Three Dimensionality in Quasi-Two Dimensional Flows: the *Barrel Effects*

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A scenario is put forward for the appearance of three-dimensionality both in quasi-2D rotating flows and quasi-2D magnetohydrodynamic (MHD) flows. We distinguish two forms of three-dimensionalities, establish how both are ignited by the presence of walls, and how they relate to each other. One form involves velocities or currents along the rotation direction or the magnetic field, while the other leads to quadratic variations of these quantities along this direction. It is shown that the common tendency of these flows to two-dimensionality and the mechanisms of the first form of three-dimensionality can be explained through a single formal analogy between rotating flows, MHD flows and a wider class of flows, whereas the second form involves a distinct mechanism. Because of this trans-disciplinary character, these phenomena are active in such diverse flows as those in atmospheres, oceans and the cooling blankets of nuclear fusion reactors.

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Rapidly rotating flows and electrically conducting flows in homogeneous static magnetic fields share a remarkable property: both tend to two-dimensionality. In the former, inertial waves propagation promotes invariance along the rotation axis [1]. In the latter, when magnetic field perturbations induced by flow motion are negligible (in the *quasi-static MagnetoHydroDynamic* (MHD) approximation), electric eddy currents damp velocity variations along the magnetic field lines [2].
This feature is crucial because 2D and 3D flows have radically different dissipative and transport properties, especially when turbulent. While in 3D turbulence, energy follows a direct cascade from large to small structures where it is dissipated by viscous friction, 2D turbulence proceeds through an inverse cascade that accumulates energy in large structures where it is dissipated by friction on the boundaries [3], a mechanism that is suppressed as three-dimensionality emerges [4]. The spectacular difference between these states places the questions of knowing whether flows obey 2D or 3D dynamics, how and when they may switch between them at the centre a vast array of problems across disciplines. These range from atmospheric and oceanic flows to the electromagnetic control of heat and fluid flows in metallurgical processes, and the dynamo problem. Despite their importance, the transitions mechanisms between 2D and 3D states, in rotating, MHD or other flows are mostly unknown and an important gap exists between theory and experiment. In MHD flows with periodic or free-slip boundaries, 3D instabilities lead to the breakdown of strictly 2D structures, and the flow can intermittently wander between 2D and 3D states [5, 6]. Because of the thin boundary layers along walls or other physical interfaces, experimental flows cannot be strictly 2D, but only quasi-2D. In these conditions, flow motion along the field can persist [7]. In wall bounded flows, variations of velocity along the main field were also observed: they led to regimes where flow structures lost quasi two-dimensionality, without being necessarily subject to 3D instabilities. When such instabilities did appear, they gave birth to steady 3D structures predicted by no current theory [8].

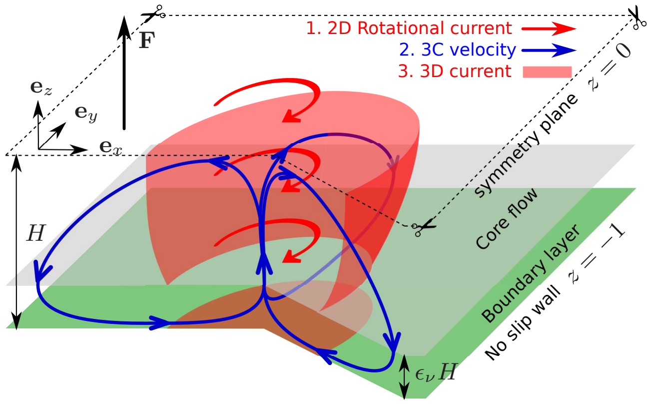

FIG. 1. Channel flow configuration. The three steps of the first barrel effect are schematically represented. In the second type of barrel effect, active in MHD flows, the steps are identical with the roles of the current and the velocity swapped.

