Computing Stabilized Decompositions

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TU Graz

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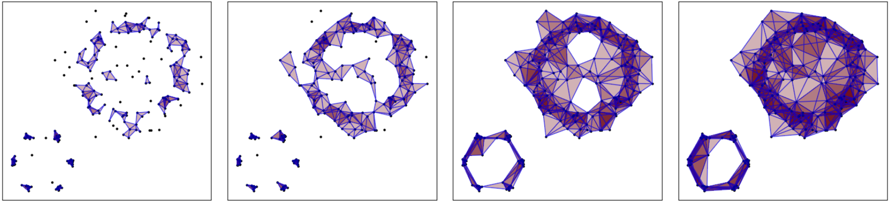
Structure

Persistence under noise

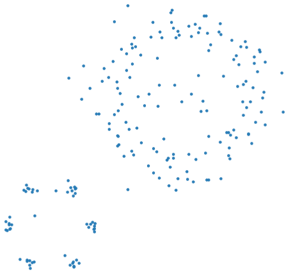
Goal

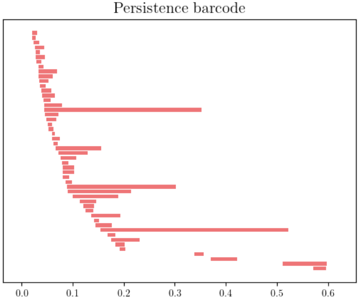
Estimate the homology groups of a noisy sample.

Alpha Complex



Persistent Homology under noise





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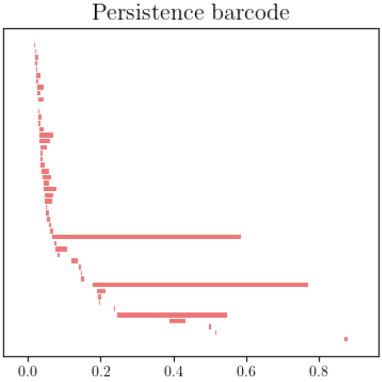
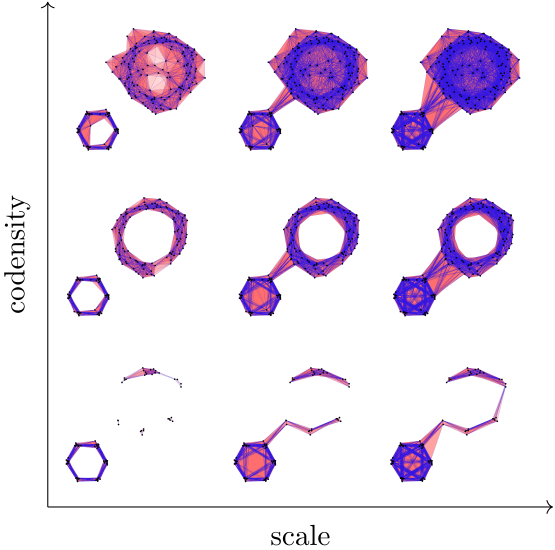


Bifiltrations



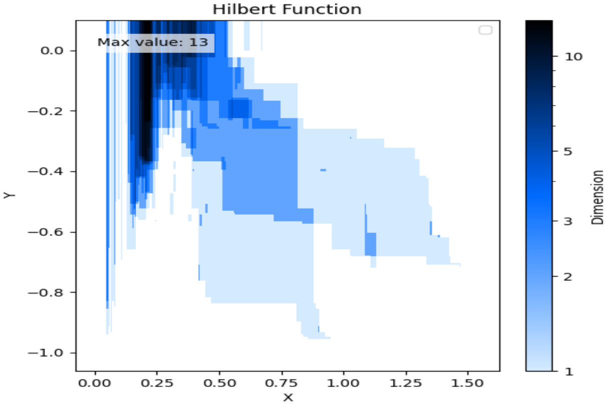
Multiparameter Persistence Modules

$$\xRightarrow{H_1(-;\mathbb{K})}
\begin{tikzcd}
\mathbb{K} \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{K}^2 \arrow[u, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[r, "{\begin{bmatrix} 0 & 1 \end{bmatrix}}"] & \mathbb{K} \arrow[u] \arrow[r] & \mathbb{K} \arrow[u] \\
\mathbb{K} \arrow[u, "{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}"] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

Definition

A *Multiparameter Persistence Module* M is a functor $M\colon \mathbb{R}^d \to \mathrm{Vect}_K$.

Hilbert Functions



Definition

Define the *thickness* $\theta(M)$ as $\sup_{\alpha \in \mathbb{R}^2} \dim_{\mathbb{K}} M_\alpha$

Decompositions

Let M be a multiparameter persistence module.

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Goal

$$\text{Find } \{M_i\}_{i \in I} \text{ such that } M \xrightarrow{\simeq} \bigoplus_{i \in I} M_i.$$

with M_i indecomposable.

Decompositions

$$
\begin{tikzcd}
\mathbb{F} \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{F}_2^2 \arrow[u, "\begin{bmatrix} 1 & 0 \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 & 1 \end{bmatrix}"] & \mathbb{F}_2 \arrow[u] \arrow[r] & \mathbb{F}_2 \arrow[u] \\
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\end{tikzcd}
$$

Decompositions

decomposes as

$$
\begin{tikzcd}
\mathbb{F}_2 \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{F}_2 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
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\end{tikzcd}
\bigoplus
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \\
\mathbb{F}_2 \arrow[r] \arrow[u] & \mathbb{F}_2 \arrow[r] \arrow[u] & \mathbb{F}_2 \arrow[u] \\
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\end{tikzcd}
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Motivation

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2. Additive algebraic constructions:
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   - Submodules
   - Resolutions
3. As an invariant itself, if the components are small.

Current Situation

What we can do

We can compute decompositions reasonably fast with AIDA.
Between $\sim n^{1.5}$ and $\sim n^3$ for density-scale bifiltrations.

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For interval-decomposable modules in $\mathcal{O}(n^3)$.

Obstructions

- Complicated topologies create large indecomposable summands.
- Every Persistence Module is close to being indecomposable.

Instability of Decompositions

Theorem (Bauer, Scoccola '22

For any $\epsilon > 0$ and finitely presented two-parameter module M, there is an indecomposable module M' that is ϵ-interleaved with M.

