L'aléatoire et l'imprévisibilité au coeur des fondements de la mécanique quantique

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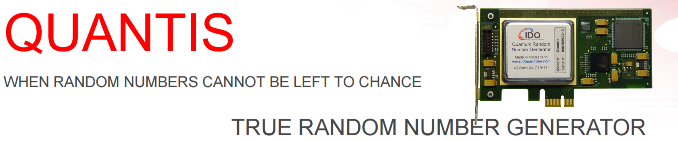
The (In)famous Quantum Randomness

"Quantum randomness" plays a crucial role in the theoretical and technical advances behind the current "second quantum revolution"



QR: The Modern Understanding of QM

"*...quantum randomness in general has a very different and intrinsic or inherent nature. Namely, even if the state of the system is pure and we know it exactly, the predictions of quantum mechanics are intrinsically probabilistic and random! Accepting quantum mechanics means making an assumption that it is correct, and consequently intrinsically random.*"[1]

"*Any classical system admits in principle a deterministic description and thus appears random to us as a consequence of a lack of knowledge about its fundamental description. Quantum theory is, on the other hand, fundamentally random.*"[2]

But what is meant by randomness here? Are these claims about *indeterminism*, or do they really go further?

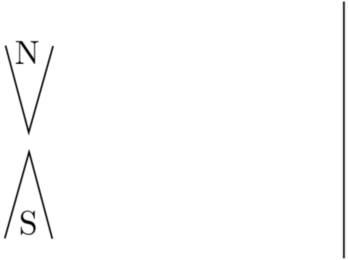
[1] M. Bera *et al.*, arXiv:1611.02176, 2016.

[2] S. Pironio *et al.*, Nature 464 (2010), p. 1021.

Outline

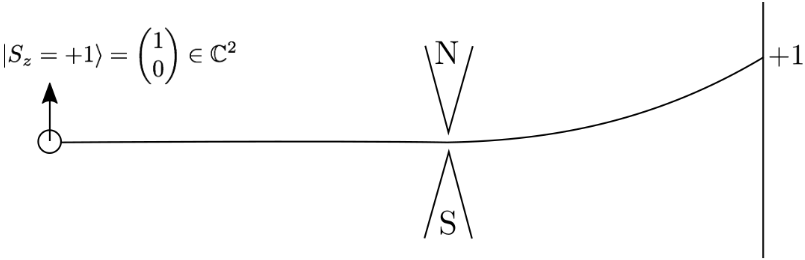
Quantum Basics

$$|S_z = +1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$$



- States are unit vectors in complex Hilbert space
- Measurement contexts are maximal sets of mutually exclusive propositions (projection operators)
- Incompatibility of measurements: $S_x \cdot S_z \neq S_z \cdot S_x$
- Quantum logic: non-distributive orthomodular lattice of propositions

Quantum Basics



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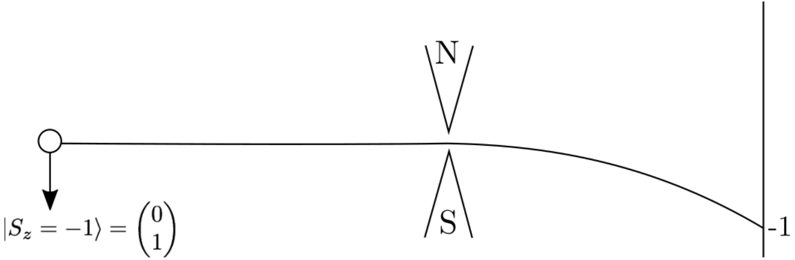

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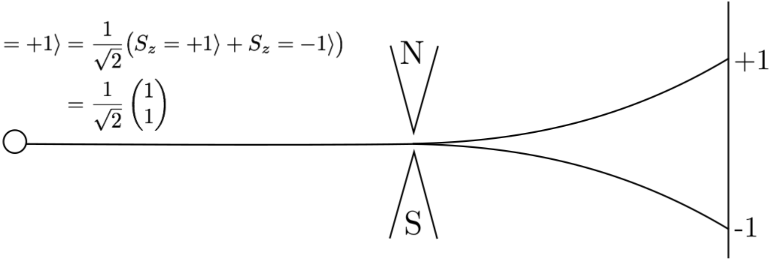
Quantum Basics

$$|S_x = +1\rangle = \frac{1}{\sqrt{2}}\big(S_z = +1\rangle + S_z = -1\rangle\big)$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$



