Vectors and Vector Spaces

Exercise 16. Project the vector $\boldsymbol{b}$ onto the line through $\boldsymbol{a}$.
Check that $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$ is perpendicular to $\boldsymbol{a}$.

1.
$$\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}.$$

Correction :
Recalling that the projection $\boldsymbol{p}$ of the vector $\boldsymbol{b}$ satisfies $\boldsymbol{p} = \widehat{\boldsymbol{x}}\boldsymbol{a}$ with
$$\widehat{\boldsymbol{x}} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{1+2+2}{1+1+1} = \frac{5}{3}.$$
then, one has
$$\boldsymbol{p} = \frac{5}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$
Moreover we have
$$\langle \boldsymbol{a}, \boldsymbol{e} \rangle = \boldsymbol{a}^T\boldsymbol{e} = \frac{1}{3}(-2+1+1) = 0,$$
and thus $\boldsymbol{e}$ is orthogonal to $\boldsymbol{a}$.

2.
$$\boldsymbol{a} = \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Correction :
Recalling that the projection $\boldsymbol{p}$ of the vector $\boldsymbol{b}$ satisfies $\boldsymbol{p} = \widehat{\boldsymbol{x}}\boldsymbol{a}$ with
$$\widehat{\boldsymbol{x}} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}} = \frac{-1-9-1}{1+9+1} = -1.$$
then, one has
$$\boldsymbol{p} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = 0.$$
Thus $\boldsymbol{e}$ is orthogonal to $\boldsymbol{a}$.

Exercise 17.

1. What multiple of $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ should be subtracted from $\boldsymbol{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ to make the result orthogonal to $\boldsymbol{a}$? Sketch a figure to show the three vectors.

   Correction :

   We have to apply the Gram-Schmidt process to the vector $\boldsymbol{b}$ (but we do not need to normalized the vector $\widetilde{\boldsymbol{b}}$). Then one has

$$\widetilde{\boldsymbol{b}} = \boldsymbol{b} - \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - \frac{4}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

   Thus we have to subtract $\boldsymbol{a}$ twice to obtain a vector orthogonal to $\boldsymbol{a}$.

2. Complete the Gram-Schmidt process.

   Correction :

   One has

$$\boldsymbol{q}_1 = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   and

$$\boldsymbol{q}_2 = \frac{\tilde{\boldsymbol{b}}}{\|\tilde{\boldsymbol{b}}\|} = \frac{1}{\sqrt{8}}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Exercise 18. Find orthogonal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ by Gram-Schmidt from $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$:

1.

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \ \text{ and } \ \boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

   Correction :

   We have $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ thus $\boldsymbol{q}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

   One has $\boldsymbol{a}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and so

$$\widetilde{\boldsymbol{q}}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T\boldsymbol{a}_2)\boldsymbol{q}_1 = \boldsymbol{a}_2 - 0 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

   Thus

$$\boldsymbol{q}_2 = \frac{\widetilde{\boldsymbol{q}}_2}{\|\widetilde{\boldsymbol{q}}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

   To finish $\boldsymbol{a}_3 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$. Then we have

$$\widetilde{\boldsymbol{q}}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^T\boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^T\boldsymbol{a}_3)\boldsymbol{q}_2 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} - \frac{9}{6}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

and

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{q}}_3}{\|\widetilde{\boldsymbol{q}}_3\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

2.

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \ \boldsymbol{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \boldsymbol{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Correction :

We have $\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ thus $\boldsymbol{q}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$.

One has $\boldsymbol{a}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ and so

$$\widetilde{\boldsymbol{q}}_2 = \boldsymbol{a}_2 - (\boldsymbol{q}_1^T \boldsymbol{a}_2)\boldsymbol{q}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Thus

$$\boldsymbol{q}_2 = \frac{\widetilde{\boldsymbol{q}}_2}{\|\widetilde{\boldsymbol{q}}_2\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

To finish $\boldsymbol{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. Then we have

$$\widetilde{\boldsymbol{q}}_3 = \boldsymbol{a}_3 - (\boldsymbol{q}_1^T \boldsymbol{a}_3)\boldsymbol{q}_1 - (\boldsymbol{q}_2^T \boldsymbol{a}_3)\boldsymbol{q}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - 0 + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 2 \\ -3 \end{pmatrix}$$

and

$$\boldsymbol{q}_3 = \frac{\widetilde{\boldsymbol{q}}_3}{\|\widetilde{\boldsymbol{q}}_3\|} = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 1 \\ 2 \\ -3 \end{pmatrix}.$$