Causal Nonseparability in Multipartite Scenarios

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joint work with
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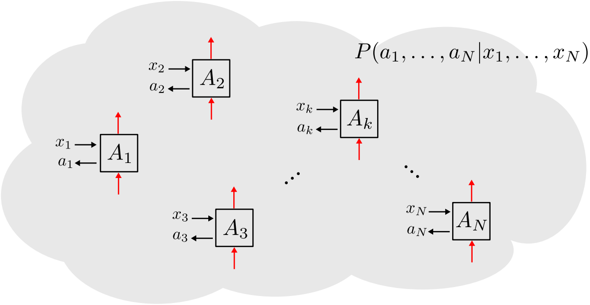
Brisbane, 9 November 2018

[arXiv:1807.10557]

Outline

General Operational Framework



- What constraints does a global causal structure impose on:
  - The correlations $P(a_1, \ldots, a_N | x_1, \ldots, x_N)$?
  - The physical resource generating the correlations?
- Assume "local quantum mechanics":
  - Input/output Hilbert spaces $\mathcal{H}^{A_I^k}$ and $\mathcal{H}^{A_O^k}$
  - Parties perform completely positive maps $\mathcal{M}_{a|x}$

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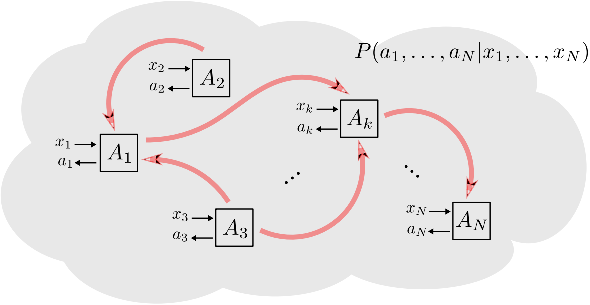

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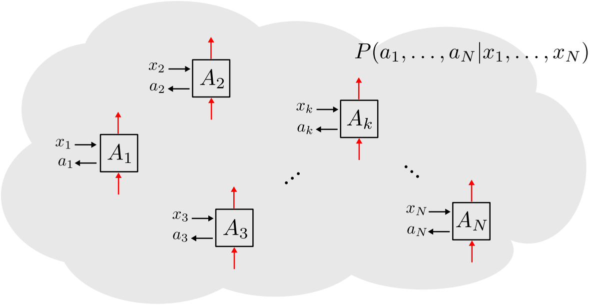

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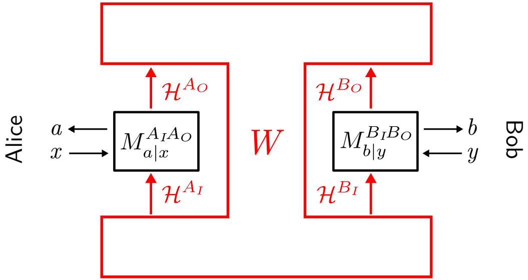
Bipartite Process Matrices



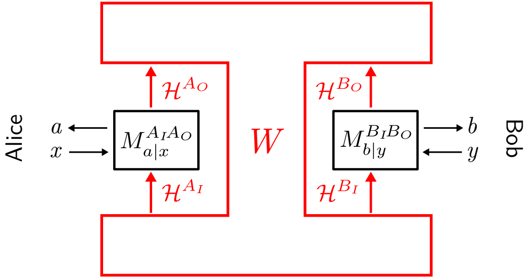
- Alice and Bob's operations: $\mathcal{M}_{a|x}$ and $\mathcal{M}_{b|y}$
  - Represent via CJ isomorphism as PSD matrices $M_{a|x}$ and $M_{b|y}$
  - $\sum_a \mathcal{M}_{a|x}$ is CPTP $\implies Tr_{A_O} \sum_a M_{a|x} = \mathbb{1}^{A_I}$

Correlations can be obtained via the generalised Born rule:

$$P(a,b|x,y) = \mathrm{Tr}\left[(M_{a|x} \otimes M_{b|y}) \cdot W\right]$$

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

Bipartite Process Matrices



Requiring $P(a,b|x,y)$ to be a valid probability distribution, even when the parties share ancillary states ρ gives:

- Positivity: $W \geq 0$
- Normalisation: $W \in \mathcal{L}^{\{A,B\}}$ and $\mathrm{Tr}\, W = d_{A_O} d_{B_O}$
- $\mathcal{L}^{\{A,B\}}$ is linear subspace of "valid" process matrices

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

Fixed Order Process Matrices

- Some processes are compatible with a fixed causal order
  - Defined in terms of signalling constraints:
    - $A \prec B$ means B cannot signal to A
  - $W^{A \prec B} = W^{A_{IO} B_I} \otimes \mathbb{1}^{B_O}$
- E.g. channel: $W^{A \prec B} = \rho^{A_I} \otimes E^{A_O B_I} \otimes \mathbb{1}^{B_O}$