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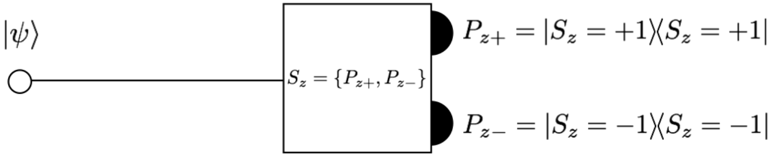

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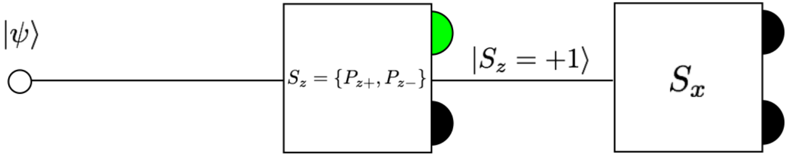

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The Born Rule

The Born rule links the abstract quantum state to the concrete observation of outcomes

The Born rule

The probability of observing outcome a_i upon measurement of the context $\{|a_i\rangle\langle a_i|\}_i$ on a state $|\psi\rangle$ is $p(a_i|\psi) = |\langle a_i|\psi\rangle|^2$. Following the observation of outcome a_i, the system 'collapses' to the state $|a_i\rangle$.

- This rule only tells us how to assign probabilities to observations, but is silent towards its physical *interpretation*
    - Large conceptual gap between this and the supposed 'intrinsic randomness' of quantum measurements
- Nor does it say anything about how this state collapse occurs
    - The standard dynamics of a quantum state are unitary and *reversible*

Interpreting the Born Rule

Epistemic interpretation:

- The probabilities express only our ignorance as to the true physical state of the system, and statements about the quantum state should be interpreted as referring to physical ensembles
  - Einstein & Schrödinger notably favoured such an approach
  - "*The statistical interpretation due to Born . . . allows, however, no real description for the individual system, rather only statistical assertions concerning ensembles of systems.*" (Einstein, 1953)

Ontic interpretation:

- The probabilities must be interpreted as physical propensities, and thus quantum measurement is fundamentally chancy and indeterministic
  - "*I myself tend to give up determinism in the atomic world.*" (Born, 1926)

Interpreting the Born Rule (2)

This ontic interpretation has become predominant, particularly in the modern information-theoretic approach to QM

Eigenstate–Eigenvalue Link[3]

A system in a state $|\psi\rangle$ has a definite value of an observable property A if and only if $|\psi\rangle$ is an eigenstate of A.

- This interpretation represents a radical departure from the classical determinism; what basis do we have for believing this is necessary?
- And does this interpretation explain common accounts of quantum randomness?
  - Need to look at how indeterminism is defined and its relation to randomness

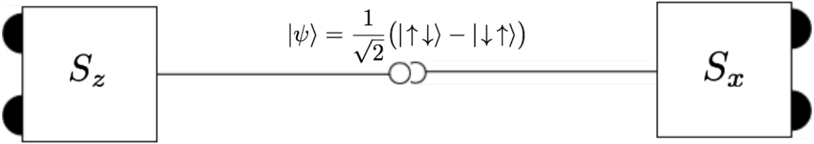
[3] M. Suárez, Br. J. Phil. Sci. 55 (2004), p. 222.

