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RIEMANN HYPOTHESIS: A NUMERICAL TREATMENT OF THE RIESZ AND HARDY-LITTLEWOOD WAVE

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Abstract. We present the results of numerical experiments in connection with the Riesz and the Hardy-Littlewood criteria for the truth of the Riemann Hypothesis (RH). The coefficients c_k of the Pochhammer expansion for the reciprocal of the Riemann Zeta function depend in our model on two parameters. The "critical functions" $c_k k^a$ (where a is some constant), whose behaviour is concerned with the possible truth of the RH, are analysed at relatively large values of k. Some cases are numerically investigated up to larger values of k, i.e. $k = 10^9$ and more.

The c_k we obtain in such a region have an oscillatory behaviour, which we call the Riesz and the Hardy-Littlewood wave. A special case is then studied numerically in some range of the critical strip. The numerical results give some evidence that the critical function is bounded for $\Re(s) > \frac{1}{2}$ and such an "evidence" is stronger in the region $\Re(s) > \frac{3}{4}$ where the wave seems to decay slowly. This give further support in favour of the absence of zeros of the Riemann Zeta function in some regions of the critical strip ($\Re(s) > \frac{3}{4}$) and a (weaker) support in the direction to believe that the RH may be true ($\Re(s) > \frac{1}{2}$).

The amplitudes and the wavelength of the wave obtained by our numerical treatment are then compared with those formulated by Baez-Duarte in his analytical approach. The agreement is satisfactory.

Finally for another special case we found that the wave appears to be bounded even though one parameter in our model grows to infinity. Our analysis suggests that RH may barely be true and it is argued that an absolute bound on the amplitudes of the waves in all cases, should be given by $|\frac{1}{\zeta(\frac{1}{2}+\epsilon)} - 1|$, with ϵ arbitrarily small positive, i.e. equal to 1.68477....

1. Introduction

Following recent works concerning the study of some well known functions appearing in the original criteria of Riesz, Hardy and Littlewood and involving the Riemann Hypothesis (RH), there is new interest in the direction of numerical experiments, where the calculations use the ideas of some modern works on the subject. These ideas concern the expansion of the reciprocal of the Riemann Zeta function in terms of the so called Pochhammer polynomials P_k, whose coefficients c_k play a central role also in the asymptotic region of very large k.

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Here we are concerned with the discrete version of the Riesz criterion which has also been studied numerically: the first numerical experiments have been announced and reported for the Riesz case. It has been found that the function c_k has an oscillatory behaviour in a region of relatively high k, in agreement with an asymptotic formula given by Baez-Duarte and involving the non-trivial zeros of the Zeta function. The agreement appears satisfactory even if only the contribution of few non-trivial zeros has been used.

In our previous paper, appeared in the first Number of this Journal, a two parameters family (parameter α and β) of Pochhammer's polynomials was introduced. This allowed the starting investigation of c_k at low values of k, but in various cases and in the so called "strong coupling" regime (high β). After the initial study at low k, our computations using the formula containing the Möbius function were easily extended to larger and larger k (up to a billion) in the strong coupling limit, with the appearance of macro-oscillations in c_k extending to larger k. This is a symptom that using such a limit the RH may eventually barely be true. In this paper we continue the numerical experiments partially using our Poisson formula already established in and which is well suited for numerical purposes.

After the formulation of the model and the definition of the statement "critical function" in Section 2, we then give in Section 3 an asymptotic formula (the Baez-Duarte formula) to compute it. This formula involves the trivial and not-trivial zeros of the Riemann Zeta function. The next three sections are dedicated to the numerical experiments. Our aim is twofold: first we will analyse the correctness of this asymptotic formula and second we will investigate the behaviour of the critical function whose boundedness will ensure the truth of the RH. In Section 4 the amplitudes of what we call the Riesz and Hardy-Littlewood wave are calculated in some cases using the Baez-Duarte formula. We then present our results for these different models up to values of k equal to one billion and observe oscillations in all cases. These oscillations are compatible with the calculated amplitudes: the agreement with the asymptotic formula of Baez-Duarte is satisfactory. In Section 5, we concentrate our study in more details by considering a special new model already proposed where $\alpha = \frac{7}{2}$ and β is increasing starting with the value equal to 4. The results show in a concrete way the "transition" from the low coupling to the "strong coupling regime": at low values of β ($\beta = 4$) we obtain up to 7 oscillation with values of k extending up to a billion. These start to deform continuously with increasing values of β approaching the infinite β limit. In such a regime, the wave is absorbed in a macroscopic region with an amplitude whose strength should be finite as already noted in our previous work. Also in these cases the agreement with the asymptotic formula is satisfactory. In Section 6 we analyse the behaviour of the critical function in the critical strip and the contribution to it of the non-trivial zeros. Moreover, the possibility that in an ideal numerical experiment (using an arbitrarily large but finite maximum value of n, say N in the formula with the Möbius function) the amplitude of the waves at finite β values should be bounded, is commented in Section 7.

2. The model

The starting point of this work is the representation of the reciprocal of the Riemann Zeta function by means of the Pochhammer polynomials $P_k(s)$ (where s is

a complex variable, $s = \sigma + it$), whose coefficients c_k have been introduced by Baez-Duarte for the Riesz case ($\alpha = \beta = 2$). For the study of the coefficients c_k, some recent analytical as well as numerical results have been obtained [1, 2, 3, 4, 5, 6, 7].

Using the Baez-Duarte approach, the representation of $\frac{1}{\zeta(s)}$ may be obtained for a family of two parameters Pochhammer polynomials (parameters $\alpha > 1$ and $\beta > 0$) and by [3] we have:

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k(\alpha, \beta) P_k(s; \alpha, \beta) \tag{2.1}$$

where

$$P_k(s; \alpha, \beta) \quad := \prod_{r=1}^{k} \left(1 - \frac{\frac{s-\alpha}{\beta}+1}{r}\right) \tag{2.2}$$

$$c_k(\alpha, \beta) \quad := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}}\right)^k \tag{2.3}$$

and $P_0(k; \alpha, \beta) = 1$.

In (2.3) the Möbius function of argument n is given by:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0, & \text{if } n \text{ contains a square.} \end{cases}$$

One has for $\Re(s) = \sigma > 1$:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{2.4}$$

so another explicit formula for the $c_k(\alpha, \beta)$ is obtained from (2.3) using the binomial coefficients and reads:

$$c_k(\alpha, \beta) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(\alpha + \beta j)} \tag{2.5}$$

As β is increasing, one may also use (especially) in the context of numerical experiments, the formula recently obtained [3] and given by:

$$c_k(\alpha, \beta) \cong \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}} \tag{2.6}$$

In such an approximation we have that

$$c_k(\alpha, \beta) \cong \sum_{p=0}^{\infty} c_p(\alpha, \beta) \frac{k^p}{p!} e^{-k} \tag{2.7}$$

which shows the emergence of a Poisson like distribution for the coefficients $c_k(\alpha, \beta)$. This should be a very satisfactory approximation [3].

