HW3 CEE 6513

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1. Weighted Residual form

$$\underbrace{\int_\Omega W_\Omega \left(EI_y \frac{\partial^4 \omega}{\partial x^4} - q \right) dx}_{①} = 0$$

$$① \Rightarrow \int EI_y \left(W_\Omega \frac{\partial^4 \omega}{\partial x^4} \right) - W_\Omega q(x)\, dx$$

$$= EI_y \left[W_\Omega \frac{\partial^3 \omega}{\partial x^3} \Big| - \int \frac{\partial^3 \omega}{\partial x^3} \frac{\partial W_\Omega}{\partial x} dx \right] - \int W_\Omega q(x)\, dx$$

$$= \underbrace{EI_y W_\Omega \frac{\partial^3 \omega}{\partial x^3} \Big|_0^{2l}}_{= W_\Omega F - 0} - \int_\Omega W_\Omega q\, dx - \left[\underbrace{\int \frac{\partial W_\Omega}{\partial x} \frac{\partial^3 \omega}{\partial x^3} dx}_{\frac{\partial W_\Omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2} \Big|_\Gamma - \int_\Gamma \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 W_\Omega}{\partial x^2} dx} \right] EI_y$$

$$= W_\Omega F - \left[\underbrace{EI_y \frac{\partial W_\Omega}{\partial x} \frac{\partial^2 \omega}{\partial x^2}}_{-M \frac{\partial W_\Omega}{\partial x} \Big|_0^{2l} = 0} \right]_\Gamma + EI_y \int \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 W_\Omega}{\partial x^2} dx - \int W_\Omega q\, dx$$

$$= -\int_\Omega W_\Omega q\, dx + EI_y \int \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 W_\Omega}{\partial x^2} dx - W_\Omega F \Big|_{x=2l} \quad ②$$

let $\omega = \sum_{I=2}^{\tilde{N}} \alpha_I \phi_I + \phi_0 \quad W_\Omega = \sum_{J=2}^{\tilde{N}} \beta_J \phi_J$ (Galerkin approximat°)

Since $\omega(x=0) = 0 \quad \omega(x=0) = \phi_0 = 0$

Thus $\omega = \sum_{I=1}^{\tilde{N}} \alpha_I \phi_I$

(notice that $J = I \in [2, N]$ since first order polynomial will be zero if differentiated twice, leading to singular matrix)

combine ④ and ⑤ and rewrite

$$④ + ⑤ = \int \left(EI_y \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 W_\Omega}{\partial x^2} - W_\Omega q \right) dx - W_\Omega(x=2l) F$$

$$= \int \left[EI_y \left(\sum_{I=1}^{\tilde{N}} \alpha_I \phi_{I,xx} \right) \left(\sum_{J=1}^{\tilde{N}} \beta_J \phi_{J,xx} \right) - \sum_{J=1}^{\tilde{N}} \beta_J \phi_J q \right] dx - \sum_{J=1}^{\tilde{N}} \beta_J \phi_J (x=2l) F$$

$$= \sum_{J=2}^{N} \beta_J \int \left[EI_y \left(\sum_{I=1}^{N} \alpha_I \phi_{I_{xx}} \phi_{J_{xx}} \right) - \phi_J q - \phi_J(x=2l) F \right.$$

$$= \sum_{J=2}^{N} \beta_J \Big[\underbrace{EI_y \sum \alpha_I \int \phi_{I_{xx}} \phi_{J_{xx}} dx - \int \phi_J q \, dx - \phi_J(x=2l) F}_{=0} \Big] = 0$$

$$\underbrace{\sum \alpha_I \int \phi_{I_{xx}} \phi_{J_{xx}} dx}_{K\vec{a}} = \underbrace{\frac{1}{EI_y} \left[\int \phi_J q \, dx \right] + \phi_J(x=2l) F}_{\vec{F}}$$

$$K_{IJ} = \int \phi_{I_{xx}} \phi_{J_{xx}} dx.$$

$a_I = \alpha_I.$ (to solve for)

$$F_J = \frac{1}{EI_y} \int \phi_J q \, dx. + \phi_J(x=2l) F$$

if $\phi_I = x^I$ assuming global basis are polynomials.

$$\phi_{I_{xx}} = I(I-1) x^{I-2}$$

$$\phi_{J_{xx}} = J(J-1) x^{J-2}$$

and thus $K_{IJ} = \int IJ(I-1)(J-1) x^{I+J-4} dx$

$$= \frac{IJ(I-1)(J-1)}{I+J-3} x^{I+J-3} \Big|_0^{2l} \quad (\text{if } I+J \neq 3).$$

$$= \frac{IJ(I-1)(J-1)}{I+J-3} (2l)^{I+J-3}$$

if $I+J=3$. $\int IJ(I-1)(J-1) x^{I+J-4} dx = 0$

$$F_J = \frac{1}{EI_y} \left[\int x^J q(x) \, dx. + (2L)^J F \right] \quad ③$$

specifically for $q(x)$, $q(x) = \begin{cases} q_0 , & 0 \le x \le l. \\ \frac{-q_0}{L} x + 2q_0 & L \le x \le 2L \end{cases}$

