Computing Persistent Laplacians

Toward Broader Applications in TDA

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ComPer 2025
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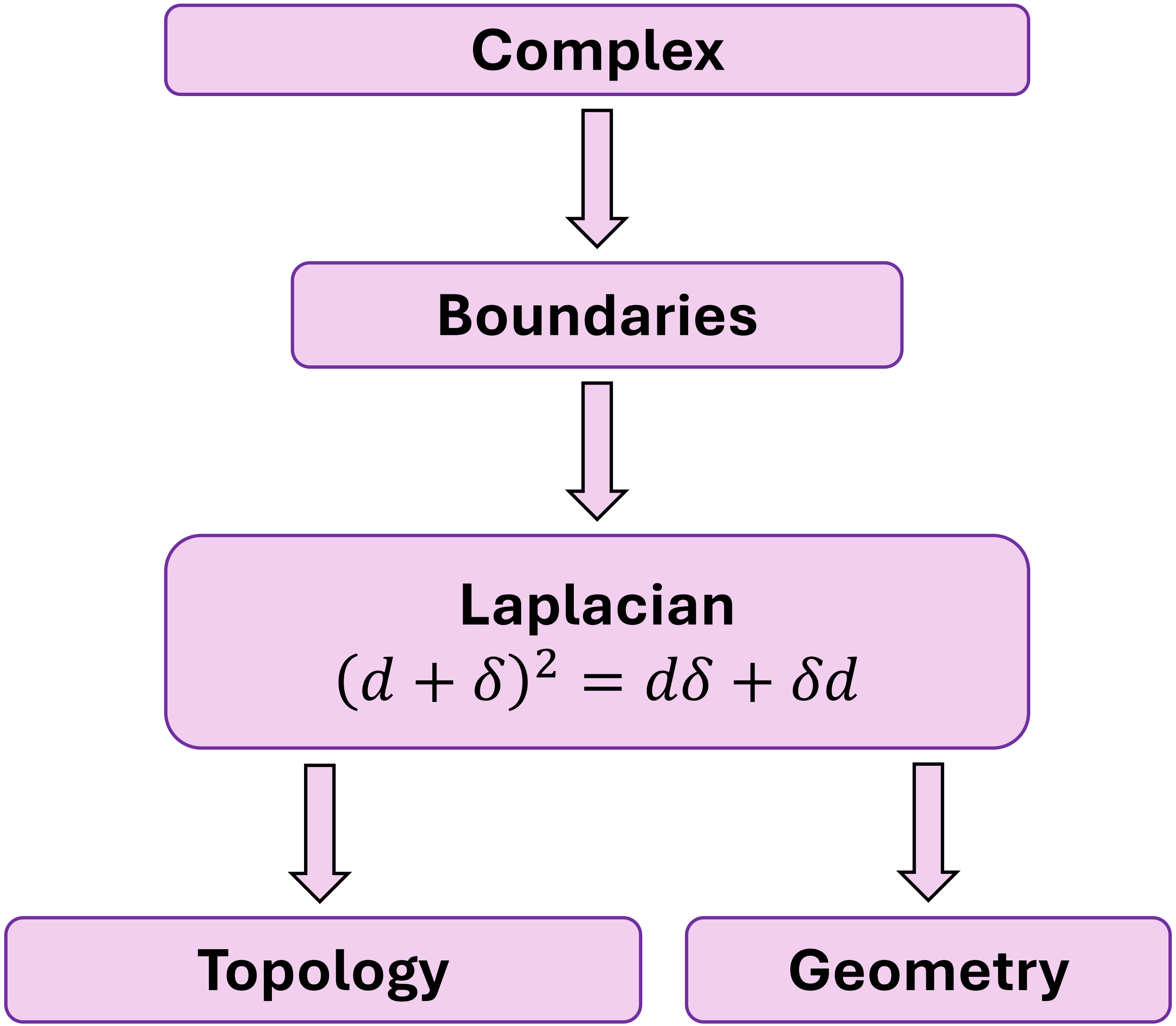
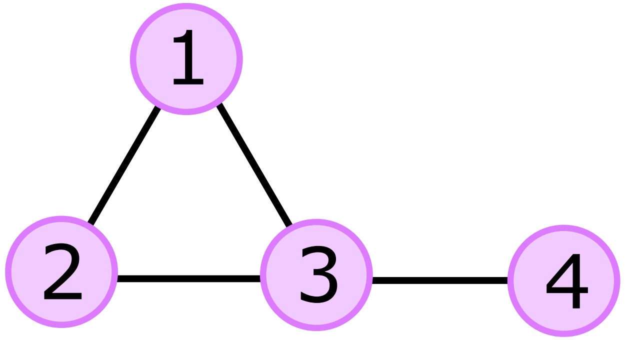
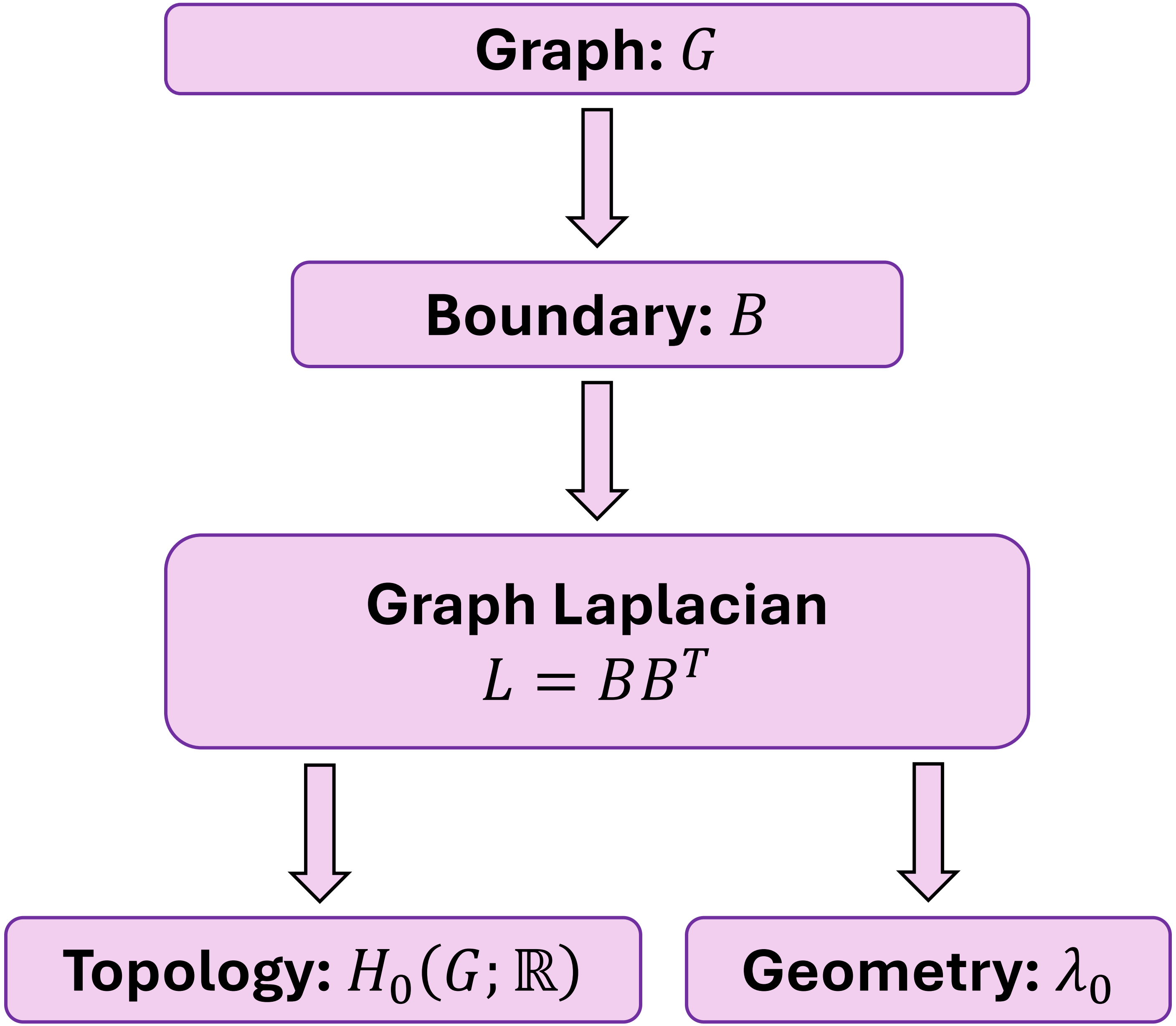
Laplacians

