# Part I
# Lagrange multiplier

To look for the optimums of a function :

$$f\left(t_1, t_2, .., t_p\right)$$

of p variables subject to :

$$l\left(t_1, t_2, ..., t_p\right) = \text{cte},$$

we compute the partial derivatives of the function :

$$g\left(t_1, t_2, .., t_p\right) = f\left(t_1, t_2, .., t_p\right) - \lambda\left(l\left(t_1, t_2, ..., t_p\right) - \text{cte}\right)$$

with respect to each variable. By canceling the $p$ partial derivatives obtained and considering the single constraint, we obtain a linear system with $p+1$ equations. The $p+1$ unknowns are the variables $(t_1, ..., t_p)$ and the Lagrange multiplier $\lambda$. The existence of solutions for the linear system is a necessary condition but not sufficient to the existence of an optimum for the function $f$.

We can generalize the previous approach when the $p$ variables are subject to $c$ constraints. We consider now a new function $g\left(t_1, t_2, ...t_p\right)$ with a linear combination of all constraints whose coefficients are the Lagrange multipliers $\lambda_1, .., \lambda_c$ and we solve the linear system with $p+c$ equations associated.

# Part II
# Derivative of a quadratic form with respect to a vector

To look for the principal components, we had to to compute the partial derivatives of both ${}^t a \Sigma a$ and ${}^t a a$ with respect to the coefficients $a_1, ..., a_p$ of the vector $a$. We denote $\frac{\partial g(a)}{\partial a} \in \mathbb{R}^p$ the vector whose components are the partial derivatives of $g\left(a\right)$ with respect to each component of the vector $a$:

$$\frac{\partial g\left(a\right)}{\partial a} = \begin{bmatrix} \frac{\partial g(a)}{\partial a_1} \\ . \\ . \\ . \\ \frac{\partial g(a)}{\partial a_j} \\ . \\ . \\ . \\ \frac{\partial g(a)}{\partial a_p} \end{bmatrix}$$

One can show that :
$$\frac{\partial\left({}^t a \Sigma a\right)}{\partial a} = 2\Sigma a.$$
Indeed :
$$\frac{\partial\left({}^t a \Sigma a\right)}{\partial a} = \begin{bmatrix} \frac{\partial\, {}^t a}{\partial a_1}\Sigma a \\ . \\ . \\ . \\ \frac{\partial\, {}^t a}{\partial a_j}\Sigma a \\ . \\ . \\ . \\ \frac{\partial\, {}^t a}{\partial a_p}\Sigma a \end{bmatrix} + \begin{bmatrix} {}^t a \Sigma \frac{\partial a}{\partial a_1} \\ . \\ . \\ . \\ {}^t a \Sigma \frac{\partial a}{\partial a_j} \\ . \\ . \\ . \\ {}^t a \Sigma \frac{\partial a}{\partial a_p} \end{bmatrix}.$$
One can notice that on each line, both elements are equal since they are of dimension $1 \times 1$ and are the transpose of each other. Thus :
$$\frac{\partial\left({}^t a \Sigma a\right)}{\partial a} = 2\frac{\partial\, {}^t a}{\partial a}\Sigma a,$$
and the derivative of ${}^t a$ with respect to $a$ is :
$$\frac{\partial\, {}^t a}{\partial a} = \begin{bmatrix} \frac{\partial\, {}^t a}{\partial a_1} \\ . \\ . \\ . \\ \frac{\partial\, {}^t a}{\partial a_j} \\ . \\ . \\ . \\ \frac{\partial\, {}^t a}{\partial a_p} \end{bmatrix} = I_p$$
which the identity matrix of dimension $p$. Similarly we can show that : $\frac{\partial\, {}^t a a}{\partial a} = 2a$