MAP531 : Statistics
PC2 – Statistical modeling

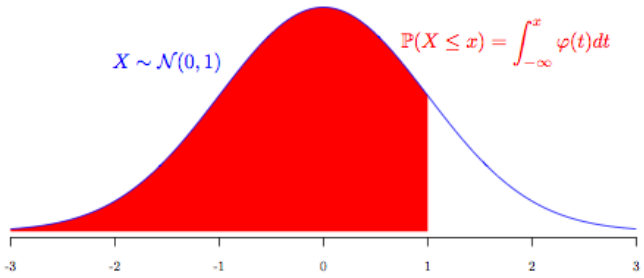
1 To warm up

Exercise 1. Small questions

1. Give a possible sample of size 4 from each of the following populations:
   - all daily newspapers published in the world,
   - all grades of students at the probability course,
   - all distances that may result when you throw a football,
   - all possible daily numbers of cars going through a crossing.
2. In 1882, Michelson and Newcomb measured the traveling time of light going from and to their lab through a mirror. Their first measurements were: 28, 26, 33, 24, 34, -44, 27, 16, 40, -2, 29, 24, 21, 25 ($*0.001 + 24.8$ in millionths of a second). Why are these measurements not identical? How do we model this variability in statistics?
3. Are the following statistical models identifiable
   - $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu + a, \sigma^2), (\mu, a, \sigma) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+\}\big)$,
   - $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}_+\}\big)$,
   - $\big((\mathcal{X} = \mathbb{R}, \mathcal{B}(\mathbb{R})), \{\mathcal{N}(\mu, \sigma^2), (\mu, \sigma) \in \mathbb{R} \times \mathbb{R}\}\big)$?

Exercise 2. Gaussian measurements
For this exercise, you can use the table of the cdf of the standard normal distribution in Figure 1. The measurement of atmospheric ozone concentration (in $\mu g/m^3$) is modeled by a random variable X with distribution $\mathcal{N}(m, \sigma^2)$ with $\sigma^2 = 3.1$.

1. Write the statistical model.
2. In many applications, data are often modeled with the Normal distribution, while often the observed values are by definition positive (e.g. weight, size, speed, duration). Can you explain why?
3. What are the units of m and σ? What do m and σ represent?
4. The ozone concentration is considered dangerous for humans when it is greater than $180 \mu g/m^3$.
   (a) Assuming that $m = 178$, what is the probability that the measurement is greater than 180? Comment the result.
   (b) Assuming that $m = 183$, what is the probability that the measurement is smaller than 180? Comment the result.
   (c) Assuming that $m = 180$, find a real number δ such that the probability $P(180 - \delta \leq X \leq 180 + \delta)$ is bigger than 95%.
5. Are the three last questions statistical or probabilistic questions? Find a question that a statistician could ask from this experience.



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Figure 1: Table of the cdf of the standard normal distribution

6. Some day, we make some measurements and we assume that this day the ozone concentration is $178\mu g/m^3$ (yet the experimenter doesn't know this concentration otherwise he wouldn't need measurements).

   (a) Compute the probability that a unique measurement is greater than 180?

   (b) What is the probability that the mean of three measurements is greater than 180 ?

   (c) How many measurements are necessary for the probability that the mean of these measurements is greater than 180 being less than 1%?

Exercise 3. We are in front of a black urn which contains N balls which are numbered from 1 to N. We don't know N the number of balls but we can draw as many balls as we wish if we put it in the urn before drawing another one.

1. Write the statistical model.

2. How can you guess the value of N with the observed numbers $x_1, \dots, x_n$ during the sampling?
3. What is the distribution of the greatest number $\hat{N} = \max(X_1, \dots X_n)$?
4. How many balls do you want to draw? Help: you can compute the probability that $\hat{N} = N$.

2 To train

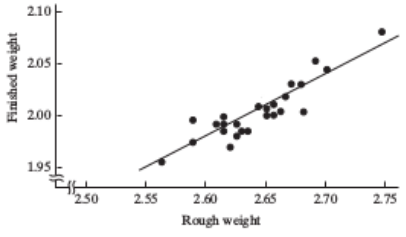
Exercise 4. Heart attack and aspirin
Devore, in his book *Probability and statistics* writes the following:
' An article in the New York times (Jan.27, 1987) reported that heart attack risk could be reduced by taking aspirin. This conclusion was based on a designed experiment involving both a control group of individuals that took a placebo (...) and a treatment group that took aspirin (...). Of the $11,034$ individuals in the control group, 189 subsequently experienced heart attacks, whereas only 104 of the $11,037$ in the aspirin group had heart attack. The incidence rate of heart attacks in the treatment group was only about half that in the control group. One possible explanation for this result is chance variation–that aspirin really doesn't have the desired effect and the observed difference is just typical variation in the same way that tossing two identical coins would usually produce different numbers of heads. However, in this case, inferential methods suggest that chance variation by itself cannot adequately explain the magnitude of the observed difference.'
Can you formalize statistically this description?

