Functions

4^{TH} WEEK

Outline

- Functions
- Surjective, Injective, Bijective Functions
- Operations
- Composite Functions
- Piecewise-defined Functions
- Inverse Functions

Definition of a Function

A *function* is a relationship between two variables such that each value of the first variable is paired with **exactly one** value of the second variable.

The *domain* is the set of permitted x values.

The *range* is the set of found values of y.

Function Vs Relation

- A relation is a correspondence between a first set, called the domain, and a second set, called the range, such that each member of the domain corresponds to at least one member of the range.
- Functions are special relations.
- Every set of ordered pairs is a relation, but every relation is not function
- Functions make up a subset of all relations.

Is it a function?

Domain (x)	Range (y)
1	-3.6
2	-3.6
3	4.2
4	4.2
5	10.7
6	12.1
52	52

For each *x*, there is only one value of *y*.

Therefore, it **IS** a function.

Is it a function?

Domain (x)	Range (y)
3	7
3	8
3	10
4	42
10	34
11	18
52	52

Three different *y*-values (7, 8, and 10) are paired with one *x*-value.

Therefore, it is **NOT** a function

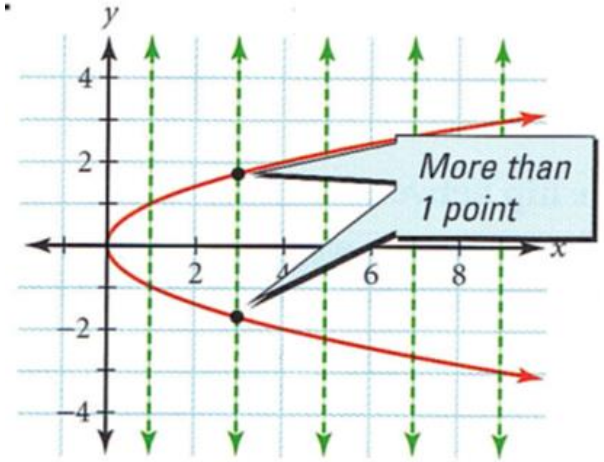
Function?

Is the relation below a function?
{(5, 8), (6, 7), (3, -1), (4, 2), (5, 9), (12, -2)}

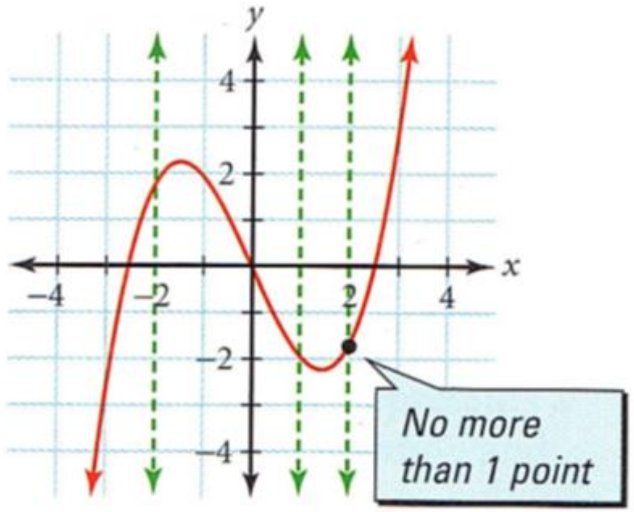
No. The *x*-value of 5 is paired with two different *y*-values.

Vertical Line Test

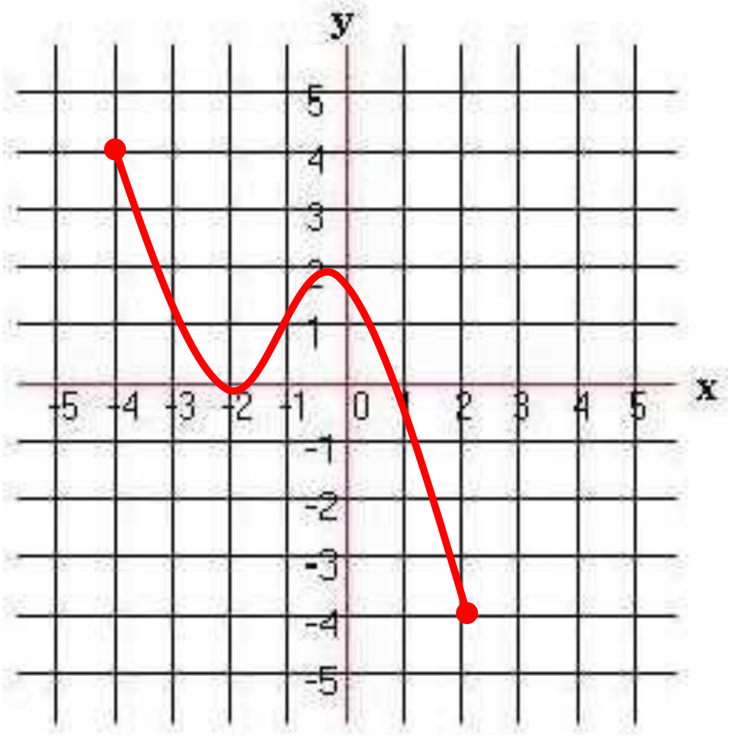
Used to determine if a graph is a function.
If a vertical line intersects the graph at more than one point, then the graph is **NOT** a function.



NOT a Function



Is it a function? Give the domain and range.

