N-Step Factorials and the Gamma Function

Adam Lawson

Brigham Young University - Idaho

July 18, 2022

Presentation Outline

Background

NSF

Future Work

Background: What is an NSF?

First we must discuss factorials and the Gamma function.

The familiar function for factorials is:

$$x! = x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot 2 \cdot 1$$

For example:

$$\begin{aligned} 5! &= 5 \cdot (5-1) \cdot (5-2) \cdot (5-3) \cdot (5-4) \\ &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

An N-Step Factorial's familiar function is:

$NSF(x, n) = x \cdot (x - 1n) \cdot (x - 2n) \cdot \ldots \cdot (x - m \cdot n)$ such that $(x - mn) > 0$ and $x - (m+1)n \leq 0$ for m in the counting numbers. $(m \in \mathbb{N})$ and $0 \leq x, 0 < n < x$.

*We can also say $m = \lfloor \frac{x}{n} \rfloor$

For example: If $x = 1.3$ and $n = 0.5$.

$$\begin{aligned} NSF(x, n) &= 1.3 \cdot (1.3 - 0.5) \cdot (1.3 - (2(0.5))) \\ &= 1.3 \cdot 0.8 \cdot 0.3 \end{aligned}$$

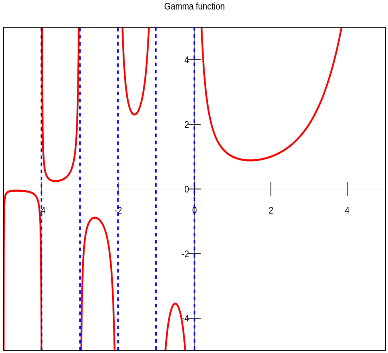
Background: The Gamma Function

1. The Gamma function is defined by:

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x}\, dx \quad t \in \mathbb{R}$$

2. When $t \in \mathbb{N}$, then:

$$\Gamma(t) = (t-1)!$$



Background: Related Research

Online Encyclopedia of Integer Sequences (OEIS):

If you were to expand the polynomial of

$$NSF(x, n) = x \cdot (x - 1n) \cdot (x - 2n) \cdot ... \cdot (x - m \cdot n)$$

we get the sequence of numbers indicated in the OEIS as coefficients in the polynomial.

Stirling Numbers

Signed Stirling Numbers relation to the coefficients of the aforementioned polynomial.

Pochhammer Symbol

Another function that will output the polynomial using Gamma functions: $\frac{\Gamma(x+n)}{\Gamma(x)}$

Presentation Outline

NSF: Definition of N-Step Factorials

Recall that

$$NSF(x, n) = x \cdot (x - 1n) \cdot (x - 2n) \cdot ... \cdot (x - m \cdot n)$$

Using this fact and looking at the NSF Function, we find that $NSF(x, n)$ is directly related to $NSF(\frac{x}{n}, 1)$.

Upon factoring out n from our first definition, we get:

$$NSF(x, n) = n^{\lceil \frac{x}{n} \rceil} \cdot NSF(\frac{x}{n}, 1)$$

NSF: Definition of N-Step Factorials Cont.

Consider the case where $x \in \mathbb{N}$

We can analyze this fact:

$$NSF(x,1) = x! = \Gamma(x+1)$$

and say: $NSF(\frac{x}{n},1) = \Gamma(\frac{x}{n}+1)$ when $\frac{x}{n} \in \mathbb{N}$. From this we can define NSF in terms of the Gamma Function when $x \in \mathbb{N}$:

