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THE DIAMETER OF SOME HECKE-FAREY MAPS

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Abstract. We study the regular maps arising from the principal congruence subgroups of the Hecke groups H_4 and H_6. We prove that they have diameter 4.

1. Introduction

The Farey map $\mathbb{F}$ has as its vertex set the extended rationals $\mathbb{Q}\cup\{\infty\}$, where we write $\infty = \frac{1}{0}$. Two rationals $\frac{a}{c}, \frac{b}{d}$ are joined by an edge if and only if $ad - bc = \pm 1$. The automorphism group of $\mathbb{F}$ is $\Gamma = PSL(2,\mathbb{Z})$, the classical modular group. In [6] and [4] we showed that $\mathbb{F}$ is the universal triangular map in that every triangular map on an orientable surface is the quotient of $\mathbb{F}$ by a subgroup of the modular group. Regular triangular maps correspond to normal subgroups of the modular group and in [3] the regular maps $\mathcal{M}_3(n)$ corresponding to the principal congruence subgroups $\Gamma(n)$ were studied. An interesting result is that the corresponding maps all have diameter 3. In [4] the authors used the Hecke groups H_q to construct the universal q-gonal maps $\hat{\mathscr{M}}_q$. We study the maps $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$. We look at a few examples, such as the regular map on Bring's surface $\mathscr{M}_4(5)$ and show that all these maps have diameter 4.

2. Hecke groups and universal q-gonal maps

In [2], Hecke introduced the groups H_q which are generated by two real Möbius transformations:

$$S(z) = \frac{-1}{z} \text{ and } T(z) = z + \lambda,$$

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where λ is a fixed positive real number. Moreover, we can represent S and T as

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Hecke showed that this group is discrete if and only if

(i) $\lambda \geq 2$, or when

(ii) $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$.

In both these cases S and T generate a Fuchsian group, but only in case (ii) does this group have a fundamental region of finite hyperbolic area. This means we get Fuchsian group of the first kind. This is the case that we shall be dealing with, so we define the Hecke group H_q to be the group generated by S and T above with $\lambda = \lambda_q$. So we define the *Hecke group* H_q to be the Fuchsian group generated by S and T above. If $R = TS$ then R is represented by the matrix

$$\begin{pmatrix} \lambda_q & -1 \\ 1 & 0 \end{pmatrix}$$

This has eigenvalues $e^{\frac{\pi}{q}i}$ and $e^{\frac{-\pi}{q}i}$ and so R is a transformation of order q. Now T is a parabolic transformation, (sometimes called a limit rotation) which can be viewed as an elliptic element of infinite order . Thus $S^2 = T^\infty = (ST)^q = I$, so H_q could be thought of as the $\Gamma(2,\infty,q)$ triangle group. The relation $T^\infty = 1$ is thought of as being vacuous so that as a group $H_q \cong C_2 * C_q$. The relation $T^\infty = 1$ can be read as "T is parabolic".

Now $\lambda_3 = 1,\ \lambda_4 = \sqrt{2},\ \lambda_6 = \sqrt{3}$. The Hecke groups H_4 and H_6 are the only Hecke groups, apart from the modular group Γ, whose elements are completely known.

For $q = 4$ and 6, H_q consists a set of all matrices of the following two types:
(i) the elements of type

$$\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \ a,b,c,d \in \mathbb{Z}\ ,\ ad - mbc = 1,$$

(ii) the elements of type

$$\begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix} \ a,b,c,d \in \mathbb{Z}\ ,\ mad - bc = 1.$$

Where $m = 2$ or 3 for $q = 4$ and 6 respectively. The elements of type (i) are called *even*, while those of type (ii) are *odd*. For $q = 4, 6$ the even elements are a subgroup of index 2 of H_q denoted by H_q^e. While the set of the odd elements is the other coset of H_q^e in H_q, denoted by H_q^o. Hence we have

$$H_q^e = \left\{ A = \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \mid A \in H_q \right\}, \tag{2.1}$$

and

$$H_q^o = \left\{ A = \begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix} \mid A \in H_q \right\}. \tag{2.2}$$

Now, for any value of $q \geq 2$ we can define the universal q-gonal map $\hat{\mathscr{M}}_q$ as follows. The edges of $\hat{\mathscr{M}}_q$ are the images of the imaginary axis I, which is the edge going from $0 = \frac{0\sqrt{m}}{1}$ to $\infty = \frac{1}{0\sqrt{m}}$. Note that the edges are hyperbolic geodesics. The vertices are the images of ∞ under H_q, The *principal face* is the q-gon that has, as its vertices, ∞, $R(\infty)$, $R^2(\infty), \cdots$, $R^{q-1}(\infty)$. The general faces are just the images of the principal face under H_q. Examples. For $q = 4$, $\lambda_4 = \sqrt{2}$ the vertices of the principal face are $\frac{1}{0}, \frac{0}{1}, \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{1}$, and for $q = 6$, $\lambda_6 = \sqrt{3}$ the vertices of the principal face are $\frac{1}{0}, \frac{0}{1}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}}{2}, \frac{2}{\sqrt{3}}, \frac{\sqrt{3}}{1}$.

Figure 1. Principal face when $q = 4$

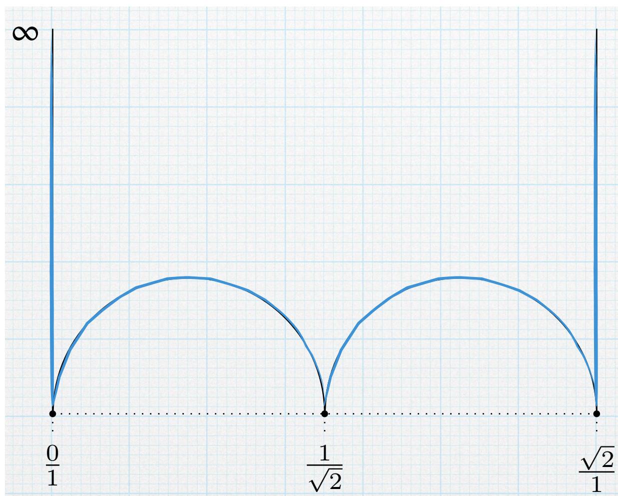
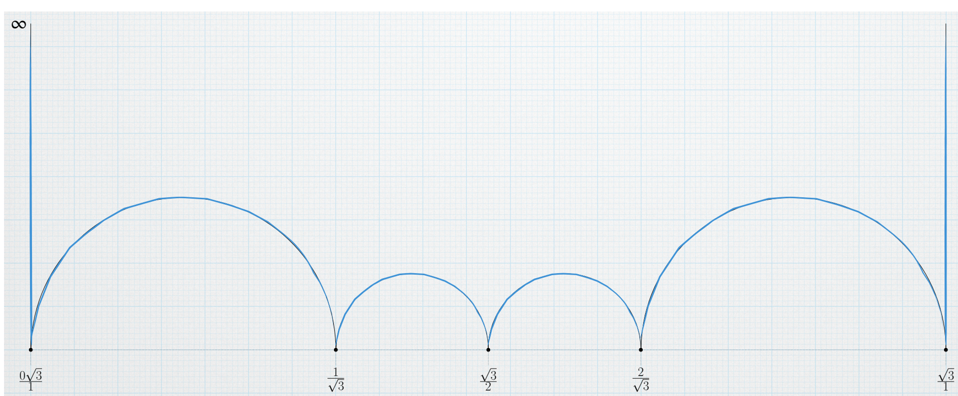


Figure 2. Principal face when $q = 6$



In Figures 3 and 4 we define the maps $\hat{\mathscr{M}}_4$ and $\hat{\mathscr{M}}_6$ and in Tables 1 and 2 we list the vertices of these maps.

