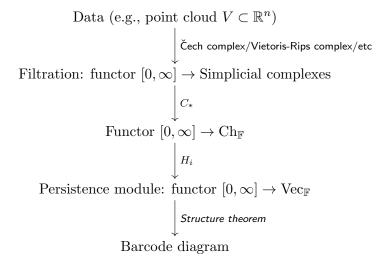
Functor calculus and multiparameter persistence

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Persistent homology



Multipersistent homology

- ▶ In multipersistent homology, persistence modules are replaced by multipersistence modules
 - ► A 1-parameter persistence module is a functor

$$F \colon [0, \infty] \to \mathsf{Vec}_{\mathbb{F}}$$

A multipersistence module is a functor

$$F \colon [0,\infty]^k \to \mathsf{Vec}_{\mathbb{F}},$$

where $[0,\infty]^k$ has the product partial order.

No barcode for multipersistence

- Unfortunately, the structure theorem does not generalize to multipersistence modules. In other words, there are multipersistence modules that do not admit an interval decomposition.
 - Hence, there is no obvious way to analyze data from multipersistent homology

$$\mathbb{F} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathbb{F}^2 \xrightarrow{(1 \ 1)} \mathbb{F}$$

$$\uparrow \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \uparrow \qquad \uparrow = 0$$

$$0 \longrightarrow \mathbb{F} \xrightarrow{=} \mathbb{F}$$

Example from: M. Botnan and M. Lesnick, An Introduction to Multiparameter Persistence (2022)

Middle exactness

- ▶ However, a pointwise finite-dimensional multipersistence module from $[0,\infty]^2$ to $\text{Vec}_{\mathbb{F}}$ is interval decomposable if it is middle-exact [1].
- ▶ A multipersistence module $F: [0, \infty]^2 \to \mathsf{Vec}_{\mathbb{F}}$ is said to be *middle-exact* if, for all $(x,y) \le (x',y')$ in $[0,\infty]^2$, the complex

$$F(x,y) \to F(x,y') \oplus F(x',y) \to F(x',y')$$

is exact in the middle.

- ► Middle-exact multipersistence modules arise naturally from interlevel set persistent homology [2].
- [1] M.B. Botnan and W. Crawley-Boevey, *Decomposition of persistence modules* (2020)
- [2] G. Carlsson, V. de Silva, and D. Morozov, Zigzag persistent homology and real-valued functions (2009)

Functor calculus: analogy from calculus

▶ In ordinary calculus, we can construct the Taylor expansion of an analytic function.

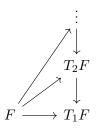
$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

This allows us to construct a sequence of degree n polynomials, $(f_n)_{n\in\mathbb{N}}$, converging to f:

$$f_n = \sum_{i=0}^n a_i x^i$$

Functor calculus: main idea

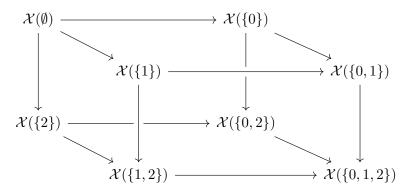
▶ Given categories \mathscr{C}, \mathscr{D} , one studies functors $F : \mathscr{C} \to \mathscr{D}$ through their tower of degree n approximations



- What degree n means varies between different variants of functor calculus
- ▶ We speak of convergence when the homotopy limit of the tower is (weakly equivalent to) F

Cubical diagrams

Let $[k-1] = \{0, \dots, k-1\}$, and let \mathcal{P}_k be the power set of [k-1] (viewed as a poset under inclusion). We call functors $\mathcal{X} : \mathcal{P}_k \to \mathscr{C}$ k-cubes.



Definition

A k-cube $\mathcal{X} \colon \mathcal{P}_k \to \mathscr{C}$ is *cocartesian* if the canonical map

$$\operatorname{colim}_{S \subsetneq [k-1]} \mathcal{X}(S) \to \mathcal{X}([k-1])$$

is an isomorphism.

Similarly, ${\mathcal X}$ is cartesian if the canonical map

$$\mathcal{X}(\emptyset) \to \lim_{S \subseteq [k-1], S \neq \emptyset} \mathcal{X}(S)$$

is an isomorphism.

Definition

A k-cube $\mathcal{X} \colon \mathcal{P}_k \to \mathscr{C}$ is strongly cocartesian if each face of dimension ≥ 2 is cocartesian. It is strongly cartesian if each face of dimension ≥ 2 is cartesian.

