Special Matrices Useful in Data Analysis

Exercise 1. Give conditions on a, b, and c for the matrix below to be positive definite.

$$\boldsymbol{N} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Correction :
The matrix is positive definite if and only if its eigenvalues are positive. Denoting by λ_1 and λ_2 its eigenvalues one has

$$\det \boldsymbol{N} = ac - b^2 = \lambda_1 \lambda_2 \quad \text{and} \quad \operatorname{Tr}(\boldsymbol{N}) = a + c = \lambda_1 + \lambda_2.$$

Thus the matrix $\boldsymbol{N}$ is positive definite if and only if $\det \boldsymbol{N} > 0$ and $\operatorname{Tr} \boldsymbol{N} > 0$, that is if and only if $ac - b^2 > 0$ and $a + c > 0$, which is equivalent to $ac - b^2 > 0$ and $a > 0$.

Exercise 5. We consider the vectors

$$\boldsymbol{a}_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{a}_2 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}.$$

Compute the projection matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ onto the lines through $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively. Multiply those projection matrices and explain why their product $\boldsymbol{P}_1\boldsymbol{P}_2$ is what it is.
Correction :
The projection matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are given by

$$\boldsymbol{P}_1 = \frac{\boldsymbol{a}_1\boldsymbol{a}_1^T}{\boldsymbol{a}_1^T\boldsymbol{a}_1} = \frac{\boldsymbol{a}_1\boldsymbol{a}_1^T}{\|\boldsymbol{a}_1\|^2} \quad \text{and} \quad \boldsymbol{P}_2 = \frac{\boldsymbol{a}_2\boldsymbol{a}_2^T}{\boldsymbol{a}_2^T\boldsymbol{a}_2} = \frac{\boldsymbol{a}_2\boldsymbol{a}_2^T}{\|\boldsymbol{a}_2\|^2}.$$

Then one has

$$\boldsymbol{P}_1 = \frac{1}{9}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\begin{pmatrix} -1 & 2 & 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$$

and

$$\boldsymbol{P}_2 = \frac{1}{9}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & 2 & -1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ 2 & 2 & 1 \end{pmatrix}.$$

With this two we find that $\boldsymbol{P}_1\boldsymbol{P}_2 = 0$. We can also see this same result by writing

$$\boldsymbol{P}_1\boldsymbol{P}_2 = \frac{\boldsymbol{a}_1\boldsymbol{a}_1^T\boldsymbol{a}_2\boldsymbol{a}_2^T}{\|\boldsymbol{a}_1\|^2\|\boldsymbol{a}_2\|^2} = \frac{\boldsymbol{a}_1\langle\boldsymbol{a}_1, \boldsymbol{a}_2\rangle\boldsymbol{a}_2^T}{\|\boldsymbol{a}_1\|^2\|\boldsymbol{a}_2\|^2}.$$

Since $\langle\boldsymbol{a}_1, \boldsymbol{a}_2\rangle = 0$ we recover that $\boldsymbol{P}_1\boldsymbol{P}_2 = 0$. Indeed, since the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are orthogonal, when we project a given vector onto $\boldsymbol{a}_1$ we produce a vector that we still be orthogonal to $\boldsymbol{a}_2$. Projecting this orthogonal vector onto $\boldsymbol{a}_2$ will result in a zero vector.

Exercise 6.

1. Project $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ by solving $\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$ and $\boldsymbol{p} = \boldsymbol{A}\widehat{\boldsymbol{x}}$:

   (a)
$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

   (b)
$$\boldsymbol{A}_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b}_2 = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}.$$

   Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$.

