From Preparation to Measurement

Through the Eyes of Entropic Uncertainty Relations

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joint work with Cyril Branciard

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ICQF, Patna, 17–21 October 2016

arXiv:1607.00261

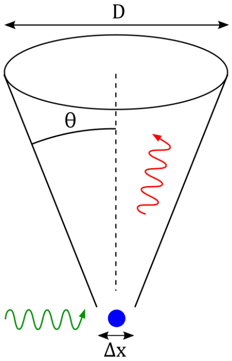
Outline

Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle (informally)

The measurement of one quantum observable introduces an irreversible disturbance into any complementary observable property of the system.

- HUP is a statement about the tradeoff between the accuracy of a measurement and the disturbance the measurement induces on the state with respect to a complementary observable.
- Historical work mostly focused on the tradeoff between how accurately a state can be prepared with respect to two complementary observables



Types of Uncertainty Relations

Preparation Uncertainty Relations

- Example: $\Delta\hat{x}\Delta\hat{p}_x \geq \frac{\hbar}{2}$
- Not about measurement *per se*

Measurement Uncertainty Relations:

- HUP: $N(\mathcal{M},\hat{x})D(\mathcal{M},\hat{p}_x) \geq \frac{\hbar}{2}$
- How to formally quantify the noise and disturbance?
- We can further distinguish two types of MUR:
  - Joint-measurement (noise-noise) relations, expressing the tradeoff between accuracy of a measurement for two complementary observables
  - Noise-disturbance relations, expressing the HUP-type tradeoff

How are all these relations related?

- They all express different, but related, consequences of non-commutativity of complementary observables

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Preparation Uncertainty Relations

Tradeoff between how uncertain the physical properties associated with two observable A and B are for any state ρ

- We will restrict ourselves to finite dimensional, non-degenerate observables: $A = \sum_a a|a\rangle\langle a|$

How to quantify this uncertainty?

- Standard Deviations: $\Delta_\rho A = \sqrt{\langle A \rangle_\rho^2 - \langle A^2 \rangle_\rho}$
  - Robertson's relation: $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$
  - Bound is state dependent
- Entropies: $H(A|\rho) = -\sum_a \mathrm{Tr}\left[|a\rangle\langle a|\, \rho\right] \log \mathrm{Tr}\left[|a\rangle\langle a|\, \rho\right]$
  - Invariant under relabelling/scaling of outcomes
  - Information theoretic flavour
  - Can use other entropies (e.g., Renyi entropies)
  - Helpful for finding state independent relations

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Entropic Uncertainty Relations

- Maassen & Uffink's relation:[1]

$$H(A|\rho) + H(B|\rho) \geq -\log \max_{a,b} |\langle a|b\rangle|^2$$

  - State *independent*
- Not generally tight (and often poor)
- When can the bound be saturated, and by what states?
  - More generally, what values of $\big(H(A|\rho), H(B|\rho)\big)$ can be obtained?

Entropic Uncertainty Region

$$E(A,B) = \Big\{\big(H(A|\rho), H(B|\rho)\big) \mid \rho \text{ is any quantum state}\Big\}$$

Goal: Characterise $E(A,B)$ to give tight uncertainty relations

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Characterising $E(A, B)$ for Qubits

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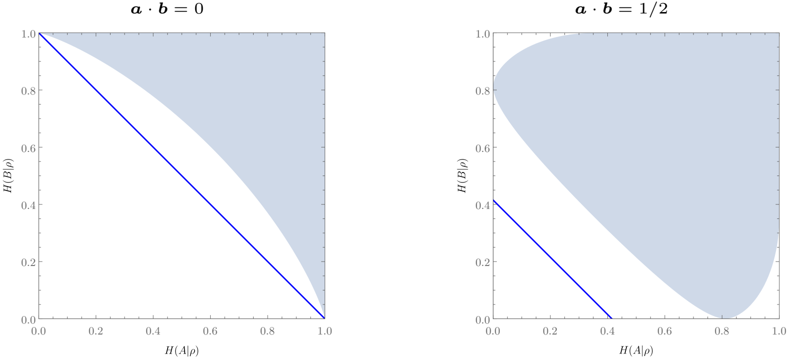

Let $h(x) = -\frac{1+x}{2}\log\left(\frac{1+x}{2}\right) - \frac{1-x}{2}\log\left(\frac{1-x}{2}\right)$ and $g(x) = h^{-1}(x)$.

$$g\left(H(A|\rho)\right)^2 + g\left(H(B|\rho)\right)^2 - 2|\boldsymbol{a}\cdot\boldsymbol{b}|\, g\left(H(A|\rho)\right) g\left(H(B|\rho)\right) \leq 1 - (\boldsymbol{a}\cdot\boldsymbol{b})^2$$

