Quantum Circuits with Classical and Quantum Control of Causal Orders

Alastair A. Abbott

joint work with
Julian Wechs, Hippolyte Dourdent and Cyril Branciard

Université de Genève, Geneva, Switzerland

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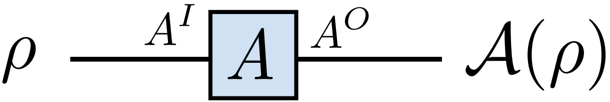
[arXiv:1807.10557 & arXiv:19??:soon™]

Quantum Channels

Quantum channels are most general physical map from quantum states to quantum states

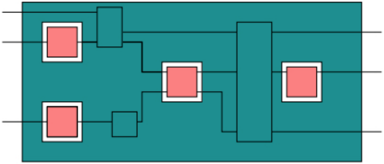
- *Completely*-positive trace preserving (CPTP) maps
- Extensively studied in quantum information



Quantum Combs

How can channels themselves be transformed?

- Quantum combs introduced as a general type of physical transformation mapping quantum channels to a quantum channel



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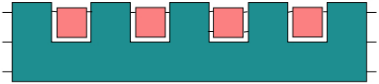


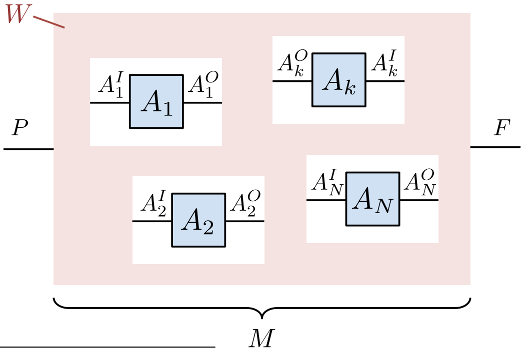
[Chiribella, D'Ariano and Perinotti, PRA 2018]

Quantum Supermaps

Quantum combs not the most general transformation on channels

Quantum Superchannel

A quantum superchannel is a transformation taking quantum channels into quantum channels even when applied to only part of channels' inputs. I.e., it is completely CPTP preserving.



[Chiribella, D'Ariano and Perinotti, EurPL 2008], [Araújo, Feix, Navascués and Brukner, Quantum 2017], [Quintino, Dong, Shimbo, Soeda, Murao, arXiv 2018]

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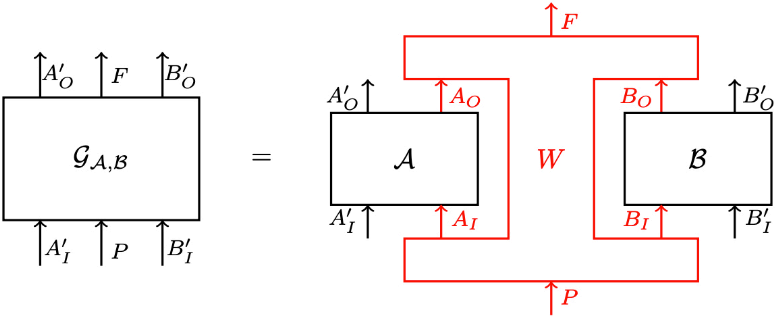
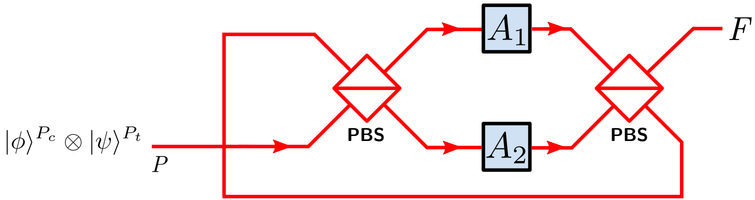


[Chiribella, D'Ariano and Perinotti, EurPL 2008], [Araújo, Feix, Navascués and Brukner, Quantum 2017], [Quintino, Dong, Shimbo, Soeda, Murao, arXiv 2018]

Superchannels and Causal Order

- Superchannels also studied as process matrices for their ability to have indefinite causal order
- Quantum switch shows quantum mechanics allows interesting ways to compose channels beyond quantum combs



Do all superchannels have a physical implementation?

[Oreshkov, Costa and Brukner, Nat. Commun. 2012], etc.

Motivation

- Quantum combs are natural, physical transformations, but unnecessarily restrictive for some applications
- We know physical superchannels beyond combs exist, but have no good systematic approach to study them
  - Examples mostly *ad hoc*: the quantum switch and its generalisation, the N-switch

Goal: find natural physical classes of superchannels and understand their relation to causal structure

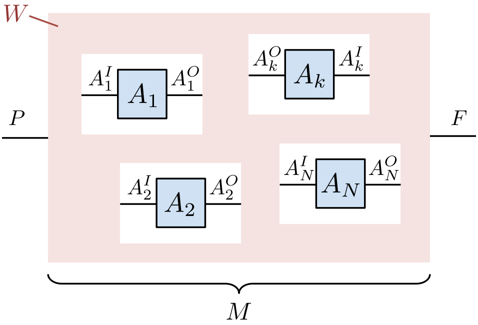
Outline

Quantum superchannels
- General setting and definition

Quantum superchannels with a fixed causal structure
- Quantum circuits (quantum combs)

Beyond superchannels with fixed causal structure
- Quantum circuits with classical control of causal order
- Quantum circuits with quantum (coherent) control of causal order

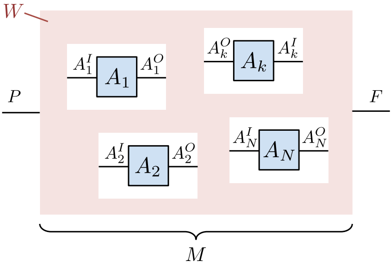
Quantum Superchannels



Work in the Choi picture

- $|\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$ is the "pure Choi isomorphism" of an identity channel
- Pure Choi isomorphism: for an operator A, $|A\rangle\rangle = \mathbb{1} \otimes A\,|\mathbb{1}\rangle\rangle$
- Choi isomorphism: for a CP map $\mathcal{M}$, $M = \mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)$

Quantum Supermaps



Superchannel represented by an operator W satisfying

$$W \geq 0, \qquad \operatorname{Tr} W = d_P d_{A_1^O} \dots d_{A_N^O}, \qquad W \in \mathcal{L}^{\mathcal{N}}$$

$$\begin{aligned} M &= \operatorname{Tr}_{A_{\mathcal{N}}^{IO}} \left[W (A_1^T \otimes \cdots \otimes A_N^T \otimes \mathbb{1}^{PF}) \right] \\ &= W * (A_1 \otimes \cdots \otimes A_N) \quad \in \; PF, \end{aligned}$$

- Output state given by $M * \rho = W * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$

[Oreshkov, Costa, Brukner, *Nat. Commun.* 2012]; [Araújo et al., Quantum 2017]; [Chiribella et al., EPL 2008]

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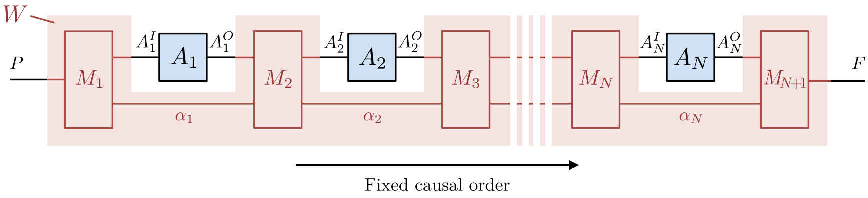
Quantum Combs as Fixed-Order Circuits