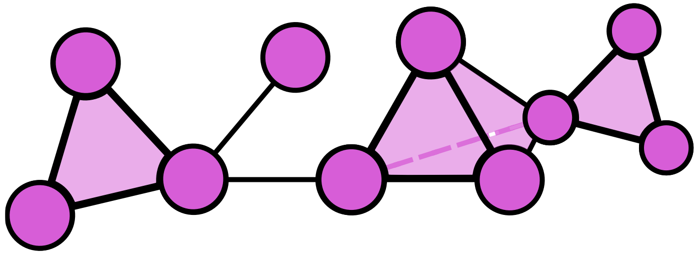

Complex
Boundaries
Laplacian
$(d+\delta)^2 = d\delta + \delta d$
Topology
Geometry

Graph Laplacian





Combinatorial Laplacian



Simplicial Complex: K

$\Downarrow$

Boundaries: d_n

$\Downarrow$

Combinatorial Laplacian

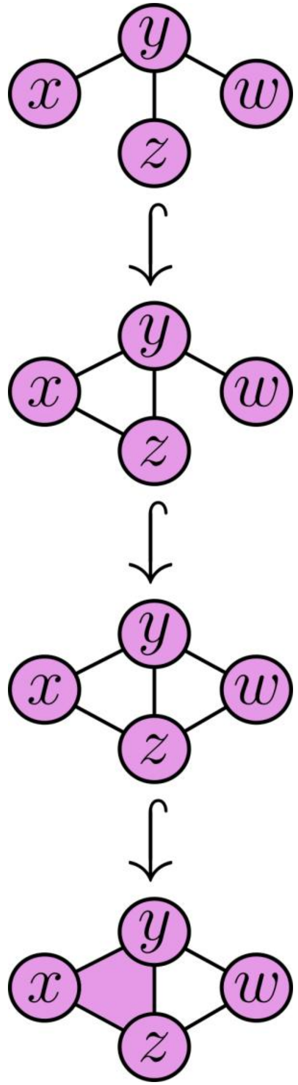
$$\Delta_n = d_{n+1} d_{n+1}^* + d_n^* d_n$$

$\Downarrow$ $\Downarrow$

Topology: $H_n(K; \mathbb{R})$

Geometry: λ_n

Persistent Laplacian



Filtered Complex: $K^a \hookrightarrow K^b$

$\Downarrow$

Boundaries: d_n^a, d_n^b

$\Downarrow$

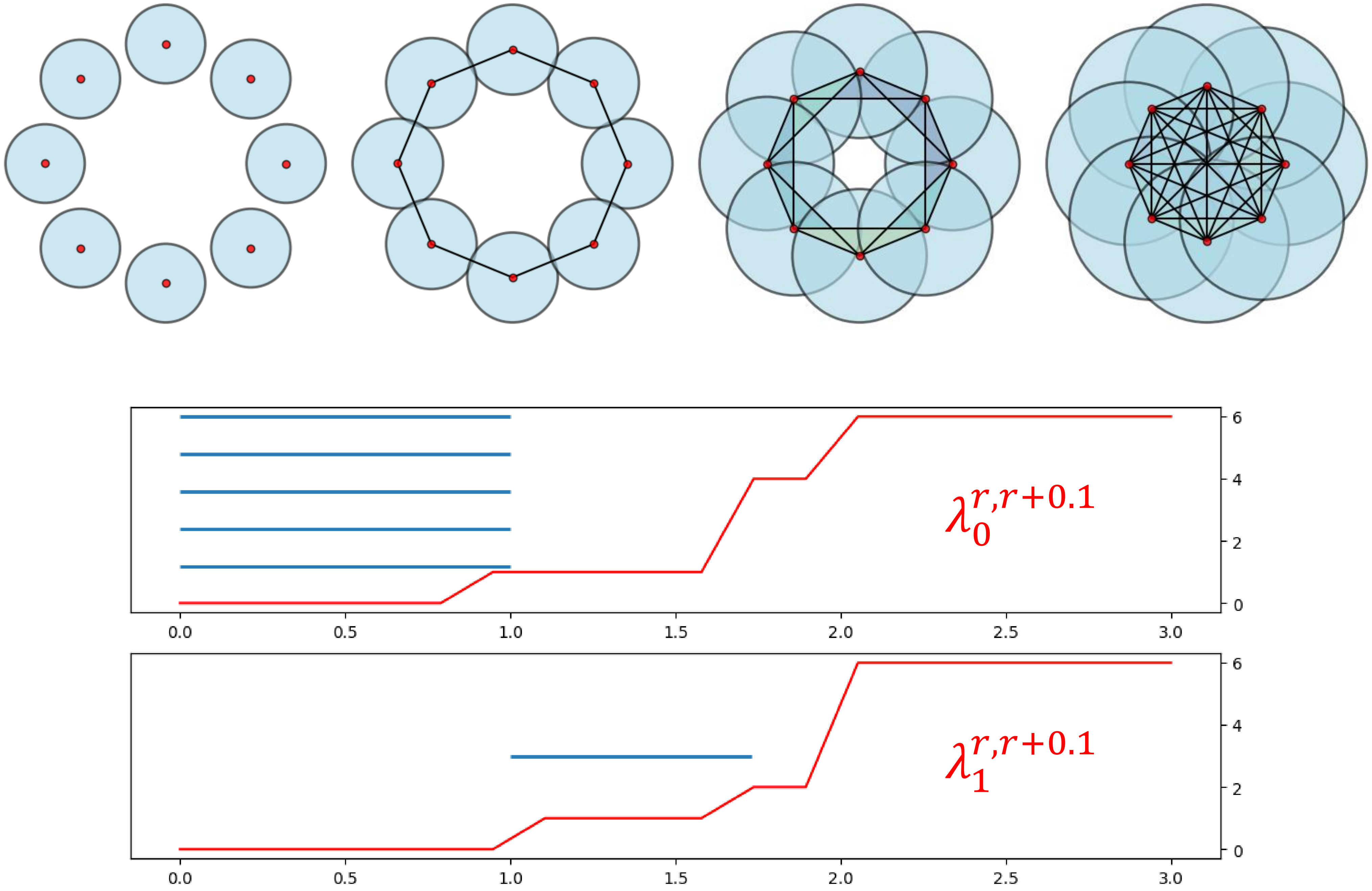
Persistent Laplacian

$$\Delta_n = d_{n+1}^{a,b}\left(d_{n+1}^{a,b}\right)^* + (d_n^{\mathrm{a}})^* d_n^a$$

$\Downarrow$ $\Downarrow$

Topology: $H_n^{a,\mathrm{b}}(K;\mathbb{R})$

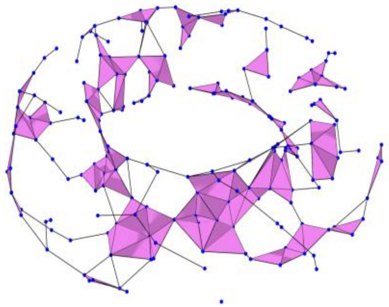
Geometry: $\lambda_n^{a,b}$



6
4
2
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
$\lambda_0^{r,r+0.1}$
$\lambda_1^{r,r+0.1}$

