A free lunch:

manifolds of positive reach can be smoothed without decreasing the reach

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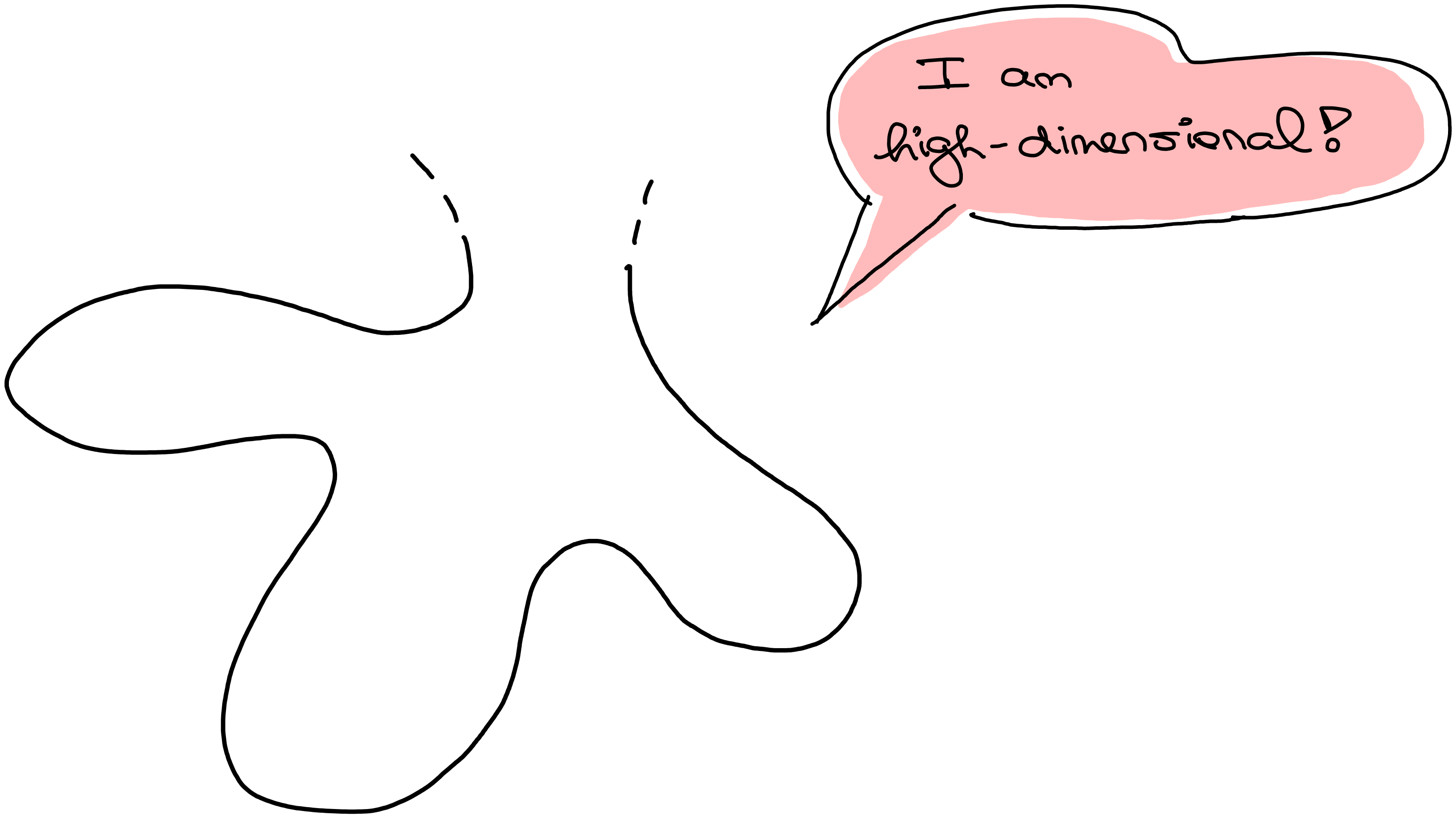
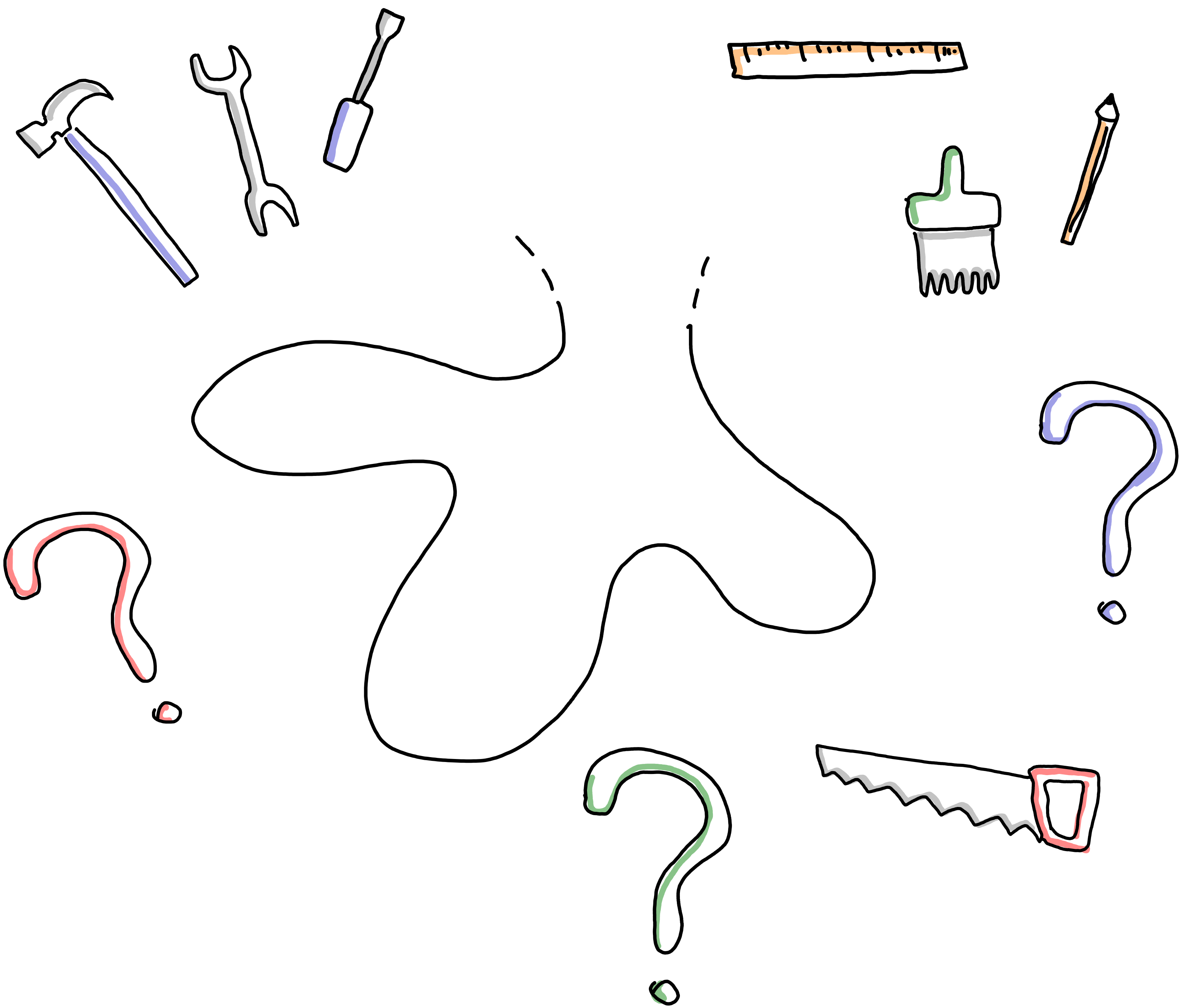
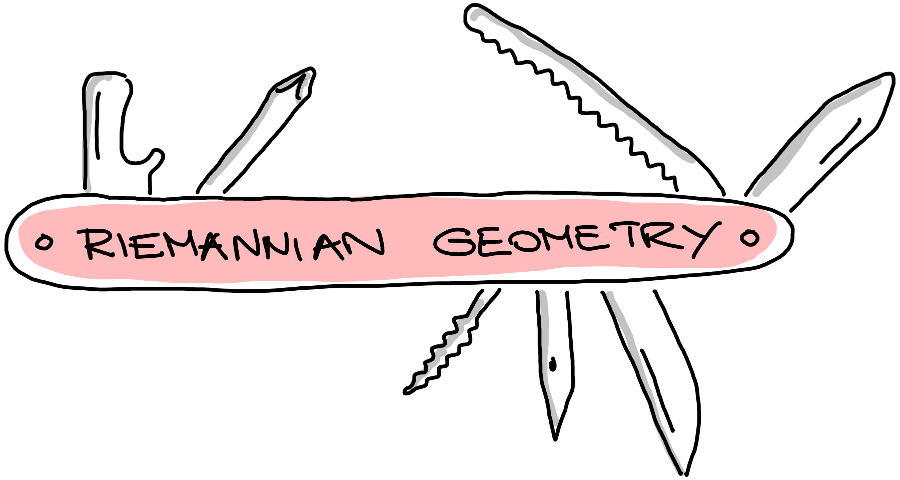
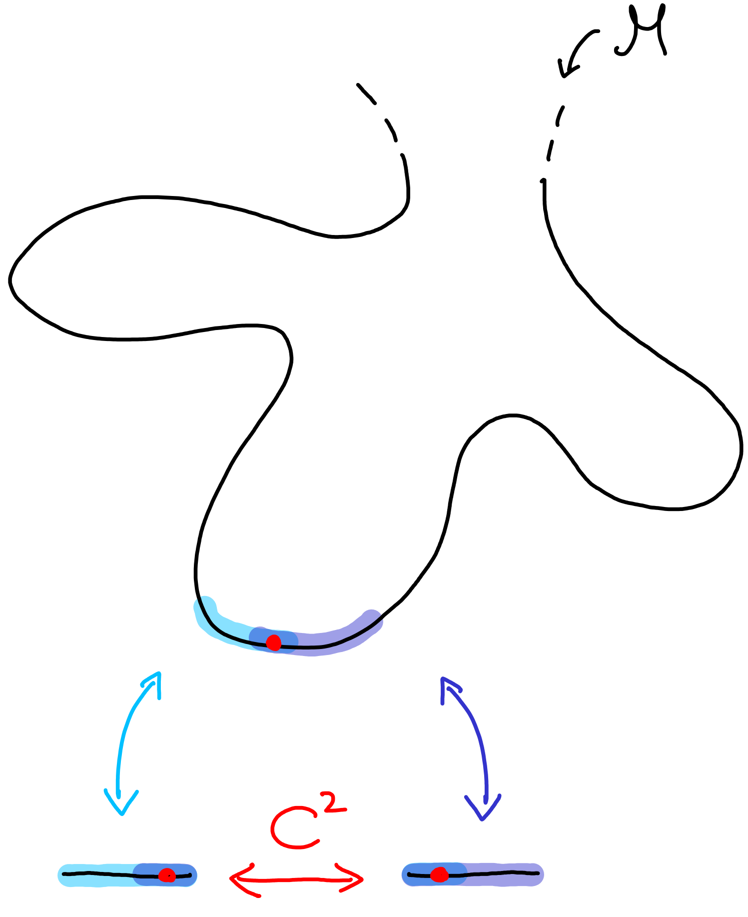

