics allows us to process information in a causally indefinite way
  - Quantum switch
  - Process matrix formalism

- Quantum circuits with classical control of causal order
  - General realisation of causally separable processes

- Quantum circuits with quantum control of causal order
  - Potential new realisable, causally nonseparable, circuits beyond the quantum switch?
  - Do QC-QCs provide new information theoretical advantages?
  - Are there other classes of physically realisable processes?

Thank you!

[paper coming soon]

Choi Isomorphism and Link Product

- $|\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$ is the "pure Choi isomorphism" of an identity channel
- Pure Choi isomorphism: for an operator A, $|A\rangle\rangle = \mathbb{1} \otimes A \, |\mathbb{1}\rangle\rangle$
- Mixed Choi isomorphism: for a CP map $\mathcal{M}$, $M = \mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)$
- Inverse Choi isomorphism given by the link product: $\mathcal{M}(\rho) = M * \rho$; $A \, |\psi\rangle = |A\rangle\rangle * |\psi\rangle$

Constraints for Process Matrix Validity

Recall the notation:

$$ {}_X W := (\mathrm{Tr}_X W) \otimes \frac{\mathbb{1}^X}{d_X}, \quad {}_1 W := W, \quad {}_{[\sum_X \alpha_X X]} W := \sum_X \alpha_X \, {}_X W, $$

Space of valid process matrices

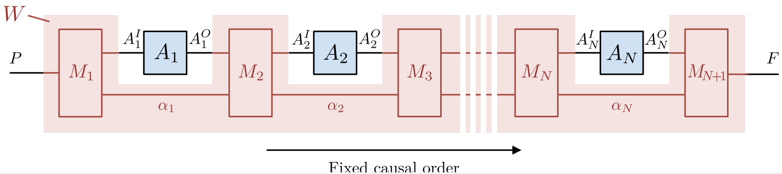
$$ W \in \mathcal{L}^{\mathcal{N}} \;\Leftrightarrow\; \forall\, \chi \subseteq \mathcal{N}, \chi \neq \emptyset, \; {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash\chi}} W = 0, $$

Space of valid process compatible with A first

$$ W \in \mathcal{L}^{A_k \prec (\mathcal{N}\backslash A_k)} $$

$$ \Leftrightarrow \; {}_{[1-A_O^k]A_{IO}^{\mathcal{N}\backslash k}} W = 0 \quad \text{and} \quad \forall\, \chi \subseteq \mathcal{N}\backslash k, \chi \neq 0, {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash k\backslash\chi}} W = 0, $$

Characterisation of QC-FOs



$$W = M_1 * M_2 * \cdots * M_{N+1} = \mathrm{Tr}_{\alpha_1 \cdots \alpha_N} \left[M_1 \cdot M_2^{T_{\alpha_1}} \cdots M_{N+1}^{T_{\alpha_N}} \right]$$

The constraint that the M_n are CPTP maps and thus satisfy $\mathrm{Tr}_{A_{n+1}^I \alpha_{n+1}} M_{n+1} = \mathbb{1}^{A_n^O \alpha_n}$ allow the W of QC-FOs to be characterised

QC-FOs compatible with order $P \prec A_1 \prec \cdots \prec A_N \prec F$

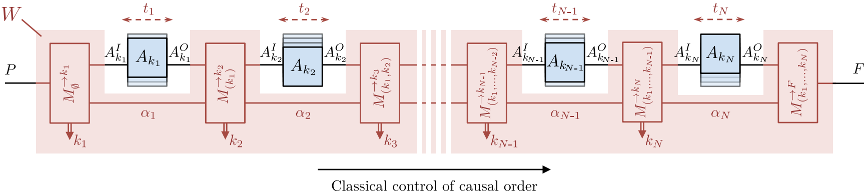
$$\mathrm{Tr}_F\, W = W_{(N)} \otimes \mathbb{1}^{A_N^O},$$

$$\mathrm{Tr}_{A_{n+1}^I}\, W_{(n+1)} = W_{(n)} \otimes \mathbb{1}^{A_n^O} \quad \forall\, n = 1, \ldots, N-1,$$

and $\quad \mathrm{Tr}_{A_1^I}\, W_{(1)} = \mathbb{1}^P.$

where $W_{(n)} := \frac{1}{d_n^O d_{n+1}^O \cdots d_N^O} \mathrm{Tr}_{A_n^O A_{\{n+1,\ldots,N\}}^{IO} F}\, W$ are reduced process matrices

