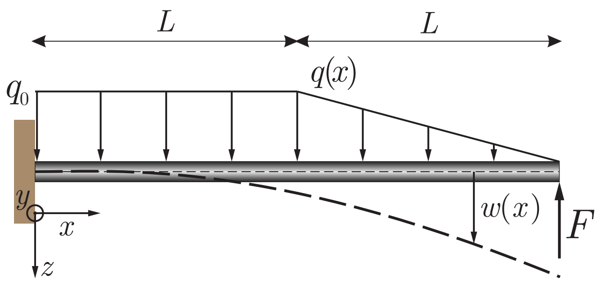
Homework 3

Problem 1:

A linear elastic cantilever beam of length $2L$ (constant Young modulus E and area moment I_y) is loaded by a distributed transverse load $q(x)$ of the sketched form with known q_0. We use Bernoulli's theory of elastic beams: the governing equation for the deflection $w(x)$ in the z-direction reads

$$E\, I_y\, w_{,xxxx}(x) = q(x).$$



The transverse force (in the positive z-direction) and the bending moment (about the y-axis) are given by

$$F_z(x) = -E\, I_y\, w_{,xxx}(x),$$
$$M_y(x) = -E\, I_y\, w_{,xx}(x),$$

respectively. In the given problem, the beam is loaded by the distributed force as well as a single force F at its free end (and the moment at the free end vanishes). Let us find an approximate solution for the deflection.

a) Formulate the Galerkin weak form of the problem.

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b) Let us find an approximate solution by assuming a polynomial of degree n with appropriate restrictions on the polynomial coefficients to satisfy essential boundary conditions. Use this approximation in a *Bubnov-Galerkin* formulation to arrive at a linear system $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{F}$ for the unknown coefficients a_i. What are matrix and vector components K_{ij} and F_i, respectively (generally written for i and j)?

c) Solve the system for the unknown coefficients for $n = 2, 3, 4, 5$ and plot the resulting deflection curve for each case. For this part, you can use the following numerical values for a beam made of steel:

$E = 210 \cdot 10^9 \text{N/m}^2, \; I_y = \frac{1}{\ } (10\text{mm})^4, \; q_0 = 100\text{N/m}, \; L = 100\text{mm}, \; F = 40\text{N}$

d) Let us show that the problem can also be solved by the *Rayleigh-Ritz* method. To this end, use variational calculus to show that the above boundary value problem derives from the potential

$$I[w] = \int_0^{2L} \left[\frac{1}{2} E\, I_y\, w_{,xx}^2(x) - q(x)\, w(x) \right] \,\mathrm{d}x + F\, w(2L).$$

e) Using variations, show that the solution indeed corresponds to a minimum of the total potential energy.

f) The error of an approximate solution can be characterized by the *energy norm* which is defined by $I[w_{\mathsf{approx}}(x) - w_{\mathsf{exact}}(x)]$. Since we do not have an exact solution, let us assume that the solution for $n = 5$ from c) is the exact solution (the error is indeed negligible). Compute the energy error for your solutions from c) for $n = 2$, $n = 3$ and $n = 4$ (using the above numerical values).