Functor calculus and multiparameter persistence

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Persistent homology

$$\begin{array}{c}
\text{Data (e.g., point cloud } V \subset \mathbb{R}^n) \\
\big\downarrow \text{ Čech complex/Vietoris-Rips complex/etc} \\
\text{Filtration: functor } [0,\infty] \to \text{Simplicial complexes} \\
\big\downarrow C_* \\
\text{Functor } [0,\infty] \to \mathrm{Ch}_{\mathbb{F}} \\
\big\downarrow H_i \\
\text{Persistence module: functor } [0,\infty] \to \mathrm{Vec}_{\mathbb{F}} \\
\big\downarrow \textit{Structure theorem} \\
\text{Barcode diagram}
\end{array}$$

Multipersistent homology

- In multipersistent homology, persistence modules are replaced by *multipersistence modules*
  - A 1–parameter persistence module is a functor

$$F\colon [0,\infty] \to \mathsf{Vec}_{\mathbb{F}}$$

  - A multipersistence module is a functor

$$F\colon [0,\infty]^k \to \mathsf{Vec}_{\mathbb{F}},$$

  where $[0,\infty]^k$ has the product partial order.

No barcode for multipersistence

- Unfortunately, the structure theorem does not generalize to multipersistence modules. In other words, there are multipersistence modules that do not admit an interval decomposition.
  - Hence, there is no obvious way to analyze data from multipersistent homology

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"] & \mathbb{F}^2 \arrow[r, "(1\ 1)"] & \mathbb{F} \\
0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\begin{pmatrix}0\\1\end{pmatrix}"] \arrow[r, "="] & \mathbb{F} \arrow[u, "="']
\end{tikzcd}$$

Example from: M. Botnan and M. Lesnick, *An Introduction to Multiparameter Persistence* (2022)

Middle exactness

- However, a pointwise finite-dimensional multipersistence module from $[0,\infty]^2$ to $\mathrm{Vec}_{\mathbb{F}}$ is interval decomposable if it is *middle-exact* [1].
- A multipersistence module $F\colon [0,\infty]^2 \to \mathrm{Vec}_{\mathbb{F}}$ is said to be *middle-exact* if, for all $(x,y) \leq (x',y')$ in $[0,\infty]^2$, the complex

$$F(x,y) \to F(x,y') \oplus F(x',y) \to F(x',y')$$

  is exact in the middle.
- Middle-exact multipersistence modules arise naturally from interlevel set persistent homology [2].

[1] M.B. Botnan and W. Crawley-Boevey, *Decomposition of persistence modules* (2020)
[2] G. Carlsson, V. de Silva, and D. Morozov, *Zigzag persistent homology and real-valued functions* (2009)

Functor calculus: analogy from calculus

- In ordinary calculus, we can construct the Taylor expansion of an analytic function.

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

- This allows us to construct a sequence of *degree* n *polynomials*, $(f_n)_{n\in\mathbb{N}}$, converging to f:

$$f_n = \sum_{i=0}^{n} a_i x^i$$

Functor calculus: main idea

- Given categories $\mathscr{C}, \mathscr{D}$, one studies functors $F\colon \mathscr{C} \to \mathscr{D}$ through their *tower of degree n approximations*

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_2F \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] & T_1F
\end{tikzcd}$$

- What *degree n* means varies between different variants of functor calculus
- We speak of convergence when the homotopy limit of the tower is (weakly equivalent to) F

Cubical diagrams

Let $[k-1] = \{0, \ldots, k-1\}$, and let $\mathcal{P}_k$ be the power set of $[k-1]$ (viewed as a poset under inclusion). We call functors $\mathcal{X} \colon \mathcal{P}_k \to \mathscr{C}$ k*–cubes*.

$$\begin{tikzcd}
\mathcal{X}(\emptyset) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{X}(\{0\}) \arrow[dr] \arrow[dd] & \\
& \mathcal{X}(\{1\}) \arrow[rr] \arrow[dd] & & \mathcal{X}(\{0,1\}) \arrow[dd] \\
\mathcal{X}(\{2\}) \arrow[rr] \arrow[dr] & & \mathcal{X}(\{0,2\}) \arrow[dr] & \\
& \mathcal{X}(\{1,2\}) \arrow[rr] & & \mathcal{X}(\{0,1,2\})
\end{tikzcd}$$

Definition

A k–cube $\mathcal{X}\colon \mathcal{P}_k \to \mathscr{C}$ is *cocartesian* if the canonical map

$$\operatorname*{colim}_{S \subsetneq [k-1]} \mathcal{X}(S) \to \mathcal{X}([k-1])$$

is an isomorphism.
Similarly, $\mathcal{X}$ is *cartesian* if the canonical map

$$\mathcal{X}(\emptyset) \to \lim_{S \subseteq [k-1], S \neq \emptyset} \mathcal{X}(S)$$

is an isomorphism.