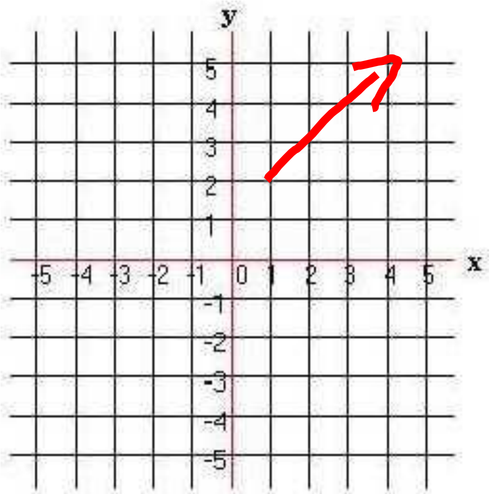


FUNCTION

$Domain: [-4,2]$

$Range: [-4,4]$

Give the Domain and Range.



$$Domain:\ x \geq 1$$

$$Range:\ y \geq 2$$

Functional Notation

- We have seen an equation written in the form y = "*some expression in x*".

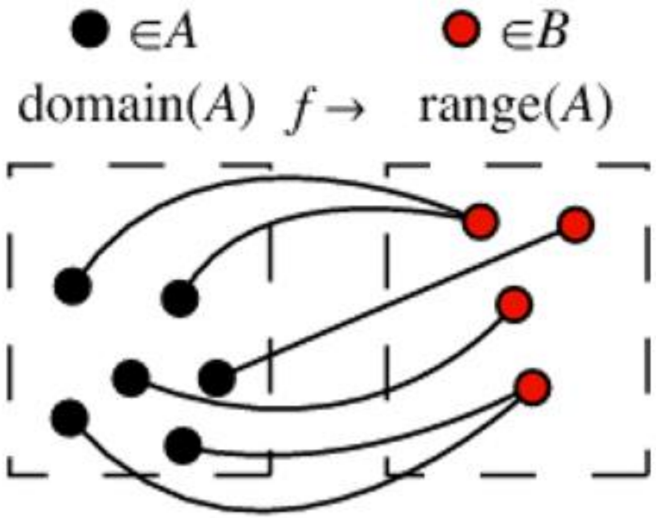
- Another way of writing this is to use ***functional notation***.

- For example, you could write $y = x^2$ as $f(x) = x^2$.

Surjective Function (Onto)

A surjective function is a function whose range is equal to its codomain. Equivalently, a function f with domain A and codomain B is surjective if for every b in B there exists at least one a in A with $f(a) = b$.

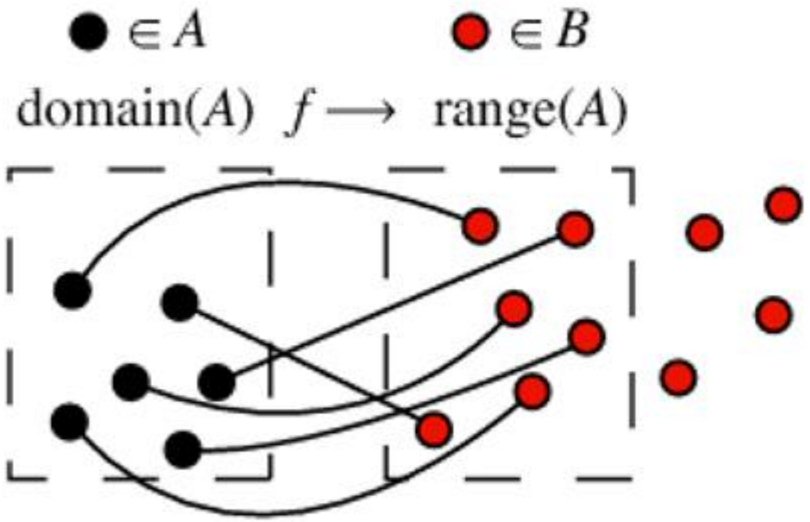
If $f: A \to B$, then f is said to be surjective if:
$\forall b \in B\ \exists a \in A,$ such that $f(a) = b$.



Injective Function (One-to-One)

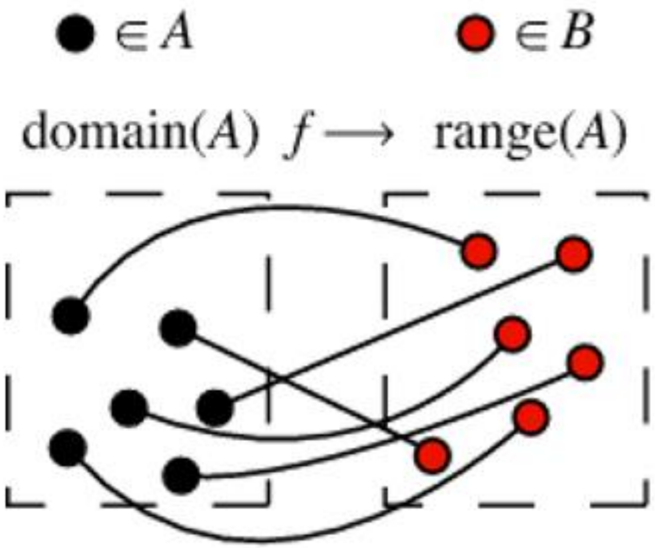
A function f is injective if and only if for all a and b in A, if $f(a) = f(b)$, then $a = b$; that is, $f(a) = f(b)$ implies $a = b$. Equivalently, if $a \neq b$, then $f(a) \neq f(b)$.

If $f: A \rightarrow B,$ then f is said to be injective if:
$\forall a, b \in A iff f(a) = f(b) \Rightarrow a = b$.



Bijective Function

A function is **bijective** (**surjective and injective**) if every element of the codomain is mapped to by *exactly* one element of the domain. (That is, the function is *both* injective and surjective.)



$$f(x) = x^2 + 2$$

Q1 What does this function do?

?

Q2 What is $f(3)$?

?

