Multipartite Causal Correlations, Polytopes and Inequalities

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joint work with

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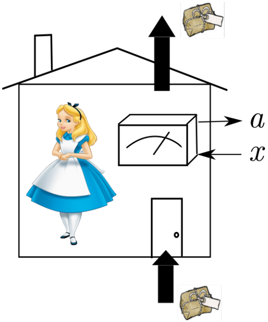
Quantum Networks, Natal, 21–25 November 2016

PRA 94, 032131 (2106) – [arXiv:1608.01528]

Outline

Operational Setting

- Consider labs that can receive a physical system, perform some operation (depending on classical input), produce a classical output, and send out a system



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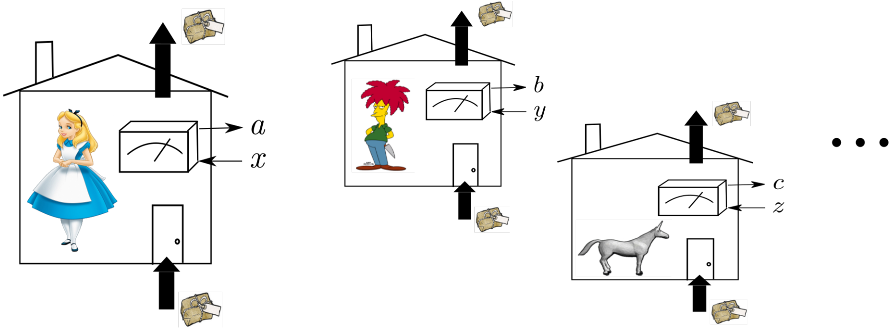

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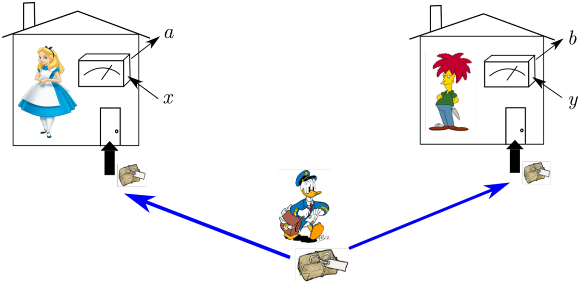
Bell-type Scenarios



$p(a,b|x,y)$

- A and B cannot signal to each other ($A \not\prec\not\succ B$)
  - $P(a|x,y) = P(a|x,y') \; \forall a,x,y,y'$
  - $P(b|x,y) = P(b|x',y) \; \forall b,y,x,x'$
- Can obtain only no-signalling correlations

What about other causal relationships? E.g., $A \prec B$?

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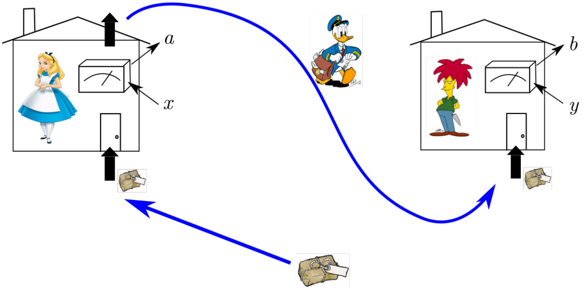

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What about other causal relationships? E.g., $A \prec B$?

Bipartite Causal Correlations

What correlations can be obtained if we only assume that A and B have *some* causal structure?

- No two-way signalling
  - Either $A \prec B$ or $B \prec A$
- In full generality, can allow choosing the causal order randomly

Bipartite Causal Correlations

A correlation $P(a, b|x, y)$ is causal iff it can be written

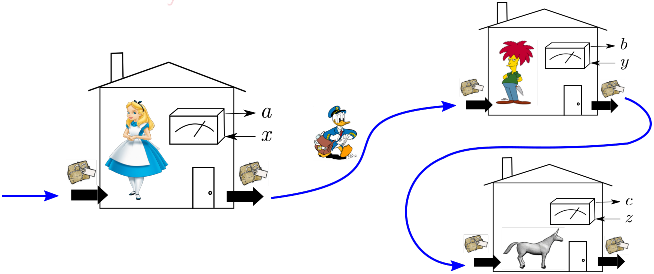
$$P(a, b|x, y) = qP^{A \prec B}(a, b|x, y) + (1 - q)P^{B \prec A}(a, b|x, y),$$

where $P^{A \prec B}(a, b|x, y) = P_A(a|x)P_B(b|y, a, x)$, and similarly for $P^{B \prec A}(a, b|x, y)$.