- This bound[2] is tight and characterises $E(A, B)$

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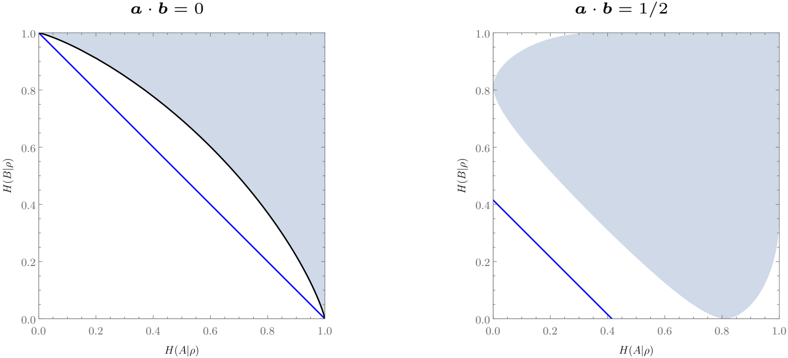

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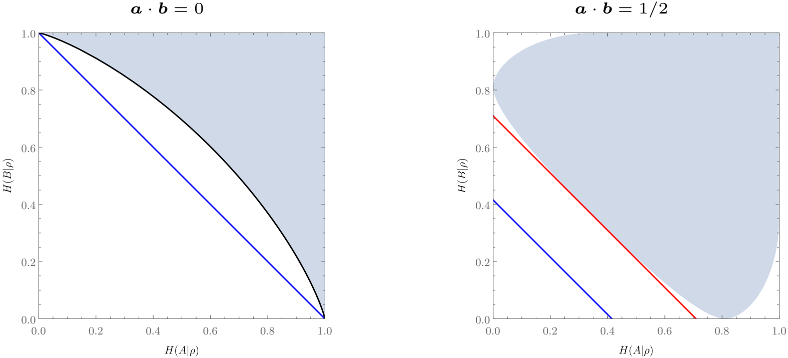

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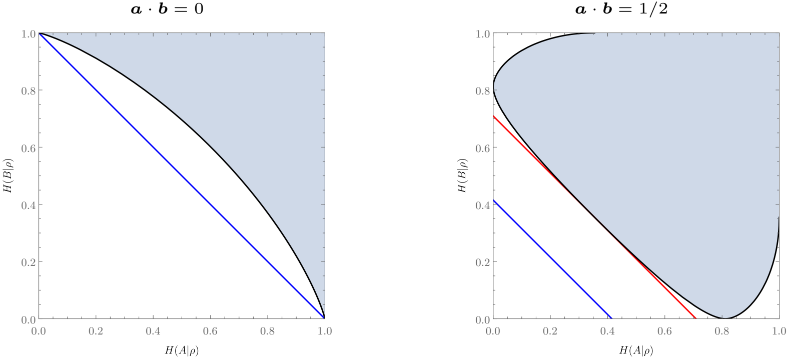

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Measurement Uncertainty Relations

- To formalise measurement uncertainty relations, we need to quantify two properties of a measurement device $\mathcal{M}$:
  - Noise: How well does $\mathcal{M}$ measure a target observable A?
  - Disturbance: How much does $\mathcal{M}$ disturb the state measured?
- Many ways one can do this:
  - Distance between target and observed distributions
  - Noise operators
- What about information theoretic approaches?
  - Quantify noise and disturbance as properties of $\mathcal{M}$ only, not for particular states

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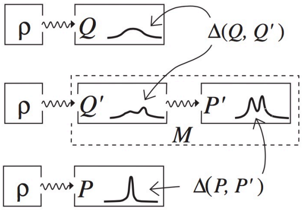

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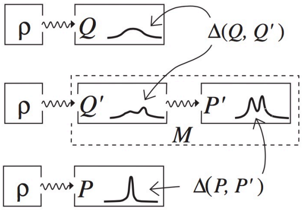

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Quantum Measurements

The most general kind of measurement device can be formalised as a quantum instrument:

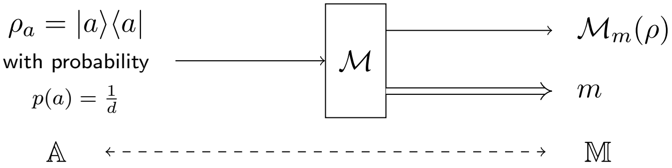
Quantum Instruments

A quantum instrument $\mathcal{M} = \{\mathcal{M}_m\}_m$ is a collection of completely positive trace-non-increasing maps $\mathcal{M}_m$ such that $\sum_m \mathcal{M}_m(\rho)$ is trace-preserving for all ρ. The probability of obtaining outcome m on input ρ is $\mathrm{Tr}[\mathcal{M}_m(\rho)]$ and the post-measurement state is $\frac{\mathcal{M}_m(\rho)}{\mathrm{Tr}[\mathcal{M}_m(\rho)]}$.