QC-CC Characterisation



Sufficient Condition for N-partite Causal Separability

$$W^{\text{sep}} = \sum_{\pi\in\Pi} \widetilde{W}_\pi, \quad \text{with}$$

1. $\widetilde{W}_\pi \geq 0$ for each permutation π of $(1,\dots,N) \equiv (A_1,\dots,A_N)$
2. For every ordered subset $(k_1,\dots,k_n)$ (with $1 \leq n \leq N$),

$$\widetilde{W}_{(k_1,\dots,k_n)} := \sum_{\pi\in\Pi_{(k_1,\dots,k_n)}} \widetilde{W}_\pi, \text{ satisfies}$$

$$\mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\dots,k_n\}}} \widetilde{W}_{(k_1,\dots,k_n)} = \left(\mathrm{Tr}_{A^O_{k_n}} \left[\mathrm{Tr}_{A^{IO}_{\mathcal{N}\setminus\{k_1,\dots,k_n\}}} \widetilde{W}_{(k_1,\dots,k_n)}\right]\right) \otimes \mathbb{1}^{A^O_{k_n}}$$

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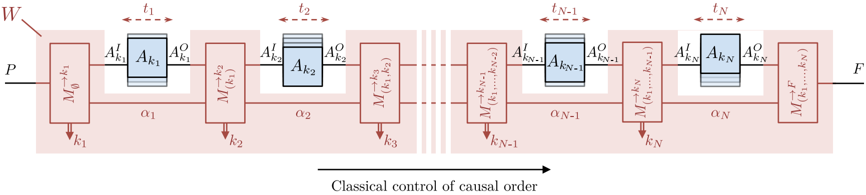


Sufficient Condition for Fourpartite Causal Separability

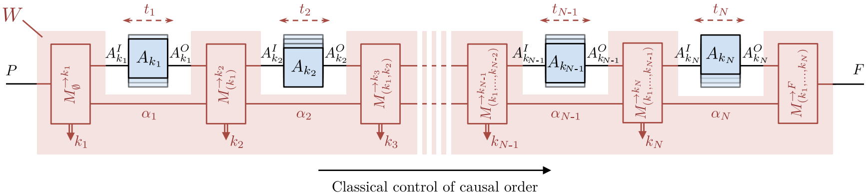
Valid process compatible with A first (up to norm.)

$$
\begin{aligned}
W &= W_{(A)} + W_{(B)} + W_{(C)} + W_{(D)} \\
&= \widetilde{W}_{(A,B)} + \widetilde{W}_{(A,C)} + \widetilde{W}_{(A,D)} + \cdots + \cdots + \cdots \\
&= \widetilde{W}_{(A,B,C,D)} + \widetilde{W}_{(A,B,D,C)} + \widetilde{W}_{(A,C,B,D)} + \widetilde{W}_{(A,C,D,B)} + \widetilde{W}_{(A,D,B,C)} + \widetilde{W}_{(A,D,C,B)} + \cdots + \cdots + \cdots
\end{aligned}
$$

For any CP map M_A, $(W_{(A,B)})_{|M_A}$ is valid, compatible with B first

For any CP maps M_A, M_B, $(W_{(A,B,C,D)})_{|M_A \otimes M_B}$ is valid, compatible with C first

QC-CC Characterisation



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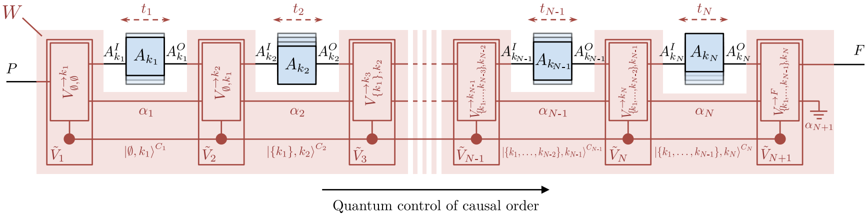
QC-QC Characterisation