$$NSF(x,n) = n^{\lceil \frac{x}{n} \rceil} \cdot \Gamma(\frac{x}{n}+1)$$

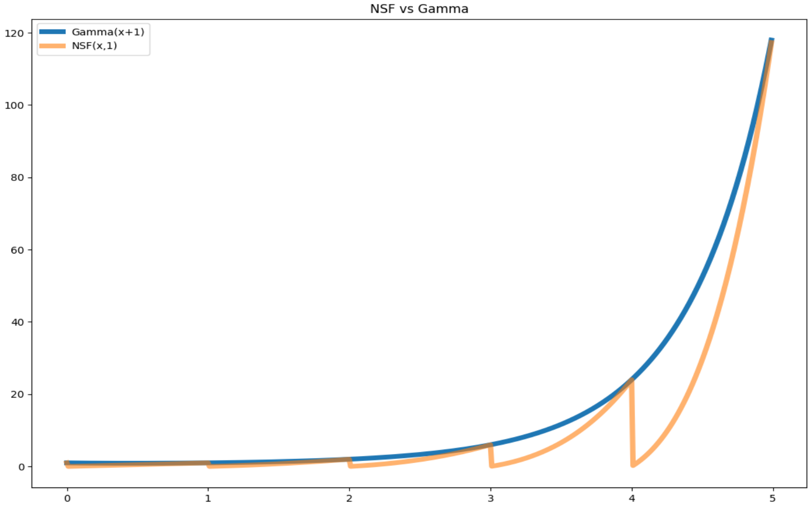
However, when $\frac{x}{n} \notin \mathbb{N}$, we have to define another function $g(x)$ to fully relate the NSF Function to the Gamma Function:

$$NSF(x,n) = n^{\lceil \frac{x}{n} \rceil} \cdot^{*} g(x\%1) \cdot \Gamma(\frac{x}{n}+1)$$

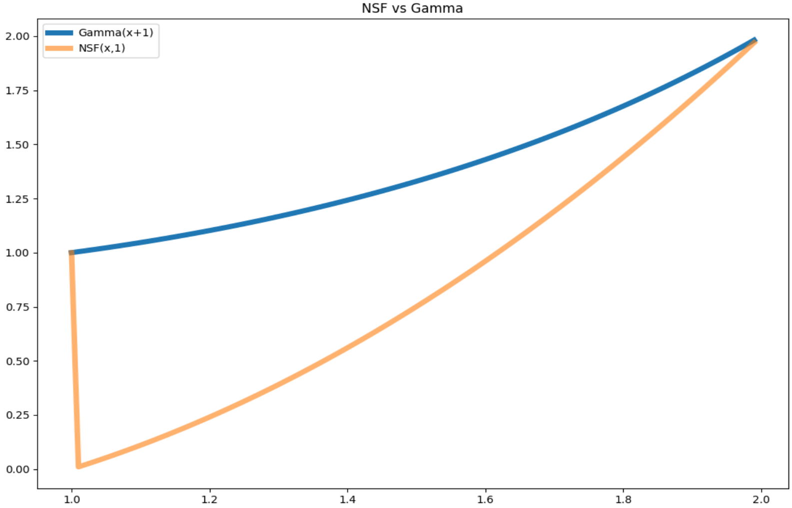
Notice that we have $g(x)$ to directly relate the N-Step Factorial to the Gamma Function and $g(x) = 1$ when $\frac{x}{n} \in \mathbb{N}$.

$^{*}g(x)$ *will be defined later*

NSF: Comparing NSF to Gamma Function



NSF: Plot Cont.



NSF: Code

Using the Definition:

$$NSF(x,n) = n^{\lceil \frac{x}{n} \rceil} \cdot NSF(\frac{x}{n},1)$$

```
import numpy as np

def nsf(base, step):
    n = np.ceil(np.divide(base/step))
    m = np.divide(base/step)
    return np.prod(np.power(step,n),nsf1(m))

#NSF of step 1
def nsf1(base):
    temp = base
    num = 1.0
    while(temp>0):
        num=np.prod(num*temp)
        temp-=1
    return num
```

NSF: What is $g(x)$?

From slide 8, we defined NSF along with $g(x)$.

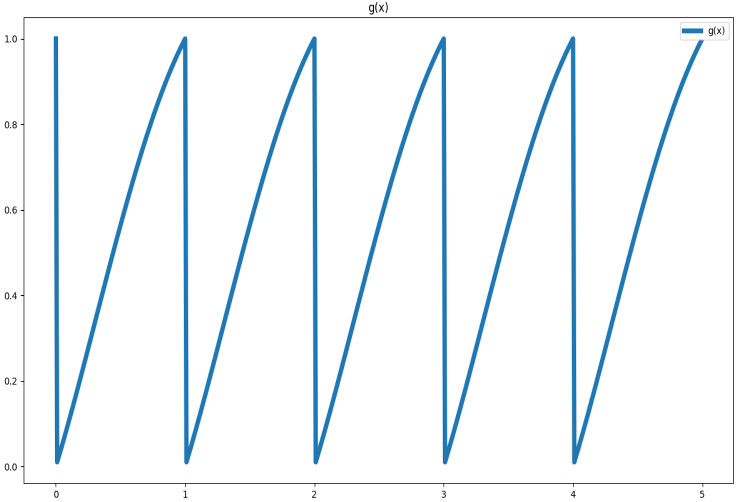
First, recall that $NSF(\frac{x}{n}, 1) = \Gamma(\frac{x}{n} + 1)$ when $x \in \mathbb{N}$, however when $x \notin \mathbb{N}$, it is not equal.

