ComPer 2025: Workshop on Computational Topology

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08:30

Computing the zero-dimensional homology over rooted trees

arXiv > math > arXiv:2411.19319

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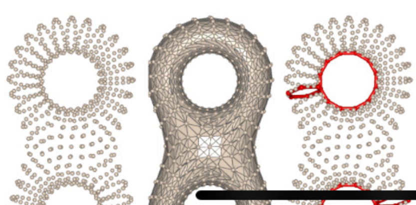
In some applications of topological data analysis one might consider only zero-dimensional homology. The zero-dimensional homology of a space is conceptually and computationally simpler than higher dimensional homology since it can be characterized as the linearization of the set of path-connected components of the space. In particular, the zero-dimensional homology modules of a filtration indexed by a poset P will in general form a proper subcategory of rep P (to the contrary, add Im H_i is dense for i greater than zero). We show that when the indexing poset P is a rooted tree, only finitely many indecomposables can be obtained as zero-dimensional homology of a filtration. In fact, all these indecomposables are reduced rooted tree modules in the sense of Kinser. Moreover, we give an algorithm for the efficient decomposition of rooted tree modules, generalizing the so-called elder rule used in TDA for linear quivers. This is a report on joint work with Riju Bindua and Luis Scoccola.



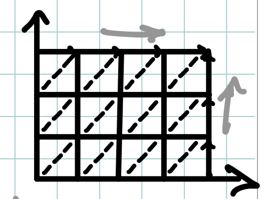
1) Setting: $\quad k$ field

$\mathcal{P}$ finite poset

$\operatorname{rep} \mathcal{P} = \{ M : \mathcal{P} \xrightarrow[\text{functor}]{} \operatorname{vec} k \}$

Example:

$\mathcal{P}$ 2d grid



($\rightsquigarrow$ multiparameter persistence)

$\boxed{\text{TDA}}$ $\quad X = (X_a)_{a \in \mathcal{P}}$ $\quad$ $\mathcal{P}$-filtered top. space:

$$X_a \subseteq X_b \text{ for } a \leq b$$

i-th homology $\downarrow$

(persistence module) $\quad M = H_i X \in \operatorname{rep} \mathcal{P}$ $\qquad H_i X_a \longrightarrow H_i X_b$ linear map

Theorem [Brodzki, Burfitt, Pirashvili 2020]

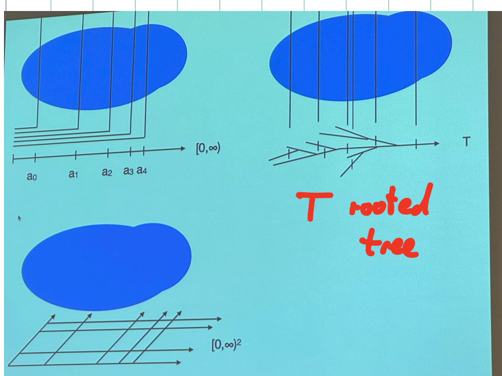
$\operatorname{Im} H_i$ is dense in $\operatorname{rep} \mathcal{P}$ for $i \geq 1$, k perfect field

2) Motivation:

<u>**Not true for H_0**</u> : $\mathrm{Im}\, H_0$ smaller than rep P

rep P is <u>wild</u> in general

<u>Question</u> : Does $\mathrm{Im}\, H_0$ become manageable when restricting to some tree $T \leq P$?



3) Result:

Theorem [Bindua, B, Scoccola 2024]

If $P = T$ is a rooted tree

then $|\operatorname{ind}(\operatorname{add}(\operatorname{Im} H_0))| < \infty$ 'H₀ is rep-finite'

P rooted tree

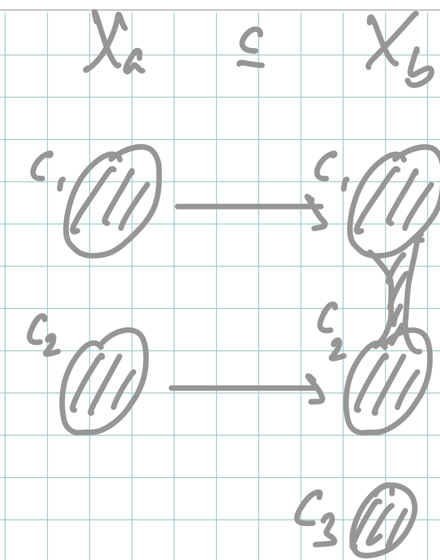
$\Leftrightarrow$ Hasse(P) is tree with unique maximum

Moreover, given a T-filtered graph G,

can compute $H_0(G) \cong U_1 \oplus \cdots \oplus U_r$ in $O(|G|^2)$ time

4) Why rooted trees?

