Sets

1^{ST} WEEK

Outline

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
  - Interval Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting unordered collection of objects
  - Programming languages have set operations.
  - Relations, set of ordered pairs representing relationships between objects
  - Graphs, sets of vertices and edges that connect vertices

Sets

- A *set* is an unordered collection of objects.
  - A pack of cards
  - A crowd of people
  - A basketball team
  - The students in this class
  - The chairs in this room
- The objects in a set are called the ***elements***, or ***members*** of the set. A set is said to ***contain*** its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster/Tabular Method

- In roster form, all elements are listed and being separated by commas and enclosed within braces {}

  $S = \{a,b,c,d\}$

- Order is not important

  $S = \{a,b,c,d\} = \{b,c,a,d\}$

- Each distinct object is either a member or not; listing more than once does not change the set.

  $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $S = \{a,b,c,d, \ldots\ldots,z\}$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$
- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$
- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots,99\}$
- Set of all integers less than 0:

  $S = \{\ldots., -3,-2,-1\}$

Some Important Sets

N = set of *natural numbers* = {0, 1,2,3….}

Z = set of *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z^+ = set of *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R^+ = set of *positive real numbers*

C = set of *complex numbers*

Q = set of *rational numbers*

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in Z^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid P(x)\}$$

- Positive rational numbers:

  $Q^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$

Interval Notation

$$[a,b] = \{x \mid a \leq x \leq b\}$$

$$[a,b) = \{x \mid a \leq x < b\}$$

$$(a,b] = \{x \mid a < x \leq b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

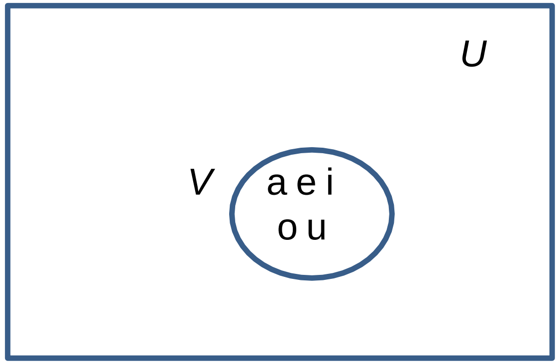
Type of Sets

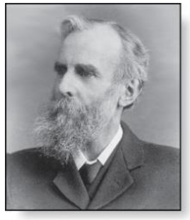
- Universal sets
- Empty sets
- Equal sets
- Subsets
- Power set
- Finite & Infinite set

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized $\emptyset$, but {} also used.

Venn Diagram





John Venn (1834-1923)
Cambridge, UK

Some things to remember

- Sets can be elements of sets.

  {N,Z,Q,R}

  {{1,2,3},a, {b,c}}

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{ \emptyset \}$

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$

- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Subsets

Definition: The set *A* is a *subset* of *B*, if and only if every element of *A* is also an element of *B*.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set *S*.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set *S*.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.
2. The set of integers with squares less than 100 is not a subset of the set of nonnegative integers.

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by

  $A = B$, iff $\forall x(x \in A \leftrightarrow x \in B)$

- Using logical equivalences we have that $A = B$ iff

  $$\forall x[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$$

- This is equivalent to

  $A \subseteq B$ and $B \subseteq A$

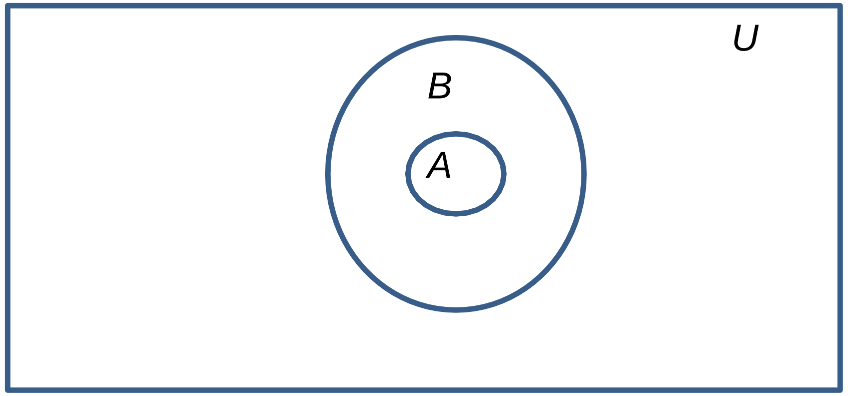
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by |*A*|, is the number of (distinct) elements of *A*.

Examples:

1. $|\emptyset| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{\emptyset\}| = 1$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted P**(***A***)**, is called the *power set* of *A*.

Example: If *A* = {a,b} then

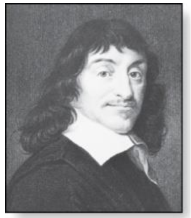
$$P(\text{A}) = \{\varnothing, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n

Tuples

- The *ordered n-tuple* $(a_1, a_2, \ldots.., a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

Cartesian Product


René Descartes
(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **Definition**: A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.