We recall that an important inequality due to Baez-Duarte [1, 2] concerning the Pochhammer polynomials of complex argument z is given by:

$$|P_k(z)| \leq C k^{-\Re(z)} \tag{2.8}$$

TABLE 1. The expected decay of c_k for different values of α and β ($\sigma = \frac{1}{2}$)

α	β	decay of $\lvert c_k \rvert$	Note
2	2	$k^{-\frac{3}{4}}$	The case of Riesz
1	2	$k^{-\frac{1}{4}}$	The case of Hardy-Littlewood
2	6	$k^{-\frac{1}{4}}$	Same decay as the Hardy-Littlewood case but numerically more convenient
$\frac{7}{2}$	4	$k^{-\frac{3}{4}}$	Same decay as the Riesz case, intensive calculations are given below
3	3	$k^{-\frac{5}{6}}$	If the Zeta function has no zero for $\Re(s) > \frac{3}{4}$ then $c_k(3,3)$ should decay at least as $k^{-\frac{3}{4}}$
4	4	$k^{-\frac{7}{8}}$	Since from the Prime Number Theorem there is no zero for $\Re(s) \geq 1$ the $c_k(4,4)$ decays at least as $k^{-\frac{3}{4}}$
2	4	$k^{-\frac{3}{8}}$	Another interesting case for calculations

The above inequality applied to our two parameters family of Pochhammer's polynomials with complex argument $z = \frac{s-\alpha}{\beta} + 1$ gives:

$$\left| P_k(\frac{s-\alpha}{\beta} + 1; \alpha, \beta) \right| \leq C k^{-\left(\frac{\sigma-\alpha}{\beta}+1\right)} \tag{2.9}$$

So that $\zeta(s)$ in (2.1) will be different from zero and thus the RH will be true for $\Re(s) = \sigma > \frac{1}{2}$ if the sequence c_k decays, at large k, as (see [3]):

$$|c_k| \leq A k^{-\frac{\alpha-\sigma}{\beta}} \tag{2.10}$$

We will also consider the "critical function" ψ defined by:

$$\psi(k; \alpha, \beta, \sigma) := c_k k^{\frac{\alpha-\sigma}{\beta}} \tag{2.11}$$

which from (2.10) is expected to be bounded by a constant A.

We now recall two original cases given in pionnering works by Riesz [8] and by Hardy-Littlewood [9]. Setting $\sigma = \frac{1}{2}$ in (2.10), for $\alpha = \beta = 2$ (Riesz case) we have that $|c_k| \leq A k^{-\frac{3}{4}}$ and for $\alpha = 1$, $\beta = 2$ (Hardy-Littlewood case), $|c_k| \leq A k^{-\frac{1}{4}}$. Other interesting cases for which we will carry out intensive numerical experiments to be presented below are summarized in Table 1.

A limiting delicate case analysed in [3] is the one where $\alpha = \frac{1}{2} + \delta$ and β grows to infinity. Here of course we do not have absolute convergence to $\frac{1}{\zeta(s)}$ (c_k may nevertheless be analysed) and from (2.10) we have that the c_k should be smaller than a constant for all k. This is what we verified with numerical experiments (not presented here) with values of k up to a billion. The value of the constant has been proposed in our previous work [3] and the conjecture was that $|c_k| \leq |\frac{1}{\zeta(\frac{1}{2})} - 1| \cong 1.68477$. However the situation is delicate ($\alpha < 1$) since Littlewood [9] has shown that, assuming RH is true, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\frac{1}{2}+\epsilon}}$ is convergent for all ϵ strictly greater than zero.

The general situation is that the "critical function" $c_k k^{\frac{\alpha-\sigma}{\beta}}$ should be bounded by a constant in absolute value as $k \to \infty$. In fact the sequence starts at zero

for $k = 0$, reaches a minimum, then starts to increase and then begins to oscillate with a "constant " amplitude as $k \to \infty$ as we will see in the experiments. In the previous work [3] we have analysed c_k in various cases but only for moderately values of k, i.e. for k not exceeding 1000, with exception of some cases at large values of β, where k reached the value of a half billion. c_k was found to have only negative values in the range considered and increasing with k. Presently, we know of recent numerical experiments in the Riesz case carried out by K. Maslanka [6] (k up to 10^6), J. Cislo and M. Wolf [4] (k up to 10^6) and M. Wolf (k up to 10^9) where the calculations indicate that the sequence c_k becomes of oscillatory type, thus assuming positive and negative values. In fact two or three oscillations with a wavelength related in first approximation to the first zero of the Riemann Zeta function has been seen. Here it should be remarked that this situation for the Riesz case is not in contraddiction with our strong coupling limit (β large) cited above (see discussion below for the case $\alpha = \frac{7}{2}$ and β increasing).

In few of these new finding, we want first analyse (in an analytical context) such a behaviour and we call this general phenomena the Riesz and the Hardy-Littlewood wave. This will be analysed using an interesting result of Baez-Duarte, i.e. an expression giving c_k for $k \to \infty$.

3. The Riesz and the Hardy-Littlewood wave

For the Riesz case, in connection with the Mellin inversion formula, the Riesz function is given (see [8] and [10]) explicitly by:

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} \tag{3.1}$$

Using the calculus of residues *F(x)* is obtained by an integration and is given by:

$$F(x) = \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(1-s)x^s}{\zeta(2s)} ds \tag{3.2}$$

where $\frac{1}{2} < a < 1$.

Recently Baez-Duarte [2], with an ingenious method found in particular an expression for the reciprocal of the Pochhammer polynomial given by:

$$\frac{1}{P_k(s)} = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{j}{s-j} \tag{3.3}$$

and one has uniformely on compact subsets:

$$\lim_{k \to \infty} P_k(s) k^s = \frac{1}{\Gamma(1-s)} \tag{3.4}$$

Moreover he was able to obtain an explicit formula connecting c_k and the set of all trivial and non-trivial zeros (let ρ denote a complex Zeta zero) under the assumption of simple zeros. For the Riesz case and for sufficiently large k the expression is given by:

$$-2kc_{k-1} = \sum_{\rho} \frac{1}{\zeta'(\rho) P_k(\frac{\rho}{2})} + o(1) \tag{3.5}$$

TABLE 2. The "amplitude" of ψ for different values of α and β, calculated with (3.8)

α	β	The function	The amplitude
2	2	$\lvert\psi(k;2,2,\frac{1}{2})\rvert = \lvert k^{\frac{3}{4}} c_k\rvert$	0.0000777506
1	2	$\lvert\psi(k;1,2,\frac{1}{2})\rvert = \lvert k^{\frac{1}{4}} c_k\rvert$	0.0000292558
2	6	$\lvert\psi(k;2,6,\frac{1}{2})\rvert = \lvert k^{\frac{1}{4}} c_k\rvert$	0.0210433
$\frac{7}{2}$	4	$\lvert\psi(k;\frac{7}{2},4,\frac{1}{2})\rvert = \lvert k^{\frac{3}{4}} c_k\rvert$	0.00841095
3	3	$\lvert\psi(k;3,3,\frac{1}{2})\rvert = \lvert k^{\frac{5}{6}} c_k\rvert$	0.00215622
4	4	$\lvert\psi(k;4,4,\frac{1}{2})\rvert = \lvert k^{\frac{7}{8}} c_k\rvert$	0.00984936
2	4	$\lvert\psi(k;2,4,\frac{1}{2})\rvert = \lvert k^{\frac{3}{8}} c_k\rvert$	0.00524454