Persistent Laplacian Pipeline

Multiscale Data

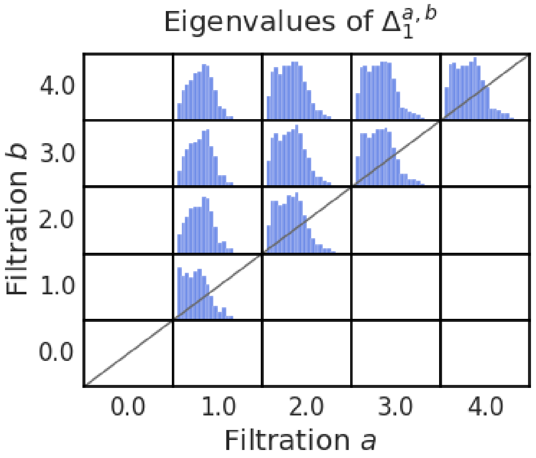


Persistent Laplacians

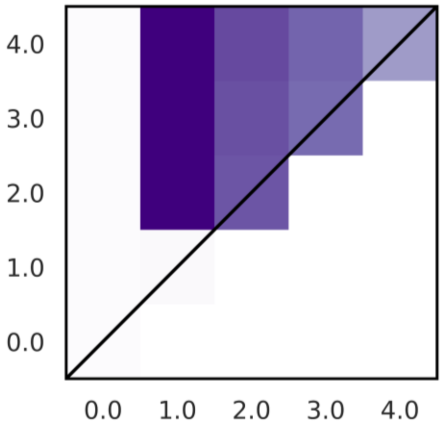
$$\{\Delta_n^{a,b} \mid a \leq b\}$$

PSD matrices

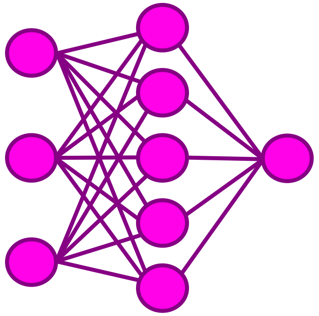
Eigenvalues



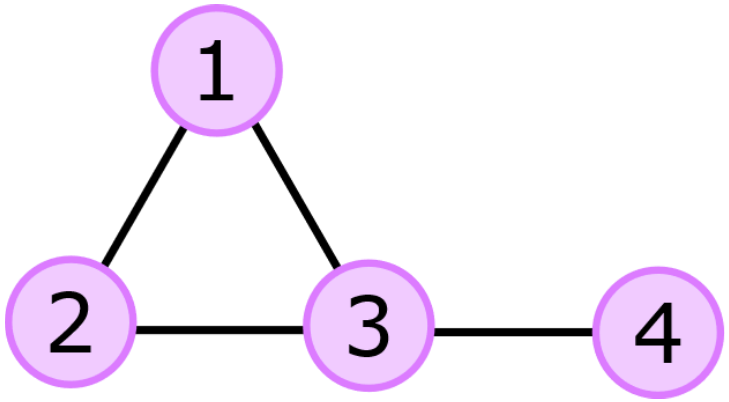
Summaries



Analysis



Graph Laplacian



$$B = \begin{array}{c c} & \begin{array}{cccc} e_{12} & e_{13} & e_{23} & e_{34} \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$L = BB^T = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

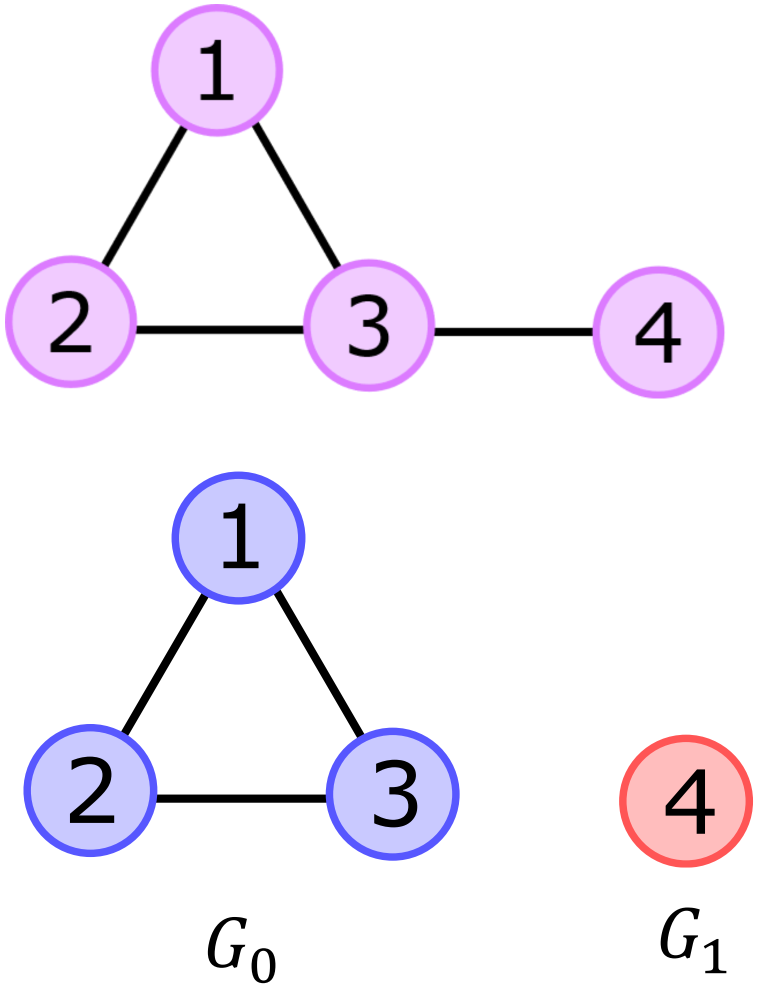
Graph Laplacian

Eigenvalues: $\{0, 1, 3, 4\}$

Topological Information:

Multiplicity of 0 = # components

Spectral Clustering



Eigenvector partition

$$L\begin{pmatrix}1\\1\\0\\-2\end{pmatrix} = 1\begin{pmatrix}1\\1\\0\\-2\end{pmatrix}\begin{matrix}v_1\\v_2\\v_3\\v_4\end{matrix}$$

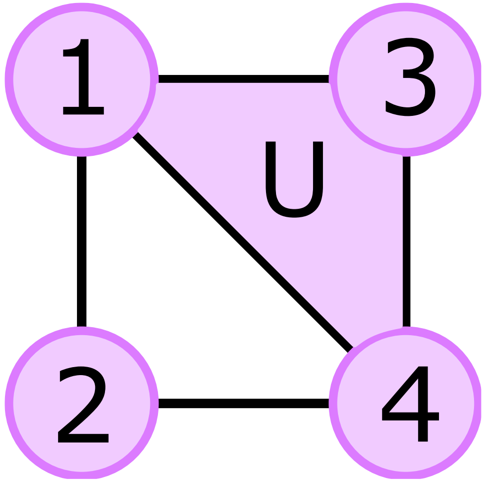
$v_i \in G_0$ if ≥ 0
$v_i \in G_1$ if < 0

Gives min-cut partition

Eigenvalues: $\{0, 1, 3, 4\}$

Geometric Information:
$\lambda = \min\{\lambda_i > 0\}$
Algebraic Connectivity

Combinatorial Laplacian



$$d_1 = \begin{array}{c|ccccc} & e_{12} & e_{13} & e_{14} & e_{24} & e_{34} \\ \hline v_1 & -1 & -1 & -1 & 0 & 0 \\ v_2 & 1 & 0 & 0 & -1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & -1 \\ v_4 & 0 & 0 & 1 & 1 & 1 \end{array}$$

$$d_2 = \begin{array}{c|c} & U \\ \hline e_{12} & 0 \\ e_{13} & 1 \\ e_{14} & -1 \\ e_{24} & 0 \\ e_{34} & 1 \end{array}$$

Eigenvalues: $\{0, 2, 3, 3, 4\}$

β_1

$$\Delta_1 = d_2 \circ d_2^* + d_1^* \circ d_1$$

$$= \begin{pmatrix} 2 & 1 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 3 & 1 & 0 \\ -1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Combinatorial Laplacian

$$C_{n+1}(K;\mathbb{R}) \xrightarrow{d_{n+1}} C_n(K;\mathbb{R}) \xrightarrow{d_n} C_{n-1}(K;\mathbb{R})$$

Inner product: $\langle \sigma_i, \sigma_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Implies $C_n \cong C^n$

$$C_{n+1}(K;\mathbb{R}) \underset{d_{n+1}^*}{\overset{d_{n+1}}{\rightleftarrows}} C_n(K;\mathbb{R}) \underset{d_n^*}{\overset{d_n}{\rightleftarrows}} C_{n-1}(K;\mathbb{R})$$

$$\begin{tikzcd} C_n(K) \arrow[loop left, "\Delta_n^{\text{up}} = d_{n+1} \circ d_{n+1}^*"] \arrow[loop right, "\Delta_n^{\text{down}} = d_n^* \circ d_n"] \end{tikzcd}$$

$$\Delta_n = d_{n+1} d_{n+1}^* + d_n^* d_n$$

$$= \Delta_n^{\text{up}} + \Delta_n^{\text{down}}$$

Properties $\Delta_n = d_{n+1} \circ d_{n+1}^* + d_n^* \circ d_n$

Hodge Decomposition:

$\ker\Delta_n \cong H_n(K; \mathbb{R})$

$C_n(K; \mathbb{R}) \cong \mathrm{Im}\, d_n^* \oplus H_n(K) \oplus \mathrm{Im}\, d_{n+1}$

de Rham Hodge Decomposition:

$\Omega^n(M) \cong \mathrm{Im}\, d^{n-1} \oplus \mathcal{H}_\Delta^n(M) \oplus \mathrm{Im}\, \delta^{n+1}$

