Topic: Differentiability

Problem 01:

A company's profit $P(x,y)$ depends on the price x and the number of units sold y. The profit function is:

$$P(x,y) = 3x^2 + 5y + 7$$

Check if the profit function is differentiable at point $(x,y) = (2,3)$ and find the partial derivates $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$.

Solution:

$$P(x,y) = 3x^2 + 5y + 7$$

Taking partial derivative w.r.t 'x'

$$P_x(x,y) = \frac{\partial}{\partial x}(3x^2 + 5y + 7)$$

$$= 6x + 0 + 0$$

$$P_x(x,y) = 6x$$

At $(x,y) = (2,3)$

$$P_x(2,3) = 6(2)$$

$$P_x(2,3) = 12$$

Taking partial derivate w.r.t 'y'

$$P_y(x,y) = \frac{\partial}{\partial y}(3x^2 + 5y + 7)$$

$$= 0 + 5 + 0$$

$$P_y(x,y) = 5$$

At $(x,y) = (2,3)$

$$P_y(2,3) = 5$$

Problem 02:

The temperature at any point (x,y,z) in a room is given by the following function:

$$T(x,y,z) = 4x^2 + 3y^2 - 2z^2 + 5xy + 3yz - 4xz$$

Find the rate of change of temperature with respect to change in direction y at point $(1,2,3)$.

$$\frac{\partial T}{\partial y} = ?$$

Solution:

$$T(x,y,z) = 4x^2 + 3y^2 - 2z^2 + 5xy + 3yz - 4xz$$

Taking partial differentiate w.r.t 'y'

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 3y^2 - 2z^2 + 5xy + 3yz - 4xz)$$

$$\frac{\partial T}{\partial y} = 0 + 6y - 0 + 5x + 3z - 0$$

$$\frac{\partial T}{\partial y} = 6y + 5x + 3z$$

At (1,2,3)

$$\frac{\partial}{\partial y}(T(x,y,z)) = 6y + 5x + 3z$$

$$\frac{\partial}{\partial y} T(1,2,3) = 6(2) + 5(1) + 3(3)$$

$$= 12 + 5 + 9$$

$$\frac{\partial}{\partial y} T(1,2,3) = 26$$

Topic :- Partial Derivate

Problem 03 :

A Rectangular garden's area $A(x,y)$ depends on its length x (meters) and width y (meters). The area is given by :

$$A(x,y) = x^2y + 2xy^2$$

Then Find $\frac{\partial A}{\partial x}$ keeping y Constant and also find $\frac{\partial A}{\partial y}$ while

Keeping x Constant.

Solution :

$$A(x,y) = x^2y + 2xy^2 \rightarrow (1)$$

Taking partial differentiate w.r.t 'x'

$$\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}(x^2y + 2xy^2)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(2xy^2)$$

$$= y\frac{\partial}{\partial x}(x^2) + 2y^2\frac{\partial x}{\partial x}$$

$$\frac{\partial A}{\partial x} = 2xy + 2y^2$$

Taking partial differentiate w.r.t 'y'

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(x^2y + 2xy^2)$$

$$= \frac{\partial}{\partial y}(x^2y) + 2x\frac{\partial}{\partial y}(y^2)$$

$$= x^2\frac{\partial y}{\partial y} + 2x \cdot 2y\frac{\partial y}{\partial y}$$

$$= x^2 + 4xy$$

Problem 04:

The profit $P(x,y)$ of a company depends on the amount x of product A and y of product B produced. The Profit function is:

$$P(x,y) = 3x^2 + 2xy + y^2$$

Find profit with respect to amount x of product A while keeping 'y' as constant & vice versa.

Solution:

$$P(x,y) = 3x^2 + 2xy + y^2$$

Taking Partial derivate w.r.t 'x'

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(3x^2 + 2xy + y^2)$$

$$= 3\frac{\partial}{\partial x}(x^2) + 2y\frac{\partial x}{\partial x} + \frac{\partial}{\partial x}(y^2)$$

$$= 6x + 2y + 0$$

$$\frac{\partial P}{\partial x} = 6x + 2y$$

Taking partial derivate w.r.t 'y'

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(3x^2 + 2xy + y^2)$$

$$= \frac{\partial}{\partial y}(3x^2) + 2x\frac{\partial y}{\partial y} + \frac{\partial}{\partial y}(y^2)$$

$$= 0 + 2x + 2y$$

$$\frac{\partial P}{\partial y} = 2x + 2y$$

Topic : Chain Rule

Problem 05:

The function $z(x,y)$ represents a System's output, where $z = x^2 + y^2$ The inputs x and y vary with time t, and are defined as $x = e^t$ and $y = \sin(t)$.