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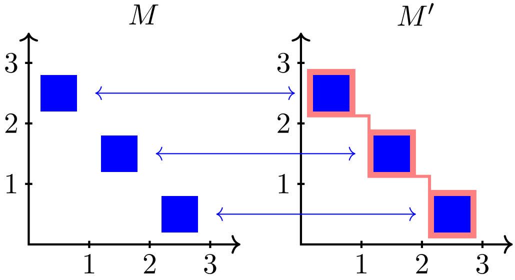


Experimental Evidence

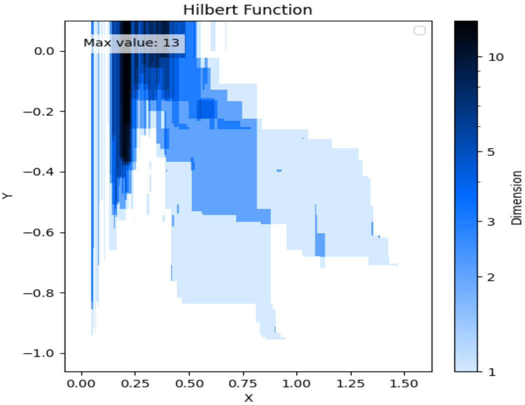


Figure: 246 Generators, 357 Relations,

Experimental Evidence

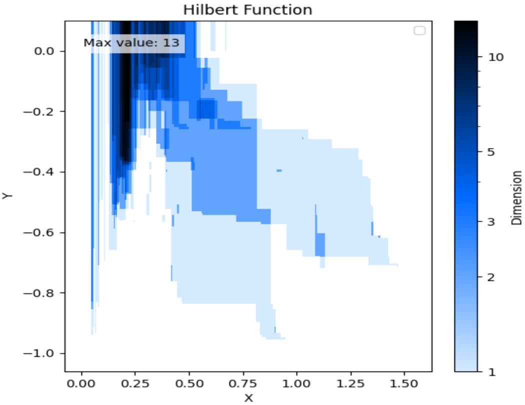
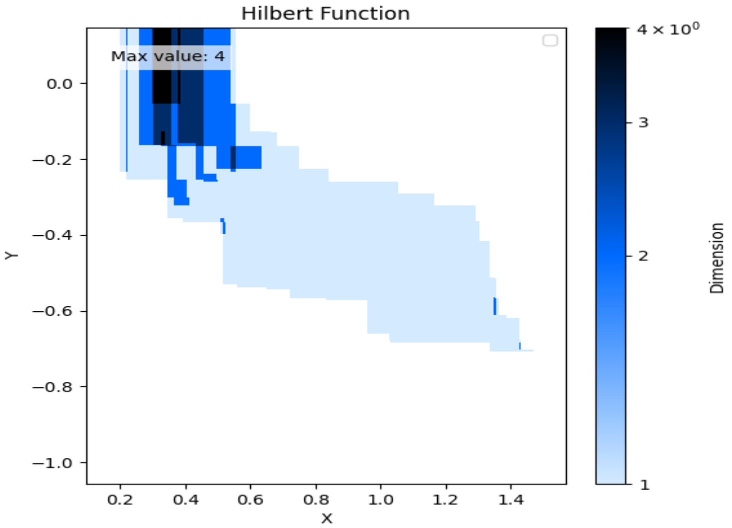
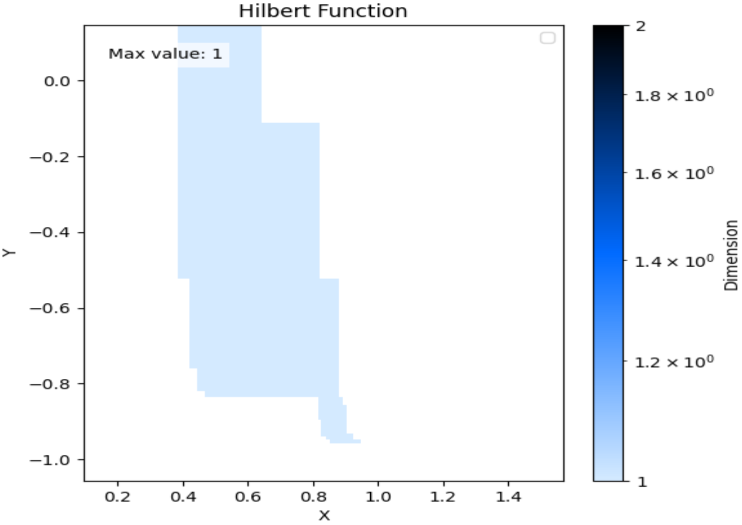
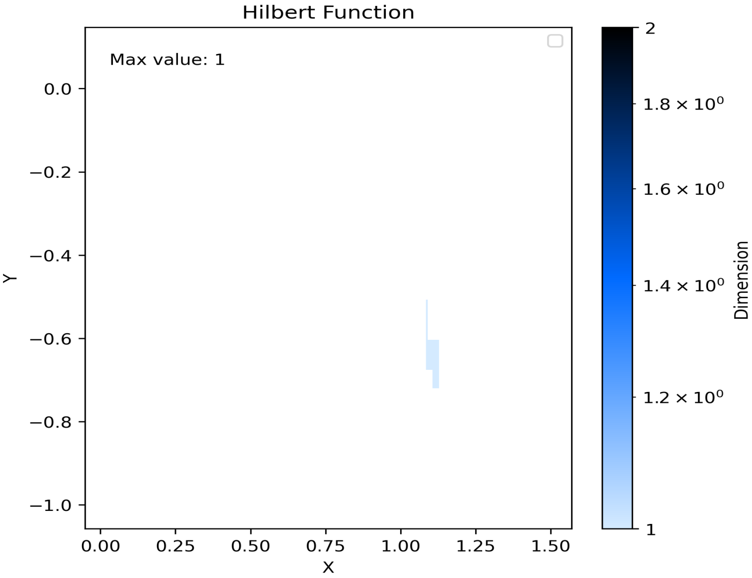
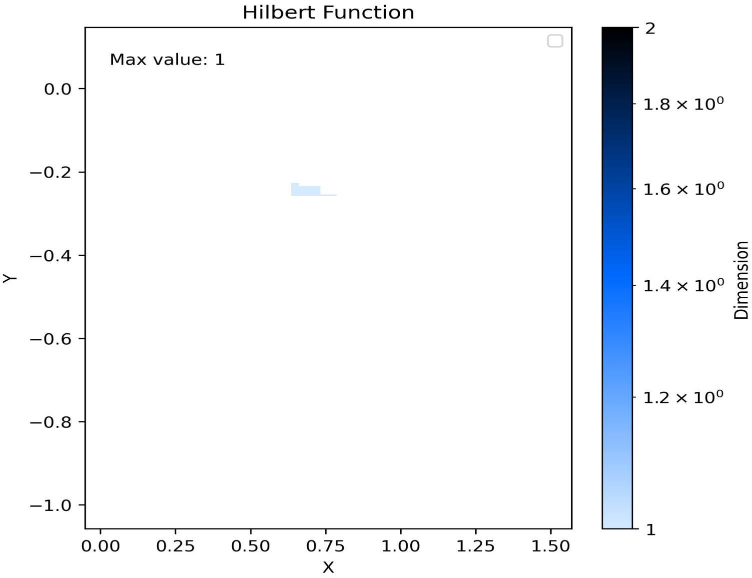