- Every instrument $\mathcal{M} = \{\mathcal{M}_m\}_m$ can be associated with a POVM $M = \{M_m\}_m$ specifying only the probabilities of each outcome

Defining Noise[3]

Let A be a discrete observable and consider the scenario:



Noise – $N(\mathcal{M}, A)$

The noise is calculated from the joint distribution $p(a, m)$:

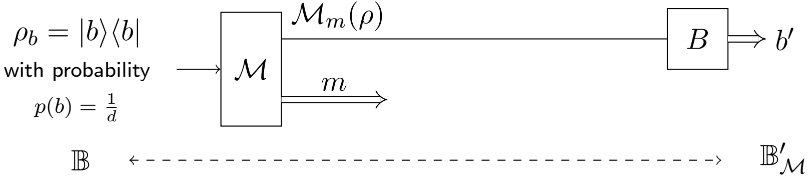
$$N(\mathcal{M}, A) = H(\mathbb{A}|\mathbb{M}) = \sum_m p(m) H(\mathbb{A}|\mathbb{M} = m).$$

- Note that $N(\mathcal{M}, A)$ depends only on the probabilities of each outcome, and not the transformation $\mathcal{M}_m$

[3] F. Buscemi, M. J. W. Hall, M. Ozawa & M. W. Wilde. PRL **112**, 050401, 2014.

Defining Disturbance

Let B be a discrete observable and consider the scenario:



Disturbance – $D(\mathcal{M}, B)$

$D(\mathcal{M}, B)$ is the uncertainty in a measurement of B following the measurement of $\mathcal{M}$ on a randomly prepared state $|b\rangle$ and the possible application of a correction $\mathcal{E}$: $D(\mathcal{M}, B) = \min_{\mathcal{E}} H(\mathbb{B}|\mathbb{B}'_{\mathcal{M},\mathcal{E}})$.

- Captures the irreversible disturbance to the state
- Note that $\mathbb{B}'_{\mathcal{M},\mathcal{E}}$ only takes d values, unlike $\mathbb{M}$

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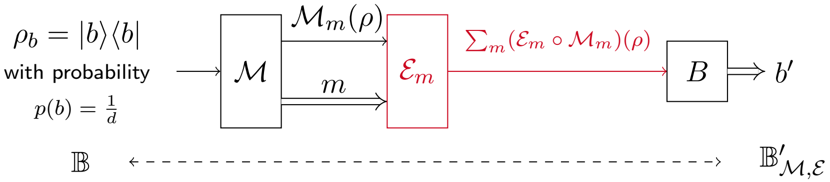

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Measurement Uncertainty Regions

The following noise-noise and noise-disturbance relations hold:

$$N(\mathcal{M}, A) + N(\mathcal{M}, B) \geq -\log \max_{a,b} |\langle a|b\rangle|^2,$$

$$N(\mathcal{M}, A) + D(\mathcal{M}, B) \geq -\log \max_{a,b} |\langle a|b\rangle|^2,$$

As for preparation relations, would like to characterise:

Noise-Noise Region

$$R_{NN}(A,B) = \big\{\big(N(\mathcal{M},A),\, N(\mathcal{M},B)\big) \mid \mathcal{M} \text{ is any instrument}\big\}$$

Noise-Disturbance Region

$$R_{ND}(A,B) = \big\{\big(N(\mathcal{M},A),\, D(\mathcal{M},B)\big) \mid \mathcal{M} \text{ is any instrument}\big\}$$

How are $E(A,B)$, $R_{NN}(A,B)$ and $R_{ND}(A,B)$ related?

- Note that all three regions depend only on A and B

From Preparation to Joint-Measurement

The noise can be rewritten in terms of measurement entropies as

$$N(\mathcal{M}, A) = \sum_m p(m) H(\mathbb{A}|\mathbb{M} = m) = \sum_m p(m) H\left(A \,\middle|\, \rho_m = \tfrac{M_m}{\mathrm{Tr}[M_m]}\right)$$

and the noise-noise region simplifies to

$$\begin{aligned} R_{NN}(A,B) = \Big\{ \sum_m p(m)\,(H(A|\rho_m),\, H(B|\rho_m)) \,\Big|\, &\{p(m), \rho_m\}_m \\ &\text{is an ensemble with } \sum_m p(m)\rho_m = \mathbb{1}/d \Big\} \\ \subseteq \Big\{ \sum_m p(m)\,(H(A|\rho),\, H(B|\rho)) \Big\} &= \mathrm{conv}\, E(A,B) \end{aligned}$$

- One can extend this trivially to more observables
- Any entropic preparation UR can thus be used to derive a valid joint-measurement UR

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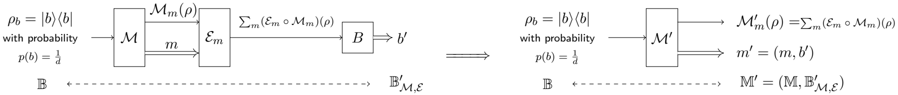