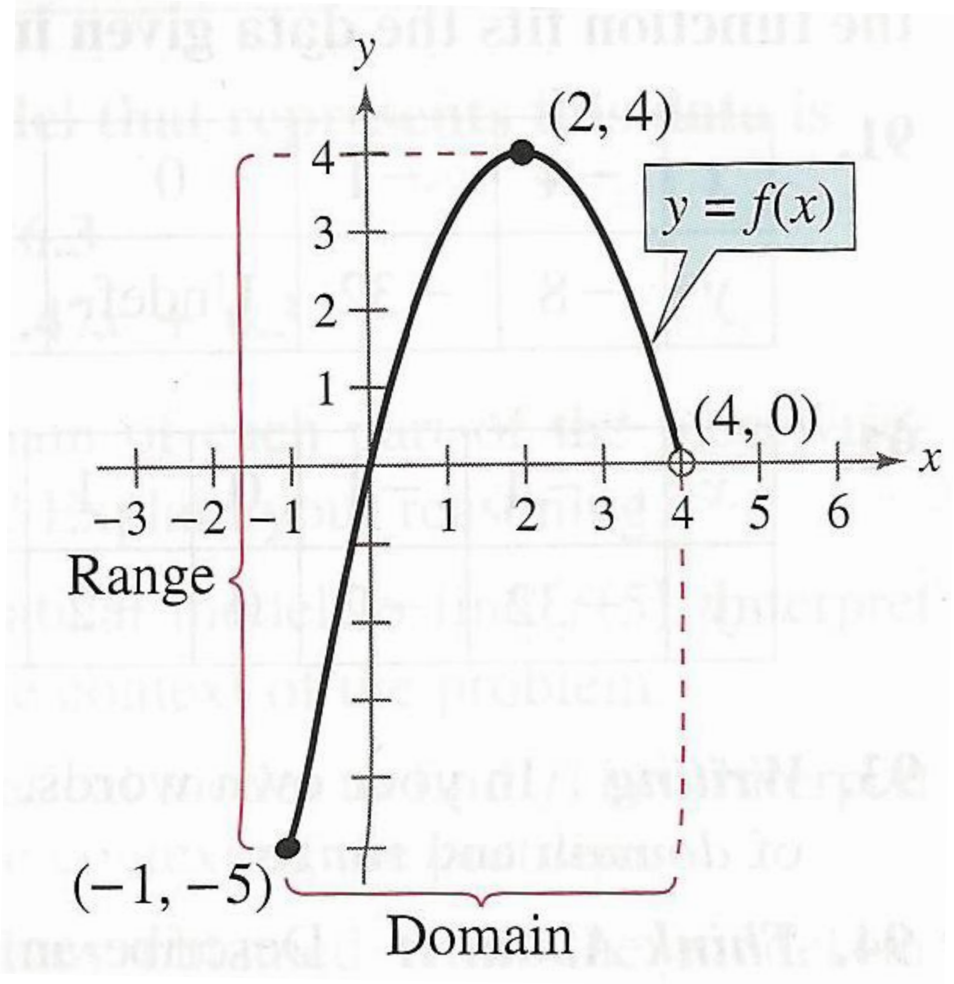
Q3 What is $f(-5)$?

?

Q4 If $f(a) = 38$, what is a?

?

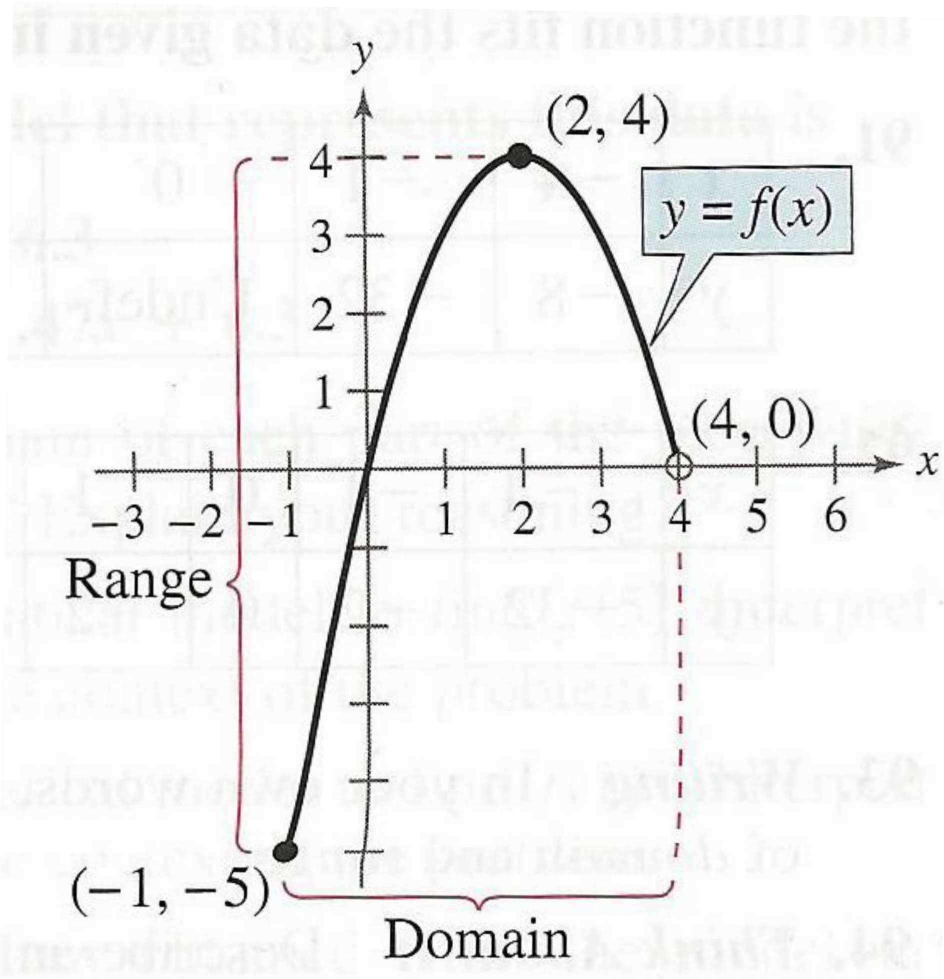
Domain & Range of a Function



What is the domain of the graph of the function f?

