MAP 531 : Statistics refresher

21 septembre 2016

1 Decision theory
- Example
- Bias

2 Minimax and Bayes estimators

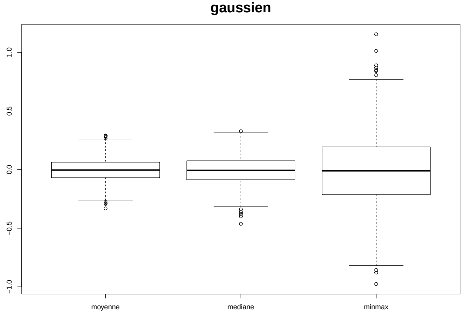
Estimation of the translation parameter

- Let $(X_1, X_2, \dots, X_n)$ be an i.i.d. n-sample with probability density function given by $p_\theta(x) = q(x-\theta)$, $\theta \in \Theta = \mathbb{R}$ where q is symmetric : $q(x) = q(-x)$
- Estimator :
  1. $\hat{\theta}_n^{(1)}(X_1, \dots, X_n) = n^{-1} \sum_{i=}^{n} X_i$ Empirical mean,
  2. $\hat{\theta}_n^{(2)}(X_1, \dots, X_n) = \mathrm{mediane}(X_1, \dots, X_n)$, median.
  3. $\hat{\theta}_n^{(2)}(X_1, \dots, X_n) = 0.5\,(\min(X_1, \dots, X_n) + \max(X_1, \dots, X_n))$.

Gaussian noise

$$q(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

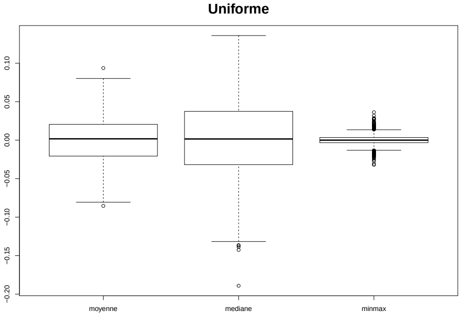
Sample : $n = 100$. Number of sets : $m = 1000$.



Uniforme noise

$$q(x) = \mathbb{1}_{[-1/2,1,2]}(x)$$

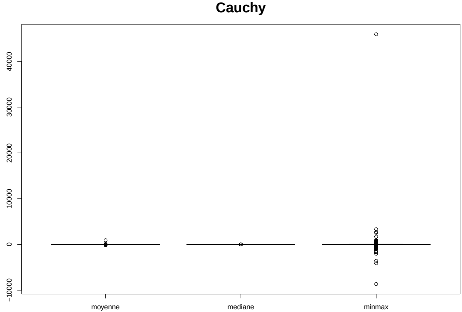
Sample : $n = 100$. Number of sets : $m = 1000$.



Cauchy noise

$$q(x) = \frac{1}{1+x^2}$$

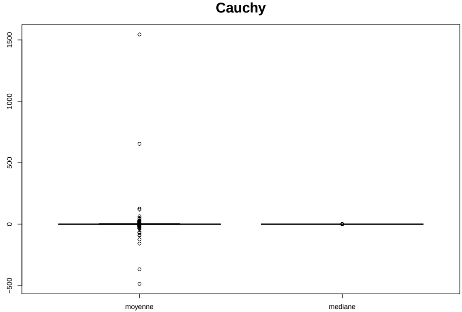
Sample : $n = 100$. Number of sets : $m = 1000$.



Cauchy noise (zoom)

$$q(x) = \frac{1}{1+x^2}$$

Sample : $n = 100$. Number of sets : $m = 1000$.



Estimator of the variance in a translation-scaling model

Let $X_1, X_2, \ldots, X_n$ n be real random variables i.i.d. with p.d.f.

$$p_\theta(x) = \frac{1}{\sigma} q\left(\frac{x-\mu}{\sigma}\right) \quad , \quad \theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+^*$$

where $\int x^2 q(x) \mathrm{d}x = 1$ and $\int x q(x) \mathrm{d}x = 0$.

