Assessing and comparing the performance of models for stream network data

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Some important features of river network data

- Confluences and 'connectedness'
- Data and networks can be *large*
- Measurements often very *sparse* in space

Methods and software available

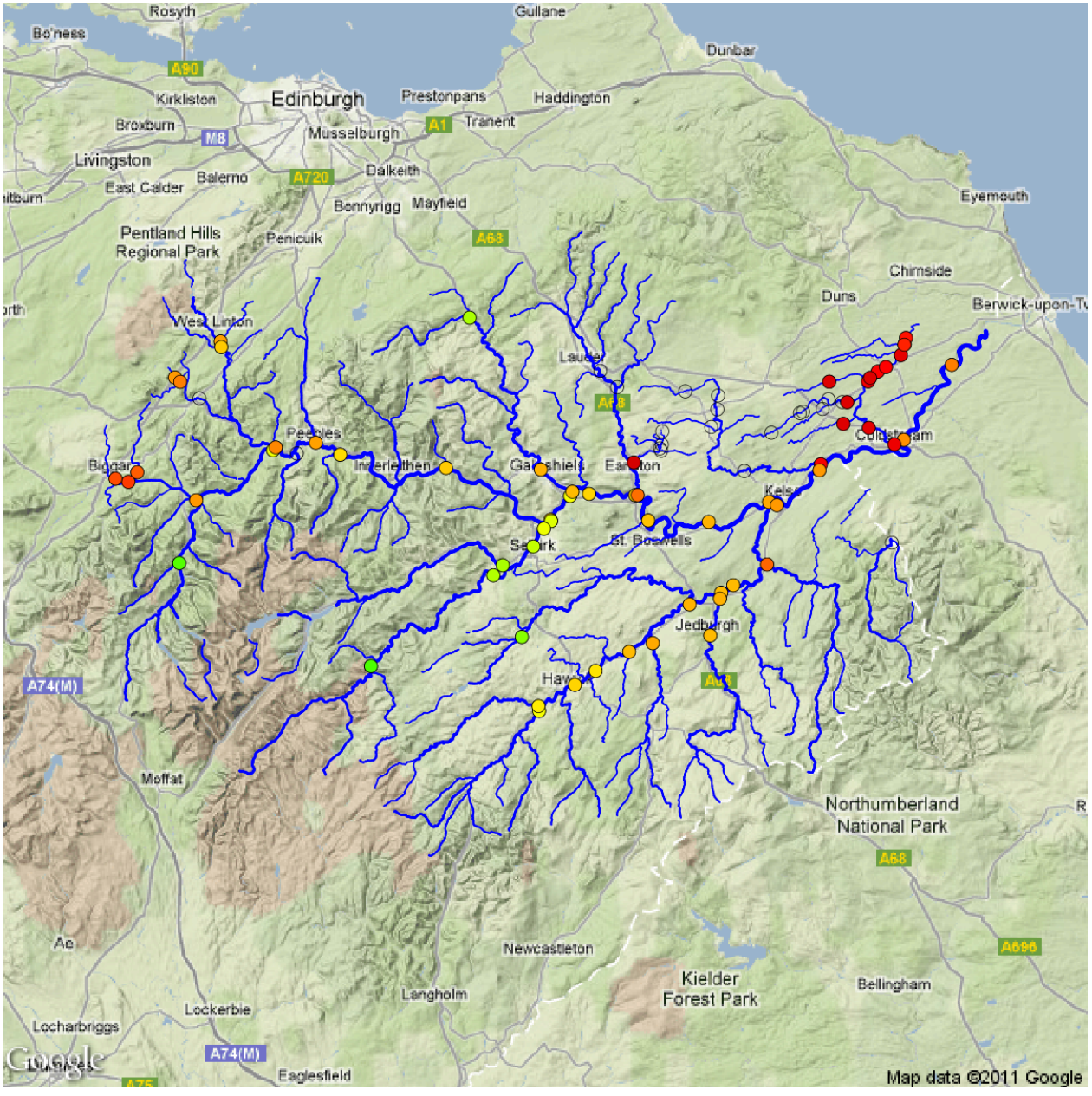
SSN

- Based on spatial covariance functions
- Ver Hoef et al. (JASA; 2010)
- R package: http://cran.r-project.org/web/packages/SSN/

SmoothNetwork

- Based on P-splines and penalised least squares
- O' Donnell et al. (JRSSC (*in press.*; 2013))
- R package: http://alastairrushworth.wordpress.com

Example: River Tweed



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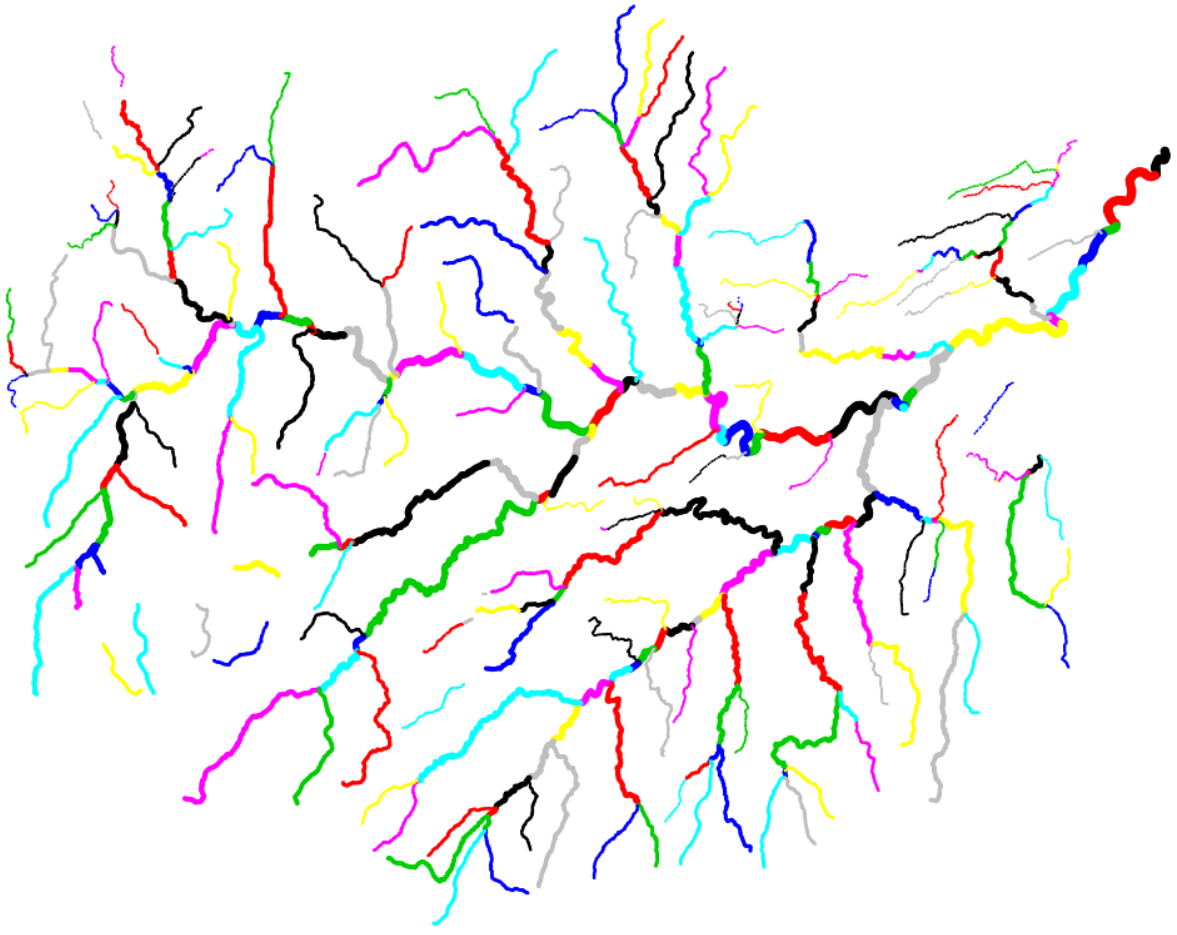