I am
high-dimensional!



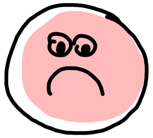
if $\mathcal{M}$ is C^2 ⇝ transition maps 2 times cont. differentiable

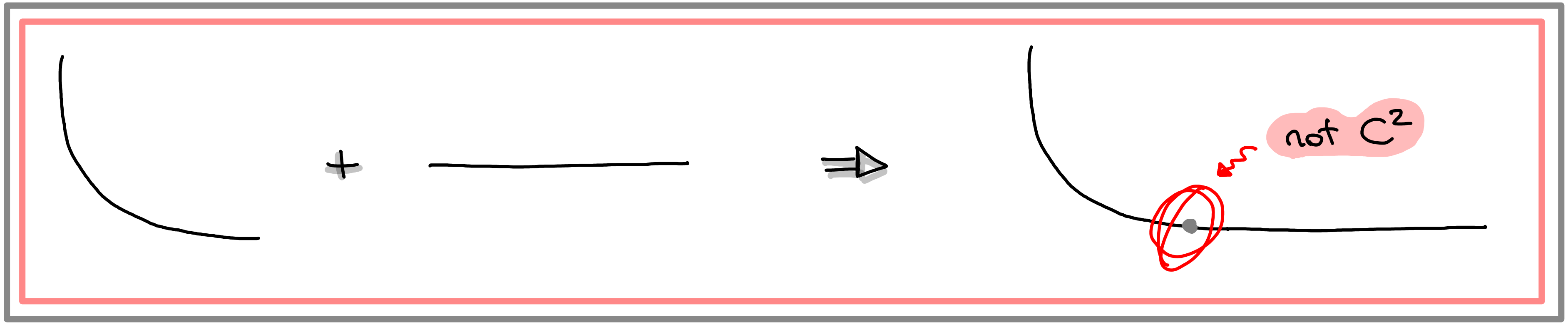
⇒



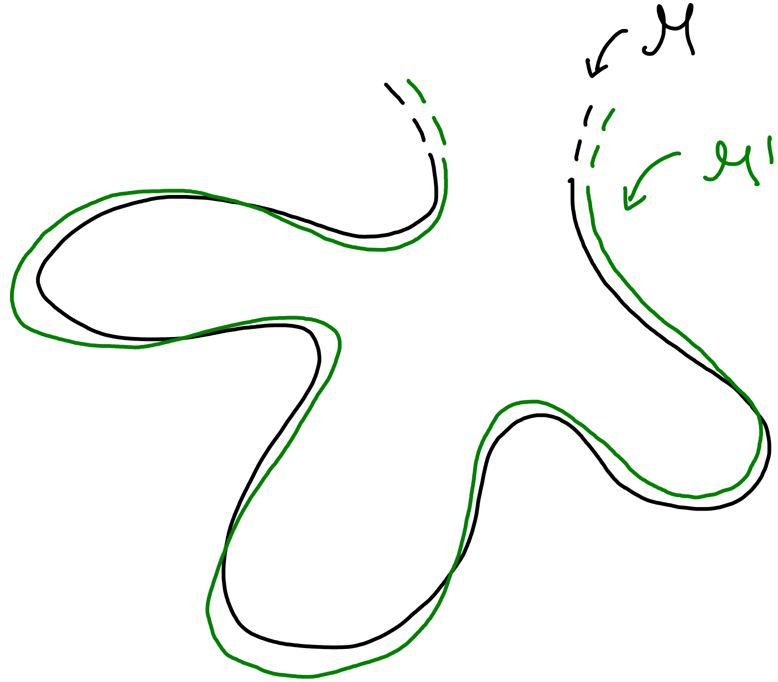


most computationally relevant manifolds are not C^2





OUR RESULT :

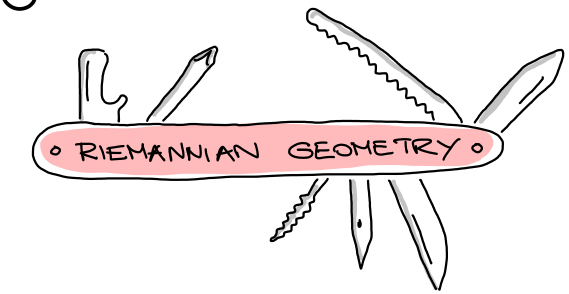


If $\mathcal{M}$ is not C^2 but looks kinda nice there exists a manifold $\mathcal{M}'$ that

- lies arbitrarily close to $\mathcal{M}$,
- also looks kinda nice, and
- is smooth.

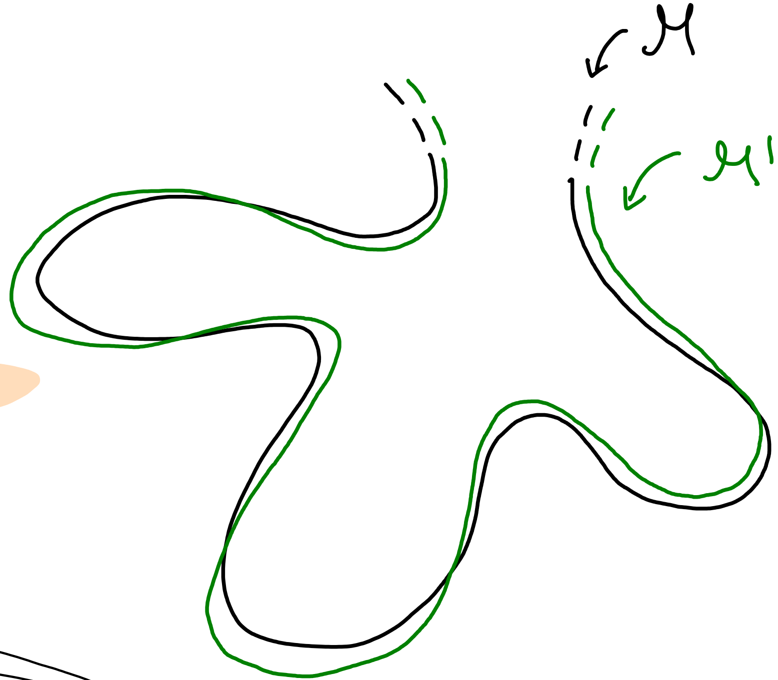
transition maps are C^∞

$\Longrightarrow$ instead of struggling with $\mathcal{M}$,

you can use 

on $\mathcal{M}'$!

AGENDA



1. kinda niceness = reach
2. being close in the C^1 sense

   NOT Hausdorff distance!

3. the theorem reisited
   - statement
   - proof sketch

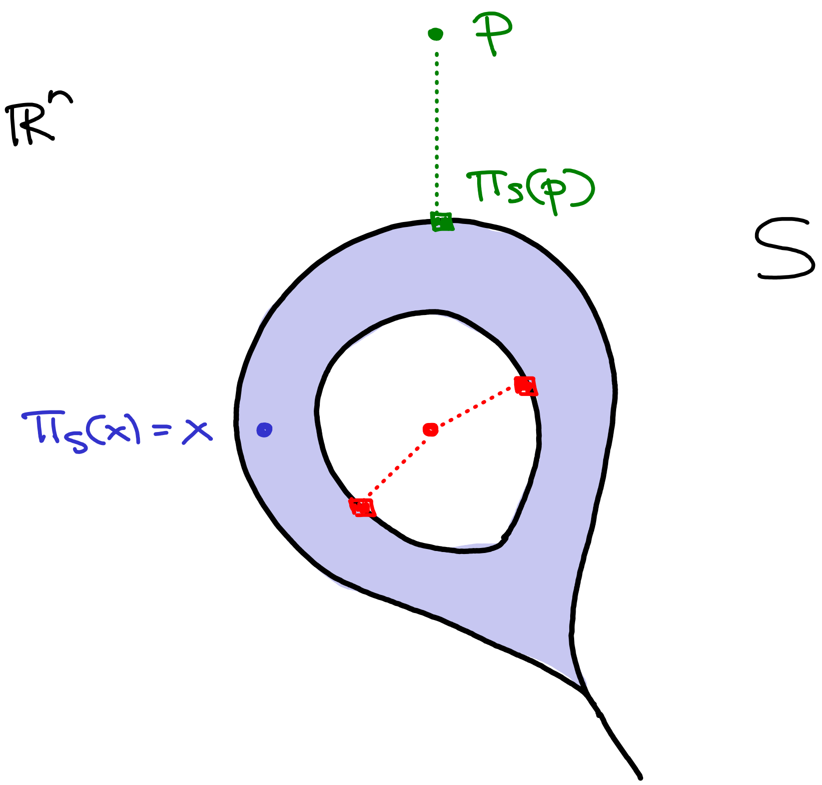
Joint work with

Mathijs Wintraecken and André Lieutier

MEDIAL AXIS AND THE REACH

- $S \subseteq \mathbb{R}^n$ set
- closest point projection : for $p \in \mathbb{R}^n$

