Final Exam (2h)

The exam is open book.

Exercise 1 [5pts]
(Hint : For this exercise you can use the formula $\int_0^{+\infty} ue^{-au}du = 1/a^2$, true for every $a > 0$.)
Let (X, Y) have joint density

$$f(x,y) = \begin{cases} ye^{-y(x+1)} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

1. Find the marginal densities f_Y and f_X.
2. Find the conditional density $f_{X|Y=y}$.
3. Prove that
$$\mathbb{E}[X|Y] = \frac{1}{Y}.$$

Answers :

1. We fix $y \geq 0$.
$$f_Y(y) = \int_{x=0}^{+\infty} ye^{-y(x+1)}dx = y\int_{x=0}^{+\infty} e^{-y(x+1)}dx = y \times \frac{e^{-y(x+1)}}{yx+1}\bigg|_{x=0}^{x=+\infty} = \frac{ye^{-y}}{y} = e^{-y},$$
which we know is a density.
We fix $x \geq 0$.
$$f_X(x) = \int_{y=0}^{+\infty} ye^{-y(x+1)}dy = \frac{1}{(x+1)^2},$$
by using the hint.
2. By the course formula,
$$f_{X|Y=y} = \frac{f(x,y)}{f_Y(y)} = \frac{ye^{-y(x+1)}}{e^{-y}} = ye^{-yx}.$$
3. According to the course formula (p.37),
$$\begin{aligned}\mathbb{E}[X|Y] &= \frac{\int_{x=0}^{+\infty} xf(x,Y)dx}{f_Y(Y)} \\ &= \frac{\int_x xYe^{-Y(x+1)}dx}{e^{-Y}} \\ &= \frac{Ye^{-Y}}{e^{-Y}}\int_x xe^{-Yx}dx \\ &= \frac{Ye^{-Y}}{e^{-Y}}1/Y^2 \qquad \text{(by the Hint)} \\ &= 1/Y.\end{aligned}$$

Exercise 2 [4pts]

1. Compute $\mathbb{E}[X^3]$, where X is a uniform random variable in $[0, 2]$. Recall that it means that X has density
$$f(x) = \frac{1}{2}\mathbf{1}_{0\leq x\leq 2}.$$
2. Find the density of X^3.

3. Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. random variables uniform in $[0,2]$. Using a result of the course, prove that

$$\frac{X_1^3 + \cdots + X_n^3}{n} \overset{\text{prob.}}{\to} 2.$$

Answers :

1.

$$\mathbb{E}[X^3] = \int_0^2 x^3/2dx = \frac{1}{2}x^4/4\Big|_{x=0}^{x=2} = \frac{1}{2}2^4/4 = 2.$$

2. Let ϕ be a continuous and bounded function.

$$\mathbb{E}[\phi(X^3)] = \frac{1}{2}\int_{x=0}^{x=2} \phi(x^3)dx = \frac{1}{2}\int_{u=0^3}^{u=2^3} \phi(u)\frac{du}{3u^{2/3}} = \int_{u=0}^{u=8} \phi(u)\frac{du}{6u^{2/3}},$$

where we have put $u = x^3$, i.e. $x = u^{1/3}$, $du/dx = 3x^2 = 3u^{2/3}$. This proves that X^3 has density $\frac{1}{6u^{2/3}}$ for $0 \le u \le 8$.

3. By the weak law of large numbers,

$$\frac{X_1^3 + \cdots + X_n^3}{n} \overset{\text{prob.}}{\to} \mathbb{E}[X_1^3] = 2.$$

Exercise 3 [6pts]

Let $(X_k)_{k\ge1}$ be a sequence of independent random variables such that for all $k \ge 1$,

$$X_k = \begin{cases} -k^2 & \text{with probability } \frac{1}{k+1}, \\ k^2 & \text{with probability } \frac{1}{k+1}, \\ 0 & \text{with probability } 1 - \frac{2}{k+1}. \end{cases}$$

We set $S_n = X_1 + \cdots + X_n$.