[Chiribella *et al.*, PRA 2009]; [Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

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- $\mathcal{L}^{A\prec B}$: subspace of valid processes compatible with $A \prec B$
- $W^{A\prec B} \in \mathcal{L}^{A\prec B}$ if:
  1. $W^{A\prec B} = \big(\mathrm{Tr}_{B_O}[W^{A\prec B}]\big) \otimes \mathbb{1}^{B_O}$
  2. $\underbrace{\mathrm{Tr}_{B_{IO}} W^{A\prec B}}_{\text{Reduced process for } A} = \big(\mathrm{Tr}_{A_O}[\mathrm{Tr}_{B_{IO}} W^{A\prec B}]\big) \otimes \mathbb{1}^{A_O}$
- Quantum circuit / channel with memory:

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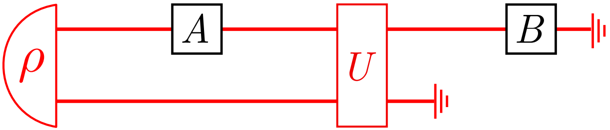

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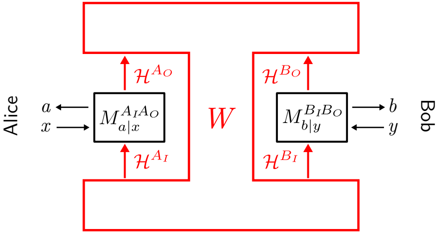
Causally Separable Process Matrices



Causally separable process matrix

$$W^{\mathsf{sep}} = q\, W^{A\prec B} + (1-q)\, W^{B\prec A},$$

(with $W^{A\prec B} \in \mathcal{L}^{A\prec B}$, $W^{B\prec A} \in \mathcal{L}^{B\prec A}$)

- Well defined causal order in every experimental run
- Causally nonseparable processes exist (in mathematics, at least)

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]

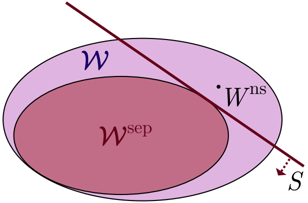
Witnesses of Causal Nonseparability

Causally separable process matrix

$$W^{\mathsf{sep}} = q\, W^{A\prec B} + (1-q)\, W^{B\prec A},$$

- Convex cone of (non-normalised) causally separable processes:

$$\mathcal{W}^{\mathsf{sep}} = (\mathcal{P} \cap \mathcal{L}^{A\prec B}) + (\mathcal{P} \cap \mathcal{L}^{B\prec A})$$



Witness of causal nonseparability

$\forall\, W^{\mathsf{ns}} \notin \mathcal{W}^{\mathsf{sep}},\ \exists S:$

$\mathrm{Tr}[S \cdot W^{\mathsf{ns}}] < 0$, and

$Tr[S \cdot W^{\mathsf{sep}}] \geq 0 \quad \forall\, W^{\mathsf{sep}} \in \mathcal{W}^{\mathsf{sep}}$

[Araújo *et al.*, NJP 2015; Branciard, Sci. Rep. 2016]

- Witnesses can be efficiently constructed by semidefinite programming (SDP)
- Witnesses can be measured experimentally

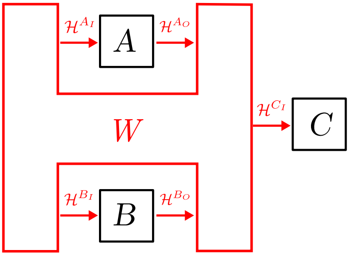
[Rubino *et al.*, Sci. Adv. 2017; Goswami *et al.*, PRL 2018]

Defining Multipartite Causal Separability

- Process matrix formalism generalises easily to N parties $\mathcal{N} = \{A_1, \ldots, A_N\}$
- Restricted tripartite scenario where C has no outgoing system
  - Only relevant orders are $A \prec B \prec C$ and $B \prec A \prec C$

Restricted Tripartite Causal Separability [Araújo *et al.*]

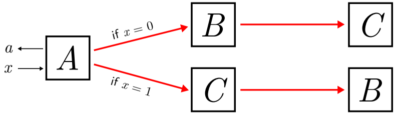
$$W^{\text{sep}} = q\, W^{A \prec B \prec C} + (1-q)\, W^{B \prec A \prec C},$$



Dynamical Causal Orders

In general, a causal process may have:

- Fixed causal orders: $A_{\sigma(1)} \prec \cdots \prec A_{\sigma(N)}$ (σ a permutation of $\{1, \ldots, N\}$)
- But also dynamical orders:



Recall recursive definition of causal correlations

Multipartite Causal Correlation [Oreshkov & Giarmatzi, NJP 2016; Abbott *et al.*, PRA 2016.]

