Algebraic Curves I

· C will be a projective, sometimes Smooth, 1-dim'l (reduced) Algebraic Variety

· <u>Defn</u>: The topological/geometric <u>genus</u> of a Smooth curve C:

of "holes" – $p_g(C)$

alternatively:

$$p_g(C) = \dim_{\mathbb{C}} H^0(C, \Omega^1_{C/\mathbb{C}})$$

· Most of the time, we will work with $p_a(C)$, the arithmetic genus. This is $\boxed{p_a(C) = 1 - \chi(C, \mathcal{O}_C)}$

· <u>Aside</u>: Depending on your background, the above defns may be confusing.

For <u>Smooth</u> C, $p_a = p_g = g$

(1) <u>Arithmetic</u>	(2) <u>Geometric</u>	(3) <u>Topological</u>
$p_a = 1 - \chi(C, \mathcal{O}_C)$ Sheaf cohomology	$p_g = h^0(C, \Omega^1_{C/\mathbb{C}})$ "Dimension of Space of holomorphic differentials"	$g = \frac{\chi(C) + 2}{2}$ or "# of holes"

⟵ More general in our context
Possibly more confusing

• Aside (aside): To see this, start with topological genus g, compute by MV $H_{1,sing}(C, \mathbb{Z}) = \mathbb{Z}^{2g}$, by duality $H^1_{sing}(C, \mathbb{C}) = \mathbb{C}^{2g}$

Now apply Hodge decomp. (can also do it yourself using Hodge to de Rham spectral sequence for Riemann surfaces) to get:

$$\underbrace{H^1(C, \underline{\mathbb{C}})}_{\text{Sh. coh.}} = \underbrace{H^{0,1}(C) \oplus H^{1,0}(C)}_{\text{Dolbeault coh.}} \quad \text{w. } \dim_{\mathbb{C}} H^{0,1}(C) = \dim_{\mathbb{C}} H^{1,0}(C)$$

$\downarrow$ De Rham thm $\qquad\qquad$ $\downarrow$ Dolbeault thm

$$\dim_{\mathbb{C}} H^1_{sing}(C, \mathbb{C}) = \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) + \dim_{\mathbb{C}} H^0(C, \Omega^1_{C/\mathbb{C}})$$

$\hookrightarrow 2g \qquad \hookrightarrow 1-\chi(C, \mathcal{O}_C) = p_a \qquad \hookrightarrow p_g$

$\square$

• Why care so much: genus is the "only" discrete invariant for curves — $\mathcal{M}_g$ is irreducible + connected.

Linear Series, Divisors:

• Studying linear series on an abstract curve is analogous to studying the representation theory of an abstract group.

While interesting on its own (e.g. BN theory),
$\rightarrow$ this will allow us to construct $\mathcal{M}_g$

- A very important correspondence:

$$\left\{ \begin{array}{c} \text{Maps} \\ C \longrightarrow \mathbb{P}^r \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Pairs} \\ (\mathcal{L}, V) \text{ where } \mathcal{L} \\ \text{a line bundle on } C, \\ V \subset H^0(C, \mathcal{L}) \text{ dim } r+1 \text{ with} \\ \text{BPF basis} \end{array} \right\}$$

- Digging a bit deeper:

$$\left\{ \begin{array}{c} \text{LB } \mathcal{L} \\ \text{on } C \end{array} \right\} \Big/ \text{iso} \longleftrightarrow \left\{ \begin{array}{c} \text{Divisors} \\ D \text{ on } C \end{array} \right\} \Big/ \begin{array}{c} \text{Lin} \\ \text{equiv.} \end{array}$$

- Defn: A divisor D on C is a formal $\mathbb{Z}$ linear combination of (closed) pts on C

$$\sum_{P \in C} a_P [P]$$

Call this free Abelian Grp $\mathrm{Weil}(C)$

- Have map:

$$\mathrm{Div}: K(C)^* \longrightarrow \mathrm{Weil}(C)$$

$$f \longmapsto \sum_{P \in C} \mathrm{val}_P(f)[P]$$

- And:

$$\mathrm{Deg}: \mathrm{Weil}(C) \longrightarrow \mathbb{Z}$$

$$\sum_{P \in C} a_P [P] \mapsto \sum_{P \in B} a_P$$

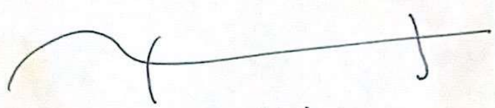
• Fact: $\mathrm{Deg} \circ \mathrm{Div} = 0$

