CEE 6513 HW2

Donglai Yang

P1. $\frac{du}{dt} + u = t$ in $\Omega = (0,1)$

$u(0) = 1$.

a). $u_h(0) = 1 = \alpha_0 + \alpha_1 t + \alpha_2 t^2 \Big|_{t=0}$

$\Rightarrow \alpha_0 = 1$.

$W_{\Omega_h}(t) = \sum_{i=1}^{N} \beta_i \delta(x - x_i) = \sum_{i=1}^{N} \beta_i \delta(t - t_i)$

$W_{q_h}(x) = \rho^2 \sum_{i=n+1}^{N} \beta \delta(t - t_i)$.

Write in weighted residual form

$$\int W_{\Omega_h}\left[\frac{du}{dt} + u - t\right] dt + \cancel{W_{q_h}(u - 1)\Big|_{t=0}} = 0$$

$$\int W_{\Omega_h}\left(\frac{du}{dt} + u - t\right) dt = 0$$

$$\int \sum_{i=1}^{N} \beta_i \delta(t - t_i)(\alpha_1 + 2\alpha_2 t + 1 + (\alpha_1 - 1)t + \alpha_2 t^2)\, dt \quad ①$$

since $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_i)\, dt = f(t_i)$

$$① \Rightarrow \sum_{i=1}^{N} \beta_i \int \delta(t - t_i)[(\alpha_1 + 1) + (\alpha_1 + 2\alpha_2 - 1)t + \alpha_2 t^2]\, dt$$

Use the identity

$$\Rightarrow \sum_{i=1}^{N} \beta_i \underbrace{[(\alpha_1 + 1) + (\alpha_1 + 2\alpha_2 - 1)t_i + \alpha_2 t_i^2]}_{=0} = 0$$

choose 2 points such as $t_1 = 0.25$, $t_2 = 0.5$

t_1: $\alpha_1 + 1 + \frac{\alpha_1}{4} + \frac{\alpha_2}{2} - \frac{1}{4} + \alpha_2 \frac{1}{16} = 0$

$\frac{5}{4}\alpha_1 + \frac{9}{16}\alpha_2 + \frac{3}{4} = 0$

t_2: $\alpha_1 + 1 + \frac{\alpha_1}{2} + \alpha_2 - \frac{1}{2} + \frac{\alpha_2}{4} = 0$

$\frac{3}{2}\alpha_1 + \frac{5}{4}\alpha_2 + \frac{1}{2} = 0$

$$\begin{bmatrix} \frac{5}{4} & \frac{9}{16} \\ \frac{3}{2} & \frac{5}{4} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}$$

Solve the system

b). using Galerkin

similarly. $\frac{du}{dt} + u - t = 0 \quad \Omega = (0,1)$.

$u_h(t) = 1 + \alpha_1 t + \alpha_2 t^2$

$W_\Omega = \sum_{i=1}^{N} \alpha_i \phi_i + \phi_0$, $W_q = \sum_{j=1}^{\tilde{N}} \beta_j \phi_j$, $\frac{du}{dt} = \alpha_1 + 2\alpha_2 t$.

Weighted R. form $= \int W_\Omega \left(\frac{du}{dt} + u - t\right) dt + \cancel{W_q (u-1)}\Big|_{t=0} = 0$ ① $\quad \to 0$

We write $u_h = \sum_{j=0}^{2} \beta_j t^j$, $\frac{du_h}{dt} = \sum_{j=0}^{2} j \beta_j t^{j-1}$ where $\beta_0 = 1$

Galerkin Method assume $W_h = u_h$

or $W_h = \sum_{i=0}^{2} \alpha_i t^i$

$$① \Rightarrow \int \sum_{i=0}^{2} (\alpha_i t^i) \sum_{j=0}^{2} (j \beta_j t^{j-1} + \beta_j t^j - t)\, dt = 0$$

$$\Rightarrow \sum_{i=0}^{2} \alpha_i \int \sum_{j=0}^{2} (j \beta_j t^{j-1} t^i + \beta_j t^j t^i - t^{i+1})\, dt$$

$$\Rightarrow \sum_{i=0}^{2} \alpha_i \underbrace{\sum_{j=0}^{2} \left(\int j \beta_j t^{j-1+i} + \beta_j t^{i+j} - t^{i+1}\, dt \right)}_{=0} = 0$$

The matrix

$\beta_0 : \int_0^1 t^i - t^{i+1}\, dt$

$\beta_1 : \int \beta_1 t^i - \beta_1 t^{i+1} - t^{i+1}\, dt$

$\beta_2 : \int 2\beta_2 t^{i+1} + \beta_2 t^{2+i} - t^{i+1}\, dt$

plug in i for $i = 0, 1, 2$

$$\begin{bmatrix} (\beta_0 + \beta_1 + \beta_2)\big|_{\alpha_1} = 0 \\ (\beta_0 + \beta_1 + \beta_2)\big|_{\alpha_2} = 0 \\ (\beta_0 + \beta_1 + \beta_2)\big|_{\alpha_3} = 0 \end{bmatrix}$$

Solve the linear system

$\vec{A}\vec{\beta} = \vec{b}$

Solve for $\vec{\beta}$

P2 $\frac{du}{dt} - u = 0 \quad \Omega = (0,T)$

$u(0) = 1$

$u_h(t) = a_0 + \alpha_1 t$

$u_h(0) = \alpha_0 = 1.$

Hence $u_h(t) = 1 + \alpha_1 t$

$$I[u] = \int_\Omega \left(\frac{du}{dt} - u\right)^2 d\Omega$$

$$\int_\Omega (\alpha_1 - 1 + \alpha_1 t)^2 dt$$

$$= \int \alpha_1^2 t^2 + (2\alpha_1^2 - 2\alpha_1) t + \alpha_1^2 - 2\alpha_1 + 1 \quad dt$$

$$= \frac{\alpha_1^2}{3} t^3 + \frac{1}{2}(2\alpha_1^2 - 2\alpha_1) t^2 + (\alpha_1 - 1)^2 t \Big|_0^T$$

$$= \frac{\alpha_1^2}{3} T^3 + (\alpha_1^2 - \alpha_1) T^2 + (\alpha_1 - 1)^2 T$$

or $$\frac{T^3}{3}\alpha_1^2 + T^2 \alpha_1^2 - T^2 \alpha_1 + \underbrace{(\alpha_1^2 - 2\alpha_1 + 1) T}_{T\alpha_1^2 - 2T\alpha_1 + T}$$

$$= \left(\frac{T^3}{3} + T^2 + T\right)\alpha_1^2 - (T^2 + 2T)\alpha_1 + T$$

u_h is found at the minimum of $I(u)$.

$$\frac{\partial}{\partial \alpha_1} I(\alpha_1) = \left(\frac{2T^3}{3} + 2T^2 + 2T\right)\alpha_1 - T^2 - 2T = 0$$

$$\alpha_1 = \left(\frac{T^2 + 2T}{\frac{2T^3}{3} + 2T^2 + 2T}\right) = \frac{T+2}{\frac{2T^2}{3} + 2T + 2}$$

plug back in $u_h(t) = 1 + \alpha_1 t$

$$= 1 + \left(\frac{T+2}{\frac{2T^2}{3} + 2T + 2}\right) t$$

as $T \to 0$

$$\lim_{T \to 0} \frac{3T + 6}{2T^2 + 6T + 6} = \lim_{T \to 0} 1 - \frac{T}{2} + \frac{T^2}{6} + O(T^4). \approx 1$$

we know $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$

so $\lim_{T \to 0} e^T = 1$

therefore they agree.