Correction Partiel - 24 octobre 2016

Exercice 1. Give a basis for null space and the column space of the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 & -1 \\ -1 & 1 & 0 & -2 \\ 2 & -3 & -1 & 5 \end{pmatrix}.$$

Correction :
Let $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathrm{Ker}(\boldsymbol{A})$, then one has

$$\boldsymbol{AX} = 0 \Leftrightarrow \begin{cases} x + 2y + 3z - t = 0 \\ -x + y \quad - 2t = 0 \\ 2x - 3y - z + 5t = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z - t = 0 \\ 3y + 3z - 3t = 0 \\ -7y - 7z + 7t = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + 3z - t = 0 \\ y + z - t = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = -z - t \\ y = -z + t \end{cases}$$

Thus the vectors $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ belong to $\mathrm{Ker}(\boldsymbol{A})$. Moreover they are linearly independent and so they form a basis for $\mathrm{Ker}(\boldsymbol{A})$.
Now, applying the rank nullity theorem we obtain

$$\dim(\mathrm{Ker}(\boldsymbol{A})) + \dim(\mathrm{Im}(\boldsymbol{A})) = 4 \Rightarrow \dim(\mathrm{Im}(\boldsymbol{A})) = 2.$$

But, we have

$$\boldsymbol{A} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix},$$

thus the vectors $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ belong to $\mathrm{Im}(\boldsymbol{A})$. Moreover they are linearly independent and so they form a basis for $\mathrm{Im}(\boldsymbol{A})$.

Exercice 2. Find the eigenvalues of the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Correction :
Let $\lambda \in \mathbb{R}$, we compute $\det(\boldsymbol{A} - \lambda \boldsymbol{I}_3)$, one has

$$\begin{aligned}
\det(\boldsymbol{A} - \lambda \boldsymbol{I}_3) &= \begin{vmatrix} 3-\lambda & -1 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 2-\lambda & 1 \end{vmatrix} \\
&= (3-\lambda)\left((2-\lambda)(3-\lambda)+1\right) + (-1-(2-\lambda)) = (3-\lambda)\left(7 - 5\lambda + \lambda^2\right) + (\lambda - 3) \\
&= (3-\lambda)\left(6 - 5\lambda + \lambda^2\right) = -(\lambda-3)^2(\lambda-2).
\end{aligned}$$

Thus the eigenvalues of $\boldsymbol{A}$ are $\lambda = 2$ and $\lambda = 3$.

Exercice 3.

1. Project $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ by solving the system $\boldsymbol{A}^T \boldsymbol{A} \widehat{\boldsymbol{x}} = \boldsymbol{A}^T \boldsymbol{b}$ and then by computing $\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}}$ with

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}.$$

Correction :
First we compute the matrix product $\boldsymbol{A}^T \boldsymbol{A}$, one has

$$\boldsymbol{A}^T \boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

and

$$\boldsymbol{A}^T \boldsymbol{b} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

Thus we search for $\widehat{\boldsymbol{x}}$ such that

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \frac{1}{15-9} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 30 \\ -18 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

We finally obtain

$$\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}.$$

2. Compute the corresponding projection matrix.

Correction :
The projection matrix $\boldsymbol{P}$ satisfies

$$\begin{aligned}
\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T \boldsymbol{A})^{-1} \boldsymbol{A}^T &= \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}.
\end{aligned}$$

3. What conditions does this matrix have to check ?
   Correction :
   The matrix $\boldsymbol{P}$ has to satisfy $\boldsymbol{P} = \boldsymbol{P}^T$, $\boldsymbol{P}^2 = \boldsymbol{P}$ and $\boldsymbol{P}\boldsymbol{b} = \boldsymbol{p}$.

Exercice 4. We consider the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

1. Using the Gram-Schmidt process, write the $\boldsymbol{QR}$ factorization of the matrix $\boldsymbol{A}$.
   Correction :
   First one has $\boldsymbol{a}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ thus $\boldsymbol{q}_1 = \frac{\boldsymbol{a}_1}{\|\boldsymbol{a}_1\|} = \frac{1}{\sqrt{6}}\boldsymbol{a}_1$.
   Then we have $\boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and so

$$\widetilde{\boldsymbol{q}}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{4}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

   Thus we obtain

$$\boldsymbol{q}_2 = \frac{\widetilde{\boldsymbol{q}}_2}{\|\widetilde{\boldsymbol{q}}_2\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

   To finish $\boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Then one has

$$\widetilde{\boldsymbol{q}}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^T\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^T\boldsymbol{a}_3)\boldsymbol{q}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{5}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

   and

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{q}}_3}{\|\widetilde{\boldsymbol{q}}_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

   Thanks to the previous computations

$$\boldsymbol{Q} = (\boldsymbol{q}_1|\boldsymbol{q}_2|\boldsymbol{q}_3) = \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

   and

$$\boldsymbol{R} = \begin{pmatrix} \|\widetilde{\boldsymbol{q}}_1\| & \langle \boldsymbol{q}_1, \boldsymbol{a}_2 \rangle & \langle \boldsymbol{q}_1, \boldsymbol{a}_3 \rangle \\ 0 & \|\widetilde{\boldsymbol{q}}_2\| & \langle \boldsymbol{q}_2, \boldsymbol{a}_3 \rangle \\ 0 & 0 & \|\widetilde{\boldsymbol{q}}_3\| \end{pmatrix} = \begin{pmatrix} \sqrt{6} & \frac{4}{\sqrt{6}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

2. Using this $\boldsymbol{QR}$ decomposition, find $\boldsymbol{x}$ such that

$$\boldsymbol{Ax} = \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}.$$

Correction :

We want to solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ with $\boldsymbol{A} = \boldsymbol{QR}$, then

$$\begin{aligned}
\boldsymbol{Ax} = \boldsymbol{b} &\Leftrightarrow \boldsymbol{QRx} = \boldsymbol{b} \Leftrightarrow \boldsymbol{Rx} = \boldsymbol{Q}^T\boldsymbol{b} \\
&\Leftrightarrow \begin{pmatrix} \sqrt{6} & \frac{4}{\sqrt{6}} & \frac{5}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix} \\
&\Leftrightarrow \begin{cases} \sqrt{6}x_1 + \dfrac{4}{\sqrt{6}}x_2 + \dfrac{5}{\sqrt{6}}x_3 = -\dfrac{10}{\sqrt{6}} \\ \dfrac{1}{\sqrt{3}}x_2 + \dfrac{2}{\sqrt{3}}x_3 = \dfrac{5}{\sqrt{3}} \\ \dfrac{1}{\sqrt{2}}x_3 = \dfrac{2}{\sqrt{2}} \end{cases} \Leftrightarrow \begin{cases} x_1 = -4 \\ x_2 = 1 \\ x_3 = 2 \end{cases}
\end{aligned}$$