Exam Session - Tuesday 25th September 2018 - 14h-16h

Exercise 1.
We consider the following matrices

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \boldsymbol{C} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

1. Compute the matrix products $\boldsymbol{AB}$ and $\boldsymbol{AC}$.

   Correction :

   We have
   $$\boldsymbol{AB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix} \text{ and } \boldsymbol{AC} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 4 & 4 & 3 \end{pmatrix}.$$
   Thus $\boldsymbol{AB} = \boldsymbol{AC}$.

2. Without computation, determine whether the matrix $\boldsymbol{A}$ is invertible or not.

   Correction :

   If $\boldsymbol{A}$ is invertible then the equality $\boldsymbol{AB} = \boldsymbol{AC}$ implies that $\boldsymbol{B} = \boldsymbol{C}$. Thus the matrix $\boldsymbol{A}$ is not invertible.

Exercise 2.

1. Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$ with
   $$\boldsymbol{a} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix}.$$

   Correction :

   Recalling that the projection $\boldsymbol{p}$ of the vector $\boldsymbol{b}$ satisfies $\boldsymbol{p} = \widehat{\boldsymbol{x}}\boldsymbol{a}$ with
   $$\widehat{\boldsymbol{x}} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{6+8+4}{4+4+1} = \frac{18}{9} = 2.$$
   Then, one has
   $$\boldsymbol{p} = 2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

2. Compute the projection matrix $\boldsymbol{P}$ onto the column space of the matrix $\boldsymbol{A}$ with

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Correction :

The projection matrix $\boldsymbol{P}$ satisfies

$$\begin{aligned}
\boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \frac{1}{15-9} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 & -1 \\ -3 & 0 & 3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}.
\end{aligned}$$

Exercise 3.

We denote by $\mathcal{B} = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ the standard basis of $\mathbb{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$f(x, y, z) = (x - y + 2z, -2x + y - 3z, -x + y - 2z), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

1. Write the matrix representation $\boldsymbol{A}$ of f in the standard basis $\mathcal{B}$.

   Correction :

   We compute $f(\boldsymbol{e}_1)$, $f(\boldsymbol{e}_2)$ and $f(\boldsymbol{e}_3)$ in function of $\boldsymbol{e}_1 = (1, 0, 0)$, $\boldsymbol{e}_2 = (0, 1, 0)$ and $\boldsymbol{e}_3 = (0, 0, 1)$. We have,

$$\begin{cases} f(\boldsymbol{e}_1) = (1, -2, -1) = \boldsymbol{e}_1 - 2\boldsymbol{e}_2 - \boldsymbol{e}_3, \\ f(\boldsymbol{e}_2) = (-1, 1, 1) = -\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3, \\ f(\boldsymbol{e}_3) = (2, -3, -2) = 2\boldsymbol{e}_1 - 3\boldsymbol{e}_2 - 2\boldsymbol{e}_3. \end{cases}$$

   Thus the matrix representation with respect to the basis $\mathcal{B}$ is

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & -3 \\ -1 & 1 & -2 \end{pmatrix}.$$

2. We recall that for any vector $\boldsymbol{u}$, $f^2(\boldsymbol{u}) = f(f(\boldsymbol{u}))$ and $f^3(\boldsymbol{u}) = f(f(f(\boldsymbol{u})))$.

   Compute $f(\boldsymbol{u})$, $f^2(\boldsymbol{u})$ and $f^3(\boldsymbol{u})$ with

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

   Correction :

   One has

$$\begin{cases} \boldsymbol{v} = f(\boldsymbol{u}) = (0, -1, 0), \\ \boldsymbol{w} = f^2(\boldsymbol{u}) = f(\boldsymbol{v}) = (1, -1, -1) \\ f^3(\boldsymbol{u}) = f(\boldsymbol{w}) = (0, 0, 0). \end{cases}$$

3. Prove that the family of vectors $\mathcal{B}' = (\boldsymbol{u}, f(\boldsymbol{u}), f^2(\boldsymbol{u}))$ is a basis for $\mathbb{R}^3$.

   Correction :

   Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ be such that $\lambda_1 \boldsymbol{u} + \lambda_2 f(\boldsymbol{u}) + \lambda_3 f^2(\boldsymbol{u}) = 0$, that is

$$\begin{cases} \lambda_1 \phantom{{}-\lambda_2} + \lambda_3 = 0 \\ \lambda_1 - \lambda_2 - \lambda_3 = 0 \\ - \lambda_3 = 0 \end{cases} \iff \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

   The three vectors of $\mathbb{R}^3$ are linearly independent in $\mathbb{R}^3$ and so it is a basis for $\mathbb{R}^3$.