1. Compute, for every k, n, $\mathbb{E}[X_k], \mathbb{E}[S_n], \mathrm{Var}(X_k), \mathrm{Var}(S_n)$.
2. Check that for every k, $\mathrm{Var}(X_k) \le 2k^3$, and deduce that for every n we have $\mathrm{Var}(S_n) \le 2n^4$.
3. Using the Chebychev inequality, prove that when $n \to +\infty$,

$$\left(\frac{S_n}{n^3}\right)_{n\ge1} \overset{\text{(prob.)}}{\to} 0.$$

4. Prove that when $k \to +\infty$,

$$(X_k)_{k\ge1} \overset{\text{(prob.)}}{\to} 0$$

but that (X_k) does not converge to 0 in L^1.

Answers :

1. We have

$$\mathbb{E}[X_k] = k^2 \times \frac{1}{k+1} + 0 \times \left(1 - \frac{2}{k+1}\right) + (-k^2) \times \frac{1}{k+1} = 0,$$
$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \ldots \mathbb{E}[X_n] = n \times 0 = 0.$$

Besides,

$$\mathbb{E}[X_k^2] = (k^2)^2 \times \frac{1}{k+1} + (-k^2)^2 \times \frac{1}{k+1} = \frac{2k^4}{k+1}.$$

We get $\mathrm{Var}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2 = \frac{2k^4}{k+1}$.
Since X_k's are independent,

$$\begin{aligned}\mathrm{Var}[S_n] &= \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n] \\ &= \frac{2\times1^4}{1+1} + \frac{2\times2^4}{2+1} + \cdots + \frac{2\times k^4}{k+1}.\end{aligned}$$

2. We have

$$\begin{aligned}\mathrm{Var}(S_n) &= \frac{2\times 1^4}{1+1} + \frac{2\times 2^4}{2+1} + \cdots + \frac{2\times n^4}{n+1}\\ &\le \frac{2\times n^4}{n+1} + \frac{2\times n^4}{n+1} + \cdots + \frac{2\times n^4}{n+1}\\ &\le n\frac{2\times n^4}{k+1} \le 2n^4.\end{aligned}$$

3. Using the Chebychev inequality

$$\begin{aligned}\mathbb{P}\left(\left|\frac{S_n}{n^3}-0\right|\ge\varepsilon\right) &= \mathbb{P}\left(\left|\frac{S_n}{n^3}-\mathbb{E}\left[\frac{S_n}{n^3}\right]\right|\ge\varepsilon\right)\\ &\le \frac{\mathrm{Var}(S_n/n^3)}{\varepsilon^2}\\ &= \frac{\mathrm{Var}(S_n)}{n^6\varepsilon^2} \qquad \text{(see equation (\$) page 14)}\\ &\le \frac{2n^4}{n^6\varepsilon^2} \qquad \text{(previous question)}\\ &\overset{n\to+\infty}{\to} 0.\end{aligned}$$

Exercise 4 [5pts]
Let $\binom{X}{Y}$ be a gaussian vector with mean $\mu = \binom{-1}{2}$ and with covariance matrix

$$C = \begin{pmatrix}4 & 2\\ 2 & 9\end{pmatrix}.$$

1. Give the mean vector μ' and the covariance matrix C' of the gaussian vector $\binom{X+Y}{2Y-X}$.
2. What is the distribution of $X+Y$?
3. Let a be a fixed constant. Compute $\mathrm{Cov}(Y, X-aY)$, and find the value of a for which Y and $X-aY$ are independent.
4. Compute $\mathbb{E}[X|Y]$.
   (Hint : Write $X = aY + (X-aY)$, where a is the answer to the previous question.)