So we can rewrite ③

$$③ \Rightarrow F_J = \frac{1}{EI_y} \left[\int_0^L x^J q_0 \, dx + \int_L^{2l} x^J \left(\frac{-q_0}{L} x + 2q_0 \right) dx + (2L)^J F \right]$$

$$= \frac{1}{EI_y} \left[\frac{q_0}{J+1} L^{J+1} - \frac{q_0}{L(J+2)} L^{J+2} (2^{J+2} - 1) + \frac{2q_0}{J+1} L^{J+1} (2^{J+1} - 1) + (2L)^J F \right]$$

d).

$$I[\omega] = \int_0^{2L} \left[\frac{1}{2} EI_y \left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 - q(x)\omega(x)\right] dx + F\omega(2L)$$

$$I[\omega+\varepsilon y] = \int_0^{2L} \left[\frac{1}{2} EI_y \left[\frac{\partial^2}{\partial x^2}(\omega+\varepsilon y)\right]^2 - q(x)(\omega+\varepsilon y)\right] dx + F[\omega(2L)+\varepsilon y]$$

$$= \int_0^{2L} \frac{1}{2} EI_y \left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2(\varepsilon y)}{\partial x^2}\right]^2 - q(x)\omega + q(x)\varepsilon y\,] dx + F[\omega(2L) + \varepsilon y]$$

$$= \int_0^{2L} \frac{1}{2} EI_y \left[\left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 + 2\frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 \varepsilon y}{\partial x^2} + \left(\frac{\partial^2 \varepsilon y}{\partial x^2}\right)^2\right) - q(x)\omega + q(x)\varepsilon y\,] dx + F[\omega(2L)+\varepsilon y]$$

$$\left.\frac{\partial I(\omega+\varepsilon y)}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^{2L} \left(\frac{1}{2} EI_y \left[2\frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 y}{\partial x^2} + 2\underbrace{\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 \varepsilon y}{\partial x^2}}_{=0 \text{ as } \varepsilon\to 0}\right] - q(x)y\right) dx + Fy$$

use integration by parts

$$\int \underbrace{\frac{\partial^2 \omega}{\partial x^2}}_{u}\underbrace{\frac{\partial^2 y}{\partial x^2} dx}_{dv} = \left.\frac{\partial^2 \omega}{\partial x^2}\frac{\partial y}{\partial x}\right|_a^b - \underbrace{\int \frac{\partial y}{\partial x}\frac{\partial^3 \omega}{\partial x^3} dx}$$

$$\Rightarrow \left. y\frac{\partial^3 \omega}{\partial x^3}\right|_a^b - \int y \frac{\partial^4 \omega}{\partial x^4} dx$$

$$= \left(\frac{\partial^2 \omega}{\partial x^2}\frac{\partial y}{\partial x} - y\frac{\partial^3 \omega}{\partial x^3}\right)\Bigg|_a^b + y\int \frac{\partial^4 \omega}{\partial x^4} dx$$

put back.

$$0 = \frac{\partial I(\omega+\varepsilon y)}{\partial \varepsilon} = \int_0^{2L} y\,\underbrace{\left[EI_y \frac{\partial^4 \omega}{\partial x^4} - q\right]}_{\text{PDE strong form}} dx + EI_y \underbrace{\left[\frac{\partial^2 \omega}{\partial x^2}\frac{\partial y}{\partial x} - y\frac{\partial^3 \omega}{\partial x^3}\right]_a^b}_{\text{B.C.}} + \underbrace{Fy}_{\text{Forcing}}$$

e) Rayleigh-Ritz. let $\omega(x) \simeq \omega_h(x) = \sum_{I=1}^{\tilde{N}} \alpha_I \varphi_I(x) + \varphi_0$.

Since $\omega(x=0) = 0$,
$\varphi_0 = 0$

for simplicity
let $\omega_h(x) = \alpha_1 x_1^2$

plug back to $I[\omega]$.

$$I(\omega) = \int_0^{2L} \left[\frac{1}{2} EI_y \left(\frac{\partial^2 \omega}{\partial x^2}\right)^2 - q(x)\omega(x)\right] dx + F\omega(2L)$$

$$= \int_0^{2L} \left[\frac{1}{2} EI_y (2\alpha_1)^2 - q(x)\alpha_1 x_1^2\right] dx + F\omega(x=2L)$$

$$= \int_0^{2L} 2EI_y \alpha_1^2 \, dx - \int_0^{2L} q(x)\alpha_1 x_1^2 \, dx + F\omega(x=2L).$$

$$= 4LEI_y\alpha_1^2 + F\alpha_1 4L^2 - \int_0^{2L} \alpha_1 q(x) x_1^2 \, dx$$

$$\frac{\partial I(\omega)}{\partial \alpha_1} = 8LEI_y\alpha_1 + 4FL^2 - \int_0^{2L} q(x) x_1^2 \, dx = 0.$$

$$\alpha_1 = \frac{\int_0^{2L} q(x)\, x \, dx - 4FL^2}{8LEI_y}$$

f). see the plot