- Quantum combs can be seen as a circuit with a fixed order $A_1 \prec \cdots A_N$

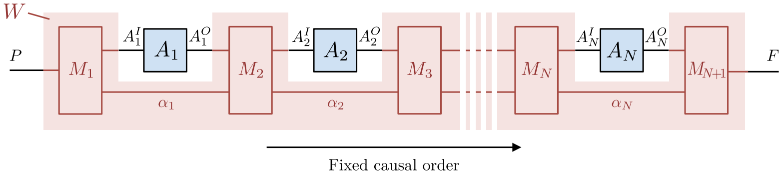


Most general quantum circuit described by CPTP maps:

- $\mathcal{M}_1 : P \to A_1^I \alpha_1$, where α_1 is an ancillary system
- $\mathcal{M}_{n+1} : A_n^O \alpha_n \to A_{n+1}^I \alpha_{n+1}$ for $1 \leq n \leq N-1$
- $\mathcal{M}_{N+1} : A_N^O \alpha_N \to F$

[Chiribella *et al.*, PRA 2009]

Quantum Circuits with Fixed Causal Order



For input ρ output is

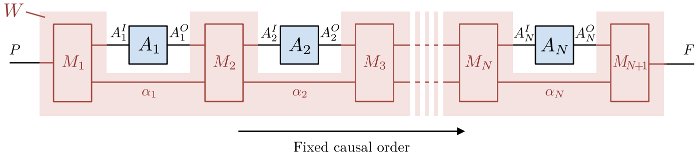
$$M_{N+1} * A_N * \cdots * M_2 * A_1 * M_1 * \rho = \underbrace{(M_1 * M_2 * \cdots * M_{N+1})}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

- W is defined uniquely by the maps M_n via the link product:

$$W = M_1 * M_2 * \cdots * M_{N+1} = \mathrm{Tr}_{\alpha_1 \cdots \alpha_N} \left[M_1 \otimes M_2^{T_{\alpha_1}} \otimes \cdots \otimes M_{N+1}^{T_{\alpha_N}} \right]$$

[Chiribella et al., PRL 2008.]

QC-FO Characterisation



The constraint that the M_n are CPTP maps and thus satisfy $\mathrm{Tr}_{A_{n+1}^I \alpha_{n+1}} M_{n+1} = \mathbb{1}^{A_n^O \alpha_n}$ allow the W of QC-FOs to be characterised

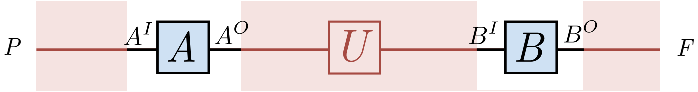
- They are precisely process matrices compatible with $P \prec A_1 \prec \cdots \prec A_N \prec F$

QC-FOs compatible with order $P \prec A_1 \prec \cdots \prec A_N \prec F$

$$\mathrm{Tr}_F W = W_{(N)} \otimes \mathbb{1}^{A_N^O},$$
$$\mathrm{Tr}_{A_{n+1}^I} W_{(n+1)} = W_{(n)} \otimes \mathbb{1}^{A_n^O} \quad \forall n = 1, \ldots, N-1,$$
$$\text{and} \quad \mathrm{Tr}_{A_1^I} W_{(1)} = \mathbb{1}^P.$$

where $W_{(n)} := \frac{1}{d_n^O d_{n+1}^O \cdots d_N^O} \mathrm{Tr}_{A_n^O A_{\{n+1,\ldots,N\}}^{IO} F} W$ are reduced process matrices

Example of a QC-FO



$$M_1 = |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|\,, \qquad M_2 = |U\rangle\rangle\langle\langle U|\,, \qquad M_3 = |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|$$

We then have

$$\begin{aligned} W &= |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{PA^I} * |U\rangle\rangle\langle\langle U|^{A^O B^I} * |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B^O F} \\ &= |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{PA^I} \otimes |U\rangle\rangle\langle\langle U|^{A^O B^I} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B^O F} \\ &= |w\rangle\rangle\langle\langle w|\,, \end{aligned}$$

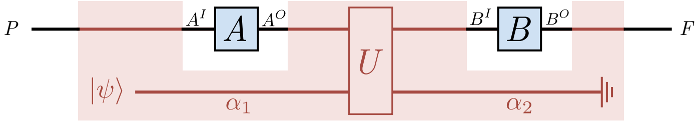
with $|w\rangle\rangle = |\mathbb{1}\rangle\rangle \otimes |U\rangle\rangle \otimes |\mathbb{1}\rangle\rangle$

- Conditions easily verified, e.g.

$$\mathrm{Tr}_F\, W = \underbrace{|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{PA^I} \otimes |U\rangle\rangle\langle\langle U|^{A^O B^I}}_{W_{(2)}} \otimes \mathbb{1}^{B^O}, \text{ etc.}$$

Example of a QC-FO

Channel with memory



$$M_1 = |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{PA^I} \otimes |\psi\rangle\langle\psi|^{\alpha_1}, \; M_2 = |U\rangle\rangle\langle\langle U|^{A^O\alpha_1 B^I \alpha_2}, \; M_3 = |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B^O F} \otimes \mathbb{1}^{\alpha_2}$$

This gives

$$W = \mathrm{Tr}_{\alpha_2}\left[|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{PA^I} \otimes \left(\langle\psi^*|U\rangle\rangle\langle\langle U|\psi^*\rangle\right)^{A^O B^I \alpha_2} \otimes |\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|^{B^O F}\right]$$

Outline

Classical Control of Causal Order

Mixing circuits with different orders is also physics

Bipartite Causal Separability [*Nat. Commun.* 2012]

A superchannel W is causally separable if it can be written

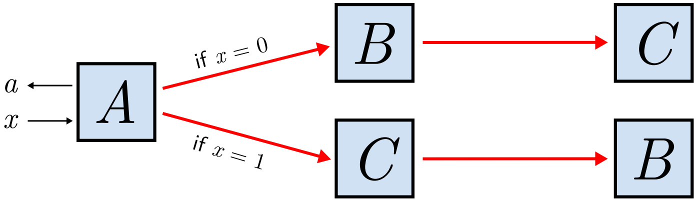
$$W = qW^{A \prec B} + (1-q)W^{B \prec A}$$

for $W^{A \prec B}, W^{B \prec A}$ QC-FOs. Otherwise it is causally nonseparable.

- The quantum switch can be proven to be causally nonseparable

Dynamical Causal Order

When considering more operations the situation is more subtle: one can have dynamical causal order

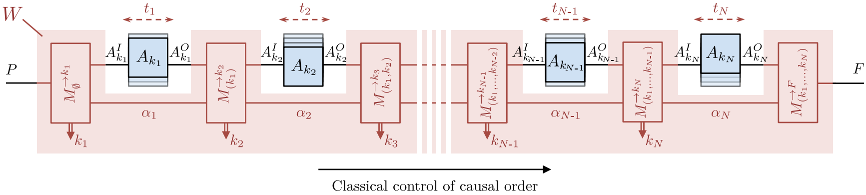


- Definition of causal separability and its characterisation is more subtle
- Can one give a circuit-type description of such superchannels?