Tripartite Causal Orders

What causal orders can three parties have?

- Sequential orders: $A \prec B \prec C$, $B \prec C \prec A$, etc.
- There are also dynamical orders



$$P(a,b,c|x,y,z) = P(a|x)\underbrace{P_{x,a}(b|y)P_{x,y,a,b}(c|z)}_{\text{causal } P_{x,a}(b,c|y,z)}$$

In general, one party acts first, then can determine (probabilistically, and as a function of their input) which party is next

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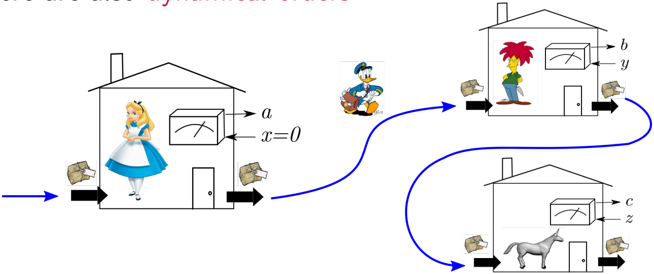

$$P(a,b,c|x,y,z) = P(a|x)\underbrace{\left[\delta_{x,0}P_{0,a}(b|y)P_{0,y,a,b}(c|z) + \delta_{x,1}P_{1,a}(c|z)P_{1,z,a,c}(b|y)\right]}_{\text{causal } P_{x,a}(b,c|y,z)}$$

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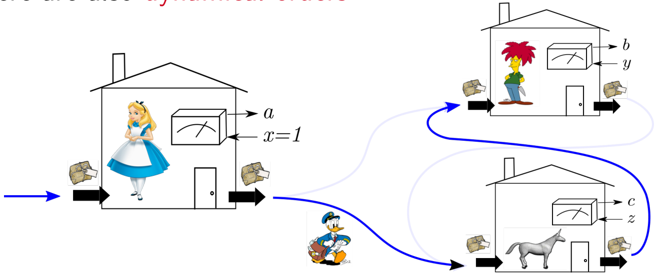

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In general, one party acts first, then can determine (probabilistically, and as a function of their input) which party is next

Multipartite Causal Correlations

- Parties $A_1, \ldots, A_N$, inputs $\vec{a} = (a_1, \ldots, a_N)$, outputs $\vec{x} = (x_1, \ldots, x_N)$

N-partite Causal Correlations[1]

1. For $N = 1$, any $P(a_1|x_1)$ is causal.
2. For $N \geq 2$, $P(\vec{a}|\vec{x})$ is causal iff it can be written

$$P(\vec{a}|\vec{x}) = \sum_k q_k P_k(a_k|x_k) P_{k,x_k,a_k}(\vec{a}_{\setminus k}|\vec{x}_{\setminus k}),$$

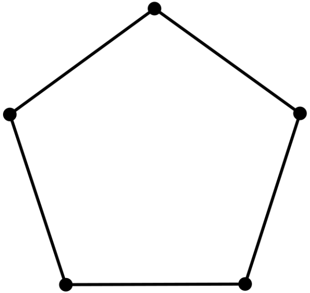
with $q_k \geq 0$, $\sum_k q_k = 1$, and all $P_{k,x_k,a_k}(\vec{a}_{\setminus k}|\vec{x}_{\setminus k})$ are causal $(N-1)$-partite correlations.

[1] Equivalent to Oreshkov & Giarmatzi. *New J. Phys.* **18**, 093020 (2016).