$$\boxed{A: \ [-1,4)}$$

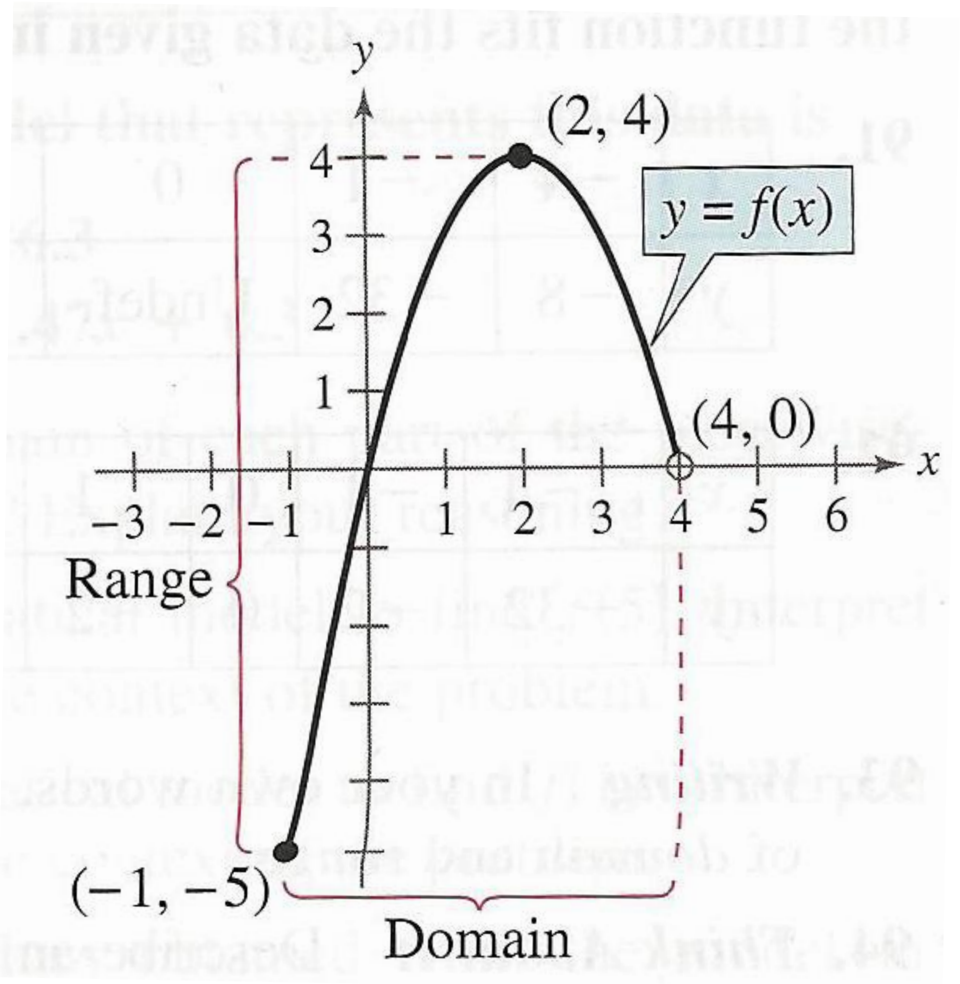
Domain & Range of a Function



What is the range of the graph of the function f?

$$\boxed{[-5,4]}$$

Domain & Range of a Function



$$Find\ \ f(-1)\, and\ \ f(2).$$

$$\boxed{f(-1) = -5}$$

$$\boxed{f(2) = 4}$$

Domain of a Function

- Every relation has a domain, the set of (input) values over which it is defined.
- If the domain is not stated, by convention we take the domain to be the largest set of (real) numbers for which the expression defining the function can be evaluated.
- We call this the "natural domain" of the function.

Domain of a Function

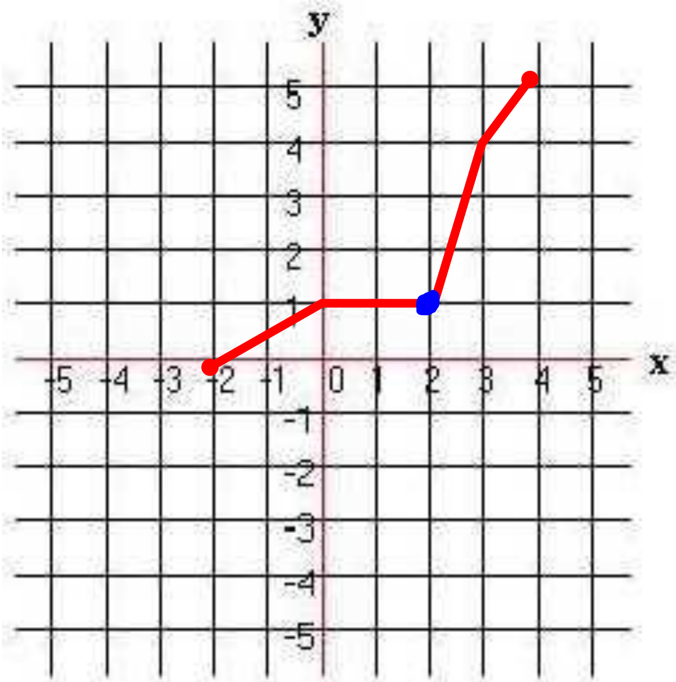
- The domain of a function can be implied by the expression used to define the function

- The function $f(x) = \frac{1}{x^2 - 4}$ has an implied domain that consists of all real numbers x other than x = ±2. The domain excludes *x*-values that result in division by zero.