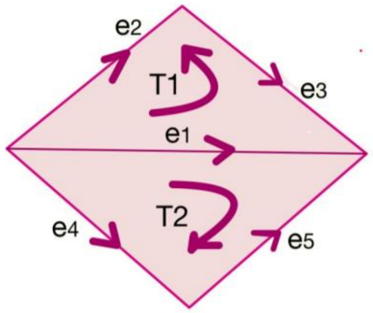
Self-adjoint: $(\Delta_n)^* = \Delta_n$

Eigenvalues ≥ 0: $\{\lambda_n^0, \lambda_n^1, \dots, \lambda_n^i\}$

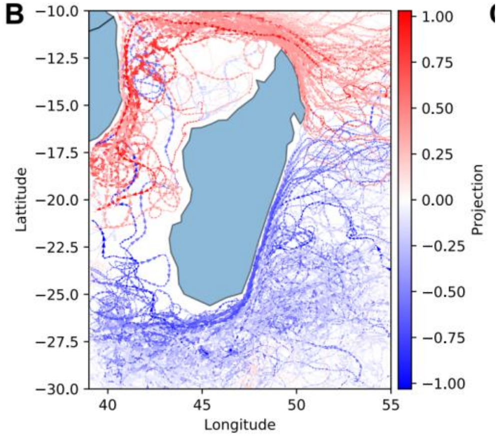
Multiplicity of 0: $\dim \ker\Delta_n = \beta_n$

Spectral gap: $\lambda_n = \min\{\lambda_n^i > 0\}$

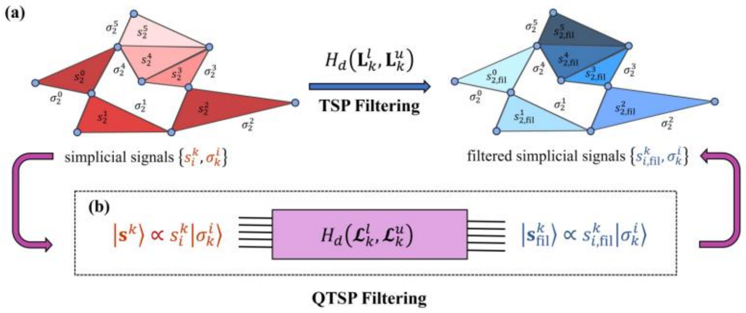
In practice



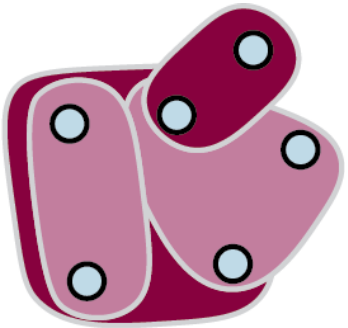
Random walks
Eidi and Mukherjee 2023



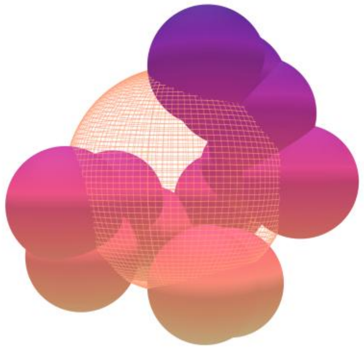
Trajectory embeddings
Schaub et al., 2019



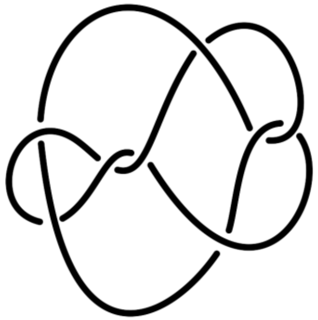
Quantum Computing
Leditto et al., 2024



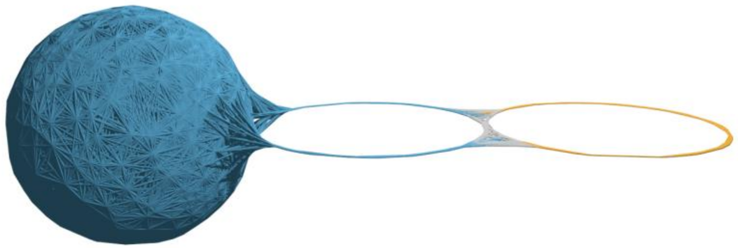
TopoNetX
Hajij et al., 2024



OAT
Henselman-Petrusek et al., 2023



Knot Theory
<u>**Jones**</u> and Wei, 2024



Topological Clustering
Ebli and Spreemann 2019

Persistent Laplacian $\Delta_n^{a,b}: C_n^a \to C_n^a$

$$
\begin{tikzcd}
C^a_{n+1} \arrow[rr, "d^a_{n+1}"] \arrow[dd, hook, "i^{a,b}_{n+1}"] & & C^a_n \arrow[r, shift left, "d^a_n"] \arrow[dl, shift left, "(d^{a,b}_{n+1})^*"] \arrow[dd, hook, "i^{a,b}_n"] & C^a_{n-1} \arrow[l, shift left, "(d^a_n)^*"] \arrow[dd, hook, "i^{a,b}_{n-1}"] \\
& C^{a,b}_{n+1} \arrow[ur, shift left, "d^{a,b}_{n+1}"] \arrow[dl, hook] & & \\
C^b_{n+1} \arrow[rr, "d^b_{n+1}"] & & C^b_n \arrow[r, "d^b_n"] & C^b_{n-1}
\end{tikzcd}
$$

Subspace: $C_{n+1}^{a,b} = \{\sigma \in C_{n+1}^b \mid d_{n+1}^b(\sigma) \in i_n^{a,b}(C_n^a)\}$

Peristent Laplacian: $\Delta_n^{a,b} = d_{n+1}^{a,b} \circ (d_{n+1}^{a,b})^* + (d_n^a)^* \circ d_n^a$

Combinatorial Laplacian

$\Delta_n : C_n \to C_n$

$\Delta_n = d_{n+1} \circ d_{n+1}^* + d_n^* \circ d_n$

Hodge Decomposition:

$\ker \Delta_n \cong H_n(K;\mathbb{R})$

$C_n(K;\mathbb{R}) \cong \operatorname{Im} d_n^* \oplus H_n(K) \oplus \operatorname{Im} d_{n+1}$

Self-adjoint: $(\Delta_n)^* = \Delta_n$

Eigenvalues ≥ 0: $\{\lambda_n^0, \lambda_n^1, \ldots, \lambda_n^k\}$

Multiplicity of 0: $\dim \ker \Delta_n = \beta_n$

Spectral gap: $\lambda_n = \min\{\lambda_n^i > 0\}$

Persistent Laplacian

$\Delta_n^{a,b} : C_n^a \to C_n^a$

$\Delta_n^{a,b} = d_{n+1}^{a,b} \circ (d_{n+1}^{a,b})^* + (d_n^a)^* \circ d_n^a$

Hodge Decomposition:

$\ker \Delta_n^{a,b} \cong H_{n+1}^{a,b}(K;\mathbb{R})$

$C_n^a(K;\mathbb{R}) \cong \operatorname{Im}(d_n^a)^* \oplus H_n^{a,b}(K) \oplus \operatorname{Im} d_{n+1}^{a,b}$

Self-adjoint: $\left(\Delta_n^{a,b}\right)^* = \Delta_n^{a,b}$

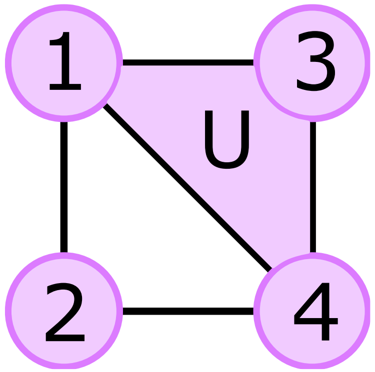
Eigenvalues ≥ 0: $\{ \left(\lambda_n^{a,b}\right)_0, \left(\lambda_n^{a,b}\right)_1, \ldots, \left(\lambda_n^{a,b}\right)_k\}$

Multiplicity of 0: $\dim \ker \Delta_n^{a,b} = \beta_n^{a,b}$

Spectral gap: $\lambda_n^{a,b} = \min\{\left(\lambda_n^{a,b}\right)_i > 0\}$

PL Example

K_a

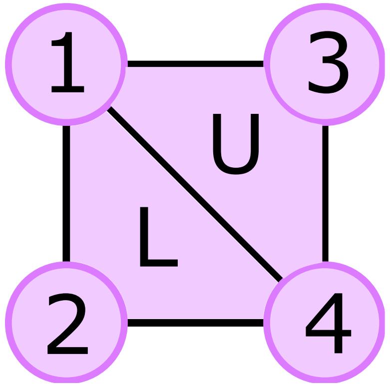


$$\Delta_{1,\mathbf{down}}^{a,b} = \begin{pmatrix} 2 & 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 & -1 \\ 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & 1 & 2 & 1 \\ 0 & -1 & 1 & 1 & 2 \end{pmatrix}$$

$$\Delta_{1,\mathbf{up}}^{a,b} = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ -1 & -1 & 2 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

Combinatorial Laplacian Eigenvalues: $\{0, 2, 3, 4, 4\}$

K_b



$$\Delta_1^{a,b} = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

Persistent Laplacian Eigenvalues: $\{2, 2, 4, 4, 4\}$

Multiplicity of 0 $= \beta_n^{a,\mathrm{b}}$

How many 0-eigenvalues should we expect?