$o(1)$ can be written explicitly in terms of the trivial zeros. It should be said that formula (3.5) of Baez-Duarte is very nice and may be used to control our numerical computations at large k to be presented below. Apparently (3.5), with some precautions, may be extended to our general model with parameters α, β (2.1) and should read:

$$-\beta k c_{k-1} = \sum_{\rho} \frac{1}{\zeta'(\rho) P_k(\frac{\rho-\alpha}{\beta}+1)} + o(1) \tag{3.6}$$

Then using (3.4) in (3.6) we can compute for large k the following approximated expression for the "critical function" (we call it the Riesz and the Hardy-Littlewood wave) ψ:

$$\psi(k;\alpha,\beta,\sigma) = k^{\frac{\alpha-\sigma}{\beta}} c_k \cong \psi_{nt}(k;\alpha,\beta,\sigma) + \psi_t(k;\alpha,\beta,\sigma) \tag{3.7}$$

where ψ_{nt} and ψ_t are the contributions of the non-trivial respectively trivial zeros of the Zeta function. They are given by:

$$\psi_{nt}(k;\alpha,\beta,\sigma) = \frac{1}{\beta} \sum_{\rho} \frac{k^{\frac{\rho-\sigma}{\beta}} \Gamma(\frac{\alpha-\rho}{\beta})}{\zeta'(\rho)} \tag{3.8}$$

and

$$\psi_t(k;\alpha,\beta,\sigma) = \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{k^{-\frac{2n+\sigma}{\beta}} \Gamma(\frac{\alpha+2n}{\beta})}{\zeta'(-2n)} \tag{3.9}$$

We concentrate our numerical research on three topics: the amplitude of the Riesz and the Hardy-Littlewood wave in the long wavelength limit (k large) for the special case $\sigma = \frac{1}{2}$, the case where we steadily increase the parameter β and finally its behaviour for different values of σ (that is inside the critical strip).

4. NUMERICAL EXPERIMENTS: THE "AMPLITUDE" OF THE CRITICAL FUNCTION

In the limit of large k, one may consider to neglect the contribution of the trivial zeros in (3.7) [2]. To prepare the comparison of (3.8) with the numerical results we write it explicitly for the various cases we will treat. In order to obtain an estimate for the amplitude of the wave in the long wavelength limit (k large) we will use in (3.8) only the first zero and its complex conjugate up to 10 decimals (β small). Setting the first derivative of (3.8) to zero and solving the equation with *Mathematica*, we obtain the data in Table 2.

The values for the amplitude of the waves in Table 2 will be compared with the results of the numerical experiments performed for the various cases using (2.3).

Our numerical experiments was carried out in more cases using the Möbius function in (2.3) up to $n = 10^6$ and we calculated c_k until $k = 10^6$ or $k = 10^8$ with a sample interval of 2500 for the k-axis. The general situation is that at moderately values of k (until some thousand) the wave given by the experimental results starts with zero amplitude, after a minimum with a negative value, increases and seems to stabilize at large values of k with oscillations displaced at larger and larger wavelength (proportional to $\log k$) and with an amplitude which seems to saturate to a constant value (given in a good approximation) by the values in Table 2. Below (Figure 1) we first give the plot of the wave for the Riesz case ($\alpha = \beta = 2$). As already remarked in [2], the first intensive calculations by K. Maslanka, J. Cislo and M. Wolf indicate the appearance of oscillations with the first one in the region $k = 20000$. Our results obtained with (2.3) confirm for such values the asymptotic limit for the wave with an amplitude in agreement with the constant obtained above ($A \cong 0.000078$).

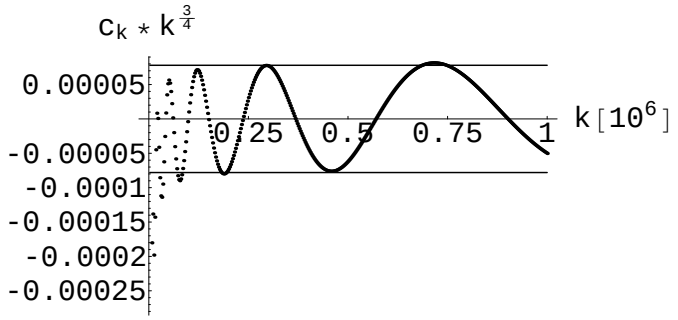

FIGURE 1. The wave $k^{\frac{3}{4}}c_k$ for the Riesz case $\alpha = \beta = 2$

Figure 2 and Figure 3 concern two cases of special interest since the decays of c_k are expected to be the same as for the Hardy-Littlewood case and for the Riesz case. In both cases there is agreement with the "amplitude" $A \cong 0.0210433$ and $A \cong 0.008411$ given above but the amplitudes are respectively 1000 and 100 time bigger than in the first case.

In Figure 4 and Figure 5 we give the plots of $k^{\frac{5}{6}}c_k(3,3)$ and $k^{\frac{7}{8}}c_k(4,4)$ where the amplitudes are still found to be in agreement with the theoretical values given above in Table 2.

The next special case is the one with $\alpha = 2$ and $\beta = 4$. Again, the experimentally detected amplitude agrees well with the theoretical value calculated above, i.e. $A = 0.0052445$ (Figure 6).

5. Numerical experiments: β increasing

For $\alpha = \frac{7}{2}$ we will now present the plots of the waves for an increasing sequence of β values i.e. 4, 8, 12, and 20 (in order to investigate the "infinite beta limit"