EPR's Incompleteness Argument

Einstein, Podolsky, and Rosen (1935) constructed an ingenious example to try and argue that quantum mechanics is incomplete, and this indeterministic interpretation incorrect

- (Completeness) "*Every element of the physical reality must have a counter part in the physical theory.*"
- "*If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.*"
- (Locality [implicit]) No physical influence can travel faster than the speed of light

EPR's Incompleteness Argument (2)



"*(1) the quantum mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantifies cannot have simultaneous reality.*"

- Insisting on an ontic interpretation of the Born rule thus requires giving up even more than immediately apparent

Hidden-Variable Theories

It was only in the 1960s that, first von Neumann and then Bell, Kochen and Specker, asked more seriously whether such a deterministic 'completion' of quantum mechanics could exist

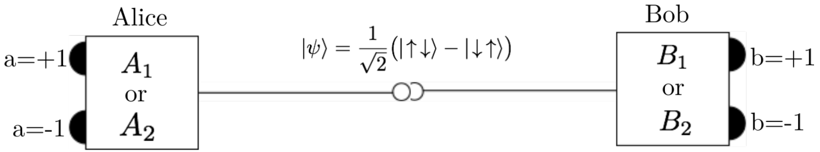
Hypothesise the existence of a hidden parameter λ completely determining the observable properties of the system:

- Preparation of $|\psi\rangle$ corresponds to preparing a state $\lambda \in \Lambda_\psi$ completely determining all measurement outcomes
- The Born rule is recovered as an ensemble average

Any deterministic extension of QM should be describable in such a way

Bell's Theorem and Inequalities

Bell applied such reasoning to the EPR scenario



- Locality: The measurement outcomes are a function only of λ and the *local* measurement choice
- Measurement Independence: Both parties can freely and independently choose what measurement to perform
- Outcome Independence: No 'retrocausal influences'

Bell's Theorem and Inequalities (2)

Under these assumptions, the statistical correlations between A and B must obey certain 'Bell inequalities', while QM predicts stronger correlations violating these inequalities

$$\sum_{a,b,i,j} \alpha_{abij} p(ab|A_i B_j) \leq \beta$$

- Crucially, can be experimentally tested and the violation confirmed
- This rules out, at a statistical level, local realism for entangled states: determinism can only be saved if locality is abandoned
- Note that, if a violation is observed, this is an indicator of nonclassical physical behaviour, whether or not quantum mechanics is complete and correct

The Kochen-Specker Theorem

Kochen and Specker showed the contradiction with determinism occurs even at the level of the Hilbert space representation, *prior* to measurement

They asked whether a hidden variable can consistently determine all measurement outcomes prior to measurement

- Noncontextuality: A measurement outcome must be independent of the other *compatible* observables measured alongside it
  - i.e. if A is compatible with both B and C, but B and C are incompatible, we should obtain the same outcome for the proposition A, whether we measure $\{A, B\}$ or $\{A, C\}$
  - A generalisation of locality

Their result applies to *individual* quantum particles (but, for technical reasons, not two-dimensional systems)

The Kochen-Specker Theorem (2)

Eigenstate-Eigenvalue Link

A system in a state $|\psi\rangle$ has a definite value of an observable property A if and only if $|\psi\rangle$ is an eigenstate of A.

The 'if' direction of the E-E Link is solid; do these results prove the 'only if'?

- Not quite: they only show that not all observables can be value definite for a state
- But possible[4] to strengthen the KS Theorem to show that every observable A must be value definite for $|\psi\rangle$ unless $|\psi\rangle$ is an eigenstate of A.

[4] A. Abbott, C. Calude & K. Svozil. J. Math. Phys. **56**, 102201 (2015).

Physical Indeterminism

Indeterminism is defined as the logical negation of determinism

"*Une intelligence qui, à un instant donné, connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la composent, si d'ailleurs elle était suffisamment vaste pour soumettre ces données à l'analyse, embrasserait dans la même formule les mouvements des plus grands corps de l'univers et ceux du plus léger atome ; rien ne serait incertain pour elle, et l'avenir, comme le passé, serait présent à ses yeux.*" (Laplace, 1814)

Determinism (Earman, Montague)

A theory is deterministic if the state of a (closed) system at time t_0 uniquely determines its future state for all $t > t_0$.