In this Letter, we examine flows in the simple configuration of a symmetric, plane channel in a transverse field, bounded by two no-slip walls distant of $2H$ (figure 1), to reveal how walls induce three-dimensionality in quasi-2D flows. The governing equations are written in a general form that encompasses both quasi-static MHD and rotating flows. In doing so, a formal analogy is defined within a class of flows with a common tendency to two-dimensionality, and a hierarchy of mechanisms is singled out that both explains how 3-Component (3C) motions are ignited and how three-dimensionality appears as a variation of physical quantities across the channel.

Following the formalism of rotating and quasi-static MHD flows, we first define a static homogeneous field $\mathbf{F} = F\mathbf{e}_z$ (vertical for convenience), orthogonal to the channel, that pervades the fluid domain and interacts with a current $\mathbf{c}$ to exert a force $\mathbf{c} \times \mathbf{F}$ on the flow. In rotating flows (*resp.* quasi-static MHD), $\mathbf{F}$ is twice the background rotation vector (*resp.* the magnetic field) and $\mathbf{c}$ is the flow current (*resp.* electric current), that appear in the Coriolis (*resp.* Lorentz) force. The current is linked to the velocity field $\mathbf{u}$ by a phenomenological law such as Ohm's law in MHD, that takes the general form $\mathbf{c} = \mathcal{F}(\mathbf{u})$. The corresponding non-dimensional equations for an incompressible fluid (density ρ, viscosity ν) depend on two dimensionless groups, built on H, the reference velocity U and reference current C: $\epsilon = \rho U^2/(HCF)$ and $\epsilon_\nu^2 = \rho\nu U/(H^2CF)$ respectively express the ratio of inertia and viscous forces to $\mathbf{c}\times\mathbf{F}$, and both are much smaller than unity. This class of analogy is summarised in table I. Denoting any quantity g in the core (*i.e.* outside the wall boundary layers) as $\check{g}$, the equations write:

$$\epsilon\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp + \nabla_\perp\check{p} = \check{\mathbf{c}}_\perp \times \mathbf{e}_z + \epsilon_\nu^2(\partial^2_{zz} + \nabla^2_\perp)\check{\mathbf{u}}_\perp, \quad (1)$$

$$\epsilon\frac{\check{d}}{dt}\check{u}_z + \partial_z\check{p} = \epsilon_\nu^2(\partial^2_{zz} + \nabla^2_\perp)\check{u}_z, \quad (2)$$

$$\partial_z\check{u}_z = -\nabla_\perp\cdot\check{\mathbf{u}}_\perp, \quad (3)$$

$$\partial_z\check{c}_z = -\nabla_\perp\cdot\check{\mathbf{c}}_\perp, \quad (4)$$

$$\check{\mathbf{c}} = \mathcal{F}(\check{\mathbf{u}}). \quad (5)$$

Vector quantities and operators are split into their components along $\mathbf{e}_z$ (subscript z) and in the (x, y) plane (subscript $\perp$).

We shall first clarify the interplay between the respective dynamics of $(\mathbf{c}_\perp, c_z)$ and $(\mathbf{u}_\perp, u_z)$ in the core. At the leading order $(\epsilon^0, \epsilon_\nu^0)$ when $\epsilon \to 0, \epsilon_\nu \to 0$, (1) and (2) readily imply that $\check{\mathbf{c}}_\perp$ and $\check{p}$ are independent of z (*i.e.* 2D). Denoting leading order quantities with the superscript g^0,

$$\check{\mathbf{c}}^0_\perp = \mathbf{e}_z \times \nabla_\perp\check{p}^0, \quad (6)$$

$$\partial_z\check{p}^0 = 0. \quad (7)$$

The problem symmetry and (4) further imply that the current is exclusively horizontal:

$$\check{c}^0_z = 0. \quad (8)$$

Two-dimensionality of the core velocity isn't ensured at this point, but depends on the nature of (5). For the time being, we shall assume that z-dependence doesn't explicitly appear in $\mathcal{F}$. In this case, $\check{\mathbf{u}}^0_\perp$ is indeed 2D. When applied to rotating and quasi-static MHD flows, this result recovers the property that both are quasi-2D in the absence of inertial and viscous effects in the core. The $z-$dependence of $\check{\mathbf{c}}_\perp$ appears from (1) and (2):

$$\partial_z\check{\mathbf{c}}_\perp = \left(\epsilon\frac{\check{d}}{dt} - \epsilon_\nu^2\nabla^2\right)\left[\nabla_\perp\check{u}_z - \partial_z\check{\mathbf{u}}_\perp\right]\times\mathbf{e}_z. \quad (9)$$