$$\pi_S(p) = \underset{x \in S}{\operatorname{argmin}}\, d(p, x) \subseteq S$$



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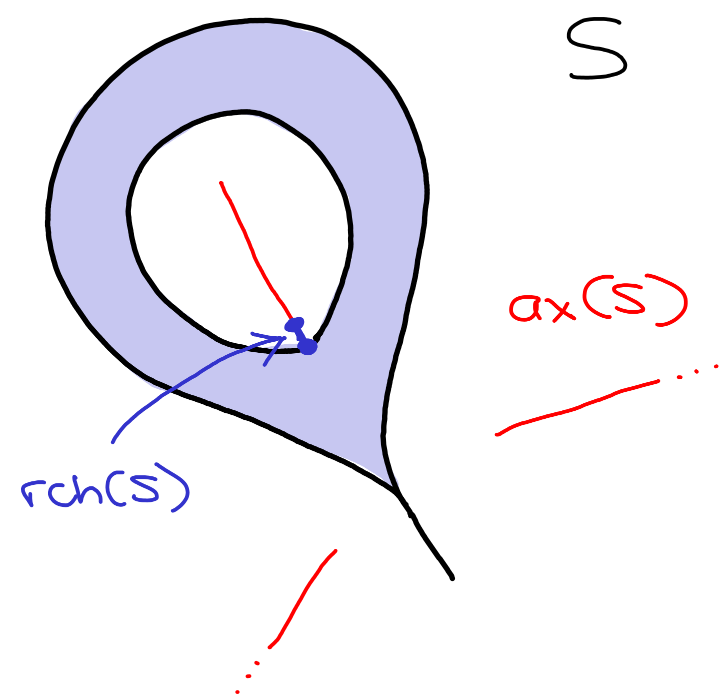


- medial axis

$$ax(S) = \{p \in \mathbb{R}^n \mid \# \pi_S(p) > 1\} \subseteq \mathbb{R}^n$$

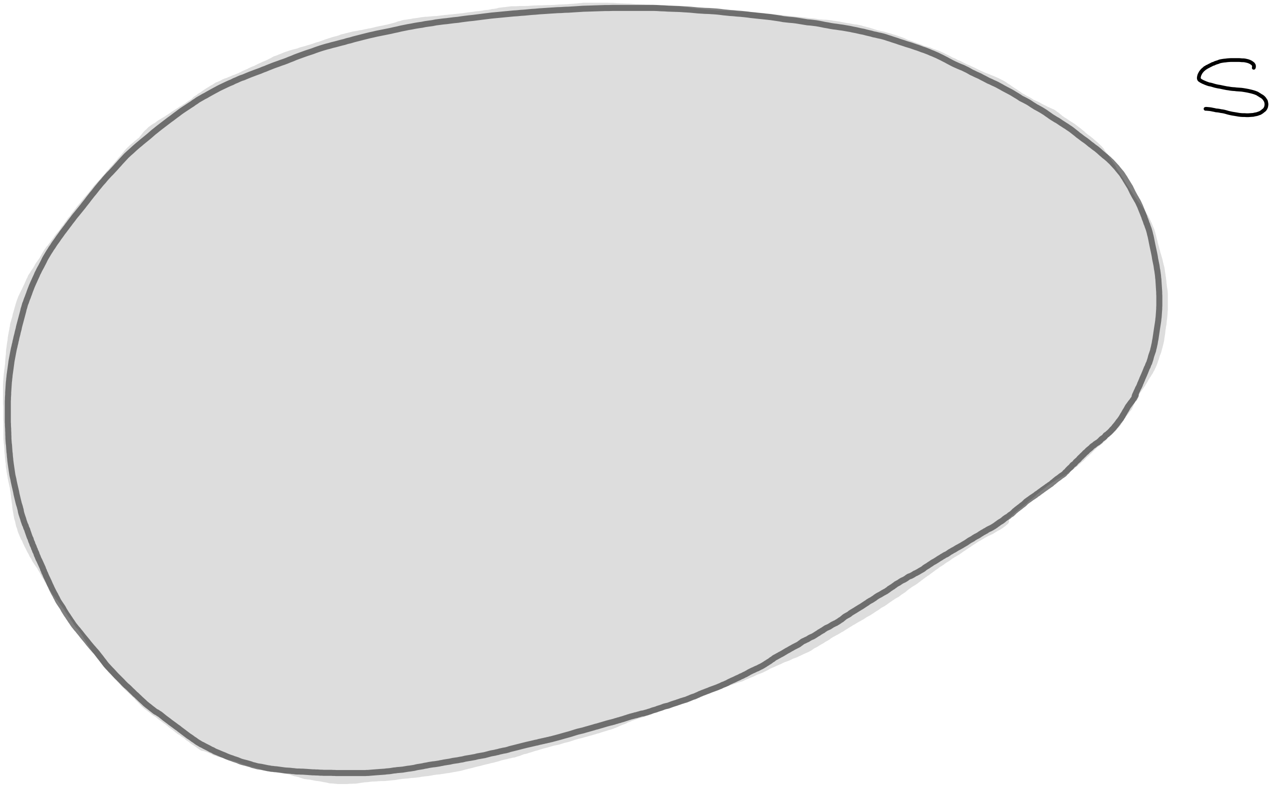
- reach

$$rch(S) = d(S, ax(S))$$

introduced by Federer in 1960's

MEDIAL AXIS AND THE REACH

THE REACH $\hat{=}$ measures "badness" of cavities



$ax(S) = \emptyset$

$rch(S) = \infty$

MEDIAL AXIS AND THE REACH

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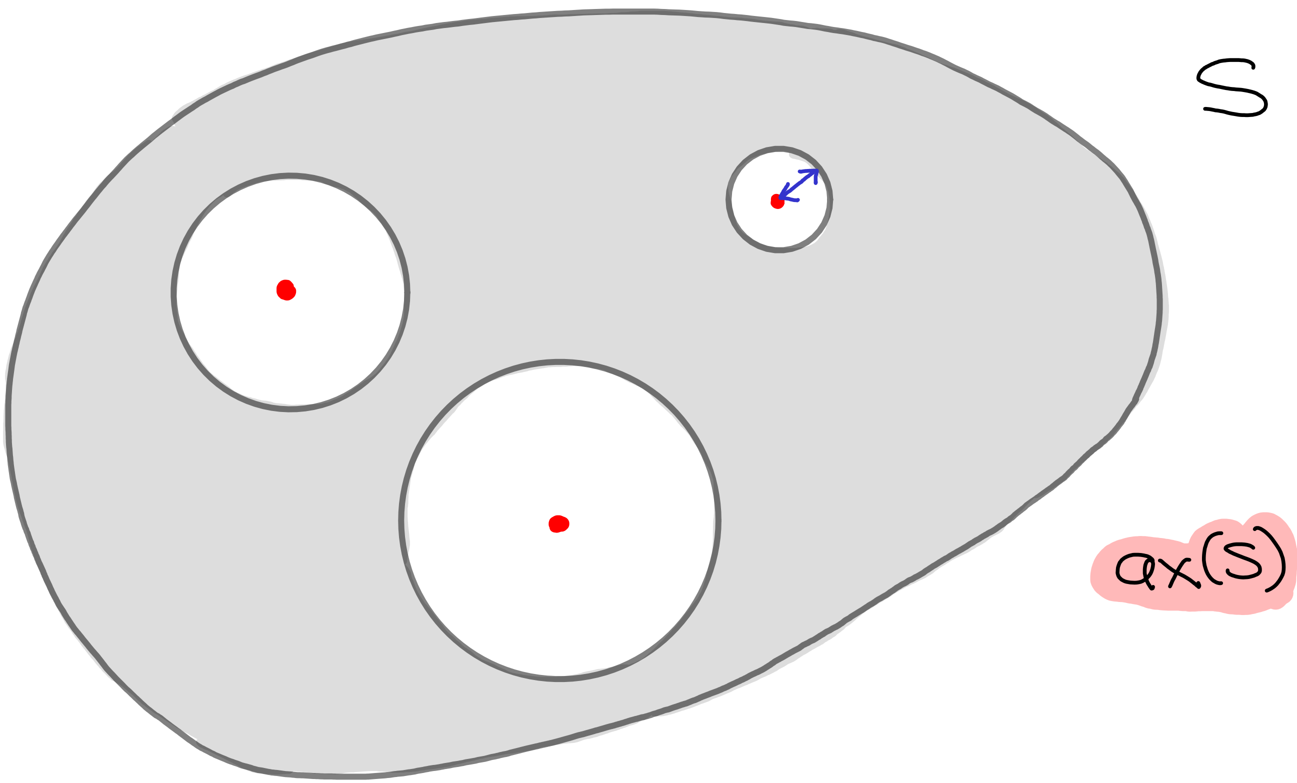