- As for QC-CCs, can characterise such W with SDP constraints

Characterisation of circuits with quantum control of causal order

W is the process matrix of a QC-QC iff $\exists$ PSD matrices $W_{(\mathcal{K},\ell)} \in P A_{\mathcal{K}}^{IO} A_\ell^I$ $\forall \mathcal{K} \subsetneq \mathcal{N}, \ell \in \mathcal{N} \setminus \mathcal{K}$ satisfying

$$\mathrm{Tr}_F\, W = \sum_{k \in \mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

$$\forall\, \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}, \quad \sum_{\ell \in \mathcal{N}\setminus\mathcal{K}} \mathrm{Tr}_{A_\ell^I}\, \widetilde{W}_{(\mathcal{K},\ell)} = \sum_{k \in \mathcal{K}} \widetilde{W}_{(\mathcal{K}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

and $$\sum_{\ell \in \mathcal{N}} \mathrm{Tr}_{A_\ell^I}\, W_{(\emptyset,\ell)} = \mathbb{1}^P.$$

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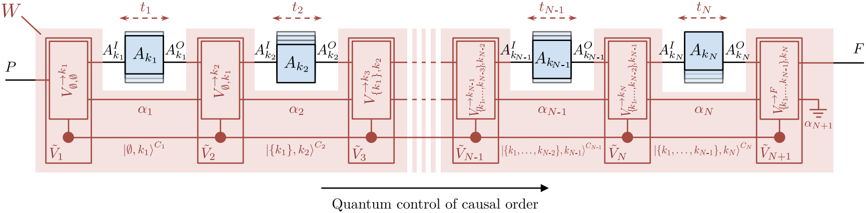

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Three-operation QC-QC Characterisation

- $\mathrm{Tr}_F\, W = \widetilde{W}_{(\{B,C\},A)} \otimes \mathbb{1}^{A^O} + \widetilde{W}_{(\{A,C\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{A,B\},C)} \otimes \mathbb{1}^{C^O}$
- $\mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A,B\},C)} = \widetilde{W}_{(\{A\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{B\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{B^I}\, \widetilde{W}_{(\{A\},B)} + \mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A\},C)} = W_{(\{\emptyset\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{A^I}\, W_{(\{\emptyset\},A)} + \mathrm{Tr}_{B^I}\, W_{(\{\emptyset\},B)} + \mathrm{Tr}_{C^I}\, W_{(\{\emptyset\},C)} = \mathbb{1}^P$

QC-QCs and Causal Correlations

Can quantum circuits with quantum control of causal order violate causal inequalities?

QC-QC correlations are causal

Let W be a QC-QC with trivial spaces $\mathcal{H}^P$ and $\mathcal{H}^F$. Then the correlations

$$P(a_1, \ldots, a_N | x_1, \ldots, x_N) = \mathrm{Tr}[W \cdot (M_{a_1|x_1}^T \otimes \cdots \otimes M_{a_N|x_N}^T)]$$

are causal for any instruments $\{M_{a_i|x_i}\}_{x_i}$.

- Can noncausal correlations be realised in nature?
  - Would require going beyond this type of generic, coherently controlled circuit
- QC-QCs nevertheless have potential for new advantages arising from indefinite causal order
  - New classes of physically realisable, causally nonseparable processes
  - Use as "quantum super-instruments", generalising quantum testers, for transformation tasks

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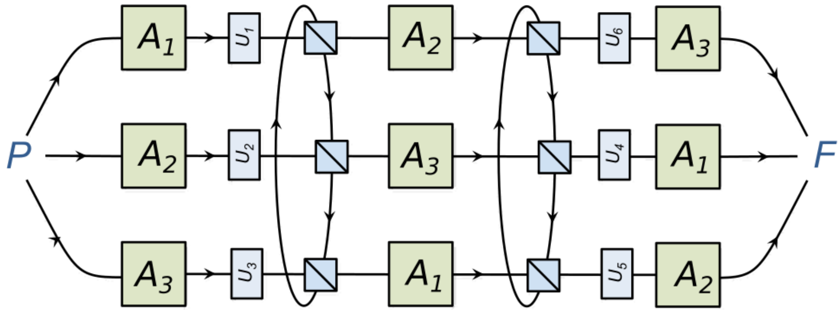
Example: New type of 3-operation QC-QC

$N = 3$, two qubit "targets" and 3-dimensional "control"