"# Poles w. mult." = "# Zeros w. mult." for Rat. function f

• Defn: $Cl(C) := \mathrm{Weil}(C) / \mathrm{Div}(K(C)^*)$

• Defn: A line bundle is a rank 1 locally free sheaf on C

$$\mathcal{L}|_{u_i} \simeq \mathcal{O}_C|_{u_i}$$



C

u_i

• ~~New~~ Defn: $\mathrm{Pic}\, C := \left\{ \begin{array}{l} \text{LB } \mathcal{L} \text{ on} \\ C \text{ up to iso.} \end{array} \right\}$

• Correspondence #2 becomes:

$$\mathrm{Pic}(C) \simeq Cl(C)$$

($\rightarrow$) Given a LB $\mathcal{L}$ and a rational section s, map $\mathcal{L}$ to $\mathrm{div}(s)$

($\leftarrow$) Given a divisor $D = \sum_{P \in C} a_P [P]$, map D to the LB $\mathcal{O}(D)$, where $\mathcal{O}(D)(U) = \{ f \in K(X)^\times : \mathrm{div}|_U f + D|_U \geq 0 \} \cup \{0\}$

Some clarifications:

- What is $\mathrm{div}(s)$ for a rational section s?

Take U_i to be a trivialization for $\mathcal{L}$, under which s_i becomes a rational function in $\mathcal{O}_C(U)$. Take $\mathrm{div}(s_i)$.

$\to$ Doing this over a trivializing cover $\{U_i\}$ for $\mathcal{L}$ gives $\mathrm{div}(s)$

Example: Take $\mathcal{O}_{\mathbb{P}}(1)$, ~~dual~~ with coordinates x, y and ~~open cover~~ a cover $\{\underbrace{\mathrm{Spec}\,\mathbb{C}[x/y]}_{U_1}, \underbrace{\mathrm{Spec}\,\mathbb{C}[y/x]}_{U_2}\}$ for $\mathbb{P}^1$. Let $s = x$. Then $\mathrm{div}|_{U_1} s = \mathrm{div}|_{U_1} x/y = [0]$ while $\mathrm{div}|_{U_2} s = \mathrm{div}_{U_2} 1 = 0 \Rightarrow$

$$\mathrm{div}(s) = [0]$$

- Proof that $\mathrm{Pic}(C) \simeq \mathrm{Cl}(C)$ for (smooth) varieties is not deep, but confusing. Have to show:

(1) (easier) $\mathcal{L} \longmapsto \mathrm{div}(s)$ is well-defined, i.e. independent of a choice of section s.

(2) (basically (1)) That $\mathrm{div}(s) = 0 \Rightarrow s$ is a rational function / regular

(3) That $\mathcal{O}(\mathrm{div}(s)) = \mathcal{L}$ (tricky!)

Continued example:

~~Take $H^0(\mathbb{P}^1, \mathcal{O}((0)))$; ...~~

Let $D = (0)$ on $\mathbb{P}^1$. Since we are "allowed" a pole at 0,

$H^0(C, \mathcal{O}((0))) = \langle 1, y/x \rangle_{\mathbb{C}}$

~~$H^0(C, \mathcal{O}((0)))$~~

$$\left.\begin{aligned} H^0(D(x), \mathcal{O}((0))) &= \langle 1 \rangle_{\mathbb{C}[y/x]} \\ H^0(D(y), \mathcal{O}((0))) &= \langle 1/(x/y) \rangle_{\mathbb{C}[\frac{x}{y}]} \end{aligned}\right\}$$

$D(x), D(y)$ Give trivializations, So $\mathcal{O}((0))$ is a LB. Moreover we see that our transition map ϕ must satisfy

$$\begin{tikzcd} \langle 1/(x/y) \rangle \arrow[r, "\phi"] \arrow[d, "\text{mult by } x/y"'] & \langle 1 \rangle \arrow[d, "\text{id}"] \\ \mathcal{O}_C(D(y)) \arrow[r] & \mathcal{O}_C(D(x)) \end{tikzcd}$$

So ϕ is multiplication by x/y. This is the same transition map for $\mathcal{O}(1) \Rightarrow \mathcal{O}((0)) \simeq \mathcal{O}(1)$.