Definition

A k-cube is strongly bicartesian if it is both strongly cartesian and strongly cocartesian.

Definition

Let \mathcal{M} be a model category. A k-cube $\mathcal{X} \colon \mathcal{P}_k \to \mathcal{M}$ is homotopy cocartesian if the canonical map

$$\operatorname{hocolim}_{S \subsetneq [k-1]} \mathcal{X}(S) \to \mathcal{X}([k-1])$$

is a weak equivalence.

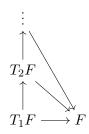
Similar for homotopy cartesian, strongly homotopy cocartesian, etc.

Functor calculus: examples

- ▶ Goodwillie calculus [1]: A functor is degree n if it sends strongly homotopy cocartesian (n+1)-cubes to homotopy cartesian (n+1)-cubes
- ▶ **Abelian calculus [2]:** A functor is degree n if it sends a specific class of strongly homotopy cocartesian (n+1)-cubes to homotopy cartesian (n+1)-cubes
- Manifold calculus [3]: A functor is degree n if it sends strongly bicartesian (n+1)-cubes (of open subsets) to homotopy cartesian (n+1)-cubes (of spaces)
- [1] T. Goodwillie, Calculus. I. The first derivative of pseudoisotopy theory (1990)
- [2] B. Johnson and R. McCarthy, Deriving calculus with cotriples (2004)
- [3] T. Goodwillie and M. Weiss, Embeddings from the point of view of immersion theory: Part ii (1999)

Functor cocalculus

- ► Functor calculus, except that the arrows in the tower go the other way [1, 2]
 - ▶ Instead of a tower, we get a *telescope* of approximations



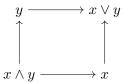
- [1] R. McCarthy, Dual calculus for functors to spectra (2001)
- [2] K. Bauer, R. Brooks, K. Hess, B. Johnson, J. Rasmusen and B. Schreiner, Constructing monads from cubical diagrams and homotopy colimits (2024)

Poset cocalculus: a functor cocalculus framework for studying functors out of posets

lacktriangle such as functors from $[0,\infty]^k$ to $\mathrm{Ch}_\mathbb{F}$

Cubes of posets

- ▶ In a poset, the colimit of a finite diagram (if it exists) is the supremum of the elements in the diagram. Likewise, the limit is given by the infimum.
- A *lattice* is a poset where the supremum $x \lor y$ and infimum $x \land y$ of two elements always exist
 - In other words, a lattice is a poset where all nonempty finite colimits and nonempty finite limits exist



A strongly bicartesian 2-cube in a lattice.

Poset cocalculus

We introduce the following definition.

Definition

Let P be a lattice and $\mathcal M$ a model category. We say that a functor $F\colon P\to \mathcal M$ is codegree n if it takes strongly bicartesian (n+1)-cubes to homotopy cocartesian (n+1)-cubes.

Dimension of an element

We restrict ourself to the case where the source poset ${\cal P}$ is a product of total orders.

Definition

Let $P=P_1\times\cdots\times P_k$, where each P_i is a total order with a minimal element 0. The *dimension* of an element (x_1,\ldots,x_k) is defined as its number of nonzero components, i.e.,

$$\dim(x_1,\ldots,x_k) = \#\{i : x_i \neq 0\}.$$

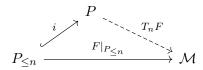
Remark: The theory also works when P is a distributive lattice.

Codegree n approximations

Let P be a product of total orders with minimal elements, \mathcal{M} a model category, and $F\colon P\to \mathcal{M}$ a functor. Let $P_{\leq n}\subseteq P$ be the subposet $P_{\leq n}=\{x\in P: \dim(x)\leq n\}.$

Definition

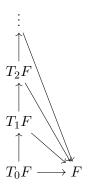
We define the codegree n approximation of F, denoted T_nF , as the homotopy left Kan extension of $F|_{P\leq n}$ along the inclusion $i\colon P_{\leq n}\hookrightarrow P$.



Explicitly,

$$T_n F(x) = \underset{y \le x, \dim(y) \le n}{\operatorname{hocolim}} F(y).$$

${\sf Codegree}\ n\ {\sf approximations}$



Codegree n approximations

Let P be a product of total orders with minimal elements, $\mathcal M$ a model category*, and $F\colon P\to \mathcal M$ a functor.

Theorem (Theorem A)

 T_nF is a codegree n functor.