Figure: 246 Generators, 357 Relations, decomposes into 192 components.


Hilbert Function
Max value: 4
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
4 × 10^0
3
2
1
Dimension


Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
2
1.8 × 10^0
1.6 × 10^0
1.4 × 10^0
1.2 × 10^0
1
Dimension


Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
X
2
1.8 × 10^0
1.6 × 10^0
1.4 × 10^0
1.2 × 10^0
1
Dimension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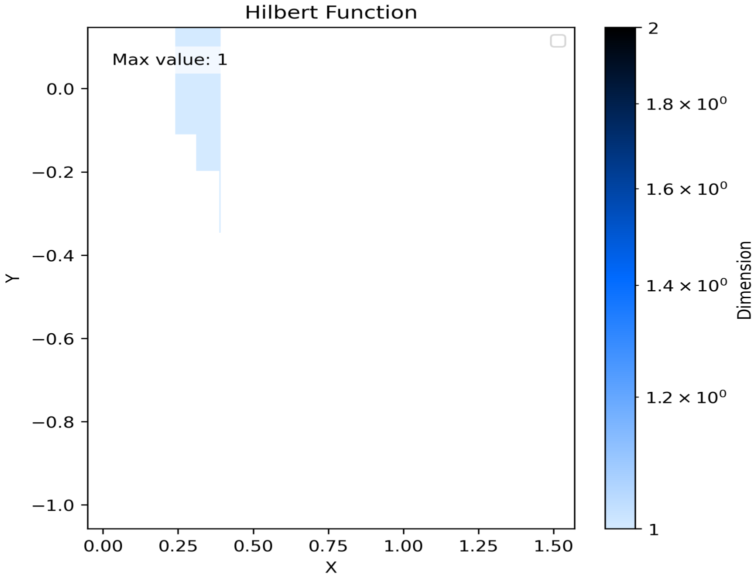
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0.75
1.00
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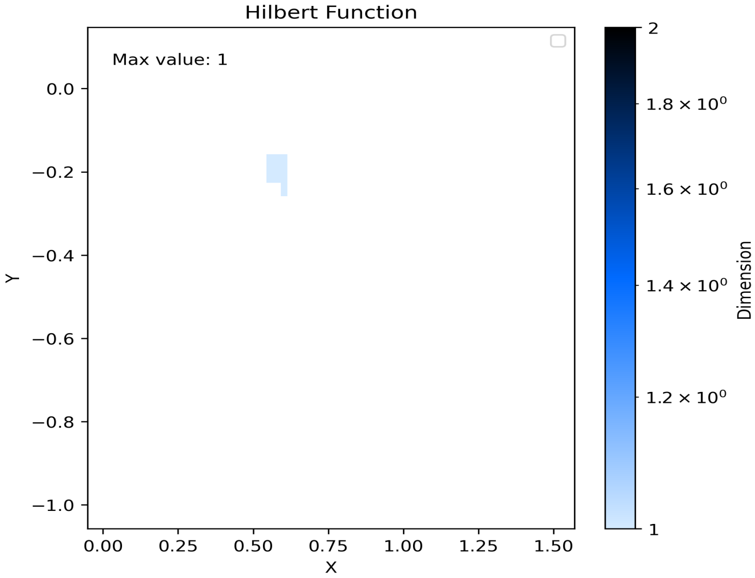
1.50
X
2
1.8 × 10⁰
1.6 × 10⁰
1.4 × 10⁰
1.2 × 10⁰
1
Dimension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0.00
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1.25
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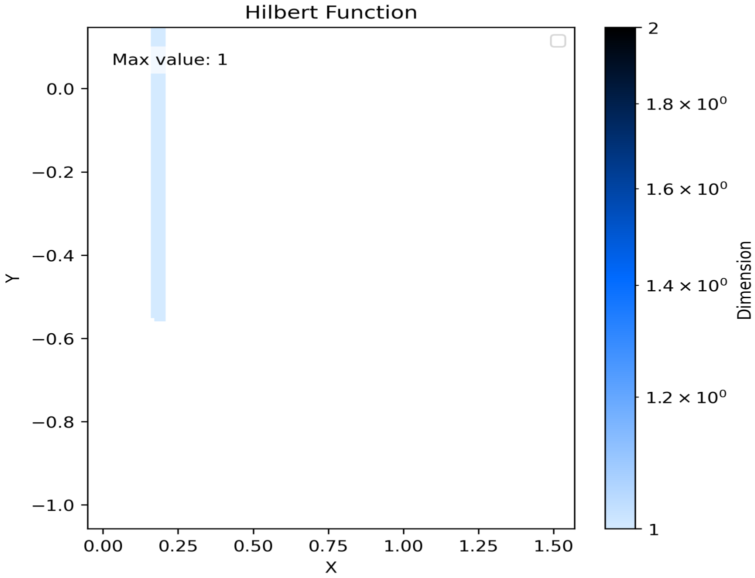
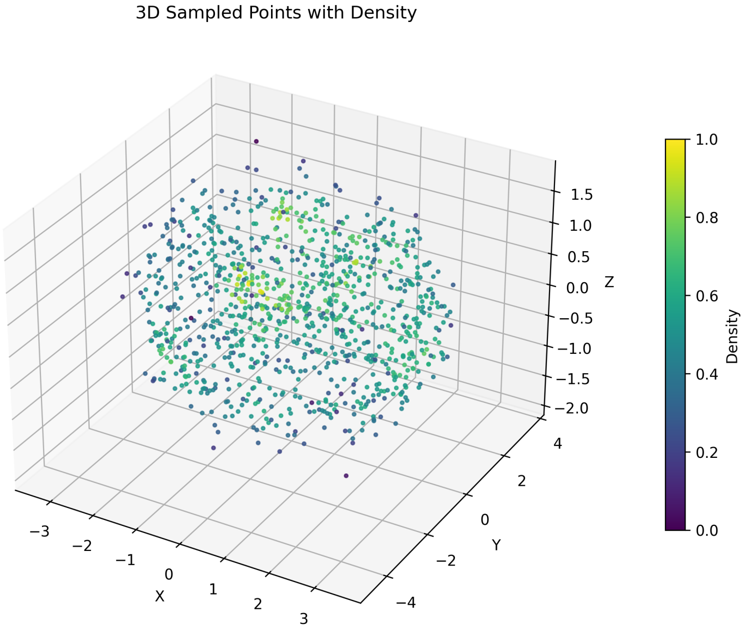
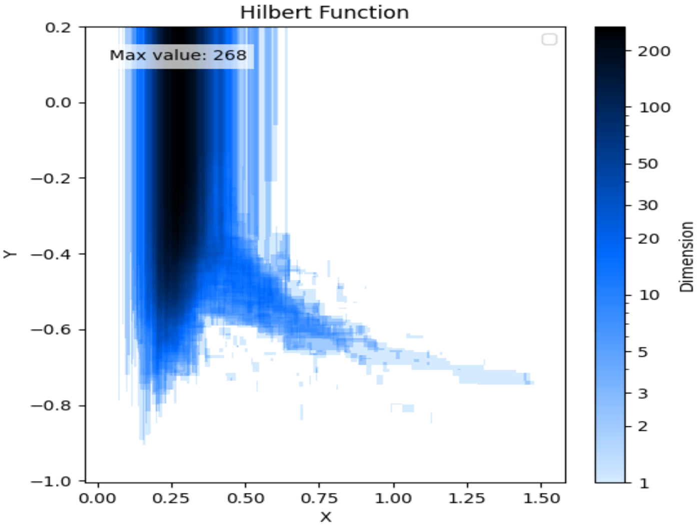
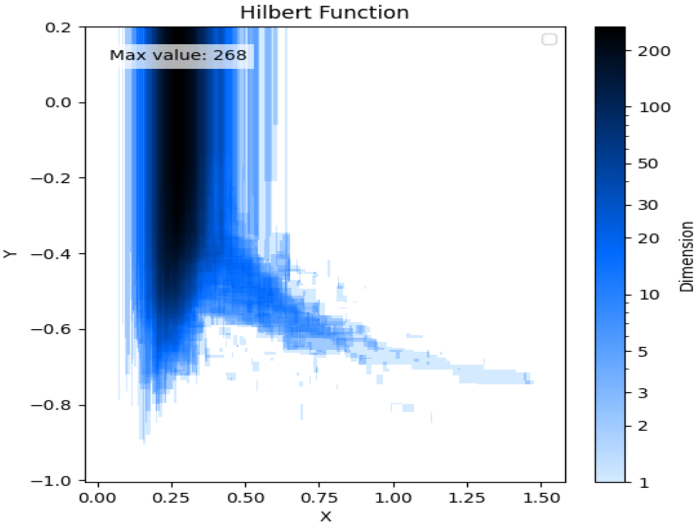
X
2
1.8×10^0
1.6×10^0
1.4×10^0
1.2×10^0
1
Dimension

