



Inverses



Identity Matrices

- Of special interest are square matrices with 1's on the main diagonal and 0's off the main diagonal, such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and so on.}$$

- A matrix of this form is called an **identity matrix** and is denoted by I . If it is important to emphasize the size, we shall write I_n for the $n \times n$ identity matrix.
- If A is an $m \times n$ matrix, then
$$A I_n = A \quad \text{and} \quad I_m A = A$$
- Recall : the number 1 plays in the numerical relationships
$$a \cdot 1 = 1 \cdot a = a .$$



Example4

Multiplication by an Identity Matrix

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

and

$$A I_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Recall : $A I_n = A$ and $I_m A = A$, as A is an $m \times n$ matrix



Definition

- If A is a square matrix, and if a matrix B of the same size can be found such that $AB=BA=I$, then A is said to be ***invertible*** and B is called an ***inverse*** of A .

Notation: $B = A^{-1}$

- If no such matrix B can be found, then A is said to be ***singular***.

Example5

Verifying the Inverse requirements

The matrix

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \text{ is an inverse of } A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

since

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$



Properties of Inverses

- *It is reasonable to ask whether an invertible matrix can have more than one inverse.* The next theorem shows that the answer is no. *An invertible matrix has exactly one inverse*.
- If B and C are both inverses of the matrix A , then $B=C$



Inverses

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad - bc} & -\frac{b}{ad - bc} \\ -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$



Inverse of a Product

- *If A and B are invertible matrices of the same size ,then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

- *The result can be extended :*

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.



Inverse of a Product

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

Applying the formula in Theorem 1.4.5, we obtain

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Also,

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Therefore, $(AB)^{-1} = B^{-1}A^{-1}$ as guaranteed by Theorem 1.4.6.



Laws of Exponents

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I$$

$$A^n = \underbrace{AA \cdots A}_{n \text{ factors}} \quad (n > 0)$$

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{n \text{ factors}}$$



Laws of Exponents

- *If A is a square matrix and r and s are integers, then*

$$A^r A^s = A^{r+s}, \quad (A^r)^s = A^{rs}$$



Laws of Exponents

■ *If A is an invertible matrix, then :*

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$ for $n = 0, 1, 2, \dots$

(c) For any nonzero scalar k , the matrix kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$.



Powers of a Matrix

Let A and A^{-1} be as in Example 7, that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$



Invertibility of a Transpose

- *If A is an invertible matrix, then A^T is also invertible and*

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$



Invertibility of a Transpose

Consider the matrices

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}, \quad (A^{-1})^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}, \quad (A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$



Exercise

- For each of these matrix A :

$$\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$$

- Find A^{-1}
- Find A^3
- Find $(A^{-1})^3$ and show that $(A^{-1})^3 = (A^3)^{-1}$
- Find A^T and show that $(A^T)^{-1} = (A^{-1})^T$