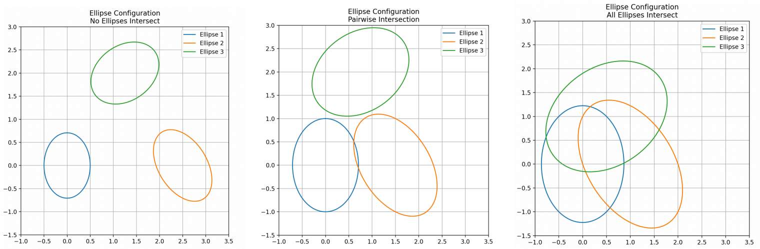
Intersection of Ellipsoidal Balls and Applications to Geometric Persistent Homology

Sean Hill
Joint work with Barbara Giunti and Felix Ye

October 11, 2025

Overview

- **Goal:** Characterize when a collection of ellipsoidal balls in $\mathbb{R}^D$ intersect.



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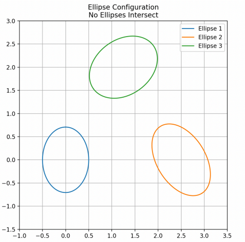
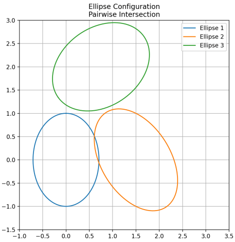
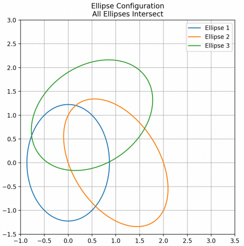
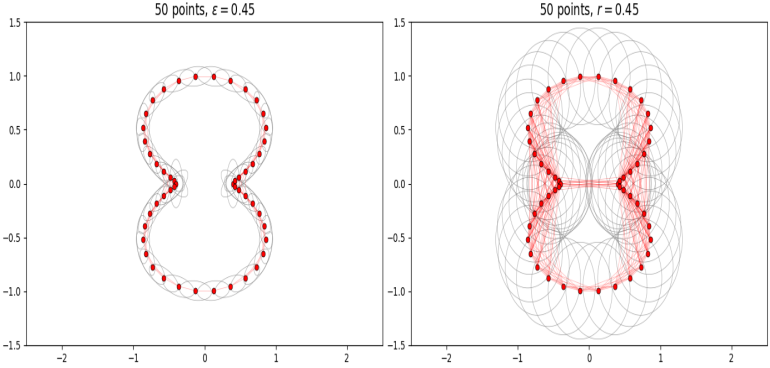




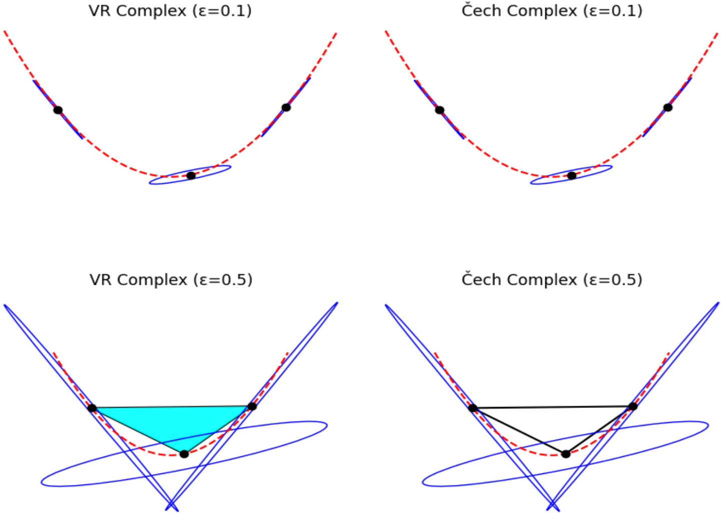


- **Motivation:** In manifold learning and persistent homology, replacing Euclidean balls with ellipsoidal ones provides a more sensitive description of local geometry.