- We have $N(\mathcal{M}', A) \leq N(\mathcal{M}, A)$ and $N(\mathcal{M}', B) \leq D(\mathcal{M}, B)$
- Thus $R_{ND}(A, B) \subseteq \mathrm{cl}\, R_{NN}(A, B)$ (cl denotes the closure under increasing either coordinate up to $\log d$)
  - Note this argument requires more outcomes. The result doesn't hold in general if the number of outcomes is fixed
  - Valid (but not generally tight) noise-disturbance URs can thus be obtained from noise-noise and entropic preparation URs
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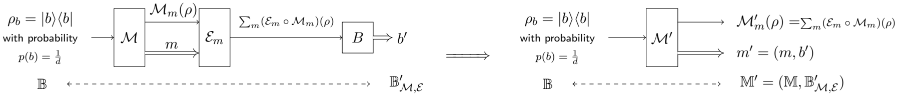

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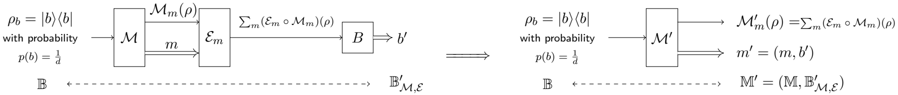

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Qubit Noise-Noise Region

For qubits, we have equality:

$$R_{NN}(A,B) = \Big\{ \sum_m p(m) \left(H(A|\rho_m),\, H(B|\rho_m) \right) \Big\} = \operatorname{conv} E(A,B)$$

- Condition that $\sum_m p(m)\rho_m = \mathbb{1}/d$ can be removed for qubits at the expense of doubling the number of outcomes
  - If one considers only dichotomic $\mathcal{M}$, $R_{NN}(A,B) = E(A,B)$
- The tight entropic preparation uncertainty relations for qubits can be used to give tight joint-measurements relations!
- Let us look at some examples for Pauli observables $A = \boldsymbol{a}\cdot\boldsymbol{\sigma}$, $B = \boldsymbol{b}\cdot\boldsymbol{\sigma}$

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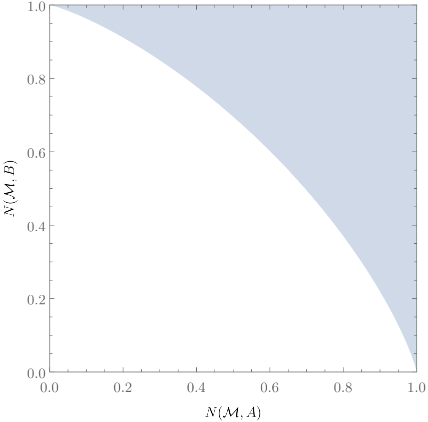
Orthogonal Spin Measurements

Consider $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ (e.g., $A = \sigma_z$, $B = \sigma_x$)

- $E(A, B)$ characterised by $g(H(A|\rho))^2 + g(H(B|\rho))^2 \leq 1$



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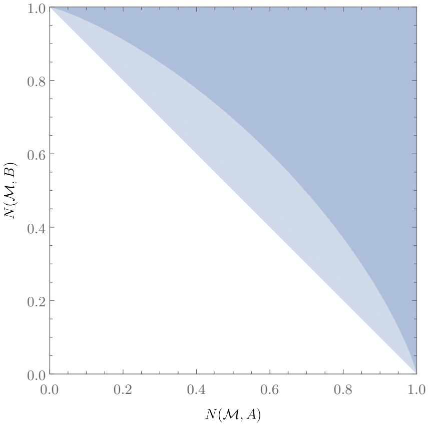

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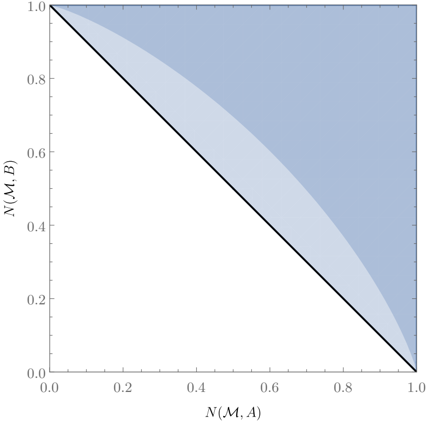

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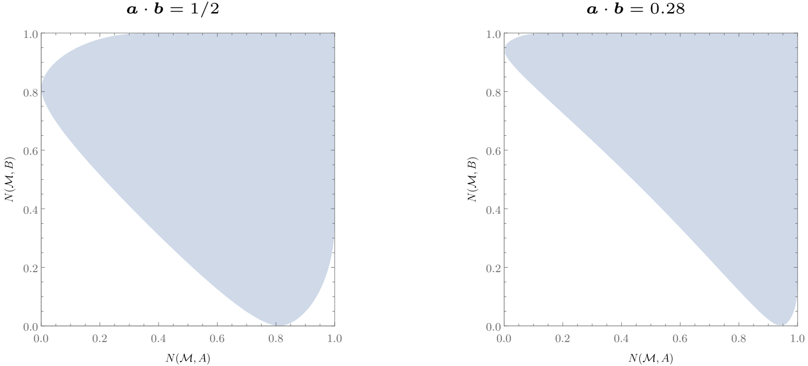
Non-Orthogonal Spin Measurements

