Matrices

Exercise 21.

1. Find orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ such that $\boldsymbol{q}_1, \boldsymbol{q}_2$ span the column space of

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}.$$

Correction :

We have $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ so $\boldsymbol{q}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Then, one has $\boldsymbol{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ thus

$$\widetilde{\boldsymbol{q}}_2 = \boldsymbol{a}_2 - \boldsymbol{q}_1^T \boldsymbol{a}_2 \boldsymbol{q}_1 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} - \frac{1-2-8}{3}\frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

We obtain

$$\boldsymbol{q}_2 = \frac{\widetilde{\boldsymbol{q}}_2}{\|\widetilde{\boldsymbol{q}}_2\|} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

We choose $\boldsymbol{a}_3$ such that the vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ are linearly independent, for example

$$\boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then we have

$$\widetilde{\boldsymbol{q}}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^T \boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^T \boldsymbol{a}_3)\boldsymbol{q}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{9}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \frac{2}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$$

and

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{q}}_3}{\|\widetilde{\boldsymbol{q}}_3\|} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}.$$

2. Solve by least squares:

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}.$$

Correction :

We want to solve $\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$ that is if we have $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$ it is equivalent to solve $\boldsymbol{R}\widehat{\boldsymbol{x}} = \boldsymbol{Q}^T\boldsymbol{b}$. Thanks to the previous computations, one has

$$\boldsymbol{Q} = (\boldsymbol{q}_1|\boldsymbol{q}_2) = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -2 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{R} = \begin{pmatrix} \|\widetilde{\boldsymbol{q}}_1\| & \langle \boldsymbol{q}_1, \boldsymbol{a}_2\rangle \\ 0 & \|\widetilde{\boldsymbol{q}}_2\| \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}.$$

Thus we obtain

$$\boldsymbol{R}\widehat{\boldsymbol{x}} = \boldsymbol{Q}^T\boldsymbol{b} \Leftrightarrow \begin{cases} 3\widehat{x}_1 - 3\widehat{x}_2 = -3 \\ 3\widehat{x}_2 = 6 \end{cases} \Leftrightarrow \widehat{\boldsymbol{x}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Exercise 22. Write the $\boldsymbol{Q}\boldsymbol{R}$ factorization of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix}.$$

Correction :

First, we have $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and so $\boldsymbol{q}_1 = \boldsymbol{a}_1$.

Then, one has $\boldsymbol{a}_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ so

$$\widetilde{\boldsymbol{q}}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

Thus, we obtain

$$\boldsymbol{q}_2 = \frac{\widetilde{\boldsymbol{q}}_2}{\|\widetilde{\boldsymbol{q}}_2\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

To finish we have $\boldsymbol{a}_3 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. Thus we have

$$\widetilde{\boldsymbol{q}}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^T\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^T\boldsymbol{a}_3)\boldsymbol{q}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - 4\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 6\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$$

and

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{q}}_3}{\|\widetilde{\boldsymbol{q}}_3\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Thanks to the previous computations, one has

$$\boldsymbol{Q} = (\boldsymbol{q}_1|\boldsymbol{q}_2|\boldsymbol{q}_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{R} = \begin{pmatrix} \|\widetilde{\boldsymbol{q}}_1\| & \langle \boldsymbol{q}_1, \boldsymbol{a}_2\rangle & \langle \boldsymbol{q}_1, \boldsymbol{a}_3\rangle \\ 0 & \|\widetilde{\boldsymbol{q}}_2\| & \langle \boldsymbol{q}_2, \boldsymbol{a}_3\rangle \\ 0 & 0 & \|\widetilde{\boldsymbol{q}}_3\| \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}.$$

Exercise 27. Let $\boldsymbol{A} = (a_{i,j})_{i,j=1,n} \in \mathcal{M}_n(\mathbb{R})$.