Find the rate of change of z with respect to time t, i.e., $\frac{dz}{dt}$ using the chain rule.

Problem 6:

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Solution:

$$Z = x^2 + y^2$$

$$x = e^t \quad , \quad y = \sin(t)$$

To find:

$$\frac{\partial z}{\partial t} = ?$$

By chain Rule:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \rightarrow (1)$$

Now

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^t)(x^2+y^2) \qquad \frac{\partial x}{dt} = \frac{d}{dt}(e^t)$$

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{dx}{dt} = e^t$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) \qquad \frac{\partial y}{dt} = \frac{d}{dt}(\sin t)$$

$$\frac{\partial z}{\partial y} = 2y \qquad \frac{\partial y}{dt} = \cos t$$

Putting values in (1)

$$\frac{\partial z}{\partial t} = 2x \cdot e^t + 2y \cdot \cos t$$

$$\frac{\partial z}{\partial t} = e^t 2x + \cos t \, 2y$$

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Problem 6:

A Company monitors the performance of a System that depends on the two variables; the Operating temperature x in (C°) and the System load y (in %). The System efficiency z is modeled by the function:

$$z(x,y) = x^2y + \ln(x+y)$$

The temperature x and y are themselves functions of two Variable (independent parameters, t and S.

$$x = e^t + S \quad , \quad y = t^2 - \sin(S)$$

To find:

$$\frac{\partial z}{\partial t} = ? \qquad \frac{\partial z}{\partial S} = ?$$

Solution:

By chain rule:

$$\frac{\partial z}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \longrightarrow (1)$$

$$\frac{\partial z}{\partial S} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial S} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial S} \longrightarrow (2)$$

Now,

- Compute partial derivatives of $z(x,y)$:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + \ln(x+y)\right)$$

$$= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(\ln(x+y))$$

$$\frac{\partial z}{\partial x} = 2xy + \frac{1}{x+y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2y + \ln(x+y)\right)$$

$$= \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(\ln(x+y))$$

$$\frac{\partial z}{\partial y} = x^2 + \frac{1}{x+y}$$

- Compute partial derivatives of x and y:

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(e^t + s) \qquad \frac{\partial x}{\partial s} = \frac{\partial}{\partial s}(e^t + s)$$

$$\frac{\partial x}{\partial t} = e^t \qquad \frac{\partial x}{\partial s} = 1$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(t^2 - \sin s) \qquad \frac{\partial y}{\partial s} = \frac{\partial}{\partial s}(t^2 - \sin s)$$

$$= 2t \qquad = -\cos s$$

Put values in equ (1)

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \rightarrow (1)$$

$$\frac{\partial z}{\partial t} = \left(2xy + \frac{1}{x+y}\right)(e^t) + \left(x^2 + \frac{1}{x+y}\right)(-\cos s)$$

$$\frac{\partial z}{\partial t} = e^t\left(2xy + \frac{1}{x+y}\right) - \cos(s)\left(x^2 + \frac{1}{x+y}\right)$$

Putting values in equ(2)

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \rightarrow (2)$$

$$\frac{\partial z}{\partial s} = \left(2xy + \frac{1}{x+y}\right) \cdot (2t) + \left(x^2 + \frac{1}{x+y}\right) \cdot (-\cos s)$$

$$\frac{\partial z}{\partial s} = 2t\left(2xy + \frac{1}{x+y}\right) - \cos(s)\left(x^2 + \frac{1}{x+y}\right)$$

Topic : Implicit functions

Problem 078

The path of a moving particle is constrained by the equation:

$$x^2 + y^2 + z^2 = 16$$

where x, y, and z represent the coordinates of the particle

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in 3D space. Determine how the particle's z-coodinate changes with respect to its x-coordinate i.e find $\frac{dz}{dx}$ using implicit differenciation.

