Γ-convergence approximation of the anisotropic perimeter

Previously

f

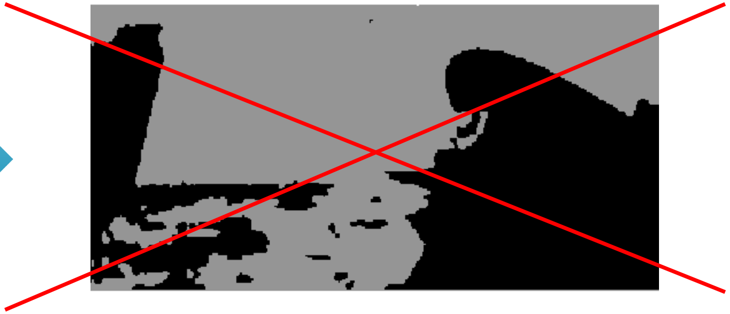
Ω_2

Ω_1

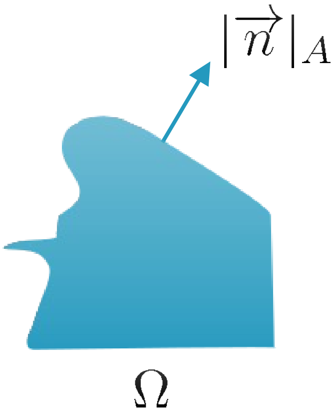
$c_1\chi_{\Omega_1} + c_2\chi_{\Omega_2}$

$$\min_{\substack{(\Omega_1,\Omega_2)\\ \text{is a partition}}} \left\| f - \sum_{i=1}^{2} c_i \chi_{\Omega_i} \right\|_{L^2}^2 + \frac{\alpha}{2}\sum_{i=1}^{2} \mathrm{Per}(\Omega_i)$$

My contribution



$$\mathrm{Per}_A(\Omega) := \int_{\partial\Omega} |\overrightarrow{n}|_A \, ds$$



Example

Consider setting $A = \begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathrm{Per} = 2$ $\quad\Rightarrow\quad$ $\mathrm{Per}_A = 20$

$\mathrm{Per} = 2$ $\quad\Rightarrow\quad$ $\mathrm{Per}_A = 2$

Results

Consider solving

$$\min_{\substack{(\Omega_1,\Omega_2,\Omega_3)\\ \text{is a partition}}} \left\| f - \sum_{i=1}^{3} c_i \chi_{\Omega_i} \right\|_{L^2}^2 + \frac{\alpha}{2} \sum_{i=1}^{3} \mathrm{Per}_A(\Omega_i)$$

Technical details

$$\min_{\substack{(\Omega_1,\dots,\Omega_n) \\ \text{is a partition}}} \left\| f - \sum_{i=1}^{n} c_i \chi_{\Omega_i} \right\|_{L^2}^2 + \frac{\alpha}{2} \sum_{i=1}^{n} \mathrm{Per}_A(\Omega_i)$$

$$= \min_{\substack{u_1+\cdots+u_n=1 \\ u_1,\dots,u_n\in\mathcal{E}}} \left\| f - \sum_{i=1}^{n} c_i u_i \right\|_{L^2}^2 + \frac{\alpha}{2} \sum_{i=1}^{n} \mathrm{TV}_A(u_i)$$

$$= \min_{\substack{u_1+\cdots+u_n=1 \\ u_1,\dots,u_n\in\mathcal{E}}} \sum_{i=1}^{n} \int |f - c_i|^2 \, u_i + \frac{\alpha}{2} \mathrm{TV}_A(u_i)$$

Theorem

Theorem: If A is constant, $\tilde{F}_\epsilon(u) := \min_{v\in H^1} \int \varepsilon|\nabla v|_A^2 + \frac{1}{\varepsilon}(v^2 + u(1-2v))$

$$\tilde{F}_\epsilon(u) \xrightarrow{\Gamma} \frac{1}{2}\mathrm{TV}_A(u) \qquad \text{as } \varepsilon \to 0$$

Corollary: For minimizing $\mathcal{I}(u_1,\ldots,u_n) := \sum_{i=1}^n \int |f-c_i|^2 u_i + \frac{\alpha}{2}\mathrm{TV}_A(u_i)$
it suffices to minimize

$$\mathcal{I}_\varepsilon(u_1,\ldots,u_n) := \min_{v_1,\ldots,v_n} \sum_{i=1}^n \int |f-c_i|^2 u_i + \alpha \int \varepsilon|\nabla v_i|_A^2 + \frac{1}{\varepsilon}(v_i^2 + u_i(1-2v_i))$$

Each cluster point for $\{\mathrm{argmin}\, \mathcal{I}_\varepsilon\}_\varepsilon$ minimizes $\mathcal{I}$.

Optimization

Let ε>0 be fixed. We should solve the minimization problem,

$$\min_{u_1,\ldots,u_n} \min_{v_1,\ldots,v_n} \sum_{i=1}^{n} \int |f - c_i|^2 u_i + \alpha \int \varepsilon |\nabla v_i|_A^2 \ + \ \frac{1}{\varepsilon}(v_i^2 + u_i(1 - 2v_i))$$

By alternating method, one should first minimize w.r.t v_i and then u_i and so on.

- Minimizing w.r.t. v_i (Euler-Lagrange eq):

$$\begin{cases} -\varepsilon^2 div(A\nabla v_i^k) + v_i^k & = u_i^{k-1} \\ A\nabla v_i^k \cdot n & = 0 \end{cases}$$

- Minimizing w.r.t. u_i : A linear programming problem

Thank you