1. Any single-partite distribution $P(a|x)$ is causal
2. For $N \geq 2$, P causal iff $P(\vec{a}|\vec{x}) = \sum_k q_k P_k(a_k|x_k) \underbrace{P_{k,x_k,a_k}(\vec{a}_{\mathcal{N}\setminus k}|\vec{x}_{\mathcal{N}\setminus k})}_{(N\text{-}1)\text{-partite causal correlation}}$

- Oreshkov & Giarmatzi [NJP 2016] considered multipartite causal separability in the general case and proposed two recursive definitions in the same spirit as for causal correlations

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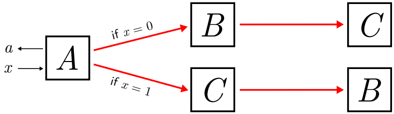

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Oreshkov & Giarmatzi's Definitions

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Idea: in any run, one party acts first and conditioned on their operation, the other parties also causally separable

- Need to define a notion of "conditional process": For a process W, party A_k and CP map M_k applied by A_k:

$$W_{|M_k} := \mathrm{Tr}_k[(M_k \otimes \mathbb{1}^{\mathcal{N}\backslash k}) \cdot W]$$

Oreshkov & Giarmatzi's Causal Separability (OG-CS):

1. Any single-partite process W is causally separable
2. For $N \geq 2$, W causally separable iff $W = \sum_k q_k \underbrace{W_{(k)}}_{\substack{\text{Valid process compatible with } A_k \prec (\mathcal{N}\backslash A_k), \\ \text{s.t. } \forall\, M_k \text{ the } (N-1)\text{-partite conditional matrix} \\ W_{|M_k} \text{ is causally separable}}}$

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- Definition natural, but allows "activation of non-causality"
  - $W^{\text{act.}}$ causally separable but $W^{\text{act.}} \otimes \rho$ nonseparable
  - Process matrix framework constructed to allow for shared ancillary systems
  - Entanglement a different kind of resource

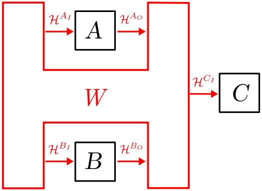
Oreshkov & Giarmatzi's "Extensible" Causal Separability (OG-ECS):

W extensibly causally separable iff $W \otimes \rho$ is causally separable $\forall\, \rho$

[Oreshkov & Giarmatzi, NJP 2016]

Weighing Up the Situation

- OG-CS $\neq$ OG-ECS in general
- In the "restricted tripartite scenario", Araújo *et al.*'s definition is equivalent to OG-ECS
  - Can have activation of causal nonseparability in this scenario too
  - $\implies$ they differ here too



OG-ECS seems like the right approach to causal separability

- Coincides with existing definitions for bipartite and restricted tripartite scenarios
- Shared ancillary states should be free resource
- Only imposes extensibility at top level of recursion

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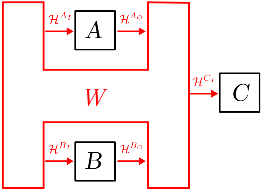

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Multipartite Causal Separability

We propose the following:

Multipartite Causal Separability

1. For $N = 1$, any W is causally separable
2. For $N \geq 2$ W is causally separable iff for all extensions $\rho \in A_{I'}^{\mathcal{N}}$

$$W \otimes \rho = \sum_{k \in \mathcal{N}} q_k W_{(k)}^{\rho}, \quad \text{where}$$

(i) $W_{(k)}^{\rho}$ is a valid process compatible with $A_k \prec (\mathcal{N} \backslash A_k)$

(ii) For any $M_k \in A_{II'O}^{k}$ $W_{|M_k}$ is causally separable

- A priori this could differ from OG-ECS
  - But it turns out they are equivalent
    - (And the decomposition $\{W_{(k)}^{\rho}\}_k$ can be taken independent of ρ)
- From now on we take this as our notion of "causally separable"

Tripartite Causal Separability

- How to check if an N-partite W is causally separable?
- Recall bipartite characterisation: $W^{\text{sep}} = W^{A \prec B} + W^{B \prec A}$

Tripartite Causal Separability [equivalent to Oreshkov & Giarmatzi, NJP 2016]

Valid process compatible with A first (up to norm.)

$$\begin{aligned} W^{\text{sep}} &= \overbrace{\widetilde{W}_{(A,B,C)} + \widetilde{W}_{(A,C,B)}}^{W_{(A)}} + \overbrace{\widetilde{W}_{(B,A,C)} + \widetilde{W}_{(B,C,A)}}^{W_{(B)}} + \overbrace{\widetilde{W}_{(C,A,B)} + \widetilde{W}_{(C,B,A)}}^{W_{(C)}} \end{aligned}$$