   Correction :

   (a) First, we compute the matrix product $\boldsymbol{A}_1^T\boldsymbol{A}_1$, one has
$$\boldsymbol{A}_1^T\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
   and
$$\boldsymbol{A}_1^T\boldsymbol{b}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$
   Thus, we search for $\widehat{\boldsymbol{x}}$ such that
$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \frac{1}{2-1}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$
   We finally obtain
$$\boldsymbol{p} = \boldsymbol{A}_1\widehat{\boldsymbol{x}} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$
   and the error is given by $\boldsymbol{e} = \boldsymbol{b}_1 - \boldsymbol{p} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

   (b) First, we compute the matrix product $\boldsymbol{A}_2^T\boldsymbol{A}_2$, one has
$$\boldsymbol{A}_2^T\boldsymbol{A}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$
   and
$$\boldsymbol{A}_2^T\boldsymbol{b}_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix}.$$
   Thus, we search for $\widehat{\boldsymbol{x}}$ such that
$$\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 14 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} = \frac{1}{6-4}\begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ 14 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -4 \\ 12 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}.$$

We finally obtain

$$\boldsymbol{p} = \boldsymbol{A}_2 \widehat{\boldsymbol{x}} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$$

and the error is given by $\boldsymbol{e} = \boldsymbol{b}_2 - \boldsymbol{p} = 0$.

2. Compute the corresponding projection matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ onto the column spaces of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ respectively.

   Verify that $\boldsymbol{p}_i = \boldsymbol{P}_i \boldsymbol{b}_i$ and $\boldsymbol{P}_i^2 = \boldsymbol{P}_i$.

   Correction :

   (a) The projection matrix $\boldsymbol{P}_1$ satisfies

$$\begin{aligned} \boldsymbol{P}_1 = \boldsymbol{A}_1 (\boldsymbol{A}_1^T \boldsymbol{A}_1)^{-1} \boldsymbol{A}_1^T &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

   We can check that we have $\boldsymbol{P}_1^2 = \boldsymbol{P}_1$ and $\boldsymbol{P}_1 \boldsymbol{b}_1 = \boldsymbol{p}$.

   (a) The projection matrix $\boldsymbol{P}_2$ satisfies

$$\begin{aligned} \boldsymbol{P}_2 = \boldsymbol{A}_2 (\boldsymbol{A}_2^T \boldsymbol{A}_2)^{-1} \boldsymbol{A}_2^T &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

   We can check that we have $\boldsymbol{P}_2^2 = \boldsymbol{P}_2$ and $\boldsymbol{P}_2 \boldsymbol{b}_2 = \boldsymbol{p}$.

Exercise 7.

1. Suppose $\boldsymbol{b}$ equals two times the first column of $\boldsymbol{A}$. What is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$?

   Correction :

   Since $\boldsymbol{b}$ is in the space spanned by the columns of the matrix $\boldsymbol{A}$, the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ will be equal to the vector $\boldsymbol{b}$. Notice that the associated projection matrix $\boldsymbol{P}$ is not the identity matrix. Indeed, if the vectors are not in the column space of $\boldsymbol{A}$ their projection is not this vector itself.

2. Compute $\boldsymbol{p}$ and $\boldsymbol{P}$ when

$$\boldsymbol{b} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix}.$$

Correction :

We have

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$$

and

$$(\boldsymbol{A}^T\boldsymbol{A})^{-1} = \frac{1}{5.5 - (-2)(-2)} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}.$$

Thus, we obtain

$$\begin{aligned} \boldsymbol{P} = \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T &= \frac{1}{21} \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \frac{1}{21} \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & 10 \\ 5 & 8 & -4 \end{pmatrix} = \frac{1}{21} \begin{pmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{pmatrix} \neq \boldsymbol{I} \end{aligned}$$

and $\boldsymbol{p} = \boldsymbol{P}\boldsymbol{b} = \boldsymbol{b}$.

Exercise 8. What linear combination of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is closest to $\boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Correction :
We search for $\widehat{\boldsymbol{x}}$ such that $\boldsymbol{A}^T\boldsymbol{A}\widehat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}.$$

We have

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}.$$

So that $\widehat{\boldsymbol{x}}$ is then given by

$$\widehat{\boldsymbol{x}} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}.$$