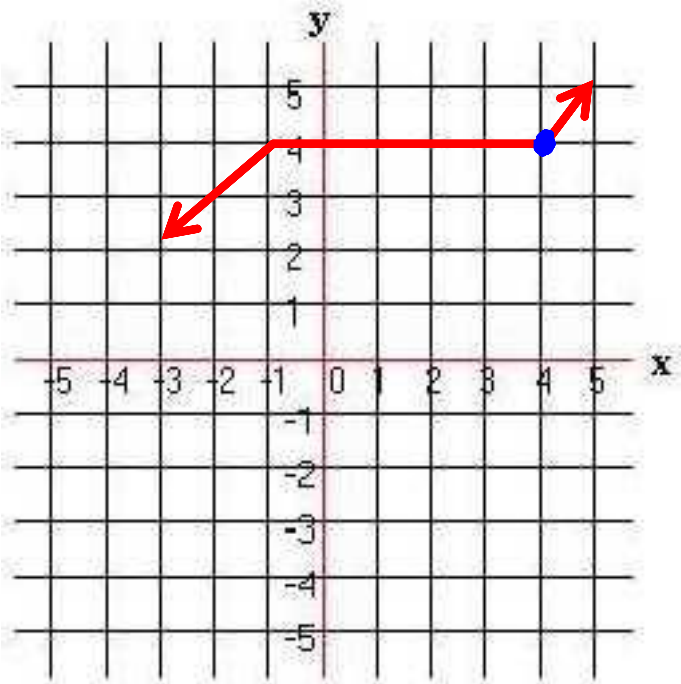
- Another common type of implied domain is that used to avoid roots of negative number. For example $f(x) = \sqrt{x}$ is defined only for $x \geq 0$

