

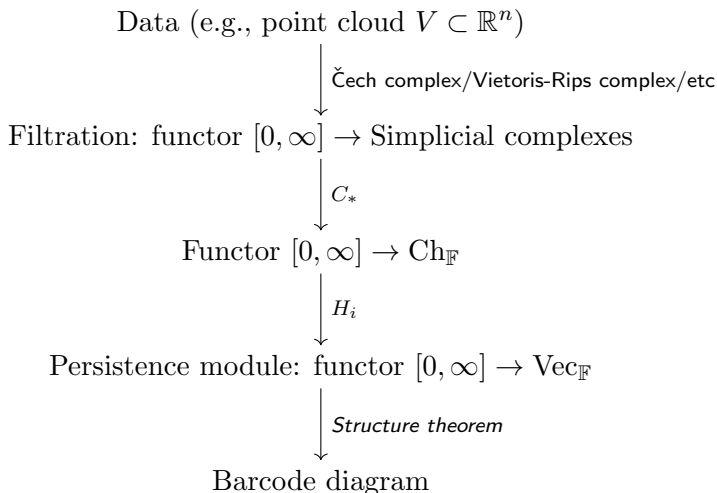
Functor calculus and multiparameter persistence

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Persistent homology



Multipersistent homology

- ▶ In multipersistent homology, persistence modules are replaced by *multipersistence modules*
 - ▶ A 1-parameter persistence module is a functor

$$F: [0, \infty] \rightarrow \text{Vec}_{\mathbb{F}}$$

- ▶ A multipersistence module is a functor

$$F: [0, \infty]^k \rightarrow \text{Vec}_{\mathbb{F}},$$

where $[0, \infty]^k$ has the product partial order.

Middle exactness

- ▶ However, a pointwise finite-dimensional multipersistence module from $[0, \infty]^2$ to $\text{Vec}_{\mathbb{F}}$ is interval decomposable if it is *middle-exact* [1].
- ▶ A multipersistence module $F: [0, \infty]^2 \rightarrow \text{Vec}_{\mathbb{F}}$ is said to be *middle-exact* if, for all $(x, y) \leq (x', y')$ in $[0, \infty]^2$, the complex

$$F(x, y) \rightarrow F(x, y') \oplus F(x', y) \rightarrow F(x', y')$$

is exact in the middle.

- ▶ Middle-exact multipersistence modules arise naturally from interlevel set persistent homology [2].

[1] M.B. Botnan and W. Crawley-Boevey, *Decomposition of persistence modules* (2020)

[2] G. Carlsson, V. de Silva, and D. Morozov, *Zigzag persistent homology and real-valued functions* (2009)

Functor calculus: analogy from calculus

- ▶ In ordinary calculus, we can construct the Taylor expansion of an analytic function.

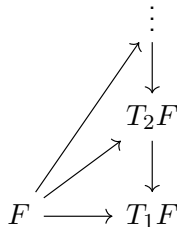
$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$

- ▶ This allows us to construct a sequence of *degree n polynomials*, $(f_n)_{n \in \mathbb{N}}$, converging to f :

$$f_n = \sum_{i=0}^n a_i x^i$$

Functor calculus: main idea

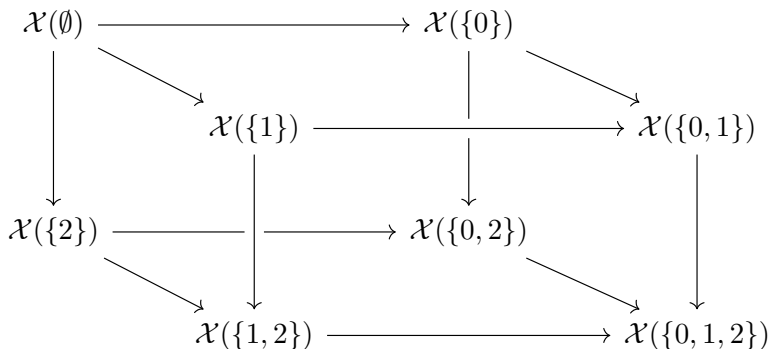
- ▶ Given categories \mathcal{C}, \mathcal{D} , one studies functors $F: \mathcal{C} \rightarrow \mathcal{D}$ through their *tower of degree n approximations*