- Initial control state $|k_1\rangle^{P_c}$ determines first party
- First party acts on first target qubit $|\psi\rangle^{P_{t_1}}$, while $|\phi\rangle^{P_{t_2}}$ encoded as ancilla in polarisation
- Unitary U_i applied jointly on both "targets": output of A_{k_1} may determine (dynamically, coherently) k_2

Can be represented in an "unravelled" form as:



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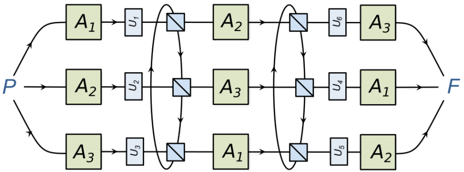


Controlled operations can be written

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_c} \, |\mathbb{1}\rangle\rangle^{P_{t_1} A^I_{k_1}} \, |\mathbb{1}\rangle\rangle^{P_{t_2} \alpha_1}$$

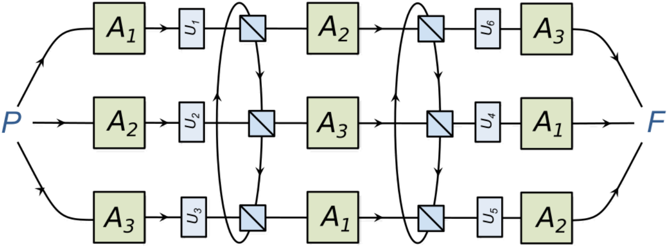
$$|V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = \big(\langle k_2 - k_1 - 1|^{\alpha_2} \, |U_{k_1}\rangle\rangle^{A^O_{k_1} \alpha_1 A^I_{k_2} \alpha_2}\big) \otimes |k_2 - k_1 - 1\rangle^{\alpha_2}$$

$$|V_{\{k_1\},k_2}^{\to k_3}\rangle\rangle = |U_{k_3+3}\rangle\rangle^{A^O_{k_2} \alpha_2 A^I_{k_3} \alpha_3}$$

$$|V_{\{k_1,k_2\},k_3}^{\to F}\rangle\rangle = |k_3\rangle^{F_c} \, |\mathbb{1}\rangle\rangle^{A^O_{k_3} F_{t_2}} \, |\mathbb{1}\rangle\rangle^{\alpha_3 F_{t_1}}$$

$$\begin{aligned} |w_{(P,A_1,A_2,A_3,F)}\rangle\rangle = {} & |1\rangle^{P_c} \, |\mathbb{1}\rangle\rangle^{P_{t_1} A^I_1} \, \big(\langle k_2 - k_1 - 1|^{\alpha_2} \, |U_{k_1}\rangle\rangle^{A^O_{k_1} P_{t_2} A^I_{k_2} \alpha_2}\big) \\ & \otimes \big(\langle k_2 - k_1 - 1|^{\alpha_2} \, |U_{k_3+3}\rangle\rangle^{A^O_{k_2} \alpha_2 A^I_{k_3} F_{t_2}}\big) \, |\mathbb{1}\rangle\rangle^{A^O_3 F_{t_1}} \, |3\rangle^{F_c} \end{aligned}$$

Example: New type of 3-operation QC-QC



As before, define $W = |w\rangle\!\rangle\langle\!\langle w|$ with $|w\rangle\!\rangle = \sum_{k_1,k_2,k_3} |w_{(P,k_1,k_2,k_3,F)}\rangle\!\rangle$

- W is easily seen to be causally nonseparable
  - Pure process, not compatible with any of A, B or C being first (after P)
- Appears qualitatively different to the quantum switch
  - Two target qubits, one of which is also used to control the causal order
- But how to prove W is fundamentally inequivalent to quantum switch?
  - Could imagine composing switches, using control of one as target for another, etc.
  - Need a more complete resource theory, e.g. generalising Taddei, Nery and

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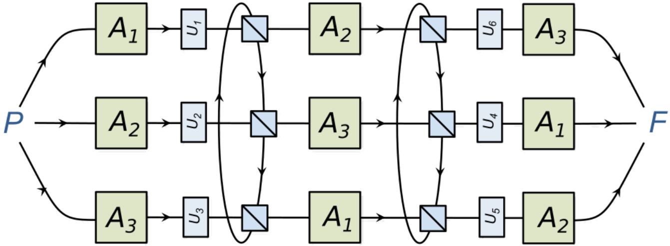

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