Exercise 5. Survival analysis and identifiability
We study a system which works as long as two machines work. The durations X_1 and X_2 of these two machines follow exponential distributions with parameters λ_1 and λ_2 : $\mathbb{P}(X_i > x) = \exp(-\lambda_i x)$.
X_1 and X_2 are assumed independent.

1. Compute the probability of the following event: "the system does not break down before time t".
   Deduce from this the probability distribution of Z, the duration of the system.
   Compute the probability of "The failure is due to machine 1".
2. Let $I = 1$ if the failure is due to machine 1 and $I = 0$ otherwise. Compute $\mathbb{P}(\{Z > t\} \cap \{I = \delta\})$, for any $t \geq 0$ and $\delta \in \{0, 1\}$. Deduce from this that Z and I are independent.
3. We are now provided with n identical systems (as in bullet '1.') which are independent from each other. Their durations are observed and given by $Z_1, ..., Z_n$.
   (a) Write the corresponding statistical model. Are the parameters λ_1 and λ_2 identifiable?
   (b) Assume now that we observe these duration and the corresponding reasons of failures $I_1, ..., I_n$, where $I_i \in \{0, 1\}$. Write the corresponding statistical model. Are the parameters λ_1 and λ_2 identifiable?

Exercise 6. Linear regression (Larsen and Marx *An introduction to mathematical statistics and its applications*)
A manufacturer of air conditioning units is having assembly problems due to the failure of a connecting rod to meet finished-weight specifications. Too many rods are being completely tooled, then rejected as overweight. To reduce that cost, the company's quality-control department wants to quantify the relationship between the weight of the finished rod, and that of the rough casting. Castings likely to produce rods that are too heavy can then be discarded before undergoing the final (and costly) tooling process. The following Figures represent the dataset that the company have.

Rod Number	Rough Weight, x	Finished Weight, y	Rod Number	Rough Weight, x	Finished Weight, y
1	2.745	2.080	14	2.635	1.990
2	2.700	2.045	15	2.630	1.990
3	2.690	2.050	16	2.625	1.995
4	2.680	2.005	17	2.625	1.985
5	2.675	2.035	18	2.620	1.970
6	2.670	2.035	19	2.615	1.985
7	2.665	2.020	20	2.615	1.990
8	2.660	2.005	21	2.615	1.995
9	2.655	2.010	22	2.610	1.990
10	2.655	2.000	23	2.590	1.975
11	2.650	2.000	24	2.590	1.995
12	2.650	2.005	25	2.565	1.955
13	2.645	2.015			



1. Propose a statistical model.
2. Propose a statistical question that the company want to answer?

3 To go further

Exercise 7. Autoregressive model
Let us consider the following observation $Z = (X_1, ..., X_n)$, where X_i are generated through an autoregressive model:

$$X_i = \theta X_{i-1} + \xi_i, \quad i = 1, ..., n, \quad X_0 = 0,$$

where ξ_i are i.i.d. $\mathcal{N}(0, \sigma^2)$ and $\theta \in \mathbb{R}$. Write the statistical model given by the observation Z.

Exercise 8. Bayesian modeling
An exam has 10 questions, each with 3 possible answers. Assume that students, who are prepared correctly, answer correctly each question with probability 0.8 and that the other students answer at random. The score S of a student is the sum of the points when 1 is attributed for each correct answer (0 otherwise). We would like to know if the student is prepared.

1. Characterize the distribution of S if the student is prepared and if he is not.
2. Give the statistical model associated to this experiment.

The previous years, 70% of the students were prepared.

3. Which prior distribution would you choose? (this defines θ, see 5.)
4. What is the posterior probability that the student is prepared given that its score is 5? Deduce the posterior distribution given the observation $S = 5$?

Another exam has 15 questions.

5. What is the posterior distribution of θ given a score 8?

We now consider a exam with n questions.

6. What is the posterior distribution of θ given a score s?
7. What happens when s increases?
8. Consider a score proportional to n: $s = cn$ with $c \in [0, 1]$. Compute the posterior distribution given a score $s = cn$. What happens when n tends to $+\infty$?