[Oreshkov & Giarmatzi, NJP 2016]; [Wechs, Abbott and Branciard, NJP 2019]

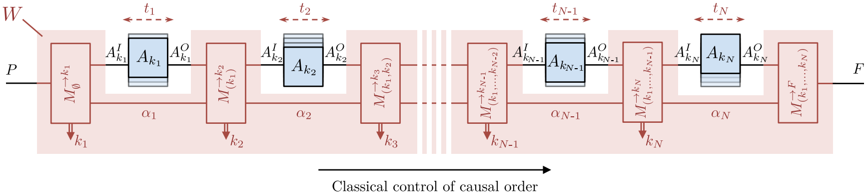
Classically Controlled Circuits

- Oreshkov & Giarmatzi [NJP, 2016] suggested causal separability corresponds to quantum circuits with classical control of causal order (QC-CCs): "classically controlled quantum circuits"



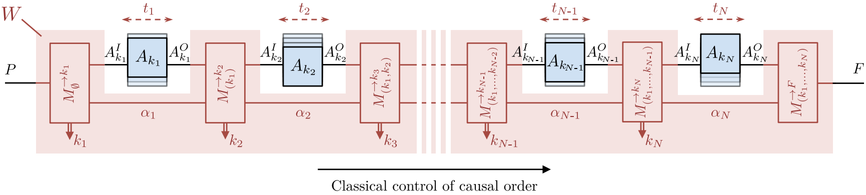
- At each time slot t_n exactly one operation A_{k_n} is applied
- Crucial requirement: each operation applied once and only once, irrespective of the operations themselves
  - Needed to ensure W gives a valid superchannel

Classically Controlled Circuits



- Outcome of instrument $\{M_{(k_1,\dots,k_n)}^{\to k_{n+1}}\}_{k_{n+1}}$ determines the $(n+1)$th operation to apply
- Technicality: the $M_{(k_1,\dots,k_n)}^{\to k_{n+1}} \in A_{k_n}^O \alpha_n A_{k_{n+1}}^I \alpha_{n+1}$ belong to different spaces
  - Can solve by embedding in common direct-sum output space

Process Matrix of a QC-CC



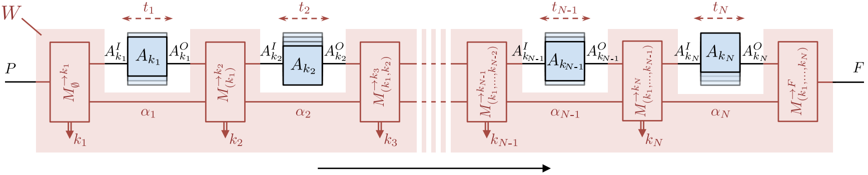
For input ρ, when operations applied in order $k_1, \dots, k_N$, output is

$$M^{\to F}_{(k_1,\dots,k_N)} * A_{k_N} * M^{\to k_N}_{(k_1,\dots,k_{N-1})} * \cdots * M^{\to k_3}_{(k_1,k_2)} * A_{k_2} * M^{\to k_2}_{(k_1)} * A_{k_1} * M^{\to k_1}_\emptyset * \rho$$
$$= \underbrace{M^{\to k_1}_\emptyset * M^{\to k_2}_{(k_1)} * M^{\to k_3}_{(k_1,k_2)} * \cdots * M^{\to k_N}_{(k_1,\dots,k_{N-1})} * M^{\to F}_{(k_1,\dots,k_N)}}_{\widetilde{W}_{(k_1,\dots,k_N,F)}} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

Process matrix of a QC-CC

$$W = \sum_{(k_1,\dots,k_N)} \widetilde{W}_{(k_1,\dots,k_N,F)}$$

QC-CC Characterisation



Characterisation of circuits with classically controlled order

W is the process matrix of a QC-CC iff $\exists$ PSD matrices $W_{(k_1,\ldots,k_N,F)}, W_{(k_1,\ldots,k_n)}$ for all $1 \leq n \leq N$ with:

- $W = \sum_{(k_1,\ldots,k_N)} W_{(k_1,\ldots,k_N,F)}$
- $\forall (k_1,\ldots,k_N),\ \mathrm{Tr}_F\, W_{(k_1,\ldots,k_N,F)} = W_{(k_1,\ldots,k_n)} \otimes \mathbb{1}^{A^O_{k_N}}$
- $\forall n = 1,\ldots,N-1,\ \forall (k_1,\ldots,k_n)$
  $\sum_{k_{n+1}} \mathrm{Tr}_{A^I_{k_{n+1}}} W_{(k_1,\ldots,k_n,k_{n+1})} = W_{(k_1,\ldots,k_n)} \otimes \mathbb{1}^{A^O_{k_n}}$
- $\sum_{k_1} \mathrm{Tr}_{A^I_{k_1}} W_{(k_1)} = \mathbb{1}^P$

QC-CC Characterisation

Characterisation of circuits with classically controlled order

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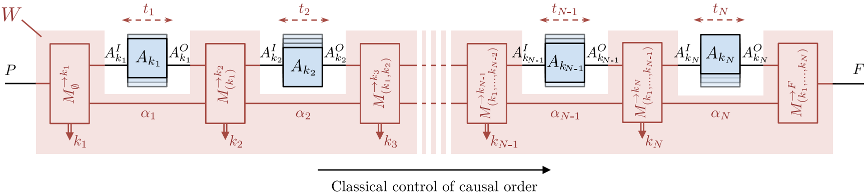
- $\forall n = 1,\ldots,N-1$, $\forall (k_1,\ldots,k_n)$
  $\sum_{k_{n+1}} \mathrm{Tr}_{A^I_{k_{n+1}}} W_{(k_1,\ldots,k_n,k_{n+1})} = W_{(k_1,\ldots,k_n)} \otimes \mathbb{1}^{A^O_{k_n}}$
- $\sum_{k_1} \mathrm{Tr}_{A^I_{k_1}} W_{(k_1)} = \mathbb{1}^P$

- Coincides with a sufficient (and possible necessary) condition for general causal separability from Wechs, AA and Branciard [NJP 2019]
  - An operational interpretation for causal separability
- Can be checked with semidefinite programming techniques

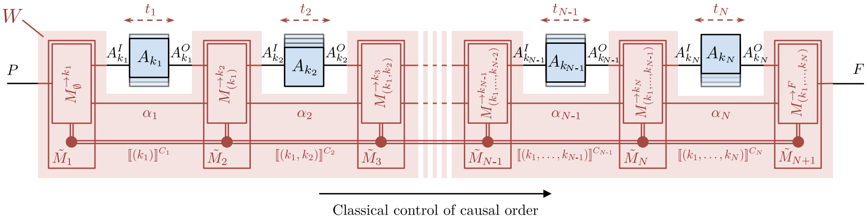
Example of a QC-CC



Classical Switch:

Alternative Descriptions of QC-CCs

- Conditioning can be included in operations by introducing (classical) control system $[\![(k_1,\dots,k_n)]\!]^{C_n} := |(k_1,\dots,k_n)\rangle\!\langle(k_1,\dots,k_n)|^{C_n}$

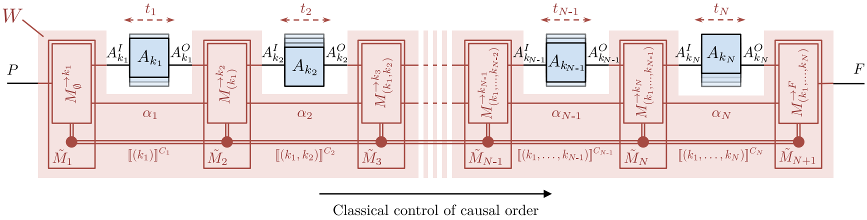


- Operations now given by the CPTP maps

$$\tilde{M}_{n+1} := \sum_{k_1,\dots,k_n,k_{n+1}} \tilde{M}_{(k_1,\dots,k_n)}^{\to k_{n+1}} \otimes [\![(k_1,\dots,k_n)]\!]^{C_n} \otimes [\![(k_1,\dots,k_n,k_{n+1})]\!]^{C_{n+1}},$$

$$\tilde{M}_1 := \sum_{k_1} \tilde{M}_{\emptyset}^{\to k_1} \otimes [\![(k_1)]\!]^{C_1}, \qquad M_{N+1} := \sum_{k_1,\dots,k_N} M_{(k_1,\dots,k_N)}^{\to F} \otimes [\![(k_1,\dots,k_N)]\!]^{C_N}$$

