Gromov–Hausdorff distance for directed spaces

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Introduction

- **[Gromov]** - Metric Structures for Riemannain and Non-Riemannian Spaces
- **Kalton & Ostrovskii** - Distances between Banach spaces, $\cdots$
- **Length structures**: $len : \mathcal{C}(I) \to [0, \infty)$ satisfying some axioms (positivity, monotonicity, aditivity, invariance under reparmetrization and 'continuity').
- **Length spaces**.

$$\begin{array}{ccc} (X, d) \text{ metric space} & \longrightarrow & \text{Metric Length structure } len \\ & & \downarrow \\ & & \text{new metric } d_{len} \text{ associated with len} \end{array}$$

Preliminaries

Let (X, d) be a metric space. The **Length** of a path γ is given by

$$l(\gamma) = \sup_{[N]} \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)), \tag{1}$$

$[N]$ = partition of I of length N. $l(\gamma) < \infty$ = **rectifiable path**.

Definition 1 (Directed spaces).

A pair $(X, \vec{P}(X))$, is called **d-space** if X is a topological space and $\vec{P}(X)$ is a collection of paths such that:

1. every constant path belongs to $\vec{P}(X)$,
2. $\vec{P}(X)$ is closed under non-decreasing reparameterization and subpath.
3. if $\gamma, \gamma' \in \vec{P}(X)$ such that $\gamma(1) = \gamma'(0)$, then the concatenation $\gamma\gamma' \in \vec{P}(X)$.

Let $\vec{X}$ and $\vec{Y}$ be d-spaces. A **d-map** $\vec{F}\colon \vec{X} \to \vec{Y}$ is a continuous map $F\colon X \to Y \mid \forall\gamma \in \vec{P}(X)$, the composition $F \circ \gamma \ \in \vec{P}(Y)$. This defines a category **dTop**. Note that a bijective d-map $\vec{F}\colon \vec{X} \to \vec{Y}$ whose inverse is a d-map is called **d-invertible**.

Example 2.

- **Discrete d-space** $(X, \vec{P}(X))$: $X \in$ **Top** and $\vec{P}(X) = \mathrm{c}^{st}$-paths.
- **Trivial d-space** $(X, \vec{P}(X))$: $X \in$ **Top** and $\vec{P}(X) = \mathcal{C}(I, X)$.

Example 3.

Partially ordered set $(X, \leq)$ and $\vec{P}(X) \subseteq \{\gamma \in \mathcal{C} : \gamma(s) \leq \gamma(t)\ \forall s \leq t\}$

A **rectifiable d-space** is a d-space $(X, \vec{P}(X))$, where X is a metric space and every d-path in $\vec{P}(X)$ is **rectifiable**.

- (M, d) metric space, then $\forall A, B \subseteq M$, their **Hausdorff distance** is defined by:

$$d_{\mathrm{H}}(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\} \tag{2}$$

- Given (X, d^X), (Y, d^Y), the **Gromov–Hausdorff distance** is defined by:

$$d_{\mathrm{GH}}(X, Y) = \inf_{f,g} d_{\mathrm{H}}(f(X), g(Y))$$

where $f\colon X \hookrightarrow Z$ and $g\colon Y \hookrightarrow Z$ are isometric embeddings.

Alternate (equivalent definitions)

- $d_{GH}(X, Y) = \inf_{d \in \mathcal{D}(d^X, d^Y)} d_H^{(X \sqcup Y, d)}(X, Y)$
- $d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R})$,
- $d_{\mathrm{GH}}(X, Y) = \frac{1}{2} \inf_{f,g} \max\{\operatorname{dis}(f), \operatorname{dis}(g), \operatorname{codis}(f, g)\}$

Here $\mathcal{R} \subseteq X \times Y$ is a **correspondence** and

$$\operatorname{dis}(\mathcal{R}) = \sup_{(x,y),(x',y') \in \mathcal{R}} |d^X(x, x') - d^Y(y, y')|.$$

A **zigzag path** btw x and $x^{'}$ is a sequence of d-paths $(\gamma_i)_{i=1}^{m} \mid \gamma_i \in \vec{P}(p_{i-1}, p_i) \cup \vec{P}(p_i, p_{i-1})$, $x = p_0$ and $x' = p_m$.