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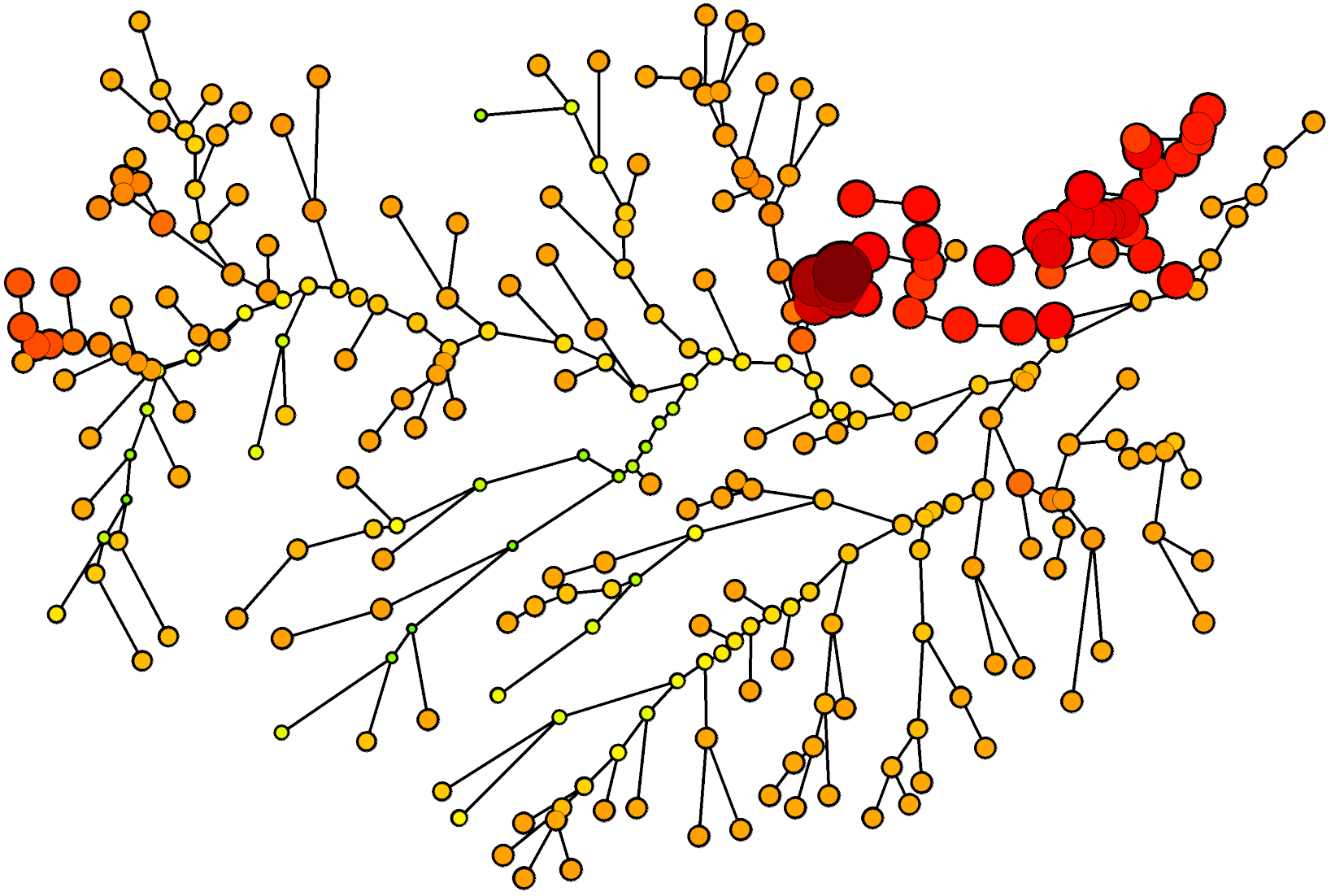


A simple model for segments

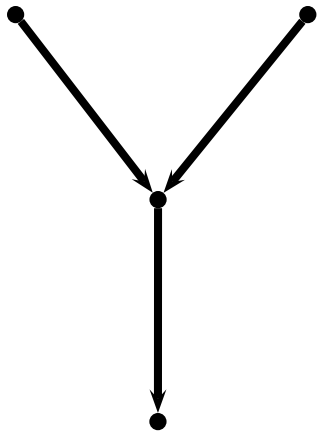
Suppose we have set of n responses each observed on one of a set of p nodes of a graph, and $\mathbf{x}$ is a covariate vector. Then

$$E(\mathbf{y}) = \underbrace{\boldsymbol{\alpha}}_{\text{intercept}} + \underbrace{\mathbf{Mb}}_{\text{Dummy variable with } p \text{ levels}}$$

where

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Confluence penalty



An idealised confluence with stream units 1, 2 **and** 3, **associated mean pollutant levels** b_1, b_2 **and** b_3.

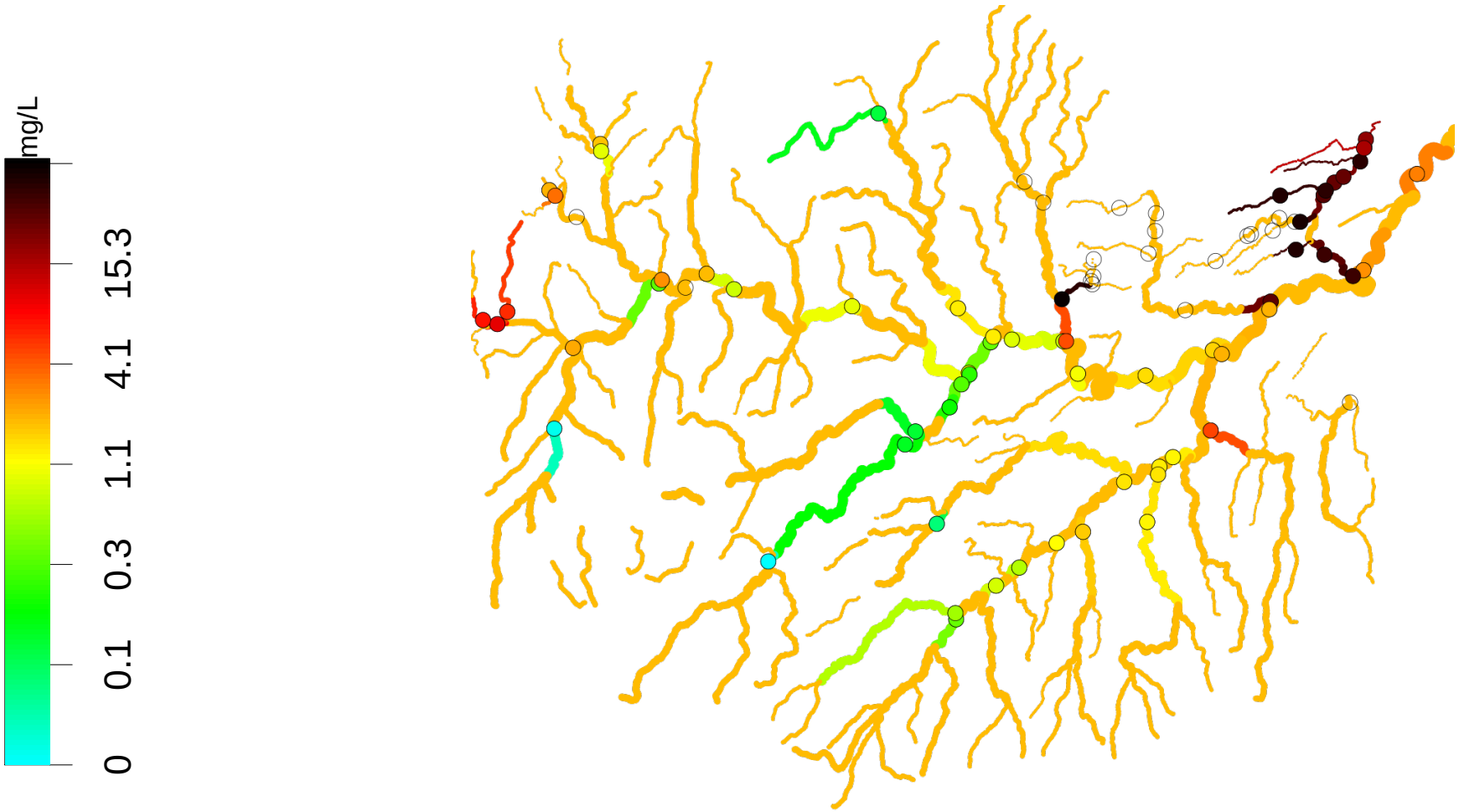
Values upstream
should be similar to those directly downstream
subject to their respective flow contributions:

$$\frac{f_1}{f_3} b_1 + \frac{f_2}{f_3} b_2 \approx b_3$$

$$\min_{\mathbf{b}} \left[(\mathbf{y} - \mathbf{M}\mathbf{b})^T (\mathbf{y} - \mathbf{M}\mathbf{b}) + \sum_{i,j \sim k} \lambda \left(\frac{f_i^2}{f_k^2} (b_i - b_k)^2 + \frac{f_j^2}{f_k^2} (b_j - b_k)^2 \right) \right]$$