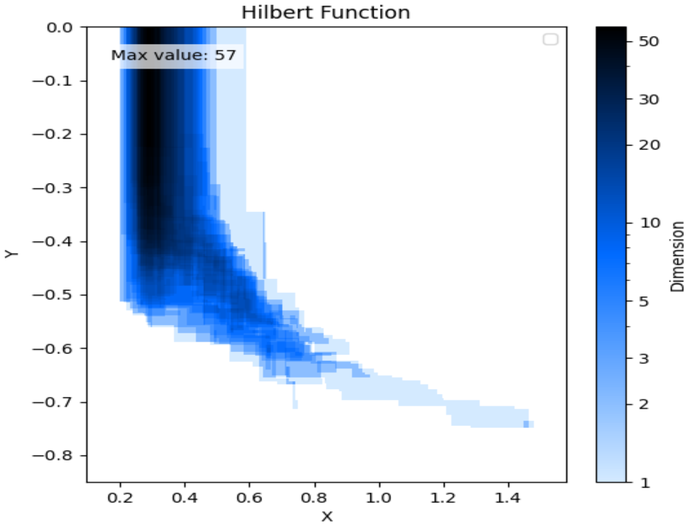

Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
X
2
1.8 × 10⁰
1.6 × 10⁰
1.4 × 10⁰
1.2 × 10⁰
1
Dimension


3D Sampled Points with Density
X
Y
Z
−3
−2
−1
0
1
2
3
4
2
0
−2
−4
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
Density
1.0
0.8
0.6
0.4
0.2
0.0


Hilbert Function
Max value: 268
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
X
200
100
50
30
20
10
5
3
2
1
Dimension



4892 generators, 3390 relations, 2742 components


Hilbert Function
Max value: 57
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
50
30
20
10
5
3
2
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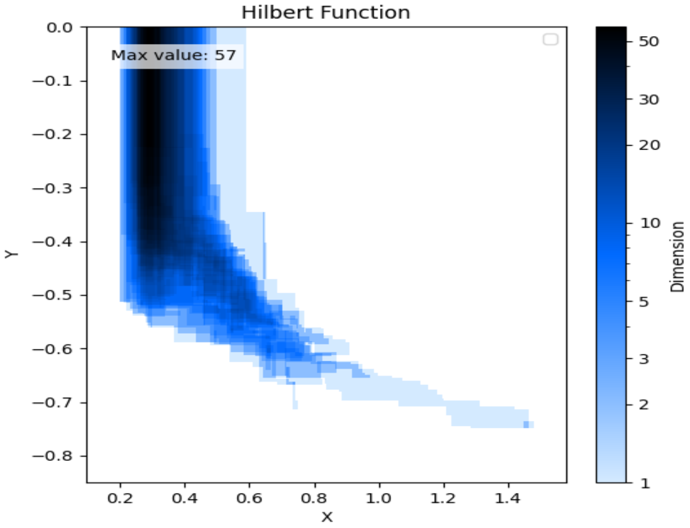
Dimension



This module is *not* close to being highly decomposable.