Solution:

$$x^2+y^2+z^2 = 16$$

Taking partial derivative w.r.t 'x'

$$\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2) = \frac{\partial 16}{\partial x}$$

$$2x\frac{\partial x}{\partial x} + 0 + 2z \cdot \frac{\partial z}{\partial x} = \frac{\partial 16}{\partial x}$$

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$2z\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z}$$

Problem 08:

The relationship between the pressure x (in pascals), temperature y (in Kelvin) and volume z (in cubic meters) of a gas

under experimental Conditions is given by:

$$2x^2yz + 6xy^3 + 9yz^2 = 16$$

Find $\frac{dz}{dy}$, using implicit differentiation.

Solution:

$$2x^2yz + 6xy^3 + 9yz^2 = 16$$

Taking implicit partial derivate w.r.t 'y'

$$\frac{\partial}{\partial y}(2x^2yz + 6xy^3 + 9yz^2) = \frac{\partial}{\partial y}(16)$$

$$2x^2\frac{\partial}{\partial y}(yz) + 6x\frac{\partial}{\partial y}(y^3) + 9\frac{\partial}{\partial y}(9z^2) = 0$$

$$2x^2\left[y\frac{\partial z}{\partial y} + z\right] + 18xy^2 + 9\left[y \cdot 2z\frac{\partial z}{\partial y} + z^2\right] = 0$$

$$2x^2y\frac{\partial z}{\partial y} + 2x^2z + 18xy^2 + 18yz\frac{\partial z}{\partial y} + 9z^2 = 0$$

$$2x^2y\frac{\partial z}{\partial y} + 18yz\frac{\partial z}{\partial y} = -9z^2 - 2x^2z - 18xy^2$$

$$(2x^2y + 18yz)\frac{\partial z}{\partial y} = -(18xy^2 + 2x^2z + 9z^2)$$

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$$\frac{\partial z}{\partial y} = \frac{-(18xy^2 + 2x^2z + 9z^2)}{(18yz + 2x^2y)}$$

Topic : Directional Derivate

Problem 09:

The temperature $T(x,y)$ at any point (x,y) on a flat surface is given by the function:

$$T(x,y) = 4x^2 + 3y^2$$

Find the rate of change of temperature θ in the direction of the vector $V = (3,4)$.

Solution:

$$T(x,y) = 4x^2 + 3y^2$$
$$\vec{V} = (3,4)$$

Step 01:

(Compute Gradient)

$$\nabla T = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} \right) \longrightarrow (1)$$

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}(4x^2+3y^2) \qquad \frac{\partial T}{\partial y} = \frac{\partial}{\partial y}(4x^2+3y^2)$$

$$\frac{\partial T}{\partial x} = 8x \qquad \frac{\partial T}{\partial y} = 6y$$

$$\nabla T = (8x\hat{i} + 6y\hat{j})$$

Step 02 :

(compute directional derivative)

$$D_v T = \nabla T \cdot \hat{n}$$

$$D_v T(x,y) = (8x\hat{i} + 6y\hat{j}) \cdot \hat{v}$$

Now,

unit vector in the direction of v is:

$$\hat{v} = \frac{\vec{v}}{|v|}$$

$$\hat{v} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{(3)^2 + (4)^2}} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{9+16}}$$

$$\hat{v} = \frac{3\hat{i} + 4\hat{j}}{\sqrt{25}} = \frac{3\hat{i} + 4\hat{j}}{5}$$

$$\hat{v} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

So,

$$D_v T = (8x\hat{i} + 6y\hat{j}) \cdot \left(\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}\right)$$

$$D_v T = (8x)\left(\frac{3}{5}\right) + 6y\left(\frac{4}{5}\right)$$

$$D_v T = \frac{24x}{5} + \frac{24y}{5}$$

Problem 10 :

The concentration $C(x,y,z)$ of a chemical Substance at any point (x,y,z) in a room is given by:

$$C(x,y,z) = x^2+2y^2+3z^2$$

Find the rate of change of Concentration at point $(1,1,1)$ in the direction of the vector $v.(1,0,-1)$

Solution :

Step 01 :

(Compute Gradient)

$$\nabla C = \left(\frac{\partial C}{\partial x}\hat{i} + \frac{\partial C}{\partial y}\hat{j} + \frac{\partial C}{\partial z}\hat{k}\right)$$

$$\nabla C = \left(\frac{\partial C}{\partial x}\hat{i} + \frac{\partial C}{\partial y}\hat{j} + \frac{\partial C}{\partial z}\hat{k}\right) \longrightarrow (1)$$