The Causal Polytope

The set of all causal correlations forms a causal (convex) polytope[2]

- Vertices are deterministic correlations
  - Both fixed-order and dynamical vertices
- Facets are causal inequalities



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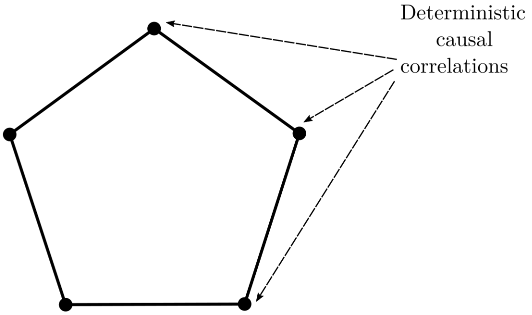

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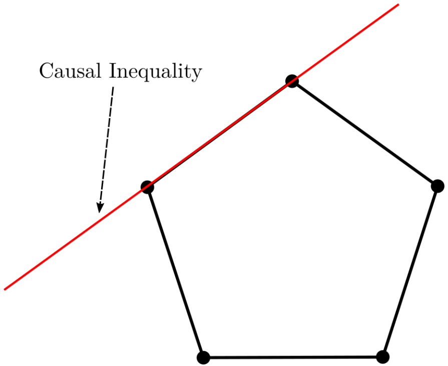

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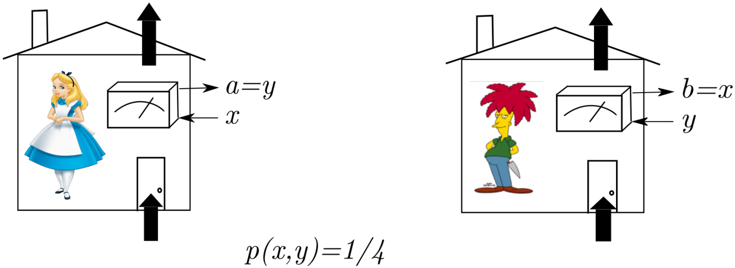
Bipartite Causal Correlations

Bipartite (binary I/O) polytope characterised by Branciard *et al.*[3]

- Two types of non-trivial causal inequality:
  - Guess your neighbour's input (GYNI):

$$p_{\text{win}} = P\left(a = y,\ b = x\right) \leq \tfrac{1}{2}$$

  - Lazy GYNI: $p_{\text{win}} = P\left(x(a \oplus y) = 0,\ y(b \oplus x) = 0\right) \leq \frac{3}{4}$



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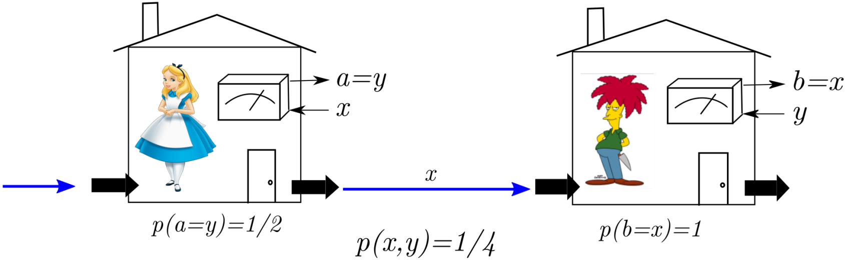

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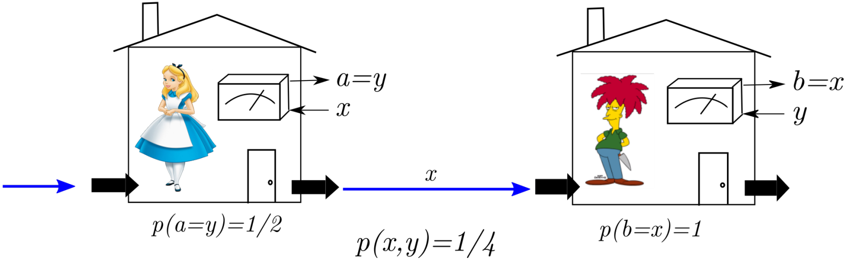

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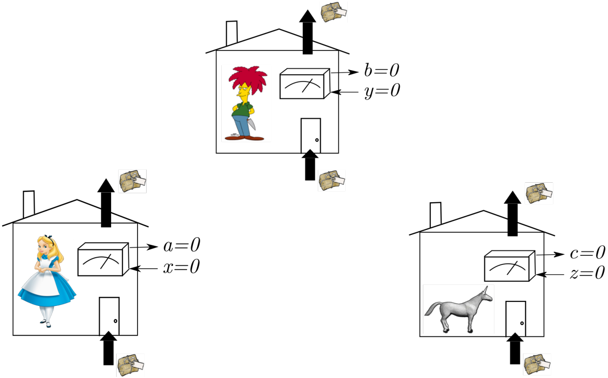
Tripartite Causal Polytope