- ▶ What *degree n* means varies between different variants of functor calculus
- ▶ We speak of convergence when the homotopy limit of the tower is (weakly equivalent to) F

Cubical diagrams

Let $[k-1] = \{0, \dots, k-1\}$, and let \mathcal{P}_k be the power set of $[k-1]$ (viewed as a poset under inclusion). We call functors $\mathcal{X}: \mathcal{P}_k \rightarrow \mathcal{C}$ *k-cubes*.



Definition

A k -cube $\mathcal{X}: \mathcal{P}_k \rightarrow \mathcal{C}$ is *cocartesian* if the canonical map

$$\operatorname{colim}_{S \subsetneq [k-1]} \mathcal{X}(S) \rightarrow \mathcal{X}([k-1])$$

is an isomorphism.

Similarly, \mathcal{X} is *cartesian* if the canonical map

$$\mathcal{X}(\emptyset) \rightarrow \lim_{S \subseteq [k-1], S \neq \emptyset} \mathcal{X}(S)$$

is an isomorphism.

Definition

A k -cube $\mathcal{X}: \mathcal{P}_k \rightarrow \mathcal{C}$ is *strongly cocartesian* if each face of dimension ≥ 2 is cocartesian. It is *strongly cartesian* if each face of dimension ≥ 2 is cartesian.

Definition

A k -cube is *strongly bicartesian* if it is both strongly cartesian and strongly cocartesian.

Definition

Let \mathcal{M} be a model category. A k -cube $\mathcal{X}: \mathcal{P}_k \rightarrow \mathcal{M}$ is *homotopy cocartesian* if the canonical map

$$\operatorname{hocolim}_{S \subsetneq [k-1]} \mathcal{X}(S) \rightarrow \mathcal{X}([k-1])$$

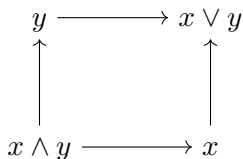
is a weak equivalence.

Similar for homotopy cartesian, strongly homotopy cocartesian, etc.

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Cubes of posets

- ▶ In a poset, the colimit of a finite diagram (if it exists) is the supremum of the elements in the diagram. Likewise, the limit is given by the infimum.
- ▶ A *lattice* is a poset where the supremum $x \vee y$ and infimum $x \wedge y$ of two elements always exist
 - ▶ In other words, a lattice is a poset where all nonempty finite colimits and nonempty finite limits exist



A strongly bicartesian 2-cube in a lattice.

Poset cocalculus

We introduce the following definition.

Definition

Let P be a lattice and \mathcal{M} a model category. We say that a functor $F: P \rightarrow \mathcal{M}$ is *codegree n* if it takes strongly bicartesian $(n+1)$ -cubes to homotopy cocartesian $(n+1)$ -cubes.

Dimension of an element

We restrict ourself to the case where the source poset P is a product of total orders.

Definition

Let $P = P_1 \times \cdots \times P_k$, where each P_i is a total order with a minimal element 0. The *dimension* of an element (x_1, \dots, x_k) is defined as its number of nonzero components, i.e.,

$$\dim(x_1, \dots, x_k) = \#\{i : x_i \neq 0\}.$$

Remark: The theory also works when P is a *distributive lattice*.

Codegree n approximations

Let P be a product of total orders with minimal elements, \mathcal{M} a model category, and $F: P \rightarrow \mathcal{M}$ a functor. Let $P_{\leq n} \subseteq P$ be the subposet $P_{\leq n} = \{x \in P : \dim(x) \leq n\}$.