rch(S) = radius of the smallest cavity

MEDIAL AXIS AND THE REACH

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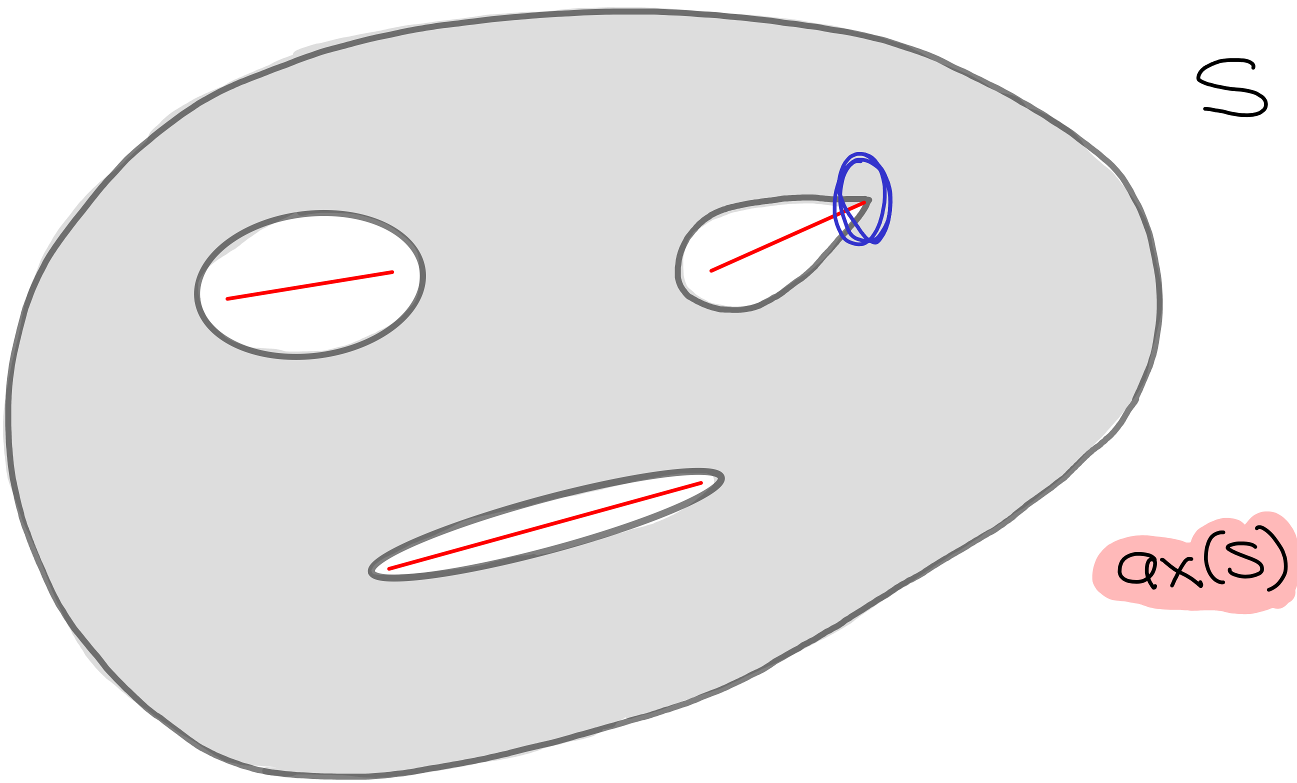

