On the connectivity of intrinsic Čech complexes of spheres

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Motivation

Motivation from applications

Consider a dataset $\mathcal{S}$. Replace each point x in $\mathcal{S}$ with a (Euclidean) ball of radius r centered at x. The union of these balls serves as a proxy for the shape of the data.

By varying r from small to large values, one can use persistent homology to get a multi-resolution summary of the shape of the data.

If the dataset $\mathcal{S}$ is sampled from a manifold M, then in the limit, as more and more points are sampled, the persistent homology of $\mathcal{S}$ converges to the persistent homology of M.

Čech complexes are among the most important tools in applied and computational topology for approximating the shape of data.

What are...Čech complexes?

The Čech complex corresponding to a dataset $\mathcal{S}$ is a combinatorial representation of data: it has a vertex for each data point of $\mathcal{S}$, an edge whenever two balls (centered around data points) intersect, a triangle whenever three balls intersect, and so on.

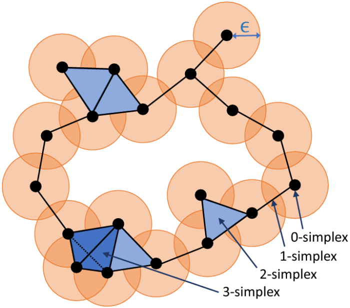


Figure: The orange discs represent Euclidean balls of radius ε, and the black and blue structures are simplices in the Čech complex at radius ε.

Intrinsic Čech complexes

Since Euclidean balls are convex, the nerve lemma implies that such a Čech complex is homotopy equivalent to the union of the balls.

Given a closed Riemannian manifold M, what if the Čech complex is defined intrinsically as the nerve of open balls that are restricted to live inside M?

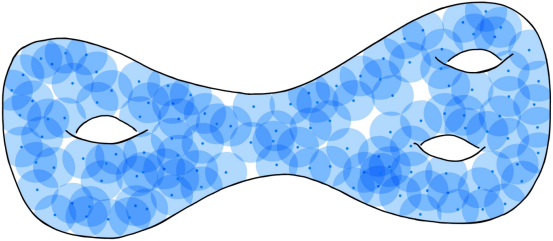


Figure: The blue discs represent balls of radius ε that live in the three-holed torus.

Motivations from theory

For small values of r, the intrinsic Čech complex of M at scale r is homotopic to M by the nerve lemma. But...once r exceeds the convexity radius of M, this need not be the case (the hypotheses of the nerve lemma may fail)!

Question

What are the homotopy types of intrinsic Čech complexes of closed Riemannian manifolds at large scales?

Extension of the nerve lemma: What are the homotopy types of the nerve of balls in a "nice" manifold when the radii are large enough so that the hypotheses of the nerve lemma are no longer satisfied?

We try to address this question for spheres.

In what follows, we work *only* with the intrinsic Čech complexes of spheres.

Čech complexes of spheres

Notations and conventions

Let S^n be the n-sphere with the geodesic metric and of diameter π.

The intrinsic Čech complex of S^n at scale $r \in (0, \pi)$, denoted $\check{\mathrm{C}}(S^n; r)$, is the nerve of all open balls of radius r in S^n.

So, $\check{\mathrm{C}}(S^n; r) := \mathrm{Nerve}(\{B_{S^n}(x, r)\}_{x \in S^n})$. Formally,

$$\check{\mathrm{C}}(S^n; r) = \left\{ \text{finite } \sigma \subset S^n \;\middle|\; \bigcap_{x \in \sigma} B_{S^n}(x, r) \neq \emptyset \right\}.$$

When $r \in (0, \pi/2]$, then $\check{\mathrm{C}}(S^n; r) \simeq S^n$ due to the nerve lemma.

So, we will work with $\check{\mathrm{C}}(S^n; \pi - \delta)$ for scales $\delta \in (0, \pi/2)$.

What is known?

In the case $n = 1$: everything!

Theorem (Adamaszek–Adams, 2017)

$$\check{\mathrm{C}}(S^1; \pi - \delta) \simeq S^{2k+1} \qquad \textit{if} \qquad \delta \in \left[\frac{\pi}{k+2}, \frac{\pi}{k+1}\right).$$

In the cases $n \geq 2$ for scales $\delta \in (0, \pi/2)$: hardly anything!

Theorem (Virk, 2020)

For $n \geq 2$ and $\delta \in (0, \pi/2)$, the Čech complex $\check{\mathrm{C}}(S^n; \pi - \delta)$ is simply connected.

Question

How does the homotopy connectivity of $\check{\mathrm{C}}(S^n; \pi - \delta)$ vary with δ?
Can we bound this connectivity as a function of the scale δ?

Main result

We can control the connectivity of $\check{\mathrm{C}}(S^n; \pi - \delta)$ in terms of the coverings of S^n.

Adams–J.–Mallick (2025): Let $\delta > 0$, and suppose the connectivity of $\check{\mathrm{C}}(S^n; \pi - \delta)$ is $k - 1$. Then S^n can be covered by $2k + 2$ balls of radius δ, but not by $k + 1$ balls of any radius less than $\frac{\delta}{2}$.

The connectivity of the Vietoris–Rips complexes of S^n was controlled in a similar way, using coverings of S^n and $\mathbb{R}P^n$, by Adams–Bush–Virk (2024).

Definition

Let $k \geq 1$ be an integer. We define the k-th covering radius of S^n as

$$\mathrm{cov}_{S^n}(k) := \inf \left\{ r \geq 0 \;\middle|\; \exists\, x_1, \ldots, x_k \in S^n \text{ such that } \bigcup_{i=1}^{k} B[x_i, r] = S^n \right\}.$$

Statement of the result

Theorem (Adams–J.–Mallick, 2025)

For $n \geq 1$ and $\delta \in (0, \pi)$, if $\operatorname{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) = k - 1$*, then*

$$\operatorname{cov}_{S^n}(2k+2) \leq \delta \leq 2 \operatorname{cov}_{S^n}(k+1).$$

In other words...