Since the leading order core velocity is 2D, $\check{\mathbf{c}}_\perp$ must in fact be 2D at least up to $\mathcal{O}(\epsilon, \epsilon_\nu^2)$, so that using (4) yields:

$$\check{c}_z = -z\check{c}_z(-1) + \mathcal{O}(\epsilon, \epsilon_\nu^2). \quad (10)$$

Similarly, the problem symmetry and (3) imply that

$$\check{u}_z = -z\check{u}_z(-1) + \mathcal{O}(\epsilon, \epsilon_\nu^2). \quad (11)$$

From (9) and (10), the appearance of $z-$ variations in $\check{\mathbf{c}}_\perp$ and $\check{c}_z$ is determined by the flow and the current injected into the core from the boundary layers that develop along the walls $\check{u}_z(-1)$ and $\check{c}_z(-1)$. Only then, can $z-$ variations in $\check{\mathbf{u}}_\perp$ appear through (5), at the same order as $\check{\mathbf{c}}_\perp$. Therefore, we now need to analyse the wall boundary layers.

In the vicinity of walls, horizontal viscous friction must balance $\mathbf{c}\times\mathbf{F}$ to achieve the no slip boundary condition, and this imposes the thickness of the resulting boundary layer to scale as ϵ_ν (respectively the *Ekman* and *Hartmann* layers in rotating and MHD flows). The governing equations are accordingly rewritten in the boundary layer near the wall $z = -1$ using stretched variable $\zeta = \epsilon_\nu^{-1}(z+1)$, to reflect that viscous friction becomes $\mathcal{O}(1)$. Denoting any quantity g in this region as $\hat{g}$,

$$\epsilon\frac{\hat{d}}{dt}\hat{\mathbf{u}}_\perp + \nabla_\perp\hat{p} = \hat{\mathbf{c}}_\perp\times\mathbf{e}_z + (\partial^2_{\zeta\zeta} + \epsilon_\nu^2\nabla^2_\perp)\hat{\mathbf{u}}_\perp, \quad (12)$$

$$\epsilon\frac{\hat{d}}{dt}\hat{u}_z + \partial_z\hat{p} = (\partial^2_{\zeta\zeta} + \epsilon_\nu^2\nabla^2_\perp)\hat{u}_z, \quad (13)$$

$$\partial_\zeta\hat{u}_z = -\epsilon_\nu\nabla_\perp\cdot\hat{\mathbf{u}}_\perp, \quad (14)$$

$$\partial_\zeta\hat{c}_z = -\epsilon_\nu\nabla_\perp\cdot\hat{\mathbf{c}}_\perp, \quad (15)$$

$$\hat{\mathbf{c}} = \mathcal{F}(\hat{\mathbf{u}}). \quad (16)$$

The no slip condition at the wall is written

$$\hat{\mathbf{u}}_\perp(-1) = 0, \quad (17)$$

while the boundary conditions for $\hat{\mathbf{c}}$ and $\hat{u}_z$ shall be left unspecified for the time being, thus allowing for possible wall-injection of current or mass into the fluid. Core and boundary layer variables must also satisfy a matching condition. Since they are not explicitly expanded in powers of ϵ, ϵ_ν, the general matching condition for any quantity g put forward by [10] simplifies to [11]:

$$\lim_{z\to-1}\check{g}(z+1) = \lim_{\zeta\to\infty}\hat{g}(\zeta). \quad (18)$$

We shall now set to the task of determining $\check{c}_z(-1)$. Integrating $\nabla_\perp\times$ (12) and using (15) leads to

$$\hat{c}_z(\zeta) - c_z^W = \epsilon_\nu\left[\partial_\zeta\omega_z^W - \partial_\zeta\hat{\omega}_z(\zeta)\right] + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^3). \quad (19)$$