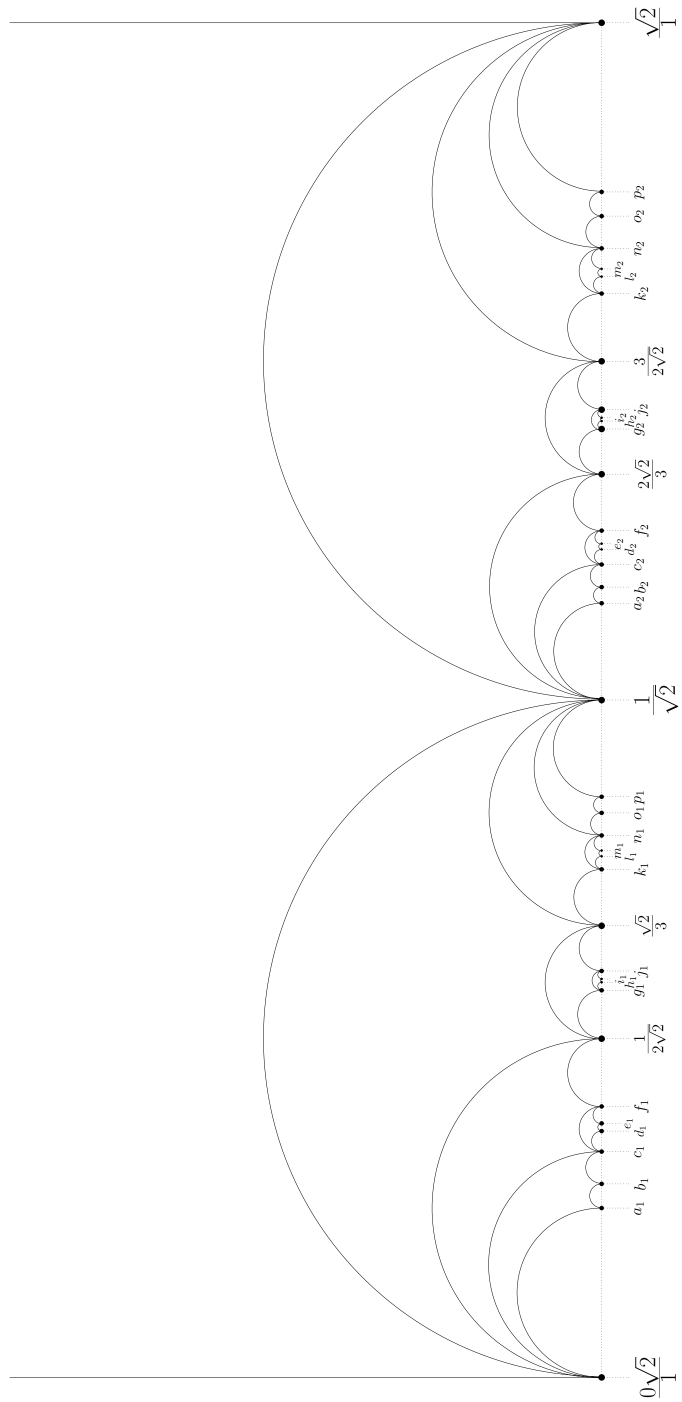


FIGURE 3. The universal 4-gonal tessellation $\hat{\mathscr{M}}_4$

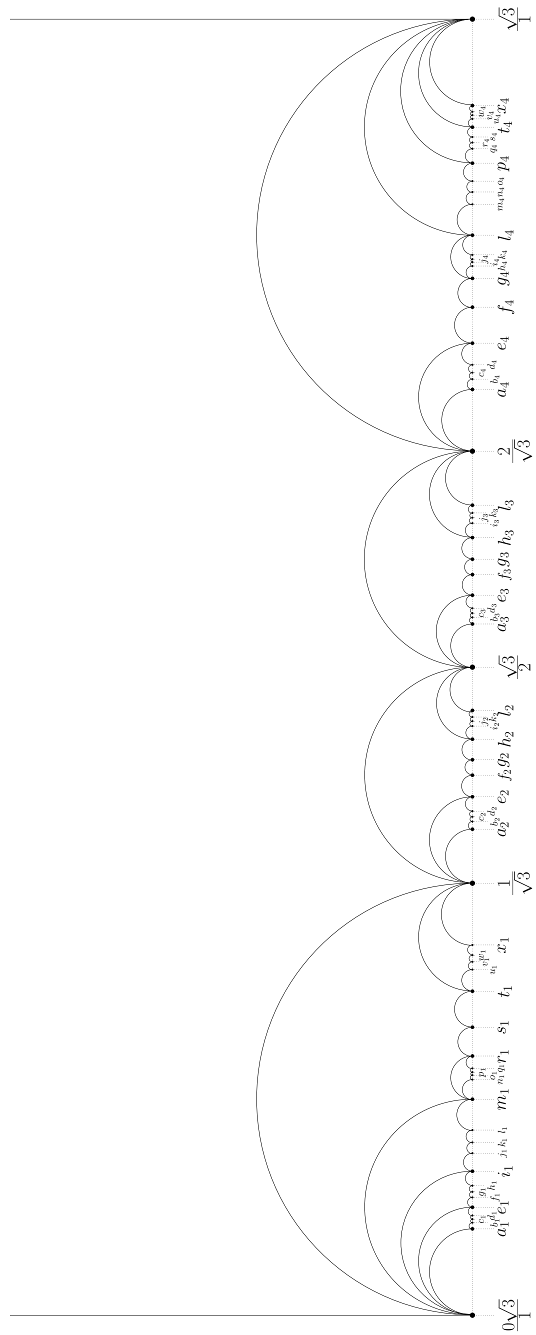


FIGURE 4. The universal 6-gonal tessellation $\hat{\mathscr{M}}_6$

TABLE 1. Table of Correspondence for $\hat{\mathscr{M}}_4$

$a_1 : \frac{1}{4\sqrt{2}}$	$i_1 : \frac{5\sqrt{2}}{17}$	$a_2 : \frac{4\sqrt{2}}{7}$	$i_2 : \frac{17}{12\sqrt{2}}$
$b_1 : \frac{\sqrt{2}}{7}$	$j_1 : \frac{3}{5\sqrt{2}}$	$b_2 : \frac{7}{6\sqrt{2}}$	$j_2 : \frac{5\sqrt{2}}{7}$
$c_1 : \frac{1}{3\sqrt{2}}$	$k_1 : \frac{3}{4\sqrt{2}}$	$c_2 : \frac{3\sqrt{2}}{5}$	$k_2 : \frac{4\sqrt{2}}{5}$
$d_1 : \frac{2\sqrt{2}}{11}$	$l_1 : \frac{5\sqrt{2}}{13}$	$d_2 : \frac{11}{9\sqrt{2}}$	$l_2 : \frac{13}{8\sqrt{2}}$
$e_1 : \frac{3}{8\sqrt{2}}$	$m_1 : \frac{7}{9\sqrt{2}}$	$e_2 : \frac{8\sqrt{2}}{13}$	$m_2 : \frac{9\sqrt{2}}{11}$
$f_1 : \frac{\sqrt{2}}{5}$	$n_1 : \frac{2\sqrt{2}}{5}$	$f_2 : \frac{5}{4\sqrt{2}}$	$n_2 : \frac{5}{3\sqrt{2}}$
$g_1 : \frac{2\sqrt{2}}{7}$	$o_1 : \frac{5}{6\sqrt{2}}$	$g_2 : \frac{7}{5\sqrt{2}}$	$o_2 : \frac{6\sqrt{2}}{7}$
$h_1 : \frac{7}{12\sqrt{2}}$	$p_1 : \frac{3\sqrt{2}}{7}$	$h_2 : \frac{12\sqrt{2}}{17}$	$p_2 : \frac{7}{4\sqrt{2}}$

TABLE 2. Table of Correspondence for $\hat{\mathscr{M}}_6$

$a_1 : \frac{1}{5\sqrt{3}}$	$m_1 : \frac{1}{2\sqrt{3}}$	$a_2 : \frac{3\sqrt{3}}{8}$	$a_3 : \frac{8}{5\sqrt{3}}$	$a_4 : \frac{5\sqrt{3}}{7}$	$m_4 : \frac{6\sqrt{3}}{7}$
$b_1 : \frac{\sqrt{3}}{14}$	$n_1 : \frac{2\sqrt{3}}{11}$	$b_2 : \frac{8}{7\sqrt{3}}$	$b_3 : \frac{7\sqrt{3}}{13}$	$b_4 : \frac{13}{6\sqrt{3}}$	$n_4 : \frac{13}{5\sqrt{3}}$
$c_1 : \frac{2}{9\sqrt{3}}$	$o_1 : \frac{5}{9\sqrt{3}}$	$c_2 : \frac{5\sqrt{3}}{13}$	$c_3 : \frac{13}{8\sqrt{3}}$	$c_4 : \frac{8\sqrt{3}}{11}$	$o_4 : \frac{7\sqrt{3}}{8}$
$d_1 : \frac{\sqrt{3}}{13}$	$p_1 : \frac{3\sqrt{3}}{16}$	$d_2 : \frac{7}{6\sqrt{3}}$	$d_3 : \frac{6\sqrt{3}}{11}$	$d_4 : \frac{11}{5\sqrt{3}}$	$p_4 : \frac{8}{3\sqrt{3}}$
$e_1 : \frac{1}{4\sqrt{3}}$	$q_1 : \frac{4}{7\sqrt{3}}$	$e_2 : \frac{2\sqrt{3}}{5}$	$e_3 : \frac{5}{3\sqrt{3}}$	$e_4 : \frac{3\sqrt{3}}{4}$	$q_4 : \frac{9\sqrt{3}}{10}$
$f_1 : \frac{\sqrt{3}}{11}$	$r_1 : \frac{\sqrt{3}}{5}$	$f_2 : \frac{5}{4\sqrt{3}}$	$f_3 : \frac{4\sqrt{3}}{7}$	$f_4 : \frac{7}{3\sqrt{3}}$	$r_4 : \frac{19}{7\sqrt{3}}$
$g_1 : \frac{2}{7\sqrt{3}}$	$s_1 : \frac{2}{3\sqrt{3}}$	$g_2 : \frac{3\sqrt{3}}{7}$	$g_3 : \frac{7}{4\sqrt{3}}$	$g_4 : \frac{4\sqrt{3}}{5}$	$s_4 : \frac{10\sqrt{3}}{11}$
$h_1 : \frac{\sqrt{3}}{10}$	$t_1 : \frac{\sqrt{3}}{4}$	$h_2 : \frac{4}{3\sqrt{3}}$	$h_3 : \frac{3\sqrt{3}}{5}$	$h_4 : \frac{17}{7\sqrt{3}}$	$t_4 : \frac{11}{4\sqrt{3}}$
$i_1 : \frac{1}{3\sqrt{3}}$	$u_1 : \frac{4}{5\sqrt{3}}$	$i_2 : \frac{5\sqrt{3}}{11}$	$i_3 : \frac{11}{6\sqrt{3}}$	$i_4 : \frac{13\sqrt{3}}{16}$	$u_4 : \frac{12\sqrt{3}}{13}$
$j_1 : \frac{\sqrt{3}}{8}$	$v_1 : \frac{3\sqrt{3}}{11}$	$j_2 : \frac{11}{8\sqrt{3}}$	$j_3 : \frac{8\sqrt{3}}{13}$	$j_4 : \frac{22}{9\sqrt{3}}$	$v_4 : \frac{25}{9\sqrt{3}}$
$k_1 : \frac{2}{5\sqrt{3}}$	$w_1 : \frac{5}{6\sqrt{3}}$	$k_2 : \frac{6\sqrt{3}}{13}$	$k_3 : \frac{13}{7\sqrt{3}}$	$k_4 : \frac{9\sqrt{3}}{11}$	$w_4 : \frac{13\sqrt{3}}{14}$
$l_1 : \frac{\sqrt{3}}{7}$	$x_1 : \frac{2\sqrt{3}}{7}$	$l_2 : \frac{7}{5\sqrt{3}}$	$l_3 : \frac{5\sqrt{3}}{8}$	$l_4 : \frac{5}{2\sqrt{3}}$	$x_4 : \frac{14}{5\sqrt{3}}$

3. THE CONGRUENCE SUBGROUPS OF THE HECKE GROUPS

Let I be an ideal of $\mathbb{Z}[\lambda_q]$. We define

$$PSL(2, \mathbb{Z}[\lambda_q], I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}[\lambda_q]) \mid a-1, b, c, d-1 \ \in I \right\}.$$

Now for any ideal I of $\mathbb{Z}[\lambda_q]$ we define the *principal congruence subgroup* of the Hecke group H_q as,

$$H_q(I) = PSL(2, \mathbb{Z}[\lambda_q], I) \cap H_q,$$

that is, the subgroup of H_q consisting of elements in $PSL(2, \mathbb{Z}[\lambda_q])$. For our interest we take the special case when $I = (n)$ and $2 \leq n \in \mathbb{Z}^+$, i.e. (n) is a principal ideal of $\mathbb{Z}[\sqrt{m}]$ where $m = 2, 3$ for $q = 4, 6$. The even principal congruence subgroup is now

$$H_q^e(n) = \left\{ \begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix} \in H_q^e \mid a \equiv d \equiv \pm 1 \bmod n,\ b \equiv c \equiv 0 \bmod n \right\}. \tag{3.1}$$

As $a\sqrt{m}$ and $d\sqrt{m}$ are not congruent to ± 1 mod n we can not have odd elements in $H_q(n)$.