Thm [classic] $\pi_0(X) = \{\text{connected cpts}\}$



$$\begin{tikzcd}
P\text{-filt.top} \arrow[r, "\pi_0"] \arrow[dr, "H_0"'] & \mathrm{set}^P \arrow[d, "\mathcal{L}"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{rep}\, P
\end{tikzcd}$$

$X = (X_a)_{a \in P}$ linearization

$\pi_0(X)$: $1 \to \bar{1} = \bar{2} \leftarrow 2$, 3

H_0 factors through sets

$H_0(X)$: $k \xrightarrow{1} k \xleftarrow{1} k$, k

$a \leq b$ in P

$\pi_0 X_a \longrightarrow \pi_0 X_b$

$\dfrac{\text{function}}{\text{between sets}}$

4) Why rooted trees?

Question: Which indecomposable representations do we get?

$P = a \to b$

$c \circledast \to c \circledast$ $\quad K \xrightarrow{1} K$

$c \circledast$ $\quad 0 \xrightarrow{0} K$

$\Bigg\} \in \mathrm{Im}\, H_0$

$\emptyset \neq X_a \subseteq X_b, \quad X_b \neq \emptyset :$ $\quad K \xrightarrow{0} 0 \quad \notin \mathrm{Im}\, H_0$

However, in $\mathrm{Im}\, H_0$ we have

$$\mathcal{L}\left(\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \rightrightarrows \circ\right) = K^2 \xrightarrow[{[1\ 1]}]{} K \cong K \xrightarrow{1} K \oplus K \xrightarrow{0} 0$$

So $\quad K \xrightarrow{0} 0 \in \mathrm{add}\, \mathrm{Im}\, H_0$

4) Why rooted trees? Which indecomposables do we get?

Observation: $\circ \leftarrow \circ \rightarrow \circ$ is <u>not</u> in Set^P,

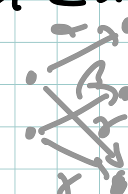
hence $K \xrightarrow{\begin{bmatrix}1\\1\end{bmatrix}} K^2 \notin \underline{\mathrm{Im}}\, H_0$.

$\Rightarrow$ For $T = $
$$\begin{tikzcd} 1 \arrow[dr] & \\ 2 \arrow[r] & 4 \\ 3 \arrow[ur] & \end{tikzcd}$$
rooted tree

, the indecomposable
$$\begin{tikzcd} K \arrow[dr, "\begin{bmatrix}1\\0\end{bmatrix}"] & \\ K \arrow[r, "\begin{bmatrix}0\\1\end{bmatrix}"'] & K^2 \\ K \arrow[ur, "\begin{bmatrix}1\\1\end{bmatrix}"'] & \end{tikzcd}$$
$\notin \mathrm{add}\, \underline{\mathrm{Im}}\, H_0$

But For $T = $
$$\begin{tikzcd} & 2 \\ 1 \arrow[ur, "\alpha"] \arrow[r, "\beta"] \arrow[dr, "\gamma"'] & 3 \\ & 4 \end{tikzcd}$$
not rooted tree

, the indecomposable
$$\begin{tikzcd} & K \\ K^2 \arrow[ur, "{[1\ 0]}"] \arrow[r, "{[0\ 1]}"'] \arrow[dr, "{[1\ 1]}"'] & K \\ & K \end{tikzcd}$$
$\in \mathrm{add}\, \underline{\mathrm{Im}}\, H_0$



In fact, for $T = $
$$\begin{tikzcd} & 2 \\ & 3 \\ 1 \arrow[uur] \arrow[ur] \arrow[dr] \arrow[ddr] & \\ & 4 \\ & 5 \end{tikzcd}$$
$\left| \mathrm{ind}\left(\mathrm{add}\, \underline{\mathrm{Im}}\, H_0\right) \right| = \infty$

5) Tree Modules KT :

$$\begin{tikzcd}[row sep=small] & & 3 \arrow[d] & & & \\ P: \quad 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

$f \uparrow$

$$T: \quad \begin{tikzcd}[row sep=small] & 3 \arrow[dr] & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \\ & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[ur] & & \\ & 2 \arrow[r] & 4 \arrow[ur] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[uuur] & \end{tikzcd}$$

$$T': \quad \begin{tikzcd}[row sep=small] & 3 \arrow[r] & 4 \arrow[dr] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \\ & 3 \arrow[dr] & & & & \\ & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[dr] & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[uuur] & \end{tikzcd}$$

Examples

$$\begin{tikzcd}[row sep=small] & & k^2 \arrow[d] & & & \\ KT: \quad k^2 \arrow[r] & k^3 \arrow[r] & k^4 \arrow[r] & k^3 \arrow[r, "{\left[\begin{smallmatrix}1&1&0\\0&0&1\end{smallmatrix}\right]}"'] & k^2 \arrow[r, "{[1\ 1]}"'] & k \end{tikzcd}$$

T is a rooted tree $/P$:

$\exists$ graph map $f: T \to P$

$KT := f^* \mathcal{L}(T) \in \operatorname{rep} P$

In fact $KT \in \operatorname{Ind} H_0$

Test : Which of KT, KT' is indecomposable?