Definition

A k–cube $\mathcal{X} \colon \mathcal{P}_k \to \mathscr{C}$ is *strongly cocartesian* if each face of dimension ≥ 2 is cocartesian. It is *strongly cartesian* if each face of dimension ≥ 2 is cartesian.

Definition

A k–cube is *strongly bicartesian* if it is both strongly cartesian and strongly cocartesian.

Definition

Let $\mathcal{M}$ be a model category. A k–cube $\mathcal{X}\colon \mathcal{P}_k \to \mathcal{M}$ is *homotopy cocartesian* if the canonical map

$$\operatorname*{hocolim}_{S \subsetneq [k-1]} \mathcal{X}(S) \to \mathcal{X}([k-1])$$

is a weak equivalence.

Similar for homotopy cartesian, strongly homotopy cocartesian, etc.

Functor calculus: examples

- **Goodwillie calculus [1]:** A functor is degree n if it sends strongly homotopy cocartesian $(n+1)$–cubes to homotopy cartesian $(n+1)$–cubes
- **Abelian calculus [2]:** A functor is degree n if it sends a specific class of strongly homotopy cocartesian $(n+1)$–cubes to homotopy cartesian $(n+1)$–cubes
- **Manifold calculus [3]:** A functor is degree n if it sends strongly bicartesian $(n+1)$–cubes (of open subsets) to homotopy cartesian $(n+1)$–cubes (of spaces)

[1] T. Goodwillie, *Calculus. I. The first derivative of pseudoisotopy theory* (1990)
[2] B. Johnson and R. McCarthy, *Deriving calculus with cotriples* (2004)
[3] T. Goodwillie and M. Weiss, *Embeddings from the point of view of immersion theory: Part ii* (1999)

Functor cocalculus

- Functor calculus, except that the arrows in the tower go the other way [1, 2]
  - Instead of a tower, we get a *telescope* of approximations

$$\begin{tikzcd}
\vdots \arrow[ddr] & \\
T_2F \arrow[u] \arrow[dr] & \\
T_1F \arrow[u] \arrow[r] & F
\end{tikzcd}$$

[1] R. McCarthy, *Dual calculus for functors to spectra* (2001)
[2] K. Bauer, R. Brooks, K. Hess, B. Johnson, J. Rasmusen and B. Schreiner, *Constructing monads from cubical diagrams and homotopy colimits* (2024)

Poset cocalculus: a functor cocalculus framework for studying functors out of posets

- such as functors from $[0, \infty]^k$ to $\mathrm{Ch}_{\mathbb{F}}$

Cubes of posets

- In a poset, the colimit of a finite diagram (if it exists) is the supremum of the elements in the diagram. Likewise, the limit is given by the infimum.
- A *lattice* is a poset where the supremum $x \vee y$ and infimum $x \wedge y$ of two elements always exist
  - In other words, a lattice is a poset where all nonempty finite colimits and nonempty finite limits exist

$$\begin{tikzcd} y \arrow[r] & x \vee y \\ x \wedge y \arrow[u] \arrow[r] & x \arrow[u] \end{tikzcd}$$

A strongly bicartesian 2–cube in a lattice.

Poset cocalculus

We introduce the following definition.

Definition

Let P be a lattice and $\mathcal{M}$ a model category. We say that a functor $F\colon P \to \mathcal{M}$ is *codegree* n if it takes strongly bicartesian $(n+1)$–cubes to homotopy cocartesian $(n+1)$–cubes.

Dimension of an element

We restrict ourself to the case where the source poset P is a product of total orders.

Definition

Let $P = P_1 \times \cdots \times P_k$, where each P_i is a total order with a minimal element 0. The *dimension* of an element $(x_1, \ldots, x_k)$ is defined as its number of nonzero components, i.e.,

$$\dim(x_1, \ldots, x_k) = \#\{i : x_i \neq 0\}.$$

Remark: The theory also works when P is a *distributive lattice*.

Codegree n approximations

Let P be a product of total orders with minimal elements, $\mathcal{M}$ a model category, and $F\colon P \to \mathcal{M}$ a functor. Let $P_{\leq n} \subseteq P$ be the subposet $P_{\leq n} = \{x \in P : \dim(x) \leq n\}$.

Definition

We define the *codegree n approximation* of F, denoted T_nF, as the homotopy left Kan extension of $F|_{P_{\leq n}}$ along the inclusion $i\colon P_{\leq n} \hookrightarrow P$.

$$\begin{tikzcd}
 & P \arrow[dr, dashed, "T_nF"] & \\
P_{\leq n} \arrow[ur, hook, "i"] \arrow[rr, "F|_{P_{\leq n}}"] & & \mathcal{M}
\end{tikzcd}$$

Explicitly,

$$T_nF(x) = \operatorname*{hocolim}_{y \leq x, \dim(y) \leq n} F(y).$$

Codegree n approximations

$$\begin{tikzcd}
\vdots \\
T_2F \arrow[u] \arrow[ddr] & \\
T_1F \arrow[u] \arrow[dr] & \\
T_0F \arrow[u] \arrow[r] & F
\end{tikzcd}$$

Codegree n approximations

Let P be a product of total orders with minimal elements, $\mathcal{M}$ a model category*, and $F\colon P \to \mathcal{M}$ a functor.

