2017-2018 **Master X-HEC – MAP 532**
Ecole polytechnique **Mathematical foundations of datascience**

Exam Session - Thursday 9th November 2017

Exercise 1.

1. Let $\boldsymbol{M} \in \mathcal{M}_n(\mathbb{R})$. Compute the following matrix product:

$$(\boldsymbol{I}_n - \boldsymbol{M})(\boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2).$$

Correction :

We have

$$(\boldsymbol{I}_n - \boldsymbol{M})(\boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2) = \boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2 - \boldsymbol{M} - \boldsymbol{M}^2 - \boldsymbol{M}^3 = \boldsymbol{I}_n - \boldsymbol{M}^3.$$

2. We consider the following matrix $\boldsymbol{M}$:

$$\boldsymbol{M} = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

(a) Compute $\boldsymbol{M}^2$, $\boldsymbol{M}^3$, and $\boldsymbol{M}^n$ for any n.

Correction :

We have

$$\boldsymbol{M}^2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{M}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we also obtain for any $n > 3$,

$$\boldsymbol{M}^n = \boldsymbol{M}^3 \boldsymbol{M}^{n-3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Deduce that the matrix $\boldsymbol{I}_n - \boldsymbol{M}$ is invertible and give its inverse.

Correction :

Since $\boldsymbol{M}^3 = \boldsymbol{0}$ the first question gives,

$$(\boldsymbol{I}_n - \boldsymbol{M})(\boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2) = \boldsymbol{I}_n.$$

Thus the matrix $\boldsymbol{I}_n - \boldsymbol{M}$ is invertible and its inverse is given by the matrix:

$$\boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2 = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Exercise 2.
We consider the three following vectors of $\mathbb{R}^3$:

$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1. Prove that the family of vectors $\mathcal{B}' = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ is a basis for $\mathbb{R}^3$.

   Correction :

   Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\lambda_1 \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2 + \lambda_3 \boldsymbol{u}_3 = 0$, that is

$$\begin{cases} \lambda_1 \quad\quad + \lambda_3 = 0 \\ -\lambda_1 + \lambda_2 + \lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases} \iff \begin{cases} \lambda_1 \quad\quad + \lambda_3 = 0 \\ \lambda_2 + 2\lambda_3 = 0 \\ \lambda_2 + \lambda_3 = 0 \end{cases} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

   The three vectors of $\mathbb{R}^3$ are linearly independent in $\mathbb{R}^3$ and so it is a basis for $\mathbb{R}^3$.

2. Write the change-of-basis matrix $\boldsymbol{P}$ from the standard basis $\mathcal{B}$ to $\mathcal{B}'$.

   Correction :

   We have,

$$\boldsymbol{u}_1 = \boldsymbol{e}_1 - \boldsymbol{e}_2, \quad \boldsymbol{u}_2 = \boldsymbol{e}_2 + \boldsymbol{e}_3, \quad \boldsymbol{u}_3 = \boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3$$

   thus

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. Write the change-of-basis matrix $\boldsymbol{P}'$ from $\mathcal{B}'$ to the standard basis $\mathcal{B}$. What is the relation between $\boldsymbol{P}$ and $\boldsymbol{P}'$?

   Correction :

   We have,

$$\boldsymbol{e}_1 = -\boldsymbol{u}_2 + \boldsymbol{u}_3, \quad \boldsymbol{e}_2 = -\boldsymbol{u}_1 - \boldsymbol{u}_2 + \boldsymbol{u}_3, \quad \boldsymbol{e}_3 = \boldsymbol{u}_1 + 2\boldsymbol{u}_2 - \boldsymbol{u}_3$$

   thus

$$\boldsymbol{P}' = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{pmatrix}.$$

   Moreover we have $\boldsymbol{P}\boldsymbol{P}' = \boldsymbol{P}'\boldsymbol{P} = \boldsymbol{I}_3$.

4. We consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$$f(x, y, z) = (-y + z, x + 2y - 3z, x + y - 2z).$$

   Write the matrix representation $\boldsymbol{A}$ of f in the standard basis.

   Correction :

   We compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$ in function of $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$ and $\boldsymbol{e}_3 = (0, 0, 1)$. We have,

$$\begin{cases} f(\boldsymbol{e}_1) = (0, 1, 1) = \boldsymbol{e}_2 + \boldsymbol{e}_3, \\ f(\boldsymbol{e}_2) = (-1, 2, 1) = -\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + \boldsymbol{e}_3, \\ f(\boldsymbol{e}_3) = (1, -3, -2) = \boldsymbol{e}_1 - 3\boldsymbol{e}_2 - 2\boldsymbol{e}_3. \end{cases}$$

Thus the matrix representation with respect to the basis $\mathcal{B}$ is

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix}.$$

5. Write the matrix representation $\boldsymbol{A}'$ of f in the basis $\mathcal{B}'$.

   Correction :

   We have,

$$\begin{cases} f(\boldsymbol{u}_1) = (1,-1,0) = \boldsymbol{u}_1, \\ f(\boldsymbol{u}_2) = (0,-1,-1) = -\boldsymbol{u}_2, \\ f(\boldsymbol{u}_3) = (0,0,0). \end{cases}$$

   Thus the matrix representation with respect to the basis $\mathcal{B}'$ is

$$\boldsymbol{A}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

6. Give a basis for $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. What is their dimension?

   Correction :

   Let $(x,y,z) \in \mathrm{Ker}(f)$ then,

$$\begin{cases} -y+z=0 \\ x+2y-3z=0 \\ x+y-2z=0 \end{cases} \iff \begin{cases} x+2y-3z=0 \\ -y+z=0 \\ y-z=0 \end{cases} \iff x=y=z.$$

   So $\dim(\mathrm{Ker}(f)) = 1$ and we have $f(\boldsymbol{u}_3) = \boldsymbol{0}$ so $\boldsymbol{u}_3 \in \mathrm{Ker}(f)$ and it is a basis for $\mathrm{Ker}(f)$.