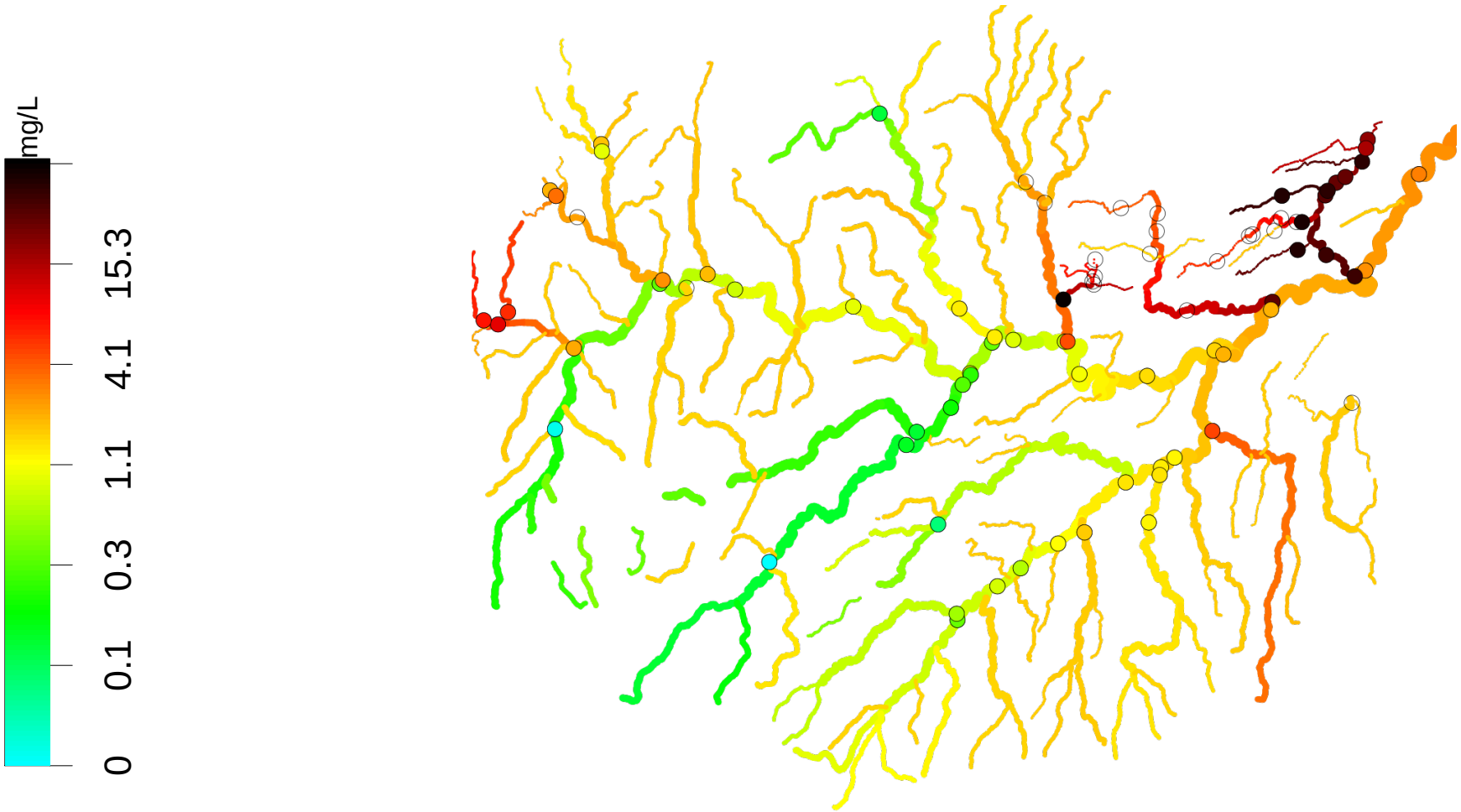
So what happens if you vary λ ?

Effect of varying the confluence penalty; $\lambda = 0$



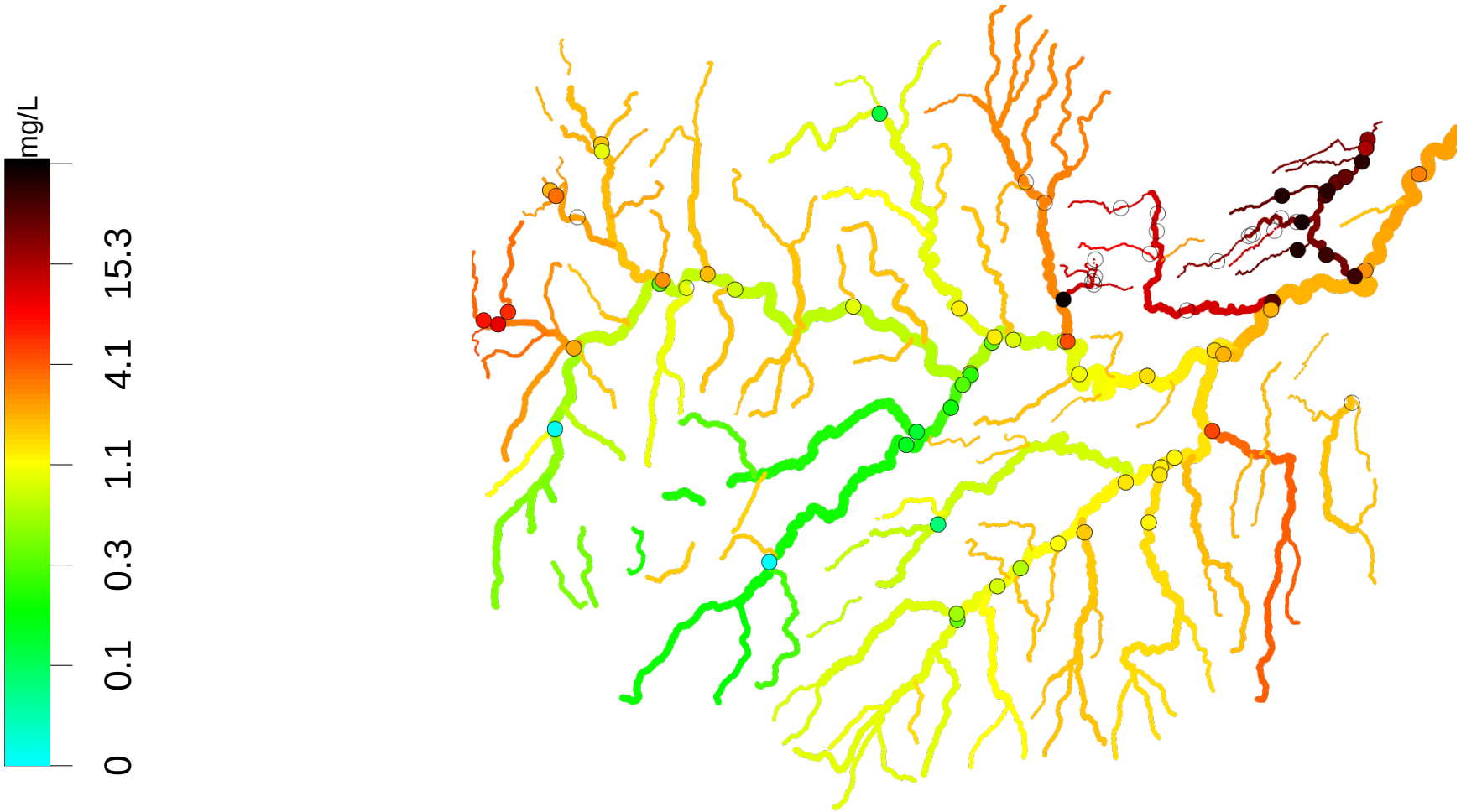
A weaker penalty allows the spatial smooth to interpolate the observed data