Not necessarily a valid process

- All terms are positive semidefinite
- $\underbrace{\mathrm{Tr}_{B_{IO}C_{IO}} W_{(A)}}_{\text{Reduced process for } A} = \left(\mathrm{Tr}_{A_O}[\mathrm{Tr}_{B_{IO}C_{IO}} W_{(A)}]\right) \otimes \mathbb{1}^{A_O}$
- $\mathrm{Tr}_{C_{IO}} \widetilde{W}_{(A,B,C)} = \left(\mathrm{Tr}_{B_O}[\mathrm{Tr}_{C_{IO}} \widetilde{W}_{(A,B,C)}]\right) \otimes \mathbb{1}^{B_O}$
- $\widetilde{W}_{(A,B,C)} = (\mathrm{Tr}_{C_O} \widetilde{W}_{(A,B,C)}) \otimes \mathbb{1}^{C_O}$ (+ permutations of these)

- For any M_A, $(\widetilde{W}_{(A,B,C)})_{|M_A}$ *is* a valid process (compatible with $B \prec C$)
- Not a convex mixture of processes; allows for dynamical orders

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Example: Fourpartite Quantum Switch

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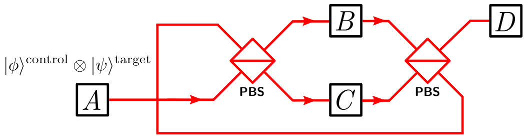
Fourpartite quantum switch: [Chiribella *et al.*, PRA 2013; Araújo *et al.*, PRL 2014]



- A "pure" 4-partite process matrix: $W^{\text{switch}} = |w\rangle\langle w|$ with

$$|w\rangle = |0\rangle^{A_O^c} |\psi\rangle^{B_I^t} |\mathbb{1}\rangle\rangle^{B_O^t C_I^t} |\mathbb{1}\rangle\rangle^{C_O^t D_I^t} |0\rangle^{D_I^c} + |1\rangle^{A_O^c} |\psi\rangle^{C_I^t} |\mathbb{1}\rangle\rangle^{C_O^t B_I^t} |\mathbb{1}\rangle\rangle^{B_O^t D_I^t} |1\rangle^{D_I^c}$$

  - Causally non-separable

- Tracing out D it becomes causally separable:

$$\underbrace{|0\rangle\langle 0|^{A_O^c} \otimes |\psi\rangle\langle\psi|^{B_I^t} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B_O^t C_I^t} \otimes \mathbb{1}^{C_O^t}}_{\widetilde{W}_{(A,B,C)} \text{ – not a valid process}} + |1\rangle\langle 1|^{A_O^c} \otimes |\psi\rangle\langle\psi|^{C_I^t} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{C_O^t B_I^t} \otimes \mathbb{1}^{B_O^t}$$

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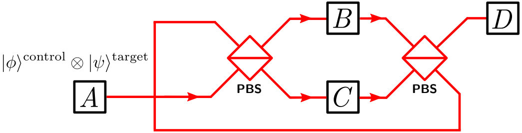

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N-partite Causal Separability

Tripartite Causal Separability

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- Can generalise condition to 4 parties and beyond:

Sufficient Condition for Fourpartite Causal Separability

Valid process compatible with A first (up to norm.)

$$\begin{aligned} W &= W_{(A)} + W_{(B)} + W_{(C)} + W_{(D)} \\ &= \overbrace{\widetilde{W}_{(A,B)} + \widetilde{W}_{(A,C)} + \widetilde{W}_{(A,D)}}^{W_{(A)}} + \overbrace{\cdots}^{W_{(B)}} + \overbrace{\cdots}^{W_{(C)}} + \overbrace{\cdots}^{W_{(D)}} \\ &= \overbrace{\widetilde{W}_{(A,B,C,D)} + \widetilde{W}_{(A,B,D,C)}}^{\widetilde{W}_{(A,B)}} + \overbrace{\widetilde{W}_{(A,C,B,D)} + \widetilde{W}_{(A,C,D,B)}}^{\widetilde{W}_{(A,C)}} + \overbrace{\widetilde{W}_{(A,D,B,C)} + \widetilde{W}_{(A,D,C,B)}}^{\widetilde{W}_{(A,D)}} + \cdots + \cdots + \cdots \end{aligned}$$

For any CP map M_A, $(W_{(A,B)})_{|M_A}$ is valid, compatible with B first

For any CP maps M_A, M_B, $(W_{(A,B,C,D)})_{|M_A \otimes M_B}$ is valid, compatible with C first