Alternative Descriptions of QC-CCs



- Defining global controlled operations $\tilde{A}_n := \bigoplus_{k_n \in \mathcal{N}} A_{k_n}$ we have

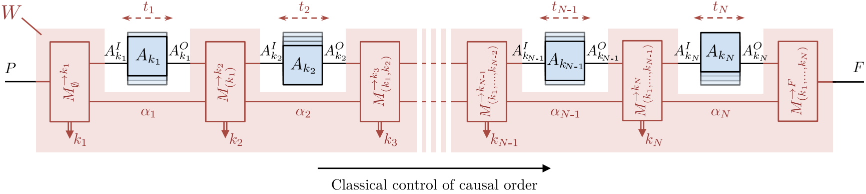
$$\tilde{M}_{N+1} * \tilde{A}_N * \tilde{M}_N * \cdots * \tilde{A}_1 * \tilde{M}_1 * \rho = \underbrace{\sum_{k_1,\ldots,k_N} W_{(k_1,\ldots,k_N,F)}}_{W} * (\rho \otimes A_1 \otimes \cdots \otimes A_N)$$

- Note that wlog we can take all operations to be purified isometries

$$M^{\to k_n}_{(k_1,\ldots,k_{n-1})} = |V^{\to k_n}_{(k_1,\ldots,k_{n-1})}\rangle\!\rangle\langle\!\langle V^{\to k_n}_{(k_1,\ldots,k_{n-1})}|$$

- Suggests natural generalisation to quantum control of causal order

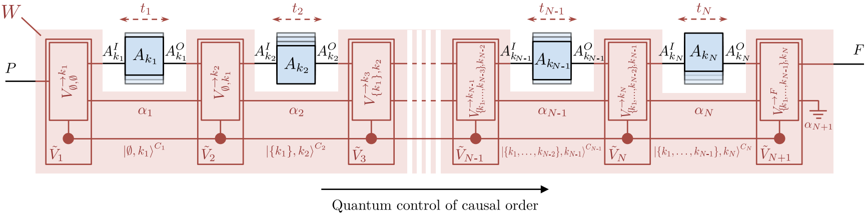
Example Rehashed



Classical Switch again:

From Classical to Coherent Control

- Relax the control state to store only *which* operations have been performed, but not their order: $|\mathcal{K}_{n-1}, k_n\rangle^{C_n}$
  - Conditioning on $\mathcal{K} = \{k_1, \dots, k_{n-1}\}$ allows different orders to interfere
  - Storing full history $|(k_1, \dots, k_n)\rangle^{C_n}$ is more restrictive and included in this case by using ancillas



- Operations given by the isometries V_n with pure CJ representation

$$|\tilde{V}_{n+1}\rangle\rangle := \sum_{\substack{\mathcal{K}_{n-1}\\ k_n, k_{n+1}}} |\tilde{V}^{\to k_{n+1}}_{\mathcal{K}_{n-1}, k_n}\rangle\rangle \otimes |\mathcal{K}_{n-1}, k_n\rangle^{C_n} \otimes |\mathcal{K}_n, k_{n+1}\rangle^{C_{n+1}},$$

$$|\tilde{V}_1\rangle\rangle := \sum_{k_1} |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\rangle \otimes |\emptyset, k_1\rangle^{C_1}, \qquad |\tilde{V}_{N+1}\rangle\rangle := \sum_{k_N} |\tilde{V}^{\to F}_{\mathcal{N}\setminus\{k_N\}, k_N}\rangle\rangle \otimes |\mathcal{N}\setminus\{k_N\}, k_N\rangle^{C_N}$$

From Classical to Coherent Control

- Relax the control state to store only *which* operations have been performed, but not their order: $|\mathcal{K}_{n-1}, k_n\rangle^{C_n}$
  - Conditioning on $\mathcal{K} = \{k_1, \ldots, k_{n-1}\}$ allows different orders to interfere
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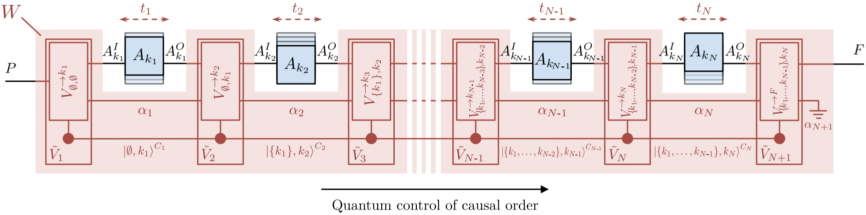

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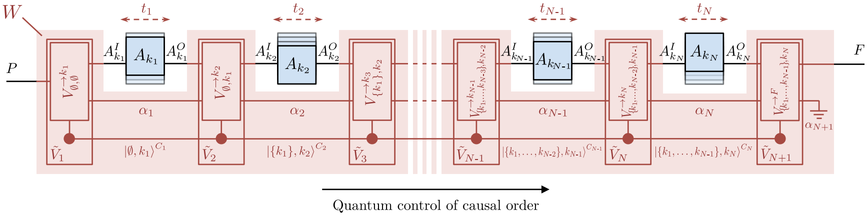
$$|\tilde{V}_1\rangle\!\rangle := \sum_{k_1} |\tilde{V}^{\to k_1}_{\emptyset,\emptyset}\rangle\!\rangle \otimes |\emptyset, k_1\rangle^{C_1} , \qquad |\tilde{V}_{N+1}\rangle\!\rangle := \sum_{k_N} |\tilde{V}^{\to F}_{\mathcal{N}\backslash\{k_N\}, k_N}\rangle\!\rangle \otimes |\mathcal{N}\backslash\{k_N\}, k_N\rangle^{C_N}$$

Coherently (Quantum) Controlled Circuits



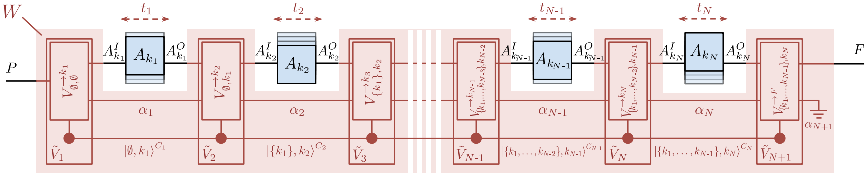
- Each $V_{\mathcal{K}_{n-1},k_n}^{\to k_{n+1}} : \mathcal{H}^{A_{k_n}^O \alpha_n} \to \mathcal{H}^{A_{k_{n+1}}^I \alpha_{n+1}}$ embedded in larger space
  - Control ensures that each party applied once and only once