Effect of varying the confluence penalty; $\lambda = +$



A weaker penalty allows the spatial smooth to interpolate the observed data

Effect of varying the confluence penalty; $\lambda = ++$



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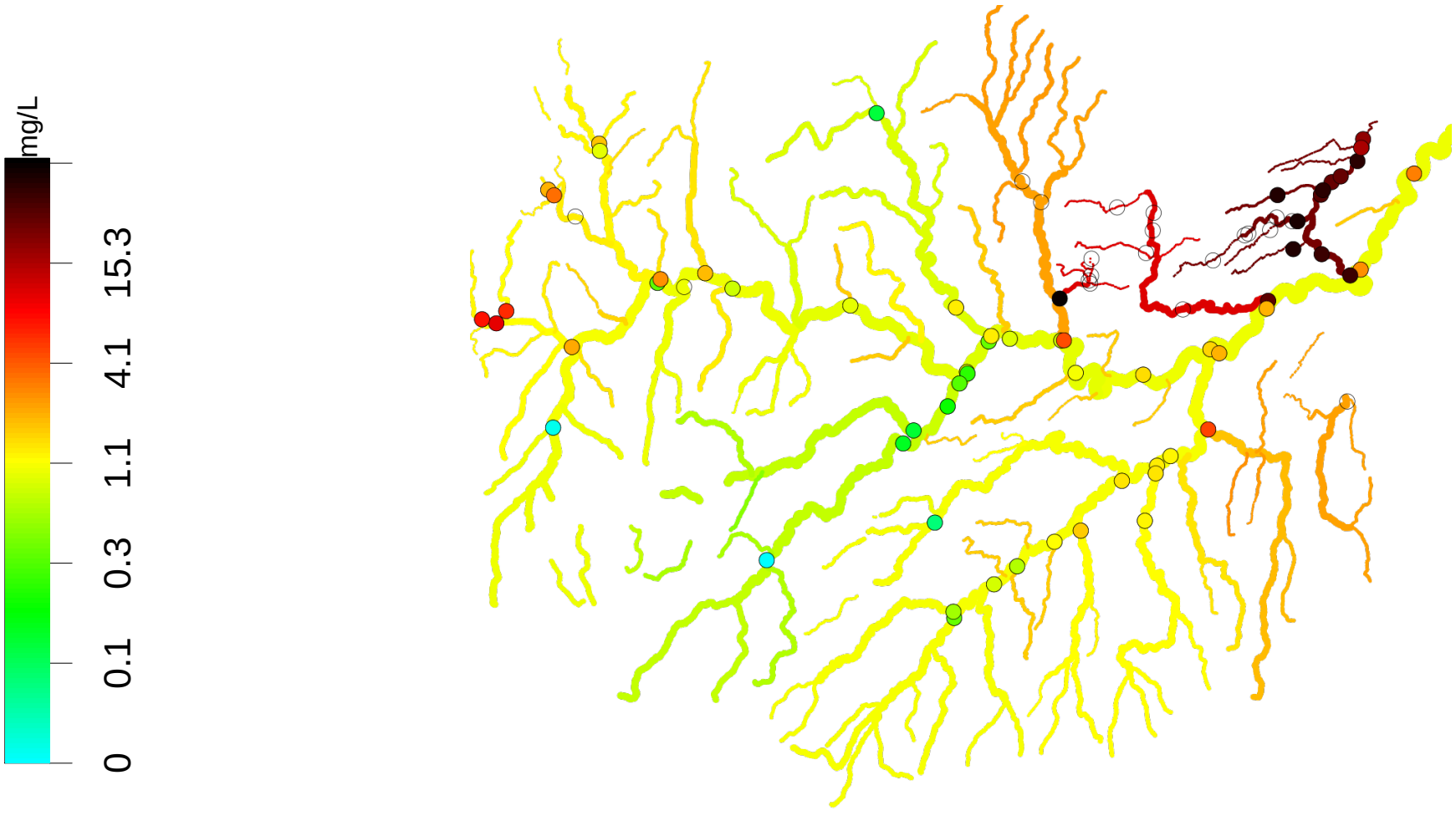
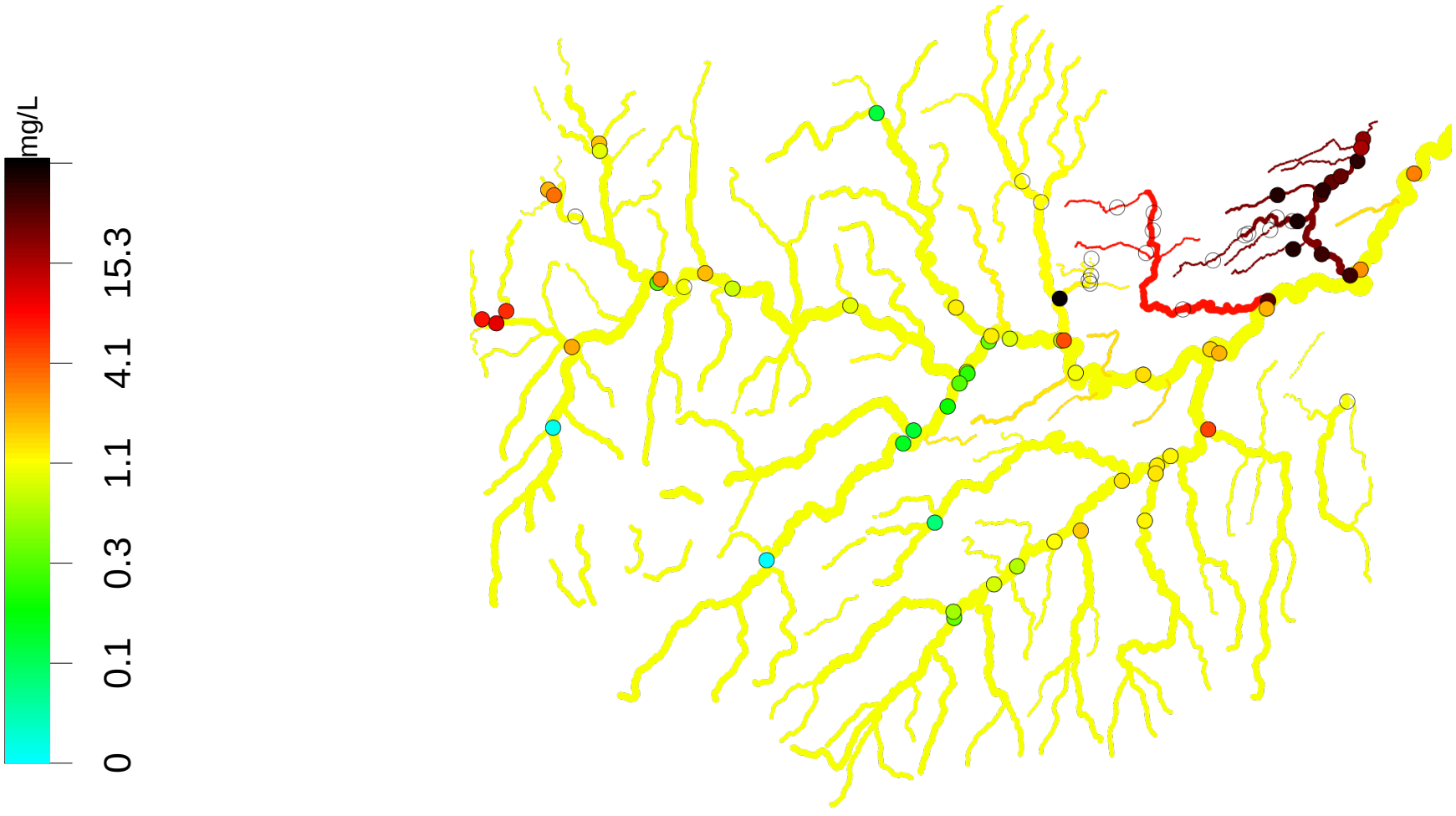