For binary I/O, the polytope has 138 304 vertices in 56 dimensions

- Too large to solve for facets

Can have interesting correlations even if some outputs are fixed

- Simplest scenario is the "lazy binary tripartite scenario" (always output 0 on input 0)



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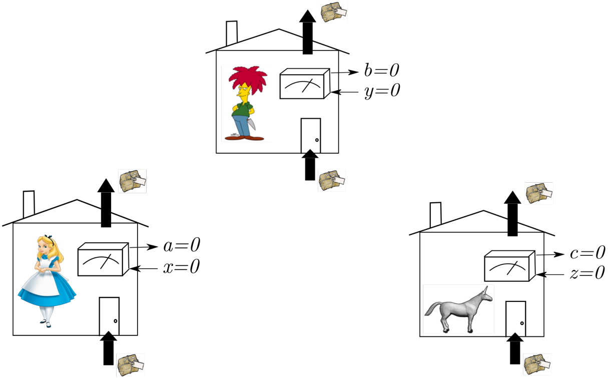

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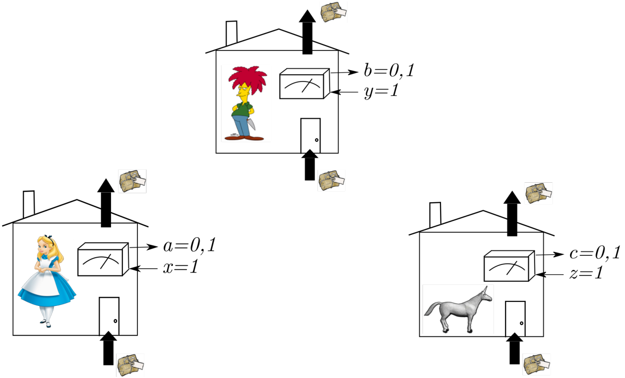


Simplest Tripartite Causal Polytope

Lazy binary tripartite causal polytope has 680 vertices (192 can only be realised by dynamical causal orders) in 19 dimensions

- Can solve completely for facets
- 305 families of inequivalent causal inequalities (of which 3 are trivial)
- Includes the lazy bipartite (conditional) GYNI inequalities

$$P\left(x(a \oplus y) = 0,\ y(b \oplus x) = 0 \,|\, z = 0\right) \leq \frac{3}{4}$$

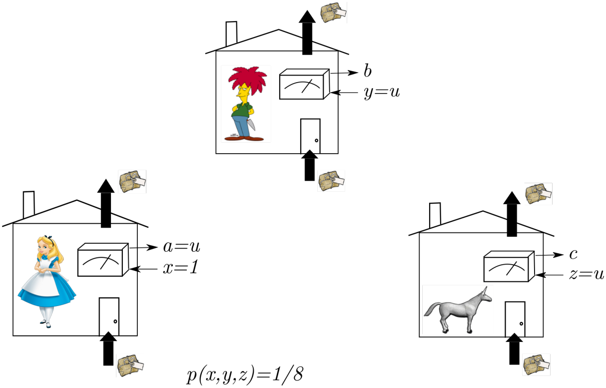
- All are also inequalities of the non-lazy tripartite polytope

Tripartite Facet Inequalities

$$P_A(1|100) + P_B(1|010) + P_C(1|001) - P_{ABC}(111|111) \geq 0$$

and the corresponding game is won with $p_{\text{win}} \leq 7/8$

- If a party receives 1 and other two get same input, he/she should guess that input



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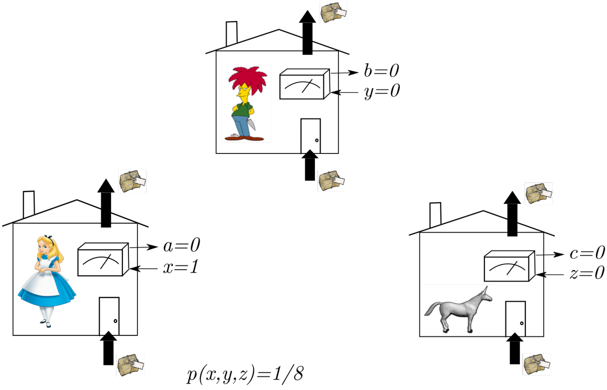