Two cases: $E(A,B)$ convex for $|\boldsymbol{a}\cdot\boldsymbol{b}| \gtrsim 0.391$, concave otherwise



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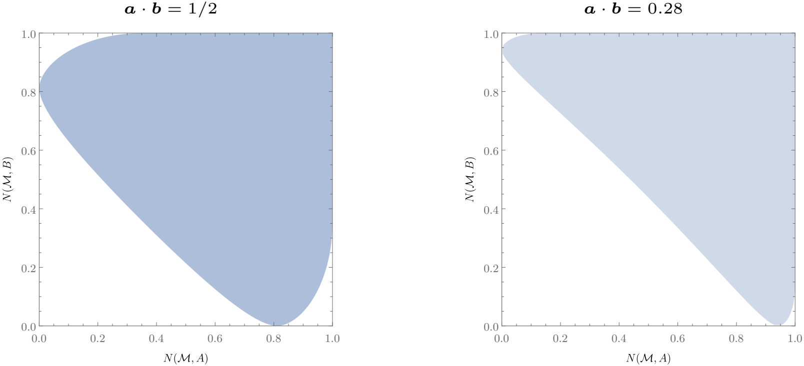

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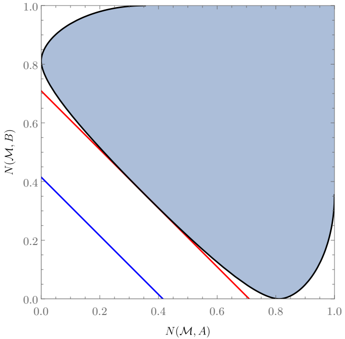

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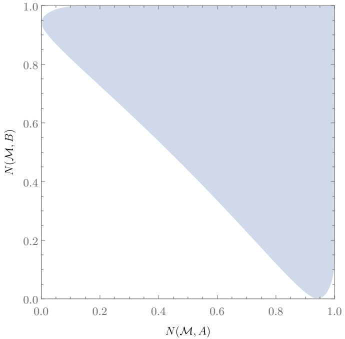

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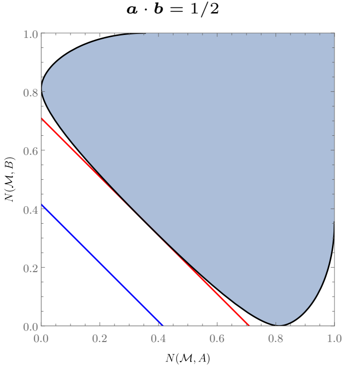

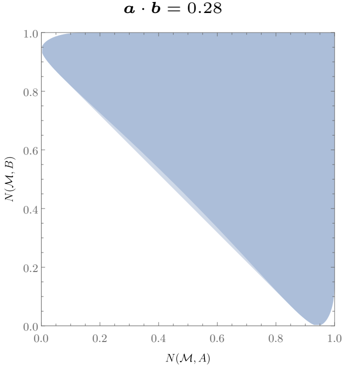

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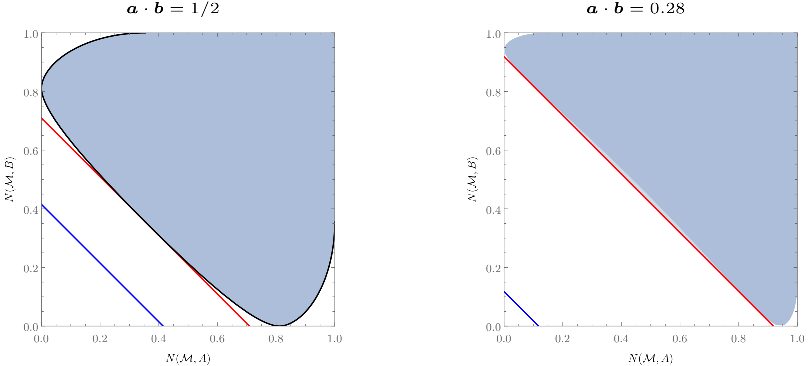

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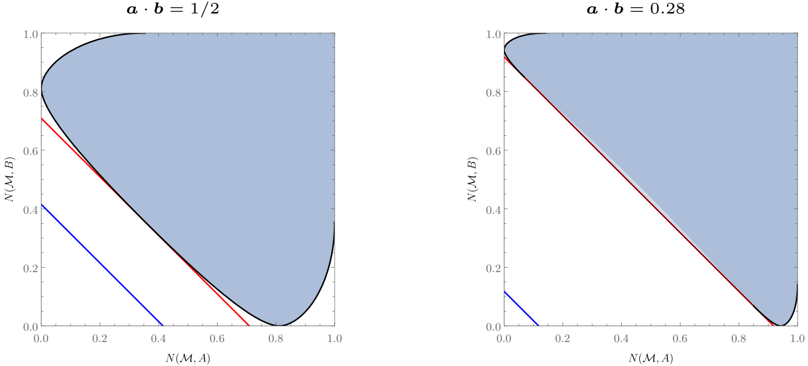