Theorem (Theorem A)

T_nF is a codegree n functor.

Theorem (Theorem B)

Suppose that P is a finite product of total orders. If F is codegree n then $T_nF \simeq F$.

(*) subject to certain mild conditions

Examples

Consider the poset $P = \{0,1\}^2$:

$$\begin{tikzcd} (0,1) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] & (1,0) \arrow[u] \end{tikzcd}$$

- A codegree 1 functor from P to $\mathrm{Ch}_{\mathbb{F}}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"'] \end{tikzcd}$$

Examples

- A codegree 2 functor, $F\colon P \to \mathrm{Ch}_{\mathbb{F}}$:

$$\begin{tikzcd} \mathbb{F} \arrow[r, "1"] & \mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"'] \end{tikzcd}$$

Examples

- A codegree 2 functor, $F\colon P \to \mathrm{Ch}_{\mathbb{F}}$:

$$\begin{tikzcd} \mathbb{F} \arrow[r, "1"] & \mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "1"'] \end{tikzcd}$$

- $T_1 F$:

$$\begin{tikzcd} \mathbb{F} \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \mathbb{F} \oplus \mathbb{F} \\ 0 \arrow[u] \arrow[r] & \mathbb{F} \arrow[u, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"'] \end{tikzcd}$$

Relation to middle exactness

Let $\mathrm{Ch}_{\mathbb{F}}$ denote the category of unbounded chain complexes over the field $\mathbb{F}$, equipped with the canonical model structure*.

Theorem

Let $F\colon [0,\infty]^2 \to \mathsf{Vec}_{\mathbb{F}}$ be a pointwise finite-dimensional functor. The following are equivalent.

(i) *F is middle-exact.*

(ii) *There exists a codegree 1 functor $\widehat{F}\colon [0,\infty]^2 \to \mathrm{Ch}_{\mathbb{F}}$ such that $F \cong H_0 \circ \widehat{F}$.*

(*) See, e.g., Theorem 2.3.13 in: M. Hovey, *Model categories* (1999)

Stability

Let d_H^Λ be the *multiplicative interleaving distance*.

Proposition

For all pairs of functors $F, G\colon [0,\infty)^k \to \mathrm{Ch}_{\mathbb{F}}$,

$$d_H^\Lambda(T_1F, T_1G) \leq d_H^\Lambda(T_2F, T_2G) \leq \cdots \leq d_H^\Lambda(F, G).$$

We can also consider functors to any category!

Filtrations of simplicial complexes

Let V be a vertex set. Given a functor $f\colon \mathcal{P}(V) \to \mathbb{R}$, we can define a *filtration* on $\mathcal{P}(V)$ as

$$\begin{aligned} F_f\colon \mathbb{R} &\to \text{Simplicial complexes} \\ t &\mapsto \{\sigma \in \mathcal{P}(V) : f(\sigma) \leq t\}. \end{aligned}$$

Observe that the poset $\mathcal{P}(V)$ is isomorphic to the product $\{0,1\}^{|V|}$. We can study the functor f using poset cocalculus.

Let $V \subset \mathbb{R}^n$ be a finite point cloud.
The *Čech filtration* of V is the filtration $F_{f_{\mathbf{C}}}$ induced by the functor

$$\begin{aligned} f_{\mathbf{C}} \colon \mathcal{P}(V) &\to \mathbb{R} \\ U &\mapsto \inf\left\{\varepsilon \;:\; \exists z \in \mathbb{R}^n \text{ such that } U \subseteq \overline{B_\varepsilon}(z)\right\}. \end{aligned}$$

The *Vietoris-Rips filtration* of V is the filtration $F_{f_{\mathbf{VR}}}$ induced by

$$\begin{aligned} f_{\mathbf{VR}} \colon \mathcal{P}(V) &\to \mathbb{R} \\ U &\mapsto \max\{d(x,y) : x, y \in U\}. \end{aligned}$$

Let $V \subset \mathbb{R}^n$ be a finite point cloud.
The *Čech filtration* of V is the filtration $F_{f_{\mathbf{C}}}$ induced by the functor

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The *Vietoris-Rips filtration* of V is the filtration $F_{f_{\mathbf{VR}}}$ induced by

$$\begin{aligned} f_{\mathbf{VR}} \colon \mathcal{P}(V) &\to \mathbb{R} \\ U &\mapsto \max \{ d(x,y) : x, y \in U \}. \end{aligned}$$

Proposition

$$f_{\mathbf{VR}} = 2 \cdot T_2 f_{\mathbf{C}}$$

Thank you!

References:

1. B.G. Hem, *Poset functor cocalculus and applications to topological data analysis* (2025) arXiv:2501.05996
2. B.G. Hem, *Decomposing multipersistence modules using functor calculus* (2025) arXiv:2510.06178