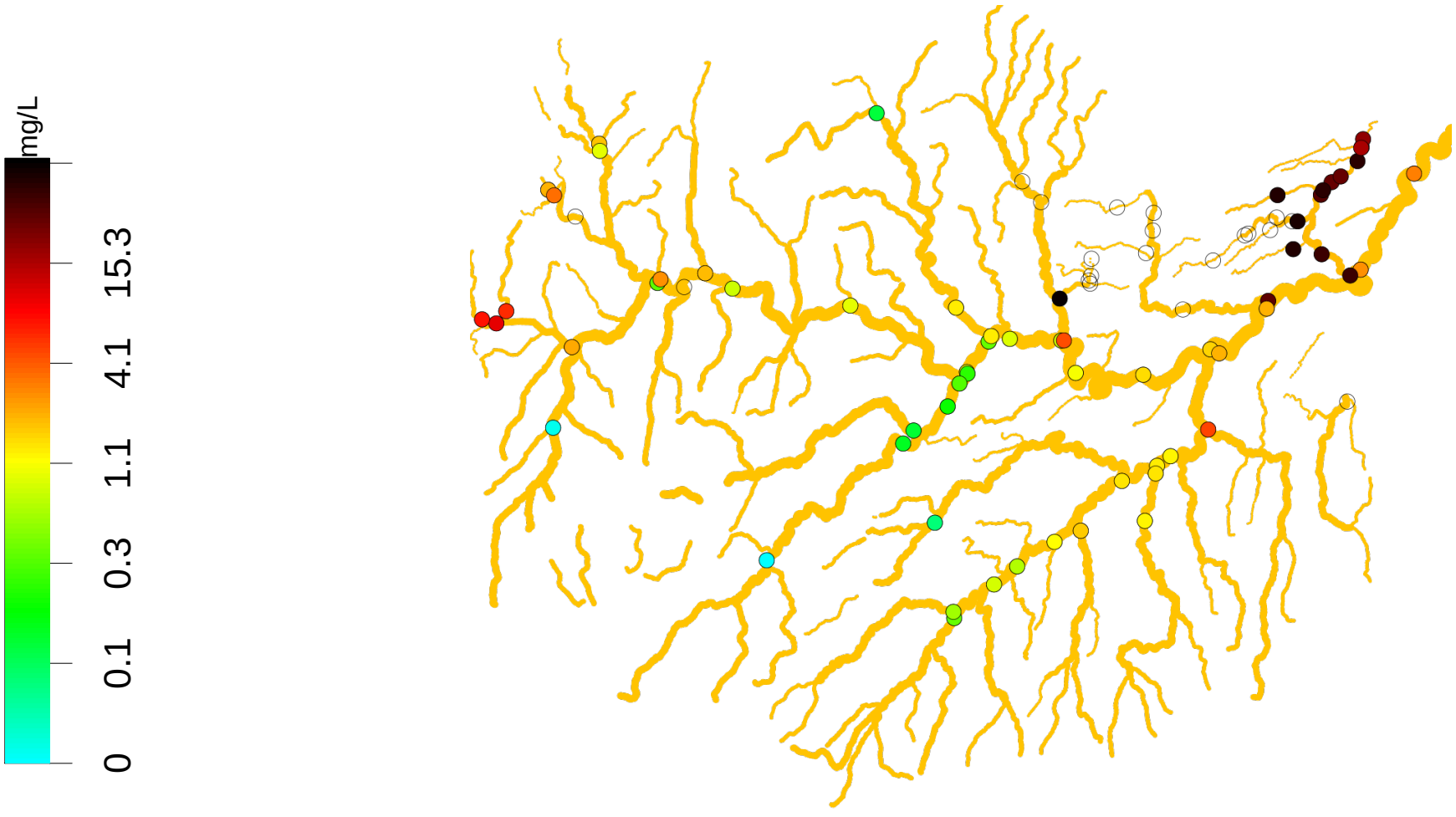
A stronger penalty tends to bring stream unit means nearer the *overall* mean

Effect of varying the confluence penalty; $\lambda = ++++$



A stronger penalty tends to bring stream unit means nearer the *overall* mean

Effect of varying the confluence penalty; $\lambda \to \infty$



A stronger penalty tends to bring stream unit means nearer the *overall* mean

Models based on P-splines: summary

- Piecewise constants for small stream units
- Estimation by PLS; very fast because of matrix sparsity
- Model easily to extend with additional smooth components

River network modelling with spatial covariance functions

Given data $\mathbf{Y} = (Y_1, \ldots, Y_n)$, covariates $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_p)$ let

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{z} + \boldsymbol{\epsilon}$$

where $\mathbf{z} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$ and $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \sigma^2\mathbf{I})$.

Cressie et al. (2006) and Ver Hoef et al. (2006) describe ways that are valid for stream networks to populate $\boldsymbol{\Sigma}$ such that $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(h|\boldsymbol{\theta})$ and where

- h is the stream distance separating two observations
- $\boldsymbol{\theta}$ is a vector of additional parameters

River network modelling with spatial covariance functions

Example: The covariance function for the 'tail up' exponential model is defined as

$$C_u(r_i, s_j|\boldsymbol{\theta}) = \begin{cases} \pi_{i,j} C_t(h|\boldsymbol{\theta}) & \text{if } r_i < s_j \text{ are flow-connected} \\ 0 & \text{if } r_i \text{ and } s_j \text{ are flow-unconnected} \end{cases}$$

where

$$C_t(h|\boldsymbol{\theta}) = \theta_v \exp(-h/\theta_r)$$

and $\pi_{i,j}$ is a weight function to account for the impact of confluences intervening two points.

River network modelling with spatial covariance functions

Key features, compared

- Very flexible: broad class of spatial structures can be captured with choice of covariance functions
- Non-Gaussian responses
- Estimation by REML; spatial effects are variance components, straightforward to add extra pieces - eg. Euclidean components.

Comparison

- Aim to answer the question

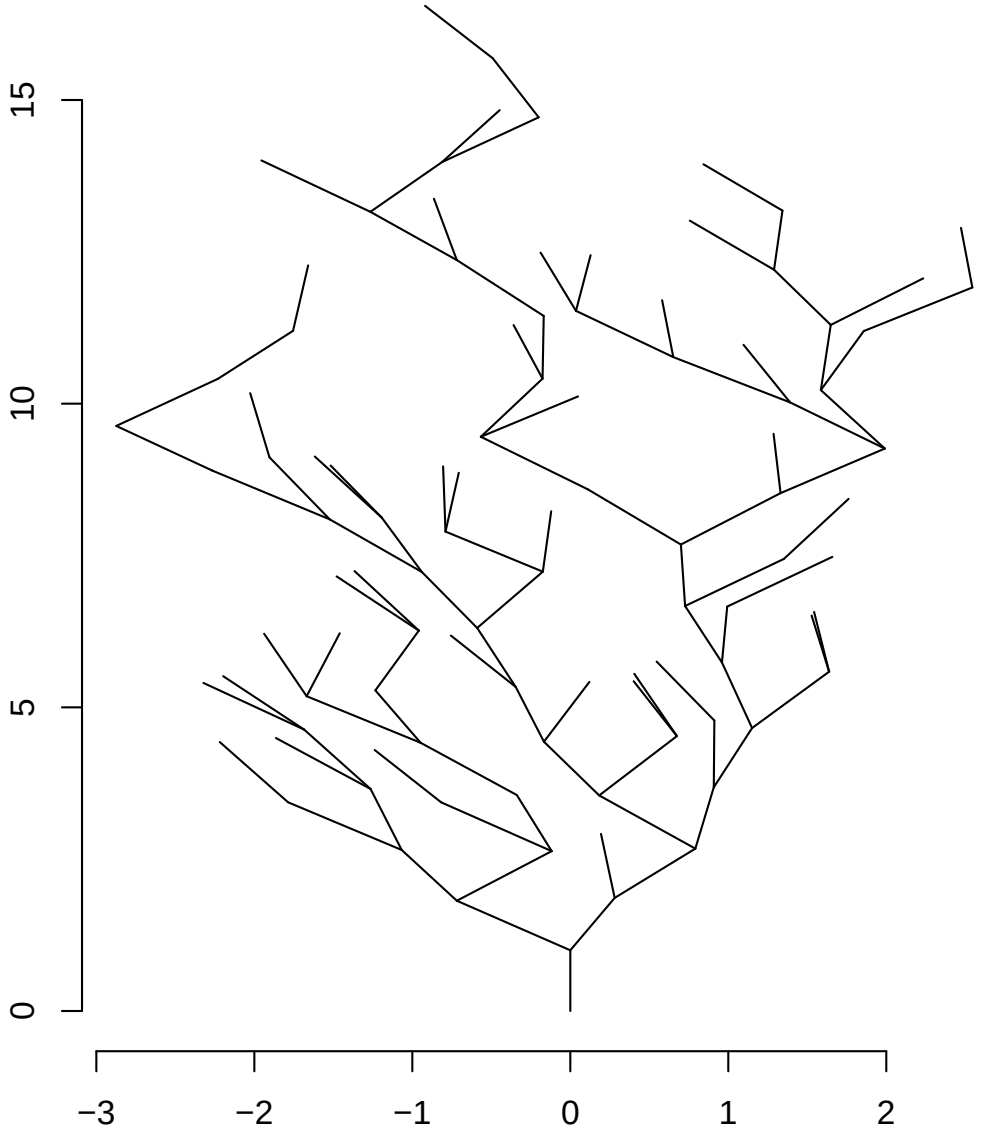
  which technique is most appropriate and when?