For input $|\psi\rangle$, circuit applies transformation

$$|\tilde{V}_{N+1}\rangle\rangle * |\tilde{A}_N\rangle\rangle * |\tilde{V}_N\rangle\rangle * \cdots * |\tilde{V}_2\rangle\rangle * |\tilde{A}_1\rangle\rangle * |\tilde{V}_1\rangle\rangle * |\psi\rangle \in \mathcal{H}^{F\alpha_{n+1}}$$

with "pure link product" $|a\rangle^{\mathsf{A}} * |b\rangle^{\mathsf{B}} := \langle\langle \mathbb{1}|^{\mathsf{A}\cap\mathsf{B}} \left(|a\rangle \otimes |b\rangle \right) = \sum_i \langle i,i|^{(\mathsf{A}\cap\mathsf{B})^{\otimes 2}} \left(|a\rangle \otimes |b\rangle \right)$

QCs with Quantum Control of Causal Order



- To identify the process matrix, note that for input $|\psi\rangle$

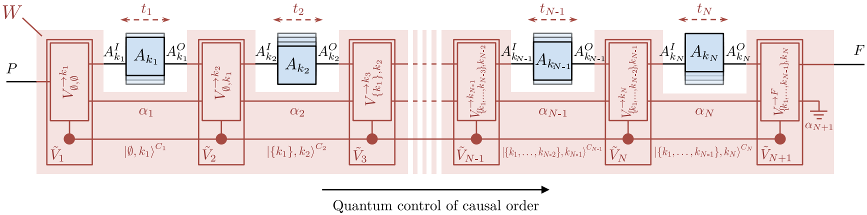
$$|\tilde{V}_{N+1}\rangle\rangle * |\tilde{A}_N\rangle\rangle * |\tilde{V}_N\rangle\rangle * \cdots * |\tilde{V}_2\rangle\rangle * |\tilde{A}_1\rangle\rangle * |\tilde{V}_1\rangle\rangle * |\psi\rangle$$

$$= \sum_{k_1,\ldots,k_N} \underbrace{|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle * |V_{\emptyset,k_1}^{\to k_2}\rangle\rangle * |V_{\{k_1\},k_2}^{\to k_3}\rangle\rangle * \cdots * |V_{\{k_1,\ldots,k_{N-1}\},k_N}^{\to F}\rangle\rangle}_{|w_{(k_1,\ldots,k_N,F)}\rangle\rangle} * (|\psi\rangle \otimes |A_1\rangle\rangle \otimes \cdots \otimes |A_N\rangle\rangle)$$

Process matrix of a QC-QC

$$W = \mathrm{Tr}_{\alpha_{N+1}} |w\rangle\rangle\langle\langle w| , \quad \text{with} \quad |w\rangle\rangle := \sum_{k_1,\ldots,k_N} |w_{(k_1,\ldots,k_N,F)}\rangle\rangle$$

QC-QC Characterisation



- As for QC-CCs, can characterise such W with SDP constraints

Characterisation of circuits with quantum control of causal order

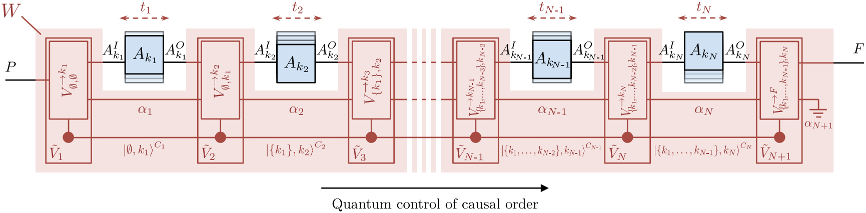
W is the process matrix of a QC-QC iff $\exists$ PSD matrices $W_{(\mathcal{K},\ell)} \in P A_{\mathcal{K}}^{IO} A_\ell^I$ $\forall \mathcal{K} \subsetneq \mathcal{N}, \ell \in \mathcal{N} \setminus \mathcal{K}$ satisfying

$$\mathrm{Tr}_F\, W = \sum_{k \in \mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

$$\forall\, \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N},\ \sum_{\ell \in \mathcal{N}\setminus\mathcal{K}} \mathrm{Tr}_{A_\ell^I}\, \widetilde{W}_{(\mathcal{K},\ell)} = \sum_{k \in \mathcal{K}} \widetilde{W}_{(\mathcal{K}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

and $$\sum_{\ell \in \mathcal{N}} \mathrm{Tr}_{A_\ell^I}\, W_{(\emptyset,\ell)} = \mathbb{1}^P.$$

QC-QC Characterisation

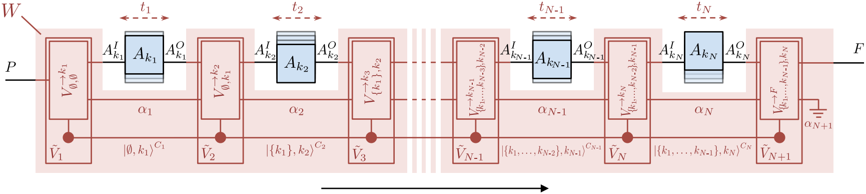


- As for QC-CCs, can characterise such W with SDP constraints

Three-operation QC-QC Characterisation

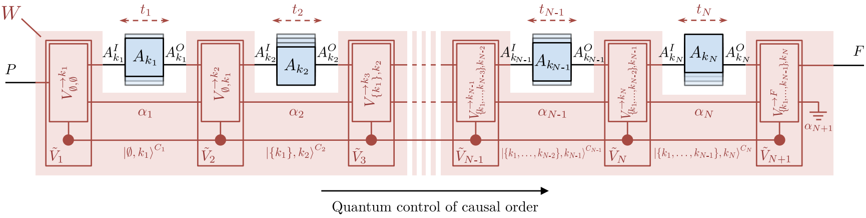
- $\mathrm{Tr}_F\, W = \widetilde{W}_{(\{B,C\},A)} \otimes \mathbb{1}^{A^O} + \widetilde{W}_{(\{A,C\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{A,B\},C)} \otimes \mathbb{1}^{C^O}$
- $\mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A,B\},C)} = \widetilde{W}_{(\{A\},B)} \otimes \mathbb{1}^{B^O} + \widetilde{W}_{(\{B\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{B^I}\, \widetilde{W}_{(\{A\},B)} + \mathrm{Tr}_{C^I}\, \widetilde{W}_{(\{A\},C)} = W_{(\{\emptyset\},A)} \otimes \mathbb{1}^{A^O}$, etc.
- $\mathrm{Tr}_{A^I}\, W_{(\{\emptyset\},A)} + \mathrm{Tr}_{B^I}\, W_{(\{\emptyset\},B)} + \mathrm{Tr}_{C^I}\, W_{(\{\emptyset\},C)} = \mathbb{1}^P$

QC-QC Summary



- QC-QCs are physically realisable, e.g., with a "quantum router"
- Realisation – in terms of the $\tilde{V}_n$ – can be effectively obtained from the any W satisfying the characterisation
  - Can be checked and obtained via SDPs, or witnesses obtained
- Classically controlled circuits are recovered as a special case
  - But QC-QCs can be causally nonseparable in general

QC-QC Summary



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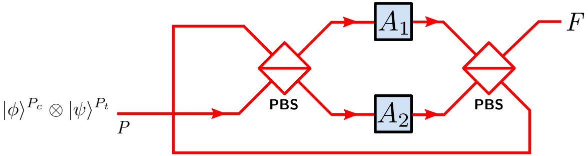
Example: Quantum Switch

$N = 2$, d-dimensional target system and 2-dimensional "control":
$\mathcal{H}^P = \mathcal{H}^{P_t} \otimes \mathcal{H}^{P_c}$ and $\mathcal{H}^F = \mathcal{H}^{F_t} \otimes \mathcal{H}^{F_c}$