- Returning to linear series; a $V \subset H^0(C, \mathcal{L})$ with generators $s_1, \dots, s_{r+1}$, not simultaneously vanishing at $p \in C$, gives a morphism

$$\begin{aligned} \pi: C &\longrightarrow \mathbb{P}^r \\ p &\longmapsto [s_1(p), \dots, s_{r+1}(p)] \end{aligned}$$

· Or more formally over ~~an~~ a trivialization $\{\mathcal{U}_i\}$

$$\mathcal{U}_i \longrightarrow \mathbb{P}^r$$

arising from:

$$\mathcal{O}_C(\mathcal{U}_i, s_i) \longleftarrow \mathcal{O}_{\mathbb{P}^r}(D(x_i))$$

$$\frac{s_j}{s_i} \longleftarrow\!\shortmid \frac{x_j}{x_i}$$

And one checks that these morphisms glue

Defn: The degree of a morphism $\pi: C \longrightarrow \mathbb{P}^n$ (or $X \longrightarrow \mathbb{P}^n$) is the number of intersections (counted w. multiplicity) of $\pi(C)$ with a general hyperplane (or a general linear subspace of complem. dim., when X arbitrary)

Fact: Since π_V for $V \in H^0(C, \mathcal{L})$ ~~lifts~~ identifies $\pi_V^* \mathcal{O}(1)$ with the subsheaf generated by V, $\deg(\pi_V) = \operatorname{Deg}(\mathcal{L}) = \operatorname{Deg}(\operatorname{div}(s))$, for $s \in V$.

· How can we use this to compute "representations" of a curve C ($\pi: C \longrightarrow \mathbb{P}^r$)

→ Need to be able to say when $\mathcal{L} / C$ has global sections,

based on $\operatorname{Deg}(\mathcal{L})$, g

Riemann - Roch:

$$H^0(C, \mathcal{L}) - H^0(C, \mathcal{L} \otimes K^\vee) = \deg(\mathcal{L}) - g + 1$$

Here; $K = \Omega^1_{C/\mathbb{C}}$ is the canonical bundle for C.

• Riemann-Roch uses Serre duality, which is <u>hard</u> to prove. But we can see why it's true if we black-box SD.

<u>Fact</u>:
• (Easy RR): $\chi(C, \mathcal{L}) = \deg \mathcal{L} + \chi(C, \mathcal{O}_C)$

<u>Pf</u>: Induct on $\sum |a_p|$, where $\operatorname{div}(s) = \sum a_p[P]$ for some ratl section s for $\mathcal{L}$. Case $\sum |a_p| = 0$ is $\mathcal{L} = \mathcal{O}_C$ and vacuous. For $\sum |a_p| = n$, tensor the closed SS exact sequence
$0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to i_{p*}\mathcal{O}_C|_P \to 0$ with $\mathcal{O}(D) \simeq \mathcal{L}$, where S, P are chosen so that $a_p > 0$ (take arbitrary s a ratl section and f a ratl function vanishing at $p \in C$, $S \cdot f$ will be defined over open, dense U as ratl. section of $\mathcal{O}(D)$) We have:

$$0 \to \mathcal{O}(D-P) \longrightarrow \mathcal{O}_C(D) \longrightarrow i_{p*}\mathcal{O}_C(D)|_P \longrightarrow 0$$

Since $\chi(C, i_{p*}\mathcal{O}_C(D)|_P) = \chi(P, \mathcal{O}_C(D)|_P)$ and P is clearly affine, $\chi(P, \mathcal{O}_C(D)|_P) = H^0(P, \mathcal{O}_C(D)|_P) = 1$. So by additivity of Euler char.:

$$\begin{aligned}\chi(C, \mathcal{O}_C(D)) &= \chi(C, \mathcal{O}_C(D-P)) + 1 \\ &= (\deg D - 1 + \chi(C, \mathcal{O}_C)) + 1 \quad \text{(inductive hypothesis)} \\ &= \deg(D) + \chi(C, \mathcal{O}_C)\end{aligned}$$