Addition

$$f(2) + g(4)$$



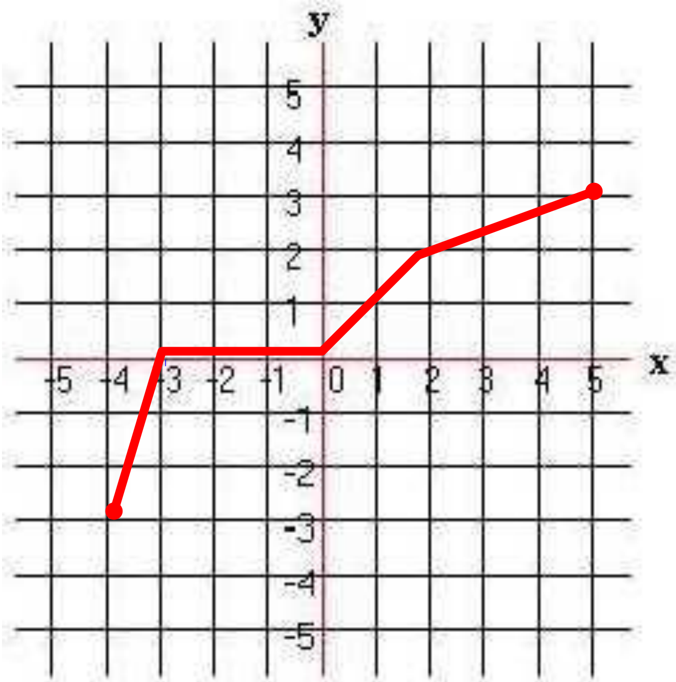
$f(x)$



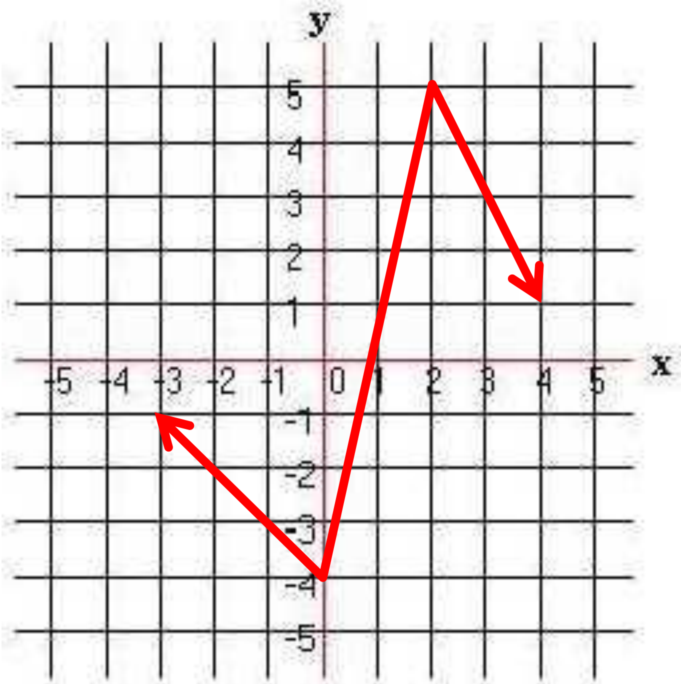
$g(x)$

Subtraction

$$f(5) - g(0)$$



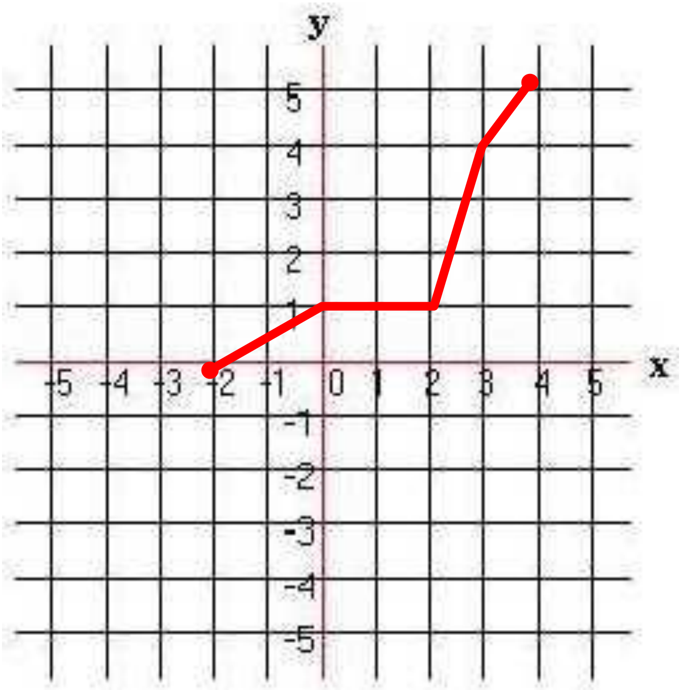
$f(x)$



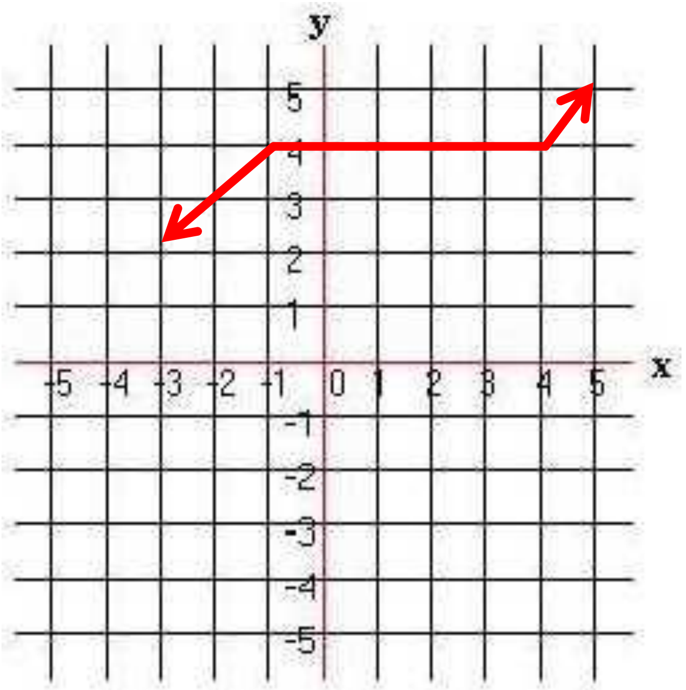
$g(x)$

Multiplication

$$f(4) \cdot g(-1)$$



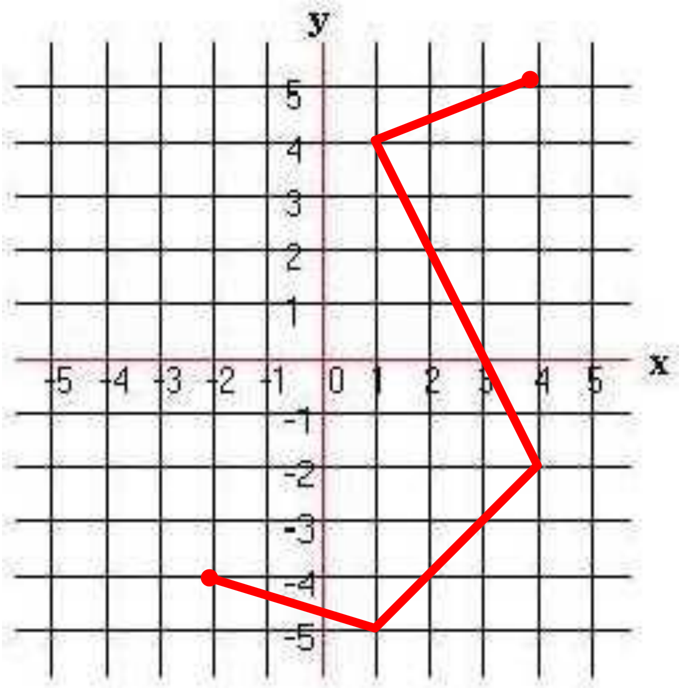
$f(x)$



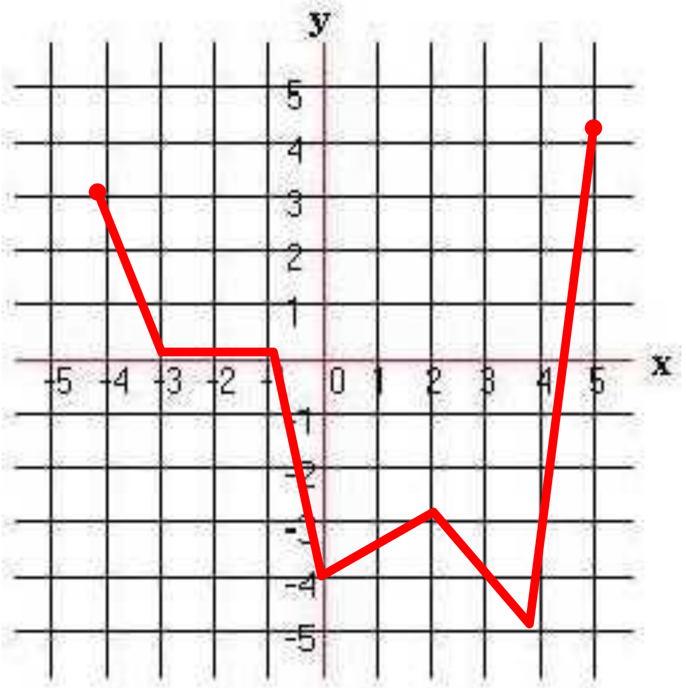
$g(x)$

Division

$$f(-2) \div g(0)$$



$f(x)$



$g(x)$

Composite Function

$$f(x) = 3x + 1$$
$$g(x) = x^2$$

Have a guess!

$$fg(2) =$$ 49? 13?