ω is the vorticity and the superscript W refers to values taken at the wall. The current injected at the wall c_z^W would be determined by a boundary condition for the current, here unspecified. By virtue of (18), and since $\partial_\zeta\check{\omega}_z(-1) = \epsilon_\nu\partial_z\check{\omega}_z(-1)$, (19) yields

$$\check{c}_z(-1) = c_z^W + \epsilon_\nu\partial_\zeta\omega_z^W + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2). \quad (20)$$

Eqs. (20) and (10) reveal the two main mechanisms that can feed eddy currents in the core: injection of current

	Field $\mathbf{F}$	current $\mathbf{c} = \mathcal{F}(\mathbf{u})$	ϵ	ϵ_ν	Wall boundary layer ($\hat{\mathbf{u}}^0_\perp =$)
Quasi-static MHD	magnetic field $\mathbf{B}$	electric current $\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u}\times\mathbf{B})$	Stuart number $N = \epsilon^{-1} = \frac{\sigma B^2 H}{\rho U}$	Hartmann number $Ha = \epsilon_\nu^{-1} = BH\sqrt{\frac{\sigma}{\rho\nu}}$	Hartmann layer [9] $\check{\mathbf{u}}^0_\perp(-1)(1-e^{-\zeta})$
Rotating flows	double rotation $2\boldsymbol{\Omega}$	flow current $\rho\mathbf{u}$	Rossby number $Ro = \epsilon = \frac{U}{2H\Omega}$	Ekman number $E = \epsilon_\nu^2 = \frac{\nu}{2H^2\Omega}$	Ekman layer [1] $\check{\mathbf{u}}^0_\perp(-1)(1-e^{-\frac{\zeta}{\sqrt{2}}}\cos\frac{\zeta}{\sqrt{2}}) + \mathbf{e}_z\times\check{\mathbf{u}}^0_\perp(-1)e^{-\frac{\zeta}{\sqrt{2}}}\sin\frac{\zeta}{\sqrt{2}}$

TABLE I. Analogy table giving the expressions of generic dimensional quantities $\mathbf{F}$, $\mathbf{c}$, and non-dimensional numbers ϵ_ν and ϵ for quasi-static MHD and rotating flows. σ and ϕ are the electric conductivity of the fluid and the electric potential.

at the wall and rotational wall friction contribute respectively at the leading order, and at $\mathcal{O}(\epsilon_\nu)$. Importantly, (9) implies that although these recirculations make $\check{\mathbf{c}}$ 3C, they do not directly affect the $z-$ dependence of $\check{\mathbf{c}}_\perp(z)$, which remains 2D up to $\mathcal{O}(\epsilon, \epsilon_\nu^2)$ at least.
Strikingly, if $c_z^W \neq 0$, (20) contradicts (8): a linearly z-dependent vertical current in the core leads to diverging horizontal currents that induce a rotational force $\check{\mathbf{c}}_\perp \times \mathbf{e}_z$, which cannot be balanced by the sole leading order pressure gradient in (6). To resolve this paradox, viscous or inertial effects must exist in the core to oppose $\check{\mathbf{c}}_\perp \times \mathbf{e}_z$. These can therefore not be neglected and, in fact, prevent two-dimensionality at the leading order. Quasi-static MHD provides a well understood manifestation of this effect: MHD flows are often driven by injecting electric current through point-electrodes embedded in an otherwise electrically conducting wall [12]. Above such an electrode develops a vortex of rotation axis $\mathbf{e}_z$, with a viscous core that is 3D at the leading order. Injecting current at the wall therefore directly prevents the flow from being quasi-2D, even at the leading order. Having now singled out this important effect, we shall assume $c_z^W = 0$ for the reminder of this Letter and focus on higher order effects.
At order $\mathcal{O}(\epsilon_\nu)$, (20) generalises two classic properties of Ekman and Hartmann boundary layers: in rotating flows, friction in the Ekman layer gives rise to secondary flows in the core $\check{u}_z(-1) \simeq \frac{\sqrt{2}}{2}E^{1/2}\check{\omega}^0_z(-1)$, by *Ekman pumping* [1]. In quasi-static MHD, $\partial_\zeta \hat{\mathbf{u}}^W_\perp = \check{\mathbf{u}}_\perp(-1) + \mathcal{O}(\epsilon_\nu, \epsilon)$, so (20) expresses that vorticity in the core drives an electric current $\check{j}_z(-1) \simeq Ha^{-1}\omega^0_z(-1)$ out of the Hartmann layers [9] (these two results are recovered using the leading order solutions of (12-16), $\hat{\mathbf{u}}_\perp$, given in table I).