The controlled operations are

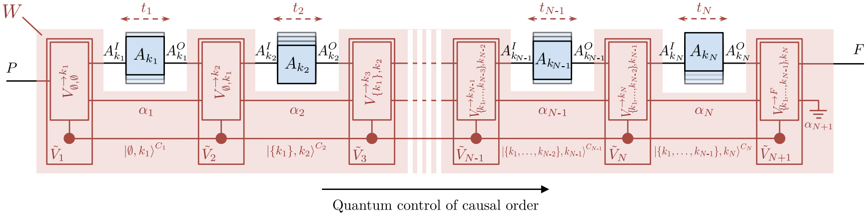
$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_\mathsf{c}} |\mathbb{1}\rangle\rangle^{P_\mathsf{t} A_{k_1}^I}, \quad |V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = |\mathbb{1}\rangle\rangle^{A_{k_1}^O A_{k_2}^I}, \quad |V_{\{k_1\},k_2}^{\to F}\rangle\rangle = |k_1\rangle^{F_\mathsf{c}} |\mathbb{1}\rangle\rangle^{A_{k_2}^O F_\mathsf{t}}$$

Process vector is then

$$\begin{aligned} |w_\mathsf{s}\rangle\rangle &:= |w_{(P,A_1,A_2,F)}\rangle\rangle + |w_{(P,A_2,A_1,F)}\rangle\rangle \\ &= |V_{\emptyset,\emptyset}^{\to A_1}\rangle\rangle * |V_{\emptyset,A_1}^{\to A_2}\rangle\rangle * |V_{\{A_1\},A_2}^{\to F}\rangle\rangle + |V_{\emptyset,\emptyset}^{\to A_2}\rangle\rangle * |V_{\emptyset,A_2}^{\to A_1}\rangle\rangle * |V_{\{A_2\},A_1}^{\to F}\rangle\rangle \\ &= |1\rangle^{P_\mathsf{c}} |\mathbb{1}\rangle\rangle^{P_\mathsf{t} A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O F_\mathsf{t}} |1\rangle^{F_\mathsf{c}} + |2\rangle^{P_\mathsf{c}} |\mathbb{1}\rangle\rangle^{P_\mathsf{t} A_2^I} |\mathbb{1}\rangle\rangle^{A_2^O A_1^I} |\mathbb{1}\rangle\rangle^{A_1^O F_\mathsf{t}} |2\rangle^{F_\mathsf{c}} \end{aligned}$$

Standard four-partite switch recovered as $W_\mathsf{switch} = |w_\mathsf{s}\rangle\rangle\langle\langle w_\mathsf{s}|$

Example: Quantum N-Switch



Beyond the Quantum Switch?

- N-partite generalisation of the quantum switch is a QC-QC
  - Essentially the extent of known “interesting” causally nonseparable processes
- Do QC-QCs offer something new, or are they all “equivalent” to the switch?
- Need a better understanding of causally nonseparable resources and free operations
  - Taddei, Nery and Aolita [arXiv:1903.06180]: local operations and controlled non-signalling operations ofr bipartite processes
  - Composition: Possible compositions severely restricted [Guérin et al., NJP 2019], but can, e.g., concatenate switches, or insert them inside other switches

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Beyond the Quantum Switch?

However, recall characterisation of QC-QCs

$$\mathrm{Tr}_F W = \sum_{k \in \mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

$$\forall \emptyset \subsetneq \mathcal{K} \subsetneq \mathcal{N}, \sum_{\ell \in \mathcal{N}\setminus\mathcal{K}} \mathrm{Tr}_{A_\ell^I} \widetilde{W}_{(\mathcal{K},\ell)} = \sum_{k \in \mathcal{K}} \widetilde{W}_{(\mathcal{K}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O},$$

and $$\sum_{\ell \in \mathcal{N}} \mathrm{Tr}_{A_\ell^I} W_{(\emptyset,\ell)} = \mathbb{1}^P.$$

- The $\widetilde{W}_{(\mathcal{N}\setminus\{k\},k)}$ need not, *a priori*, be valid process matrices
  - $\mathrm{Tr}_F W$ not necessarily a mixture of valid process matrices compatible with fixed last parties, i.e.

$$\mathrm{Tr}_F W \neq q_1 W^{\{A_1,A_2\}\prec A_3} + q_2 W^{\{A_1,A_3\}\prec A_2} + q_3 W^{\{A_2,A_3\}\prec A_1}$$

- Seems like such W can't be obtained by composing switches
  - Numerically, such processes seem to exist: further study needed to find (and interpret) nice examples
  - In particular, can one obtain new types of advantages with QC-QCs – which, by construction, are realisable

QC-QCs and Causal Correlations

Can quantum circuits with quantum control of causal order violate causal inequalities?

QC-QC correlations are causal

Let W be a QC-QC with trivial spaces $\mathcal{H}^P$ and $\mathcal{H}^F$. Then the correlations

$$P(a_1,\dots,a_N|x_1,\dots,x_N) = \mathrm{Tr}[W\cdot(M_{a_1|x_1}^T\otimes\cdots\otimes M_{a_N|x_N}^T)]$$

are causal for any instruments $\{M_{a_i|x_i}\}_{x_i}$.

- Can noncausal correlations be realised in nature?
  - Would require going beyond this type of generic, coherently controlled circuit
- QC-QCs nevertheless have potential for new advantages arising from indefinite causal order
  - New classes of physically realisable, causally nonseparable processes
  - Use as "quantum super-instruments", generalising quantum testers, for transformation tasks

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Summary & Outlook

- Circuit-based (physical) families of superchannels of increasing power
- Quantum circuits with classical control of causal order
  - Coincide with sufficient condition for causal separability
- Quantum circuits with quantum control of causal order
  - Potential new realisable, causally nonseparable, circuits beyond the quantum switch?
  - Do QC-QCs provide new information theoretical advantages?
  - Need for resource theoretical treatment for such processes
  - Are there other classes of physically realisable processes?

[arXiv:1807.10557 + new paper soon]

Choi Isomorphism and Link Product

- $|\mathbb{1}\rangle\rangle = \sum_i |i\rangle \otimes |i\rangle$ is the "pure Choi isomorphism" of an identity channel
- Pure Choi isomorphism: for an operator A, $|A\rangle\rangle = \mathbb{1} \otimes A\,|\mathbb{1}\rangle\rangle$
- Mixed Choi isomorphism: for a CP map $\mathcal{M}$, $M = \mathcal{I} \otimes \mathcal{M}(|\mathbb{1}\rangle\rangle\langle\langle\mathbb{1}|)$
- Inverse Choi isomorphism given by the link product: $\mathcal{M}(\rho) = M * \rho$; $A\,|\psi\rangle = |A\rangle\rangle * |\psi\rangle$

Constraints for Process Matrix Validity

Recall the notation:

$$ {}_X W := (\mathrm{Tr}_X W) \otimes \frac{\mathbb{1}^X}{d_X}, \quad {}_1 W := W, \quad {}_{[\sum_X \alpha_X X]} W := \sum_X \alpha_X \, {}_X W, $$

Space of valid process matrices

$$ W \in \mathcal{L}^{\mathcal{N}} \;\Leftrightarrow\; \forall\, \chi \subseteq \mathcal{N}, \chi \neq \emptyset, \; {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash\chi}} W = 0, $$

Space of valid process compatible with A first

$$ W \in \mathcal{L}^{A_k \prec (\mathcal{N}\backslash A_k)} $$

$$ \Leftrightarrow \; {}_{[1-A_O^k]A_{IO}^{\mathcal{N}\backslash k}} W = 0 \quad \text{and} \quad \forall\, \chi \subseteq \mathcal{N}\backslash k, \chi \neq 0, {}_{\prod_{i\in\chi}[1-A_O^i]A_{IO}^{\mathcal{N}\backslash k \backslash \chi}} W = 0, $$