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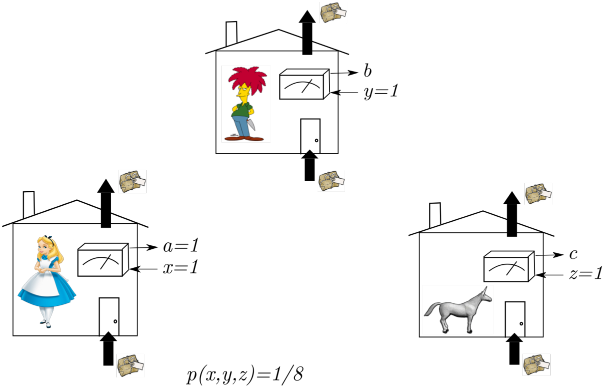

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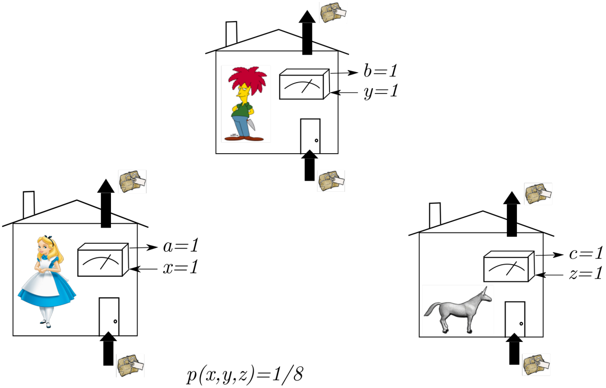

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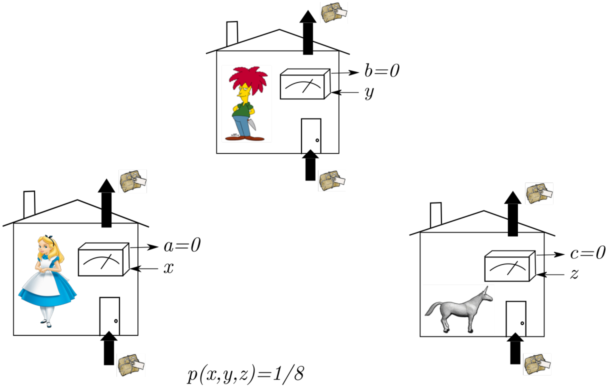

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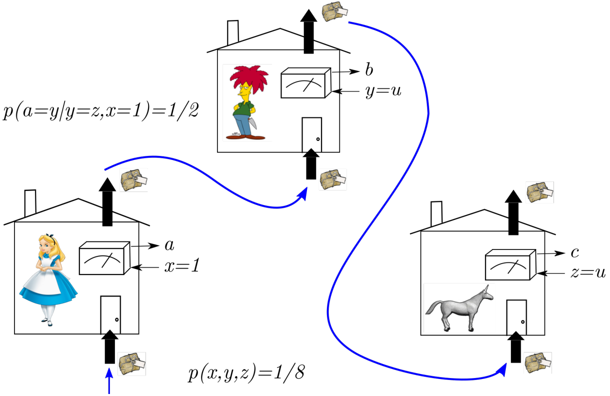

Tripartite Facet Inequalities

$$P_{AB}(11|110) + P_{BC}(11|011) + P_{AC}(11|101) - P_{ABC}(111|111) \geq 0$$

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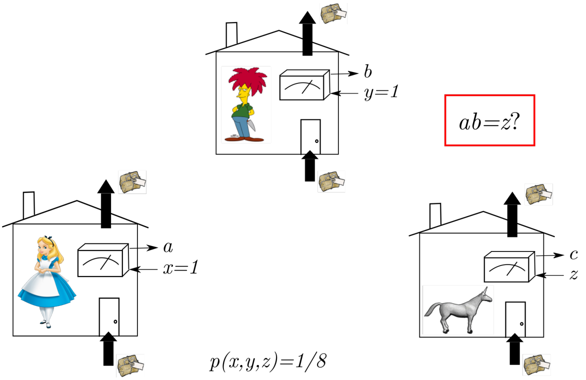
- If two parties get input 1, product of their outputs should be 3rd party's input