If $\operatorname{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) = k - 1$, then S^n can be covered by $2k+2$ balls of radius δ, but it cannot be covered by $k+1$ balls of radius $\frac{\delta}{2} - \varepsilon$ for any small $\varepsilon > 0$.

Understanding the theorem

On the homotopy types

Definition

Let $\delta > 0$ be a scale. We define the δ-covering number of S^n as

$$\mathrm{numCover}_{S^n}(\delta) := \min \left\{ m \geq 1 \;\middle|\; \exists\, x_1, \ldots, x_m \in S^n \text{ such that } \bigcup_{i=1}^{m} B[x_i, \delta] = S^n \right\}$$

Corollary

For $n \geq 1$ and $\delta \in (0, \pi)$, the connectivity of $\check{\mathrm{C}}(S^n; \pi - \delta)$ satisfies

$$\tfrac{1}{2}\,\mathrm{numCover}_{S^n}(\delta) - 2 \leq \mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \leq \mathrm{numCover}_{S^n}\left(\tfrac{\delta}{2}\right) - 2.$$

Corollary

For $n \geq 1$, the homotopy type of the Čech complex $\check{\mathrm{C}}(S^n; \pi - \delta)$ changes infinitely many times as δ varies over the range $(0, \pi)$.

The case of the circle

In the case $n = 1$, our upper bound to $\mathrm{conn}(\check{\mathrm{C}}(S^n; r))$ is sharp. However, our lower bound is off by a factor of 4.

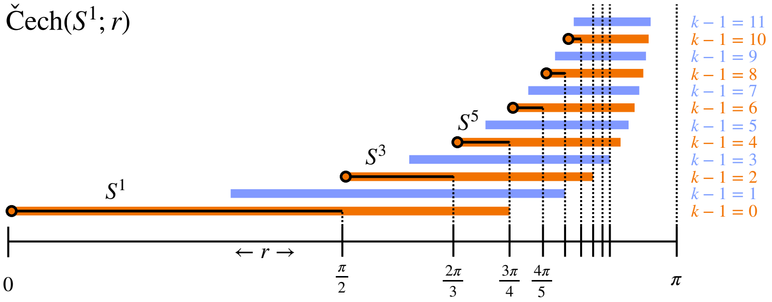


Figure: Black bars: homotopy types of $\check{\mathrm{C}}(S^1; r)$ as r varies. Colored bars: intervals where $\check{\mathrm{C}}(S^1; r)$ may have connectivity $k-1$ as implied by our theorem. Left endpoints of orange bars (when $k-1$ is even) are tight.

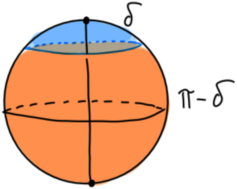
Proof of the theorem

Towards the upper bound

Definition (Erdős–Hajnal, 1967)

The Borsuk graph of S^n for scale $\delta > 0$, denoted $\mathrm{Bor}(S^n; \delta)$, is a simple graph whose vertex set is S^n and whose edges are all pairs of vertices $x, y \in S^n$ with $d(x, y) > \delta$.

Key idea: The Čech complex $\check{\mathrm{C}}(S^n; \pi - \delta)$ is the same as the neighborhood complex $N(\mathrm{Bor}(S^n; \delta))$ of the Borsuk graph $\mathrm{Bor}(S^n; \delta)$.



Proof: $\{x\} \in \check{\mathrm{C}}(S^n; \pi - \delta) \iff \exists\, v \in S^n$ with $d(x, v) < \pi - \delta$
$\iff \exists -v \in S^n$ with $d(x, -v) > \delta \iff \{x\} \in N(\mathrm{Bor}(S^n; \delta))$.

The upper bound

Theorem (Lovász, 1978)

For the chromatic number of any graph G, $\chi(G) \geq \operatorname{conn}(N(G)) + 3$.

In particular, $\chi(\mathrm{Bor}(S^n;\delta)) \geq \operatorname{conn}(N(\mathrm{Bor}(S^n;\delta))) + 3$.

Proof of the upper bound.

If $\delta > 2\operatorname{cov}_{S^n}(k+1)$ for some $k \geq 1$, then

$$\operatorname{conn}(\check{\mathrm{C}}(S^n;\pi-\delta)) = \operatorname{conn}(N(\mathrm{Bor}(S^n;\delta))) \leq \chi(\mathrm{Bor}(S^n;\delta)) - 3 \leq k-2$$

since $\check{\mathrm{C}}(S^n;\pi-\delta) = N(\mathrm{Bor}(S^n;\delta))$. □

The right-most inequality can be obtained from the work of Moy (2024).

The lower bound

Corollary (To Barmak, 2023)

Let L be a simplicial complex. If each collection of $2k + 2$ number of vertices of L is contained in a simplex of L, then L is k-connected.

Proof of the lower bound.

Assume $0 < \delta < \mathrm{cov}_{S^n}(2k + 2)$.

Step 1: Choose $\varepsilon > 0$ such that $\delta + \varepsilon < \mathrm{cov}_{S^n}(2k + 2)$. Take any finite ε-dense set $X \subset S^n$. We prove $\mathrm{conn}(\check{\mathrm{C}}(X; \pi - \delta)) \geq k$ using the above corollary.

Step 2: We use some homotopy-theoretic techniques to conclude that $\mathrm{conn}(\check{\mathrm{C}}(S^n; \pi - \delta)) \geq k$ as well. $\square$

Thank You

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Principal references

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A complete list of references can be found in the highlighted paper.