Causal Separability: Necessary Conditions

- Explicit necessary conditions can be obtained by choosing specific CP maps and ancillas at each level of the recursive definition
- Ognyan and Giarmatzi showed how such a choice proves sufficient conditions also necessary in tripartite case
  1. ρ: maximally entangled state for each pair of parties
  2. M_{A_k}: $|\Phi^+\rangle\!\langle\Phi^+|$ – M.E.S. between A^{IO} and half of ancilla between A_k and some $A_{k'}$
- "Teleports" A_k's system on A_k^{IO} to $A_{k'}^{I'}$

$$\underbrace{W_{(k)}^{\rho}}_{N\text{-partite, } A_k \text{ first}} \longrightarrow \underbrace{W_{(k)}^{A_k^{IO} \to A_{k'}^{I'}}}_{(N-1)\text{-partite, formally equivalent to } W_{(k)}} \otimes \rho' := (W_{(k)}^{\rho})_{|M_{A_k}}$$

- Any constraints obeyed by $W_{(k)}^{A_k^{IO} \to A_{k'}^{I'}}$ must be obeyed by $W_{(k)}$ once Hilbert spaces relabelled
  - Can repeat for each $k' \neq k$

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Necessary condition for N-partite causal separability

An N-partite $W^{\text{sep}} \in \mathcal{W}^{\text{sep}}$ must have a decomposition $W = \sum_{k\in\mathcal{N}} W_{(k)}$ where:

1. $W_{(k)}$ is a valid process compatible with $A_k \prec (\mathcal{N}\backslash A_k)$
2. For each $k' \neq k$, $W_{(k)}^{A_k^{IO}\to A_{k'}^{I'}}$ is an $(N-1)$-partite causally separable process
   - $\implies$ obeys the necessary conditions for $(N-1)$-partite processes

- Coincides with separable condition for $N = 3$ [Oreshkov & Giarmatzi, NJP 2016]
- Also reduced 4-partite scenario (no output for D, c.f. quantum switch)
- Note that decomposition may differ for each k'
  - Satisfying these conditions with a unique decomposition would imply the sufficient conditions
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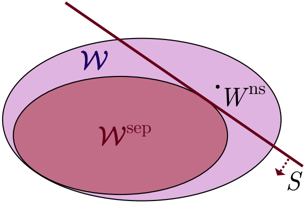
Witnesses of Causal Nonseparability

Causally separable process matrix

$$W^{\text{sep}} = q\, W^{A \prec B} + (1-q)\, W^{B \prec A},$$

- Convex cone of (non-normalised) causally separable processes:

$$\mathcal{W}^{\text{sep}} = (\mathcal{P} \cap \mathcal{L}^{A \prec B}) + (\mathcal{P} \cap \mathcal{L}^{B \prec A})$$



Witness of causal nonseparability

$\forall\, W^{\text{ns}} \notin \mathcal{W}^{\text{sep}},\ \exists S:$

$\mathrm{Tr}[S^T \cdot W^{\text{ns}}] < 0$, and

$Tr[S^T \cdot W^{\text{sep}}] \geq 0 \quad \forall\, W^{\text{sep}} \in \mathcal{W}^{\text{sep}}$

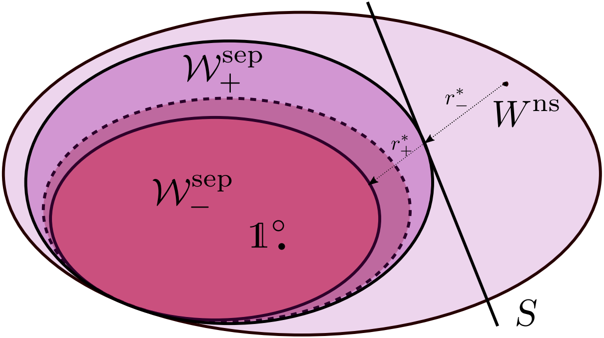
[Araújo *et al.*, NJP 2015; Branciard, Sci. Rep. 2016]

- Witnesses can be efficiently constructed by semidefinite programming (SDP)
- Witnesses can be measured experimentally

[Rubino *et al.*, Sci. Adv. 2017; Goswami *et al.*, PRL 2018]

Witnessing Causal Nonseparability

- Both necessary and sufficient conditions define convex cones $\mathcal{W}_+^{\mathsf{sep}}$, $\mathcal{W}_-^{\mathsf{sep}}$ of (non-normalised) process matrices



- Membership can be tested with SDP
- Dual SDP from necessary condition gives causal witnesses
- So far no numerical evidence that $\mathcal{W}_-^{\mathsf{sep}} \neq \mathcal{W}_+^{\mathsf{sep}}$, but. . .

Cones $\mathcal{W}^{\text{sep}}$ and $\mathcal{S}$ for tripartite case

Adopt the notation $\mathcal{L}_X = \{W|{}_X W = 0\}$.

$$
\begin{aligned}
\mathcal{W}^{\text{sep}} = \mathcal{L}_{[1-A_O]B_{IO}C_{IO}} &\cap \big(\mathcal{P} \cap \mathcal{L}_{[1-B_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]B_{IO}} \cap \mathcal{L}_{[1-B_O]}\big) \\
+ \mathcal{L}_{[1-B_O]A_{IO}C_{IO}} &\cap \big(\mathcal{P} \cap \mathcal{L}_{[1-A_O]C_{IO}} \cap \mathcal{L}_{[1-C_O]} + \mathcal{P} \cap \mathcal{L}_{[1-C_O]A_{IO}} \cap \mathcal{L}_{[1-A_O]}\big) \\
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\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S} = &\Big(\mathcal{L}^{\perp}_{[1-A_O]B_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-B_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]B_{IO}} + \mathcal{L}^{\perp}_{[1-B_O]})\Big) \\
\cap &\Big(\mathcal{L}^{\perp}_{[1-B_O]A_{IO}C_{IO}} + (\mathcal{P} + \mathcal{L}^{\perp}_{[1-A_O]C_{IO}} + \mathcal{L}^{\perp}_{[1-C_O]}) \cap (\mathcal{P} + \mathcal{L}^{\perp}_{[1-C_O]A_{IO}} + \mathcal{L}^{\perp}_{[1-A_O]})\Big) \\
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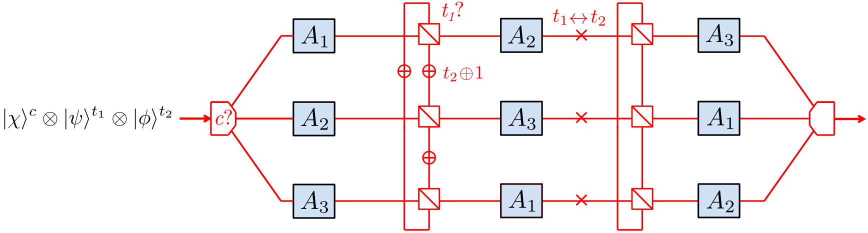
\end{aligned}
$$

Example: New type of 3-operation QC-QC

$N = 3$, two qubit "targets" and 3-dimensional "control"

- Initial control state $|k_1\rangle^{P_c}$ determines first party
- First party acts on first target qubit $|\psi\rangle^{P_{t_1}}$
- Output of A_{k_1} determines (dynamically, coherently) k_2 and conditions a flip on second target $|\phi\rangle^{P_{t_2}}$, which is swapped to become "active" target *after* A_{k_2}
- A_{k_3} then acts on this second target qubit