For q =4 or 6 and for $n > 2$, the index of $H_q(n)$ in H_q was found by Parson [5, Theorem 2.3].

Theorem 1.

$$\mu_q(n) = |H_q : (H_q)(n)| = \begin{cases} n^3 \prod_{p|n} (1 - \frac{1}{p^2}) & if\ (n, m) = 1 \\ n^3 (1 - \frac{1}{m}) \prod_{p|n, p \neq m} (1 - \frac{1}{p^2}) & if\ (n, m) = m. \end{cases} \tag{3.2}$$

4. Pricincipal congruence maps

Definition 1. If $\hat{\mathscr{M}}_q$ is the universal q-gonal map, and I is an ideal of $\mathbb{Z}[\lambda_q]$ then a *principal congruence map* $\mathscr{M}_q(I)$ or a *PC-map* is a map of the form $\hat{\mathscr{M}}_q/H_q(I)$.

The maps $\mathscr{M}_q(n)$ for $q = 4$ and 6 lie on the Riemann surface $\mathbb{H}^*/H_q(n)$, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. The automorphism group of $\hat{\mathscr{M}}_q/H_q(n)$ is $H_q/H_q(n)$ of order $\mu_q(n)$. The elements of this group can be considered to be matrices

$$\begin{pmatrix} a & b\sqrt{m} \\ c\sqrt{m} & d \end{pmatrix}$$

$a, b, c, d \in \mathbb{Z}_n$, $ad - mbc \equiv 1 \bmod n$ together with

$$\begin{pmatrix} a\sqrt{m} & b \\ c & d\sqrt{m} \end{pmatrix}$$

$a, b, c, d \in \mathbb{Z}_n$, $mad - bc \equiv 1 \bmod n$.

The vertices of $\hat{\mathscr{M}}_q$, for $q = 4, 6$ come from the first columns of the matrices of H_q. In exactly the same way the vertices of $\mathscr{M}_q(n)$ come from the first columns of the matrices of $H_q/H_q(n)$ that is, they have the form $\frac{a}{c\sqrt{m}}$, or $\frac{a\sqrt{m}}{c}$, where $a, c \in \mathbb{Z}_n$ and $m = 2$ for $q = 4$ and $m = 3$ for $q = 6$.

4.1. **Vital Statistics for PC-maps $\mathscr{M}_4(n)$.** The vertices of $\mathscr{M}_4(n)$ are classified as even and odd vertices. The even vertices of $\mathscr{M}_4(n)$ form the orbit of $\infty = \frac{1}{0\sqrt{2}}$ under even elements of $H_4/H_4(n)$. Thus an even vertex is

$$\begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix} \frac{1}{0\sqrt{2}} = \frac{a}{c\sqrt{2}}.$$

As $ad - 2bc \equiv 1 \bmod n$ it follows that a is odd whenever n is even. Thus even vertices have the form $\frac{a}{c\sqrt{2}}$ where $(a, c, n) = 1$, and a is odd whenever n is even.

The odd vertices form the orbit of $0 = \frac{0\sqrt{2}}{1}$ under even elements of $H_4/H_4(n)$. Odd vertices have the form $\frac{b\sqrt{2}}{d}$ where $(b, d, n) = 1$ and as above we can show that d is odd whenever n is even.

4.1.1. *The bipartite nature of* $\mathscr{M}_4(n)$. The even and odd vertices give a bipartite structure on $\mathscr{M}_4(n)$. For the edges are defined as joining vertices of the form $\frac{a}{\sqrt{2}c}$ and $\frac{\sqrt{2}b}{d}$ whenever $ad - 2bc \equiv \pm 1 \bmod n$. That is even vertices are joined to odd vertices, and clearly even vertices or odd vertices cannot be joined by an edge.

Using Theorem 1 we can work out out the numbers of darts, edges, faces and vertices of $\mathscr{M}_4(n)$,

The number of darts =	$\mu_4(n)$
The number of edges =	$\mu_4(n)/2$
The number of faces =	$\mu_4(n)/4$
The number of vertices =	$\mu_4(n)/n$

If $g_4(n)$ is the genus of the map $\mathscr{M}_4(n)$ then the Euler characteristic is given by

$$2 - 2g_4(n) = \mu_4(n)(\frac{1}{n} - \frac{1}{2} + \frac{1}{4}) = \mu_4(n)(\frac{4-n}{4n}),$$

$$g_4(n) = \mu_4(n)(\frac{n-4}{8n}) + 1,$$

from which we deduce the following formulae for the genus of $\mathscr{M}_4(n)$

$$g_4(n) = \begin{cases} 0 & if\ n = 2 \\ 1 + \frac{n^2}{8}(n-4)\prod_{p|n}(1 - \frac{1}{p^2}) & if\ (n,2) = 1 \\ 1 + \frac{n^2}{16}(n-4)\prod_{p|n, p\neq 2}(1 - \frac{1}{p^2}) & if\ (n,2) = 2. \end{cases} \tag{4.1}$$

$g_4(n)$ is also the genus of the surface $\mathbb{H}^*/H_4(n)$ which carries the map $\mathscr{M}_4(n)$.

Let us see what these vertices are for low values of n, say $n = 2, 3, ..., 8$. As shown in Figures 6, 7, 8 and 5, these maps give interesting geometric shapes.

The corresponding maps $\mathscr{M}_4(n)$ for these values of n are as follows:

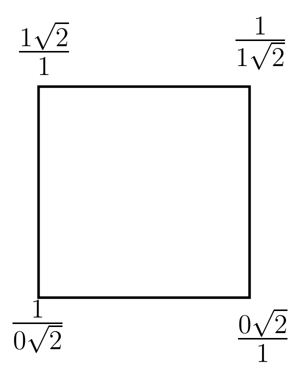


Figure 5. $\mathscr{M}_4(2)$; Square embedded in a sphere

TABLE 3. $\mathscr{M}_4(n)$ vertices for low values of n

n	$\mathscr{M}_4(n)$ vertices
2	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}$
3	$\frac{1}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{2\sqrt{2}}{1}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}$
4	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{2\sqrt{2}}{1}$
5	$\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{2}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{4}{1\sqrt{2}}, \frac{0}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{2}{2\sqrt{2}}, \frac{4}{2\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{2}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{2}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{4}, \frac{2\sqrt{2}}{0}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{2}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{4}$
6	$\frac{1}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{1}{3\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{5}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{5}, \frac{3\sqrt{2}}{1}$
7	$\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}, \frac{0}{1\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{2}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{4}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{6}{1\sqrt{2}}, \frac{0}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{2}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{4}{2\sqrt{2}}, \frac{5}{2\sqrt{2}},$ $\frac{6}{2\sqrt{2}}, \frac{0}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{2}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{5}{3\sqrt{2}}, \frac{6}{3\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{2}, \frac{0\sqrt{2}}{3}, \frac{1\sqrt{2}}{0}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{2}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{4}, \frac{1\sqrt{2}}{5}, \frac{1\sqrt{2}}{6}, \frac{2\sqrt{2}}{0}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{2}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{4}, \frac{2\sqrt{2}}{5},$ $\frac{2\sqrt{2}}{6}, \frac{3\sqrt{2}}{0}, \frac{3\sqrt{2}}{1}, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{3}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{5}, \frac{3\sqrt{2}}{6}$
8	$\frac{1}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}, \frac{1}{1\sqrt{2}}, \frac{3}{1\sqrt{2}}, \frac{5}{1\sqrt{2}}, \frac{7}{1\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}, \frac{5}{2\sqrt{2}}, \frac{7}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}, \frac{5}{3\sqrt{2}}, \frac{7}{3\sqrt{2}}, \frac{1}{4\sqrt{2}}, \frac{3}{4\sqrt{2}},$ $\frac{0\sqrt{2}}{1}, \frac{0\sqrt{2}}{3}, \frac{1\sqrt{2}}{1}, \frac{1\sqrt{2}}{3}, \frac{1\sqrt{2}}{5}, \frac{1\sqrt{2}}{7}, \frac{2\sqrt{2}}{1}, \frac{2\sqrt{2}}{3}, \frac{2\sqrt{2}}{5}, \frac{2\sqrt{2}}{7}, \frac{3\sqrt{2}}{1}, \frac{3\sqrt{2}}{3}, \frac{3\sqrt{2}}{5}, \frac{3\sqrt{2}}{7}, \frac{4\sqrt{2}}{1}, \frac{4\sqrt{2}}{3}$

