CEE 6513 PS 1

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a) $u_h = \alpha_0 + \alpha_1 x + \alpha_2 x^2$

$$\int w_h u \, d\Gamma_u = w_h(\alpha_0 + \alpha_1 x + \alpha_2 x^2)\Big|_{x=0} = 0$$

$w_h \alpha_0 = 0$.

Thus $\alpha_0 = 0$

$$R_\Omega = \frac{d^2u}{dx^2} + u + x$$
$$R_q = \frac{du}{dx}$$
$$R_u = u - 1$$

$$\int_\Omega w_R R_\Omega \, dx + \underbrace{\int_{\Gamma_q} w_q R_q \, d\Gamma_q}_{w_q R_q |_{x=1}} + \underbrace{\int_{\Gamma_u} w_u R_u \, d\Gamma_u}_{w_u R_u |_{x=0}} = 0$$

Satisfied automatically

Galerkin assumptions

- $u = \bar{u}$ at Γ_u is satisfied.
- $w_\Omega = w$, $w_q = w$

Thus ① $\Rightarrow$ $\displaystyle\int_\Omega w\left(\frac{d^2u}{dx^2} + u + x\right) d\Omega + w_q R_q\Big|_{x=1} = 0$

integrate by parts

$$= -\int_\Omega \frac{dw}{dx}\frac{du}{dx}\,dx + \cancel{w\frac{du}{dx}\Big|_0}^{0} + \int_\Omega w(u+x)\,dx + \cancel{w R_q}\Big|_{x=1} = 0 \quad ①$$

$$= -\int_\Omega \frac{dw}{dx}\frac{du}{dx}\,dx + \int_\Omega w(u+x)\,dx = 0$$

Bubnov-Galerkin $u \approx u_h(x) = \alpha_1 x + \alpha_2 x^2$

$w \approx w_h(x) = \beta_1 x + \beta_2 x^2$ (same basis).

$$u_h = \sum_{i=1}^{2} \alpha_i x^i$$

$$w_h = \sum_{j=1}^{2} \beta_j x^j$$

$$-\int_\Omega \left(\frac{dw}{dx}\frac{du}{dx}\right)dx = -\int_\Omega \left(\sum_{j=1}^{2} \beta_j \frac{\partial x^j}{\partial x}\right)\left(\sum_{i=1}^{2} \alpha_i \frac{\partial x^i}{\partial x}\right)dx = -\sum_{j=1}^{2}\beta_j\left(\sum_{i=1}^{2}\alpha_i \int \frac{\partial x^j}{\partial x}\frac{\partial x^i}{\partial x}\,dx\right)$$

$$\int_\Omega w(u+x)\,dx = \int_\Omega \left(\sum_{j=1}^{2}\beta_j x^j\right)\left(\sum_{i=1}^{2}\alpha_i x^i + x\right)dx$$

$$= \sum_{j=1}^{2}\beta_j\left(\sum_{i=1}^{2}\alpha_i \int_\Omega x^j x^i \, dx\right) + \sum_{j=1}^{2}\beta_j \int_\Omega x\,dx$$

Thus $\displaystyle\sum_{j=1}^{2}\beta_j\left(\sum_{i=1}^{2}\alpha_i K_{ji} - F_j\right) = 0$

where $\displaystyle K_{ji} = \int_\Omega \left(\frac{\partial x^j}{\partial x}\frac{\partial x^i}{\partial x} - x^j x^i \, dx\right)$

and $\displaystyle F_j = \int_\Omega x\,dx$

$$K_{ji} = \int_\Omega \left(\frac{\partial x^j}{\partial x}\frac{\partial x^i}{\partial x} - x^j x^i \, dx \right)$$

and $F_j = \int_\Omega x\,dx$

if we care, it can be further simplied since

$$\frac{\partial x^j}{\partial x} = j x^{j-1}, \quad \frac{\partial x^i}{\partial x} = i x^{i-1}, \quad x^j x^i = x^{i+j}, \quad \frac{\partial x^j}{\partial x}\frac{\partial x^i}{\partial x} = ij\, x^{i+j-2}$$

$$F_j = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Thus. $K_{11} \overset{j=1,\, i=1}{=} \int x^0 - x^2\, dx = \int_0^1 1 - x^2\, dx = \frac{2}{3}$

$$K_{12} \overset{j=1,\, i=2}{=} \int_0^1 2x - x^3\, dx = \frac{3}{4}$$

$$K_{21} \overset{j=2,\, i=1}{=} \int_0^1 2x - x^3\, dx = \frac{3}{4}$$

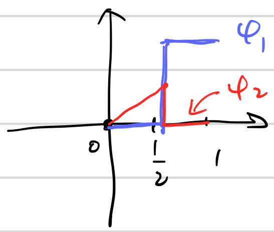
$$K_{22} \overset{j=2,\, i=2}{=} \int_0^1 4x^2 - x^4\, dx = \frac{17}{15}$$

$$K_{ji} = \begin{bmatrix} \frac{2}{3} & \frac{3}{4} \\ \frac{3}{4} & \frac{17}{15} \end{bmatrix}$$

$$F_j = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus $\vec{\alpha} = \vec{K}^{-1}\vec{F} = \begin{bmatrix} 0.99 \\ -0.22 \end{bmatrix}$

2) For φ_1 and φ_2



The general solution is the same as in 1)

To find if it's admissible.

we need it to satisfy $u=1$ at $\Gamma_u = \{0\}$

In the weighted residual form

$$\int w u \, d\Gamma_u = 0 \Rightarrow (\beta_1\varphi_1 + \beta_2\varphi_2)u\Big|_{x=0} = (\beta_1\varphi_1 + \beta_2\varphi_2)(\alpha_0 + \alpha_1 x + \alpha_2 x)\Big|_{x=0}$$

$$(\alpha_0\beta_1\varphi_1 + \alpha_0\beta_2\varphi_2)\Big|_{x=0} = 0$$

It's admissible

$$F_{ji} = \int_\Omega \left(\frac{\partial \varphi_j}{\partial x}\frac{\partial x^i}{\partial x} - \varphi_j x^i \right) dx$$

$$\frac{\partial \varphi_j}{\partial x} = \begin{cases} \frac{\partial \varphi_1}{\partial x} = 0 & (j=1) \\ \frac{\partial \varphi_2}{\partial x} = \begin{cases} 1 & \text{for } 0 \le x \le \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \le x \le 1 \end{cases} \end{cases}$$

$$K_{11} \overset{j=1,\, i=1}{=} \int_{\frac{1}{2}}^1 -1 \cdot x^1\, dx = -\frac{3}{8}$$

$$K_{12} \overset{j=1,\, i=2}{=} \int_{\frac{1}{2}}^1 -1 \cdot x^2\, dx = -\frac{7}{24}$$

$$K_{21} \overset{j=2,\, i=1}{=} \int_0^{\frac{1}{2}} 1 \cdot 1 - x \cdot x\, dx = \frac{11}{24}$$

$$K_{22} \overset{j=2,\, i=2}{=} \int_0^{\frac{1}{2}} 1 \cdot 2x - x \cdot x^2\, dx = \frac{15}{64}$$

Hence $\vec{K} = \begin{bmatrix} -\frac{3}{8} & -\frac{7}{24} \\ \frac{11}{24} & \frac{15}{64} \end{bmatrix}$

$$\vec{\alpha} = \vec{K}^{-1}\vec{F} = \begin{bmatrix} 5.7441 \\ -9.0995 \end{bmatrix}$$