Tangent-Normal ellipsoids respect geometry



Ellipsoidal-VR complex can create spurious features



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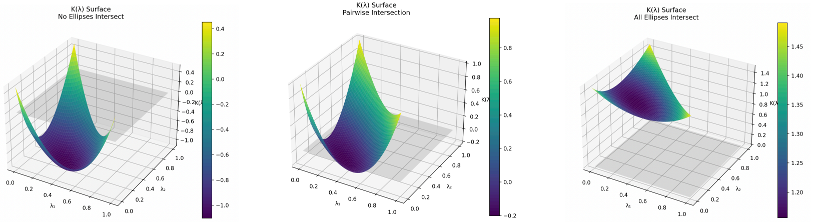

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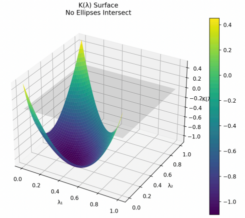

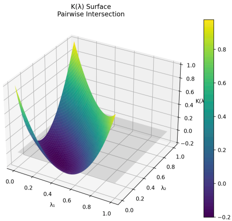

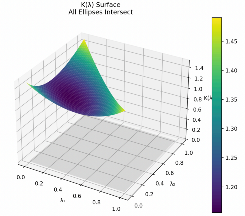

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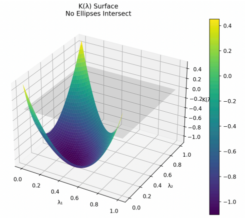

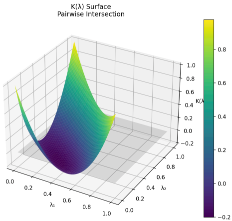

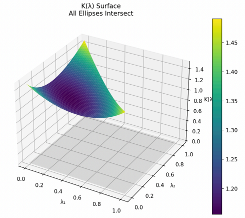

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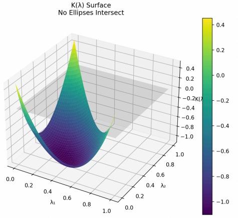

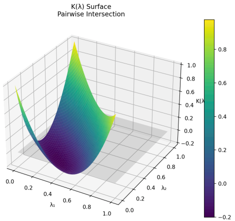

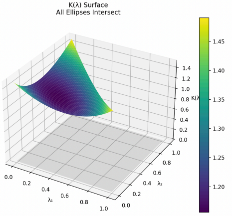

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- **Approach:** Use geometric intuition and convex optimization to pinpoint the minimal intersection radius ϵ^*.
- Results are known for $k = 2$ ellipses. See Gilitschenski [1].
- We generalize this for $k > 2$ and relate it to a LP-type problem for ϵ^*.

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- We derive an intersection criterion for a given subset S of ellipses.
- Checking the intersection criterion once allows us to also compute the optimal radius $\epsilon^* = f(S)$ (via KKT conditions).
- We show $f(S)$ is a LP-type problem function.

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- Kališnik et al [3]: Ellipsoidal Rips filtrations better capture anisotropy and curvature, improving signal in bottlenecks.
- Challenge: Pairwise intersection checks can create spurious topological features (e.g., holes from pairwise intersecting ellipses).

Differences from the Euclidean case

- By the symmetry of Euclidean distance, the intersection of k Euclidean balls at radius ϵ is equivalent to the minimum volume enclosing ball (MVEB) having radius at most ϵ.

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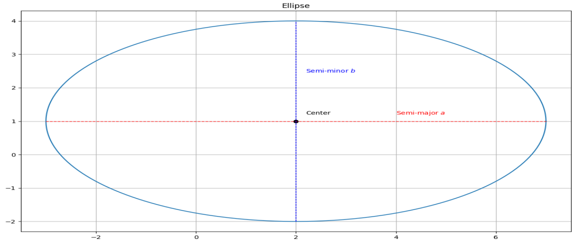
- This equivalence fails for non-homogeneous ellipsoidal balls.

Counterexample (H., Giunti)

Let $S = \{x_1, x_2\}$ with any two distinct ellipsoidal balls centered at them that do not intersect. Then the affine hull of S is a line and the MVEEB is ill-posed: flatten an enclosing ellipsoidal ball in directions orthogonal to the line; its volume goes to zero.

