Persistent Cohomology Operations and Gromov–Hausdorff Estimates

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Part I

Computing Steenrod Barcodes

(Review)

Structure on cohomology

As graded vector spaces

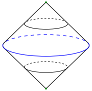
$$\mathrm{H}^\bullet(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \cong \mathrm{H}^\bullet(S^1 \vee S^2;\mathbb{F}_2).$$

Similarly, as graded abelian groups

$$\mathrm{H}^\bullet(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) \cong \mathrm{H}^\bullet(S^2 \vee S^4;\mathbb{Z}).$$

These can be distinguished by the product structure $\mathrm{H}^\bullet \otimes \mathrm{H}^\bullet \to \mathrm{H}^\bullet$.

Let Σ denote suspension, for example $\Sigma(S^1)$ is



As graded rings

$$\mathrm{H}^\bullet(\Sigma(\mathbb{C}\mathrm{P}^2)) \cong \mathrm{H}^\bullet(\Sigma(S^2 \vee S^4)).$$

These can be distinguished by the Steenrod square $\mathrm{Sq}^2\colon \mathrm{H}^\bullet \to \mathrm{H}^\bullet$.

Alexander–Whitney diagonal

The product structure comes from a chain-level approximation

$$\Delta_0 \colon \mathrm{C}(\mathbb{\Delta}^n) \to \mathrm{C}(\mathbb{\Delta}^n) \otimes \mathrm{C}(\mathbb{\Delta}^n)$$

to the diagonal map.

Example

$$\Delta_0[0,1,2] = [0] \otimes [0,1,2] + [0,1] \otimes [1,2] + [0,1,2] \otimes [2].$$

The cup product on cohomology is defined by

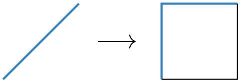
$$\begin{aligned} [\alpha][\beta] &= \big[(\alpha \smile_0 \beta)\big] \\ &\overset{\text{def}}{=} \big[(\alpha \otimes \beta) \circ \Delta_0\big]. \end{aligned}$$

Steenrod construction

Unlike the diagonal of spaces, Δ_0 is **not** invariant under

$$x \otimes y \overset{T}{\mapsto} y \otimes x.$$

For example, $\Delta_0 \colon \mathrm{C}(\mathbb{\Delta}^1) \to \mathrm{C}(\mathbb{\Delta}^1) \otimes \mathrm{C}(\mathbb{\Delta}^1)$ looks like



To correct homotopically the breaking of this symmetry, Steenrod introduced **explicit** maps

$$\Delta_i \colon \mathrm{C}(\mathbb{\Delta}^n) \to \mathrm{C}(\mathbb{\Delta}^n)^{\otimes 2} \quad \text{satisfying} \quad \partial \Delta_i = \big(1 - T\big)\Delta_{i-1}.$$

These define the Steenrod squares explicitly by

$$\begin{aligned} \mathrm{Sq}^k \colon \mathrm{H}^\bullet(X; \mathbb{F}_2) &\to \mathrm{H}^\bullet(X; \mathbb{F}_2) \\ [\alpha] &\mapsto \big[\alpha \smile_i \alpha\big] = \big[(\alpha \otimes \alpha) \circ \Delta_i\big] \end{aligned}$$

A new description of Steenrod's cup-i products

Example

$$\Delta_0[0,1,2] = [0] \otimes [0,1,2] + [0,1] \otimes [1,2] + [0,1,2] \otimes [2]$$
$$= \Big(d_{12} \otimes \mathsf{id} + d_2 \otimes d_0 + \mathsf{id} \otimes d_{01}\Big)[0,1,2]^{\otimes 2}.$$

Notation

$$\forall n, q \in \mathbb{N}, \quad \mathrm{P}^n_q = \big\{U \subseteq \{0,\dots,n\} : |U| = q\big\}$$
$$\forall U = \{u_1 < \cdots < u_q\} \in \mathrm{P}^n_q, \quad d_U = d_{u_1} \cdots d_{u_q}$$

Defn.(Med.) For a basis element $x \in \mathrm{C}_n(X, \mathbb{F}_2)$

$$\Delta_i(x) = \sum d_{U^0}(x) \otimes d_{U^1}(x)$$

where the sum is over P^n_{n-i} and $U^\varepsilon = \{u_j \in U \mid u_j + j \equiv \varepsilon \text{ mod } 2\}$.