What is this “geometric information”?

- **Short answer:** we don’t know
- **Medium answer:** analogies and machine learning
- **Long answer:** Cheeger-type inequalities
  - Manifolds

$$\frac{h_M^2}{4} \leq \lambda_M$$

  - Graphs

$$\frac{h_G^2}{2\max_{v\in V}\deg(v)} \leq \lambda_G \leq 2h_G$$

  - Non-branching filtered simplicial complexes (Botnan & Dong, 2025)

$$\min_{j\leq h+r} \frac{\widehat{\mathbf{A}}(P_j)}{\mathbf{V}(P_j)} \leq \lambda_{\min}^{\mathcal{K},\mathcal{L}} \leq \min_{j\leq h+r} \frac{\mathbf{A}(P_j)}{\mathbf{V}(P_j)}$$

Algorithms

- Projection - R. Wang et al., 2019
- Reduction - Mémoli, Wan, Y. Wang 2020
- **Schur Complement - MWW 2020**
- Non-branching – Dong 2024

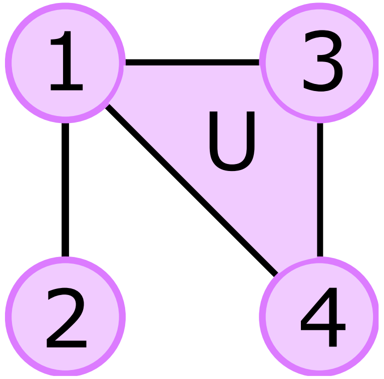
$$\Delta_{n,\mathbf{up}}^{b} = \overbrace{\begin{array}{cc} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \\ \begin{array}{cc} \underbrace{\qquad\qquad}_{n_n^a} & \underbrace{\qquad}_{n_n^b - n_n^a} \end{array} \end{array}}^{n_n^b}$$

$$\Delta_n^{a,b} = \Delta_{n,\mathbf{up}}^{b} / \Delta_{n,\mathbf{up}}^{b}\left(I_a^b, I_a^b\right) = A - BD^{-1}C$$

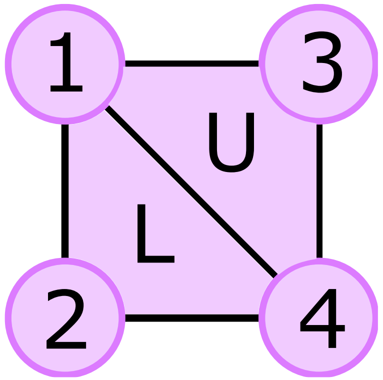
$$= O\left(\left(n_n^b\right)^3\right)$$

PL Example 2

K_a



K_b



$$\Delta^b_{1,\mathrm{up}} = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\Delta^a_{1,\mathbf{down}} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

Eigenvalues: $\{0, 1, 3, 4\}$

$$\boldsymbol{\Delta^b_{n,\mathrm{up}} / \Delta^b_{n,\mathrm{up}}\left(I^b_a, I^b_a\right) = A - BD^{-1}C}$$

$$= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} (1)^{-1} \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

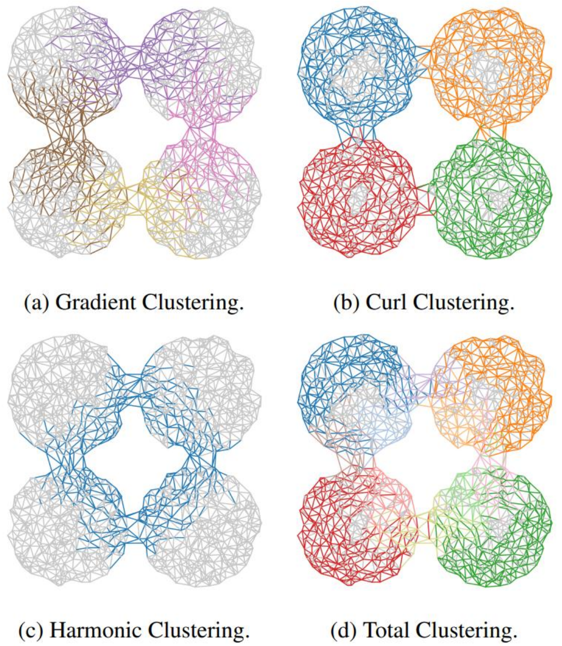
Eigenvalues: $\{0, 0, 0, 3\}$

Persistent Laplacian: $\Delta^{a,b}_1 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

Eigenvalues: $\{1, 3, 3, 4\}$

Applications

Topological Spectral Clustering
(Grande and Schaub, 2024)

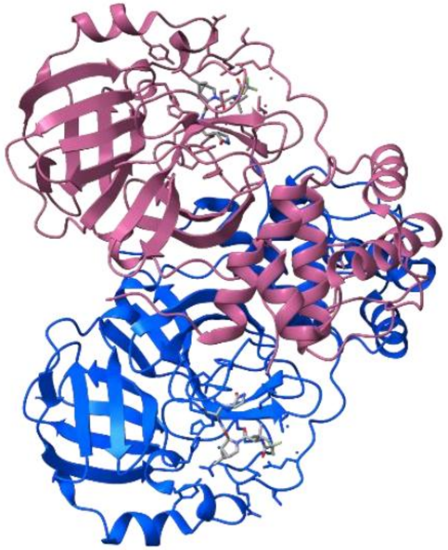


(a) Gradient Clustering. (b) Curl Clustering.

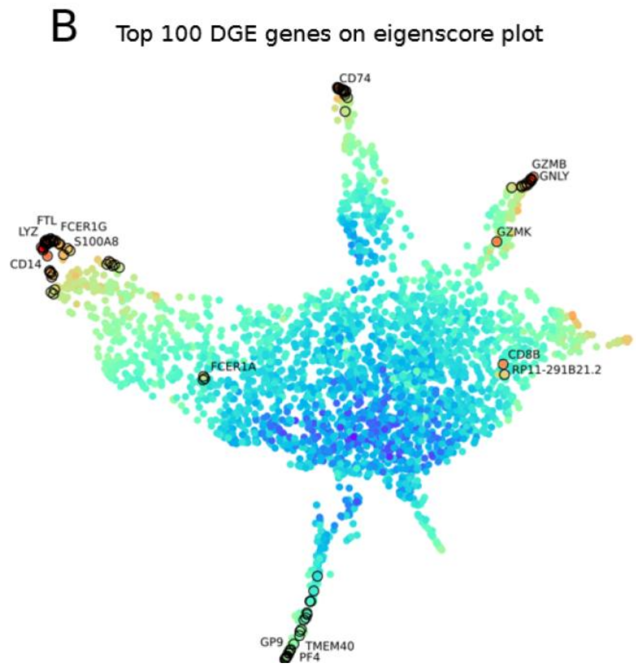
(c) Harmonic Clustering. (d) Total Clustering.