- Conditions on terms given by linear constraints with same interpretation as before

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- Can generalise condition to 4 parties and beyond:

Sufficient Condition for N-partite Causal Separability

$$W^{\mathsf{sep}} = \sum_{\pi \in \Pi} \widetilde{W}_\pi, \quad \text{with}$$

1. $\widetilde{W}_\pi \geq 0$ for each permutation π of $(1, \ldots, N) \equiv (A_1, \ldots, A_N)$
2. For every ordered subset $(k_1, \ldots, k_n)$ (with $1 \leq n \leq N$),

$$\widetilde{W}_{(k_1,\ldots,k_n)} := \sum_{\pi \in \Pi_{(k_1,\ldots,k_n)}} \widetilde{W}_\pi, \text{ satisfies}$$

$$\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}} \widetilde{W}_{(k_1,\ldots,k_n)} = \left(\mathrm{Tr}_{A_O^{k_n}} \left[\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}} \widetilde{W}_{(k_1,\ldots,k_n)}\right]\right) \otimes \mathbb{1}^{A_O^{k_n}}$$

- This constraint is sufficient for W^{sep} to be causally separable, is it necessary?

Causal Separability: Necessary Conditions

- Explicit necessary conditions can be obtained by choosing specific CP maps and ancillas at each level of the recursive definition
- Ognyan and Giarmatzi showed how such a choice proves sufficient conditions also necessary in tripartite case
  1. ρ: maximally entangled state for each pair of parties
  2. M_{A_k}: $|\Phi^+\rangle\!\langle\Phi^+|$ – M.E.S. between A_{IO} and half of ancilla between A_k and some $A_{k'}$
- "Teleports" A_k's system on A_{IO}^k to $A_{I'}^{k'}$

$$\underbrace{W_{(k)}^{\rho}}_{N\text{-partite, } A_k \text{ first}} \longrightarrow \underbrace{W_{(k)}^{A_{IO}^k \to A_{I'}^{k'}}}_{(N-1)\text{-partite, formally equivalent to } W_{(k)}} \otimes \rho' := (W_{(k)}^{\rho})_{|M_{A_k}}$$

- Any constraints obeyed by $W_{(k)}^{A_{IO}^k \to A_{I'}^{k'}}$ must be obeyed by $W_{(k)}$ once Hilbert spaces relabelled
  - Can repeat for each $k' \neq k$

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Necessary Condition for Causal Separability

Necessary condition for N-partite causal separability

An N-partite $W^{\text{sep}} \in \mathcal{W}^{\text{sep}}$ must have a decomposition $W = \sum_{k\in\mathcal{N}} W_{(k)}$ where:

1. $W_{(k)}$ is a valid process compatible with $A_k \prec (\mathcal{N}\backslash A_k)$
2. For each $k' \neq k$, $W_{(k)}^{A_{IO}^k \to A_{I'}^{k'}}$ is an $(N-1)$-partite causally separable process
   - $\implies$ obeys the necessary conditions for $(N-1)$-partite processes

- Coincides with separable condition for $N = 3$ [Oreshkov & Giarmatzi, NJP 2016]
- Also reduced 4-partite scenario (no output for D, c.f. quantum switch)
- Note that decomposition may differ for each k'
  - Satisfying these conditions with a unique decomposition would imply the sufficient conditions

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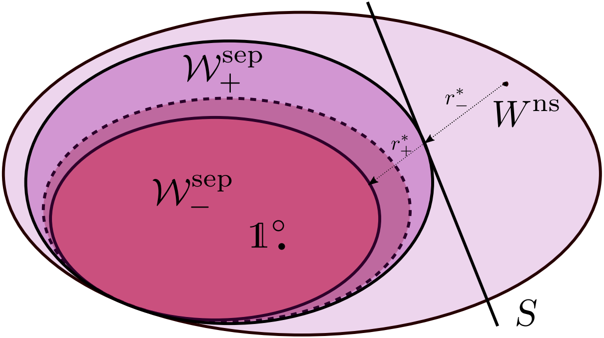
Witnessing Causal Nonseparability

- Both necessary and sufficient conditions define convex cones $\mathcal{W}_+^{\mathsf{sep}}$, $\mathcal{W}_-^{\mathsf{sep}}$ of (non-normalised) process matrices



- Membership can be tested with SDP
- Dual SDP from necessary condition gives causal witnesses
- So far no numerical evidence that $\mathcal{W}_-^{\mathsf{sep}} \neq \mathcal{W}_+^{\mathsf{sep}}$, but...