6) Rooted Tree Modules

Prop [BBS] Let P be a rooted tree. Then

$$\operatorname{add} \operatorname{Im} H_0 = \operatorname{add}\left(kT \mid T \xrightarrow{f} P \text{ rooted tree}/P \right)$$

rooted tree module

Question: When is kT indecomposable?

Ringel: no answer (1995; study tree modules)

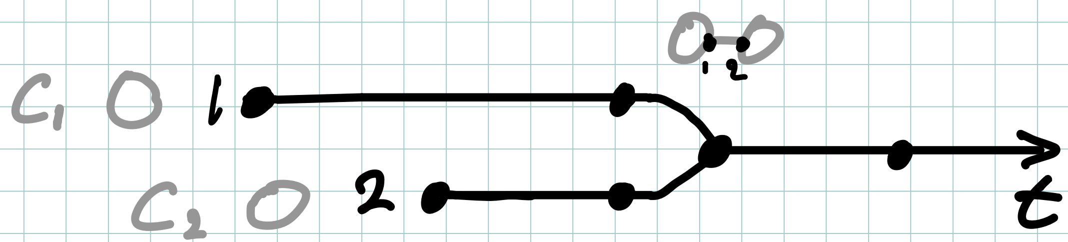
Crawley-Boevey: In case $T = P$, kT is always indecomposable

Kinser [2010]: kT indecomposable $\Leftrightarrow$ T reduced

Def: T reduced if ... (recursive def of $\leq$ on reduced trees used to define when T is reduced, recursively)

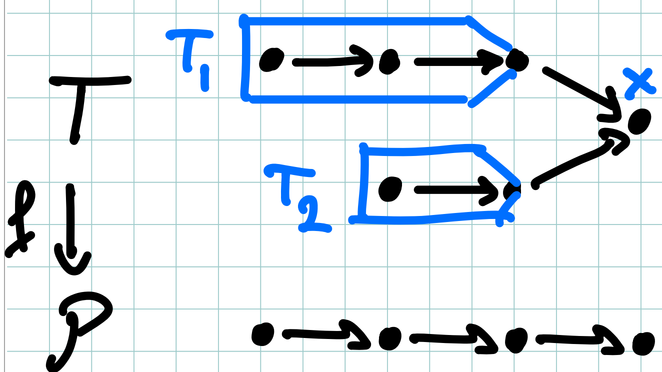
7) The elder rule

TDA $P = \mathbb{R}_{\geq 0}$



T

elder rule: When 2 components merge, kill the younger one!



T_1, T_2 subtrees merging at x

$T = \mathrm{merge}(T_1, T_2)$

$$\left.\begin{array}{l}\exists \text{ morphism } T_2 \to T_1 \\ (\text{as trees over } P)\end{array}\right\} \Longrightarrow \quad kT \cong kT_2 \oplus k(T_1 \to x)$$

in rep P

8) Generalized elder rule

Thm [BBS] Let P be a rooted tree,
and $T = \mathrm{merge}(T_1, \dots, T_\ell)$ a rooted tree over P.

(a) If $\exists$ morphism $T_2 \to T_1$, then $kT \cong kT_2 \oplus k(T \setminus T_2)$
(as trees over P) in rep P

(b) kT indecomposable $\overset{\text{[Kinser]}}{\iff}$ T reduced
$\iff \mathrm{Hom}_{\mathrm{tree}/P}(T, T) = \{\mathrm{id}_T\}$

(c) $\exists$ $O(|T|^2)$-time algorithm decomposing kT

$\rightsquigarrow$ examples over $\mathbb{E}_7$!

Examples:

$$S = \{s_1, \dots, s_j\} \subseteq \{1, \dots, n\}$$

$$P = \begin{tikzcd} 1 \arrow[ddr] & \\ 2 \arrow[dr] & \\ \vdots \quad \vdots & n+1 \\ n \arrow[ur] & \end{tikzcd} \qquad \text{Then } T_S = \begin{tikzcd} s_1 \arrow[dr] & \\ \vdots & n+1 \\ s_j \arrow[ur] & \end{tikzcd} \text{ is reduced,}$$

so $\left| \operatorname{ind}\left(\operatorname{add}(\operatorname{Im} H_0)\right) \right| \geq 2^n$

$$P: \quad \underbrace{1 \to 2 \to \cdots n-1}_{T_1 \text{ with root at } n-1, \text{ or empty}} \to n \qquad T = \operatorname{merge}(T_1)$$

$n-1 \quad + \quad 1$ trees with root n

9) Finiteness

If P is a rooted tree
then $\left|\operatorname{ind}(\operatorname{add}(\operatorname{Im} H_0))\right| < \infty$ 'Ho is rep-finite'

Proof: Let T be a reduced rooted tree over P.

Write $T = \operatorname{merge}(T_1, \dots, T_\ell)$. Then each T_i is reduced (otherwise, T_i would split off a summand).

By induction, there are only finitely many possibilities for each T_i, and ℓ must be bounded by the product of these numbers (otherwise there are repetitions, hence morphisms) $\square$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