4. Write the matrix representation $\boldsymbol{A}'$ of f in the basis $\mathcal{B}'$.

   Correction :

   Thanks to the definition of the vector is the basis $\mathcal{B}'$ we immediately obtain the matrix representation with respect to the basis $\mathcal{B}'$

$$\boldsymbol{A}' = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

5. Write the change-of-basis matrix $\boldsymbol{P}$ from the standard basis $\mathcal{B}$ to $\mathcal{B}'$.

   Correction :

   We have

$$\boldsymbol{u} = \boldsymbol{e}_1 + \boldsymbol{e}_2, \quad f(\boldsymbol{u}) = -\boldsymbol{e}_2, \quad f^2(\boldsymbol{u}) = \boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3.$$

   Thus we obtain

$$\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

6. Write the matrix $\boldsymbol{P}^{-1}$.

   Correction :

   The matrix $\boldsymbol{P}^{-1}$ is the change-of-basis matrix from the basis $\mathcal{B}'$ to the standard basis $\mathcal{B}$. One has

$$\boldsymbol{e}_1 = \boldsymbol{u} + f(\boldsymbol{u}), \quad \boldsymbol{e}_2 = -f(\boldsymbol{u}), \quad \boldsymbol{e}_3 = \boldsymbol{u} + 2f(\boldsymbol{u}) - f^2(\boldsymbol{u})$$

   thus

$$\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}.$$

7. Give a basis for $\mathrm{Ker}(\boldsymbol{A})$ and $\mathrm{Im}(\boldsymbol{A})$. What is their dimension?

   Correction :

   Thanks to the shape of the matrix $\boldsymbol{A}'$, we easily obtain

$$\dim(\mathrm{Ker}(\boldsymbol{A})) = \dim(\mathrm{Ker}(\boldsymbol{A}')) = 1.$$

   Moreover the vector $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ belongs to $\mathrm{Ker}(\boldsymbol{A})$, thus it is a basis for $\mathrm{Ker}(\boldsymbol{A})$.

Then, thanks to the rank nullity theorem we have $\operatorname{rank}(\boldsymbol{A}) = 2$. Thus if we denote by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ the columns of the matrix $\boldsymbol{A}$ one has

$$\operatorname{Im}(\boldsymbol{A}) = \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \operatorname{span}(\boldsymbol{a}_1, \boldsymbol{a}_2),$$

because $\boldsymbol{a}_3 = \boldsymbol{a}_1 - \boldsymbol{a}_2$.

Thus, the family of vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ is a basis for $\operatorname{Im}(\boldsymbol{A})$.

8. Do we have $\operatorname{Ker}(\boldsymbol{A}) \oplus \operatorname{Im}(\boldsymbol{A}) = \mathbb{R}^3$?

   Correction :

   The vector $\boldsymbol{a}_2 \in \operatorname{Im}(\boldsymbol{A})$ (by definition) but this vector also belongs to $\operatorname{Ker}(\boldsymbol{A})$.

   Thus $\operatorname{Ker}(\boldsymbol{A}) \cap \operatorname{Im}(\boldsymbol{A}) \neq \{0\}$ and we do not have $\operatorname{Ker}(\boldsymbol{A}) \oplus \operatorname{Im}(\boldsymbol{A}) = \mathbb{R}^3$.

Exercise 4.
Let $\boldsymbol{A} \in \mathcal{M}_{n,p}(\mathbb{R})$.

1. Prove that $\operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A}) = \operatorname{Ker}(\boldsymbol{A})$.

   Correction :

   We remark that $\boldsymbol{A}^T\boldsymbol{A} \in \mathcal{M}_p(\mathbb{R})$ thus $\operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A}) \subset \mathbb{R}^p$ and $\operatorname{Ker}(\boldsymbol{A}) \subset \mathbb{R}^p$.

   Let $\boldsymbol{X} \in \operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A})$, then one has $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{X} = 0$. Thus we obtain

   $$0 = \left\langle \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{X}, \boldsymbol{X} \right\rangle = \langle \boldsymbol{A}\boldsymbol{X}, \boldsymbol{A}\boldsymbol{X} \rangle = \|\boldsymbol{A}\boldsymbol{X}\|^2 \Rightarrow \boldsymbol{A}\boldsymbol{X} = 0,$$

   that is $\boldsymbol{X} \in \operatorname{Ker}(\boldsymbol{A})$ and $\operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A}) \subset \operatorname{Ker}(\boldsymbol{A})$.

   Let $\boldsymbol{X} \in \operatorname{Ker}(\boldsymbol{A})$, then

   $$\boldsymbol{A}\boldsymbol{X} = 0 \Rightarrow \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{X} = 0 \Rightarrow \boldsymbol{X} \in \operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A}),$$

   and so $\operatorname{Ker}(\boldsymbol{A}) \subset \operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A})$.

2. Deduce that $\operatorname{rank}(\boldsymbol{A}^T\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A})$.

   Correction :

   Thanks to the rank nullity theorem one has

   $$\dim\left(\operatorname{Ker}(\boldsymbol{A})\right) + \operatorname{rank}(\boldsymbol{A}) = p \quad \text{and} \quad \dim\left(\operatorname{Ker}(\boldsymbol{A}^T\boldsymbol{A})\right) + \operatorname{rank}(\boldsymbol{A}^T\boldsymbol{A}) = p.$$

   Thus we deduce from the previous question that $\operatorname{rank}(\boldsymbol{A}^T\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A})$.