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$$\begin{aligned}
\mathbb{E}_\theta\left[\sum_{i=1}^n (X_i - \bar{X}_n)^2\right] &= \mathbb{E}_\theta\left[\sum_{i=1}^n (X_i - \mu - (\bar{X}_n - \mu))^2\right] \\
&= \mathbb{E}_\theta\left[\sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X}_n - \mu)^2\right] \\
&= n\sigma^2 - \sigma^2 = (n-1)\sigma^2 \ .
\end{aligned}$$

Estimator of the variance in a translation-scaling model

Let $X_1, X_2, \dots, X_n$ n be real random variables i.i.d. with p.d.f.

$$p_\theta(x) = \frac{1}{\sigma} q\left(\frac{x-\mu}{\sigma}\right) \quad , \qquad \theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}_+^*$$

where $\int x^2 q(x) \mathrm{d}x = 1$ and $\int x q(x) \mathrm{d}x = 0$.

$$\mathbb{E}_\theta \left[\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right] = (n-1)\sigma^2$$

Therefore

$$S_n^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

is an unbiased estimator of σ^2

Unbaised estimation

In the scaling estimation issue, it is easy to build an unbaised estimator

$$
\begin{aligned}
\mathbb{E}_\theta[(X_1-\bar{X}_n)^2] &= \mathbb{E}_\theta[(X_1-\mu-(\bar{X}_n-\mu))^2] \\
&= \mathbb{E}_\theta[(X_1-\mu)^2] - 2\mathbb{E}_\theta[(X_1-\mu)(\bar{X}_n-\mu)] + \mathbb{E}_\theta[(\bar{X}_n-\mu)^2] \\
&= \mathbb{E}_\theta[(X_1-\mu)^2] - \frac{2}{n}\mathbb{E}_\theta[(X_1-\mu)^2] + \mathbb{E}_\theta[(\bar{X}_n-\mu)^2] \\
&= \sigma^2 - 2\frac{\sigma^2}{n} + \frac{\sigma^2}{n} \\
&= \frac{n-1}{n}\sigma^2 \,.
\end{aligned}
$$

Unbaised estimation

$$\mathbb{E}_\theta[(X_1 - \bar{X}_n)^2] = \frac{n-1}{n}\sigma^2$$

Therefore

$$\frac{n}{n-1}(X_1 - \bar{X}_n)^2$$

is an unbiased estimator of σ^2

Unbaised estimation

Unbaised estimator

$$\frac{n}{n-1}(X_1 - \bar{X}_n)^2$$

Biased estimator

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$$

Which one is the best ?

Unbaised estimation

A biased estimator is not necessarely a “bad” estimator. Being unbaised means : the errors are symmetric around the true value.

Estimation of a uniform distribution support

Let $X_1, X_2, \ldots, X_n$ be independent random variables following $\mathrm{Unif}([0,\theta])$, where $\theta \in \Theta = \mathbb{R}_+^*$. We denote $X_{n:n} = \max(X_1, \ldots, X_n)$.
For all $\theta \in \Theta$ and $x \in [0,\theta]$, we have

$$\mathbb{P}_\theta(X_{n:n} \leq x) = \mathbb{P}_\theta(\max(X_1, \ldots, X_n) \leq x) = \prod_{i=1}^{n} \mathbb{P}_\theta(X_i \leq x) = (x/\theta)^n \; .$$