Definition

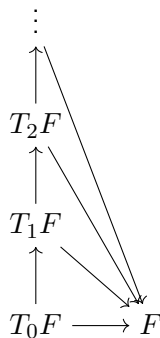
We define the *codegree n approximation* of F , denoted $T_n F$, as the homotopy left Kan extension of $F|_{P_{\leq n}}$ along the inclusion $i: P_{\leq n} \hookrightarrow P$.

$$\begin{array}{ccc}
 & P & \\
 i \nearrow & & \searrow T_n F \\
 P_{\leq n} & \xrightarrow{F|_{P_{\leq n}}} & \mathcal{M}
 \end{array}$$

Explicitly,

$$T_n F(x) = \operatorname{hocolim}_{y \leq x, \dim(y) \leq n} F(y).$$

Codegree n approximations



Codegree n approximations

Let P be a product of total orders with minimal elements, \mathcal{M} a model category*, and $F: P \rightarrow \mathcal{M}$ a functor.

Theorem (Theorem A)

$T_n F$ is a codegree n functor.

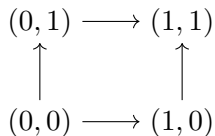
Theorem (Theorem B)

Suppose that P is a finite product of total orders. If F is codegree n then $T_n F \simeq F$.

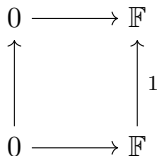
(*) subject to certain mild conditions

Examples

Consider the poset $P = \{0, 1\}^2$:



- A codegree 1 functor from P to $\text{Ch}_{\mathbb{F}}$:



Relation to middle exactness

Let $\text{Ch}_{\mathbb{F}}$ denote the category of unbounded chain complexes over the field \mathbb{F} , equipped with the canonical model structure*.

Theorem

Let $F: [0, \infty]^2 \rightarrow \text{Vec}_{\mathbb{F}}$ be a pointwise finite-dimensional functor. The following are equivalent.

- (i) F is middle-exact.
- (ii) There exists a codegree 1 functor $\hat{F}: [0, \infty]^2 \rightarrow \text{Ch}_{\mathbb{F}}$ such that $F \cong H_0 \circ \hat{F}$.

(*) See, e.g., Theorem 2.3.13 in: M. Hovey, *Model categories* (1999)

Stability

Let d_H^Λ be the *multiplicative interleaving distance*.

Proposition

For all pairs of functors $F, G: [0, \infty)^k \rightarrow \mathbf{Ch}_\mathbb{F}$,

$$d_H^\Lambda(T_1 F, T_1 G) \leq d_H^\Lambda(T_2 F, T_2 G) \leq \cdots \leq d_H^\Lambda(F, G).$$

We can also consider functors to any category!

Filtrations of simplicial complexes

Let V be a vertex set. Given a functor $f: \mathcal{P}(V) \rightarrow \mathbb{R}$, we can define a *filtration* on $\mathcal{P}(V)$ as

$$F_f: \mathbb{R} \rightarrow \text{Simplicial complexes} \\ t \mapsto \{\sigma \in \mathcal{P}(V) : f(\sigma) \leq t\}.$$

Observe that the poset $\mathcal{P}(V)$ is isomorphic to the product $\{0, 1\}^{|V|}$. We can study the functor f using poset cocalculus.

Let $V \subset \mathbb{R}^n$ be a finite point cloud.

The *Čech filtration* of V is the filtration F_{f_C} induced by the functor

$$f_C: \mathcal{P}(V) \rightarrow \mathbb{R}$$

$$U \mapsto \inf \{ \varepsilon : \exists z \in \mathbb{R}^n \text{ such that } U \subseteq \overline{B_\varepsilon(z)} \}.$$

The *Vietoris-Rips filtration* of V is the filtration $F_{f_{VR}}$ induced by

$$f_{VR}: \mathcal{P}(V) \rightarrow \mathbb{R}$$

$$U \mapsto \max \{ d(x, y) : x, y \in U \}.$$

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Proposition

$$f_{VR} = 2 \cdot T_2 f_C$$

Thank you!

References:

1. B.G. Hem, *Poset functor cocalculus and applications to topological data analysis* (2025) arXiv:2501.05996
2. B.G. Hem, *Decomposing multipersistence modules using functor calculus* (2025) arXiv:2510.06178