Can be represented in an "unravelled" form as:



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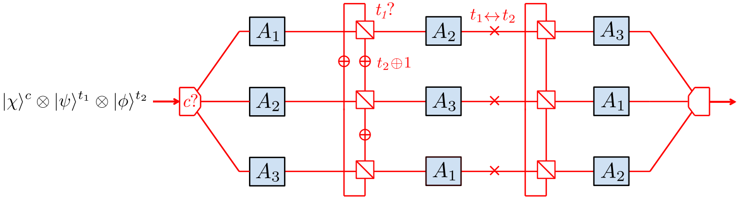

Controlled operations can be written

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\rangle = |k_1\rangle^{P_c} \, |\mathbb{1}\rangle\rangle^{P_{t_1} A^I_{k_1}} \, |\mathbb{1}\rangle\rangle^{P_{t_2} \alpha_1}$$

$$|V_{\emptyset,k_1}^{\to k_2}\rangle\rangle = |0\rangle^{A^O_{k_1}} \, |0\rangle^{A^I_{k_2}} \, |\mathbb{1}\rangle\rangle^{\alpha_1 \alpha_2}, \text{ and } |V_{\emptyset,k_1}^{\to k_2'}\rangle\rangle = |1\rangle^{A^O_{k_1}} \, |1\rangle^{A^I_{k_2'}} \, |X\rangle\rangle^{\alpha_1 \alpha_2} \text{ for } k_2 \neq k_2'$$

$$|V_{\{k_1\},k_2}^{\to k_3}\rangle\rangle = |\mathbb{1}\rangle\rangle^{A^O_{k_2} \alpha_3} \, |\mathbb{1}\rangle\rangle^{\alpha_2 A^I_{k_3}}$$

$$|V_{\{k_1,k_2\},k_3}^{\to F}\rangle\rangle = |k_3\rangle^{F_c} \, |\mathbb{1}\rangle\rangle^{A^O_{k_3} F_{t_2}} \, |\mathbb{1}\rangle\rangle^{\alpha_3 F_{t_1}}$$

giving (with cyclic permutations for $k_1 = 2, 3$)

$$|w_{(P,A_1,A_2,A_3,F)}\rangle\rangle = |1\rangle^{P_c} \, |\mathbb{1}\rangle\rangle^{P_{t_1} A^I_1} \, |00\rangle^{A^O_1 A^I_2} \, |\mathbb{1}\rangle\rangle^{A^O_2 F_{t_1}} \, |\mathbb{1}\rangle\rangle^{P_{t_2} A^I_3} \, |\mathbb{1}\rangle\rangle^{A^O_3 F_{t_2}} \, |3\rangle^{F_c}$$

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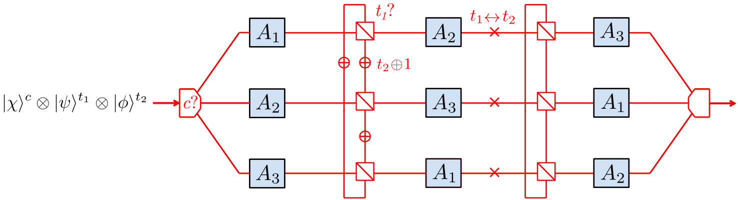
Example: New type of 3-operation QC-QC



Controlled operations can be written

$$|V_{\emptyset,\emptyset}^{\to k_1}\rangle\!\rangle = |k_1\rangle^{P_c} |\mathbb{1}\rangle\!\rangle^{P_{t_1} A_{k_1}^I} |\mathbb{1}\rangle\!\rangle^{P_{t_2} \alpha_1}$$

$$|V_{\emptyset,k_1}^{\to k_2}\rangle\!\rangle = |0\rangle^{A_{k_1}^O} |0\rangle^{A_{k_2}^I} |\mathbb{1}\rangle\!\rangle^{\alpha_1 \alpha_2}, \text{ and } |V_{\emptyset,k_1}^{\to k_2'}\rangle\!\rangle = |1\rangle^{A_{k_1}^O} |1\rangle^{A_{k_2'}^I} |X\rangle\!\rangle^{\alpha_1 \alpha_2} \text{ for } k_2 \neq k_2'$$

$$|V_{\{k_1\},k_2}^{\to k_3}\rangle\!\rangle = |\mathbb{1}\rangle\!\rangle^{A_{k_2}^O \alpha_3} |\mathbb{1}\rangle\!\rangle^{\alpha_2 A_{k_3}^I}$$

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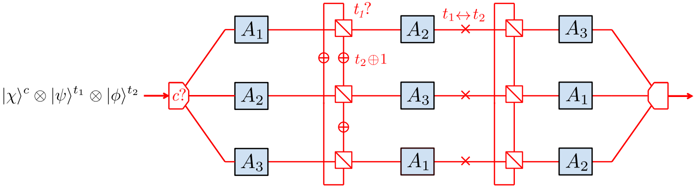

As before, define $W = |w\rangle\rangle\langle\langle w|$ with $|w\rangle\rangle = \sum_{k_1,k_2,k_3} |w_{(P,k_1,k_2,k_3,F)}\rangle\rangle$

- W is easily seen to be causally nonseparable
  - Pure process, not compatible with any of A, B or C being first (after P)
- Appears qualitatively different to the quantum switch
  - Two target qubits, one of which is also used to control the causal order
- But how to prove W is fundamentally inequivalent to quantum switch?
  - Could imagine composing switches, using control of one as target for another, etc.
  - Need a more complete resource theory, e.g. generalising Taddei, Nery and Aolita's proposal for bipartite processes [arXiv:1903.06180]

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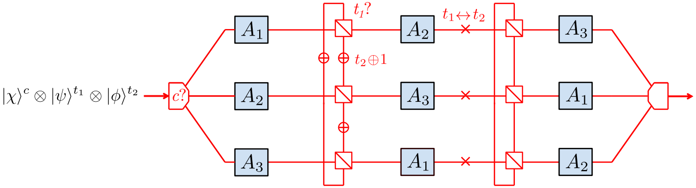

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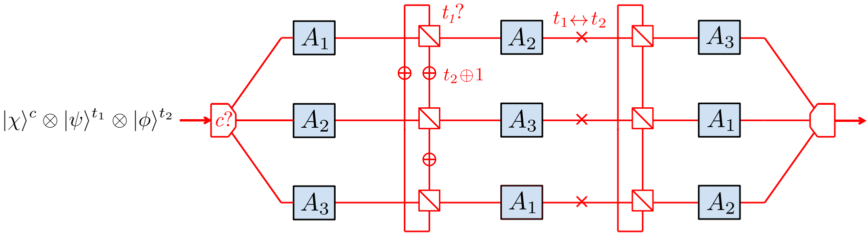

- Crucial difference: $\operatorname{Tr}_F W$ is causally nonseparable and cannot be written as a mixture of valid process matrices with fixed last parties, i.e.

$$\operatorname{Tr}_F W \neq q_1 W^{\{A_1,A_2\}\prec A_3} + q_2 W^{\{A_1,A_3\}\prec A_2} + q_3 W^{\{A_2,A_3\}\prec A_1}$$

- Recall characterisation: $\operatorname{Tr}_F W = \sum_{k\in\mathcal{N}} \widetilde{W}_{(\mathcal{N}\setminus\{k\},k)} \otimes \mathbb{1}^{A_k^O}$
  - the $\widetilde{W}_{(\mathcal{N}\setminus\{k\},k)}$ need not be valid process matrices
- Seems like no composition of quantum switches could give this property!