Value Indefiniteness to Indeterminism

Kochen-Specker type value indefiniteness expresses precisely this kind of indeterminism: the outcome of the measurement is not determined prior to the act of measurement

- The Eigenstate-Eigenvalue link can be reduced to the assumption that any ‘elements of physical reality’ must behave classically, i.e. non-contextually and locally
- The indeterminism of all non-trivial quantum measurements can indeed be reduced to more plausible assumptions about how a physical theory should be
- This indeterminism is not a consequence of the formal structure of QM, and must hold for any extension thereof

Alternative Explanations

Such indeterminism is nonetheless not 'unconditional', as quantum randomness is often claimed to be: it rests on our belief in the constrains imposed on the λ (e.g., noncontextuality) and other assumptions

- Physically, these are very reasonable, but certainly not untouchable
- The idea of a consistent contextual, nonlocal HVT is indeed consistent: Bohmian mechanics is one such an example with a physical interpretation
    - But *any* such theory necessarily involves either physical properties that are *in principle* immeasurable, or instantaneous signalling, and the paradoxes associated therewith

Is Randomness Indeterminism?

Does quantum indeterminism amount to quantum randomness? More generally, can randomness be equated with indeterminism and/or chance?

- Such a view was commonplace until recently dates perhaps to Laplace's assertion than "rien ne serait incertain" (and thus random) in a deterministic world
- This identification is still often implicitly adopted in the physical sciences

"*The result is completely random because in such a measurement the elementary system carries no information whatsoever about the measurement result.*"[5]

[5] A. Zeilinger, Found. Phys 29 (1999), p. 636.

Indeterminism, Chance and Randomness

It is instructive to analyse this identification in two parts:

(IC) An outcome happens by chance iff it is indeterministic

(CT) An outcome is random iff it happens by chance

What is meant by 'chance'?[6]

- Refers to possibility (as opposed to necessity), e.g. as "graded possibility" (Leibniz)
- Has a quantitative aspect, representable as probabilities
- Regulates rational belief in line with Lewis' Principal Principle:

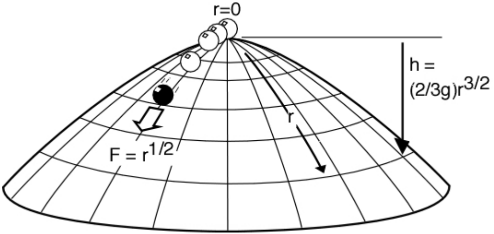
$$\mathsf{Cr}(A|\text{“}P(A|E) = x\text{”} \wedge E) = x$$

- Should relate to frequency realisation in similar circumstances

[6] E.g. J. Schaffer, Brit. J. Phil. Sci 54 (2003), 27–41; D. Mellor, Aus. J. Phil. 78 (2006), 16–27.

IC has been challenged in several ways in recent years

- Arguments for deterministic chance, mostly based on existence of probabilities that objectively play the role of chance in theories despite underlying determinism[7]
- Indeterminism in (even classical) physics that doesn't play the role of chance, e.g. Norton's dome



[7] Eg. B. Loewer, Stud. Hist. Phil. Mod. Phys. 32 (2001), pp. 609–620; Eagle, Noûs 45:2 (2011), 269–299.

Quantum Measurement and Chance

The negative definition of indeterminism leaves the future "*completely unspecified. [. . .] Randomness such as described by the Born rule of quantum mechanics, on the other hand, is not a matter of 'anything goes', but a highly constrained affair: exactly one of a number of possible outcomes will occur, and the probabilities of all the outcomes are given beforehand.*"[8]

- So quantum measurements seems to fit the bill for being considered chancy

Müller goes on to argue that quantum randomness is precisely this kind of constrained indeterminism

- This argument, and other more casual ones, for quantum randomness nonetheless appeal implicitly to the CT

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[8] T. Müller, *(Quantum) randomness as limited indeterminism.*(2015).