Thm.(Med.) All cup-i constructions in the literature are isomorphic, i.e.,

For each $i \in \mathbb{N}$, either $\triangle_i = \triangle'_i$ or $\triangle_i = T\triangle'_i$.

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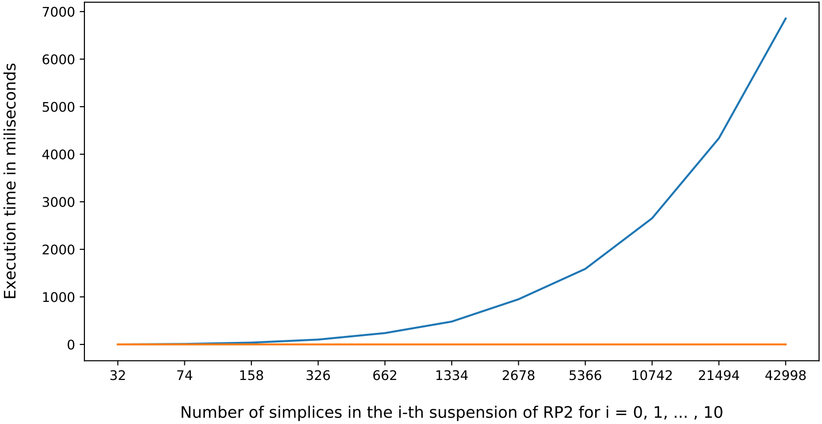
Fast computation of Steenrod squares

Comparing with SAGE: (algorithm based on EZ-AW contraction)

Sq^1 on $\Sigma^i \mathbb{R}P^2$ (i^{th} suspension of the real projective plane)



Vietoris–Rips filtration

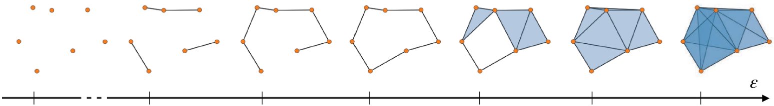
Let $(\mathcal{X}, d)$ be a compact metric space, e.g. a point cloud.

For any $t \in \mathbb{R}$ let $\mathrm{VR}_t(\mathcal{X})$ be the simplicial complex with a d-simplex $[x_0, x_1, \ldots, x_d]$ iff

$$\forall i, j, \quad d(x_i, x_j) \leq t.$$

The *filtered simplicial complex* $\mathrm{VR}(\mathcal{X})$ is the Vietoris–Rips filtration of $\mathcal{X}$.

For a point cloud we have



$$\mathrm{VR}_{t_0}(\mathcal{X}) \subset \mathrm{VR}_{t_1}(\mathcal{X}) \subset \cdots \subset \mathrm{VR}_{t_n}(\mathcal{X}).$$

Steenrod barcodes

Given a filtered simplicial complex

$$X_0 \to X_1 \to \cdots \to X_n.$$

Cohomology induces a persistence module.

$$\mathrm{H}^\bullet(X_n;\mathbb{F}_2) \longrightarrow \cdots \longrightarrow \mathrm{H}^\bullet(X_1;\mathbb{F}_2) \longrightarrow \mathrm{H}^\bullet(X_0;\mathbb{F}_2)$$

Steenrod barcodes

Given a filtered simplicial complex

$$X_0 \to X_1 \to \cdots \to X_n.$$

Cohomology induces a persistence module.

A Steenrod square induces an endomorphism

$$\begin{tikzcd}
\mathrm{H}^\bullet(X_n;\mathbb{F}_2) \arrow[r] & \cdots \arrow[r] & \mathrm{H}^\bullet(X_1;\mathbb{F}_2) \arrow[r] & \mathrm{H}^\bullet(X_0;\mathbb{F}_2) \\
\mathrm{H}^\bullet(X_n;\mathbb{F}_2) \arrow[r] \arrow[u, "\mathrm{Sq}^k"] & \cdots \arrow[r] & \mathrm{H}^\bullet(X_1;\mathbb{F}_2) \arrow[r] \arrow[u, "\mathrm{Sq}^k"] & \mathrm{H}^\bullet(X_0;\mathbb{F}_2). \arrow[u, "\mathrm{Sq}^k"]
\end{tikzcd}$$

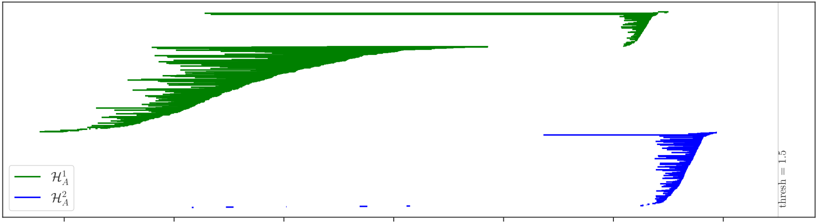
The $\operatorname{img}\mathrm{Sq}^k$-barcode of X is the barcode of the image of this map.