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Qubit Noise-Disturbance Relations

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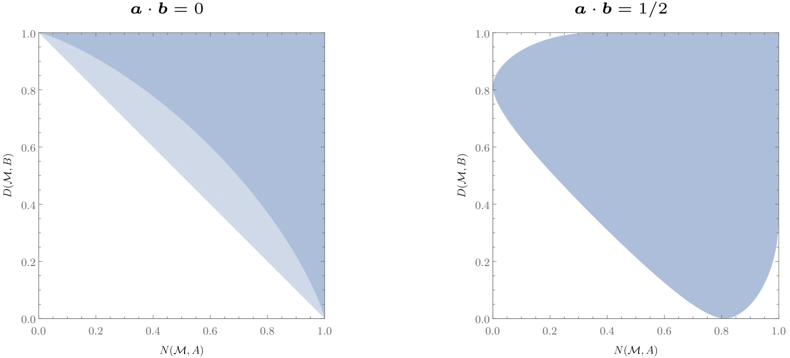

- Noise-noise uncertainty relations restricted to the lower boundary of $R_{NN}(A,B)$ are also noise-disturbance relations
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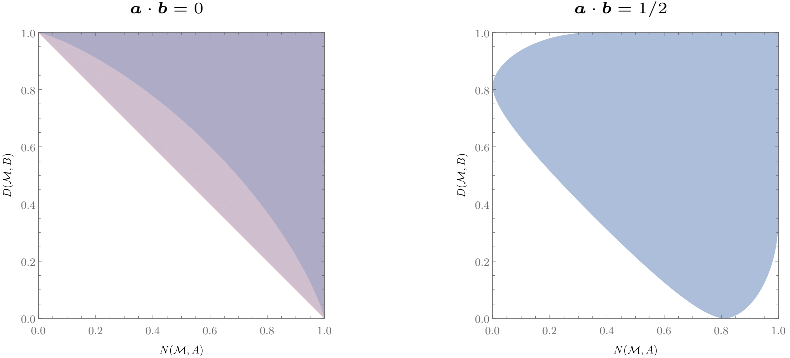

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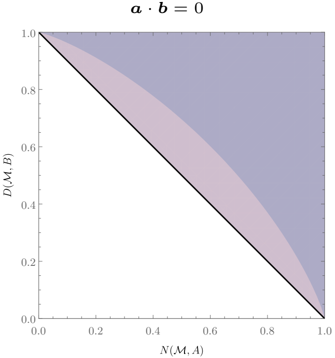

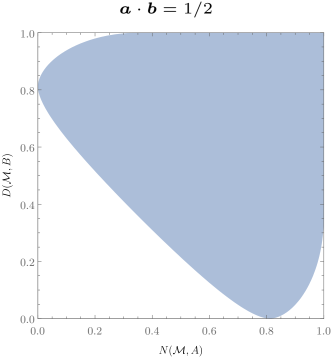

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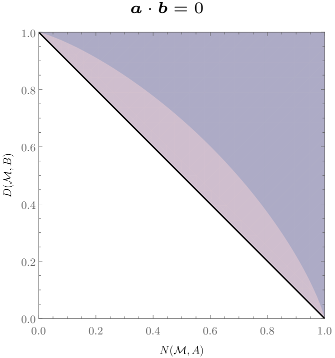

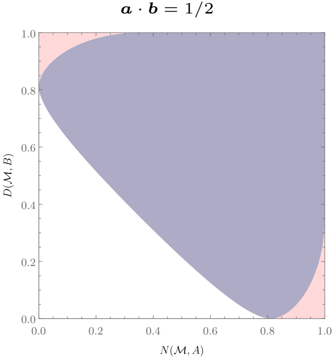

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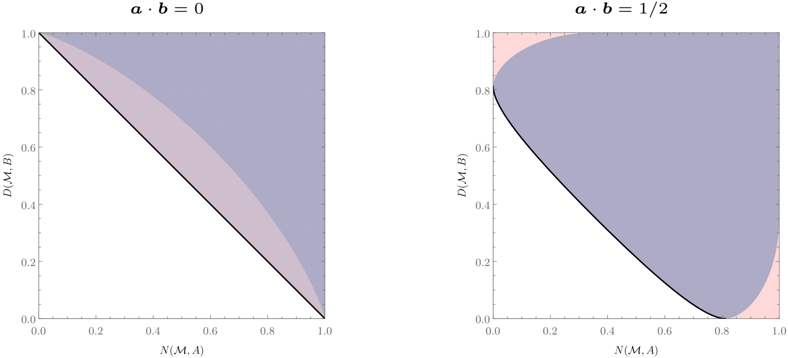

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Characterising $R_{ND}(A, B)$