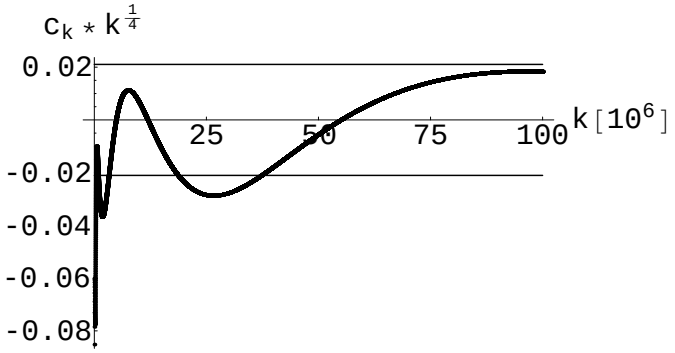

FIGURE 2. The wave $k^{\frac{1}{4}}c_k$ for the case $\alpha = 2, \beta = 6$

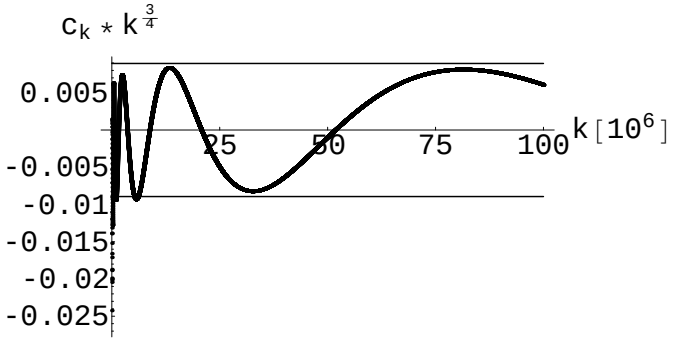

FIGURE 3. The wave $k^{\frac{3}{4}}c_k$ for the case $\alpha = \frac{7}{2}, \beta = 4$

already introduced in our previous work [3]). We will compute the function

$$\psi(k; \frac{7}{2}, \beta, \frac{1}{2}) = k^{\frac{3}{\beta}} c_k(\frac{7}{2}, \beta) \tag{5.1}$$

which will also be compared with the expression given by the Baez-Duarte formula (3.8) in the asymptotic region $k \to \infty$, i.e. $\psi_{nt}(k; \frac{7}{2}, \beta, \frac{1}{2})$. Here we will take into account only the contribution of the groundstate of the spectrum i.e. $\rho = \frac{1}{2} + 14.134725141i$ and its complex conjugate $\overline{\rho}$. It is then convenient to introduce the new variable $x = \log k$. This allow us to control more efficiently the wavelength and the amplitude of the wave in the region to be considered (x runs from 8 to 22, so k up to 3.6×10^9).

In the Figures 7-10 we present our numerical results for increasing β values, which we call the "strong coupling limit". In the range $\log k > 10$ the two waves are walking close together arm in arm at least for $\beta = 4$ (Figure 7). This confirms

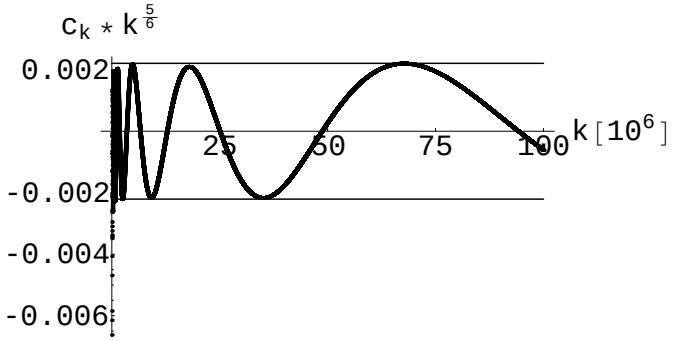

FIGURE 4. The wave $k^{\frac{5}{6}}c_k$ for the case $\alpha = \beta = 3$

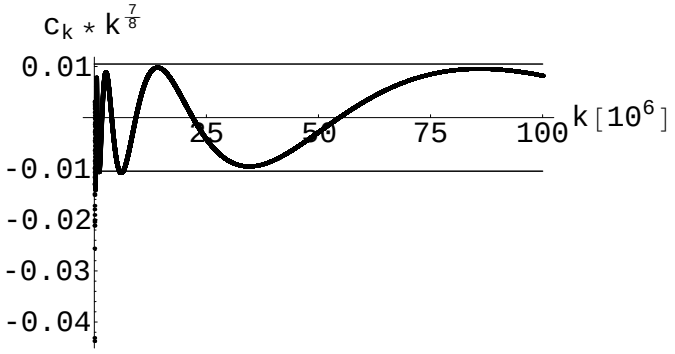

FIGURE 5. The wave $k^{\frac{7}{8}}c_k$ for the case $\alpha = \beta = 4$

that the contribution of the first zero ($\rho = \frac{1}{2} + 14.134725141i$ and $\overline{\rho}$) appears to be dominant at low values of β.

At the same time one can see that in this case $|c_k|$ itself is smaller than (c_k is not the critical function!):

$$\left| \frac{1}{\zeta(\frac{7}{2})} - 1 \right| \cong 0.11247897 \tag{5.2}$$

at least for the case $\beta = 4$ as already discussed in our previous work [3] concerning only very low values of k. Figure 11 confirms this behaviour also for large values of k. In this example the region of annihilation of the "eincoming" wave extends up to larger and larger values of k. It should be noted that for the critical function ψ (5.1) the situation is more delicate since the value of a possible bound on ψ depends on β.

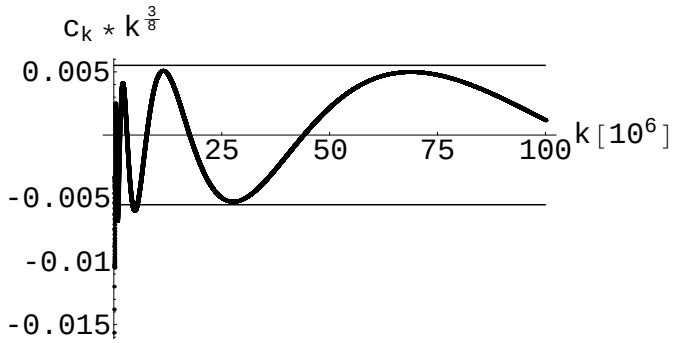

FIGURE 6. The wave $k^{\frac{3}{8}}c_k$ for the case $\alpha = 2, \beta = 4$

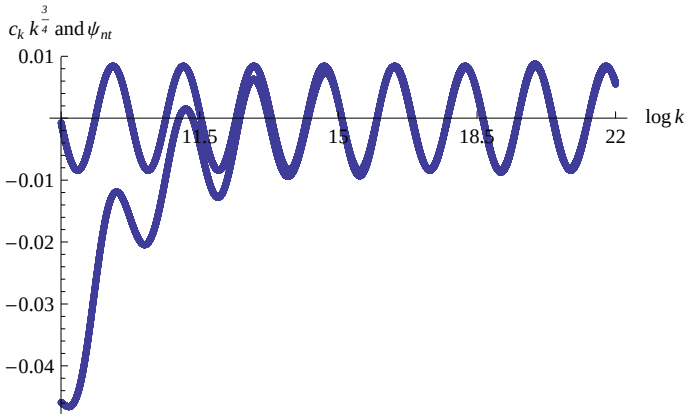

FIGURE 7. The wave $k^{\frac{3}{4}}c_k$ (lowest curve) and the approximation ψ_{nt} (highest curve), $\alpha = \frac{7}{2}$ and $\beta = 4$

6. Numerical experiments: the critical function in the critical strip

As above, in this numerical study it is convenient to introduce the variable $x = \log k$, in term of which we define the critical function corresponding to α and β. With the help of (2.6), this is given in the next calculations by:

$$\psi(x;\alpha,\beta,\sigma) = e^{\frac{\alpha-\sigma}{\beta}x}\sum_{n=1}^{2000}\frac{\mu(n)}{n^{\alpha}}e^{-\frac{e^x}{n^{\beta}}} =: \psi_{\sigma}(x) \tag{6.1}$$

2000 is the maximum argument N used in these experiments. For the special case we treat ($\alpha = \frac{15}{2}$ and $\beta = 4$) ψ will be calculated up to $x = 30$ (this corresponds to $k = e^{30} = 1.06865 \times 10^{13}$).