Structure

The Pruning Construction

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Theorem

Let $\varepsilon \geq 0$, and let M and N be ε-interleaved pfd modules with $\theta(M) < \infty$. Then $\mathrm{Pru}_\varepsilon(M)$ is a $2\varepsilon\theta(M)$-refinement of both M and N. In particular $d_I(M, \mathrm{Pru}_\varepsilon(M)) \leq 2\varepsilon\theta(M)$.

Algebraic Preliminaries

Definition

Let M be a persistence module and $\alpha \in \mathbb{R}^2$. We define its *shift* $M[\alpha]$ via

$$M[\alpha]_\beta = M_{\alpha+\beta} \qquad M[\alpha]_{\beta\to\gamma} = M_{(\alpha+\beta)\to(\alpha+\gamma)}.$$

The structure maps induce natural homomorphisms

$$\mathrm{sh}^\alpha_M \colon M \to M[\alpha].$$

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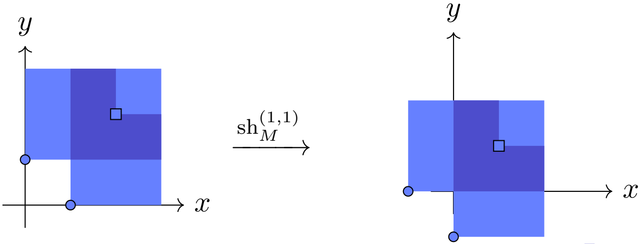

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$$\mathrm{Ker}_\alpha(M) \coloneqq \ker \mathrm{sh}^\alpha_M$$
$$\mathrm{Im}_\alpha(M) \coloneqq \operatorname{im} \mathrm{sh}^\alpha_{M[-\alpha]}$$

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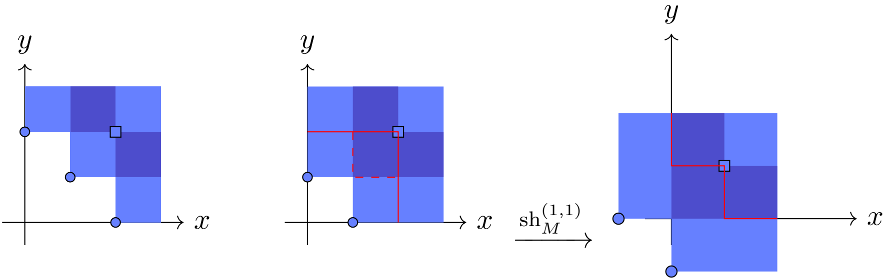

Erosion

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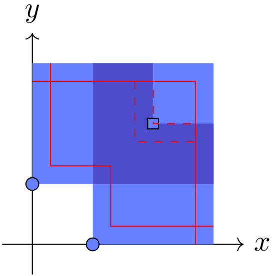

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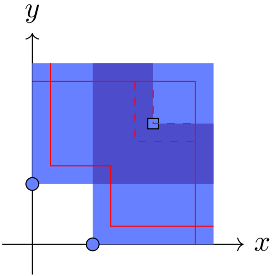

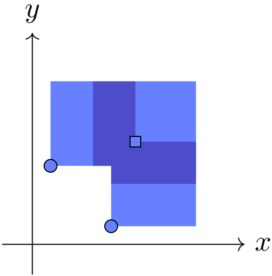

The Pruning Construction

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Define I to be the largest submodule of M such that for any $f\colon M \to M[2\epsilon]$ it holds that $f(I) \subseteq \mathrm{sh}_M^{2\epsilon}(I)$.

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$\mathrm{Pru}_\varepsilon(M) \coloneqq I/K$.

The Pruning Construction

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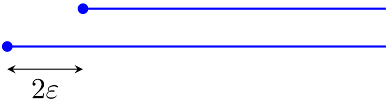

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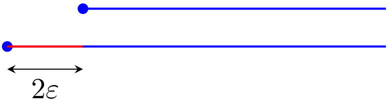

Structure

Computing the Pruning

Lemma

Set $I_0 \coloneqq X$, $K_0 \coloneqq 0$. We define iteratively

$$I_i \coloneqq \bigcap_{f \in \mathrm{End}(X)_{2\epsilon}} f^{-1} \circ \mathrm{sh}_X^{2\epsilon}(I_{i-1}) \text{ and} \tag{1}$$

$$K_i \coloneqq \sum_{f \in \mathrm{End}(X)_{2\epsilon}} \left(\mathrm{sh}_I^{2\epsilon}\right)^{-1} \circ f(K_{i-1}). \tag{2}$$

Then $I = \bigcap^{\infty} I_i$ and $K = \bigcup^{\infty} K_i$.

Computing the Pruning

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$\Rightarrow$ This implies an algorithm:
Choose a basis $\{f_i\}$ for $\mathrm{Hom}(M, M[2\varepsilon])$ and compute the above.

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Then $I = \bigcap^{\infty} I_i$ and $K = \bigcup^{\infty} K_i$.

$\Rightarrow$ This implies an algorithm:

Choose a basis $\{f_i\}$ for $\mathrm{Hom}(M, M[2\varepsilon])$ and compute the above.

$\dim_{\mathbb{K}} \mathrm{Hom}(M, M[2\varepsilon]) \in \mathcal{O}(n^2)$.