Drug Resistance
(Chen, Liu, Du, **Jones**, Wee, Wang, Chen, Shen, Wei, 2025)

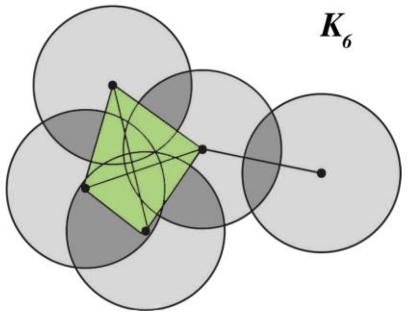
SARC-CoV M^{pro} - PAXLOVID



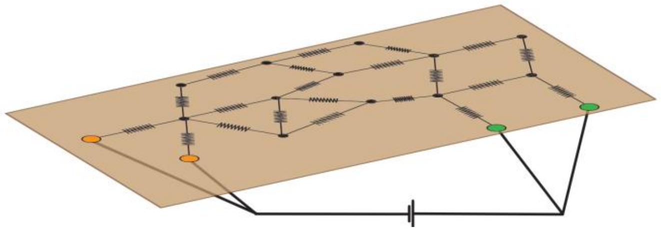
Single-cell Differential Gene Expression
(R. Hoekzema, et al., 2022)



Variants



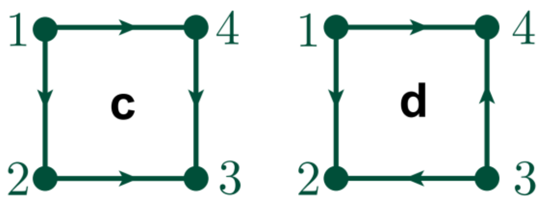
Simplicial Complexes
R. Wang, D.-D. Nguyen, and G.-W. Wei 2019



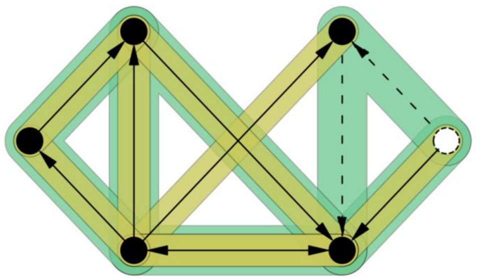
Weighted Simplicial Complexes
F. Mémoli, Z. Wan, Y. Wang 2020

$$
\begin{array}{c|ccc}
 & \mathcal{F}_0 & \mathcal{F}_1 & \mathcal{F}_2 \\
\hline
\mathcal{F}_{01} & -\mathcal{F}_{0\leqslant 01} & \mathcal{F}_{1\leqslant 01} & 0 \\
\mathcal{F}_{02} & -\mathcal{F}_{0\leqslant 02} & 0 & \mathcal{F}_{2\leqslant 02} \\
\mathcal{F}_{12} & 0 & -\mathcal{F}_{1\leqslant 12} & \mathcal{F}_{2\leqslant 12}
\end{array}
$$

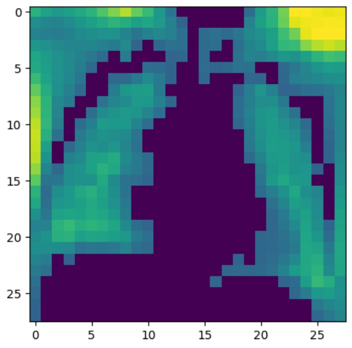
Cellular sheaves
X. Wei and G.-W. Wei 2021



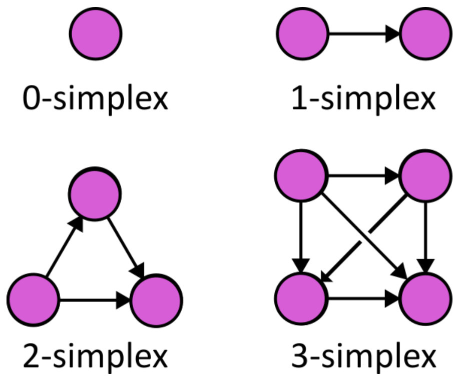
Path complexes
R. Wang and G.-W. Wei 2022



Directed hypergraphs
D. Chen, J. Liu, J. Wu, G.-W. Wei 2023

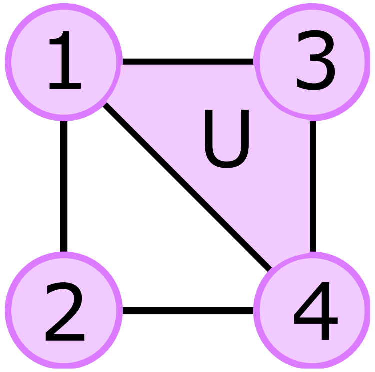


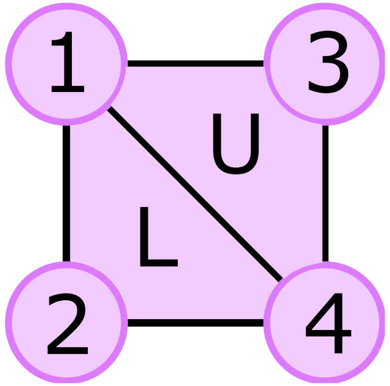
Cubical complexes
R. Dong 2024



Directed flag complexes
Jones and G.-W. Wei 2024

PETLS: PErsistent Topological Laplacian Software






```python
from gudhi import SimplexTree
from petls import Complex

stree = SimplexTree()
stree.insert([1,3,4], filtration=0.0)
stree.insert([1,2], filtration=0.0)
stree.insert([2,4], filtration=0.0)
stree.insert([1,2,4], filtration=1.0)

complex = Complex(simplex_tree=stree)
print(complex.spectra(dim=1,a=0,b=1))
```

