Relations

3RD WEEK

Outline

- Ordered Pair
- Cartesian Product
- Relations
- Domain and Range
- Inverse Relations
- Type of Relations

Ordered Pair

Definition: An ordered pair (x, y) is a single element consisting of a pair of elements in which

- ✓ x is the first element (coordinate)
- ✓ y is the second element (coordinate)

Note:

If $\{a, b\}$ is a set, then $\{a, b\} = \{b, a\}$

If (a, b) is an ordered pair, then $(a, b) \neq (b, a)$

Two ordered pair (x, y) and (w, z) will be equal if $x = w$ and $y = z$

Cartesian Product

Definition: The cartesian product of two sets A and B is the set of all ordered pairs (a, b) with $a \in A \; and \; b \in B$

$$A \times B = \{(a, b) |\, a \in A \; and \; b \in B\}$$

Note (in general):

$$A \times B \neq B \times A$$
$$|A \times B| \neq |B| \times |A|$$

Relations

Definition: A relation R from set A to set B is a subset of A x B

- If (a, b) $\in$ R, we say that "a is related to b" and write aRb
- If (a, b) $\notin$ R, we say that "a is not related to b" and write $a\not{R}b$
- If A = B, we often say that R is a relation on A

Domain & Range

Definition: The domain of relation R is the set of all first elements of the ordered pairs which belong to R, denoted by Dom(R).
Definition: The range is the set of second elements of the ordered pairs which belong to R, denoted by Ran(R).

Example:
A = {1, 2, 3} and B = {x, y, z},
and consider the relation R = {(1, y), (1, z), (3, y)}

The domain of R is Dom(R) = {1, 3}
The range of R is Ran(R) = {y, z}
The codomain of R = {x, y, z}

Inverse Relations

Definition: Let R be any relation from set A to B. The inverse of R, denoted by R^{-1}, is the relation from B to A denoted by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

- ✓ If R is any relation, then $(R^{-1})^{-1} = R$.
- ✓ The domain and range of R^{-1} are equal to the range and domain of R, respectively.
- ✓ If R is a relation on A, then R^{-1} is also a relation on A.

Composition of Relations

Definition: Suppose A, B and C are sets, and

- ✓ R is a relation from A to B
- ✓ S is a relation from B to C
- ✓ Then the composition of R and S, denoted by $R \circ S$, is a relation from A to C defined by

$$R \circ S = \{(a, c) | \exists\, b \in B, for\ which\ (a, b) \in R\ and (b, c) \in S\}$$

Type of Relations

- ✓ Reflexive and Irreflexive Relations
- ✓ Symmetric and Antisymmetric Relations
- ✓ Transitive Relations

Reflexive and Irreflexive Relations

Definition: A relation R on a set A is reflexive if $(a,a) \in R$ for all $a \in A$
Thus R is irreflexive if there exists $a \in A$ such that $(a,a) \notin R$

Example: Consider the following relations on the set $A = \{1, 2, 3\}$
Determine which relation is reflexive:

- ✓ $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
- ✓ $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- ✓ $R_3 = \{\}$

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Symmetric Relations

Definition: A relation R on a set A is symmetric if whenever $(a, b) \in R$ then $(b, a) \in R$.
Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Example: Consider the following relations on the set A = {1, 2, 3}
Determine which relation is symmetric :

✓ $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$

✓ $R_2 = \{(1, 1), (1, 2), (2, 2)\}$

Antisymmetric Relations

Definition: A relation R on a set A is antisymmetric if whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$

Equivalently:
For all $a, b \in A$, if $(a,b) \in R$ and $a \neq b$, then $(b,a) \in R$ must not hold.

Example: Consider the following relations on the set $A = \{1, 2, 3\}$
Determine which relation is symmetric :
- ✓ $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
- ✓ $R_2 = \{(1, 1), (1, 2), (2, 2)\}$
- ✓ $R_3 = \{(1, 1), (2, 2), (3, 3)\}$

Note: not symmetric ≠ antisymmetric

Transitive Relations

Definition: A relation R on a set A is transitive if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Thus R is not transitive if there exist $a, b, c \in A$ such that $(a, b) \in R$ and $(b, c) \in R$ but $(a, c) \notin R$

Example: Consider the following relations on the set $A = \{1, 2, 3\}$

Determine which relation is transitive :

- ✓ $R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$
- ✓ $R_2 = \{(1, 1), (1, 2), (2, 2), (2, 3)\}$
- ✓ $R_3 = \{(1, 1), (1, 2), (1, 3), (3, 3)\}$

Equivalence Relations

Definition: A relation R on a set A is called an equivalence relations if R is reflexive, symmetric, and transitive

It follows 3 properties:

- ✓ For every $a \in A$, $(a, a) \in R$
- ✓ If $(a, b) \in R$ then $(b, a) \in R$
- ✓ If $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Example: Consider the following relations on the set $A = \{1, 2, 3, 4\}$
Determine whether this relation is equivalence or not:
$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 3), (3, 3), (4, 4)\}$

Exercise

Consider these relations on the set of integers

- ✓ $R_1 = \{(a, b) | a \leq b\}$
- ✓ $R_2 = \{(a, b) | a > b\}$
- ✓ $R_3 = \{(a, b) | a = b \text{ or } a = -b\}$
- ✓ $R_4 = \{(a, b) | a = b\}$
- ✓ $R_5 = \{(a, b) | a = b + 1\}$
- ✓ $R_6 = \{(a, b) | a + b \leq 3\}$

Determine whether each of these relations are equivalence