Size of $\mathrm{Hom}(M, M[2\varepsilon])$

	Random Points - 78 g			Random Points - 254 g		
ε	0.00	0.01	0.02	0.00	0.01	0.02
$\mathrm{End}(M)_\varepsilon$	713	914	1111	5010	7632	10383

	Two circles - 357 g			Torus - 647 g		
ε	0.00	0.01	0.02	0.00	0.01	0.02
$\mathrm{End}(M)_\varepsilon$	16192	17089	17844	7230	9307	11635

Observation

Let $f = \mathrm{sh}_M^{2\varepsilon} \circ g$ where $g \colon M \to M$,
then f can be skipped in the construction of $\mathrm{Pru}_\varepsilon(M)$.

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Proposition

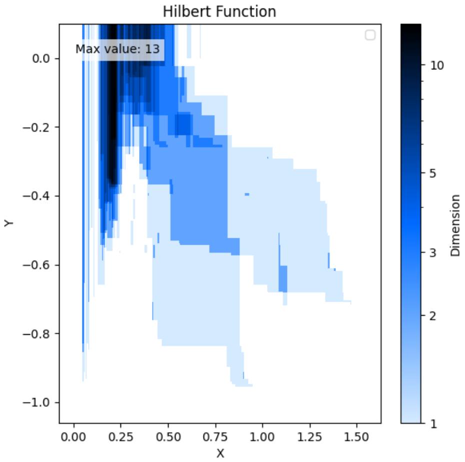
Can instead consider a basis for

$$\mathrm{Hom}(M, M[2\varepsilon]) \Big/ \left(\mathrm{sh}_M^{2\varepsilon}\right)_* \mathrm{Hom}(M, M)$$

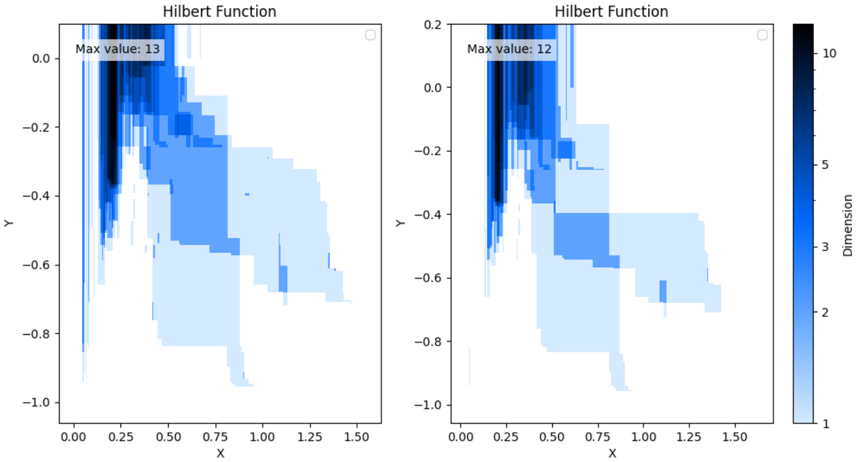
Size of $\mathrm{Hom}(M, M[2\varepsilon])$

	Two circles			Torus		
2ε	0.005	0.01	0.02	0.005	0.01	0.02
$\mathrm{End}(M)_0$	16192	16192	16192	7230	7230	7230
$\mathrm{End}(M)_{2\varepsilon}$	16645	17089	17844	8241	9307	11635
$\mathrm{End}(M)_{2\varepsilon/0}$	453	897	1652	1011	2077	4405

Results

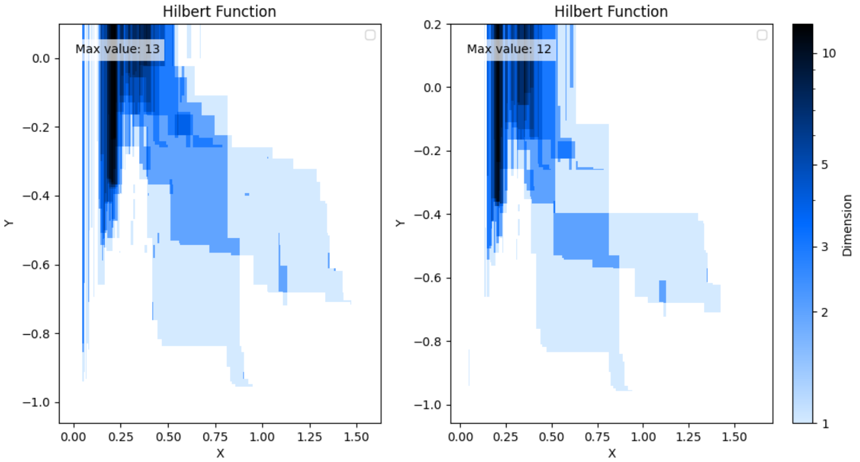

Hilbert Function
Max value: 13
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
X
10
5
3
2
1
Dimension

Results



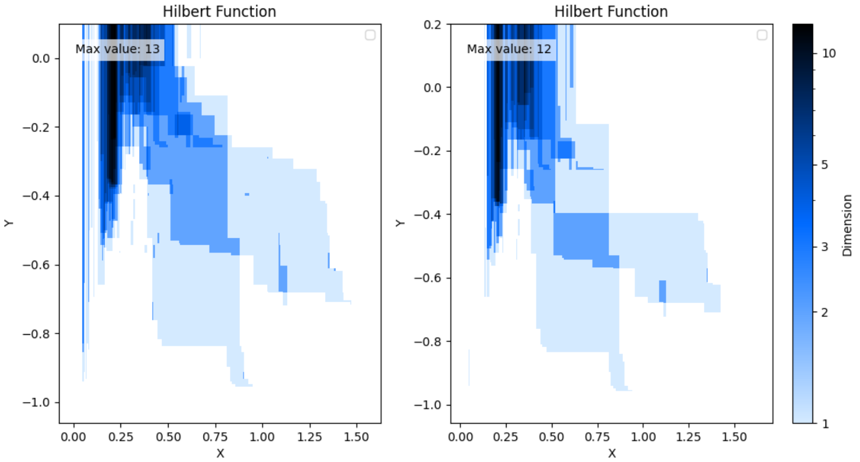
Hilbert Function
Max value: 13
Y
X
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
Hilbert Function
Max value: 12
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Dimension
10
5
3
2
1