The Commonplace Thesis

(CT) An outcome is random iff it happens by chance

For CT to be tenable, chance needs to satisfy what we expect of randomness

The qualities of randomness can be divided into two groups: those dealing with objects, and those dealing with processes

- Product randomness: is a particular sequence of results of coin flips random?
- Process randomness: is the coin-flip process random?
- Chance is clearly a process notion, so already clear that the CT can only be valid for a process notion of randomness

Product Randomness

The notion of randomness for finite strings and, especially, infinite sequences has been extensively studied in mathematics

- Frequency based approaches
  - Must content with existence of Borel-normal sequences, e.g. 0100011011000001010011100101110111...
- Von Mises (1941): All 'admissibly chosen' subsequences should have limiting frequency of 0 equal to $1/2$
- Kolmogorov (1965), Martin-Löf (1966), Chaitin (1966): All finite prefixes of a sequence should by incompressible by a universal prefix-free Turing machine

The K-ML-C notion of randomness is largely accepted. Crucially, one can show that absolute randomness is impossible.

Process Randomness

Less consensus, but often seen as related to unpredictability

- Idea already present in Laplacian indeterminism
- Popper,[9] although arguing that indeterminism *is* unpredictability can be read in this way

  "*. . . indeterministic in the sense that it implies the impossibility of predicting certain kinds of physical events.*"
- Eagle goes a step further defined randomness as "maximal unpredictability"

Defining randomness as a form of unpredictability inserts a subjective element into randomness: unpredictable for whom? And under what conditions?

[9] Br. J. Phil. Sci, 1 (1950), 117.

The Commonplace Thesis (2)

Both product and process randomness seem legitimate and distinct

- One may hope that a random process only produces random output, but this turns out to be deeply problematic
- At best, one can hope for this with high probability
- So an unconditioned CT seems to be ruled out

Eagle (and others) have argued at length that the CT is problematic even for product randomness

- Highly biased chance seems to fail intuition about randomness
- Classical process, such as chaos, seem to satisfy many properties of randomness, but are not chancy

Quantum Randomness as Unpredictability?

Quantum randomness is often justified by an appeal to such unpredictability

"An immediate consequence of this is objective randomness. [...] we can in principle only make a probabilistic prediction of, say, the position of a particle in a future measurement."[10]

- How to handle subjective element of 'randomness as unpredictability', while quantum randomness is 'absolute'?

Note that even if unpredictability has a subjective component, this does not mean it isn't intrinsic, or that it can't have an objective origin

[10] J. Kofler & A. Zeilinger. Eur. Rev. 18 (2010), p. 470.

Formalising Unpredictability

For this approach to defining randomness to be complete, we need to clarify precisely what it means for an event to be predictable

- Laplace's "intelligence qui … connaîtrait toutes les forces dont la nature est animée et la situation respective des êtres qui la composent, si d'ailleurs elle était suffisamment vaste pour soumettre ces données à l'analyse" is hardly physical
- Popper attempted to *effectivise* such an agent, demanding that the demon be "embodied" in the physical world as "*a classical mechanical calculating and predicting machine which is so constructed as to produce permanent records of some kind*" in order to solve a *finite prediction task*: "*the task of predicting, with some chosen and specified degree of precision, some event occurring in a finite mechanical system, sufficiently isolated…*"[11]

[11] K. Popper, Br. J. Phil. Sci. 1 (1950), 118,124.

Formalising Unpredictability (2)

More modern attempts at formalising unpredictability have also been attempted

- Wolpert[12] defines physical "inference devices", more abstract version of Popper's predictors, but whose applicability to specific physical situations is obscure
- Eagle defines a more appropriate notion for these purposes: "*A prediction function $P(S,t)$ takes as input the current state S of a system described by a theory T as discerned by a predictor P and an elapsed time t, and yields a temporally indexed probability distribution Pr_t over the space of possible states of the system. A prediction is a specific use of some prediction function by some predictor on some initial state S_0 and time t_0 who adopts Pr_t as its posterior credence function conditionally on the evidence and the theory.*"[13]

[12] D. Wolpert, Physica D 237 (2008), 1257–1281.

[13] A. Eagle, Brit. J. Phil. Sci. 56 (2005), 769.