Before we present the results of our numerical experiments for various values of σ (for $\sigma = 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{3}{10}$) it is important to give the explicit expression of the contribution of the non-trivial (ψ_{nt}) and also of the trivial zeros (ψ_t) to the critical

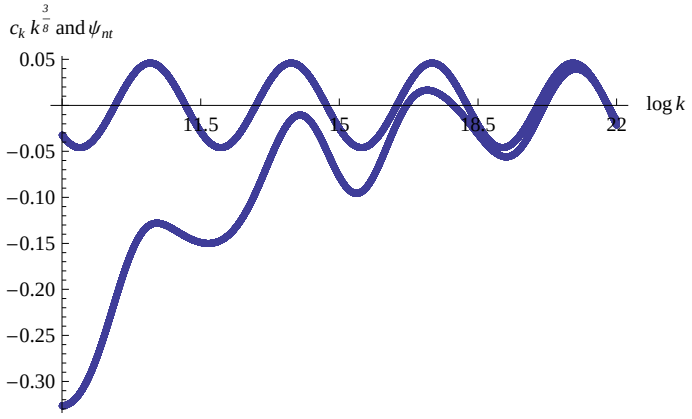

FIGURE 8. The wave $k^{\frac{3}{8}}c_k$ (lowest curve) and the approximation ψ_{nt} (highest curve), $\alpha = \frac{7}{2}$ and $\beta = 8$

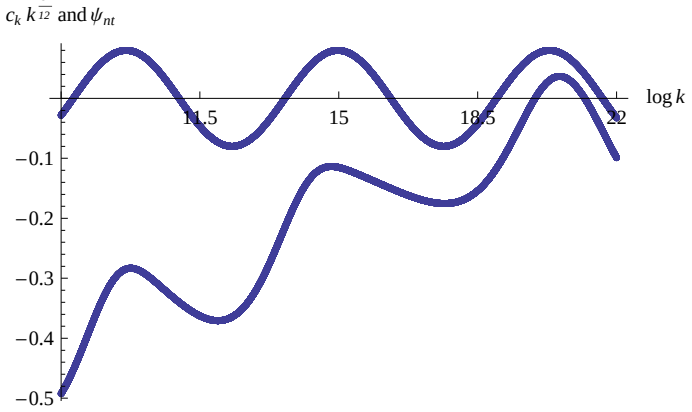

FIGURE 9. The wave $k^{\frac{3}{12}}c_k$ (lowest curve) and the approximation ψ_{nt} (highest curve), $\alpha = \frac{7}{2}$ and $\beta = 12$

function defined above for the general case α and β. For the non-trivial zeros at σ, it is given by:

$$\psi_{nt}(x;\alpha,\beta,\sigma) = \frac{1}{\beta}\sum_{j=1}^{2}\frac{e^{\frac{i\Im(\rho_j)}{\beta}x}\Gamma(-\frac{\rho_j-\alpha}{\beta})}{\zeta'(\rho_j)} \tag{6.2}$$

where ρ_j is a non-trivial zero of $\zeta(s)$. In our experiments we will limit to the contribution of the first two lower zeros given experimentally by $\rho_1 = \frac{1}{2}+14.134725i$ and $\rho_2 = \frac{1}{2} + 21.022040i$ and the complex conjugate of them. The corresponding contribution will be denoted by $r_1(x)$ (from ρ_1 and $\overline{\rho}_1$) and $r_2(x)$ (from ρ_2 and $\overline{\rho}_2$).

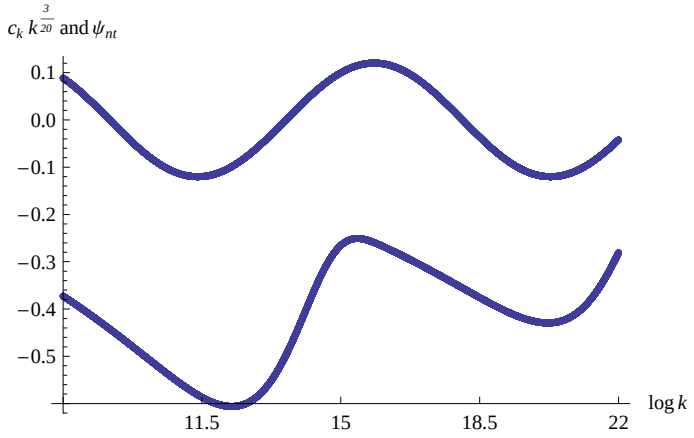

FIGURE 10. The wave $k^{\frac{3}{20}}c_k$ (lowest curve) and the approximation ψ_{nt} (highest curve), $\alpha = \frac{7}{2}$ and $\beta = 20$

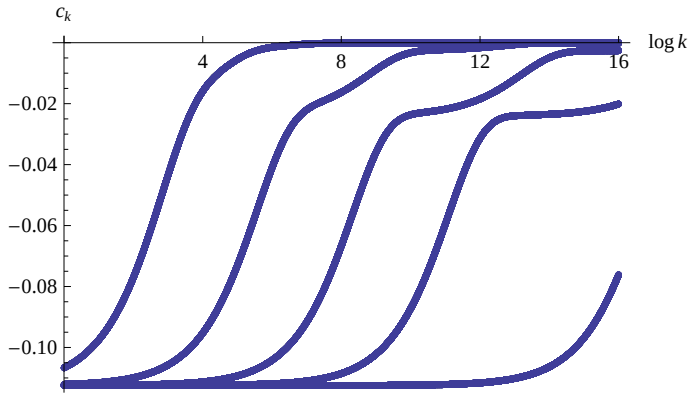

FIGURE 11. $c_k(\frac{7}{2}, \beta)$ for $\beta = 4, 8, 12, 16, 24$ (from left to right)

The contribution of the trivial zeros $\rho = -2n$ to the critical function is given by:

$$\psi_t(x; \alpha, \beta, \sigma) = \frac{1}{\beta} \sum_{n=1}^{20} \frac{e^{-\frac{2n+\sigma}{\beta}x} \Gamma(\frac{\alpha+2n}{\beta})}{\zeta'(-2n)} \tag{6.3}$$

where a summation until $N = 20$ will be sufficient.

So, in our calculations we will set $\alpha = \frac{15}{2}$ and $\beta = 4$ in the above formulas, for any value of σ we shall consider. The contribution ψ_t for σ will be indicated with $g_\sigma(x)$. Below we present the results of our numerical experiments performed using *Mathematica*. The fluctuation's errors in the Möbius function around the maximum index $N = 2000$ will be specified in Section 7.