Results



192 components

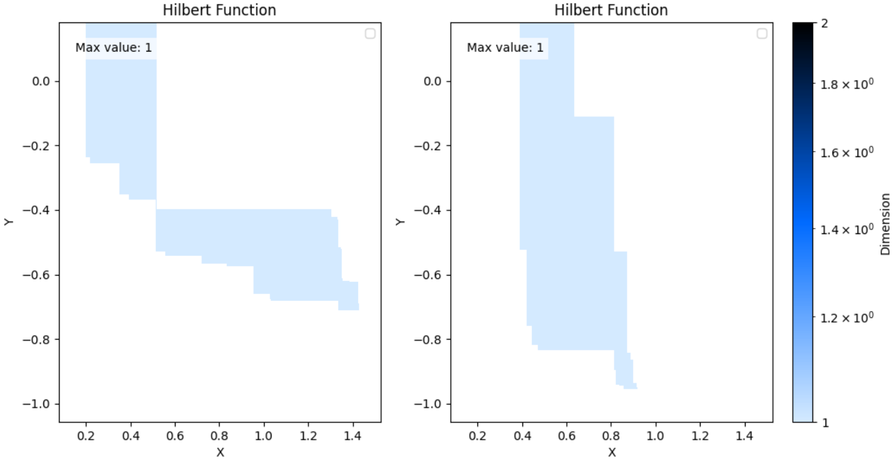
Results



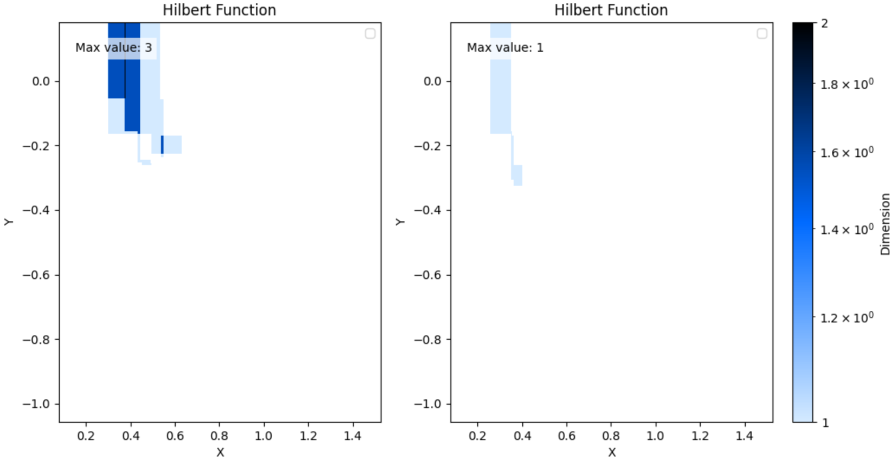
192 components

72 components

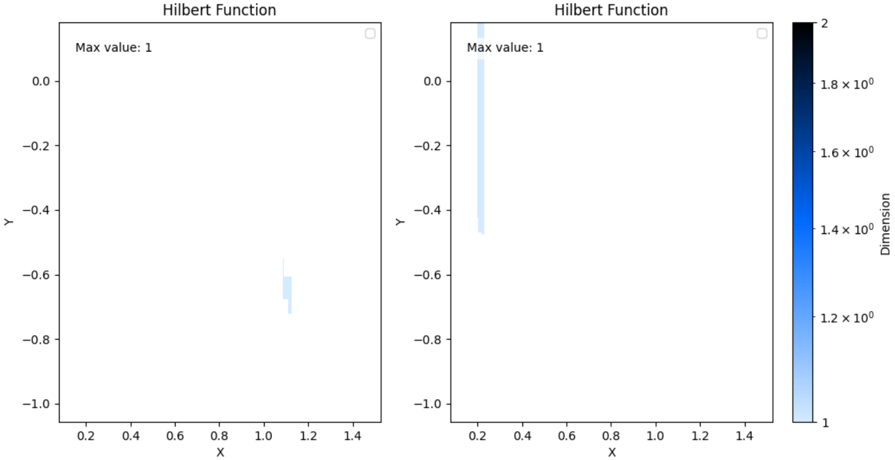
Results


Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
Hilbert Function
Max value: 1
2
1.8×10^0
1.6×10^0
1.4×10^0
1.2×10^0
1
Dimension

Results

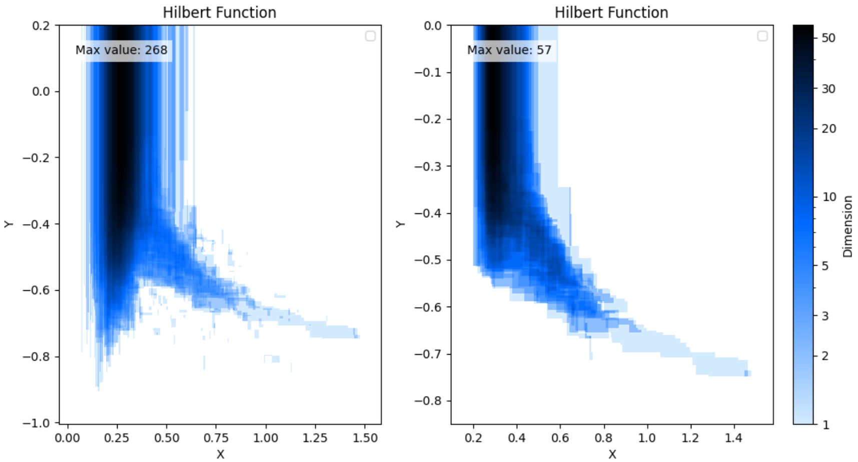

Hilbert Function
Max value: 3
Y
X
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Hilbert Function
Max value: 1
Dimension
2
1.8 × 10^0
1.6 × 10^0
1.4 × 10^0
1.2 × 10^0
1