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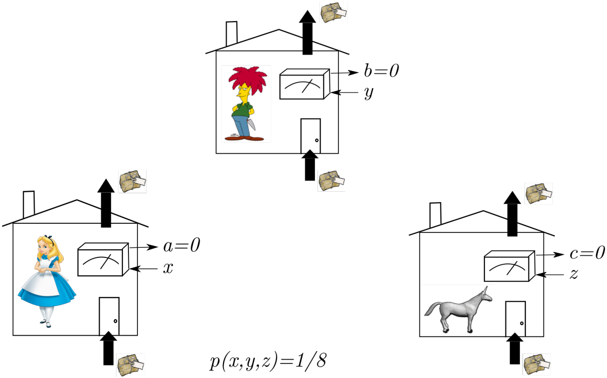
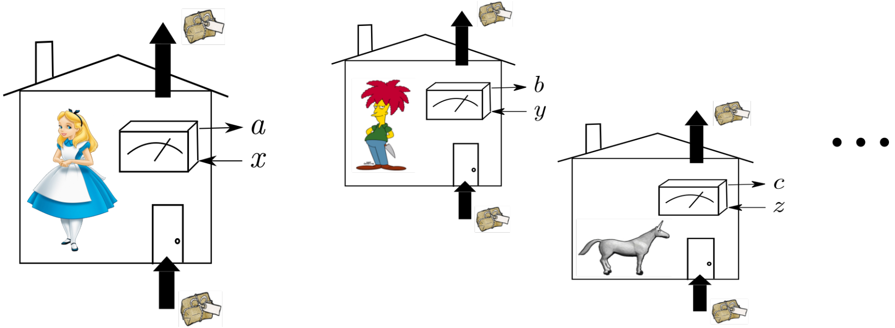


Process Matrix Formalism

- Can we violate causal inequalities if global causality dropped?
- QM holds locally for each agent
  - Agent's action modelled by quantum instrument
- No assumption of *global* causal order, only global consistency of probabilities



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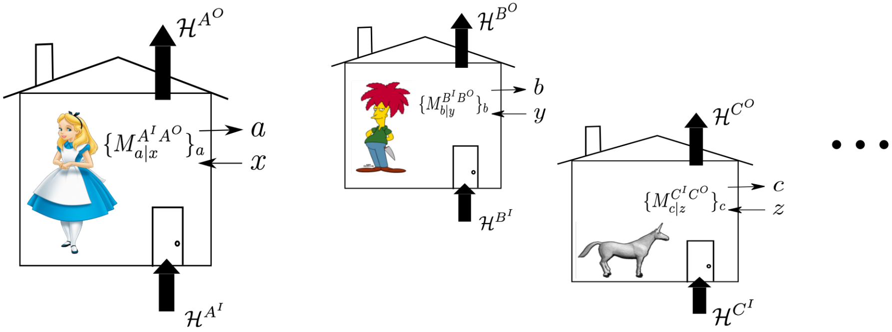


Process Matrix Formalism

Process matrix: $W \in \mathcal{L}(\mathcal{H}^{A^I} \otimes \mathcal{H}^{A^O} \otimes \mathcal{H}^{B^I} \otimes \mathcal{H}^{B^O} \otimes \mathcal{H}^{C^I} \otimes \mathcal{H}^{C^O})$

- $P(a,b,c|x,y,z) = \mathrm{tr}\left[\left(M_{a|x}^{A^I A^O} \otimes M_{b|y}^{B^I B^O} \otimes M_{c|z}^{C^I C^O}\right) \cdot W\right]$
- Valid processes defined by requirements that:
  - Positivity of probabilities: $W \geq 0$
  - Normalisation of probabilities: $\sum_{a,b,c} P(a,b,c|x,y,z) = 1$ for all operations $\{M_{a|x}\}_a$, $\{M_{b|y}\}_b$, $\{M_{c|z}\}_c$

Crucially, there exist process matrices violating causal inequalities[4]

- Some noncausal processes can be implemented, but open question whether causal inequalities can be violated in the lab
- Can search for violation of inequalities by generalising SDP approach applied to bipartite case

[4] Oreshkov, Costa & Brukner. *Nat. Commun.* **3**, 1092 (2012).