   OR

   Thanks to question 5. we see that $f(\boldsymbol{u}_1) = \boldsymbol{u}_1$, $f(-\boldsymbol{u}_2) = \boldsymbol{u}_2$ and $f(\boldsymbol{u}_3) = \boldsymbol{0}$ thus we have $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathrm{Im}(f)$ and $\boldsymbol{u}_3 \in \mathrm{Ker}(f)$. Moreover we easily see that the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly independent in $\mathbb{R}^3$ (they are not collinear), thus we get:

$$\dim(\mathrm{Ker}(f)) \geq 1 \quad \text{and} \quad \dim(\mathrm{Im}(f)) \geq 2.$$

   Furthermore, thanks to the rank nullity theorem we have:

$$\dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = 3,$$

   thus

$$\dim(\mathrm{Ker}(f)) = 1 \quad \text{and} \quad \dim(\mathrm{Im}(f)) = 2.$$

   To conclude, the vector $\boldsymbol{u}_3$ is a basis for $\mathrm{Ker}(f)$ and the family of vectors $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is a basis for $\mathrm{Im}(f)$.

7. Do we have $\mathrm{Ker}(f) \oplus \mathrm{Im}(f) = \mathbb{R}^3$?

   Correction :

   First, we have

$$\dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f)) = 3.$$

Moreover, let $\boldsymbol{X} = (x, y, z) \in \mathrm{Ker}(f) \cap \mathrm{Im}(f)$ then there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\boldsymbol{X} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} \beta \\ -\beta + \gamma \\ \gamma \end{pmatrix} \implies \alpha = \beta = \gamma = -\beta + \gamma \implies \alpha = \beta = \gamma = 0.$$

Thus $\boldsymbol{X} = \boldsymbol{0}$ that is $\mathrm{Ker}(f) \cap \mathrm{Im}(f) = \{\boldsymbol{0}\}$ and so $\mathrm{Ker}(f) \oplus \mathrm{Im}(f) = \mathbb{R}^3$.

Exercise 3.

1. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{R})$ be an idempotent and symmetric matrix. Prove that $\boldsymbol{A}$ is positive semi-definite.

   Correction :

   We have $\boldsymbol{A}^T = \boldsymbol{A}$ and $\boldsymbol{A}^2 = \boldsymbol{A}$. Let $\boldsymbol{X} \in \mathbb{R}^n$, then one has

   $$\langle \boldsymbol{AX}, \boldsymbol{X} \rangle = \langle \boldsymbol{A}^2\boldsymbol{X}, \boldsymbol{X} \rangle = \langle \boldsymbol{AX}, \boldsymbol{A}^T\boldsymbol{X} \rangle = \langle \boldsymbol{AX}, \boldsymbol{AX} \rangle = \|\boldsymbol{AX}\|^2 \geq 0$$

   and so $\boldsymbol{A}$ is positive semi-definite.

2. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{R})$. Prove that $\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T$ is symmetric positive definite.

   Correction :

   First we prove that $\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T$ is symmetric:

   $$(\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T)^T = \boldsymbol{I}_n^T + (\boldsymbol{A}\boldsymbol{A}^T)^T = \boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T.$$

   Furthermore, let $\boldsymbol{X} \in \mathbb{R}^n$ then we have,

   $$\begin{aligned} \big\langle (\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T)\boldsymbol{X}, \boldsymbol{X} \big\rangle &= \langle \boldsymbol{X}, \boldsymbol{X} \rangle + \langle \boldsymbol{A}\boldsymbol{A}^T\boldsymbol{X}, \boldsymbol{X} \rangle = \langle \boldsymbol{X}, \boldsymbol{X} \rangle + \langle \boldsymbol{A}^T\boldsymbol{X}, \boldsymbol{A}^T\boldsymbol{X} \rangle \\ &= \|\boldsymbol{X}\|^2 + \|\boldsymbol{A}^T\boldsymbol{X}\|^2 \geq 0. \end{aligned}$$

   Thus the matrix $\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T$ is positive semi-definite. Moreover, we have a sum of non-negative terms and so the sum is zero if and only if

   $$\|\boldsymbol{X}\|^2 = \|\boldsymbol{A}^T\boldsymbol{X}\|^2 = 0,$$

   that is $\boldsymbol{X} = 0$ and so the matrix $\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T$ is positive definite.

3. Show that if λ is an eigenvalue of an orthogonal matrix, then so is $\frac{1}{\lambda}$.

   Correction :

   Let $\boldsymbol{Q}$ be an orthogonal matrix that is $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$. Let λ be an eigenvalue of $\boldsymbol{Q}$ and $\boldsymbol{X}$ an associated eigenvector, then:

   $$\boldsymbol{QX} = \lambda\boldsymbol{X} \implies \boldsymbol{Q}^T\boldsymbol{QX} = \lambda\boldsymbol{Q}^T\boldsymbol{X} \implies \boldsymbol{X} = \lambda\boldsymbol{Q}^T\boldsymbol{X}.$$

   Since $\boldsymbol{Q}$ is an orthogonal matrix, $\lambda \neq 0$ thus:

   $$\boldsymbol{Q}^T\boldsymbol{X} = \frac{1}{\lambda}\boldsymbol{X},$$

   and $\frac{1}{\lambda}$ is an eigenvalue of $\boldsymbol{Q}^T$. We conclude by noting that $\boldsymbol{Q}$ and $\boldsymbol{Q}^T$ have the same eigenvalues.