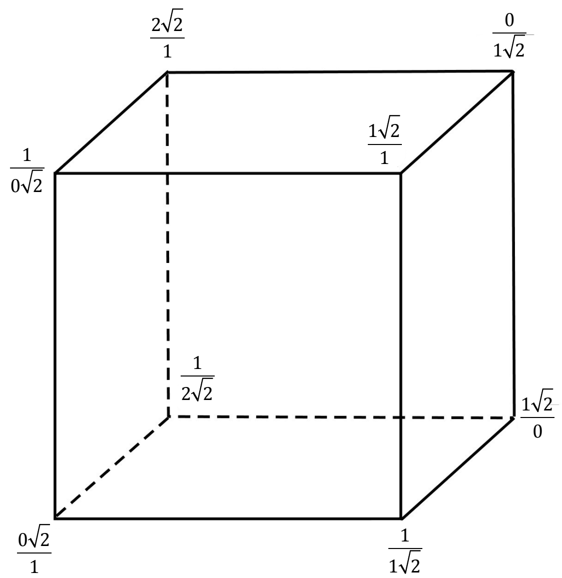


FIGURE 6. $\mathscr{M}_4(3)$; Cube embedded in a sphere

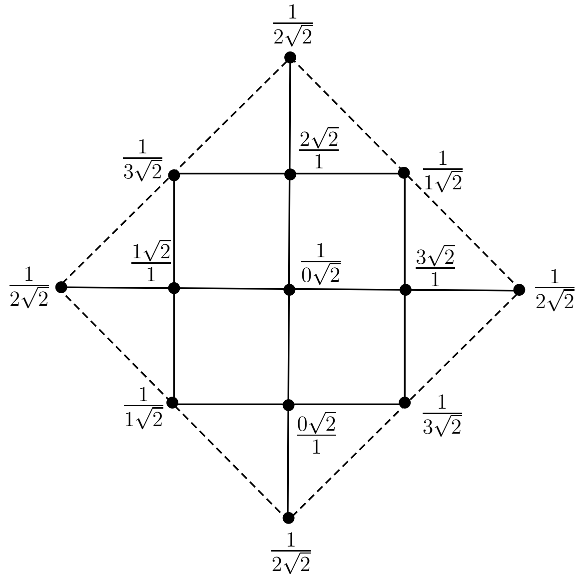


FIGURE 7. $\mathscr{M}_4(4)$; $\{4,4\}_{2,2}$ embedded in a square torus of genus 1

[The square torus is obtained by identifying the opposite sides of the outer square].

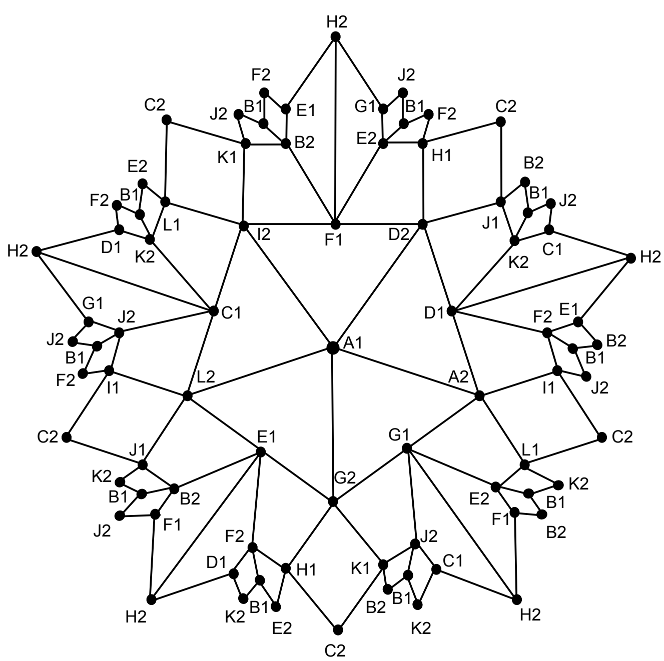


FIGURE 8. $\mathscr{M}_4(5)$; $\{5,4\}$ embedded in Bring's curve of genus 4

TABLE 4. Table of Correspondence for $\mathscr{M}_4(5)$

$A_1 : \frac{1}{0\sqrt{2}}$	$B_1 : \frac{2}{0\sqrt{2}}$	$C_1 : \frac{0}{1\sqrt{2}}$	$D_1 : \frac{1}{1\sqrt{2}}$
$E_1 : \frac{2}{1\sqrt{2}}$	$F_1 : \frac{3}{1\sqrt{2}}$	$G_1 : \frac{4}{1\sqrt{2}}$	$H_1 : \frac{0}{2\sqrt{2}}$
$I_1 : \frac{1}{2\sqrt{2}}$	$J_1 : \frac{3}{2\sqrt{2}}$	$K_1 : \frac{2}{2\sqrt{2}}$	$L_1 : \frac{4}{2\sqrt{2}}$
$A_2 : \frac{0\sqrt{2}}{1}$	$B_2 : \frac{0\sqrt{2}}{2}$	$C_2 : \frac{1\sqrt{2}}{0}$	$D_2 : \frac{1\sqrt{2}}{1}$
$E_2 : \frac{1\sqrt{2}}{2}$	$F_2 : \frac{1\sqrt{2}}{3}$	$G_2 : \frac{1\sqrt{2}}{4}$	$H_2 : \frac{2\sqrt{2}}{0}$
$I_2 : \frac{2\sqrt{2}}{1}$	$J_2 : \frac{2\sqrt{2}}{2}$	$K_2 : \frac{2\sqrt{2}}{3}$	$L_2 : \frac{2\sqrt{2}}{4}$

The following table provides the vital statistics for some examples of maps $\mathscr{M}_4(n)$

n	$\mu_4(n)$	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert F\rvert$	$g_4(n)$
2	8	4	4	2	0
3	24	12	8	6	0
4	32	16	8	8	1
5	120	60	24	30	4
6	96	48	16	24	5
7	336	168	48	84	19
8	256	128	32	64	17

TABLE 5. Vital statistics for some maps $\mathscr{M}_4(n)$

4.2. **Bring's Surface.** When considering regular triangular maps $\hat{\mathscr{M}}_3$, maps of type $\{n, 3\}$ the most interesting, is often thought of as Klein's surface, $\mathscr{M}_3(7)$, of genus 3 with $PSL(2,7)$, of order 168 as its automorphism group. This is the lowest genus surface admitting the Hurwitz bound of $84(g-1)$ automorphisms. The underlying complex algebraic curve is the Klein quartic $x^3y + y^3z + z^3x = 0$.

For regular maps of type $\{n, 4\}$, one of the most interesting is probably the map $\mathscr{M}_4(5)$ of type $\{5, 4\}$, Figure 8, of genus 4 that underlies Bring's curve. This has S_5 as automorphism group. To get this surface we start with the triangle group $\Gamma(2, 5, 4)$. This has a presentation

$$\langle X, Y, Z | X^2 = Y^5 = Z^4 = XYZ = 1\rangle.$$

There is an epimorphism $\theta : \Gamma \longrightarrow S_5$ (of order 120) defined by

$\theta(X) = x = (1, 5)$

$\theta(Y) = y = (5, 4, 3, 2, 1)$

$\theta(Z) = z = (2, 3, 4, 5)$.

As θ preserves the orders of elements of finite order it follows that the kernel of θ is a surface group N whose genus can be computed from the Riemann-Hurwitz formula:

$$2g - 2 = 120(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5})$$

giving $g_4(5) = 4$, Table 5.

By the work of Conder [1] there is a uniqued regular map of genus 4 with S_5 as automorphism group. The underlying Riemann surface is *Bring's surface.* this is the Riemann surface of *Bring's curve.*

There is a homomorphism from H_4 to $\Gamma(2,5,4)$. Our normal subgroup N pulls back to a normal subgroup of H_4 of index 120. By the uniqueness of the regular map of genus 4 with S_5 as automorphism group, it follows that this subgroup must be conjugate to $H_4(5)$. Thus the quotient space of $H_4(5)$ must be Bring's surface.

Bring's surface is the Riemann surface of Bring's curve which is the complete intersection of three hypersurfaces.

$$\sum_{i=1}^{5} x_i = 0, \ \sum_{i=1}^{5} x_i^2 = 0, \ \sum_{i=1}^{5} x_i^3 = 0.$$

For Bring's curve, see [8], where it is shown that the regular map comes from an imbedding of Kepler's small stellated icosahedron in a surface of genus 4.

4.3. **Vital Statistics for PC-maps $\mathscr{M}_6(n)$.** Discussion of the vital statistics for $\mathscr{M}_6(n)$ is analogous to that for $\mathscr{M}_4(n)$, with slight differences. The vertices of $\hat{\mathscr{M}}_6$ are partitioned into even and odd vertices, as in $\mathscr{M}_6(n)$. The even vertices of $\mathscr{M}_6(n)$ are the orbit of $\infty = \frac{1}{0\sqrt{3}}$ under even elements of H_6. For $a,b,c,d \in \mathbb{Z}_n$, even vertices have the form $\frac{a}{c\sqrt{3}}$ where $(a,c,n) = 1$, and $3 \nmid a$ whenever $3|n$. The odd vertices are the orbit of $0 = \frac{0\sqrt{3}}{1}$ under even elements of H_6. Odd vertices have the form $\frac{b\sqrt{3}}{d}$ where $(b,d,n) = 1$ and $3 \nmid d$ whenever $3|n$. Edges and faces are exactly the same as described in $\mathscr{M}_4(n)$. As in case $q = 4$, the edges never join two vertices of the same type this implies that the graph is bipartite. Finding the numbers of darts, edges, faces and vertices of $\mathscr{M}_6(n)$, using Theorem 1 we have;

The number of darts =	$\mu_6(n)$
The number of edges =	$\mu_6(n)/2$
The number of faces =	$\mu_6(n)/6$
The number of vertices =	$\mu_6(n)/n$

If $g_6(n)$ is the genus of the map $\mathscr{M}_6(n)$ then the Euler characteristic is given by

$$2 - 2g_6(n) = \mu_6(n)(\frac{1}{n} - \frac{1}{2} + \frac{1}{6}) = \mu_6(n)(\frac{3-n}{3n}),$$

$$g_6(n) = 1 + \mu_6(n)(\frac{n-3}{6n}),$$

from which we deduce the following formulae for the genus of $\mathscr{M}_6(n)$

$$g_6(n) = \begin{cases} 0 & if\ n = 2 \\ 1 + \frac{n^2}{6}(n-3)\prod_{p|n}(1-\frac{1}{p^2}) & if\ (n,3) = 1 \\ 1 + \frac{n^2}{9}(n-3)\prod_{p|n, p\neq 3}(1-\frac{1}{p^2}) & if\ (n,3) = 3. \end{cases} \tag{4.2}$$