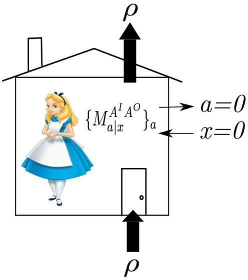
Violating the Inequalities

$$I_1 = P_A(1|100) + P_B(1|010) + P_C(1|001) - P_{ABC}(111|111) \geq 0$$

- With qubits, able to find a W_1 giving violation $I_1 \approx -0.3367 < 0$
  - No simple form for W_1
  - All parties use the instruments:

$$\left\{M_{0|0} = |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|\right\}, \left\{M_{0|1} = |0\rangle\langle 0| \otimes |1\rangle\langle 1|,\ M_{1|1} = |1\rangle\langle 1| \otimes |0\rangle\langle 0|\right\}.$$

- Open question whether higher dimensional systems can give better violation



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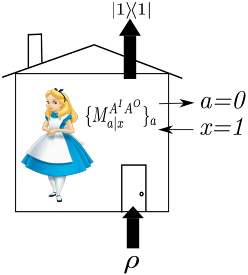

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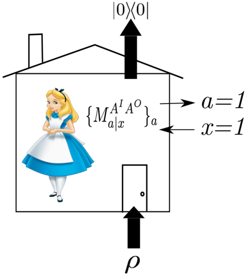

Violating the Inequalities

$$I_2 = P_{AB}(11|110) + P_{BC}(11|011) + P_{AC}(11|101) - P_{ABC}(111|111) \geq 0$$

Violated by the classical process matrix

$$W_2 = \frac{1}{8}\Big[\mathbb{1}^{\otimes 6} - \frac{1}{2}\big[Z\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}Z\mathbb{1}\mathbb{1}\mathbb{1} + \mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}Z\mathbb{1} - Z\mathbb{1}Z\mathbb{1}Z\mathbb{1}$$
$$+ (\mathbb{1}-Z)Z(\mathbb{1}-Z)ZZ\mathbb{1} + Z\mathbb{1}(\mathbb{1}-Z)Z(\mathbb{1}-Z)Z + (\mathbb{1}-Z)ZZ\mathbb{1}(\mathbb{1}-Z)Z\big]\Big]$$

and instruments

$$\big\{M'_{0|0} = \frac{1}{2}(\mathbb{1} + ZZ) = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|\big\},$$
$$\big\{M_{0|1} = |0\rangle\langle 0| \otimes |1\rangle\langle 1|,\ M_{1|1} = |1\rangle\langle 1| \otimes |0\rangle\langle 0|\big\}.$$

Gives $I_2 = -1 < 0$

- Maximum algebraic violation with classical process matrices[5]
  - Impossible in bipartite scenario[6]

[5] Baumeler, Feix & Wolf. *Phys. Rev. A* **90**, 042106 (2014).

[6] Oreshkov, Costa & Brukner. *Nat. Commun.* **3**, 1092 (2012).

Summary of Violations

Violation found for all 302 non-trivial inequalities with qubits

	Non algebraic max	Algebraic max	
Can be violated classically	219	65	284
No classical violation found	18	0	18
			302

Of the 284 that can be violated classically, larger violation can be found with non-classical processes for 202

- Algebraic maximum only obtained by classical processes
- Same instruments used to violate all inequalities

Summary of Violations

Violation found for all 302 non-trivial inequalities with qubits

	Non algebraic max	Algebraic max	
Can be violated classically	219	65	284
No classical violation found	18	0	18
			302

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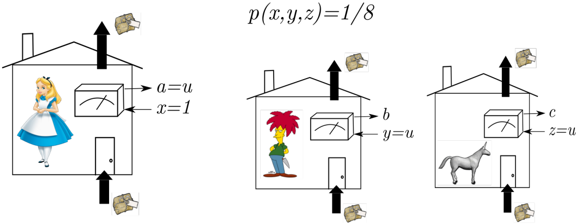
Generalisations