3. Do we have $\operatorname{Ker}(\boldsymbol{A}\boldsymbol{A}^T) = \operatorname{Ker}(\boldsymbol{A})$? Justify.

   Correction :

   We have $\operatorname{Ker}(\boldsymbol{A}) \subset \mathbb{R}^p$ and $\boldsymbol{A}\boldsymbol{A}^T \in \mathcal{M}_n(\mathbb{R})$ that is $\operatorname{Ker}(\boldsymbol{A}\boldsymbol{A}^T) \subset \mathbb{R}^n$. Thus we cannot obtain $\operatorname{Ker}(\boldsymbol{A}\boldsymbol{A}^T) = \operatorname{Ker}(\boldsymbol{A})$.

Exercise 5.
Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}_n(\mathbb{R})$. Prove that the eigenvalues of $\boldsymbol{BA}$ are the same as those of $\boldsymbol{ABA}$ when $\boldsymbol{A}$ is idempotent.
Correction :
Let λ be an eigenvalue of $\boldsymbol{BA}$ and let $\boldsymbol{X} \neq 0$ be an associated eigenvector. One has

$$\boldsymbol{BAX} = \lambda\boldsymbol{X} \Rightarrow \boldsymbol{ABAX} = \lambda\boldsymbol{AX} \Rightarrow \boldsymbol{ABA}^2\boldsymbol{X} = \lambda\boldsymbol{AX} \Rightarrow \boldsymbol{ABA}(\boldsymbol{AX}) = \lambda\boldsymbol{AX},$$

where we use that $\boldsymbol{A}^2 = \boldsymbol{A}$.

- If $\boldsymbol{AX} \neq 0$, then $\boldsymbol{AX}$ is an eigenvector of the matrix $\boldsymbol{ABA}$ associated to the eigenvalue λ.
- If $\boldsymbol{AX} = 0$, then $\lambda = 0$ (because $\boldsymbol{X} \neq 0$) and $\boldsymbol{ABAX} = 0 = \lambda\boldsymbol{X}$, that is $\boldsymbol{X}$ is an eigenvector of the matrix $\boldsymbol{ABA}$ associated to the eigenvalue $\lambda = 0$.

Now we assume that λ is an eigenvalue of $\boldsymbol{ABA}$ and $\boldsymbol{X} \neq 0$ is an associated eigenvector. Using that $\boldsymbol{A}^2 = \boldsymbol{A}$ we obtain

$$\boldsymbol{ABAX} = \lambda\boldsymbol{X} \Rightarrow \boldsymbol{A}^2\boldsymbol{BAX} = \lambda\boldsymbol{AX} \Rightarrow \boldsymbol{ABAX} = \lambda\boldsymbol{AX} \Rightarrow \boldsymbol{BA}(\boldsymbol{BAX}) = \lambda\boldsymbol{BAX}.$$

Thus we conclude that

- If $\boldsymbol{BAX} \neq 0$, then $\boldsymbol{BAX}$ is an eigenvector of the matrix $\boldsymbol{BA}$ associated to the eigenvalue λ.
- If $\boldsymbol{BAX} = 0$, then $\lambda = 0$ (because $\boldsymbol{X} \neq 0$) and $\boldsymbol{BAX} = 0 = \lambda\boldsymbol{X}$, that is $\boldsymbol{X}$ is an eigenvector of the matrix $\boldsymbol{BA}$ associated to the eigenvalue $\lambda = 0$.

Exercise 6.
We consider the matrix $\boldsymbol{A} = n\boldsymbol{I}_n - \mathbf{1}_n$ where the matrix $\mathbf{1}_n \in \mathcal{M}_n(\mathbb{R})$ is the matrix whose all coefficients are equal to 1.

1. Is the matrix $\boldsymbol{A}$ symmetric positive semi-definite?

   Correction :

   Let $\boldsymbol{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_n(\mathbb{R})$, then

   $$\langle \boldsymbol{AX}, \boldsymbol{X} \rangle = n\|\boldsymbol{X}\|_2^2 - \left(\sum_{i=1}^n x_i\right)^2 = n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2.$$

   Thanks to the Cauchy-Schwarz inequality, one has

   $$\left|\sum_{i=1}^n x_i\right| = \left|\left\langle \boldsymbol{X}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle\right| \leq \|\boldsymbol{X}\|_2 \left\| \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\|_2 = \sqrt{n}\|\boldsymbol{X}\| = \sqrt{n}\left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}.$$

   Thus we deduce that $\langle \boldsymbol{AX}, \boldsymbol{X} \rangle \geq 0$ and the matrix $\boldsymbol{A}$ is symmetric positive semi-definite.

2. Is the matrix $\boldsymbol{A}$ symmetric positive definite?

   Correction :

   We consider the vector $\boldsymbol{X} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. We have

$$\boldsymbol{A}\boldsymbol{X} = n\boldsymbol{X} - \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix} = 0,$$

   and the matrix $\boldsymbol{A}$ is not symmetric positive definite.