The vertices for low values of n, say $n = 2, 3, ..., 6$ are;

TABLE 6. $\mathscr{M}_6(n)$ vertices for low values of n

n	$\mathscr{M}_6(n)$ vertices
2	$\frac{1}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}$
3	$\frac{1}{0\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{1}, \frac{2\sqrt{3}}{1}$
4	$\frac{1}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{1}{2\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{2\sqrt{3}}{1}$
5	$\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{0}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{4}{2\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{2}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{2\sqrt{3}}{0}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{4}$
6	$\frac{1}{0\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{5}{2\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}$ $\frac{0\sqrt{3}}{1}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{5}, \frac{3\sqrt{3}}{1}, \frac{3\sqrt{3}}{2}$
7	$\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}, \frac{3}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{6}{1\sqrt{3}}, \frac{0}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{2}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{4}{2\sqrt{3}}, \frac{5}{2\sqrt{3}},$ $\frac{6}{2\sqrt{3}}, \frac{0}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}, \frac{6}{3\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{2}, \frac{0\sqrt{3}}{3}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{1\sqrt{3}}{6}, \frac{2\sqrt{3}}{0}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{2}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{4}, \frac{2\sqrt{3}}{5},$ $\frac{2\sqrt{3}}{6}, \frac{3\sqrt{3}}{0}, \frac{3\sqrt{3}}{1}, \frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{3}, \frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{5}, \frac{3\sqrt{3}}{6}$
8	$\frac{1}{0\sqrt{3}}, \frac{3}{0\sqrt{3}}, \frac{0}{1\sqrt{3}}, \frac{1}{1\sqrt{3}}, \frac{2}{1\sqrt{3}}, \frac{3}{1\sqrt{3}}, \frac{4}{1\sqrt{3}}, \frac{5}{1\sqrt{3}}, \frac{6}{1\sqrt{3}}, \frac{7}{1\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{3}{2\sqrt{3}}, \frac{5}{2\sqrt{3}}, \frac{7}{2\sqrt{3}}, \frac{0}{3\sqrt{3}}, \frac{1}{3\sqrt{3}},$ $\frac{2}{3\sqrt{3}}, \frac{3}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}, \frac{6}{3\sqrt{3}}, \frac{7}{3\sqrt{3}}, \frac{1}{4\sqrt{3}}, \frac{3}{4\sqrt{3}},$ $\frac{0\sqrt{3}}{1}, \frac{0\sqrt{3}}{3}, \frac{1\sqrt{3}}{0}, \frac{1\sqrt{3}}{1}, \frac{1\sqrt{3}}{2}, \frac{1\sqrt{3}}{3}, \frac{1\sqrt{3}}{4}, \frac{1\sqrt{3}}{5}, \frac{1\sqrt{3}}{6}, \frac{1\sqrt{3}}{7}, \frac{2\sqrt{3}}{1}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{5}, \frac{2\sqrt{3}}{7}, \frac{3\sqrt{3}}{0}, \frac{3\sqrt{3}}{1},$ $\frac{3\sqrt{3}}{2}, \frac{3\sqrt{3}}{3}, \frac{3\sqrt{3}}{4}, \frac{3\sqrt{3}}{5}, \frac{3\sqrt{3}}{6}, \frac{3\sqrt{3}}{7}, \frac{4\sqrt{3}}{1}, \frac{4\sqrt{3}}{3}$

The corresponding $\mathscr{M}_6(n)$ maps for these values of n;

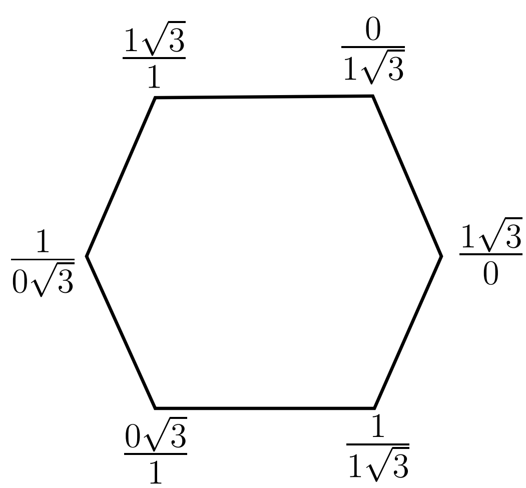


FIGURE 9. $\mathscr{M}_6(2)$; Hexagon embedded in a sphere

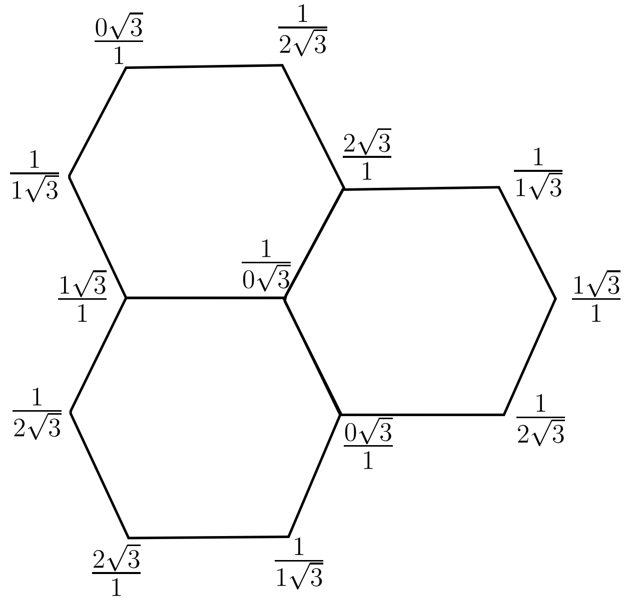


FIGURE 10. $\mathscr{M}_6(3)$; $\{3, 6\}$ embedded in a hexagonal torus

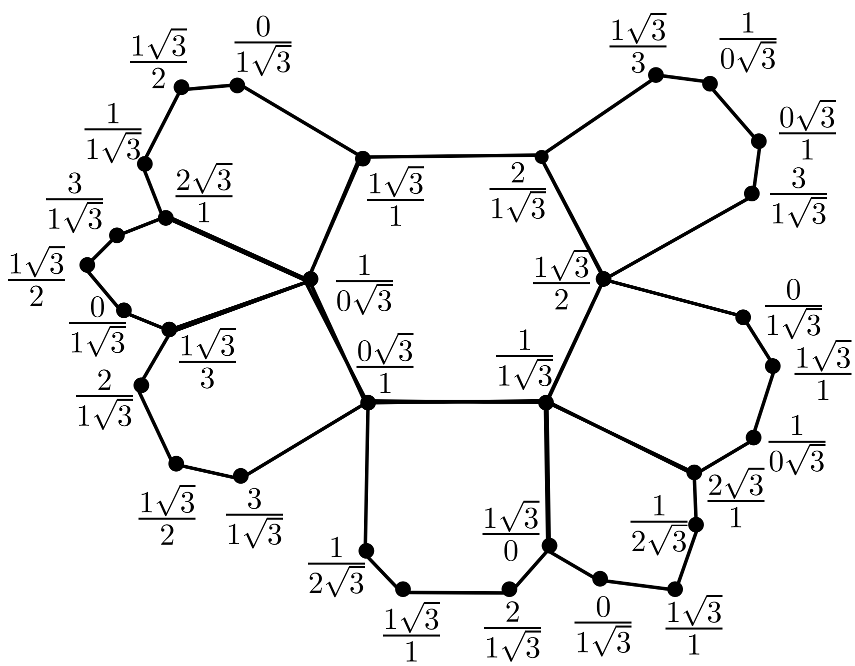


FIGURE 11. $\mathscr{M}_6(4)$; $\{4, 6\}$ of genus 3

The following table provides the vital statistics for some examples of maps $\mathscr{M}_6(n)$

n	$\mu_6(n)$	$\lvert E\rvert$	$\lvert V\rvert$	$\lvert F\rvert$	$g_6(n)$
2	12	6	6	2	0
3	18	9	6	3	1
4	48	24	12	8	3
5	120	60	24	20	9
6	108	54	18	18	37
7	336	168	48	56	33
8	384	192	48	64	41

TABLE 7. Vital statistics for some maps $\mathscr{M}_6(n)$

5. POLES OF $\mathscr{M}_q(n)$

A *pole* of $\mathscr{M}_3(n)$ is a point of the form $\frac{a}{0}$ where $(a,n)=1$. In [7] it was also shown that the distance between two poles of $\mathscr{M}_3(n)$ is equal to 3, the diameter of the map. For this reason, it is important to study poles when considering the diameter of a Hecke-Farey map. We call the points of the form $\frac{a}{0\sqrt{m}}$ or $\frac{b\sqrt{m}}{0}$ the *even-poles* and *odd-poles* of $\mathscr{M}_q(n)$ respectively.

Lemma 2. *(i) If n is even then $\mathscr{M}_4(n)$ has no odd-poles.*
(ii) If $3|n$ then $\mathscr{M}_6(n)$ has no odd-poles.

Proof. (i) The odd elements of $\mathscr{M}_4(n)$ have the form $\frac{b\sqrt{2}}{d}$ with $(b,d,n)=1$ and d is odd if n is even (Section 4.1). Hence there can be no odd-poles when n is even. (ii) Similar to (i) but for $\mathscr{M}_6(n)$ the odd elements have the form $\frac{b\sqrt{3}}{d}$ with $(b,d,n)=1$ and $3\nmid d$ if $3|n$ (Section 4.3). As $3|0=d$, then there can be no odd-poles if $3|n$. $\square$

Even-poles and odd-poles have the form $\frac{a}{0\sqrt{m}}$ and $\frac{b\sqrt{m}}{0}$ respectively, where $(a,0,n)=1$ and $(b,0,n)=1$ implies $(a,n)=1$ and $(b,n)=1$. Since the Euler function $\phi(n)$ counts all the integers that are relatively prime to n, and we identify a with $-a$ and b with $-b$ therefore the number of even-poles in $\mathscr{M}_q(n)$ is $\phi(n)/2$ for $n>2$. Similarly the number of odd-poles, if they are exist, is $\phi(n)/2$. So the total number of poles is either $\phi(n)$ if both even and odd-poles exist or $\phi(n)/2$ if there are no odd-poles.