- Performance measured by
  - Predictions: RMSPE, Prediction interval coverage, bias...
  - Fixed effects: RMSE, Confidence region coverage, bias...
  - Time to fit model

- Need some mechanism for generating data...

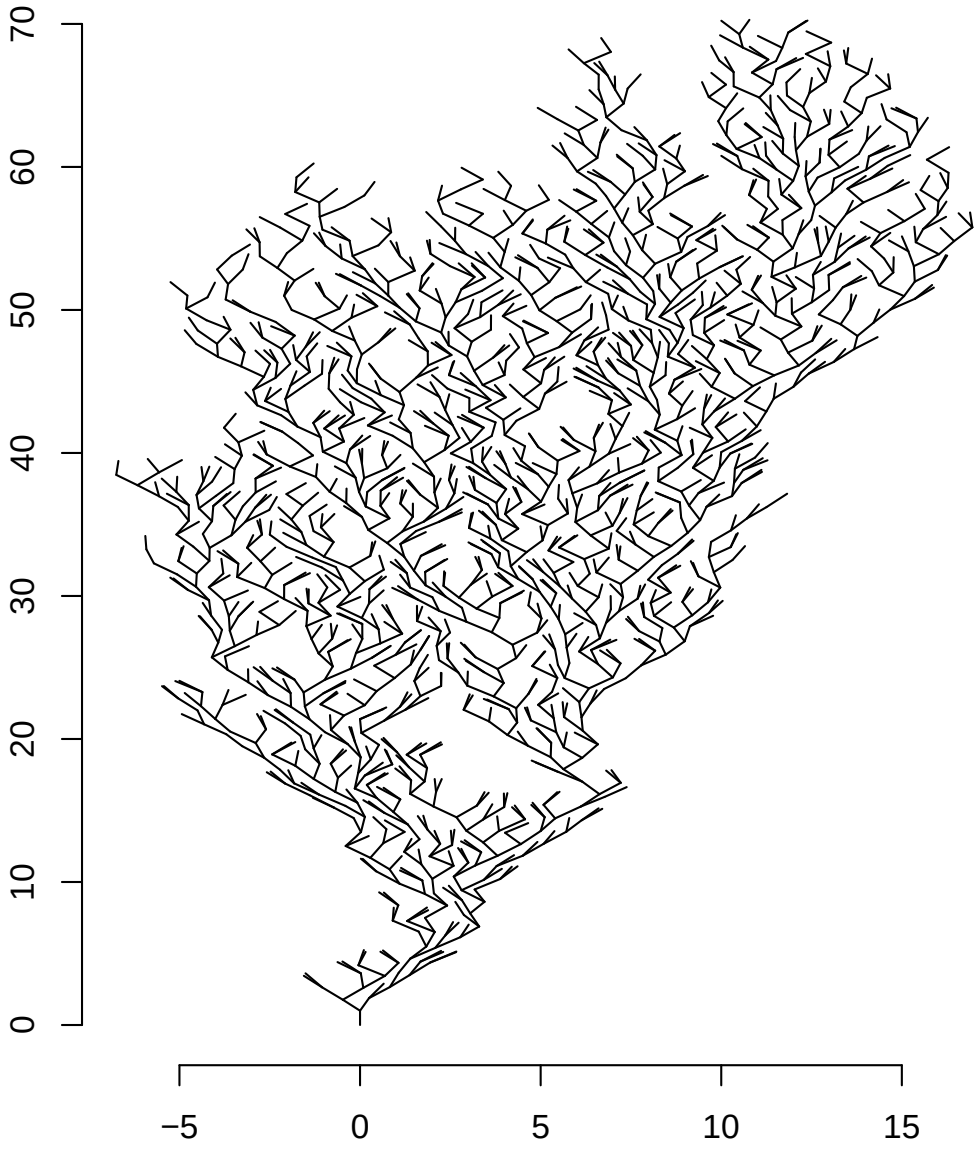
Comparison study - generating data

From the R package **SSN**, `createSSN()`, 100 stream segments



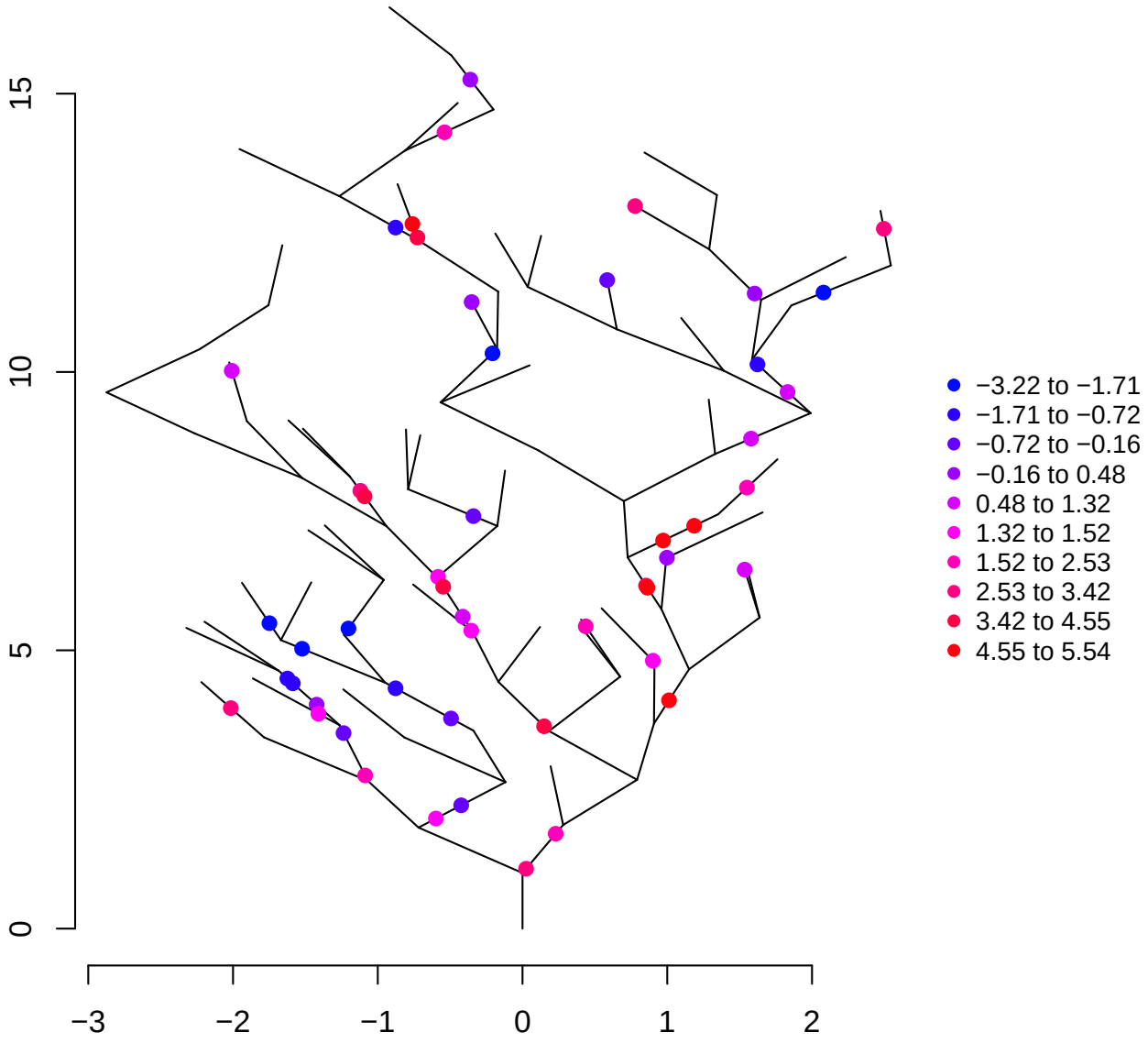
Comparison study - generating data

From the R package **SSN**, `createSSN()`, 2000 stream segments



Comparison study- generating data