With *U. Lupo* and *G. Tauzin* we developed `steenroder` to compute these.

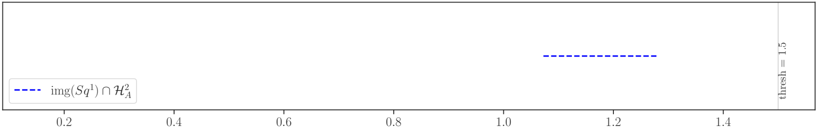
Space of conformations of C_8H_{16}

Points in $\mathbb{R}^{24}$ (positions of 8 carbons in $\mathbb{R}^3$)

H^1 (green) and H^2 (blue) barcodes of (part of) this point cloud



Sq^1-barcode



Consistent with a Klein bottle.

Part II

Persistent Cohomology Operations and Gromov–Hausdorff Distance

(New work with Ling Zhou)

Persistent cohomology operations

Let $\Bbbk$ be a field.

A $\Bbbk$-linear cohomology operation θ is a natural transformation

$$\theta\colon \mathrm{H}^{\ell}(-;\Bbbk) \to \mathrm{H}^{m}(-;\Bbbk).$$

If X is a functor from $\mathbb{R}$ to cellular spaces,

$$\mathrm{img}_{\theta}(X) \leq \mathrm{H}^{m}(X;\Bbbk), \qquad \ker_{\theta}(X) \leq \mathrm{H}^{\ell}(X;\Bbbk)$$

are persistent modules. Focus on $\mathrm{img}_{\theta}(X)$, all statements hold for both.

We will focus mostly on the case when X is the Vietoris–Rips filtration $\mathrm{VR}(\mathcal{X})$ of a metric space $\mathcal{X}$. In this case, we write

$$\mathrm{H}_{m}^{\mathrm{VR}}(\mathcal{X}) \overset{\mathrm{def}}{=} \mathrm{H}_{m}(\mathrm{VR}(\mathcal{X});\Bbbk), \qquad \mathrm{img}_{\theta}^{\mathrm{VR}}(\mathcal{X}) \overset{\mathrm{def}}{=} \mathrm{img}_{\theta}(\mathrm{VR}(\mathcal{X})).$$

Decomposition theorems

Thm.(Med.–Zhou) For pointed metric spaces, with the gluing metric,

$$\mathrm{img}_\theta^{\mathrm{VR}}(\mathcal{X} \vee \mathcal{Y}) \cong \mathrm{img}_\theta^{\mathrm{VR}}(\mathcal{X}) \oplus \mathrm{img}_\theta^{\mathrm{VR}}(\mathcal{Y}).$$

Proof sketch. [Adamaszek et al. 20] $\mathrm{VR}_r(\mathcal{X}) \vee \mathrm{VR}_r(\mathcal{Y}) \to \mathrm{VR}_r(\mathcal{X} \vee \mathcal{Y})$ is a natural homotopy equivalence for the gluing metric. Cohomology operations respect the natural direct sum split of cohomology on wedges.

Thm.(Med.–Zhou) For metric spaces, with the ℓ^∞ metric,

$$\mathrm{img}_\mathrm{P}^{\mathrm{VR}}(\mathcal{X} \times \mathcal{Y}) \cong \mathrm{img}_\mathrm{P}^{\mathrm{VR}}(\mathcal{X}) \otimes \mathrm{img}_\mathrm{P}^{\mathrm{VR}}(\mathcal{Y}),$$

where P is a total Steenrod operation (over any field $\mathbb{F}_p$).

Proof sketch. [Adamaszek–Adams 17, Lim et al. 24] There is a natural isomorphism $H^{\mathrm{VR}}(\mathcal{X}) \otimes H^{\mathrm{VR}}(\mathcal{Y}) \cong H^{\mathrm{VR}}(\mathcal{X} \times \mathcal{Y})$. The Cartan formula provides a natural isomorphism.