- Characterising exactly $R_{ND}(A, B)$ is more difficult than $R_{NN}(A, B)$ since the transformation of the state under $\mathcal{M}_m$ and the optimal correction $\mathcal{E}_m$ must be taken into account
  - Recall the noise depends only on the POVM outcomes (i.e., outcome probabilities)
- Although the disturbance for a *given* $\mathcal{M}$ requires determining the optimal correction $\mathcal{E}$, one can ignore the correction in order to characterise the *lower boundary* of $R_{ND}(A, B)$
  - Can always incorporate $\mathcal{E}$ into $\mathcal{M}$ to give a new measurement $\mathcal{M}'$
  - Similarly, can also restrict oneself to *purity preserving* instruments:

$$\mathrm{cl}\, R_{ND}(A, B) = \mathrm{cl}\{(N(\mathcal{M}, A), H(\mathbb{B}|\mathbb{B}'_{\mathcal{M},\mathcal{I}})) \mid \mathcal{M} \text{ is purity preserving}\}$$

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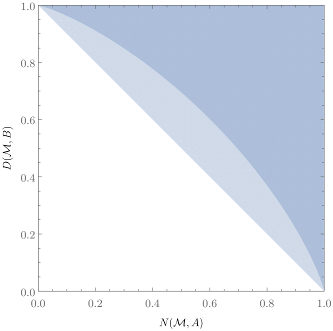
Numerically Studying $R_{ND}(\sigma_z, \sigma_x)$

Most random instruments are far from the boundary: need to sample carefully

- Generate random POVMs $M = \{M_m\}_m$ and consider transformations of form $\mathcal{M}_m(\rho) = U_m\sqrt{M_m}\rho\sqrt{M_m}U_m^\dagger$
  - Look for optimal $\{U_m\}_m$ for each such M



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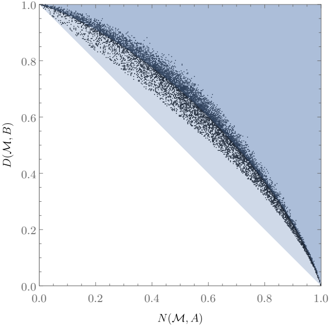

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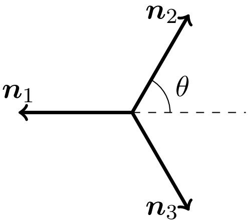
Saturating the $R_{ND}(\sigma_z, \sigma_x)$ bound

The apparent numerical bound can be parameterised by a class of 3-outcome measurements:

- Consider the POVM $M = \{M_m = p_m(\mathbb{1} + \boldsymbol{n}_m \cdot \boldsymbol{\sigma})\}_m$ where p_m which must satisfy $\sum p_m = 1$ and $\sum p_m \boldsymbol{n}_m = \mathbf{0}$
- For $\theta \in [0, \pi/2]$ take $\boldsymbol{n}_m$ as follows:



- Following measurement outcome m, the system is in state with Bloch-vector $\boldsymbol{n}_m$
- Perform the correction mapping $\boldsymbol{n}_2, \boldsymbol{n}_3 \to \boldsymbol{x}$ and leaving $\boldsymbol{n}_1 = -\boldsymbol{x}$ unchanged

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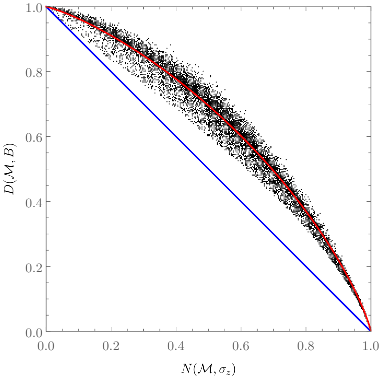
Conjecture

$R_{ND}(\sigma_z, \sigma_x)$ is bounded by the parametric curve $(N(\mathcal{M}, \sigma_z), D(\mathcal{M}, \sigma_x)) = \left(\frac{\cos\theta + h(\sin\theta)}{1+\cos\theta}, \frac{h(\cos\theta)}{1+\cos\theta}\right)$ for $0 \le \theta \le \frac{\pi}{2}$.

Note that (cf. noise-noise case):

- This region is non-convex
- Asymmetrical
- Three-outcome measurement appear to be optimal
- Makes use of non-trivial corrections/transformations for the measurements



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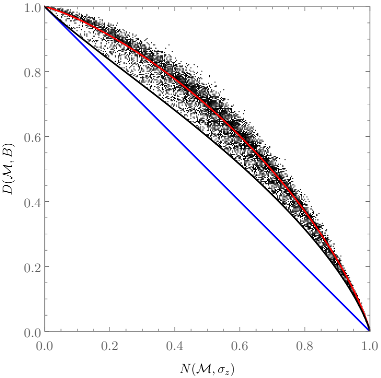