Outline

Process Matrix Formalism

- Bipartite process matrices & causal separability

Multipartite Causal Nonseparability

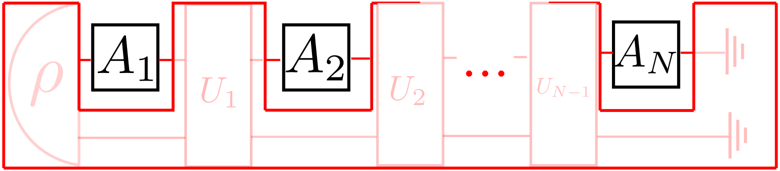
- Defining multipartite causal separability
- Characterising causal separability
- Witnessing causal nonseparability

Quantum Circuits with Classical and Quantum Controls of Causal Order

- Classically controlled circuits
- Coherently (quantum) controlled circuits

Quantum Circuits $\equiv$ Fixed-Order Processes

- Quantum circuits can be seen as fixed order processes
  - Equivalently: quantum supermap, quantum combs



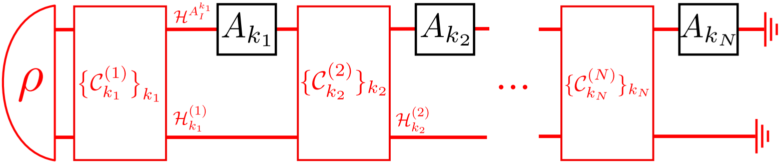
- Conversely: any W compatible with a fixed order $A_1 \prec \cdots \prec A_N$ can be implemented as a quantum circuit
  - i.e., $\mathcal{W}^{A_1 \prec \cdots \prec A_N}$ precisely characterises quantum circuits

$$\underbrace{\mathrm{Tr}_{A_{IO}^{k+1} \cdots A_{IO}^{N}} W^{A_1 \prec \cdots \prec A_N}}_{W^{A_1 \prec \cdots \prec A_k}} = \left(\mathrm{Tr}_{A_O^k} \left[\mathrm{Tr}_{A_{IO}^{k+1} \cdots A_{IO}^{N}} W^{A_1 \prec \cdots \prec A_N} \right] \right) \otimes \mathbb{1}^{A_O^k}$$

[Chiribella *et al.*, PRA 2009]

Classically Controlled Circuits

- Causally separable processes go beyond quantum circuits
- Can they be realised? How?
  - Oreshkov & Giarmatzi [NJP, 2016] suggested they correspond to quantum circuits with classical control of causal order: "classically controlled quantum circuits"

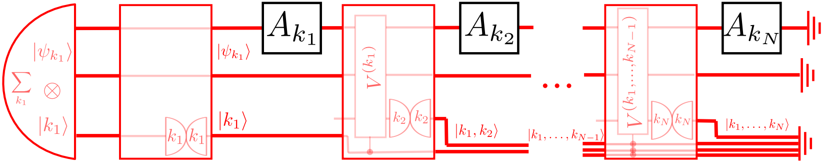


- Outcome of operation $\{\mathcal{C}^{(i)}_{k_i}\}_{k_i}$ determines the ith party
  - Can depend on previous parties and operations $\rightarrow$ allows dynamical causal order
- Fairly easy to see such circuits are causally separable
  - What about the converse?

Classically Controlled Circuits

Characterisation of classically controlled circuits

A process W^{sep} represents a classically controlled circuit iff it satisfies the sufficient conditions for causal separability.



- Isometries $V^{(k_1,\dots,k_i)} = \sum_{k_{i+1}} V_{k_{i+1}}^{(k_1,\dots,k_i)} \otimes |k_{i+1}\rangle$
  - $V_{k_{i+1}}^{(k_1,\dots,k_i)}$ can be constructed from $\widetilde{W}_{(k_1,\dots,k_i)}$
- Can constructively give the circuit from any $W^{\text{sep}} \in \mathcal{W}_{-}^{\text{sep}}$

Recall characterisation: $W^{\text{sep}} = \sum_{\pi\in\Pi} \widetilde{W}_\pi$ with $\widetilde{W}_{(k_1,\dots,k_n)} := \sum_{\pi\in\Pi_{(k_1,\dots,k_n)}} \widetilde{W}_\pi$

$$\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\dots,k_n\}}} \widetilde{W}_{(k_1,\dots,k_n)} = \left(\mathrm{Tr}_{A_O^{k_n}} [\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\dots,k_n\}}} \widetilde{W}_{(k_1,\dots,k_n)}]\right) \otimes \mathbb{1}^{A_O^{k_n}}$$

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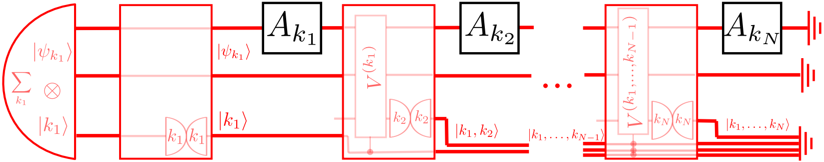