In Figure 12 we present the plot of the two functions $\psi(x; \frac{15}{2}, 4, \frac{1}{2}) - g_{1/2}(x)$ and $r_1(x) + r_2(x)$ up to $x = 30$ which shows not only a good agreement but also the oscillatory behaviour of the contribution of the first two non-trivial zeros. Note

that for the Riesz case ($\alpha = \beta = 2$), the contribution of the trivial zeros to the c_k have been treated by K. Maslanka using the Rice integrals [6].

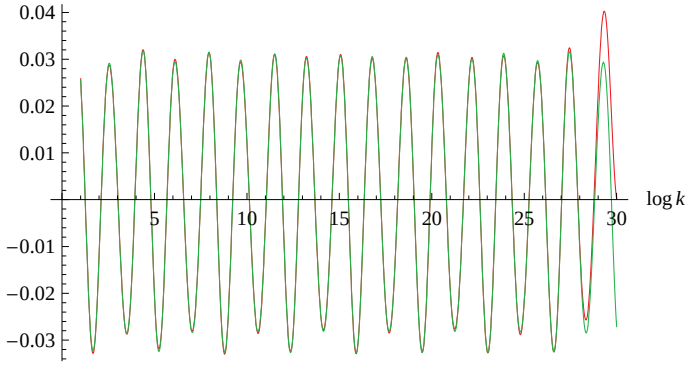

FIGURE 12. Plot of $\psi_{1/2}(x) - g_{1/2}(x)$ [red] and $r_1(x) + r_2(x)$ [green] up to $x = 30$

In the next Figure 13 we present the plots of some critical functions ψ_σ corresponding to different values of σ using (6.1) and this without any comparison with the Baez-Duarte asymptotic expansion considered above. It is to be noted that all functions ψ_σ has the same zeros and we observe that there is a well marked evidence that for $\sigma > \frac{1}{2}$ increasing to 1 the amplitudes decay while for $\sigma < \frac{1}{2}$ the amplitudes grow. These functions have been indicated with ψ_1, $\psi_{7/8}$, $\psi_{3/4}$, $\psi_{5/8}$, $\psi_{1/2}$, $\psi_{3/8}$, $\psi_{3/10}$ respectively.

It should be said that $\psi_{3/8}$ and $\psi_{3/10}$, we have considered, have no relation with the representation of $\frac{1}{\zeta(s)}$ which is valid only for $\Re(s) > \frac{1}{2}$. The two functions help only to visualize that $\psi_{1/2}$ is the borderline for the critical functions decaying for $\Re(s) > \frac{1}{2}$ as suggested by our numerical experiments up to $x = 30$. It should also be added that from the duality relation (Riemann's symmetry of the Zeta function), given by:

$$\frac{1}{\zeta(1-s)} = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \frac{1}{\zeta(s)} \tag{6.4}$$

it follows that the right hand side of (6.4) ensures a representation of $\frac{1}{\zeta(s)}$ via the Pochhammer polynomials in the region $0 < \Re(s) < \frac{1}{2}$.

Here there is more evidence that the amplitude of the wave at $\sigma = \frac{1}{2}$ is decreasing with $x = \log k$. The experiments of Figure 13 give in any cases a stronger evidence: for $\sigma > \frac{3}{4}$ the amplitudes of the waves are decaying, and thus are bounded in amplitude by a constant. This is a symptom of the absence of non-trivial zeros in the critical segment $\frac{3}{4} < \sigma < 1$.

In Figure 14 we present the result for a special case where we allow a slower decrease in the critical function (see addendum in the exponent of the critical function), which is the same as to say that we ask only for a slower decay of c_k, at $\sigma = \frac{1}{2}$ i.e. of the type $c_k = \frac{A \log k}{k^{\frac{7}{4}}}$ for the case considered. This is not the same as to ask that the RH is true or that the RH is true with non-trivial zeros which are simple [2]. It is a case in between the two.

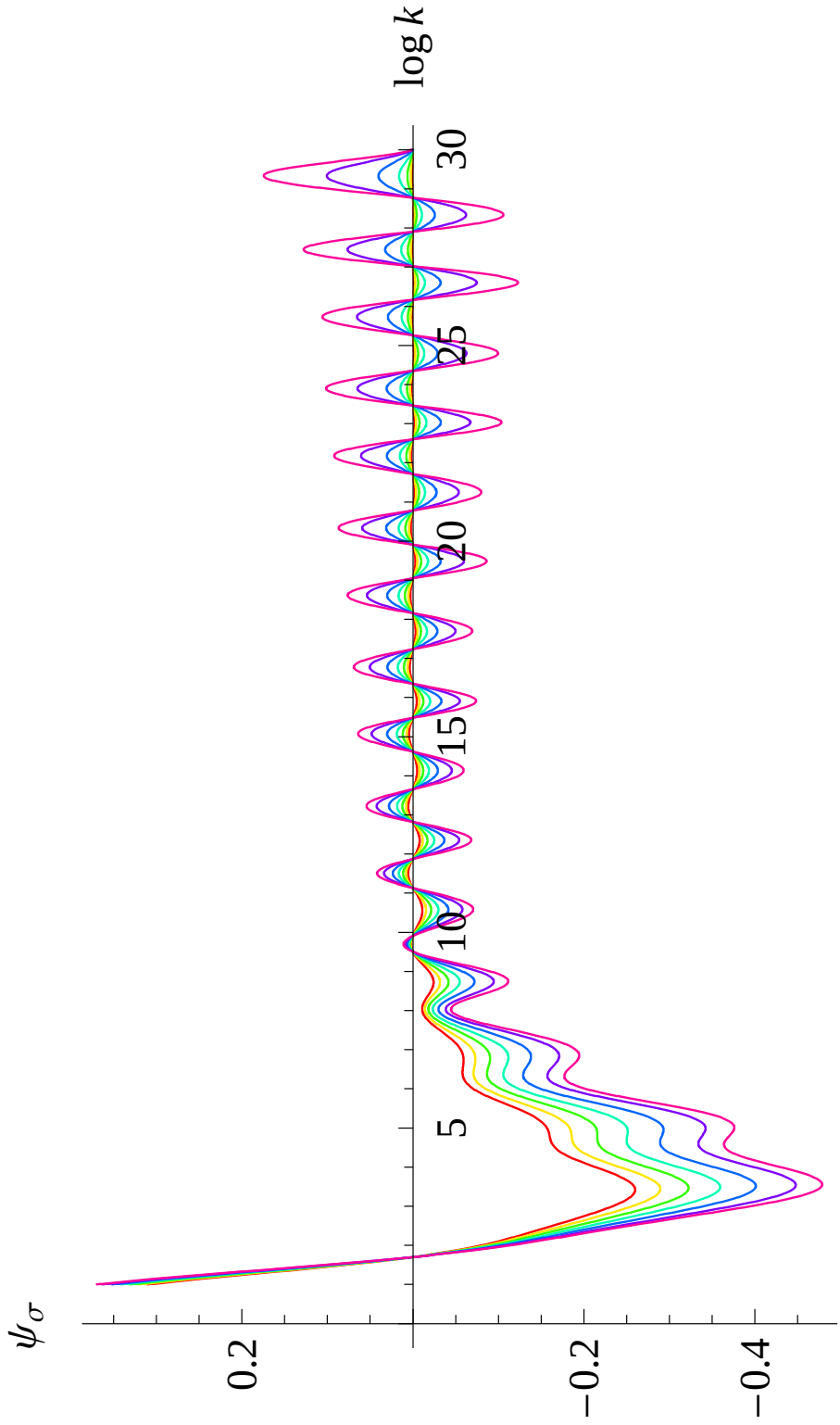

FIGURE 13. Plot of ψ_σ for $\sigma = 1, \frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}, \frac{3}{10}$ up to $x = 30$, in order of increasing amplitudes