Now,

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}(x^2+2y^2+3z^2)$$

$$\frac{\partial C}{\partial x} = 2x+0+0$$

$$\boxed{\frac{\partial C}{\partial x} = 2x}$$

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}(x^2+2y^2+3z^2)$$

$$\frac{\partial C}{\partial y} = 4y$$

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$$\frac{\partial C}{\partial z} = \frac{\partial}{\partial z}(x^2+2y^2+3z^2)$$

$$= \frac{\partial C}{\partial z} = 6z$$

Putting values in equation (1)

$$\nabla C = (2x\,\hat{i} + 4y\,\hat{j} + 6z\,\hat{k})$$

Step 02:

(Compute Directional Derivate)

we know that

$$D_v C = \nabla C \cdot \hat{v}$$

$$D_v C = (2x\hat{i} + 4y\hat{j} + 6z\hat{k}) \cdot \hat{v}$$

Now,

The unit vector in the direction of v is

$$\hat{v} = \frac{\vec{v}}{|v|}$$

$$\hat{v} = \frac{(\hat{i}+0\hat{j}-\hat{k})}{\sqrt{(1)^2+(0)^2+(-1)^2}} = \frac{\hat{i}-\hat{k}}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}$$

so,

$$D_v C = (2x\,\hat{i} + 4y\,\hat{j} + 6z\,\hat{k}) \cdot \left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}\right)$$

$$D_v C = (2x)\left(\frac{1}{\sqrt{2}}\right) + (4y)(0) + 6z\left(-\frac{1}{\sqrt{2}}\right)$$

$$D_v C = \frac{2x}{\sqrt{2}} + \left(-\frac{6z}{\sqrt{2}}\right)$$

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$$D_v C = \frac{2x}{\sqrt{2}} - \frac{6z}{\sqrt{2}}$$

At point (1,1,1):

$$D_v C(1,1,1) = \frac{2(1)}{\sqrt{2}} - \frac{6(1)}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} - \frac{6}{\sqrt{2}}$$

$$= \frac{2-6}{\sqrt{2}}$$

$$= \frac{-4}{\sqrt{2}} = \frac{-2.2}{\sqrt{2}} = \frac{-2.\cancel{\sqrt{2}}\sqrt{2}}{\cancel{\sqrt{2}}}$$

$$D_v C(1,1,1) = -2\sqrt{2}$$

Topic: Relative Minima & Relative Maxima.

Problem 11:

A Company manufactures two types of products, x and y. The total Cost of the production is modeled as:

$$C(x,y) = x^2 + y^2 - 4x - 6y + 10$$

Determine the production levels (x, y) that minimize the cost using the second derivate test.

Solution:

Step 01: $\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 - 4x - 6y + 10)$

$$\frac{\partial C}{\partial x} = 2x - 4 \longrightarrow (*)$$

Taking partial derivate w.r.t 'y'

$$\frac{\partial C}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 - 4x - 6y + 10)$$

$$\frac{\partial C}{\partial y} = 2y - 6 \longrightarrow (**)$$

Step 02:

(Critical Point)

Set partial derivatives to zero:

$2x - 4 = 0$

$2x = 4$

$x = \cancel{4}^{2}/\cancel{2}$

$x = 2$

$2y - 6 = 0$

$2y = 6$

$y = \cancel{6}^{3}/\cancel{2}$

$y = 3$

Critical point : $(x, y) = (2, 3)$

Step 03:

(Hessian Determinant)

$$D = (f_{xx} \cdot f_{yy} - (f_{xy})^2)$$

or $D = \frac{\partial^2 c}{\partial x^2} \cdot \frac{\partial^2 c}{\partial y^2} - \left(\frac{\partial^2 c}{\partial x \partial y}\right)^2 \longrightarrow \odot$

Now,

Taking derivate w.r.t 'x' of equ ⊛

$$\frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}(2x-4)$$

$$\frac{\partial^2 c}{\partial x^2} = 2$$

Taking partial derivate w.r.t 'y' of equ ⊛

$$\frac{\partial^2 c}{\partial y^2} = \frac{\partial}{\partial y}(2y-6)$$

$$\frac{\partial^2 c}{\partial y^2} = 2$$

Taking partial derivate w.r.t y of equ ⊛

$$\frac{\partial^2 c}{\partial x \partial y} = \frac{\partial}{\partial y}(2x-4)$$

$$\frac{\partial^2 c}{\partial x \partial y} = 0$$

Putting values in equ ⊙

$$D = 2 \cdot 2 \cdot (0)^2$$

$$D = 4$$

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Since $D>0$ and $\frac{\partial^2 C}{\partial x^2}>0$, the Critical point is relative minimum.