$\mathrm{rch}(S) = 0$

MEDIAL AXIS AND THE REACH

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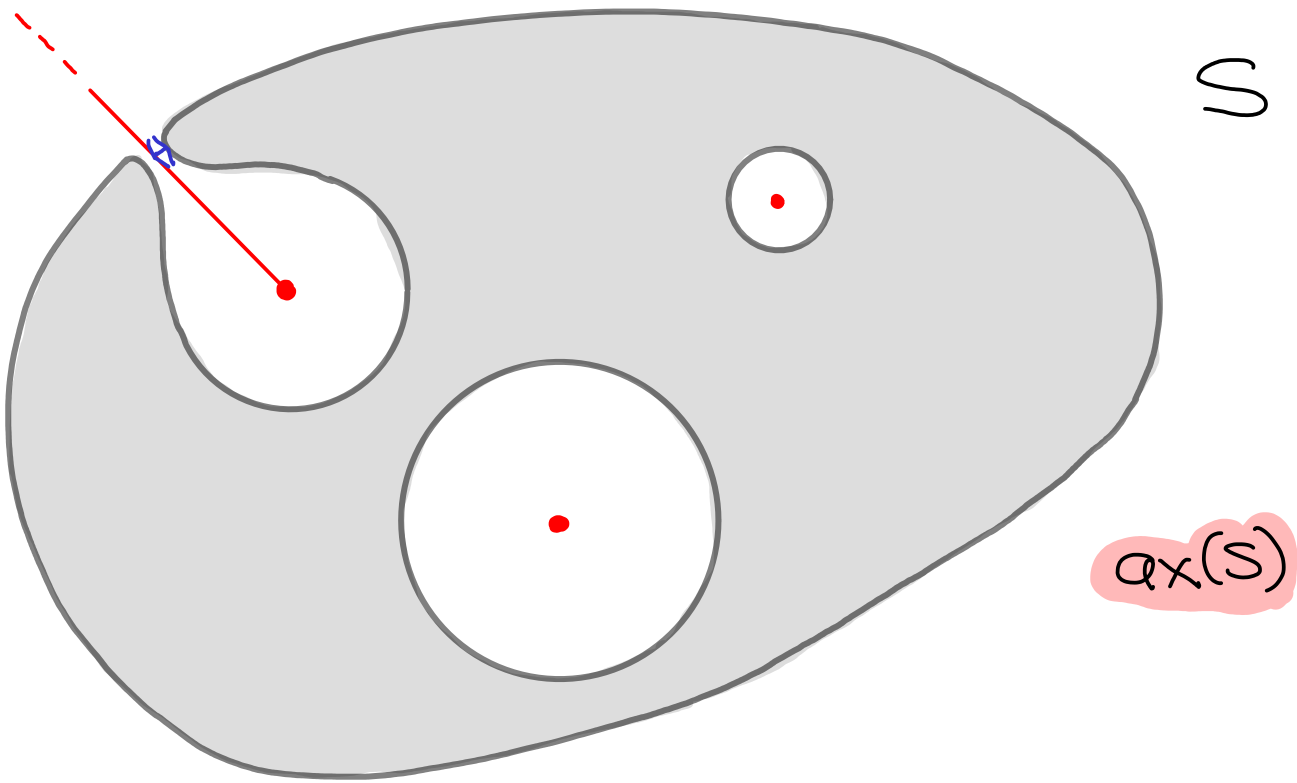
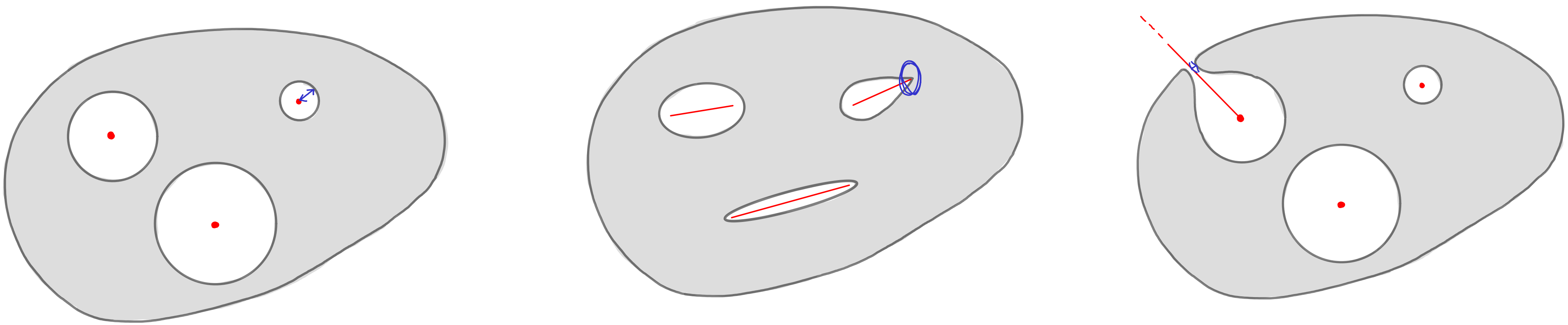


rch(S) = size of the bottleneck

MEDIAL AXIS AND THE REACH

THE REACH ≙ measures "badness" of cavities



the bigger the reach, the better!

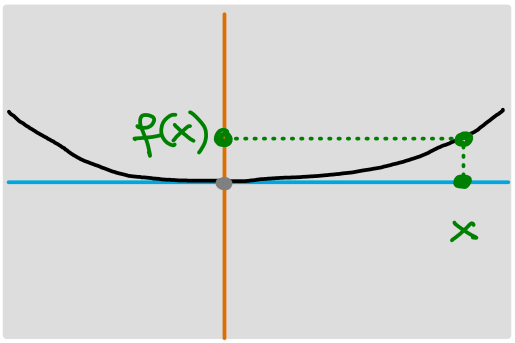
reach zero = trouble!

SETTINGS

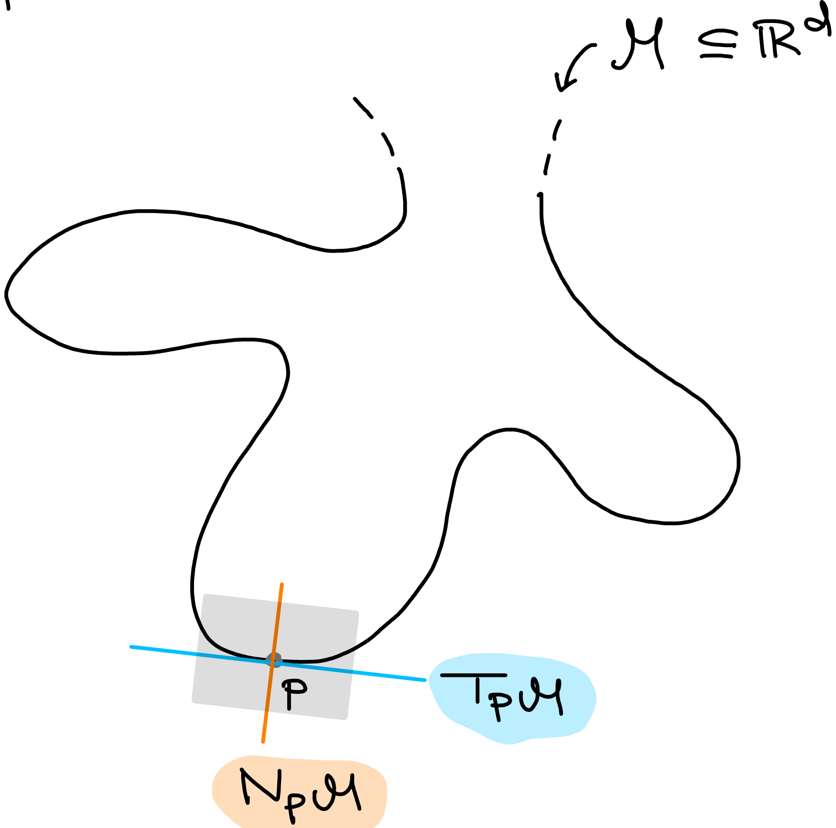
- $\dim \mathcal{M} = n \Rightarrow$ locally, $\mathcal{M}$ is a graph of

$$f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{d-n}$$

$$\mathbb{R}^{d-n} \cong N_p\mathcal{M}$$



$$U \subseteq \mathbb{R}^n \cong T_p\mathcal{M}$$



once cont. differentiable, derivatives are Lipschitz

- $\mathrm{rch}(\mathcal{M}) > 0 \Rightarrow f$ at least $C^{1,1}$ [Lytchak '04]