The p.d.f. of $X_{n:n}$ is therefore given by

$$n\frac{x^{n-1}}{\theta^n}$$

for $x \in [0, \;\theta]$. Thus :

$$\mathbb{E}_\theta[X_{n:n}] = \int_0^\theta x.n\frac{x^{n-1}}{\theta^n}\mathrm{d}x = \frac{n}{n+1}\frac{\theta^{n+1}}{\theta^n} = \frac{n}{n+1}\theta \; ,$$

$$\mathbb{E}_\theta[X_{n:n}^2] = \int_0^\theta x^2.n\frac{x^{n-1}}{\theta^n}\mathrm{d}x = \frac{n}{n+2}\frac{\theta^{n+2}}{\theta^n} = \frac{n}{n+2}\theta^2 \; .$$

Estimation of a uniform distribution support

Let $X_1, X_2, \ldots, X_n$ be independent random variables following $\mathrm{Unif}([0,\theta])$, where $\theta \in \Theta = \mathbb{R}_+^*$. We denote $X_{n:n} = \max(X_1, \ldots, X_n)$.

$$\mathbb{E}_\theta[X_{n:n}] = \frac{n}{n+1}\theta \; , \quad \mathbb{E}_\theta[X_{n:n}^2] = \frac{n}{n+2}\theta^2 \; .$$

The estimator $(n+1)/nX_{n:n}$ is an unbaised estimator of θ.
The quadratic risk of $a_n X_{n:n}$ is

$$\begin{aligned}\mathbb{E}_\theta[(a_n X_{n:n} - \theta)^2] &= a_n^2 \mathbb{E}_\theta[X_{n:n}^2] - 2a_n\theta\mathbb{E}_\theta[X_{n:n}] + \theta^2 \\ &= \frac{na_n^2}{n+2}\theta^2 - \frac{2a_n n}{n+1}\theta^2 + \theta^2 = \theta^2\left\{\frac{na_n^2}{n+2} - \frac{2a_n n}{n+1} + 1\right\}\end{aligned}$$

The minimum is reached for $a_n = (n+2)/(n+1)$ and equals

$$\mathbb{E}_\theta\left[\left(\frac{n+2}{n+1}X_{n:n} - \theta\right)^2\right] = \frac{\theta^2}{(n+1)^2}$$

Estimation of a uniform distribution support

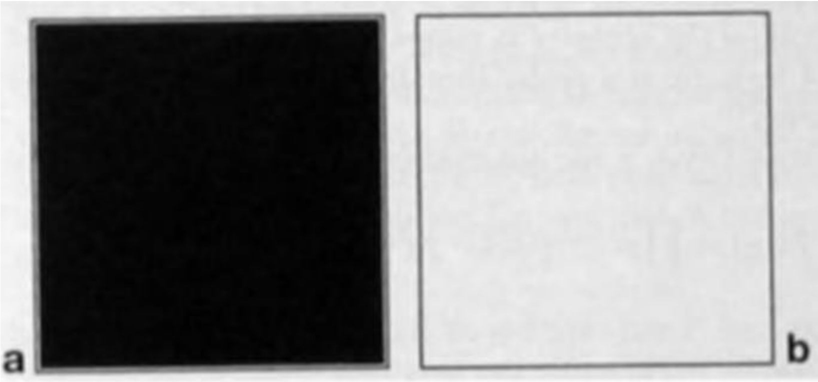
Let $X_1, X_2, \ldots, X_n$ be independent random variables following $\mathrm{Unif}([0,\theta])$, where $\theta \in \Theta = \mathbb{R}_+^*$. We denote $X_{n:n} = \max(X_1, \ldots, X_n)$. The estimator $(n+1)/nX_{n:n}$ is unbaised but for all $\theta \in \Theta$

$$\mathbb{E}_\theta\left[\left(\frac{n+2}{n+1}X_{n:n} - \theta\right)^2\right] \leq \mathbb{E}_\theta\left[\left(\frac{n+1}{n}X_{n:n} - \theta\right)^2\right]$$