• Taking "easy" RR and plugging in defn. for $g = 1 - \chi(C, \mathcal{O}_C)$

$$\chi(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

or:

~~$\chi(C,\mathcal{L})$~~

$$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$$

Serre Duality: Suppose X is a proper, smooth variety. Then for any locally free $\mathcal{F}$ on X, there is an isomorphism

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, K_X \otimes \mathcal{F}^\vee)^\vee$$

where $\dim X = n$, K_X is the canonical bundle

* Serre duality holds under weaker assumptions; X is Cohen-Macaulay. In this case ω_X denotes the dualizing sheaf (K_X not well-defined!) This is important for, e.g., nodal curves

• We now return the original RR:

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\vee \otimes K_C) = \deg \mathcal{L} + 1 - g$$

Morphisms & Embeddings of Low Genus Curves:

• $g = 0$:

- What is a smooth genus 0 curve over $\mathbb{C}$?

-> Take $p \in C$. By RR $D = [p]$ has a 2 dim'l space of GS. Choose $g, f \in H^0(C, \mathcal{O}(D))$ and give a map to $\mathbb{P}^1$. This is degree 1, so must be a closed embedding. But then $C \simeq \mathbb{P}^1$

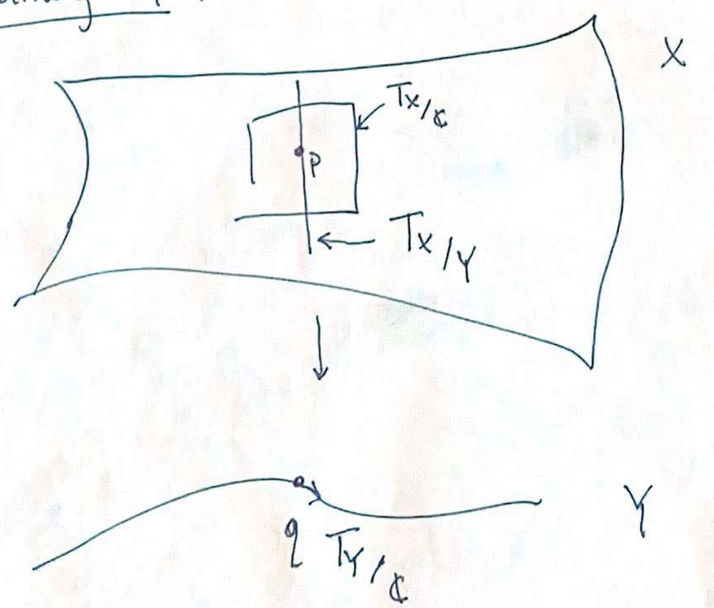
• Given smooth curves C, C', there is a natural exact sequence of Kahler differentials for $\pi : C \to C'$

$$\pi^* \Omega^1_{C'/\mathbb{C}} \xrightarrow{d\pi} \Omega^1_{C/\mathbb{C}} \longrightarrow \Omega^1_{C/C'} \longrightarrow 0$$

You should view this, when π is smooth, as the dual of

$$0 \to T_{C/C'} \longrightarrow T_{C/\mathbb{C}} \longrightarrow \pi^* T_{C'/\mathbb{C}} \longrightarrow 0$$

For arbitrary X, Y, the picture is:



• Fact: For smooth C, C', $\Omega^1_{C/C'}$ is supported at R, where R is the ramification locus. In particular:

$$\Omega^1_{C/C'} \otimes k(P) = \Omega^1_{A/A'} \otimes k(P) = \Omega^1_{A \otimes k(P)/k(P)} = \bigoplus_{P_i : \pi(P_i) = P} \Omega^1_{A \otimes k(P_i)/k(P)}$$

$\dim_{k(P)} \Omega^1_{A \otimes k(P_i)/k(P)} = e_{P_i} - 1$, for e_{P_i} the ram. index

$$\boxed{P \in C' \quad \begin{matrix} \operatorname{Spec}(A) \subset C \\ \downarrow \\ \operatorname{Spec}(A') \subset C' \end{matrix}}$$