Cartesian Product

Definition: The cartesian products of the sets A_1,A_2,......,A_n, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered *n*-tuples (a_1,a_2,......,a_n) where a_i belongs to A_i for *i* = 1, … n.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is *A* × *B* × C where *A* = {0,1}, *B* = {1,2} and *C* = {0,1,2}

Solution: *A* × *B* × C **=** {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)}

Union

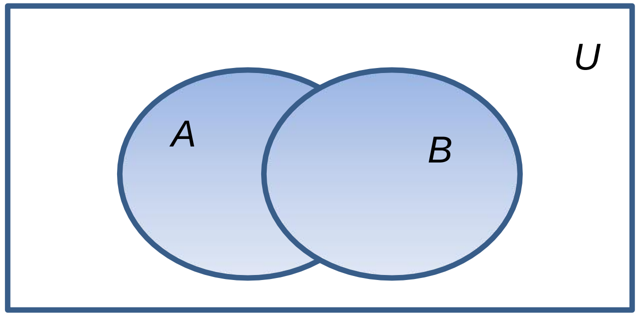
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set:

$$\{x|x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for *A* ∪ B



Intersection

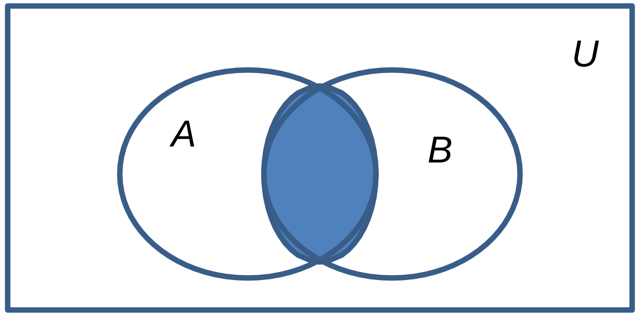
- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is
$$\{x | x \in A \wedge x \in B\}$$
- Note if the intersection is empty, then A and B are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:**What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

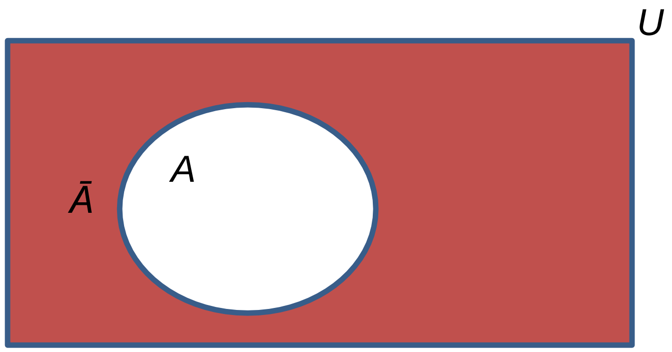
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c.)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

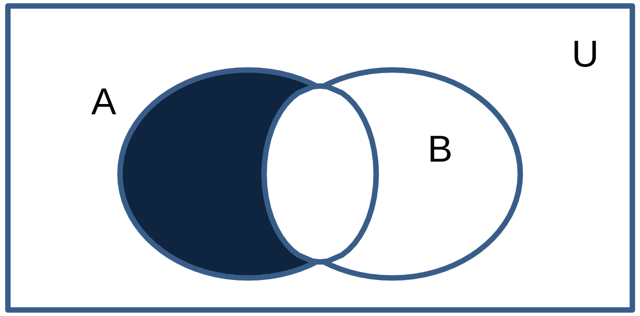
Venn Diagram for Complement



Difference

- **Definition:** Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\} = A \cap \overline{B}$$



Venn Diagram for $A - B$

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

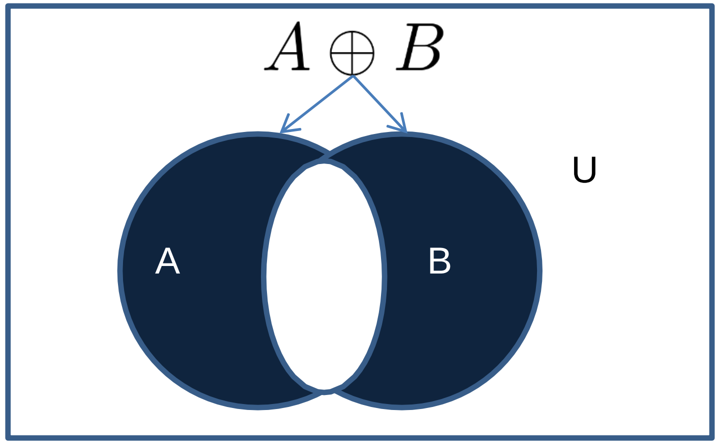
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

– **Solution**: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Exercises

List the members of these sets.

a) $\{x \mid x \text{ is a real number such that } x2 = 1\}$
b) $\{x \mid x \text{ is a positive integer less than } 12\}$
c) $\{x \mid x \text{ is the square of an integer and } x < 100\}$
d) $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

Use set builder notation to give a description of each of these sets.

a) $\{0, 3, 6, 9, 12\}$
b) $\{-3, -2, -1, 0, 1, 2, 3\}$
c) $\{m, n, o, p\}$

For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
b) the set of people who speak English, the set of people who speak Chinese
c) the set of flying squirrels, the set of living creatures that can fly

Exercises

Determine whether each of these pairs of sets are equal

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}, \{5, 3, 1\}$
b) $\{\{1\}\}, \{1, \{1\}\}$
c) $\emptyset, \{\emptyset\}$

Determine whether each of these statements is true or false.

a) $0 \in \emptyset$
b) $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$
d) $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$
f) $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subseteq \{\emptyset\}$

What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

Exercises

Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.

a) the set of sophomores taking discrete mathematics in your school
b) the set of sophomores at your school who are not taking discrete mathematics
c) the set of students at your school who either are sophomores or are taking discrete mathematics
d) the set of students at your school who either are not sophomores or are not taking discrete mathematics

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3,4,5\}$, $B=\{4,5,6,7,8\}$
Determine:

a) $A \cup B$
b) $A \cap B$
c) $\bar{A}$
d) $A - B$
e) $B - A$
f) $A \oplus B$