1. Prove that
$$\|\boldsymbol{A}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{i,j}| \quad \text{and} \quad \|\boldsymbol{A}\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{i,j}|.$$
2. Prove that
$$\|\boldsymbol{A}\|_2 = \sqrt{\rho(\boldsymbol{A}^T\boldsymbol{A})} \quad \text{and} \quad \|\boldsymbol{A}^T\|_2 = \sqrt{\rho(\boldsymbol{A}\boldsymbol{A}^T)} = \|\boldsymbol{A}\|_2.$$

Correction :

1. • By definition one has
$$\|\boldsymbol{A}\|_\infty = \sup_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_\infty=1} \|\boldsymbol{A}\boldsymbol{x}\|_\infty.$$
For any $j \in \{1, \cdots, n\}$, we have
$$\left|\sum_{j=1}^n a_{i,j}x_j\right| \le \sum_{j=1}^n |a_{i,j}||x_j| \le \|\boldsymbol{x}\|_\infty \sum_{j=1}^n |a_{i,j}|.$$
Thus, if $\|\boldsymbol{x}\|_\infty = 1$ we obtain
$$\|\boldsymbol{A}\boldsymbol{x}\|_\infty = \max_{i=1,\cdots,n} \left|\sum_{j=1}^n a_{i,j}x_j\right| \le \max_{i=1,\cdots,n} \sum_{j=1}^n |a_{i,j}|.$$
Now, we want to prove that there exists $\boldsymbol{x} \in \mathbb{R}^n$, with $\|\boldsymbol{x}\|_\infty = 1$, such that
$$\|\boldsymbol{A}\boldsymbol{x}\|_\infty = \max_{i=1,\cdots,n} \sum_{j=1}^n |a_{i,j}|.$$
To this end we consider the index i_0 such that
$$\sum_{j=1}^n |a_{i_0,j}| = \max_{i=1,\cdots,n} \sum_{j=1}^n |a_{i,j}|$$
and we choose $\boldsymbol{x}_0 = (\mathrm{sign}(\mathrm{a_{i_0,j}}))_j$.
Thus we have $\|\boldsymbol{x}_0\|_\infty = 1$ and
$$\|\boldsymbol{A}\boldsymbol{x}_0\|_\infty = \max_{i=1,\cdots,n} \left|\sum_{j=1}^n a_{i,j}\mathrm{sign}(\mathrm{a_{i_0,j}})\right| \ge \left|\sum_{j=1}^n a_{i_0,j}\mathrm{sign}(\mathrm{a_{i_0,j}})\right| = \sum_{j=1}^n |a_{i_0,j}| = \max_{i=1,\cdots,n} \sum_{j=1}^n |a_{i,j}|.$$
• By definition we have
$$\|\boldsymbol{A}\|_1 = \sup_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|_1=1} \|\boldsymbol{A}\boldsymbol{x}\|_1$$
and
$$\|\boldsymbol{A}\boldsymbol{x}\|_1 = \sum_{i=1}^n \left|\sum_{j=1}^n a_{i,j}x_j\right| \le \sum_{j=1}^n |x_j| \left(\sum_{i=1}^n |a_{i,j}|\right) \le \max_{j=1,\cdots,n} \sum_{i=1}^n |a_{i,j}| \sum_{j=1}^n |x_j|.$$
Since $\sum_{j=1}^n |x_j| = 1$, we obtain
$$\|\boldsymbol{A}\|_1 \le \max_{j=1,\cdots,n} \sum_{i=1}^n |a_{i,j}|.$$

Now, we want to prove that there exists $\boldsymbol{x} \in \mathbb{R}^n$, with $\|\boldsymbol{x}\|_1 = 1$, such that

$$\|\boldsymbol{A}\boldsymbol{x}\|_1 = \max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{i,j}|.$$

To this end it is sufficient to consider the vector $\boldsymbol{x} \in \mathbb{R}^n$ defined by $x_{j_0} = 1$ and $x_j = 0$ if $j \neq j_0$, where j_0 is such that

$$\sum_{i=1}^{n} |a_{i,j_0}| = \max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{i,j}|.$$

Then, it is easy to check that we have $\|\boldsymbol{A}\boldsymbol{x}\|_1 = \max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{i,j}|$.