Stability

Recall the stability of persistent homology:

$$d_{\mathrm{I}}\big(\mathrm{H}_m^{\mathrm{VR}}(\mathcal{X}),\, \mathrm{H}_m^{\mathrm{VR}}(\mathcal{Y})\big) \leq 2 \cdot d_{\mathrm{GH}}(\mathcal{X},\mathcal{Y}),$$

where d_{I} and d_{GH} are the interleaving and Gromov–Hausdorff distances.

Thm.(Med.–Zhou) For any linear cohomology operation θ:

$$d_{\mathrm{I}}\big(\operatorname{img}_\theta^{\mathrm{VR}}(\mathcal{X}),\, \operatorname{img}_\theta^{\mathrm{VR}}(\mathcal{Y})\big) \leq 2 \cdot d_{\mathrm{GH}}(\mathcal{X},\mathcal{Y}).$$

Proof sketch. [Blumberg–Lesnick 23] Their homotopy interleaving is s.t.

$$d_{\mathrm{HI}}\big(\mathrm{VR}(\mathcal{X}), \mathrm{VR}(\mathcal{Y})\big) \leq 2 \cdot d_{\mathrm{GH}}(\mathcal{X},\mathcal{Y}).$$

(2) Some work shows

$$d_{\mathrm{I}}\big(\operatorname{img}_\theta^{\mathrm{VR}}(\mathcal{X}),\, \operatorname{img}_\theta^{\mathrm{VR}}(\mathcal{Y})\big) \leq d_{\mathrm{HI}}\big(\mathrm{VR}(\mathcal{X}), \mathrm{VR}(\mathcal{Y})\big).$$

Question: Does the additional homotopical information of cohomology operations manifest itself as sharper bounds?

Answer

For every $n > 1$ the following inequalities hold:

1. For any $m \in \mathbb{N}$

$$d_{\mathrm{I}}\Big(\mathrm{H}_m^{\mathrm{VR}}(\mathbb{RP}^n),\, \mathrm{H}_m^{\mathrm{VR}}(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^n)\Big) < \frac{\pi}{4}.$$

2. There is $k \in \mathbb{N}$ such that

$$d_{\mathrm{I}}\Big(\mathrm{img}_{\mathrm{Sq}^k}^{\mathrm{VR}}(\mathbb{RP}^n),\, \mathrm{img}_{\mathrm{Sq}^k}^{\mathrm{VR}}(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^n)\Big) \geq \frac{\pi}{3},$$

   where Sq^k is the k^{th} Steenrod square.

Here $\mathbb{RP}^n$, with diameter π, is the quotient of a round sphere under the antipodal action, and each sphere in the wedge sum is round of diameter π.

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Proof ingredients (time permitting)

(1) Connection between Kuratowski embedding and VR filtrations. ([Hausmann 95, Lim et al. 24]).

(2) Filling radius of $\mathbb{S}^n$ and $\mathbb{RP}^n$ are known. ([Gromov 83, Katz 83]).

(3) $\exists$ natural homotopy equivalence $\mathrm{VR}_r(\mathbb{S}^n) \to \mathbb{S}^n$ for $r \in (0, \zeta_n]$. ([Adamaszek et al. 18, Gil 24]).

(4) Group G acting "nicely" on $\mathcal{X}$. $\exists$ natural homotopy equivalence $\mathrm{VR}_r(\mathcal{X}_G) \to \mathrm{VR}_r(\mathcal{X})_G$ for small r. ([Adams et al. 22, Barham 24]).

Key lemma. Given "nice" G-action on an equatorial system

$$\mathbb{S}^{n_1} \to \mathbb{S}^{n_2} \to \mathbb{S}^{n_3} \to \cdots .$$

If $\mathrm{Rad}_{\mathrm{fill}}\left(\mathbb{S}^{n_i}_G\right)$ is non-decreasing as a function of i, then, for any $i \leq j$,

$$\mathrm{Rad}_{\mathrm{H}^{n_i}}\left(\mathbb{S}^{n_j}_G\right) \leq \mathrm{Rad}_{\mathrm{fill}}\left(\mathbb{S}^{n_j}_G\right).$$

Remark. These ingredients could also be used to get a similar statement for Lens spaces, but we do not know their filling radii.

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