Example 3. *From Table 3, for $n=6$, the number of poles in $\mathscr{M}_4(6)$ is $\phi(6)/2=1$, namely $\frac{1}{0\sqrt{2}}$, and there are no odd-poles.*
For $n=7$, the number of poles in $\mathscr{M}_4(7)$ is $\phi(7)=6$, and these are $\frac{1}{0\sqrt{2}}$, $\frac{2}{0\sqrt{2}}$, $\frac{3}{0\sqrt{2}}$, $\frac{1\sqrt{2}}{0}$, $\frac{2\sqrt{2}}{0}$, $\frac{3\sqrt{2}}{0}$. From Table 6, for $n=6$, the number of poles in $\mathscr{M}_6(6)$ is $\phi(6)/2=1$, namely $\frac{1}{0\sqrt{3}}$, and there are no odd-poles. For $n=5$, the number of poles in $\mathscr{M}_6(5)$ is $\phi(5)=4$, and these are $\frac{1}{0\sqrt{3}}$, $\frac{2}{0\sqrt{3}}$, $\frac{1\sqrt{3}}{0}$, $\frac{2\sqrt{3}}{0}$.

In a connected graph the *distance* $\delta(x,y)$ between two vertices x,y is defined as the least number of edges in a path from x to y. The *diameter* of a graph or map is the maximum distance between two of its vertices. In [7, Theorem 11], we showed that the diameter of the PC-maps $\mathscr{M}_3(n)$ is equal to 3, for $n\geq 5$. In Theorem 14 we will prove that for $q=4,6$ and $n>6$, the diameter of the maps $\mathscr{M}_q(n)$ is equal to 4.

Lemma 4. *Paths in a bipartite graph must be of even length if they are connecting two vertices in the same part and they must be of odd length if they are connecting vertices in different parts.*

Theorem 5. *If odd-poles exist, then the distance between any even-pole and any odd-pole of $\mathscr{M}_q(n)$ is equal to 3, for $n \geq 5$.*

Proof. For the existence of the odd-poles, we take $2 \nmid n$ and $3 \nmid n$ for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ respectively. By regularity of $\mathscr{M}_q(n)$ we may assume that one of the even-poles is $\frac{1}{0\sqrt{m}}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.1, Section 4.3), then $\delta(\frac{1}{0\sqrt{m}}, \frac{a\sqrt{m}}{0})$ is odd, by Lemma 4. This even pole $\frac{1}{0\sqrt{m}}$ is not adjacent to an odd-pole $\frac{a\sqrt{m}}{0}$, so $\delta(\frac{1}{0\sqrt{m}}, \frac{a\sqrt{m}}{0}) \neq 1$. However, we can always find an odd vertex and even vertex to construct a path of length 3 between $\frac{1}{0\sqrt{m}}$ and $\frac{a\sqrt{m}}{0}$ of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{0\sqrt{m}}{1} \longleftrightarrow \frac{1}{y\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}, \tag{5.1}$$

where $y \equiv \pm(am)^{-1} \bmod n$. As a^{-1} is the inverse of a modulo n exists because $(a, 0, n) = 1$, hence $(a, n) = 1$. Also m^{-1} is the inverse of m modulo n exists because $(m, n) = 1$ then y has a solution. □

Example 6. *1- Using Figure 8 of the map $\mathscr{M}_4(5)$ and Table 3, when $n = 5$ and $x = 0$, then $\delta(\frac{1}{0\sqrt{2}}, \frac{1\sqrt{2}}{0}) = 3$, using (5.1) we have*

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{2\sqrt{2}} \longleftrightarrow \frac{1\sqrt{2}}{0} \quad \textit{or} \quad A1 \longleftrightarrow A2 \longleftrightarrow I1 \longleftrightarrow C2$$

where $y = 2$. (See Figure 12.)
2- Using Figure 11 of the map $\mathscr{M}_6(4)$ and Table 6, when $n = 4$ and $x = 0$, then $\delta(\frac{1}{0\sqrt{3}}, \frac{1\sqrt{3}}{0}) = 3$, using (5.1) we have $\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{1\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{0}$, where $y = 1$. (See Figure 12.)

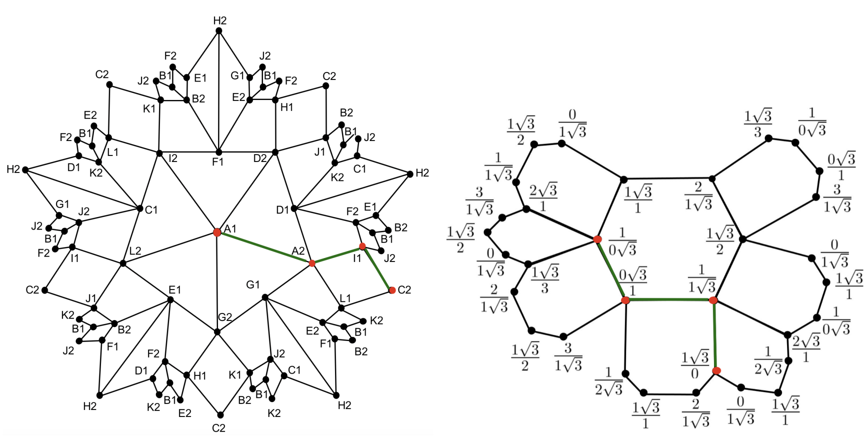


FIGURE 12. Left: $\delta(\frac{1}{0\sqrt{2}}, \frac{1\sqrt{2}}{0}) = d(A1, C2) = 3$. Right: $\delta(\frac{1}{0\sqrt{3}}, \frac{1\sqrt{3}}{0}) = 3$

Theorem 7. *The distance between any two distinct even-poles in $\mathscr{M}_q(n)$ is equal to 4, for $n > 6$.*

Proof. By regularity we may assume that one of the even-poles is $\frac{1}{0\sqrt{m}}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.1, Section 4.3), then $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}})$ is even by Lemma 4, where $a \not\equiv \pm 1 \bmod n$. This immediately excludes $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}}) = 1$ and 3. Also there is no path of length 2 between $\frac{1}{0\sqrt{m}}$ and $\frac{a}{0\sqrt{m}}$, for otherwise there would be $x, y \in \mathbb{Z}_n$ such that we have the following path

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{x\sqrt{m}}{y} \longleftrightarrow \frac{a}{0\sqrt{m}}. \tag{5.2}$$

Then $y \equiv \pm 1 \bmod n$ and $ay \equiv \pm 1 \bmod n$. As $(a, n) = 1$ then a^{-1} exists, therefore y is the inverse of $a \bmod n$. Thus $ay \equiv a \equiv \pm 1 \bmod n$, implies $a \equiv \pm 1 \bmod n$, which is a contradiction. Thus $\delta(\frac{1}{0\sqrt{m}}, \frac{a}{0\sqrt{m}}) \neq 2$.

However, we can always construct a path of length 4 between $\frac{1}{0\sqrt{m}}$ and $\frac{a}{0\sqrt{m}}$, of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{0\sqrt{m}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{m}} \longleftrightarrow \frac{m^{-1}(a+1)\sqrt{m}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{m}}, \tag{5.3}$$

where a^{-1} is the inverse of a modulo n, and m^{-1} is the inverse of m modulo n. The inverse a^{-1} exists because $(a, 0, n) = 1$ and hence $(a, n) = 1$. The inverse m^{-1} exists if $(m, n) = 1$, meaning that whenever n is odd for $q = 4$, $m = 2$ and $3 \nmid n$ for $q = 6$, $m = 3$.

If m^{-1} does not exist i.e. $(m, n) = m$, we have the following cases.

(i) For $q = 4, m = 2$ and n is an even integer, then a is odd, so we can construct a path of length 4 between $\frac{1}{0\sqrt{2}}$ and $\frac{a}{0\sqrt{2}}$, of the form

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{2}} \longleftrightarrow \frac{\frac{1}{2}(a+1)\sqrt{2}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{2}}. \tag{5.4}$$

(ii) For $q = 6, m = 3$ and $3|n$, then $3 \nmid a$. Let $a \equiv 1 \bmod 3$, so we can construct a path of length 4 between $\frac{1}{0\sqrt{3}}$ and $\frac{a}{0\sqrt{3}}$, of the form

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{3}} \longleftrightarrow \frac{\frac{1}{3}(a-1)\sqrt{3}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{3}}. \tag{5.5}$$

If $3 \nmid a$ and $a \equiv 2 \bmod 3$, we can construct a path of length 4 between $\frac{1}{0\sqrt{3}}$ and $\frac{a}{0\sqrt{3}}$, of the form

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{a^{-1}\sqrt{3}} \longleftrightarrow \frac{\frac{1}{3}(a+1)\sqrt{3}}{a^{-1}} \longleftrightarrow \frac{a}{0\sqrt{3}}. \tag{5.6}$$

Thus for all $n > 6$ the distance between two distinct even-poles in $\mathscr{M}_q(n)$ is equal to 4. □

Theorem 8. *If odd-poles exist then the distance between any two distinct odd-poles in $\mathscr{M}_q(n)$ is equal to 4 for $n \geq 5$.*

Proof. For the existence of the odd-poles, we take $2 \nmid n$ and $3 \nmid n$ for $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ respectively. By regularity we may assume that one of the odd-poles is $\frac{1\sqrt{m}}{0}$. As $\mathscr{M}_q(n)$ is a bipartite graph (Section 4.1, Section 4.3), then $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0})$ is even by Lemma 4, where $a \not\equiv \pm 1 \bmod n$. This immediately excludes $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0}) = 1$ and 3. Also there is no path of length 2 between $\frac{1\sqrt{m}}{0}$ and $\frac{a\sqrt{m}}{0}$, for otherwise there would be $x, y \in \mathbb{Z}_n$ such that we have the following path

$$\frac{1\sqrt{m}}{0} \longleftrightarrow \frac{x}{y\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}. \tag{5.7}$$