Definition 4 (Zigzag distance).

(X, d) metric space and $\vec{X}$ d-space. The **zigzag distance** induced by d is defined by

$$d_{ZZ}(x, x') = \inf_{\gamma \in \vec{P}_{ZZ}(x,x')} l_{ZZ}(\gamma).$$

Proposition 1.

1. *Let $\vec{X}$ be a d-space and d_{ZZ} the zigzag distance induced by d. Then*

$$d_{ZZ}(x, x') \geq d(x, x').$$

2. *The zigzag distance, d_{ZZ}, on $\vec{X}$ is an extended metric.*
3. *Every path γ is C^0 wrt d_{ZZ}.*

Given a directed metric space $(\vec{X}, d_{ZZ})$, we can construct a new directed metric space $(\vec{X}, (d_{ZZ})_{zz})$ where the d-paths are continuous with respect to d_{ZZ}. This construction can be iterated.

Proposition 2 (Idempotence).

Let $(\vec{X}, d_{ZZ})$ be a directed metric space. Then, $(d_{ZZ})_{zz} = d_{ZZ}$.

Change of convention! Topology now is wrt d_{ZZ}.

Definition 5.

Consider two directed metric spaces $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$. A d-map $\vec{F} \colon \vec{X} \to \vec{Y}$ is called a **d-isometry** if for any x, x' in X,

$$d^X_{ZZ}(x, x') = d^Y_{ZZ}(\vec{F}(x), \vec{F}(x')).$$

- Two d-spaces $\vec{X}$ and $\vec{Y}$ are **d-isometric** if there is a bijective d-isometry $\vec{F}\colon \vec{X} \to \vec{Y}$ such that its inverse $(F^{-1}\colon Y \to X)$ is a d-map.
- The **directed Hausdorff distance** of $\vec{X}$ and $\vec{Y}$ is defined by

$$\vec{d}_H(\vec{X}, \vec{Y}) = d_H((X, d_{ZZ}), (Y, d_{ZZ})). \tag{3}$$

Definition 6 (GH-distance version 1).

The **directed Gromov–Hausdorff distance** between two directed metric spaces $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ is given by:

$$\vec{d}_{GH}(\vec{X}, \vec{Y}) = \inf_{\vec{F}, \vec{G}} \vec{d}_H(f(\vec{X}), g(\vec{Y})), \tag{4}$$

where $\vec{F}\colon \vec{X} \to \vec{Z}$ and $\vec{G}\colon \vec{Y} \to \vec{Z}$ are directed isometries into some directed metric space $(\vec{Z}, d^Z_{ZZ})$.

Theorem 7.

Let $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ be directed metric spaces. Then

$$d_{GH}((X, d^X_{ZZ}), (Y, d^Y_{ZZ})) = \vec{d}_{GH}((\vec{X}, d^X_{ZZ}), (\vec{Y}, d^Y_{ZZ}))$$

Corollary 1.

The directed Gromov–Hausdorff distance is a metric on the space of isometry classes of compact directed metric spaces.

Note Corollary 1 requires compactness of the directed metric space $(\vec{X}, d_{ZZ})$.

Further results

- For any zigzag connected d-space, $\vec{d}_{GH}(\vec{X}, \overleftarrow{X}) = 0$.
- Let $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ be a compact directed metric space. Then
  1. $\vec{d}_{GH}(\vec{X}, \vec{Y}) \leq \frac{1}{2} \max\{\mathrm{Diam}(\vec{X}), \mathrm{Diam}(\vec{Y})\}$.
  2. $\vec{d}_{GH}(\vec{X}, \vec{Y}) = \frac{1}{2}\mathrm{Diam}(\vec{Y})$, if $X = \{x_0\}$.

Conjecture 1.