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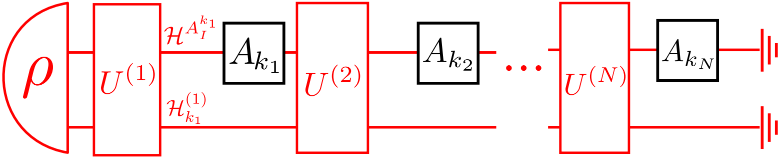
Coherently (Quantum) Controlled Circuits

- This suggests a generalisation: quantum circuits with coherent control of causal order



- Unitaries $U^{(i)}$ coherently route subspaces to different parties
  - Constraints on $U^{(i)}$: Each A_{k_j} acts exactly once, irrespective of what operations performed

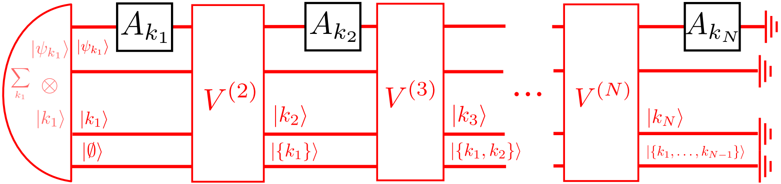
- Generalisation of quantum-switch-type circuits
- Allows coherent, dynamical orders

Coherently (Quantum) Controlled Circuits

Characterisation of coherently controlled circuits

A process W represents a coherently controlled circuit iff there are PSD matrices $\widetilde{W}_{\mathcal{K}}^{\ell}$ (for $\mathcal{K} \subsetneq \mathcal{N}$, $\ell \in \mathcal{N}\backslash\mathcal{K}$) satisfying $W = \sum_{k\in\mathcal{N}} \widetilde{W}_{\mathcal{N}\backslash\{k\}}^{k}$ and, $\forall \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}$,

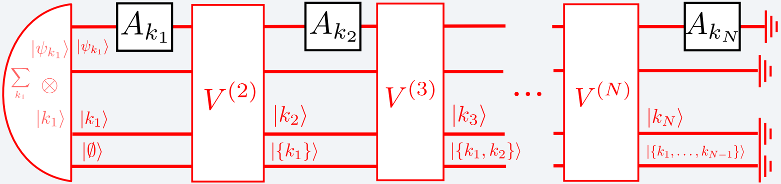
$$\mathrm{Tr}_{A_{IO}^{\mathcal{N}\backslash\mathcal{K}}} \Big[\sum_{\ell\in\mathcal{N}\backslash\mathcal{K}} \widetilde{W}_{\mathcal{K}}^{\ell} \Big] = \sum_{k\in\mathcal{K}} \Big(\mathrm{Tr}_{A_O^k} \big[\mathrm{Tr}_{\mathcal{N}\backslash\mathcal{K}} \widetilde{W}_{\mathcal{K}\backslash\{k\}}^{k} \big] \Big) \otimes \mathbb{1}^{A_O^k}$$



$$V^{(i)} = \sum_{\substack{\{k_1,\ldots,k_{i-2}\},\\ k_{i-1},k_i}} V_{\{k_1,\ldots,k_{i-2}\}}^{k_{i-1}\to k_i} \otimes |k_i\rangle\langle k_{i-1}| \otimes |\{k_1,\ldots,k_{i-1}\}\rangle\langle\{k_1,\ldots,k_{i-2}\}|$$

- $V_{\{k_1,\ldots,k_{i-2}\}}^{k_{i-1}\to k_i}$ can be constructed from $\widetilde{W}_{\{k_1,\ldots,k_{i-1}\}}^{k_i}$

Coherently (Quantum) Controlled Circuits



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- Includes classically controlled circuits as a special case
  - These coincide for $N = 2$, but for $N \geq 3$ can be causally nonseperable (e.g., quantum switch)
- Class of generalised switch-type circuits that we can constructively realise
- No such process can violate a causal inequality!
- Can we do anything new/interesting with these circuits?