In this case the critical function (indicated with $\psi_{1/2+}$) is explicitly given by:

$$\psi_{1/2+}(x) = e^{\frac{7}{4}x - \log x} \sum_{n=1}^{2000} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}} \tag{6.5}$$

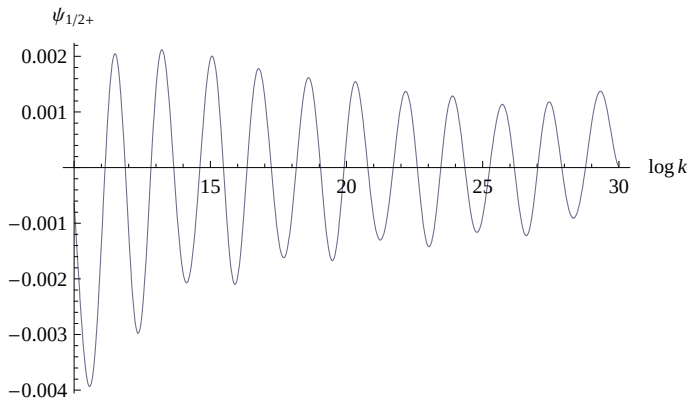

FIGURE 14. Plot of $\psi_{1/2+}$

In the last experiment we set $\sigma = \frac{3}{4}$ and compare $\psi_{3/4}$ with the asymptotic expression of Baez-Duarte: for the trivial zeros we set $\sigma = \frac{3}{4}$ in the above formula, for the non-trivial zeros (the two we consider) we keep the same value of $\Im(\rho_{1,2})$ but we assume that their real part is $\Re(\rho_{1,2}) = \frac{3}{4}$. The plot in Figure 15 of the function $\psi_{3/4}(x)$ and of $g_{3/4}(x) + r_1(x) + r_2(x)$ are clearly different: in $\psi_{3/4}$ there is the trace via the Möbius function of where the non-trivial zeros are located and thus the amplitude is decaying. In the second function, the two considered zeros are supposed to have $\Re(s) = \frac{3}{4}$ and the wave which appears seems to have a constant amplitude as in the case $\psi_{1/2}$ which of course would be sufficient to ensure the truth of the RH.

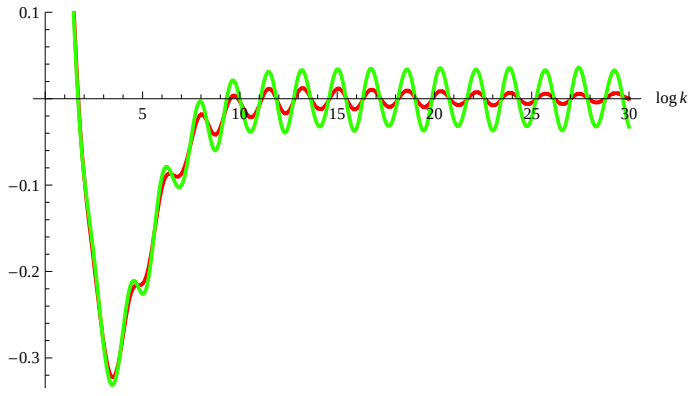

FIGURE 15. Plots of the functions $\psi_{3/4}(x)$ [red] and $g_{3/4}(x) + r_1(x) + r_2(x)$ [green]

In the next Section we analyse a (weak) stability property of our results obtained with $N = 2000$ in the Möbius function and give some indications why the waves for $\sigma = \frac{3}{4}$ should be decaying, thus ensuring more credibility on the absence of zeros of the Riemann Zeta function in the segment $\frac{3}{4} < \sigma < 1$.

7. Numerical considerations

In the context of the numerical experiments performed so far, it is helpful to obtain a crude inequality concerning a bound on the critical function. This is simply obtained by setting $|\mu(n)| = 1$ in (2.3). We consider the critical function for $\sigma = \frac{1}{2}$ given by:

$$k^{\frac{\alpha-\frac{1}{2}}{\beta}} c_k \cong k^{\frac{\alpha-\frac{1}{2}}{\beta}} \sum_{n=1}^{N} \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k =: \psi_k(\alpha, \beta, N)$$

where N is the maximum value of the argument in the Möbius function considered in an ideal numerical experiment (N finite). Introducing the variable $x = \log k$ we have that:

$$|\psi_k(\alpha, \beta, N)| \leq e^{\frac{\alpha-\frac{1}{2}}{\beta} x} (\zeta(\alpha) - 1) e^{\log(1-\frac{1}{N^\beta}) e^x}$$

For large N we have:

$$|\psi_k(\alpha, \beta, N)| \leq (\zeta(\alpha) - 1)\, e^{\frac{\alpha-\frac{1}{2}}{\beta} x - \frac{1}{N^\beta} e^x}$$

As an example we consider the case $\alpha = \frac{7}{2}$ and $\beta = 4$ (Section 4). Remembering that from Table 2 the amplitude calculated only with the first non-trivial zero is about 0.008411, we may ask: for what N and k, $|\psi_k(\alpha, \beta, N)|$ is bounded by the value 0.008411? For example the inequality is satisfied for the followig pairs:

$N = 1000$ and $x > 31$, or
$N = 10^6$ and $x > 60$.

As a second example we consider the Riesz case ($\alpha = \beta = 2$). From Table 2, the amplitude (still restricting to the contribution of the first zero) is 0.000078. The inequality is satisfied as follows:

$N = 1000$ and $x > 17$, or
$N = 10^6$ and $x > 31$, or
$N = 10^9$ and $x > 87.2$.

This inequality may be helpful to control the numerical computations in the experiments.