Results

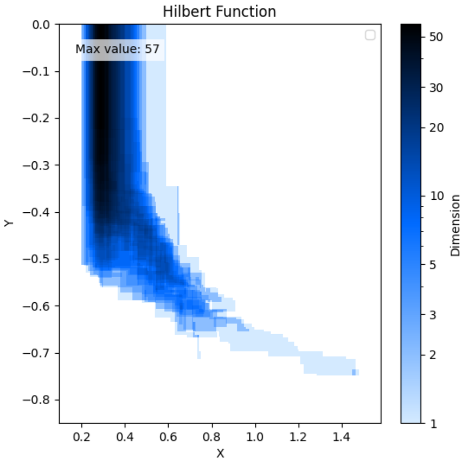

Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
Hilbert Function
Max value: 1
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
2
1.8 × 10^0
1.6 × 10^0
1.4 × 10^0
1.2 × 10^0
1
Dimension

Results

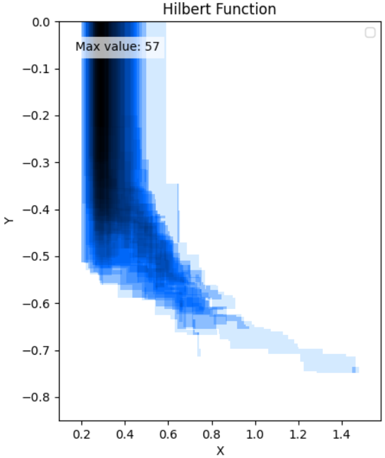
and its largest component

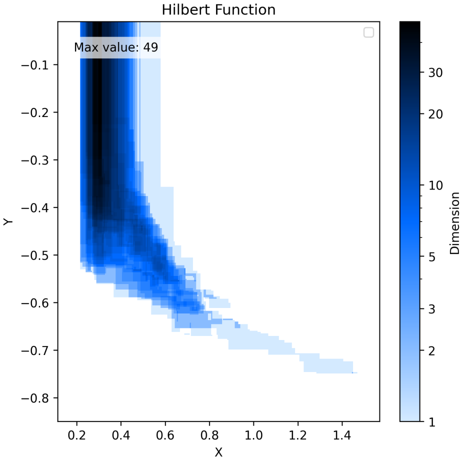


Results

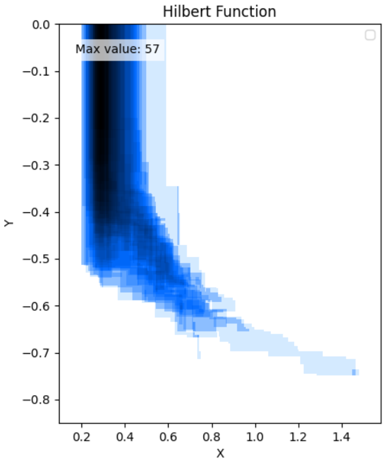

Hilbert Function
Max value: 57
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
50
30
20
10
5
3
2
1
Dimension

Results

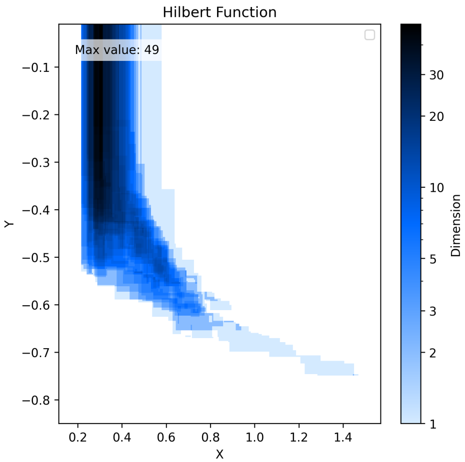

Hilbert Function
Max value: 57
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X


Hilbert Function
Max value: 49
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
30
20
10
5
3
2
1
Dimension

Results

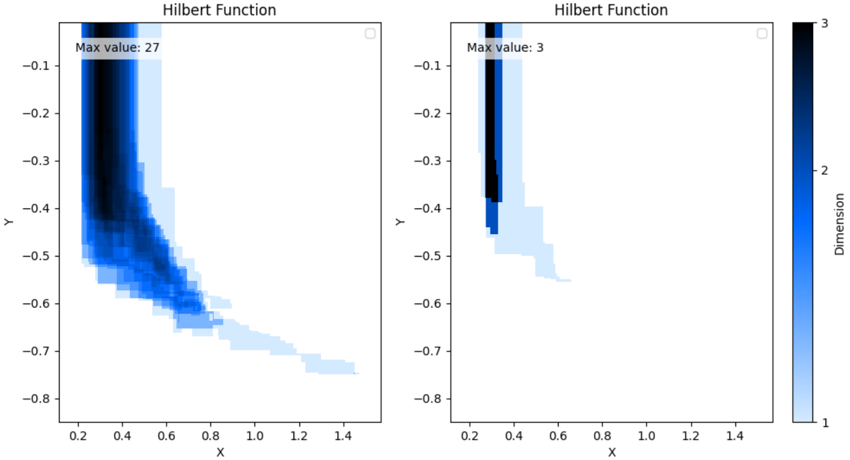


1 component



69 components

Results


Hilbert Function
Max value: 27
Y
X
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Hilbert Function
Max value: 3
Dimension
3
2
1

Open Problems

- The number of iterations is bounded by $\theta(M)$ - can we do better?

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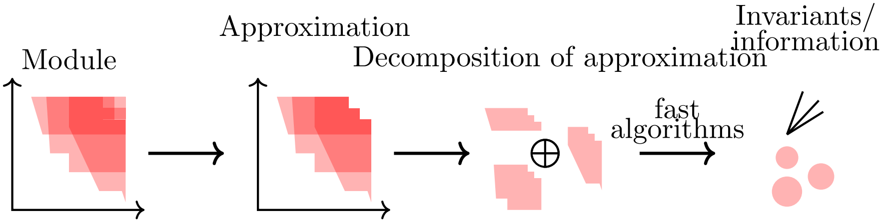
- The number of iterations is bounded by $\theta(M)$ - can we do better?
- Notice that

$$\mathrm{Hom}(M, M[2\varepsilon]) \Big/ \left(\mathrm{sh}_M^{2\varepsilon}\right)_* \mathrm{Hom}(M, M) = \left(\mathrm{End}(M) \otimes {}^{F[2\varepsilon]}/_{F}\right)_0 .$$

Can you compute this faster than $\mathrm{Hom}(M, M[2\varepsilon])$?

Thank you!



Code

https://github.com/JanJend/Stable-Decomposition