Formalising Unpredictability (3)

- An event is unpredictable for a predictor if the best available prediction function yields a posterior credence that gives a probability for the event different to 0 or 1
- Eagle defines randomness as maximal unpredictability, i.e., the posterior credence (again conditioned on the evidence and theory) of the event is equal to the prior probability of the event

Eagle's notion lacks the effectivity that was crucial to Popper's and Wolpert's approaches, and is deliberately theory dependent and (partially) subjective

- Can these two approaches be combined to provide an effective definition suitable for understanding quantum randomness?

An Effective Model of Unpredictability

We proposed a model based on a predictor acting *via computable means* on finite information extracted *locally* from the system in question *via measurement*[14]

More precisely, for an experiment/prediction task E, consider

- An extraction technique ξ used to obtain *finite, locally accessible* information $\xi(\lambda)$ from the ontological state λ
- A computable function P_E which computes a prediction using the information extracted: i.e. $P_E(\xi(\lambda))$
- E is predictable if we can give such a ξ, P_E that always give the correct predictions

[14] A. Abbott, C. Calude & K. Svozil, Information 6 (2015), 773.

Randomness as Maximal Unpredictability

Following Eagle, we can define randomness as maximum unpredictability with respect to this model

Randomness as Maximal Unpredictability

E is random iff there is no predictor extractor pair ξ, P_E, such that the posterior credence of E, given the prediction $P_E(\xi(\lambda))$, is uniform over a set of $k > 1$ outcomes.

- Although we require here uniformity for randomness, one can easily modify this definition to quantify degrees of randomness

Note that even if a prediction is subjective (i.e., made by a particular predictor), unpredictability is objective in the sense that E is unpredictable for *any* predictor

Quantum Randomness

Does this notion of randomness correctly explain quantum randomness?

- It is not difficult to show that quantum value indefiniteness (i.e., indeterminism) implies unpredictability
- Events predicted by the Born rule to be uniformly distributed are thus random, as expected

Seems to match the kind of informal account of quantum randomness mentioned earlier, but also definitions motivated by more practical scenarios:

- "*Measurement outcomes are intrinsically random events in the sense that they are uncorrelated with any prior events.*"[15]
- "*Any attempt to better explain the outcomes of quantum measurements [in terms of correlations] is destined to fail.*"[16]

[15] J. Bub, Entropy 17 (2015), p. 7377.

[16] R. Colbeck & R. Renner, Nat. Commun. 2 (2011), 414.

Quantum Randomness

Correctly grounding such claims is important, since explains bold claims in contemporary applications of quantum theory in cryptography, etc.

- Essential to these applications are violations of Bell inequalities. As pointed out by Cavalcanti and Wiseman, "*if it is impossible to signal faster than light, then it is impossible to predict the outcomes of experiments that violate Bell inequalities, even if those outcomes might be determined by an underlying hidden-variable model.*"[17]
  - So in a *device independent* way (i.e., from measurement statistics alone), and some reasonable assumptions, one can *certify* that a quantum device is random
  - QRNGs constructed on this principle are not, e.g., susceptible to 'memory-stick attacks', whereas a product notion of randomness can make no such guarantee
  - But one must assume that the predictors can make free choices

[17] E. Cavalcanti & H. Wiseman, Found. Phys. 42 (2012), 1330.

Summary

- Common accounts of quantum randomness have been based on quantum indeterminism and invoked the Commonplace Thesis to deem measurements random
- Quantum indeterminism is well grounded in the theorems of Bell and Kochen and Specker
- The process notion of randomness applicable to quantum measurements can be formalised rigorously via lack of effective predictability
- This corresponds well to common claims about quantum randomness, and the advantages it provides in quantum information
- However, one must be more cautious about claims of 'true' or 'absolute' randomness

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Chaotic Randomness

Is chance a necessary condition for this type of randomness? How would this effect the status of quantum randomness?