Then $my \equiv \pm 1 \bmod n$ and $amy \equiv \pm 1 \bmod n$. Thus my is the inverse of $a \bmod n$, so $amy \equiv a \equiv \pm 1 \bmod n$ implies $a \equiv \pm 1 \bmod n$, which is a contradiction. Therefore $\delta(\frac{1\sqrt{m}}{0}, \frac{a\sqrt{m}}{0}) \neq 2$. However, we can always construct a path of length 4 between $\frac{1\sqrt{m}}{0}$ and $\frac{a\sqrt{m}}{0}$, of the form

$$\frac{1\sqrt{m}}{0} \longleftrightarrow \frac{1}{m^{-1}\sqrt{m}} \longleftrightarrow \frac{a^{-1}\sqrt{m}}{a^{-1}+1} \longleftrightarrow \frac{a^{-1}-1}{a^{-1}m^{-1}\sqrt{m}} \longleftrightarrow \frac{a\sqrt{m}}{0}, \tag{5.8}$$

where a^{-1} is the inverse of a modulo n, and m^{-1} is the inverse of m modulo n. The inverse a^{-1} exists because $(a, 0, n) = 1$ and hence $(a, n) = 1$, also the inverse m^{-1} exist because $(m, n) = 1$. $\square$

Example 9. *1- Consider Figure 8 of the map $\mathscr{M}_4(5)$ and Table 3, when $n = 5$. As $\phi(5)/2 = 2 > 1$, we have two distinct even-poles and odd-poles, thus (5.3) is applicable and we have $\delta(\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}) = 4$. Here we have $a^{-1} = 3$ and $m^{-1} = 3$.*

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{4}{2\sqrt{2}} \longleftrightarrow \frac{1\sqrt{2}}{2} \longleftrightarrow \frac{2}{0\sqrt{2}}$$

or

$$A1 \longleftrightarrow A2 \longleftrightarrow L1 \longleftrightarrow E2 \longleftrightarrow B1.$$

(See Figure 13.)

2- Consider Table 3, when $n = 8$. As $\phi(8)/2 = 2 > 1$, we have two distinct even-poles in $\mathscr{M}_4(8)$, thus (5.4) is applicable and we have $\delta(\frac{1}{0\sqrt{2}}, \frac{3}{0\sqrt{2}}) = 4$. Here $a^{-1} = 3$, and we have the following path:

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{0\sqrt{2}}{1} \longleftrightarrow \frac{1}{3\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{3} \longleftrightarrow \frac{3}{0\sqrt{2}}.$$

3- When $q = 6$ and $n = 9$. As $\phi(9)/2 = 3 > 1$, we have three distinct even-poles in $\mathscr{M}_6(9)$, thus (5.6) is applicable and we have $\delta(\frac{1}{0\sqrt{3}}, \frac{2}{0\sqrt{3}}) = 4$. Here $a^{-1} = 5$ and $a = 2 \equiv 2 \bmod 3$. We have the following path:

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{0\sqrt{3}}{1} \longleftrightarrow \frac{1}{5\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{5} \longleftrightarrow \frac{2}{0\sqrt{3}}.$$

Example 10. *1- Consider Figure 8 of the map $\mathscr{M}_4(5)$ and Table 3, when $n = 5$. As $\phi(5)/2 = 2 > 1$, we have two distinct odd-poles, thus (5.8) is applicable and $\delta(\frac{1\sqrt{2}}{0}, \frac{2\sqrt{2}}{0}) = 4$. Here $a^{-1} = 3$ and $m^{-1} = 3$. We have the following path:*

$$\frac{1\sqrt{2}}{0} \longleftrightarrow \frac{4}{2\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{1} \longleftrightarrow \frac{3}{1\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{0}$$

or

$$C2 \longleftrightarrow L1 \longleftrightarrow I2 \longleftrightarrow F1 \longleftrightarrow H2.$$

(See Figure 13.)

2- Consider Table 6, when $3 \nmid n = 7$. As $\phi(7)/2 = 3 > 1$, we have three distinct *odd-poles in* $\mathscr{M}_6(7)$, *thus* (5.8) *is applicable and we have* $\delta(\frac{1\sqrt{3}}{0}, \frac{2\sqrt{3}}{0}) = 4$. *Here* $a^{-1} = 4$ *and* $m^{-1} = 5$. *We have the following path:*

$$\frac{1\sqrt{3}}{0} \longleftrightarrow \frac{6}{2\sqrt{3}} \longleftrightarrow \frac{3\sqrt{3}}{2} \longleftrightarrow \frac{4}{1\sqrt{3}} \longleftrightarrow \frac{2\sqrt{3}}{0}.$$

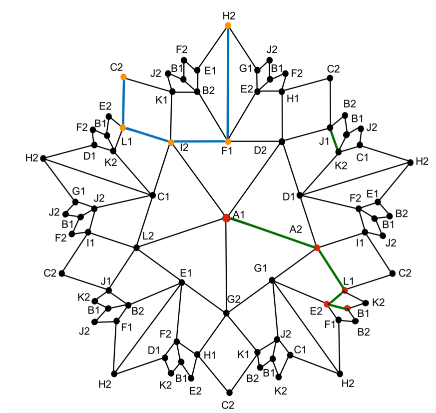


Figure 13. $\delta(\frac{1}{0\sqrt{2}}, \frac{2}{0\sqrt{2}}) = \delta(A1, B1) = 4$, $\delta(\frac{1\sqrt{2}}{0}, \frac{2\sqrt{2}}{0}) = \delta(C2, H2) = 4$ in $\mathscr{M}_4(5)$

Lemma 11 ([7, Lemma 10])**.** *Let* b, d, n *be integers such that* $(b, d, n) = 1$. *Then there exists an integer* k *so that* $(b + dk, n) = 1$.

Proof. Let $p_1, \cdots, , p_r$ be a list, without repetition, of those prime divisors of n that are not divisors of b or d.

Let $k = p_1 p_2 \cdots p_r$. Suppose that q is a prime divisor of n. If $q = p_i$ for some i then q is not a divisor of b so it is not a divisor of $b + dk$ for any k.

If q is not equal to p_i for any i then it is coprime to k and so it is a divisor of one but not both of b and d. If q were a divisor of both b and d then $(b, d, n) \neq 1$ contrary to hypothesis. if q is a divisor of b then it cannot divide $b + dk$ as q is coprime to dk. If q is a divisor of d then q does not divide b and so does not divide $b + dk$. Hence $b + dk$ and n are coprime . $\square$

Note: If all prime divisors of n divide b or d then $k = 1$.

Theorem 12. *Given an odd vertex* $\frac{b\sqrt{m}}{d}$ *in* $\mathscr{M}_q(n)$, $n \geq 5$, *then* $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d}) = 3$ *if and only if* $d \not\equiv \pm 1$ *mod* n.

Proof. By Lemma 4, the distance $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d})$ is odd. If $\delta(\frac{1}{0\sqrt{m}}, \frac{b\sqrt{m}}{d}) = 1$, then $\frac{b\sqrt{m}}{d}$ is adjacent to $\frac{1}{0\sqrt{m}}$ implies $d \equiv \pm 1$ mod n, which is a contradiction. If $d \equiv 0$ mod n then $\delta = 3$, by Theorem 2. Now for the case when $d \not\equiv 0$ mod n, there are no odd-poles, by Lemma 2 (i.e. n is even for $q = 4$ and $3|n$ for $q = 6$). We want to construct a path of length 3 between $\frac{1}{0\sqrt{m}}$ and $\frac{b\sqrt{m}}{d}$ of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{k\sqrt{m}}{1} \longleftrightarrow \frac{u}{v\sqrt{m}} \longleftrightarrow \frac{b\sqrt{m}}{d}, \tag{5.9}$$

where k is as in Lemma 11 and $u, v \in \mathbb{Z}_n$. Thus we have two simultaneous congruences:

$$mkv - u \equiv \pm 1 \bmod n$$

$$ud - mvb \equiv \pm 1 \bmod n$$

By Lemma 11 we know that $(dk - b)$ is coprime to n and therefore has a multiplicative inverse modulo n. Hence the congruence in (5.9) can be solved for v which, in turn, determines u. $\square$

Example 13. *(i) The distance between* $\frac{1}{0\sqrt{2}}$ *and* $\frac{2\sqrt{2}}{9}$ *in* $\mathscr{M}_4(18)$ *is 3 and we have the following path*

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{3\sqrt{2}}{1} \longleftrightarrow \frac{11}{2\sqrt{2}} \longleftrightarrow \frac{2\sqrt{2}}{9},$$

where $v = 2$, $u = 11$ *and* $x = 3$.
(ii) The distance between $\frac{1}{0\sqrt{3}}$ *and* $\frac{3\sqrt{3}}{7}$ *in* $\mathscr{M}_6(15)$ *is 3 and we have the following path*

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{2\sqrt{3}}{1} \longleftrightarrow \frac{11}{12\sqrt{3}} \longleftrightarrow \frac{3\sqrt{3}}{7},$$

where $v = 12$, $u = 11$ *and* $x = 2$.

6. THE DIAMETER OF $\mathscr{M}_q(n)$

Theorem 14. *The diameter of* $\mathscr{M}_q(n)$ *is equal to 4 for all* $n > 6$.

Proof. The distance between any odd vertex and even vertex in $\mathscr{M}_q(n)$ is equal 1, if they are adjacent. Otherwise, $\delta = 3$, by Theorems 5 and 12. Also, the distance between any two distinct vertices of the same type is ≤ 4, by Theorems 7 and 8. Thus the diameter of $\mathscr{M}_q(n)$ is equal to 4 for all $n > 6$. Refer to Table 8 for the diameters of $\mathscr{M}_q(n)$ for $n \leq 5$. $\square$

TABLE 8. The diameters of $\mathscr{M}_4(n)$ and $\mathscr{M}_6(n)$ for $n \leq 6$

n	$q = 4$		$q = 6$	
	$\mathscr{M}_4(n)$	Diameter	$\mathscr{M}_6(n)$	Diameter
2	Square Figure 5	2	Hexagon Figure 9	3
3	Cube Figure 6	3	$\{3,6\}$ Figure 10	3
4	$\{4,4\}_{2,2}$ Figure 7	2	$\{4,6\}$ Figure 11	3
5	$\{5,4\}$ Figure 8	4	$\{5,6\}$	4
6	$\{6,4\}$	3	$\{6,6\}$	3

Definition 2. If $\frac{a}{c\sqrt{m}}$ and $\frac{b}{d\sqrt{m}}$ are two distinct even vertices or if $\frac{a\sqrt{m}}{c}$ and $\frac{b\sqrt{m}}{d}$ are two distinct odd vertices in $\mathscr{M}_q(n)$, then $\Delta = (ad - bc)\sqrt{m}$. If $\frac{a}{c\sqrt{m}}$ and $\frac{b\sqrt{m}}{d}$ are two vertices of different types, then we define $\Delta = \pm(ad - mbc)$.