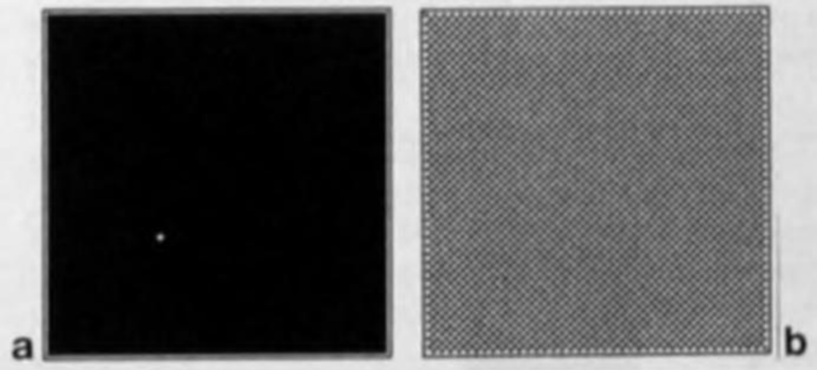
We say that $(n+1)/nX_{n:n}$ is inadmissible for the quadratic risk.

Observation

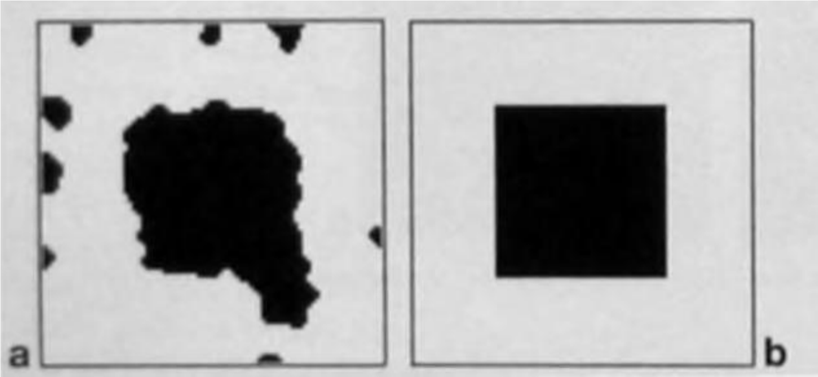
Solution triviale avec uniquement l'apriori



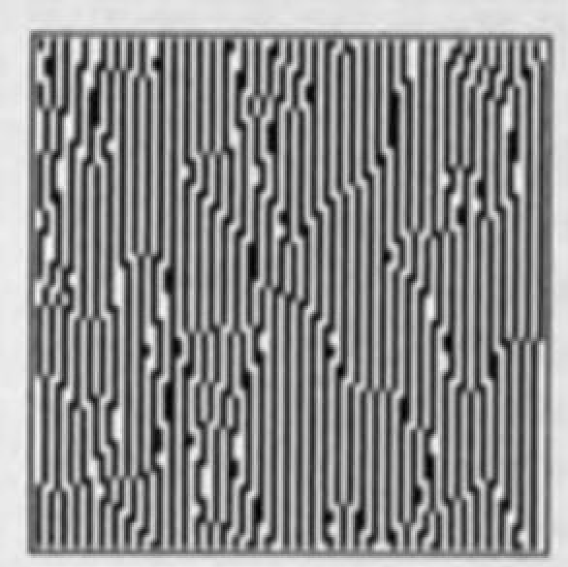
Images ne respectant pas les contraintes