Review: ellipses via high school geometry



Recall that an ellipse centered at the origin in the plane is the set of all points (x, y) such that

$$x^2/a^2 + y^2/b^2 = 1.$$

A point (x, y) lies in the interior of the ellipse if

$$x^2/a^2 + y^2/b^2 < 1.$$

Ellipses via linear algebra

The above equation can be written using linear algebra:

$$[x, y] \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

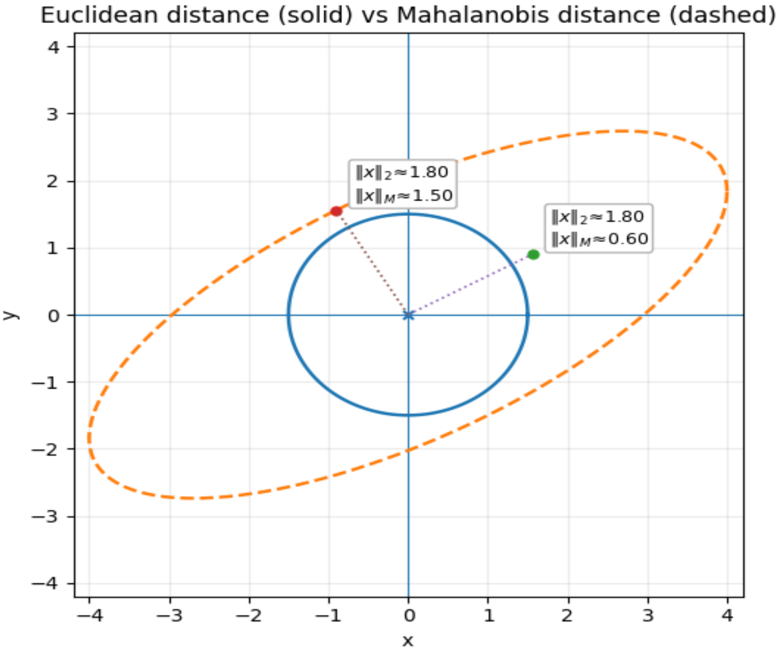
$$= x^2/a^2 + y^2/b^2.$$

This generalizes ellipses to any dimension, axis and center $\vec{x}$ using the Mahalanobis distance

$$\|\vec{y} - \vec{x}\|^2_{A(\vec{x})^{-1}} := (\vec{y} - \vec{x})^T A(\vec{x})^{-1} (\vec{y} - \vec{x})$$

where A is a PD matrix, i.e. the above expression is always > 0 for $\vec{y} \neq \vec{x}$.

Mahalanobis distance



Ellipsoidal balls

Let A_i be a real PD symmetric matrix and $c_i \in \mathbb{R}^d$. Define, for $\epsilon \geq 0$,

$$E_i(\epsilon) := \{x \in \mathbb{R}^d : (x - c_i)^\top A_i^{-1}(x - c_i) \leq \epsilon^2\},$$

the ellipsoidal closed ball with orientation and scale determined by the matrix A_i centered at c_i. We emphasize that the ellipses are **not** homogeneous in orientation and scale.

The optimal radius

Let $I_k = \{1, 2, \ldots, k\}$ and define, for any $S \subset I_k$,

$$f(S) = \inf\{\epsilon \geq 0 : \cap_{i \in S} E_i(\epsilon) \neq \emptyset\}.$$

Thus, $f(S)$ is the first radius at which the ellipsoidal balls at x_i intersect for the subset of indices $i \in S$.