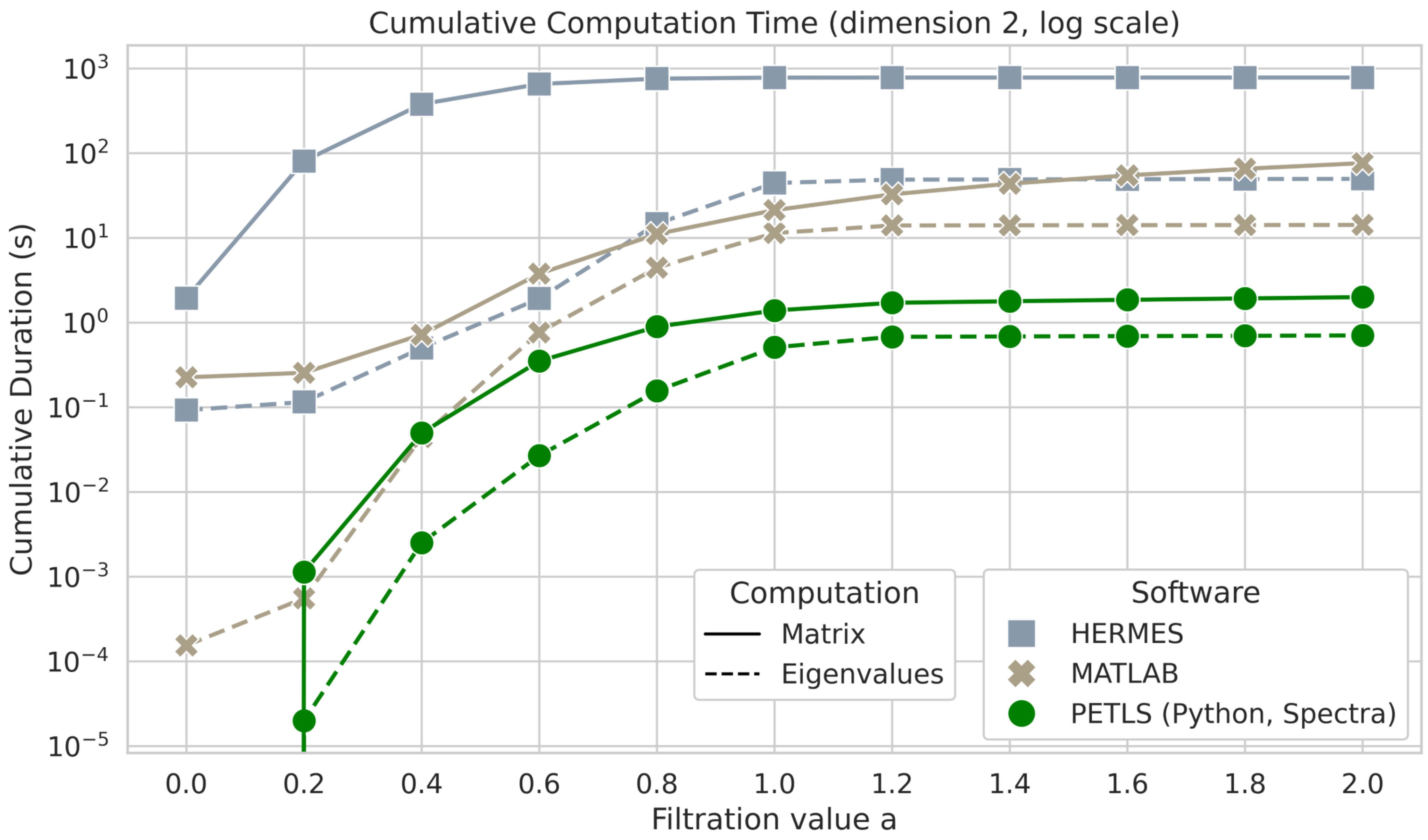
✓ 0.0s

```
[2. 2. 4. 4. 4.]
```

Highlights

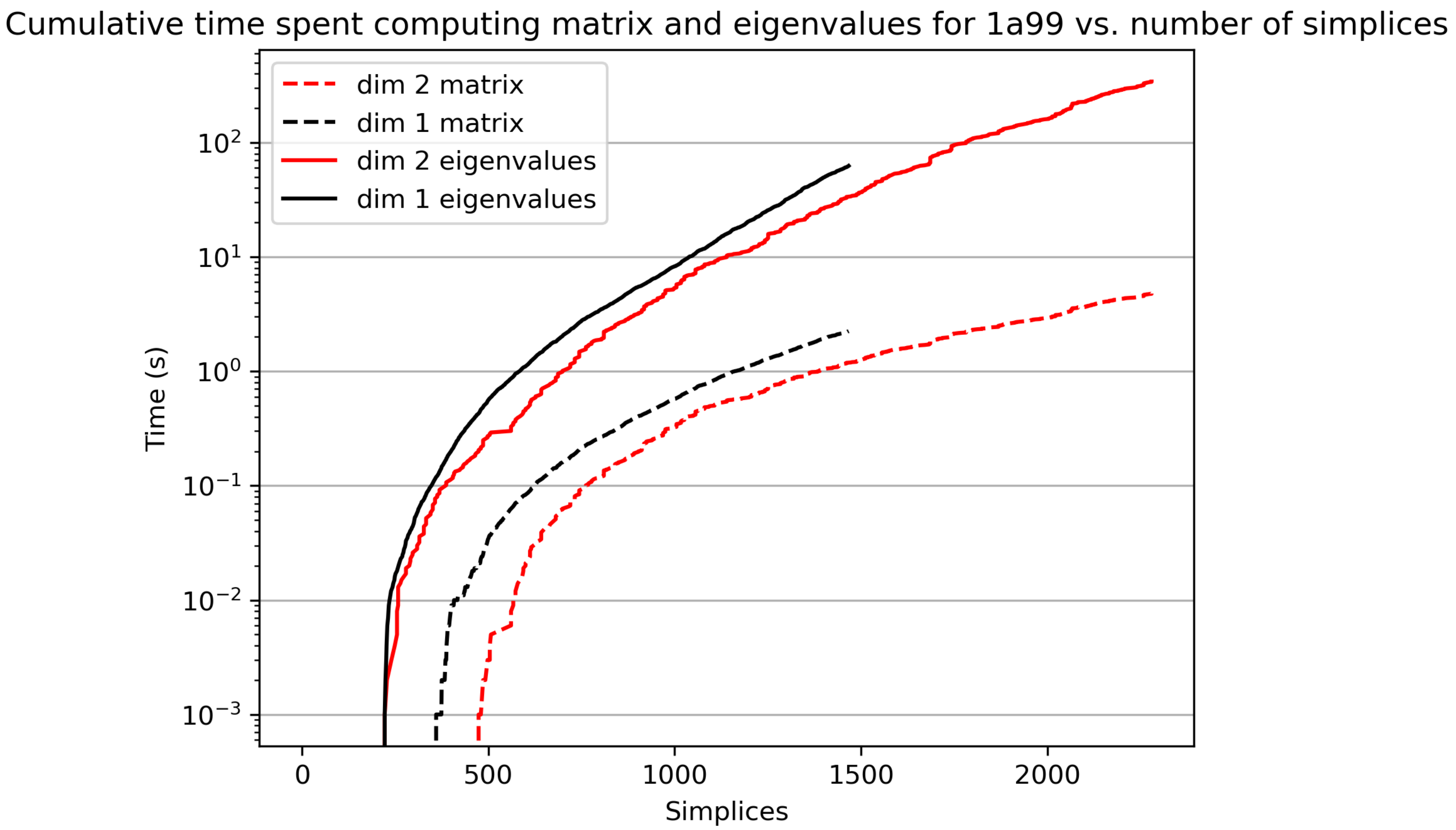
- C++ backend (fast)
- Python frontend (easy)
- Modular algorithms for
  - Up-Laplacian
  - Eigenvalues

… or any filtered boundary matrices

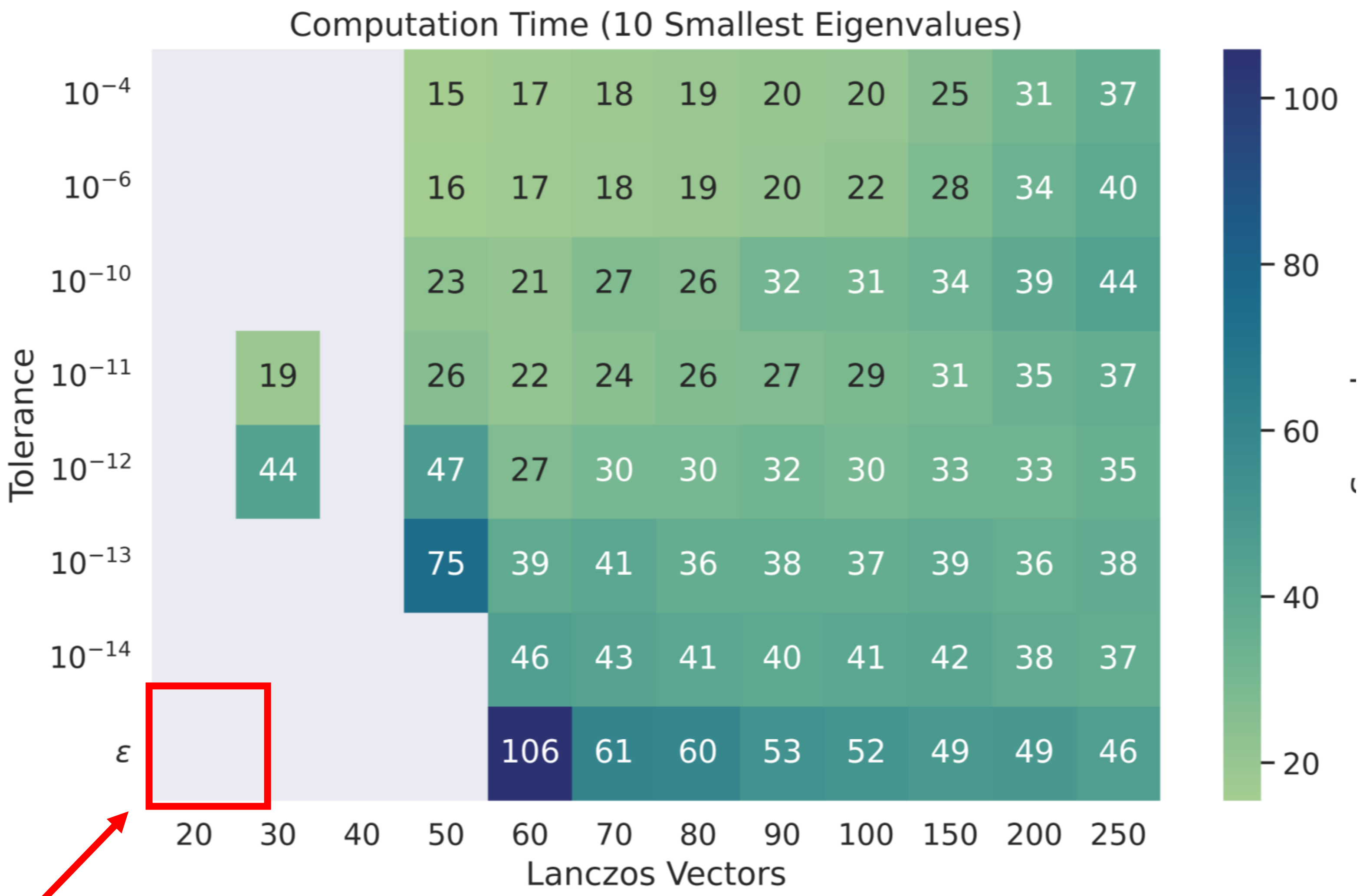


Cumulative Computation Time (dimension 2, log scale)
Cumulative Duration (s)
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Filtration value a
Computation
Matrix
Eigenvalues
Software
HERMES
MATLAB
PETLS (Python, Spectra)