Let $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ be directed compact metric spaces, then

$$d_{GH}(X, Y) \leq \vec{d}_{GH}(\vec{X}, \vec{Y}).$$

If $\vec{F}\colon (\vec{X}, d_{ZZ}^X) \to (\vec{Y}, d_{ZZ}^Y)$ is a d-map, we define its distortion by

$$\operatorname{dis}(\vec{F}) = \sup_{x,x' \in \vec{X}} |d_{ZZ}^X(x,x') - d_{ZZ}^Y(\vec{F}(x), \vec{F}(x'))|. \tag{5}$$

Codistorsion is defined analogously.

Definition 8 (GH-distance version 2).

The **distortion distance** between directed metric spaces $(\vec{X}, d_{ZZ}^X)$ and $(\vec{Y}, d_{ZZ}^Y)$ is defined as

$$\vec{d}_{\text{dis}}(\vec{X}, \vec{Y}) = \frac{1}{2} \inf_{\vec{F},\vec{G}} \max\{\operatorname{dis}(\vec{F}), \operatorname{dis}(\vec{G}), \operatorname{codis}(\vec{F}, \vec{G})\}, \tag{6}$$

Properties

- **Triangle Inequality:** $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ directed metric spaces. Then,

$$\vec{d}_{\mathsf{dis}}(\vec{X}, \vec{Z}) \leq \vec{d}_{\mathsf{dis}}(\vec{X}, \vec{Y}) + \vec{d}_{\mathsf{dis}}(\vec{Y}, \vec{Z}).$$

- $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ compact directed metric spaces. Then,

$$\vec{d}_{\mathrm{dis}}(\vec{X}, \vec{Y}) \leq \frac{1}{2} \max\{\mathrm{diam}(\vec{X}), \mathrm{diam}(\vec{Y})\},$$

where $\mathrm{diam}(\vec{X}) = \mathrm{diam}(X, d^X_{ZZ})$ and $\mathrm{diam}(\vec{Y}) = \mathrm{diam}(Y, d^Y_{ZZ})$

Some Theorems

Theorem 9.

D-maps $\vec{F} : \vec{X} \to \vec{Y}$ *and* $\vec{G} : \vec{Y} \to \vec{X}$ *for which* $\mathrm{dis}(\vec{F}) = \mathrm{dis}(\vec{G}) = \mathrm{codis}(\vec{F}, \vec{G}) = 0$ *exist if and only if spaces* $\vec{X}$ *and* $\vec{Y}$ *are d-isometric.*

Proposition 3.

For any bounded directed metric spaces $\vec{X}$ *and* $\vec{Y}$*,*

$$\vec{d}_{GH}(\vec{X}, \vec{Y}) \leq \vec{d}_{\mathrm{dis}}(\vec{X}, \vec{Y}).$$

- $\vec{d}_{\mathrm{dis}}$ is an **extended pseudo-metric** on the space of d-isometry classes of directed metric spaces.

A **d-relation** $\mathcal{R}$ between $\vec{X}$ and $\vec{Y}$ is a relation between X and Y | $\forall (x, y)$ and $(x', y') \in \mathcal{R}$

- $\forall \gamma_1 \in \vec{P}(x, x')\ \exists \gamma_2 \in \vec{P}(y, y')$,
- similarly, $\forall \gamma_2 \in \vec{P}(y, y')\ \exists \gamma_1 \in \vec{P}(x, x')$.

d-correspondence = d-relation + correspondence.

Definition 10 (GH-distance version 3).

The **d-correspondence distortion distance** between directed metric spaces $(\vec{X}, d^X_{ZZ})$ and $(\vec{Y}, d^Y_{ZZ})$ is defined as

$$\vec{d}_{\text{c-dis}}(\vec{X}, \vec{Y}) = \frac{1}{2} \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R}), \tag{7}$$

where $\mathcal{R}$ varies over all the d-correspondences between $\vec{X}$ and $\vec{Y}$.

Proposition 4.

For every two compact directed metric spaces $\vec{X}$ and $\vec{Y}$,

$$\vec{d}_{GH}(\vec{X}, \vec{Y}) \leq \vec{d}_{\text{c-dis}}(\vec{X}, \vec{Y}).$$

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