Criterion for intersection (generalization of Gilitschenski [1])

Theorem (H., Giunti)

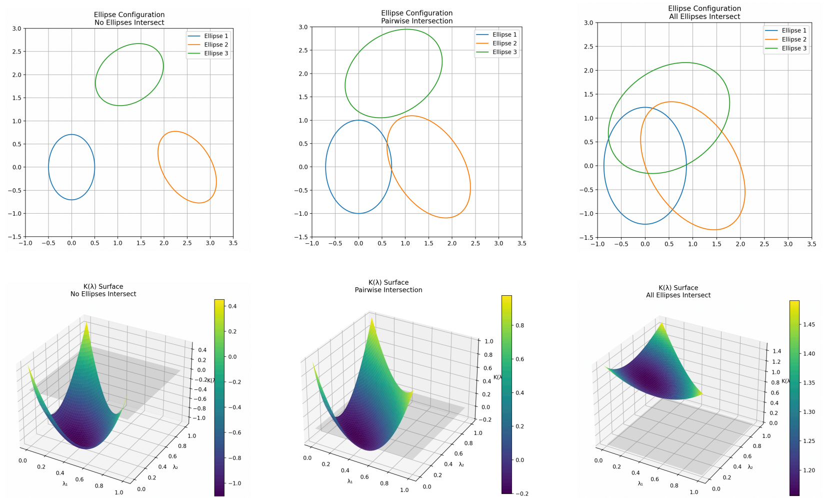
$$\cap_{i \in S} E_i(\epsilon) \neq \emptyset \iff \min_{\lambda \in \Delta^{|S|}} K(\lambda) > 0$$

where $K(\lambda) = \epsilon^2 - C(\lambda)$, *with*

$$C(\lambda) = \sum_{i \in S} \lambda_i (c_i - m(\lambda))^T A_i^{-1} (c_i - m(\lambda))$$

and $m(\lambda)$ *is a precision-weighted centroid of the* x_i.

Visualization of criterion



Computing the optimal radius

Theorem (H., Giunti)

Let λ^* minimize $K(\lambda)$ on the probability simplex. Then the optimal radius $\epsilon^* = f(S)$ is given by

$$f(S) = \max_{1 \leq i \leq |S|} \|m(\lambda^*) - x_i\|_{A_i^{-1}}.$$

Further, the distance in the maximum is constant for $i \in S$ such that $\lambda_i^* > 0$.

KKT Conditions: properties of the optimal centroid

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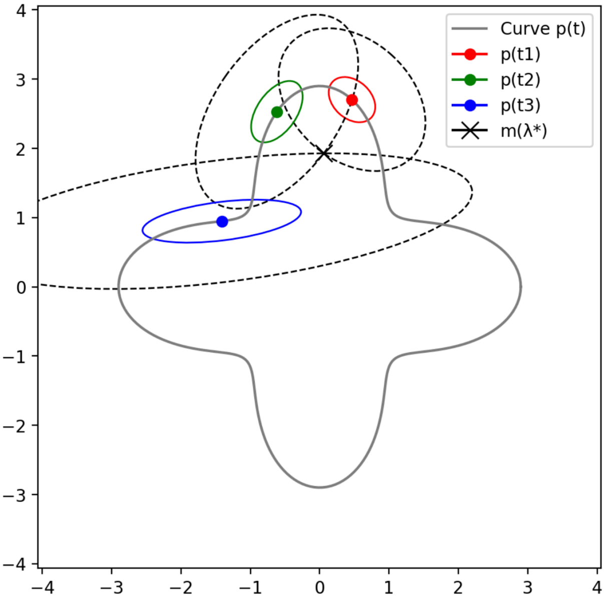
- This common Mahalanobis distance is our optimal radius ϵ^*.
- If $\lambda_i^* = 0$ then $m(\lambda^*)$ lies in the interior of $E_i(\epsilon^*)$.
- Otherwise if all $\lambda_i^* > 0$ then $\cap_i E_i(\epsilon^*) = \{m(\lambda^*)\}$.