Summary of Characterisations

Quantum circuits with fixed causal order

$$\underbrace{\mathrm{Tr}_{A_{IO}^{k+1}\cdots A_{IO}^{N}} W^{A_1\prec\cdots\prec A_N}}_{W^{A_1\prec\cdots\prec A_k}} = \left(\mathrm{Tr}_{A_O^k}[\mathrm{Tr}_{A_{IO}^{k+1}\cdots A_{IO}^{N}} W^{A_1\prec\cdots\prec A_N}]\right)\otimes \mathbb{1}^{A_O^k}$$

Quantum circuits with classical control of causal order

$W^{\mathsf{sep}} = \sum_{\pi\in\Pi}\widetilde{W}_\pi$ with $\widetilde{W}_{(k_1,\ldots,k_n)} := \sum_{\pi\in\Pi_{(k_1,\ldots,k_n)}}\widetilde{W}_\pi$

$$\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}}\widetilde{W}_{(k_1,\ldots,k_n)} = \left(\mathrm{Tr}_{A_O^{k_n}}[\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\{k_1,\ldots,k_n\}}}\widetilde{W}_{(k_1,\ldots,k_n)}]\right)\otimes\mathbb{1}^{A_O^{k_n}}$$

Quantum circuits with coherent control of causal order

$W = \sum_{k\in\mathcal{N}}\widetilde{W}^k_{\mathcal{N}\setminus\{k\}}$ and, $\forall\emptyset\subsetneq\mathcal{K}\subsetneq\mathcal{N}$,

$$\mathrm{Tr}_{A_{IO}^{\mathcal{N}\setminus\mathcal{K}}}\Big[\sum_{\ell\in\mathcal{N}\setminus\mathcal{K}}\widetilde{W}^\ell_{\mathcal{K}}\Big] = \sum_{k\in\mathcal{K}}\Big(\mathrm{Tr}_{A_O^k}\big[\mathrm{Tr}_{\mathcal{N}\setminus\mathcal{K}}\widetilde{W}^k_{\mathcal{K}\setminus\{k\}}\big]\Big)\otimes\mathbb{1}^{A_O^k}$$

Summary & Outlook

- Definition of multipartite causal (non)separability
- Characterisation of causally separable process matrices
  - Separate necessary and sufficient conditions
  - Coincide for $N = 2, 3$; in general?
  - Necessary condition allows construction of witnesses of causal nonseparability
- Quantum circuits with classical control of causal order
  - Coincides with sufficient condition for causal separability
- Quantum circuits with quantum control of causal order
  - Generalisation of implementable quantum switch type circuits
  - Are there other physically realisable processes?

[arXiv:1807.10557 + new paper soon(ish)]

Constraints for Process Matrix Validity

Recall the notation:

$$ {}_X W := (\mathrm{Tr}_X W) \otimes \frac{\mathbb{1}^X}{d_X}, \quad {}_1 W := W, \quad {}_{[\sum_X \alpha_X X]} W := \sum_X \alpha_X \, {}_X W, $$

Space of valid process matrices

$$ W \in \mathcal{L}^{\mathcal{N}} \;\Leftrightarrow\; \forall\, \chi \subseteq \mathcal{N}, \chi \neq \emptyset, \; {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash\chi}} W = 0, $$

Space of valid process compatible with A first

$$ W \in \mathcal{L}^{A_k \prec (\mathcal{N}\backslash A_k)} $$

$$ \Leftrightarrow \; {}_{[1-A_O^k]A_{IO}^{\mathcal{N}\backslash k}} W = 0 \quad \text{and} \quad \forall\, \chi \subseteq \mathcal{N}\backslash k, \chi \neq 0, {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash k\backslash\chi}} W = 0, $$

Cones $\mathcal{W}^{\text{sep}}$ and $\mathcal{S}$ for tripartite case

Adopt the notation $\mathcal{L}_X = \{W | {}_X W = 0\}$.

$$\begin{aligned}
\mathcal{W}^{\text{sep}} = \mathcal{L}_{[1-A_O]B_{IO}C_{IO}} &\cap \big(\mathcal{P} \cap \mathcal{L}_{[1-B_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]B_{IO}} \cap \mathcal{L}_{[1-B_O]}\big) \\
+ \mathcal{L}_{[1-B_O]A_{IO}C_{IO}} &\cap \big(\mathcal{P} \cap \mathcal{L}_{[1-A_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]A_{IO}} \cap \mathcal{L}_{[1-A_O]}\big) \\
+ \mathcal{L}_{[1-C_O]A_{IO}B_{IO}} &\cap \big(\mathcal{P} \cap \mathcal{L}_{[1-A_O]B_{IO}} \cap \mathcal{L}_{[1-B_O]} + \mathcal{P} \cap \mathcal{L}_{[1-B_O]A_{IO}} \cap \mathcal{L}_{[1-A_O]}\big),
\end{aligned}$$

$$\begin{aligned}
\mathcal{S} = &\Big(\mathcal{L}^{\perp}_{[1-A_O]B_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-B_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]B_{IO}} + \mathcal{L}^{\perp}_{[1-B_O]})\Big) \\
\cap &\Big(\mathcal{L}^{\perp}_{[1-B_O]A_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-A_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]A_{IO}} + \mathcal{L}^{\perp}_{[1-A_O]})\Big) \\
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\end{aligned}$$