Can these inequalities be generalised to N-parties?

$$P_A(1|100) + P_B(1|010) + P_C(1|001) - P_{ABC}(111|111) \geq 0$$



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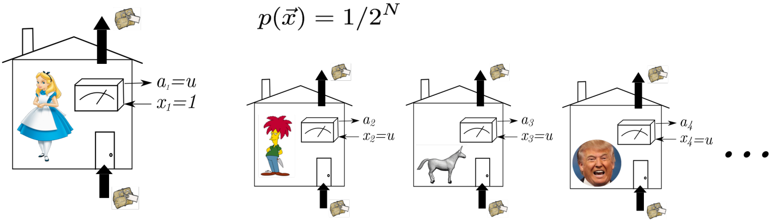

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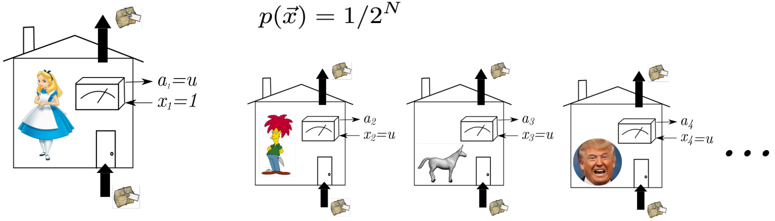
$$p(\vec{x}) = 1/2^N$$



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- Proven to be facet for $N = 4$
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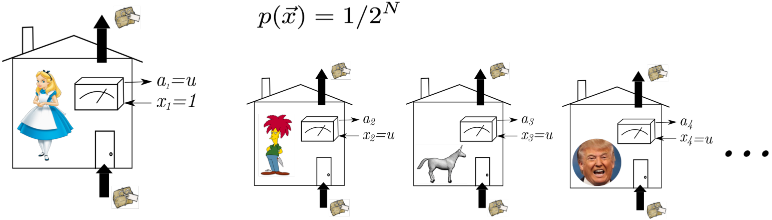

- $p_{\text{win}} \leq 1 - 2^{-N} \xrightarrow{N\to\infty} 1$
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Generalisations

Similarly,

$$P_{AB}(11|110) + P_{BC}(11|011) + P_{AC}(11|101) - P_{ABC}(111|111) \geq 0$$

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$$\sum_{k \in \mathcal{N}} P_{\mathcal{N}\setminus k}\big(1,\ldots,1\,|\,x_k = 0,\, \vec{x}_{\setminus k} = (1,\ldots,1)\big) - P_{\mathcal{N}}\big(1,\ldots,1\,|\,1,\ldots,1\big) \;\geq\; 0\,,$$

with $p_{\text{win}} \leq 1 - 2^{-N}$

For binary I/O (non-lazy) scenario, GYNI generalises[7] to "Guess your left-hand neighbour's input when an even number of parties receive input 1"

- Facet for 3 parties with $p_{\text{win}} \leq 3/4$

[7] M. Araújo & A. Feix, *private communication* (2016).

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Summary

Take home messages:

- Causal correlations form a polytope, where vertices are the deterministic correlations obtainable by either *fixed-order* or *dynamic causal orders*
  - Facets are *causal inequalities*
- Completely characterised the simplest tripartite example
  - 302 nontrivial causal inequalities
  - All can be violated by process matrices, most by *classical* process matrices

Open Questions

- Can any process violating these inequalities be physically implemented?
- Do these inequalities detect *genuine* multipartite noncausality?

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Thank You!

Paper Reference: PRA 94, 032131 (2106) – [arXiv:1608.01528]

Looking for Violations of Inequalities

We generalise the approach used by Branciard *et al.*[8] to find violations of bipartite inequalities

- Given all but one of W, $\{M_{a|x}\}_a$, $\{M_{b|y}\}_b$, $\{M_{c|z}\}_c$, finding the last one giving maximal violation of a causal inequality is an SDP problem
- Start with random instruments and use iterative 'see-saw' approach
    - Generalises to N-parties

[8] Branciard, Araújo, Feix, Costa & Brukner. *New J. Phys.* **18**, 013008 (2016).