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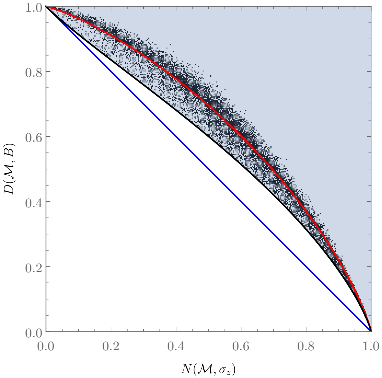

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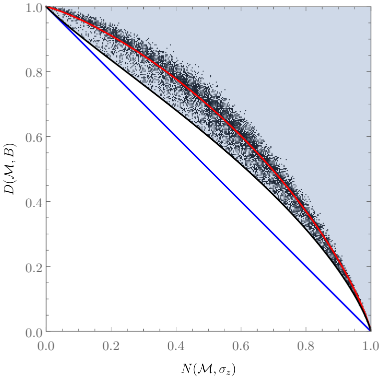

Relations for Lüders Instruments

The fact that non-trivial dynamics seem necessary to saturate $R_{ND}(\sigma_z, \sigma_x)$ raises the question: What noise-disturbance values can be obtained by dynamics implementing simple dynamics

- In particular, for Lüders instruments that implement $\mathcal{M}(\rho) = \sqrt{M_m}\rho\sqrt{M_m}$
- Analytic analysis simplified, can calculate "$\mathcal{I}$-disturbance" directly from $\sum_m \mathcal{M}_m(\rho)$

Theorem

For Lüders instruments with no post-measurement correction (on qubits), $R_{ND_{\mathcal{I}}}(A, B) = E(A, B)$ and for orthogonal spins we have the tight relation

$$g(N(\mathcal{M}, \sigma_z))^2 + g(D_{\mathcal{I}}(\mathcal{M}, \sigma_x))^2 \leq 1.$$

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Dichotomic Measurements

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- Perhaps surprisingly, no.

There is a measurement $\mathcal{M}$ giving $g(N(\mathcal{M},\sigma_z))^2 + g(D(\mathcal{M},\sigma_x))^2 \approx 1.024$.

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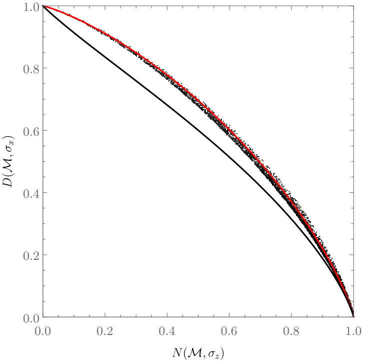

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Summary

- Relations between preparation and measurement URs
  - $R_{ND}(A,B) \subseteq \mathrm{cl}\, R_{NN}(A,B)$ & $R_{NN}(A,B) \subseteq \mathrm{conv}\, E(A,B)$
  - Entropic preparation uncertainty relations (lower bounds for $E(A,B)$) can immediately give joint-measurement and noise-disturbance relations
  - Inclusions not necessarily tight
- Tight joint-measurement uncertainty relations for qubits
  - Four-outcome POVMs needed for "optimal" measurements when $E(A,B)$ non-convex
  - Can readily generalise to 3 or more observables
- Conjectured tight characterisation of noise-disturbance region
  - Three-outcome measurements with non-trivial corrections or measurement dynamics needed for "optimal" measurements

Outlook and References

Several points remain to explore:

- Noise-disturbance for non-orthogonal Pauli measurements
- Measurements that are optimal with respect to both tradeoffs
- Higher dimensional systems
- Relation to other notions of noise/disturbance

Thank You!

Further information:

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Saturating Noise-Noise Bound

- Any point $\big(N(\mathcal{M},A),N(\mathcal{M},B)\big)$ in $\{(H(A|\rho),\, H(B|\rho))|\rho\}$ can be obtained by a POVM projecting onto ρ, i.e.,

$$\{\tfrac{1}{2}(\mathbb{1}+\boldsymbol{r}\cdot\boldsymbol{\sigma}),\ \tfrac{1}{2}(\mathbb{1}-\boldsymbol{r}\cdot\boldsymbol{\sigma})\}$$

- Let (u_1,v_1) and (u_2,v_2) be two points obtained by projections onto ρ_1 and ρ_2
- For any $q\in[0,1]$ the POVM

$$\{\tfrac{q}{2}(\mathbb{1}+\boldsymbol{r}_1\cdot\boldsymbol{\sigma}),\ \tfrac{q}{2}(\mathbb{1}-\boldsymbol{r}_1\cdot\boldsymbol{\sigma}),\tfrac{1-q}{2}(\mathbb{1}+\boldsymbol{r}_2\cdot\boldsymbol{\sigma}),\ \tfrac{1-q}{2}(\mathbb{1}-\boldsymbol{r}_2\cdot\boldsymbol{\sigma})\}$$

  gives

$$\begin{aligned}\big(N(\mathcal{M},A),N(\mathcal{M},B)\big) &= q(u_1,v_1)+(1-q)(u_2,v_2)\\ &= (qu_1+(1-q)u_2, qv_1+(1-q)v_2)\end{aligned}$$