We shall now turn our attention to the determination of $\check{u}_z(-1)$, which controls $\partial_z\check{\mathbf{c}}_\perp(z)$. First, from (13) and (14), the pressure is constant across the boundary layer:

$$\nabla_\perp\hat{p}(\zeta) = \nabla_\perp\check{p}(-1) + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2)$$
$$= \check{\mathbf{c}}_\perp(-1)\times\mathbf{e}_z - \epsilon\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp(-1) + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2) \quad (21)$$

The horizontal pressure gradient in the boundary layer thus results from the balance of forces in the core. (12) expresses that each of these forces alters the local balance between viscous forces and $\hat{\mathbf{c}}\times\mathbf{e}_z$ in the boundary layer, and thereby drives horizontal jets. Should these jets be horizontally divergent, they in turn induce a vertical flow from the boundary layer to the core. The equation for $\hat{u}_z$ thus follows from $\nabla_\perp\cdot$(12) and (14):

$$\epsilon\epsilon_\nu\nabla_\perp\cdot\left[\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp(-1) - \frac{\hat{d}}{dt}\hat{\mathbf{u}}_\perp\right]$$
$$+\epsilon_\nu\nabla_\perp\times[\hat{\mathbf{c}}_\perp - \check{\mathbf{c}}_\perp(-1)]\cdot\mathbf{e}_z = \partial^3_{\zeta\zeta\zeta}\hat{u}_z + \mathcal{O}(\epsilon_\nu^3), \quad (22)$$

and $\check{u}_z(-1)$ follows by integration, using (18) and (11):

$$(1-\epsilon_\nu)\check{u}_z(-1) = u_z^W + \epsilon_\nu u_z^C + \epsilon_\nu\epsilon u_z^I + \mathcal{O}(\epsilon_\nu^3), \quad (23)$$
$$u_z^C = \int_0^\infty\int_\zeta^\infty\int_{\zeta_2}^\infty \nabla_\perp\times[\hat{\mathbf{c}}_\perp - \check{\mathbf{c}}_\perp(-1)]\cdot\mathbf{e}_z d\zeta d\zeta_2 d\zeta_1,$$
$$u_z^I = \int_0^\infty\int_\zeta^\infty\int_{\zeta_2}^\infty \nabla_\perp\cdot\left[\frac{\check{d}}{dt}\check{\mathbf{u}}_\perp(-1) - \frac{\hat{d}}{dt}\hat{\mathbf{u}}_\perp\right] d\zeta d\zeta_2 d\zeta_1.$$