Case 1: $\lambda_j^* > 0$ for all j: the ellipses



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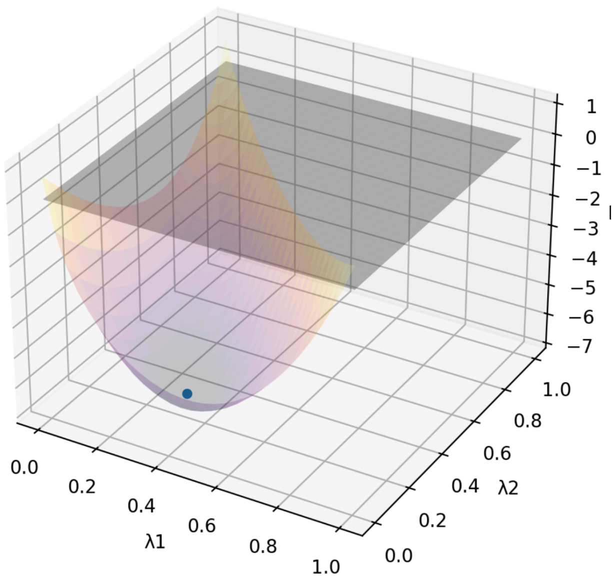

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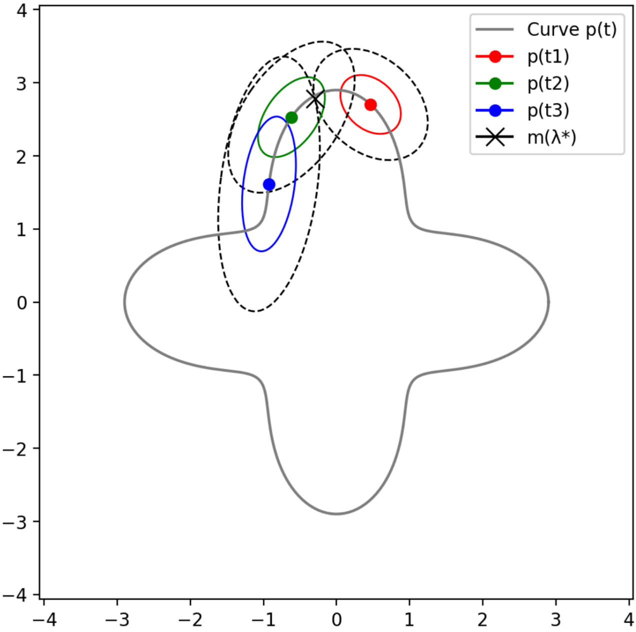

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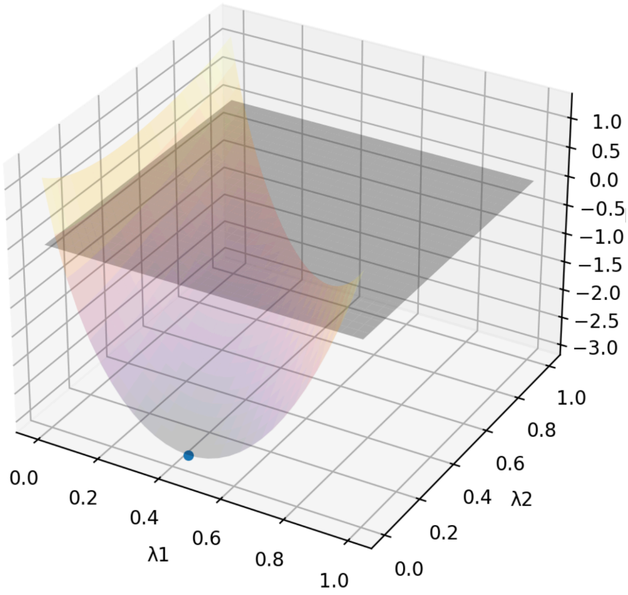

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1. *Monotonicity: if $A \subset B \subset I_k$ then $f(A) \leq f(B) \leq f(I_k)$.*
2. *Locality: Suppose $A \subset B \subset I_k$ and $i \in I_k$. If $f(A) = f(B) = f(A \cup \{i\})$ then $f(B) = f(B \cup \{i\})$.*

