Exam Session - Thursday 9th November 2017

Exercise 1.

1. Let $\boldsymbol{M} \in \mathcal{M}_n(\mathbb{R})$. Compute the following matrix product:
$$(\boldsymbol{I}_n - \boldsymbol{M})(\boldsymbol{I}_n + \boldsymbol{M} + \boldsymbol{M}^2).$$
2. We consider the following matrix $\boldsymbol{M}$:
$$\boldsymbol{M} = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$
   (a) Compute $\boldsymbol{M}^2$, $\boldsymbol{M}^3$, and $\boldsymbol{M}^n$ for any n.
   (b) Deduce that the matrix $\boldsymbol{I}_n - \boldsymbol{M}$ is invertible and give its inverse.

Exercise 2.
We consider the three following vectors of $\mathbb{R}^3$:
$$\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1. Prove that the family of vectors $\mathcal{B}' = (\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ is a basis for $\mathbb{R}^3$.
2. Write the change-of-basis matrix $\boldsymbol{P}$ from the standard basis $\mathcal{B}$ to $\mathcal{B}'$.
3. Write the change-of-basis matrix $\boldsymbol{P}'$ from $\mathcal{B}'$ to the standard basis $\mathcal{B}$. What is the relation between $\boldsymbol{P}$ and $\boldsymbol{P}'$?
4. We consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:
$$f(x, y, z) = (-y + z, x + 2y - 3z, x + y - 2z).$$
   Write the matrix representation $\boldsymbol{A}$ of f in the standard basis.
5. Write the matrix representation $\boldsymbol{A}'$ of f in the basis $\mathcal{B}'$.
6. Give a basis for $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. What is their dimension?
7. Do we have $\mathrm{Ker}(f) \oplus \mathrm{Im}(f) = \mathbb{R}^3$?

Exercise 3.

1. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{R})$ be an idempotent and symmetric matrix. Prove that $\boldsymbol{A}$ is positive semi-definite.
2. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{R})$. Prove that $\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{A}^T$ is symmetric positive definite.
3. Show that if λ is an eigenvalue of an orthogonal matrix, then so is $\frac{1}{\lambda}$.