• Now, we have ~~by adjunction~~ twisting by $\Omega^1_{C/\mathbb{C}}{}^\vee$:

$$\pi^*\Omega^1_{C'/\mathbb{C}} \otimes \Omega^1_{C/\mathbb{C}}{}^\vee \to \mathcal{O}_C \to \Omega^1_{C/\mathbb{C}} \to 0$$

But this is the CSS exact sequence for R, the ramification locus; we have a surjection $\pi^*\Omega^1_{C'/\mathbb{C}} \otimes \Omega^1_{C/\mathbb{C}}{}^\vee \twoheadrightarrow \mathcal{O}(-R)$ of LB, i.e. an isomorphism. This gives us, after dualizing and taking degrees

$$\deg R = \deg(\pi^*\Omega^1_{C'/\mathbb{C}}{}^\vee) + \deg(\Omega^1_{C/\mathbb{C}})$$

$$\deg R = \deg(\pi)(2g'-2) + 2g-2$$

RR with $\mathcal{L} = \mathcal{O}_C$ gives $\deg K_C = 2g-2$

$\boxed{\text{Riemann - Hurwitz}}$

• ~~See back to $g=0$. Take any rat'l function f on C and extend f. Take $p \in C$~~

• ~~Back to $g=0$~~

• Similarly for the conormal sequence; with $\pi: X \hookrightarrow Y$ smooth;

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to \pi^*\Omega^1_{Y/\mathbb{C}} \to \pi_*\Omega^1_{X/\mathbb{C}} \to 0$$

• When $X \hookrightarrow Y$ smooth, we may do something similar

Conormal seq:

$$0 \to \mathcal{I}_X/\mathcal{I}_X^2 \to i^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to 0$$

Giving:

$$K_X \simeq i^*K_Y \otimes \Lambda^n \mathcal{I}_X/\mathcal{I}_X^2{}^\vee$$

• Back to $g=0$: (i.e. $C = \mathbb{P}^1$)

• For morphisms $\mathbb{P}^1 \to \mathbb{P}^1$: this is simply a branched cover of $\mathbb{P}^1$; the only data are the degree and ramif. locus. For a degree d $\mathcal{L}$ (which will have ≥ 2 dim'l space of GS for $d>0$), we have by Riemann-Hurwitz:

$$\deg R = d(2) - 2$$

• For morphisms $\mathbb{P}^1 \to \mathbb{P}^2$ deg d

Lemma:

* Suppose $C \hookrightarrow \mathbb{P}^2$ is a plane curve. Then from adjunction
deg k, genus g

$$K_C \simeq i^*K_Y \otimes (\mathcal{I}_C/\mathcal{I}_C^2)^\vee$$

$$\simeq i^*(K_Y \otimes \mathcal{I}_C^\vee)$$

$$\simeq i^*(\mathcal{O}(-2-1) \otimes \mathcal{O}(k))$$

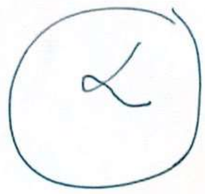
$\Rightarrow$ ~~$\deg K_C = 2g-2 = -3k \Rightarrow g = \frac{2-3k}{2}$~~

$$(-3k+k)k = 2g-2$$

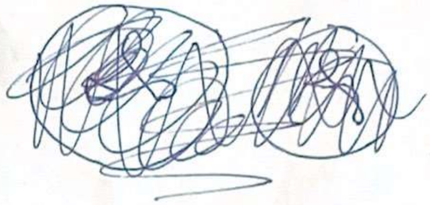
$$\Rightarrow \frac{(k-2)(k-1)}{2} = 2g-2$$

So if

- If $d=2$, then $\deg \mathcal{L} = 3$ and $|\mathcal{L}|$ gives an embedding
- If $d \geq 3$, then by genus deg. formula image cannot be genus 0 curve and it is not an embedding

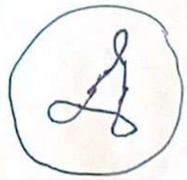


deg 3



Why? Take Nodal exact seq. $0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{p \in \text{nodes}(C)} \mathcal{O}_p \to 0$ and LES in ~~h~~ cohomology. Conclude

$\#$ nodes $= p_a$



deg 4