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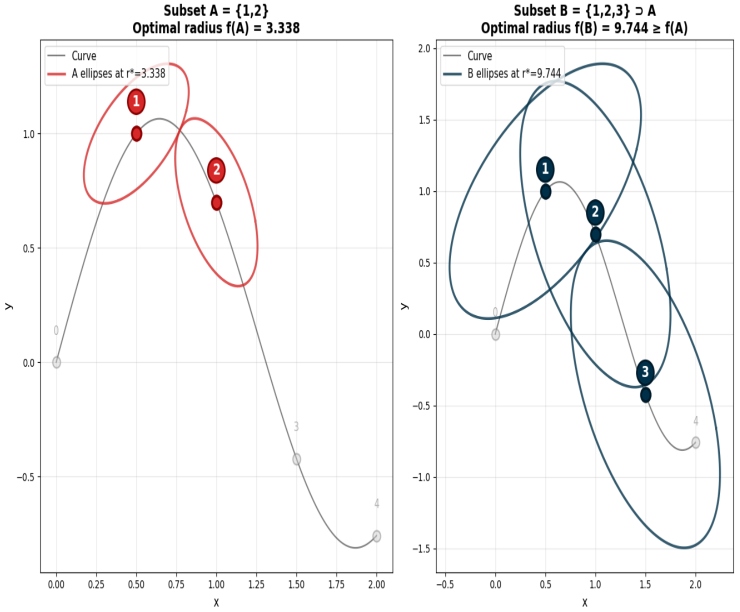
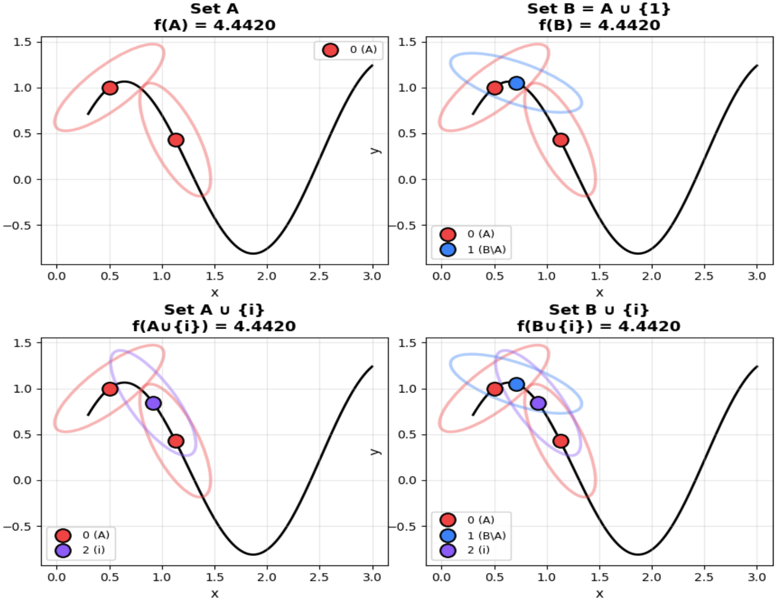


Locality: $A \subset B \subset I_k$, $i \in I_k$, $f(A) = f(B) = f(A \cup \{i\})$ $\implies f(B) = f(B \cup \{i\})$.



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- Plan for the future: study these algorithms and implement an efficient algorithm for building ellipsoidal Čech complexes.

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- The set function $f(S)$ (first intersection radius) is LP-type: monotone and local. This enables output-sensitive computation from small "basis" subsets.
- For PH, ellipsoids respect tangent/normal anisotropy and curb spurious features that plague Euclidean-ball VR/Čech at comparable radii.

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References I

[1] Igor Gilitschenski and Uwe D. Hanebeck. "A robust computational test for overlap of two arbitrary-dimensional ellipsoids in fault-detection of Kalman filters". In: *2012 15th International Conference on Information Fusion*. 2012, pp. 396–401.

[2] Sara Kališnik and Domen Lešnik. "Finding the Homology of Manifolds Using Ellipsoids". In: *Journal of Applied and Computational Topology* 8 (2024), pp. 193–238. DOI: 10.1007/s41468-023-00145-6. URL: https://doi.org/10.1007/s41468-023-00145-6.

[3] Sara Kališnik, Bastian Rieck, and Ana Žegarac. *Persistent Homology via Ellipsoids*. 2024. arXiv: 2408.11450 [math.AT]. URL: https://arxiv.org/abs/2408.11450.