$$\boxed{\|f\|_1 = \max\{\|f\|, \|Df\|\}}$$

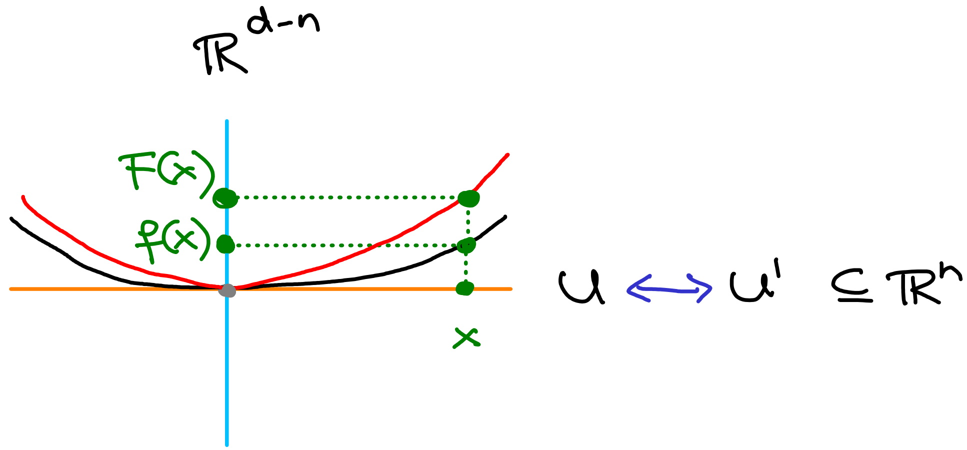
supremum norm

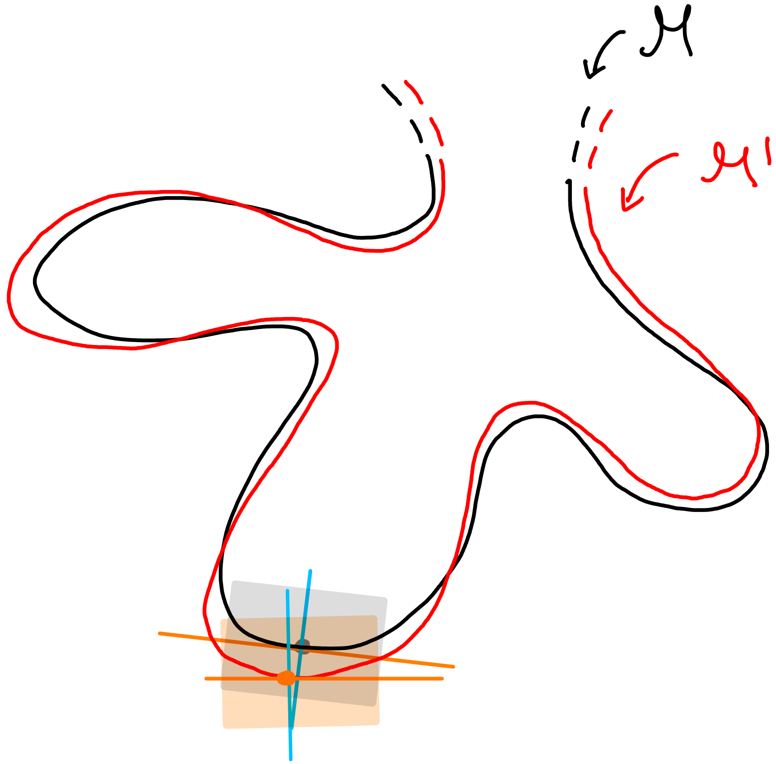
operator norm

SETTINGS

- $\mathcal{M}$ and $\mathcal{M}'$ homeomorphic,

$$f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{d-n} \quad \text{graph of } \mathcal{M}$$
$$\updownarrow$$
$$F : \mathcal{U}' \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{d-n} \quad \text{graph of } \mathcal{M}'$$



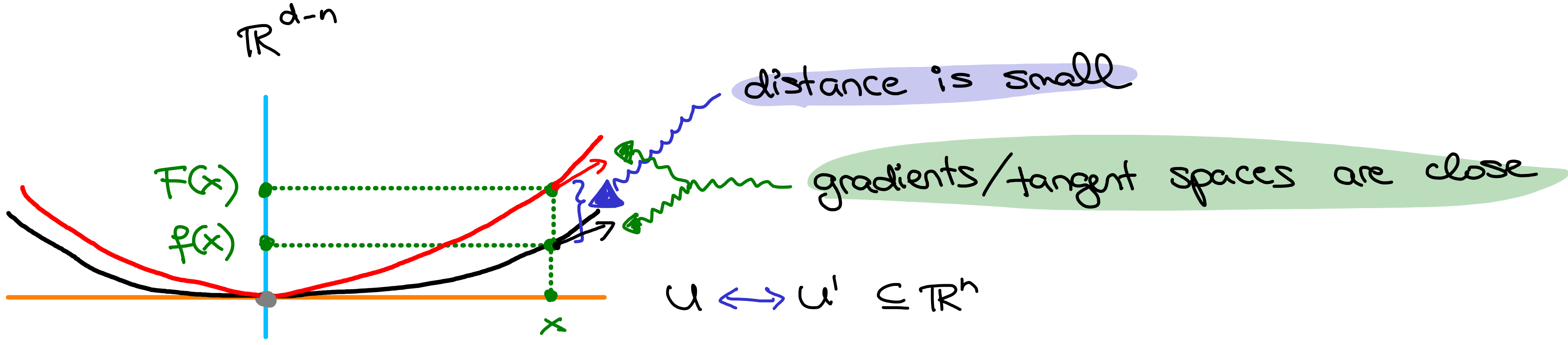


$\Rightarrow$ $\mathcal{M}$ and $\mathcal{M}'$ ε-close in the C^1 sense if for all neighbourhoods

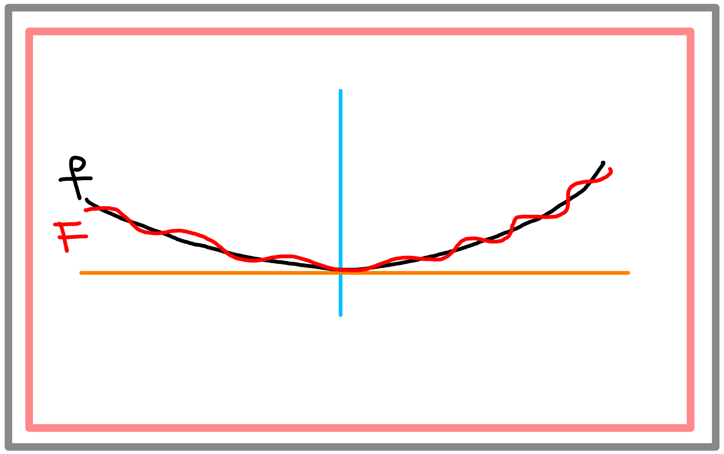
$$\boxed{\| f - F \|_1 < \varepsilon}$$

SETTINGS

... in other words ...