An Unexpected Problem: Eigenvalues



Iterative Methods Don't Always Work



Standard parameters fail

Reduction by Persistent Homology

$\boldsymbol{\Delta_n^{a,b}}$ **= PH + "non-harmonic"**

"non-harmonic" $=$ $\boldsymbol{\Delta_n^{a,b}}$ **– PH**

PH representatives as columns of

$\mathrm{N} = [\alpha_1 \; \dots \alpha_n]$ = basis for ker $\Delta_n^{a,b}$

Let X = basis for im $\Delta_n^{a,b}$ (e.g. random)

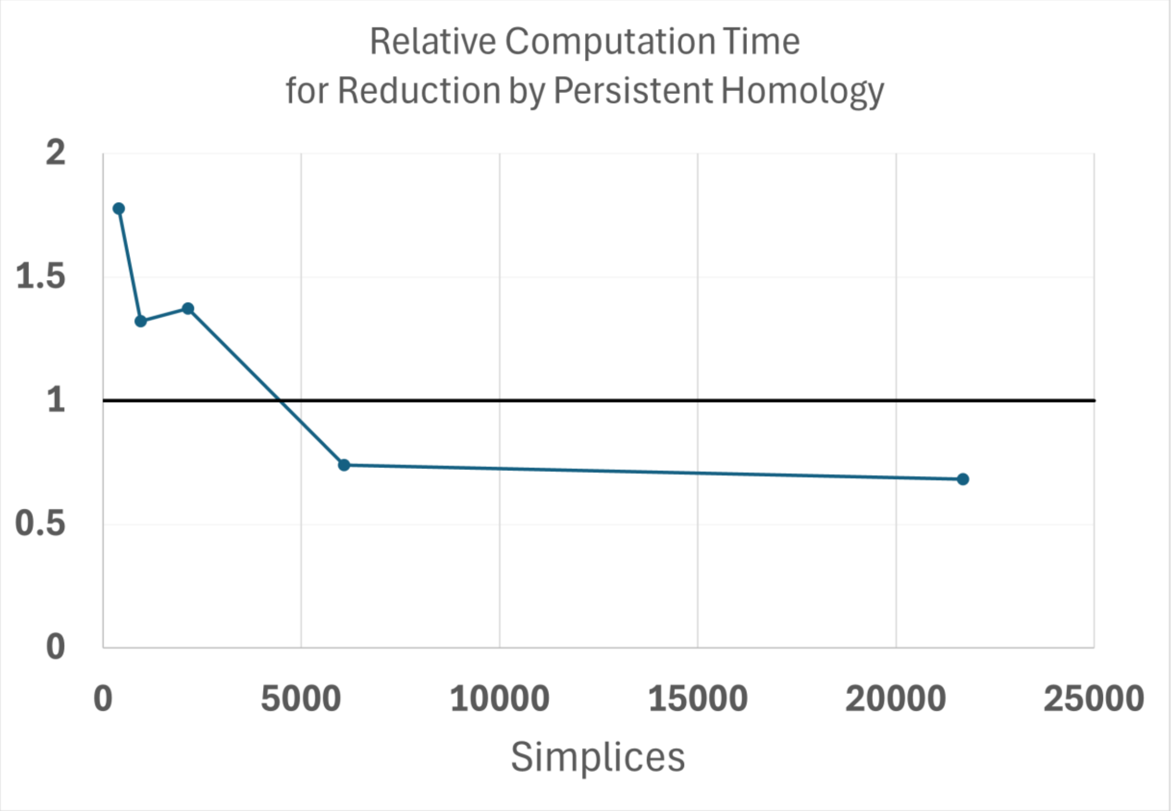
$$S = [X \quad N]$$

$$L = S^{-1}\Delta_n^{a,b}S = [S^{-1}LX \quad 0]$$

Let A =Top left $(\dim C_n^a - \beta_n^{a,b})$ square of L

Properties of A:

1) Same nonzero eigenvalues as $\Delta_n^{a,b}$
2) Smaller than $\Delta_n^{a,b}$
3) Positive <u>definite</u> (no zero-eigenvalues)



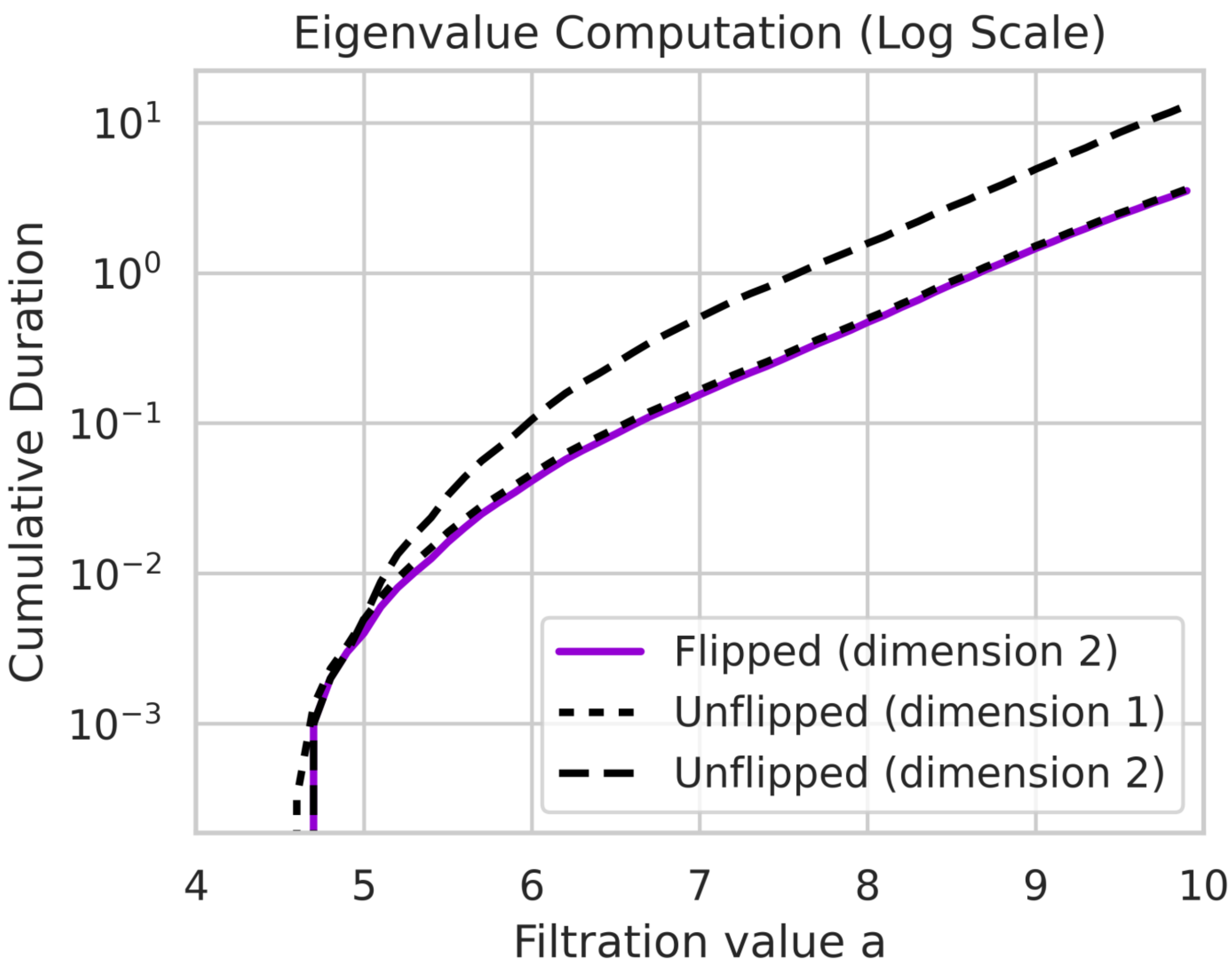
Synthetic data,
without optimization,
saves ~20% computation time

Faster via (standard) linear algebra

Nonzero eigenvalues(XY) =
Nonzero eigenvalues (YX)

Apply to $\Delta_N^{a,b} = (B_N^a)^T B_N^a$

Saves 70% of computation time



Recommended Resources

- “Persistent Spectral Graph” - R. Wang, et al. (2020)
- “Persistent Laplacians: properties, algorithms, and implications” - Mémoli, Wan, Y. Wang (2022)
- “Persistent sheaf Laplacians” - X. Wei and G.-W. Wei (2025)
- “PETLS: PErsistent Topological Laplacian Software” - Jones and Wei (2025)
- “Disentangling the Spectral Properties of the Hodge Laplacian: Not All Small Eigenvalues Are Equal” - Grande and Schaub (2024)

Thank you!

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BenJones-Math.com/software/PETLS/
github.com/bdjones13

```
> pip install petls
```