Correction :

1. We have clearly $\mu' = \binom{1}{5}$. For the covariance, we have by linearity

$$\begin{aligned}\mathrm{Var}(X+Y) &= \mathrm{Var}(X)+\mathrm{Var}(Y)+2\mathrm{Cov}(X,Y) = 4+9+2\times 2 = 17.\\ \mathrm{Cov}(X+Y,2Y-X) &= 2\mathrm{Cov}(X,Y)-\mathrm{Cov}(X,X)+2\mathrm{Cov}(Y,Y)-\mathrm{Cov}(Y,X) = 2\times 2-4+2\times 9-2 = 16\\ \mathrm{Var}(2Y-X) &= 4\mathrm{Var}(Y)+\mathrm{Var}(X)+2\mathrm{Cov}(-X,2Y) = 4\times 9+4-4\times 2 = 32.\end{aligned}$$

   Finally,

$$C' = \begin{pmatrix}17 & 16\\ 16 & 32\end{pmatrix}.$$

2. $X+Y \sim \mathcal{N}(1,17)$.
3.

$$\mathrm{Cov}(Y,X-aY) = \mathrm{Cov}(Y,X) - a\mathrm{Cov}(Y,Y) = 2-9a.$$

   Therefore we have $\mathrm{Cov}(Y,X-aY)=0$ for $a=2/9$. Now, Y and $X-aY = X-2Y/9$ are the two components of a gaussian vector with a null covariance. Therefore they are independent (Proposition 4.4).
4. Since $X = 2Y/9 + (X-2Y/9)$, we have by linearity of the conditional expectation

$$\begin{array}{rccc}\mathbb{E}[X|Y] & = & \mathbb{E}[2Y/9|Y] & + \;\mathbb{E}[X-2Y/9|Y]\\ & = & 2Y/9 & + \;\mathbb{E}[X-2Y/9]\\ & & \text{('taking out what is known')} & \text{(independence)}\end{array}$$

   Finally, $\mathbb{E}[X-2Y/9] = \mathbb{E}[X] - 2\mathbb{E}[Y]/9 = -1-2\times 2/9 = -13/9$. We get

$$\mathbb{E}[X|Y] = 2Y/9 - 13/9.$$

Bonus Exercise

Exercise 5 [3pts]
I want to sell my house and I have decided to accept the first offer Z exceeding s euros. I make the assumptions that offers are i.i.d. random variables $X_1, X_2, \dots,$ with common cumulative distribution function F. Denote by $N \geq 1$ the number of offers before I sell the house. For example, if $s = 350000$ euros and

$$(X_1, X_2, X_3, X_4, X_5, \dots,) = (340000, 310000, 335000, 365000, 340000, \dots)$$

then $N = 4$ and $Z = X_4 = 365000$.

1. For any $t \geq 0$, compute $\mathbb{P}(Z \leq t; N = n)$.
2. Find the cumulative distribution and the density of Z.

Correction :

1. If $t < s$ then $\mathbb{P}(Z \leq t) = 0$. If $t \geq s$,

$$\begin{aligned}\mathbb{P}(Z \leq t; N = n) &= \mathbb{P}(X_1 < s, X_2 < s, \dots, X_{n-1} < s, s \leq X_n \leq t)\\ &= F(s)^{n-1}(F(t) - F(s)).\end{aligned}$$

2.

$$\begin{aligned}\mathbb{P}(Z \leq t) &= \sum_{n\geq 1} \mathbb{P}(Z \leq t; N = n)\\ &= \sum_{n\geq 1} F(s)^{n-1}(F(t) - F(s))\\ &= (F(t) - F(s)) \sum_{n\geq 1} F(s)^{n-1}\\ &= \frac{F(t) - F(s)}{1 - F(s)},\end{aligned}$$

where we use

$$\sum_{n\geq 1} x^{n-1} = \sum_{p\geq 0} x^p = \frac{1}{1-x}.$$

We can differentiate $\mathbb{P}(Z \leq t)$ in order to get the density f_Z of Z :

$$f_Z = \frac{\partial}{\partial t} \frac{F(t) - F(s)}{1 - F(s)} = \frac{f_X(t)}{1 - F(s)},$$

where f_X is the density of X.