"*Il peut arriver que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux; une petite erreur sur les premières produirait une erreur énorme sur les derniers. La prédiction devient impossible.*"[18]

"*Il faut donc bien que le hasard soit autre chose que le nom que nous donnons à notre ignorance.*"[19]

- Even if this 'unpredictability' is not due to chance, the role of the 'interval of measure' is essential and thus the unpredictability is nonetheless intrinsic

[18] H. Poincaré, Science et Methode, Flammarion (1908), 68–69.

[19] H. Poincaré, Calcul des probabilités, Gauthier-Villars (1912).

Chaotic Randomness

For a given prediction task for a chaotic system (e.g., after a fixed time):

- Assuming classical mechanics, a sufficiently good extractor exists and thus renders the task predictable, and hence not random
- But given a fixed limit on measurement accuracy (a restriction on the extractors), such tasks may exhibit *relativised* unpredictability
- If such there was a fundamental, physical such limit, such processes *would be random*
  - For such systems, external perturbations pose a further, eventually more important, limit

Thus, in principle, one may have deterministic, intrinsic randomness.

What is Measurement?

Even if the Born rule is to be interpreted as representing real chance, a lingering problem remains: QM described unitary, reversible, dynamics, while measurement thus involves irreversible, acausal change. How does this change occur?

"*On the information-theoretic interpretation, the "big" measurement problem is a pseudo-problem. If the universe is genuinely indeterministic and measurement outcomes are intrinsically random, then it isn't possible to provide a dynamical explanation of how a system produces a definite outcome when it's measured—that's what it means for the measurement outcomes to be intrinsically random.*"[20]

But if QM is truly complete, it should be applicable to the measurement apparatus itself, which appears to still pose a problem

[20] J. Bub, Entropy 17 (2015), p. 7377.

The Measurement Problem

- Gisin[21] argues that to avoid the conclusion that "there are several kinds of stuffs out there, i.e. physical dualism, some stuff that respects the superposition principle and some that doesn't, or there are special configurations of atoms and photons where the superposition principle breaks down", QM must be extended to describe the dynamical collapse of the quantum state
- Everettians argue that measurement is subjective, a result of the observer becoming entangled with the system being measured
- Similarly, Quantum Bayesianism (QBism) takes the view that the Born rule should be interpreted in a strictly Bayesian sense
  - Quantum randomness must be understood differently in such frameworks where one does not have indeterminism; can it still be given a consistent meaning?

[21] N. Gisin, arXiv:1701.08300 (2017).

Kochen-Specker Definitions

Definition

A *value assignment function* on $\mathcal{O}$ is a partial two-valued function $v : \mathcal{O} \times \mathcal{C}_{\mathcal{O}} \to \{0,1\}$.

Definition

An observable $P \in \mathcal{O}$ is noncontextual under v if, for all $C, C' \in \mathcal{C}_{\mathcal{O}}$ with $P \in C, C'$, we have $v(P,C) = v(P,C')$.

Definition

A value assignment function v is admissible if for every context $C \in \mathcal{C}_{\mathcal{O}}$:

(a) if there exists a $P \in C$ with $v(P) = 1$, then $v(P') = 0$ for all $P' \in C \setminus \{P\}$;

(b) if there exists a $P \in C$ with $v(P') = 0$ for all $P' \in C \setminus \{P\}$, then $v(P) = 1$.

Kochen-Specker and Bell Theorems

Kochen-Specker Theorem

Let $n \geq 3$ and $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$ be states such that $0 < |\langle\psi|\phi\rangle| < 1$. Then there is a finite set of observables $\mathcal{O}$ containing P_ψ and P_ϕ for which there is no admissible value assignment function on $\mathcal{O}$ such that $v(P_\psi) = 1$ and P_ϕ is value definite.

Bell-CHSH Inequality

For two parties A and B, who can each choose between two measurements labelled $x, y = \pm 1$ producing outputs $a, b = \pm 1$, any locally realist model must satisfy

$$|C(a,b|0,0) + C(a,b|0,1) + C(a,b|1,0) - C(a,b|1,1)| \leq 2,$$

where $C(a,b|x,y) = \sum_{a,b}(-1)^{ab} p(a,b|x,y)$ is the correlation between a and b on inputs x and y.