PC7 – Hypothesis testing

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Exercise 6: Urn

A urn contains N balls, numbered from 1 to N where $N \in \{99, 101\}$ is unknown. The following balls, which were placed back in the urn after each drawing, were drawn:

12	52	48	60	17	03	25	98	15	51	12	38	76
25	64	87	23	19	05	82	46	73	09	50	48	34.

Propose a rule to test H_0: $N = 99$ versus H_1: $N = 101$, following all the steps described in Exercise 1. Besides compute the probability of type II error of your decision rule.

Let $x_1, x_2, \cdots, x_n$ be the number of balls drawn from the urn n times independently. We assume that they are realizations of i.i.d. random variables $X_1, X_2, \cdots, X_n$, distributed as Discrete Uniform distribution $\mathcal{U}\{1, N\}$, where N is the unknown maximum number on balls. To test H_0: $N = 99$ versus H_1: $N = 101$, a natural estimator (a test statistic) to approximate the parameter N is $T(X) = \max\{X_1, \cdots, X_n\}$. We can write the cumulatice distribution function of $T(X)$ as:

$$\begin{aligned}\mathbb{P}(T(X) \leq t) &= \mathbb{P}(\max\{X_1, \cdots, X_n\} \leq t) = \mathbb{P}(X_1 \leq t, X_2 \leq t, \cdots, X_n \leq t) \\ &= [\mathbb{P}(X_1 \leq t)]^n = \begin{cases} 1 & \text{if } t < 1 \\ \left(\frac{t}{N}\right)^n & \text{if } t \in [1, N] \\ 0 & \text{otherwise} \end{cases} .\end{aligned}$$

We want to reject H_0 if the maximum observed number is large so that the rejection region can be right-sided $\{N_c, N_c + 1, \cdots\}$. We decide to choose N_c such that the probability of wrongly rejecting the true null hypothesis H_0, i.e, the probability of the type I risk, is smaller than the significance level $\alpha = 0.05$.

$$\mathbb{P}_{H_0}(T(X) \geq N_c) = 1 - \left(\frac{N_c - 1}{99}\right)^{26}$$

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Nc = 99
1 - ((Nc-1)/99)^26
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## [1] 0.231997
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Nc = 100
1 - ((Nc-1)/99)^26
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## [1] 0
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So we choose $N_c = 100$.

Finally, we reject H_0 when the maximum observed number on balls is larger or equal to 100. With this rule, we will never reject H_0 when H_0 is true. And the probability of type II error is:

$$1 - \mathbb{P}_{H_1}(T(X) \geq N_c) = \left(\frac{99}{101}\right)^{26} = 0.59$$

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(99/101)^26
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## [1] 0.5945102
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Remark: If we choose a different shape for rejection region, for example a single point $\{N_c\}$, then we choose the value of N_c s.t. the probability of the type I risk, as close as possible and smaller than the significance level $\alpha = 0.05$:

$$\mathbb{P}_{H_0}(T(X) = N_c) = \left(\frac{N_c}{99}\right)^{26} - \left(\frac{N_c - 1}{99}\right)^{26}$$

```r
Nc = 94
(Nc/99)^26 - ((Nc-1)/99)^26
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```
## [1] 0.06309423
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```r
Nc = 93
(Nc/99)^26 - ((Nc-1)/99)^26
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## [1] 0.04822471
```

So we choose $N_c = 93$. And the type II error is:

$$1 - \mathbb{P}_{H_1}(T(X) = N_c) = 1 - \left[\left(\frac{93}{101} \right)^{26} - \left(\frac{93-1}{101} \right)^{26} \right] = 0.97 > 0.59$$

```r
1-((93/101)^26 - ((93-1)/101)^26)
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## [1] 0.9713299
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We conclude that this test has larger type II error, i.e., it is not as powerful as the previous one.

In fact, according to the Neyman-Pearson Theorem, the likelihood ratio test is uniformly most powerful. You can try to formulate this test. Be careful, it is not easy to compute the rejection region here because the test statistic, i.e, the likelihood ratio contains two indicaters for $\max\{X_1, X_2, \cdots, X_n\}$.