Eq. (23) singles out three possible origins of secondary flows between core and boundary layer: the flow directly injected at the wall u_z^W doesn't interact with the boundary layer and is integrally transmitted to the core where it affects the flow at the leading order. The symmetric boundary conditions at the walls imply through (11) that it creates two symmetric recirculations there. Should no flow be injected at the wall ($u_z^W = 0$), secondary flows can result from a rotational current as $\check{u}_z(-1) = \epsilon_\nu u_z^C + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2)$. Unlike wall-injected flow, the corresponding vertical flow builds up from horizontally divergent jets in the boundary layer but recirculates in the core in the same way. It leads to secondary flows at order ϵ_ν there, a scaling they inherit from the thickness of the boundary layer where they are created. Finally, if the core current is curl-free at the leading order, then $u_z^C = 0$. Inertial effects of order ϵ then take over as the dominant mechanism that drives jets in the boundary layers, resulting in a secondary flow $\check{u}_z(-1) = \epsilon\epsilon_\nu u_z^I + \mathcal{O}(\epsilon_\nu^3)$.
In rotating flows, $\mathbf{c} = \mathbf{u}$ (non-dimensionally) and $\nabla\times\mathbf{c}_\perp = \omega_z\mathbf{e}_z \neq 0$ (see table I), so the mechanism responsible for secondary flows is the Ekman pumping mentioned earlier when analysing eddy currents fed by $\check{c}_z(-1)$. Still, application of (23) again recovers the well-known result that $\check{u}_z(-1) = \epsilon_\nu u_z^C + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2) = \frac{\sqrt{2}}{2}E^{1/2}\check{\omega}_z(-1) + (RoE^{1/2}, E)$[1].
In quasi-static MHD, on the other hand, the integrand in the expression of u_z^C is 0 at the leading order so in the inertialess theory of the Hartmann layer, no flow escapes

to the core [9]. When inertia is taken into account though, (23) yields $\check{u}_z(-1) = \epsilon\epsilon_\nu u_z^I + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^3) = -(5/6)Ha^{-1}N^{-1}\nabla\cdot\left[\check{\mathbf{u}}_\perp^0(-1)\cdot\nabla\check{\mathbf{u}}_\perp^0(-1)\right]$ [13].

Having expressed the current and mass flow that feed into the core, we are now in position to return to the core flow equations analysed at the beginning of this Letter and determine the conditions of appearance of $z-$dependence in core quantities. First, since $\check{\mathbf{c}}_\perp$ appears at a higher order than $\check{\mathbf{u}}_\perp$ in (1-3), and since we further assumed that $z-$ dependence didn't appear explicitly in $\mathcal{F}$, z-dependence cannot appear at lower order in $\check{\mathbf{u}}_\perp$ than in $\check{\mathbf{c}}_\perp$. Consequently, at the first order at which it appears, (9) simplifies into:

$$\partial_z\check{\mathbf{c}}_\perp = \left(\epsilon\frac{d}{dt} - \epsilon_\nu^2\nabla_\perp^2\right)\nabla_\perp\check{u}_z\times\mathbf{e}_z + \mathcal{O}(\check{u}_z\epsilon, \check{u}_z\epsilon_\nu^2). \quad (24)$$

By virtue of the symmetric boundary conditions, and depending on which of the three terms in (23) is non-zero of highest order, three-dimensionality appears under either of the three corresponding forms:

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon\frac{z^2}{2}\frac{d}{dt}\nabla_\perp u_z^W + \mathcal{O}(\epsilon\epsilon_\nu, \epsilon_\nu^2), \quad (25)$$

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon\epsilon_\nu\frac{z^2}{2}\frac{d}{dt}\nabla_\perp u_z^C + \mathcal{O}(\epsilon^2\epsilon_\nu, \epsilon_\nu^3), \quad (26)$$

$$\check{\mathbf{c}}_\perp = \check{\mathbf{c}}_\perp(0) + \epsilon^2\epsilon_\nu\frac{z^2}{2}\frac{d}{dt}\nabla_\perp u_z^I + \mathcal{O}(\epsilon_\nu^2). \quad (27)$$

Physically, these equations express that the z-linear vertical flow created in the core by the mass flow ejected from the boundary layers builds up a z-quadratic pressure in the core. A quadratic current must in turn be drawn in the core, for the force $\check{\mathbf{c}}_\perp\times\mathbf{e}_z$ to be able to balance the corresponding quadratic component of the horizontal pressure gradient. The z-dependence of $\check{\mathbf{u}}_\perp$ is determined by the nature of $\mathcal{F}$: in rotating flows for example, $\mathcal{F}$ is a linear functional of $\check{\mathbf{u}}_\perp$. Consequently, $\check{\mathbf{u}}_\perp$ also depends quadratically on z, because $\check{\mathbf{c}}_\perp$ does. Core structures are thus not columnar as usually assumed in quasi-2D flows but rather *barrel*-shaped.