Theorem (Theorem B)

Suppose that P is a finite product of total orders. If F is codegree n then $T_nF \simeq F$.

(*) subject to certain mild conditions

Examples

Consider the poset $P = \{0, 1\}^2$:

$$(0,1) \longrightarrow (1,1)$$

$$\uparrow \qquad \uparrow$$

$$(0,0) \longrightarrow (1,0)$$

▶ A codegree 1 functor from P to $Ch_{\mathbb{F}}$:

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{F} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{F} \end{array}$$

Examples

▶ A codegree 2 functor, $F: P \to \operatorname{Ch}_{\mathbb{F}}$:



Examples

▶ A codegree 2 functor, $F: P \to \operatorname{Ch}_{\mathbb{F}}$:

$$\begin{bmatrix}
\mathbb{F} & \xrightarrow{1} & \mathbb{F} \\
\uparrow & & \uparrow \\
0 & \longrightarrow \mathbb{F}
\end{bmatrix}$$

 $ightharpoonup T_1F$:

$$\mathbb{F} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \mathbb{F} \oplus \mathbb{F}$$

$$\uparrow \qquad \qquad \uparrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Relation to middle exactness

Let $\mathrm{Ch}_\mathbb{F}$ denote the category of unbounded chain complexes over the field \mathbb{F} , equipped with the canonical model structure*.

Theorem

Let $F: [0,\infty]^2 \to \mathsf{Vec}_{\mathbb{F}}$ be a pointwise finite-dimensional functor. The following are equivalent.

- (i) F is middle-exact.
- (ii) There exists a codegree 1 functor $\widehat{F}:[0,\infty]^2\to\operatorname{Ch}_{\mathbb{F}}$ such that $F\cong H_0\circ\widehat{F}$.

(*) See, e.g., Theorem 2.3.13 in: M. Hovey, Model categories (1999)

Stability

Let d_H^{Λ} be the multiplicative interleaving distance.

Proposition

For all pairs of functors $F, G: [0, \infty)^k \to \operatorname{Ch}_{\mathbb{F}}$,

$$d_H^{\Lambda}(T_1F, T_1G) \le d_H^{\Lambda}(T_2F, T_2G) \le \dots \le d_H^{\Lambda}(F, G).$$

We can also consider functors to any category!

Filtrations of simplicial complexes

Let V be a vertex set. Given a functor $f\colon \mathcal{P}(V)\to \mathbb{R}$, we can define a *filtration* on $\mathcal{P}(V)$ as

$$F_f \colon \mathbb{R} \to \text{Simplicial complexes}$$

 $t \mapsto \{ \sigma \in \mathcal{P}(V) : f(\sigma) \le t \}.$

Observe that the poset $\mathcal{P}(V)$ is isomorphic to the product $\{0,1\}^{|V|}$. We can study the functor f using poset cocalculus.

Let $V \subset \mathbb{R}^n$ be a finite point cloud.

The Čech filtration of V is the filtration $F_{f_{\mathbf{C}}}$ induced by the functor

$$f_{\mathbf{C}} \colon \mathcal{P}(V) \to \mathbb{R}$$

$$U \mapsto \inf \left\{ \varepsilon : \exists z \in \mathbb{R}^n \text{ such that } U \subseteq \overline{B_{\varepsilon}}(z) \right\}.$$

The Vietoris-Rips filtration of V is the filtration $F_{f_{\mathbf{V}\mathbf{R}}}$ induced by

$$f_{\mathbf{VR}} \colon \mathcal{P}(V) \to \mathbb{R}$$

 $U \mapsto \max \{ d(x, y) : x, y \in U \}.$

Let $V \subset \mathbb{R}^n$ be a finite point cloud.

The Čech filtration of V is the filtration $F_{f_{\mathbf{C}}}$ induced by the functor

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The Vietoris-Rips filtration of V is the filtration $F_{f_{\mathbf{V}\mathbf{R}}}$ induced by

$$f_{\mathbf{VR}} \colon \mathcal{P}(V) \to \mathbb{R}$$

 $U \mapsto \max \{ d(x, y) : x, y \in U \}.$

Proposition

$$f_{\mathbf{VR}} = 2 \cdot T_2 f_{\mathbf{C}}$$

Thank you!

References:

- 1. B.G. Hem, Poset functor cocalculus and applications to topological data analysis (2025) arXiv:2501.05996
- 2. B.G. Hem, *Decomposing multipersistence modules using functor calculus* (2025) arXiv:2510.06178