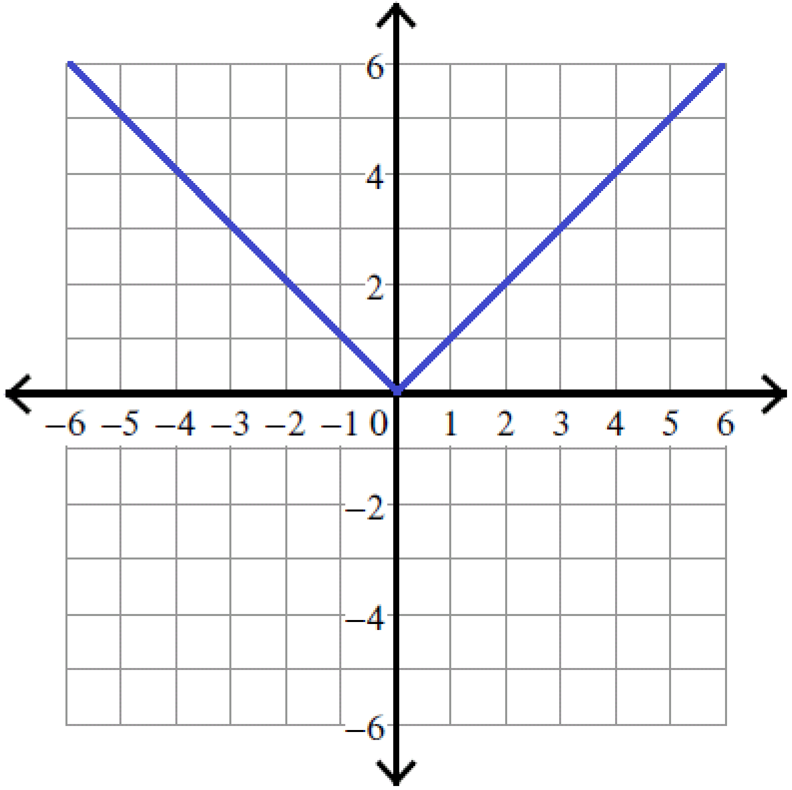
$fg(2)$ means $f\big(g(2)\big)$, i.e. "f of g of 2".
We therefore apply the functions to the input in sequence from right to left.

Piecewise-defined Functions

A piecewise-defined function is a function that is defined by two or more equations over a specified domain.

The absolute value function $f(x) = |x|$ can be written as a piecewise-defined function.