From the R package **SSN**, `SimulateOnSSN()`, 100 stream segments



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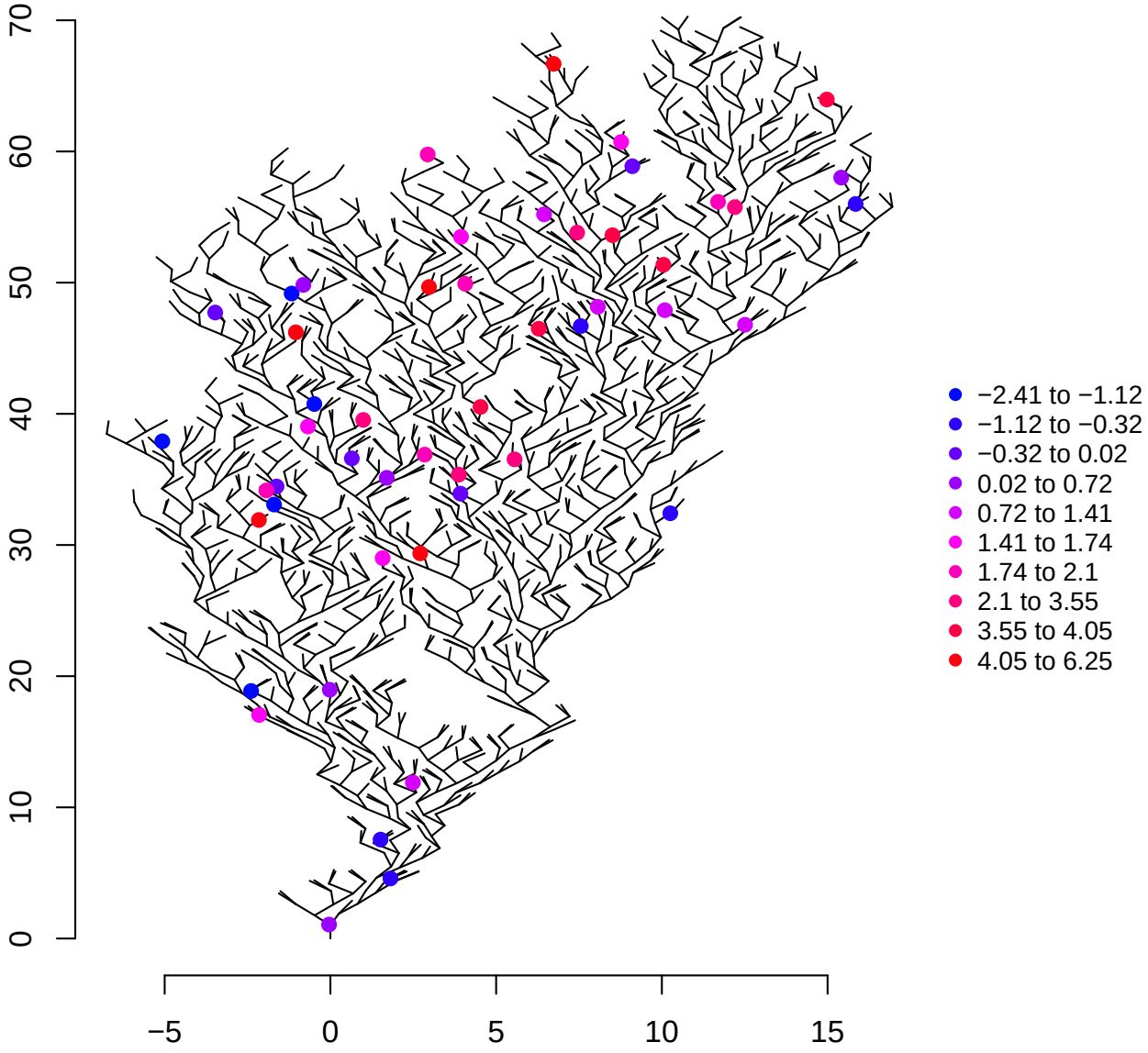


Comparison study - design

Sparsity and abundance of data

- Network size: small (100 stream segments) vs. large (2000 stream segments)
- 50 data points vs. 500

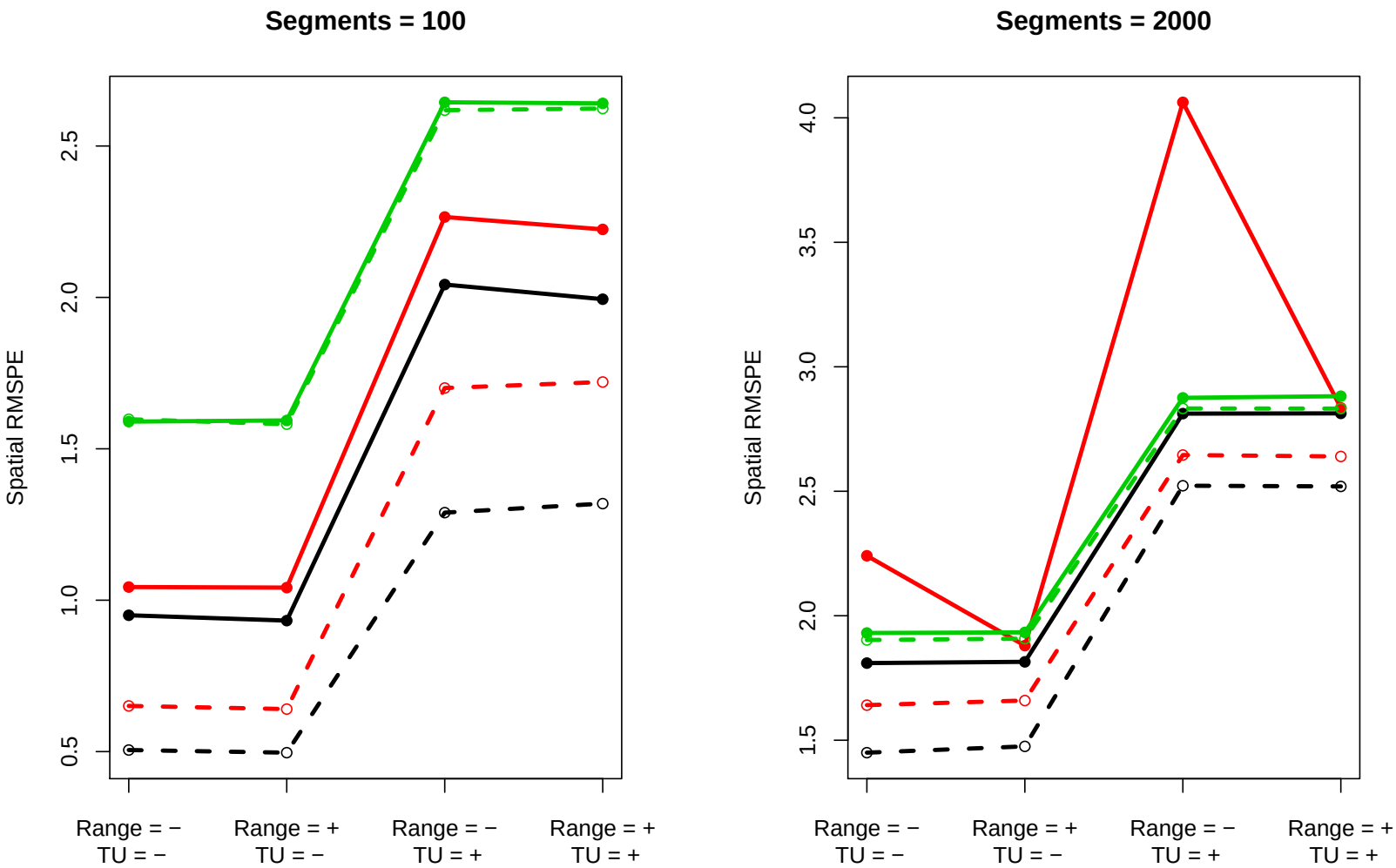
Data spatial structure

- Large or small range parameter
- Different levels of spatial strength (relative to fixed effects)

Fixed effects

- One strong and observed, one strong and unobserved and one weak and observed.