To fully relate the NSF Function to the Gamma Function, we decided to create a $g(x)$, which is defined by the ratio of the NSF of step 1 and the Gamma Function:

$$g(x) = \frac{\mathrm{NSF}(\frac{x}{n}, 1)}{\Gamma(\frac{x}{n} + 1)}$$

Graphing $g(x)$, it seems to be periodic.

NSF: g(x) Plot



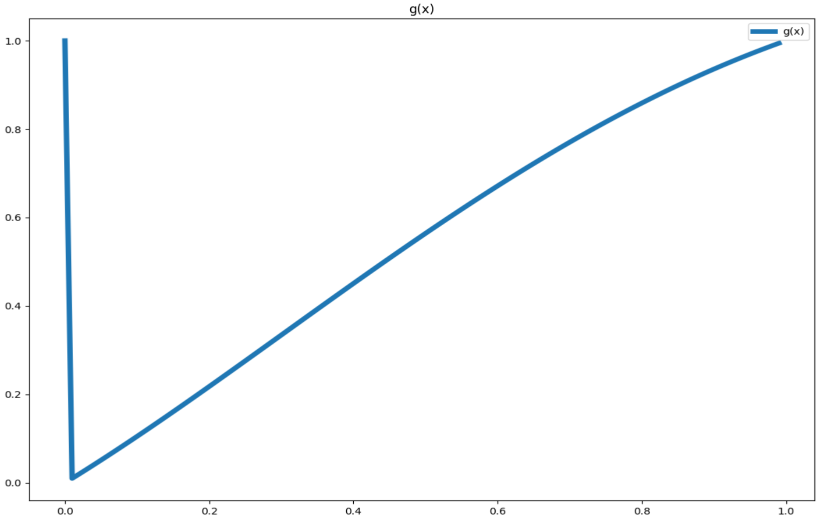
NSF: Short Proof that $g(x)$ is Periodic

We know from the definition of NSF:
$\mathrm{NSF}(x+1,1) = (x+1)\mathrm{NSF}(x,1)$.
We also know by definition of the Gamma Function:
$\Gamma(x+2) = (x+1)\Gamma(x+1)$.* Then,

$$\begin{aligned} g(x+1) &= \frac{\mathrm{NSF}(x+1,1)}{\Gamma(x+2)} \\ &= \frac{(x+1)\mathrm{NSF}(x,1)}{(x+1)\Gamma(x+1)^*} \\ &= \frac{\mathrm{NSF}(x,1)}{\Gamma(x+1)} = g(x) \end{aligned}$$

*source: https://www.cantorsparadise.com/the-beautiful-gamma-function-and-the-genius-who-discovered-it-8778437565dc

NSF: g(x) Plot Cont.



NSF: Approximating $g(x)$

Taylor Series:

Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point.

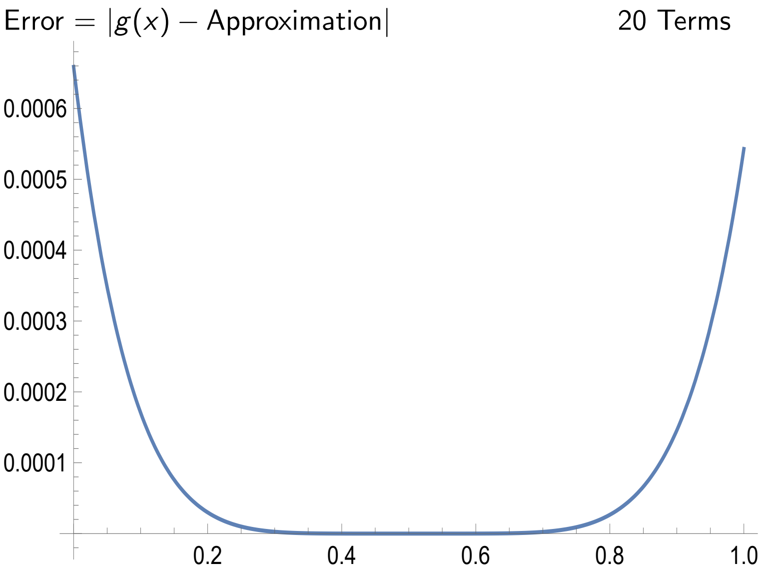
Fourier Series:

Fourier series is a sum that represents a periodic function as a sum of sine and cosine waves.

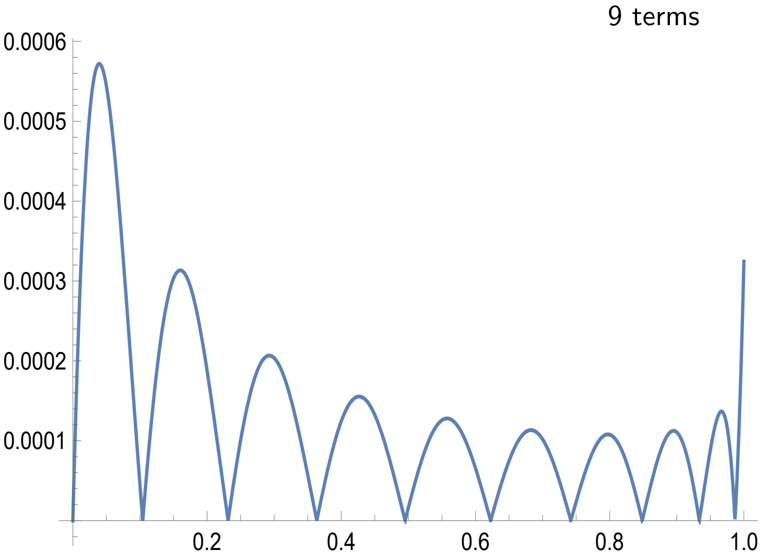
We can use any of these to approximate $g(x)$ and make the error as small as we want.

$$
\begin{aligned}
&0.56419 + 1.10779\,(x-0.5) - \\
&0.304502\,(x-0.5)^2 - \\
&0.439103\,(x-0.5)^3 + \\
&0.200585\,(x-0.5)^4 + \\
&0.0298893\,(x-0.5)^5 - \\
&0.0388487\,(x-0.5)^6 + \\
&0.00767326\,(x-0.5)^7 + \\
&0.00156537\,(x-0.5)^8 - \\
&0.00103455\,(x-0.5)^9 + \\
&0.000165035\,(x-0.5)^{10} + \\
&0.0000184068\,(x-0.5)^{11} - \\
&0.0000128187\,(x-0.5)^{12} + \\
&2.18517\times 10^{-6}\,(x-0.5)^{13} + \\
&1.33749\times 10^{-8}\,(x-0.5)^{14} - \\
&8.05838\times 10^{-8}\,(x-0.5)^{15} + \\
&1.66378\times 10^{-8}\,(x-0.5)^{16} - \\
&1.06539\times 10^{-9}\,(x-0.5)^{17} - \\
&2.30753\times 10^{-10}\,(x-0.5)^{18} + \\
&7.05829\times 10^{-11}\,(x-0.5)^{19} - \\
&8.05472\times 10^{-12}\,(x-0.5)^{20} + O[x-0.5]^{21}
\end{aligned}
$$

NSF: Difference Plots - Taylor



NSF: Difference Plots - Fourier



Presentation Outline

Future Work: Potential Points of Interests

Represent NSF as an integral; similar to the Gamma Function.

Taking the limit of $NSF(x, n)$ as n approaches 0.

NSF at negative integers and it's behavior compared to the Gamma Function.

How it behaves with Complex Numbers $(a + bi)$ and its relation to the Riemann Zeta Function.

Learning Experiences

1. Using Python to graph and assist in approximating $g(x)$

2. Using Mathematica to verify work and approximating $g(x)$

3. Desmos may be a helpful graphing tool, but is extremely limited for research purposes

4. Computational tools are essential for exploring hard problems.