Lastly, we shall illustrate a second type of "barrel effect", where $\check{\mathbf{u}}_\perp$ depends quadratically on z. Unlike previously, it shall be triggered by an explicit z-dependence in $\mathcal{F}$. Such an example is found in MHD flows, where z appears in the core electric potential $\check{\phi}$ at order $\epsilon_\nu = Ha^{-1}$. In MHD, the non-dimensional expression of $\mathcal{F}$ is given by Ohm's law $\mathbf{c} = \mathbf{j} = -\nabla\phi + \mathbf{u}\times\mathbf{e}_z$ (table I). At the leading order, z-dependence is still absent in $\mathcal{F}$ [13], so all core quantities remain 2D. Application of (10) and (20) provide the expression of the vertical current there: $\check{c}_z = Ha^{-1}z\omega_z^0 + \mathcal{O}(Ha^{-2}, N^{-1})$. According to Ohm's law, this vertical electric current induces a quadratic component of the electric potential $\phi = (z^2/2)Ha^{-1}\check{\omega}_z^0 + \mathcal{O}(Ha^{-2}, N^{-1})$, which introduces an explicit z-dependence in $\mathcal{F}$ at this order. Since, however, (9) still implies that $\check{\mathbf{c}}_\perp$ must be 2D at $\mathcal{O}(Ha^{-1})$, Ohm's law demands that $\check{\mathbf{u}}_\perp$ be quadratic at $\mathcal{O}(Ha^{-1})$ [13]. This mechanism is analogous to the barrel effect identified previously, with j_z, ϕ and Ohm's law taking over the respective roles of u_z, p and (2). The first barrel effect was driven by vertical *flows* out of the boundary layer into the core, induced by 2D *rotational currents*. The MHD barrel effect, by contrast, is driven by vertical *currents* out of the boundary layer, due to 2D *vorticity*. This second analogy underlines that although one common formalism explains why rotating and MHD flows are either 2D or 3C, the origins of their three-dimensionality are still formally analogous but involve a distinct analogy.

This Letter stresses that whenever no-slip walls are present in flows with finite inertia or viscosity, not only is strict two dimensionality but also strict quasi-two dimensionality (*i.e.* wall boundary layers and a strictly 2D core flow) precluded in a large class of flows that include rotating and MHD flows. 2D structures induce eddy currents and flow recirculations, in turn responsible for variations of physical quantities along the main field. We recently observed precisely these effects in liquid metal MHD flows [8], as well as strongly 3D structures, near the 2D-3D transition. Existing transition scenarios based on the instability of strictly 2D structures [5, 6] fail to predict these phenomena, but the weakly 3D structures found in this Letter should provide a more realistic platform to understand the complete 2D-3D transition mechanisms.

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[1] J. Pedlosky, *Geophysical Fluid Dynamics* (Springer Verlag, 1987)
[2] P. Roberts, *Introduction to Magnetohydrodynamics* (Longmans, 1967)
[3] P. Tabeling, Phys. Rep. **362**, 1 (2002)
[4] M. Shats, D. Byrne, and H. Xia, Phys. Rev. Lett., 264501(2010)
[5] A. Thess and O. Zikanov, J. Fluid Mech. **579**, 383 (2007)
[6] T. Boeck, D. Krasnov, and A. Thess, Phys. Rev. Lett., 244501(2008)
[7] R. Akkermans, L. Kamp, H. Clercx, and G. V. Heijst, Europhys. Lett. **83**, 24001 (2008)
[8] R. Klein and A. Pothérat, Phys. Rev. Let. **104** (2010)
[9] R. Moreau, *Magnetohydrodynamics* (Kluwer, 1990)
[10] S. Kaplun, ZAMP **V-9**, 111 (1954)
[11] A. Pothérat, J. Sommeria, and R. Moreau, Phys. Fluids, 403(2002)
[12] J. Sommeria, J. Fluid Mech. **189**, 553 (1988)
[13] A. Pothérat, J. Sommeria, and R. Moreau, J. Fluid Mech. **424**, 75 (2000)