Result:

The cost is minimized at $(x,y)=(2,3)$ and the minimum Cost is:

$$C(2,3) = (2)^2+(3)^2-4(2)-6(3)+10$$
$$= 4+9-8-18+10$$
$$C(2,3) = -3$$

Problem 12:

A company's Revenue $R(x,y)$ from two products is given by:

$$R(x,y) = 3x^2+2y^2-2xy+6x-8y$$

Find the critical points $R(x,y)$ and determine whether they Correspond to a relative maximum, minimum, or Saddle point.

Solution:

Step 01: (Partial derivate)

$$R(x,y) = 3x^2+2y^2-2xy+6x-8y \longrightarrow (1)$$

Taking partial derivate w.r.t 'x'

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}(3x^2 + 2y^2 - 2xy + 6x - 8y)$$

$$\frac{\partial R}{\partial x} = 6x - 2y + 6 \rightarrow (*)$$

Taking partial derivative w.r.t 'y'

$$\frac{\partial^2 R}{\partial y} = \frac{\partial}{\partial y}(3x^2 + 2y^2 - 2xy + 6x - 8y)$$

$$\frac{\partial R}{\partial y} = 4y - 2x - 8 \rightarrow (**)$$

Step 02: (critical point)

Set equation (*) and (**) equals to zero.

$$6x - 2y + 6 = 0$$
$$6x = 2y - 6$$
$$x = \frac{2y-6}{6}$$
$$x = \frac{\cancel{2}(y-3)}{\cancel{6}3}$$
$$x = \frac{y-3}{3} \rightarrow (2)$$

$$4y - 2x - 8 = 0$$
$$4y = 2x + 8$$
$$4y = 2(x+4)$$
$$y = \frac{\cancel{2}(x+4)}{\cancel{4}2}$$
$$y = \frac{x+4}{2} \rightarrow (3)$$

Put $x = \frac{y-3}{3}$ in (3)

$$y = \frac{\left(\frac{y-3}{3} + 4\right)}{2}$$

$$y = \frac{\left(y-3+12/3\right)}{2}$$

$$y = \frac{y-3+12}{6}$$

$$6y = y+9$$

$$6y - y = 9$$

$$5y = 9$$

$$y = 9/5$$

Put $y = 9/5$ in equ (2)

$$x = \frac{y-3}{3} \rightarrow (2)$$

$$x = \frac{9/5 - 3}{3}$$

$$x = \frac{9 - 15/5}{3}$$

$$x = \frac{-\cancel{6}\,2}{\cancel{15}\,5}$$

$$x = \frac{-2}{5}$$

Critical point : $(x, y) = \left(\frac{-2}{5}, \frac{9}{5}\right)$

Day:

Step 03: (Hessian Determinant)

$$D = \frac{\partial^2 R}{\partial x^2} \cdot \frac{\partial^2 R}{\partial y^2} - \left(\frac{\partial^2 R}{\partial x \partial y}\right)^2 \longrightarrow (4)$$

Now.

$$\frac{\partial^2 R}{\partial x^2} = \frac{\partial}{\partial x}(6x - 2y + 6) \qquad \frac{\partial^2 R}{\partial y^2} = \frac{\partial}{\partial y}(4y - 2x - 8)$$

$$\frac{\partial^2 R}{\partial x^2} = 6 \qquad \frac{\partial^2 R}{\partial y^2} = 4$$

$$\frac{\partial^2 R}{\partial x \partial y} = \frac{\partial}{\partial y}(6x - 2y + 6)$$

$$\frac{\partial^2 R}{\partial x \partial y} = -2$$

Putting values in equ (4)

$$D = (6)(4) - (-2)^2$$

$$= 24 - 4$$

$$D = 20$$

$D > 0$ and $\frac{\partial^2 R}{\partial x^2} > 0$, so the critical point is relative minimum.

Result:

The revenue has a relative minimum at $(x, y) = \left(-\frac{2}{5}, \frac{9}{5}\right)$.