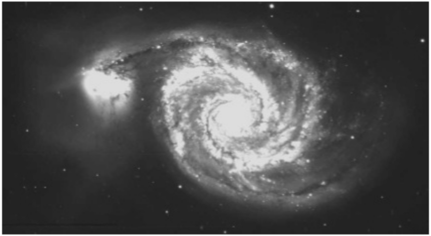
Une solution et l'image réelle

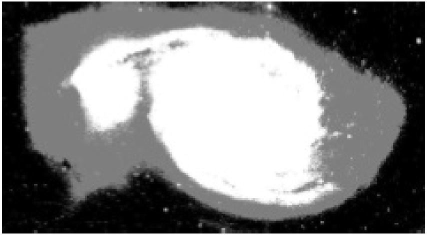


En cas de mauvais apriori

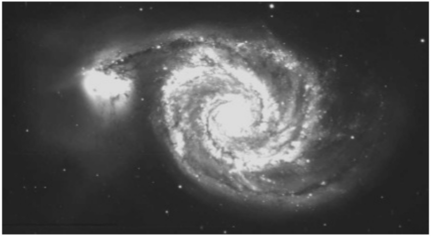


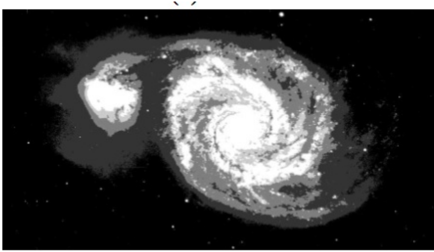
Autre exemples en imagerie : Segmentation utilisant des HMM





Segmentation utilisant des HMM

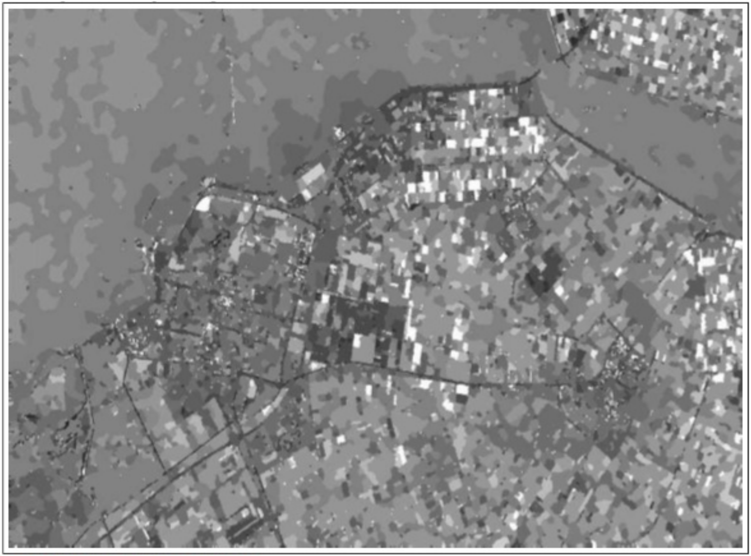




Segmentation

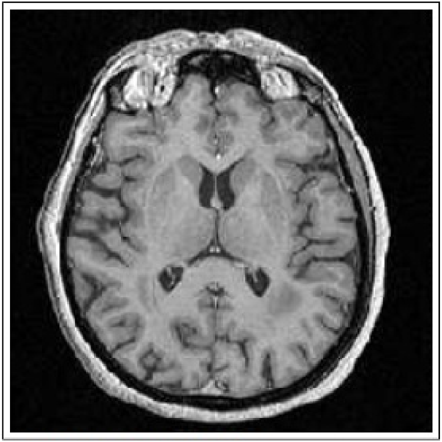
a. Image radar originale ©ERS-1

Segmentation

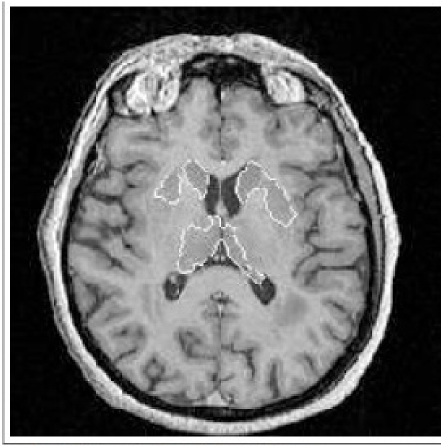


b. Image segmentée en régions

Segmentation



a. Image cérébrale IRM.



Résultat de la segmentation