Theorem 15. *Let $\frac{a}{c\sqrt{m}}$ be an even vertex and $\frac{b\sqrt{m}}{d}$ be an odd vertex in $\mathscr{M}_q(n)$. Then*

$$\delta(\frac{a}{c\sqrt{m}}, \frac{b\sqrt{m}}{d}) = \begin{cases} 1 & if\ and\ only\ if\ |\Delta| \equiv 1 \bmod n, \\ 3 & otherwise. \end{cases} \tag{6.1}$$

Proof. Let $\frac{a}{c\sqrt{m}}$, $\frac{b\sqrt{m}}{d}$ be an even and an odd vertices respectively. By Lemma 4, δ is odd. If $|\Delta| \not\equiv 1 \bmod n$, then the two vertices are not adjacent, thus we must have $\delta = 3$ by Theorem 14. Now to see if an even vertex $\frac{a}{c\sqrt{m}}$ and an odd vertex $\frac{b\sqrt{m}}{d}$ are distance three apart when $|\Delta| \not\equiv 1 \bmod n$, let $T \in H_q$ be the transformation where $T\binom{a}{c\sqrt{m}} = \binom{1}{0\sqrt{m}}$. Apply this transformation to the odd vertex, $T\binom{b\sqrt{m}}{d} = \binom{x\sqrt{m}}{y}$ and then we can apply Theorem 12. Otherwise, if $T\binom{a}{c\sqrt{m}} = \binom{1}{0\sqrt{m}}$ and $T\binom{b\sqrt{m}}{d} = \binom{x\sqrt{m}}{0}$, then we can apply Theorem 5. □

Example 16. *(i) Let us check if $\delta(\frac{4}{1\sqrt{2}}, \frac{2\sqrt{2}}{3}) = \delta(G1, K2) = 3$ as in Figure 8. Let $T = \begin{pmatrix} 1 & 0\sqrt{2} \\ 1\sqrt{2} & 1 \end{pmatrix} \in H_q$, then $T\binom{4}{1\sqrt{2}} = \binom{1}{0\sqrt{2}} = A1$, and $T\binom{2\sqrt{2}}{3} = \binom{2\sqrt{2}}{2} = J2$. Then we can apply Theorem 12, to get $\delta(\frac{1}{0\sqrt{2}}, \frac{2\sqrt{2}}{2}) = \delta(A1, J2) = 3$.*

(ii) Applying the same T we can check the distance between $\frac{4}{1\sqrt{2}} = G1$ and $\frac{2\sqrt{2}}{1} = I2$. We have $T\binom{4}{1\sqrt{2}} = \binom{1}{0\sqrt{2}} = A1$, and $T\binom{2\sqrt{2}}{1} = \binom{2\sqrt{2}}{0} = H2$, so $\delta(\frac{1}{0\sqrt{2}}, \frac{2\sqrt{2}}{0}) = \delta(A1, H2) = 3$ by applying Theorem 5.

Theorem 17. *If x and y are two distinct vertices that are either both odd or both even in $\mathscr{M}_q(p)$ where $p > 3$ is a prime, then*

$$\delta(x, y) = \begin{cases} 4 & if\ and\ only\ if\ \Delta = 0, \\ 2 & otherwise. \end{cases} \tag{6.2}$$

Proof. By Lemma 4, $\delta(x, y) = 2$ or 4 but not greater that 4, by Theorem 14. Let $x = \frac{a}{c\sqrt{m}} = \frac{1}{0\sqrt{m}}$ and $y = \frac{b}{d\sqrt{m}}$ be two distinct even vertices such that $\Delta \neq 0$. Then we can always construct a path of length 2 between them of the form

$$\frac{1}{0\sqrt{m}} \longleftrightarrow \frac{(b \pm 1)m^{-1}d^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{b}{d\sqrt{m}}, \tag{6.3}$$

where m^{-1} and d^{-1} are always exist because $(m, p) = 1$ and $(d, p) = 1$. Now if $\Delta = 0$, then x and y must be two distinct even-poles, thus $\delta(x, y) = 4$, by Theorem 7. Otherwise, $x = \frac{0}{1\sqrt{m}}$ and $y = \frac{0}{d\sqrt{m}}$ (i.e. $a \equiv b \equiv 0 \bmod p$). We can always construct a path of length 4 between $\frac{0}{1\sqrt{m}}$ and $\frac{0}{d\sqrt{m}}$ of the form

$$\frac{0}{1\sqrt{m}} \longleftrightarrow \frac{m^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{1}{0\sqrt{m}} \longleftrightarrow \frac{d^{-1}m^{-1}\sqrt{m}}{1} \longleftrightarrow \frac{0}{d\sqrt{m}}, \tag{6.4}$$

where m^{-1} and d^{-1} are always exist because $(m, p) = 1$ and $(d, p) = 1$. Next we want to show that if $\delta(x, y) = 4$ then $\Delta = 0$. Apply the transformation $T \in H_q$, such that $T\binom{a}{c\sqrt{m}} = \binom{1}{0\sqrt{m}}$ and $T\binom{b}{d\sqrt{m}} = \binom{e}{f\sqrt{m}}$. Now consider these two new points, i.e. $\delta(\frac{1}{0\sqrt{m}}, \frac{e}{f\sqrt{m}}) = 4$ if $\Delta = 0$. As $\delta(x, y) = 4$ which therefore can happen when $f = 0$ then we can apply Theorem 7. The determinant of these two points $\frac{1}{0\sqrt{m}}$ and $\frac{e}{0\sqrt{m}}$, is equal to 0. As T preserves the determinants of points, thus the determinant of $\frac{a}{c\sqrt{m}}$ and $\frac{b}{d\sqrt{m}}$ is also equal 0. Similarly, we can prove the theorem if given any two distinct odd vertices $\frac{a\sqrt{m}}{c}$ and $\frac{b\sqrt{m}}{d}$. $\square$

This theorem does not cover the primes $p = 2$ and $p = 3$. However from Figure 5, we can see that $\mathscr{M}_4(2)$ is a square, and in Figure 6 we see that $\mathscr{M}_4(3)$ is a cube. By looking at the even and odd vertices we see directly that the distance between even vertices is equal to 2 and the distance between odd vertices is equal to 2. In Figure 9, we see that $\mathscr{M}_6(2)$ is a hexagon and in Figure 10 we see that $\mathscr{M}_6(3)$ is the hexagonal map on the torus. Again in these cases we see directly that the distance between even vertices is equal to 2 and the same for odd vertices.

In the above theorem we have made this restriction from n to $p > 3$ as for the first case when $\delta(x, y) = 4$, the odd-poles and the even vertices of the form $\frac{0}{c\sqrt{m}}$ only exist if n is odd for $q = 4$ and $3 \nmid n$ for $q = 6$, by Lemma 2 and Sections 4.1, 4.1. Thus it is better to restrict n to p to satisfy that $(0, c, n) = 1$. Now for the second case if we replace p by some composite numbers n, then $\delta(x, y) = 2$ does not apply as described in the following counter-example.

Example 18. *(i) The distance between* $\frac{1}{0\sqrt{2}}$ *and* $\frac{2}{5\sqrt{2}}$ *in* $\mathscr{M}_4(25)$ *is equal to 4 while* $\Delta = 5\sqrt{2}$*, as we can find a path of length 4 of the form*

$$\frac{1}{0\sqrt{2}} \longleftrightarrow \frac{21\sqrt{2}}{1} \longleftrightarrow \frac{16}{1\sqrt{2}} \longleftrightarrow \frac{3\sqrt{2}}{2} \longleftrightarrow \frac{2}{5\sqrt{2}}.$$

(ii) The distance between $\frac{1}{0\sqrt{3}}$ *and* $\frac{5}{11\sqrt{3}}$ *in* $\mathscr{M}_6(24)$ *is equal to 4 while* $\Delta = 11\sqrt{3}$*, as we can find a path of length 4 of the form*

$$\frac{1}{0\sqrt{3}} \longleftrightarrow \frac{20\sqrt{3}}{1} \longleftrightarrow \frac{13}{1\sqrt{3}} \longleftrightarrow \frac{1\sqrt{3}}{2} \longleftrightarrow \frac{5}{11\sqrt{3}}.$$

REFERENCES

1. M. Conder and P. Dobcsányi, *Determination of all regular maps of small genus*, J. Combin. Theory Ser. B **81** (2001), no. 2, 224–242. MR 1814906
2. E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung*, Math. Ann. **112** (1936), no. 1, 664–699. MR 1513069
3. I. Ivrissimtzis and D. Singerman, *Regular maps and principal congruence subgroups of Hecke groups*, European J. Combin. **26** (2005), no. 3-4, 437–456. MR 2116181
4. D. Kattan and D. Singerman, *Universal q-gonal tessellations and their petrie paths.*
5. L. A. Parson, *Generalized Kloosterman sums and the Fourier coefficients of cusp forms*, Trans. Amer. Math. Soc. **217** (1976), 329–350. MR 0412112
6. D. Singerman, *Universal tessellations*, Rev. Mat. Univ. Complut. Madrid **1** (1988), no. 1-3, 111–123. MR 977044
7. D. Singerman and J. Strudwick, *The Farey maps modulo n*, Acta Math. Univ. Comen., New Ser. **89** (2020), no. 1, 39–52 (English).
8. M. Weber, *Kepler's small stellated dodecahedron as a Riemann surface*, Pacific J. Math. **220** (2005), no. 1, 167–182. MR 2195068