2. By definition one has

$$\|\boldsymbol{A}\|_2^2 = \sup_{\boldsymbol{x}\in\mathbb{R}^n,\ \|\boldsymbol{x}\|_2=1} \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}\rangle = \sup_{\boldsymbol{x}\in\mathbb{R}^n,\ \|\boldsymbol{x}\|_2=1} \langle \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}\rangle.$$

Since $\boldsymbol{A}^T\boldsymbol{A}$ is a symmetric positive definite matrix (because $\langle \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}\rangle = \langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x}\rangle \geq 0$), there exists an orthonormal basis $(\boldsymbol{f}_i)_{i=1,\cdots,n}$ and eigenvectors $(\lambda_i)_{i=1,\cdots,n}$, with $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ such that $\boldsymbol{A}\boldsymbol{f}_i = \lambda_i \boldsymbol{f}_i$ for any $i \in \{1, \cdots, n\}$.

Let $\boldsymbol{x} = \sum_{i=1}^{n} \alpha_i \boldsymbol{f}_i \in \mathbb{R}^n$, we have

$$\langle \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x}\rangle = \left\langle \sum_{i=1}^{n} \lambda_i \alpha_i \boldsymbol{f}_i, \sum_{i=1}^{n} \alpha_i \boldsymbol{f}_i \right\rangle = \sum_{i=1}^{n} \lambda_i \alpha_i^2 \leq \lambda_n \|\boldsymbol{x}\|_2^2.$$

Thus we deduce that $\|\boldsymbol{A}\|_2^2 \leq \rho(\boldsymbol{A}^T\boldsymbol{A})$.

To obtain the equality, we consider the vector $\boldsymbol{x} = \boldsymbol{f}_n$. Indeed, one has $\|\boldsymbol{f}_n\|_2 = 1$, and $\|\boldsymbol{A}\boldsymbol{f}_n\|_2^2 = \langle \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{f}_n, \boldsymbol{f}_n\rangle = \lambda_n = \rho(\boldsymbol{A}^T\boldsymbol{A})$.

Exercise 28. For $\boldsymbol{A} = (a_{i,j})_{i,j=1,n} \in \mathcal{M}_n(\mathbb{R})$, we set $\|\boldsymbol{A}\|_F = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{\frac{1}{2}}$.
Show that $\|\boldsymbol{A}\|_F^2 = \mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{A})$. Deduce $\|\boldsymbol{A}\|_2 \leq \|\boldsymbol{A}\|_F \leq \sqrt{n}\|\boldsymbol{A}\|_2$ and $\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\|_F\|\boldsymbol{x}\|_2$, for any $\boldsymbol{x} \in \mathbb{R}^n$.
Correction :
By noticing that $(\boldsymbol{A}^T\boldsymbol{A})_{i,i} = \sum_{k=1}^{n} a_{k,i}^2$, we easily obtain

$$\mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{A}) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{k,i}^2 = \|\boldsymbol{A}\|_F^2.$$

Moreover, $\|\boldsymbol{A}\|_2^2 = \rho(\boldsymbol{A}^T\boldsymbol{A}) = \lambda_n$ where λ_n is the largest eigenvalue of the matrix $\boldsymbol{A}^T\boldsymbol{A}$. But the trace of a diagonalizable matrix is also the sum of its eigenvalues. Thus we have

$$\|\boldsymbol{A}\|_2^2 \leq \sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{A}).$$

We conclude that

$$\|\boldsymbol{A}\|_2 \leq \|\boldsymbol{A}\|_F.$$

Furthermore, $\|\boldsymbol{A}\|_F^2 = \mathrm{Tr}(\boldsymbol{A}^T\boldsymbol{A}) \leq n\rho(\boldsymbol{A}^T\boldsymbol{A})$ and so $\|\boldsymbol{A}\|_F \leq \sqrt{n}\|\boldsymbol{A}\|_2$.
Finally, since $\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\|_2\|\boldsymbol{x}\|_2$, we deduce that $\|\boldsymbol{A}\boldsymbol{x}\|_2 \leq \|\boldsymbol{A}\|_F\|\boldsymbol{x}\|_2$.