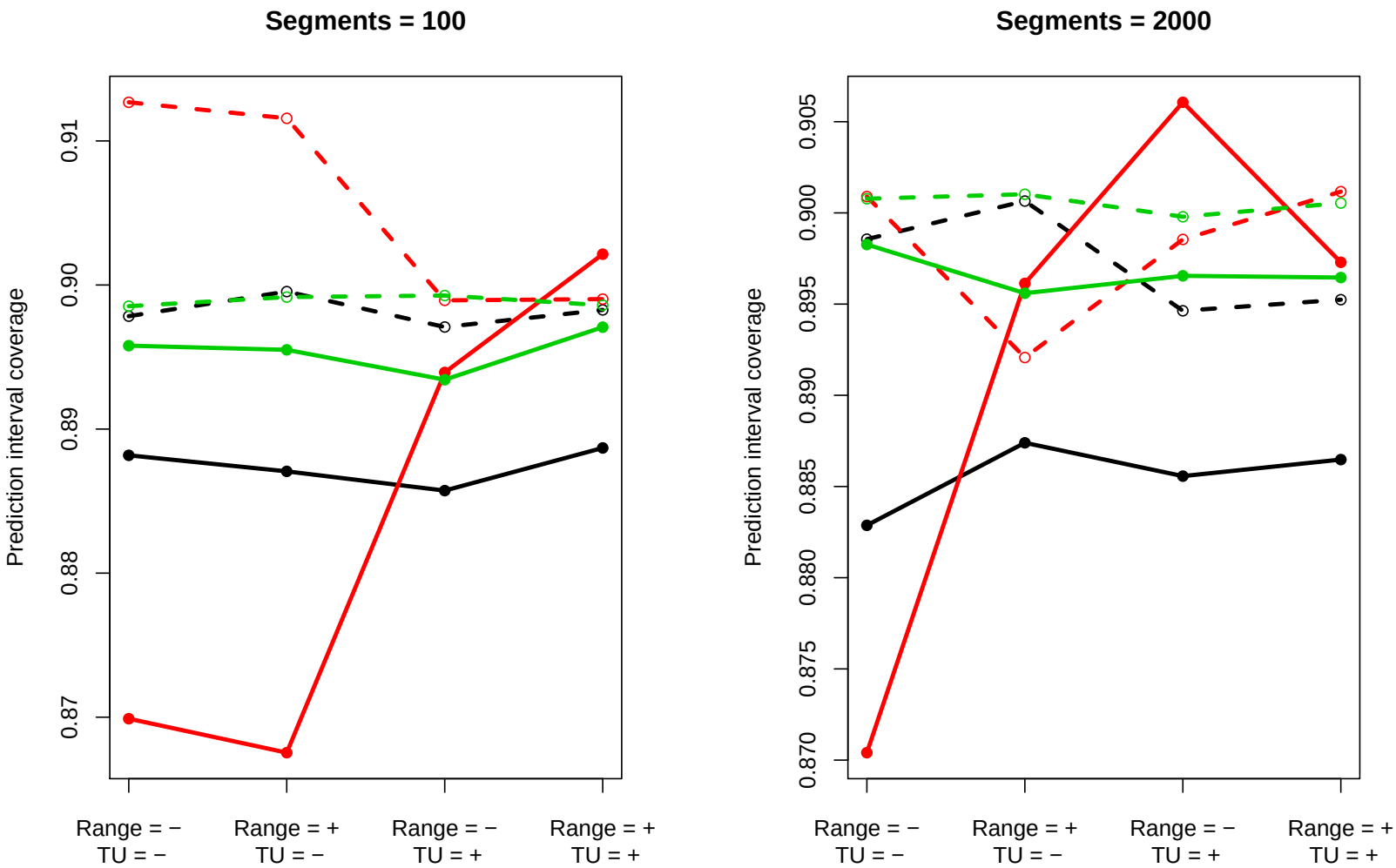
Prediction Error



Dashed lines: $n = 500$; solid lines: $n = 50$.

Black: SSN model; Red: spline model; Green: Linear model.

Prediction Interval coverage



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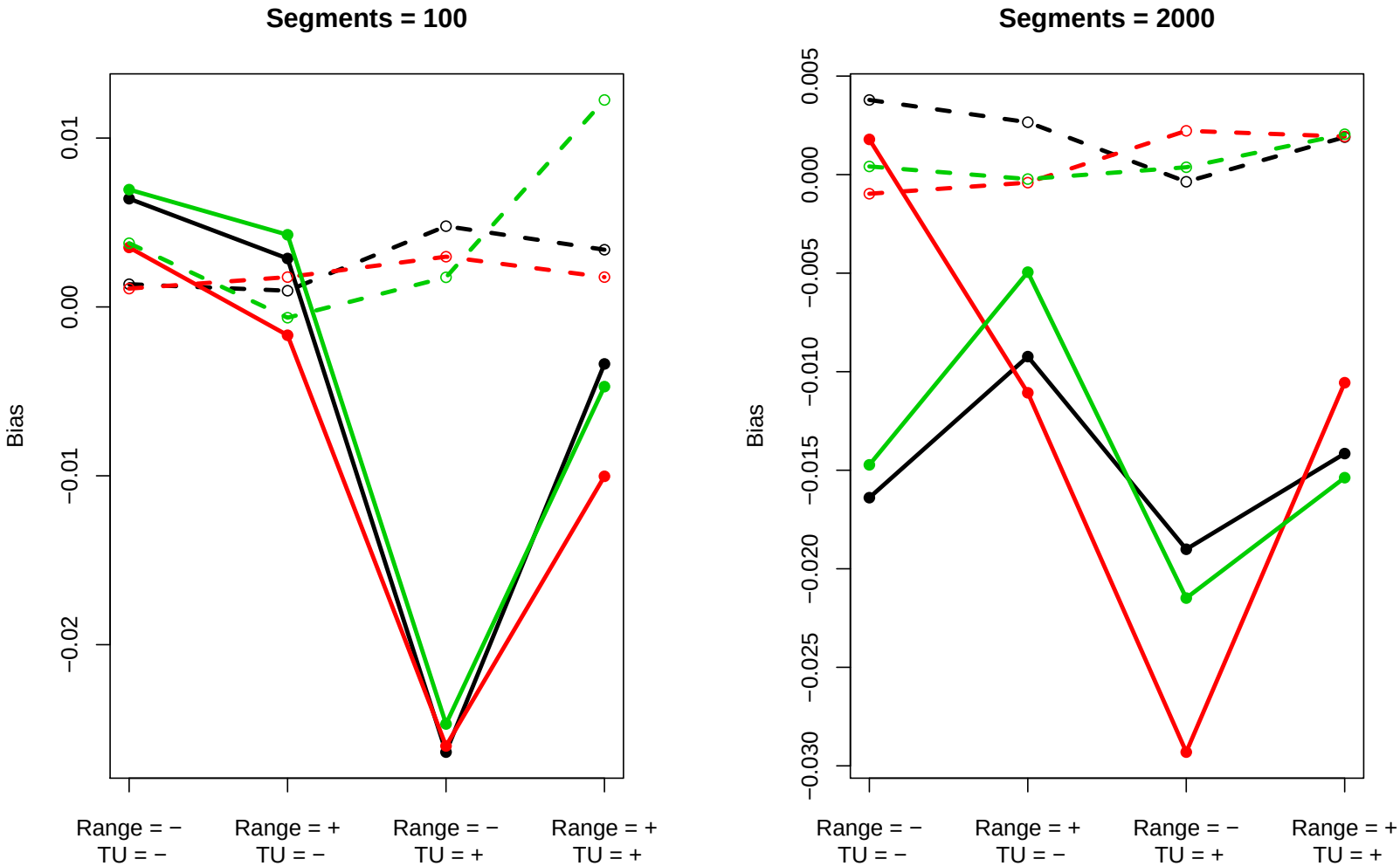
Prediction bias



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Summary

Prediction errors are smallest under SSN model

- Interval coverage is relatively poor for spline model, best for SSN; improves for larger n
- Strong dependance at small scales is the most challenging to capture well

Comparison is a work in progress

- Compare performance with other types of simulated spatial structures eg. Euclidean
- Sensitivity of SSN to misspecification
- Spatio-temporal data; smooth covariate effects eg. altitude

Extending the spline approach...

λ **selection**: Uncertainty not accounted for: Bayesian approach?

Further idea: Smooth basis functions *on* the network, rather than piecewise constant

- Natural extension, suggested first by Cressie and O'Donnell (2010).
- Sparsity and computational feasibilty should be preserved

Treat flow rates as parameters

- Bayesian analysis inevitable; uncertainty accounted for
- Priors based on proxies/ partial data

Thanks for your attention!