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

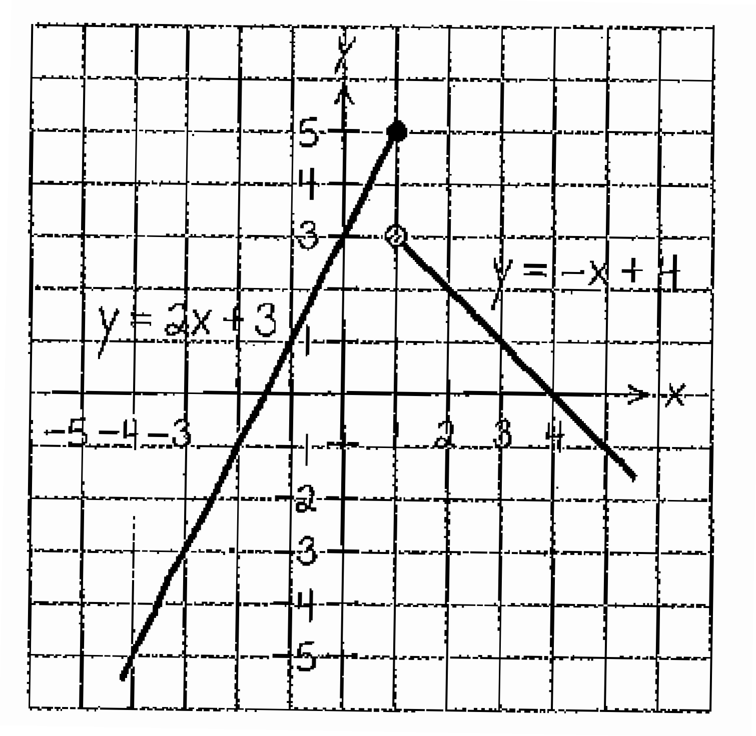


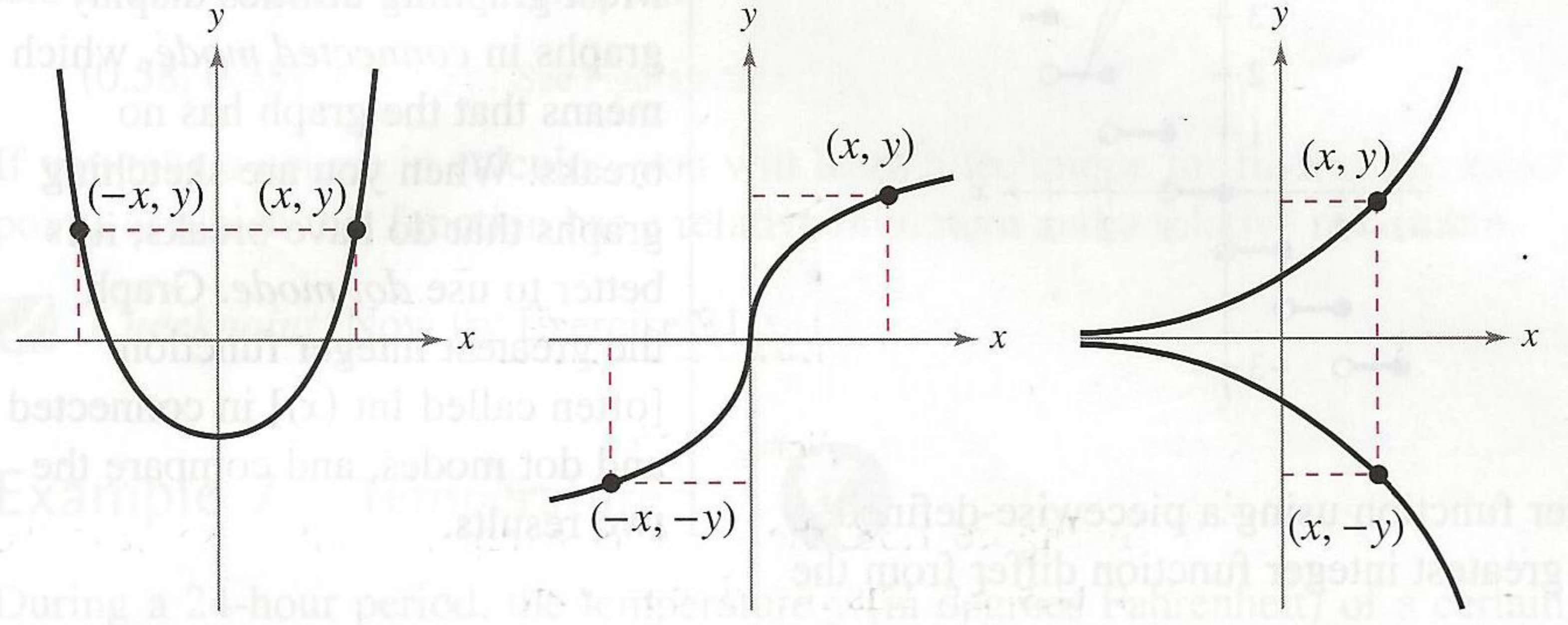
Graph of a Piecewise-Defined Function

Sketch the graph of

$$f(x)=\begin{cases} 2x+3, & x\le 1 \\ -x+4, & x>1 \end{cases}$$

Notice when open dots and closed dots are used. Why?





Symmetric to y-axis
Even function

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

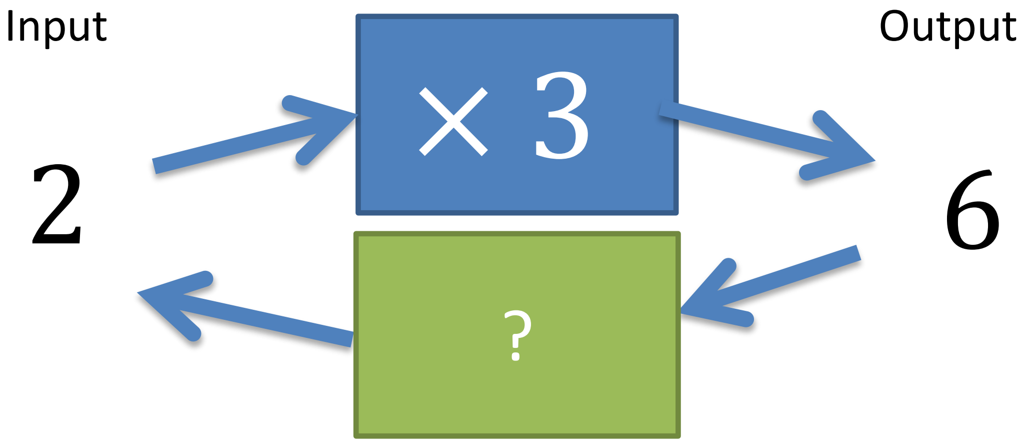
Figure 1.34

$$f(-x) = f(x)$$

$$f(-x) = -f(x).$$

Inverse Functions

A function takes and input and produces an output.
The inverse of a function does the **opposite: it describes how we get from the output back to the input.**



So if $f(x) = 3x$, then the inverse function is :

$$\boldsymbol{f^{-1}(x)} = \boxed{?}$$

Exercise

If $g(x) = 3x - 1$, determine:
(a) $g(x - 1)$
(b) $g(2x)$
(c) $g(x^3)$

If $f(x) = 2x - 1$ determine:
(a) $f(2x)$
(b) $f(x^2)$
(c) $f(2x - 1)$
(d) $f\big(1 + 2f(x - 1)\big)$
(e) Solve $f(x + 1) + f(x - 1) = 0$

Exercise

Draw the graph of this piecewise-defined function

$$f(x) = \begin{cases} 5 & for\ x \leq 0 \\ 5 + 4x - x^2 & for\ 0 \leq x \leq 5 \\ 0 & for\ x \geq 5 \end{cases}$$

Exercise

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$
b. $h(x) = x^2 + 1$
c. $f(x) = x^3 - 1$
d. $j(x) = |x|$

Exercise

Find the inverse functions

$$f(x) = \frac{x+1}{x-2}$$

$$f(x) = 3x - 1$$

$$f(x) = \sqrt{x} + 3$$

$$f(x) = \frac{1}{x}$$

Exercise

$$f(x) = 2x + 1$$
$$g(x) = \frac{1}{x}$$

Determine:

$fg(x)$
$gf(x)$
$ff(x)$
$gg(x)$