Another numerical consideration will be the following. We consider the critical function $\psi_{3/4}(x)$ obtained with $N = 2000$ (maximum argument in the Möbius function appearing in the Baez-Duarte definition of the c_k). We will suppose that the numerical results are given with good accuracy. We now ask: if we increase N from 2000 up to 10^6 in a ideal experiment, what will be the change of the critical function in the range $x < 30$?

$$\psi_{3/4}(x; N = 2000) = e^{\frac{27}{16} x} \sum_{n=1}^{2000} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}}$$

$$\psi_{3/4}(x; N = 10^6) = e^{\frac{27}{16} x} \sum_{n=1}^{10^6} \frac{\mu(n)}{n^{\frac{15}{2}}} e^{-\frac{e^x}{n^4}}$$

The difference Δ between the two functions is bounded by:

$$\Delta \leq e^{\frac{27}{16}x} \sum_{n=2000}^{10^6} \frac{1}{n^{\frac{15}{2}}} e^{-\frac{e^x}{10^{24}}}$$

If we ask that Δ will be smaller than say 10^{-6} time 0.015 which is about the value of the amplitude of the wave in the range $x \leq 30$, obtained with $N = 2000$, we have:

$$\Delta \leq e^{\frac{27}{16}x} e^{-\frac{e^x}{10^{24}}} (\zeta(\frac{15}{2}) - \zeta(\frac{15}{2}; N = 2000)) \leq 0.015 \cdot 10^{-6}$$

The difference between the Zetas is estimated to:

$$\int_{2000}^{\infty} \frac{1}{x^{\frac{15}{2}}} dx = \frac{2}{13} 2000^{-\frac{13}{2}} = \frac{2}{65} 10^{-26}$$

and the inequality takes the form:

$$\frac{27}{16}x - e^{\frac{x}{10^{24}}} + \log(\frac{2}{65}) - 26\log(10) + 6\log(10) - \log(0.015) \leq 0$$

with the solution $x \leq 27$. Thus for $x \leq 27$, the amplitudes will change at most 10^{-6} time of its value 0.015. This shows some stability in the numerical experiments as N increases in a ideal experiment. Of course this is independent of how many zeros are employed in the Baez-Duarte estimation.

The third remark deals with the formula for the c_k we have used in our experiments and given by [3]:

$$\hat{c}_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} e^{-\frac{k}{n^{\beta}}}$$

instead of the correct formula:

$$c_k = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}}\right)^k$$

Again, as above, the crude inequality $|\mu(n)| \leq 1$ may be used to show that the difference between the two sequences becomes smaller as k get bigger and depends on α and β. In fact it behaves unconditionally as:

$$\frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}} \tag{7.1}$$

To see this, let $\Delta = |\hat{c}_k - c_k|$ then:

$$\Delta \leq \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^{\alpha}} \left(e^{-\frac{k}{n^{\beta}}} - \left(1 - \frac{1}{n^{\beta}}\right)^k\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \left(e^{-\frac{k}{n^{\beta}}} - \left(1 - \frac{1}{n^{\beta}}\right)^k\right)$$

since $e^{-\frac{k}{n^{\beta}}} \geq (1 - \frac{1}{n^{\beta}})^k$. Passing to the continuous variable x, the contribution of the second integral is given by [3]:

$$\int_1^{\infty} \frac{1}{x^{\alpha}} \left(1 - \frac{1}{x^{\beta}}\right)^k dx = \frac{1}{\beta} \frac{\Gamma(\frac{\alpha-1}{\beta})\Gamma(k+1)}{\Gamma(\frac{\alpha-1}{\beta} + k + 1)}$$

while the first is given by:

$$\int_1^\infty \frac{e^{-\frac{k}{x^\beta}}}{x^\alpha} dx = \frac{1}{\beta k^{\frac{\alpha-1}{\beta}}} \Gamma(\frac{\alpha-1}{\beta})$$

Using Stirling's formula, at large k the difference behaves like:

$$\Delta \leq \frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}}.$$

For the model under consideration the decay is as $\frac{C}{k^{\frac{21}{8}}}$ and is stronger then in the usual Riesz case ($\alpha = \beta = 2$) where an early more detailed calculation gives a decay like $\frac{C}{k^{\frac{3}{2}}}$ [4].

Finally it should be added that the general upper bound for Δ is related to the discrete derivative of the Baez-Duarte coefficients given by:

$$\begin{aligned} c_k(\alpha,\beta) - c_{k+1}(\alpha,\beta) &= \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} \left(\left(1 - \frac{1}{n^\beta}\right)^k - \left(1 - \frac{1}{n^\beta}\right)^{k+1} \right) \\ &= \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta}\right)^k \left(1 - 1 + \frac{1}{n^\beta}\right) = c_k(\alpha+\beta,\beta) \end{aligned}$$

which unconditionally are bounded by $\frac{C}{k^{\frac{\alpha+\beta-1}{\beta}}}$ as above.

In the same way

$$-\frac{d}{dk} \sum_{n=1}^\infty \frac{\mu(n)}{n^\alpha} e^{-\frac{k}{n^\beta}} = \sum_{n=1}^\infty \frac{\mu(n)}{n^{\alpha+\beta}} e^{-\frac{k}{n^\beta}}$$

which gives the same decay since the function is equal to $c_k(\alpha+\beta,\beta)$ as above.

At large k we also have [3]:

$$c_k \approx \sum_{p=0}^\infty \frac{c_p k^p e^{-k}}{p!}$$

a Poisson like distribution for the coefficients c_k.

8. Conclusions

In this work, we have used the expansion in terms of the Pochhammer polynomials for the reciprocal of the Zeta function. Our expansion contains two parameters α and β so that our analysis was possible for different functions, called "critical functions". The boundedness of the critical function would ensure the truth of the RH.

In a numerical context we have first presented an extensive treatment of the critical functions via the Möbius function. Then we have compared the amplitudes of the "Riesz, Hardy and Littlewood waves" using an extension of the formula of Baez-Duarte (the formula contains the contribution of the trivial and non-trivial zeros of the Riemann Zeta function): the agreement with the treatment using the Möbius function seems satisfactory even if we have considered only very few zeros in the Baez-Duarte formula for the coefficients c_k.

For a special case where $\alpha = \frac{7}{2}$ and $\beta = 4$, we have then considered different values of $\Re(s)$, i.e. values in the critical segment from 1 to $\frac{3}{10}$: the critical functions

appear decaying, starting from the right border $\Re(s) = 1$ to reach near $\Re(s) = \frac{1}{2}$ a behaviour still bounded, with oscillations of a nearby constant amplitude. This is not in contradiction with the possible truth of the RH. Finally we have remarked some stability property of the amplitudes of the waves involved in the experiments in the asymptotic region (increasing values of $\log k$).

The numerical results up to a maximum value of $\log k = 30$ (i.e. $k = 1.06865 \times 10^{13}$) go more in the direction to believe that the critical functions do not increase with $\log k$ and that they should reach a behaviour with a stable amplitude of the waves which is smaller that the maximum conjectured value given by $1.68477\ldots$.

In the context of validity of our numerical results, our analysis gives further indication that the RH may barely be true as indicated by our two parameter models in the week as well as in the "strong coupling regime".

So the open question is still the following: the critical function at large value of k is growing, stabilizing to a "periodic pure wave" with constant amplitude or decaying with a zero amplitude? From the results of our numerical treatment we are more in favour of the last two cases.

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