here, f and F are NOT close in C^1

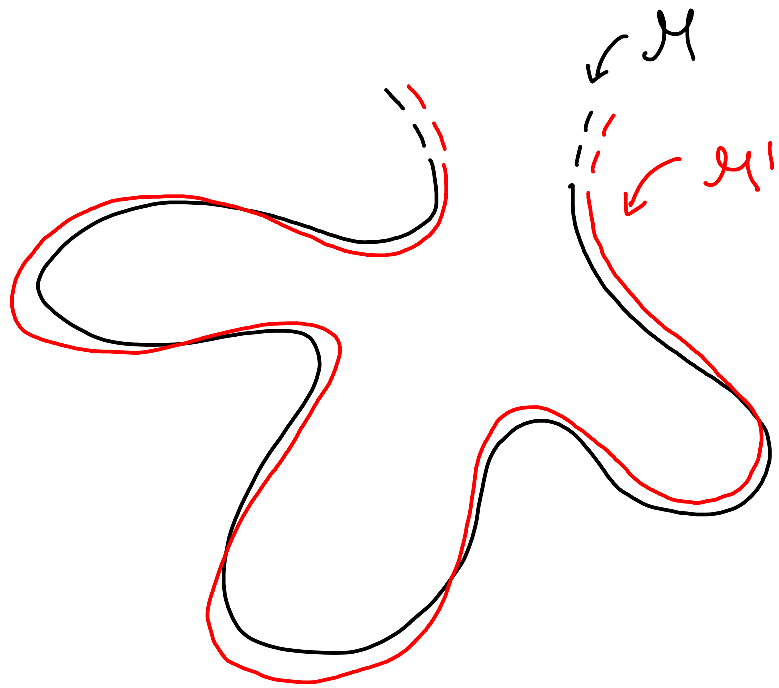


OUR THEOREM

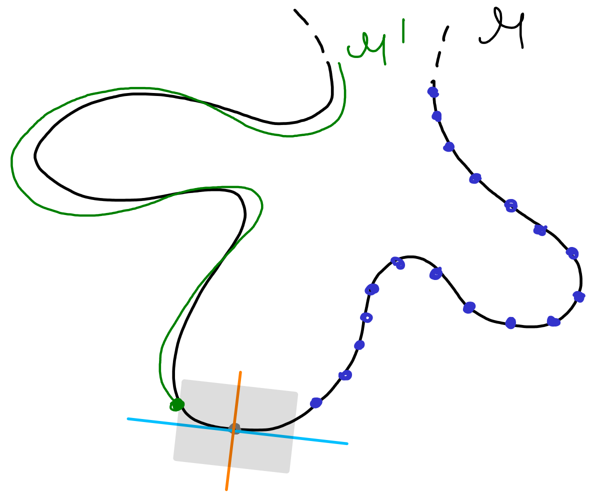
- $\mathcal{M} \subseteq \mathbb{R}^d$ compact manifold of $\mathrm{rch}(\mathcal{M}) > 0$
- $\varepsilon > 0$

Then there exists a manifold $\mathcal{M}' \subseteq \mathbb{R}^d$ such that :

- $\mathcal{M}'$ is C^∞
- $\mathcal{M}$ and $\mathcal{M}'$ are ε-close in the C^1 sense
- $\mathrm{rch}(\mathcal{M}') \geq \mathrm{rch}(\mathcal{M}) - \varepsilon$

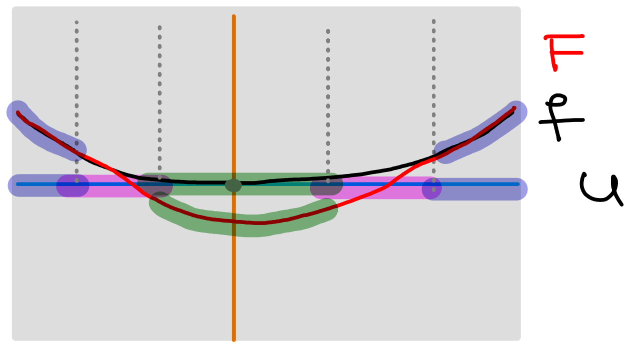


PROOF SKETCH



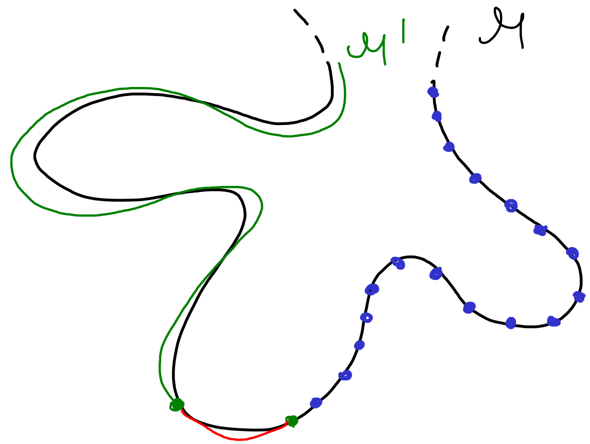
① Sample $\mathcal{M}$ depending on the given ε
$\Rightarrow$ work with one neighbourhood at a time

② - Split $\mathcal{U}$ into three sets:
- $\mathcal{U}_1$ "close to the origin"
- $\mathcal{U}_3$ "close to $\partial\mathcal{U}$"
- $\mathcal{U}_2$ "in between"



- Use KERNEL-BASED SMOOTHING to smooth $f|_{\mathcal{U}_1} \rightsquigarrow F|_{\mathcal{U}_1}$
- Use partition of unity to achieve $F|_{\mathcal{U}_3} = f|_{\mathcal{U}_3}$

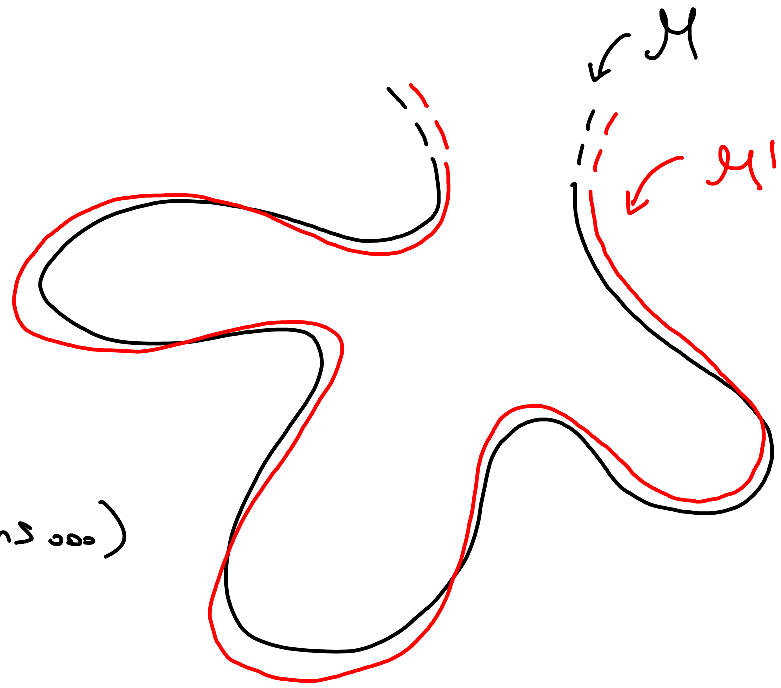
③ - Replace the graph of f with the graph of F
- Control the reach using Federer's work



④ Repeat steps ② and ③ iteratively for all points of the sample

SUMMARY

If you're struggling with a manifold ...



- the reach is worth keeping an eye on!
  (the higher, the better)

- viewing it as a map is worth trying!
  (... if you're brave enough to face the norms...)

- if not, apply our theorem and you'll get a smooth manifold (almost) for free!

THANK YOU FOR YOUR ATTENTION!