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Inverted pendulum model Linear–Quadratic Regulator (LQR)

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ABSTRACT

The paper presents the control system, used to move pendulum from its hanging-down position to the upright, unstable position and then hold it there with LQR. The algorithm was implemented in LabVIEW, which resulted in application for automatic control of the pendulum. The elaborated simulations with the pendulum physical model verified the controllability and stability of the implemented controller.

Keywords: rotary inverted pendulum, control system, LQR

1. INTRODUCTION

Mathematical modeling and computer simulations play an important role in modern scientific research. They allow you to check the properties and functioning of the device before creating its physical equivalent. Doing so reduces the loss, effort and risk associated with the production of the wrong model or incorrect control of the object.

The development of theoretical research in the context of the rotary pendulum was dictated by the need to develop controllers to balance the parameters of the rocket during takeoff. The reverse pendulum system similar to the rocket during launch is highly unstable and requires a continuous correction mechanism to stay in a vertical position. With this model, you can also analyze the stabilization of the airplane in a turbulent air flow, automatic landing of the aircraft, stabilization of the cabin on the ship, etc.

The inverted pendulum was developed at the Tokyo Institute of Technology by Katsuhisa Furuta and his associates in 1992 [4, 12]. In later years, many articles and works based on this study were created, which were used to demonstrate linear and nonlinear control methods [1, 2, 9, 17]. Despite the huge interest in the subject, few of the resulting publications use the full dynamics of the system. Many works take into account only the rotational moment of inertia of the pendulum for the main axis, as in the article [13], written by M. Iwase and K. Furut. J. Akesson and K. J. Aström in the publication [2] assumed that the moments of inertia of the pendulum for the two main axes are equal and the others are zero. In publications [5, 7] took into account all the moments of inertia and used Lagrange formalism.

You can use different methods to design the controller. Lai K. in the publication [16] used the PID controller. In the article [15] the theory of fuzzy logic is used. In addition, sliding control, pole shifting method as well as control based on artificial neural networks can also be used. The linear-square regulator (LQR) is also used to design the inverted pendulum regulator. It arouses great interest, which is why a lot of different publications on this subject arose, among others: [3, 18-25].

The inverted pendulum model used in the study is a non-linear, unstable object with two degrees of freedom, as well as undersigned, because the controlled quantities are more than inputs in the system. The LabVIEW environment, in which regulation of this object has been implemented, is used to control the DC motor and read the position of the pendulum. In order to stabilize the object, a linear-square regulator was used.

2. PHYSICAL MODEL OF A ROTATING INVERTED PENDULUM

The subject of further considerations is the development of an algorithm for controlling the physical model of an inverted pendulum rotor, which is a product of the cooperating companies National Instruments [24] and Quanser [23] and its implementation in the LabVIEW environment. The control algorithm includes regulators, the pendulum swings from a stable equilibrium position, lift the unstable vertical position up and there will be countered by using the optimization based on linear-quadratic regulator.

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The model of the inverted pendulum is made of a driven and controlled arm and attached pendulum. The arm can be rotated horizontally, while the pendulum can rotate freely in a vertical plane perpendicular to the pivot arm.

The movement of the system is unambiguously defined by the angle θ of the arm displacement from the reference point and the pendulum's inclination angle α with respect to the vertical plane. The angle θ takes positive values when the arm rotates counter-clockwise and negative when the movement is in the other direction. Analogously, the angle α takes positive and negative values depending on the direction of rotation. In the upper position the swing angle of the pendulum reaches +/- 180°, while in the lower position it is 0°.

A pendulum is an unstable object with two degrees of freedom, undersigned (more controlled than inputs) and highly nonlinear, which is due to the force of gravity and coupling resulting from Coriolis and centripetal forces, therefore it must be actively balanced to stay in a vertical position. Table 1 lists and describes basic parameters related to the pendulum model [23].

Table 1. Physical model details.

No.	Symbol	Description	Value
1	M_p	The weight of the pendulum	0,127 [kg]
2	L_p	The length of the pendulum	0,337 [m]
3	J_p	Moment of inertia of the pendulum for the center of mass	0,0012 [kg·m^2]
4	B_p	Damping coefficient for the pendulum	0,0024 [N·m·s/rad]
5	M_r	Weight of the swing arm	0,257 [kg]
6	L_r	Length of the swivel arm	0,216 [m]
7	J_r	Moment of inertia of the pivot arm	0,001 [kg·m^2]
8	B_r	Damping factor for the arm	0,0024 [N·m·s/rad]
9	K	Encoder resolution	4096 [imp/obr]

These parameters and their values will be used in the mathematical model. The LabVIEW environment, in which object regulation has been implemented, is used to control the DC motor and read the position of the pendulum.

2.1 Mathematical model of the pendulum

The mathematical model of the rotary pendulum was derived using the Lagrange formalism, which is used to determine kinematic motion equations for more complex systems. Equations describing the movement of the arm and the pendulum in relation to the voltage on the motor, i.e. the description of the dynamics, were obtained using the Euler-Lagrange equations. To simplify differential equations, Newton's notation was accepted. The simplified model of the mechanism under consideration is presented in Figure 1.

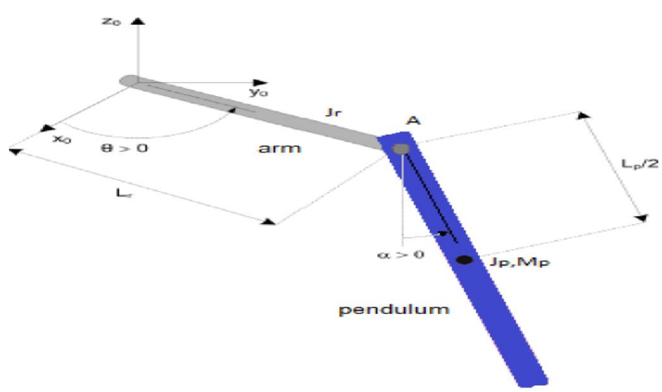


Figure 1. Parameters of the inverted pendulum, included in the calculation model.

A linear state space description of the rotary pendulum used for further research is the following:

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{M_p{}^2L_p{}^2L_rg}{J_T} & -\dfrac{B_r\left(M_pL_p{}^2+4J_p\right)}{J_T} & -\dfrac{2M_pL_pL_rB_p}{J_T} \\ 0 & -\dfrac{2M_p{}^2L_pg\left(J_r+M_pL_r{}^2\right)}{J_T} & -\dfrac{2M_pL_pL_rB_r}{J_T} & -\dfrac{4B_p\left(M_pL_r{}^2+J_r\right)}{J_T} \end{bmatrix},$$

$$B=\begin{bmatrix} 0 \\ 0 \\ \dfrac{M_pL_p{}^2+4J_p}{J_T} \\ \dfrac{4M_pL_pL_r}{J_T} \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

where: $J_T=4J_rJ_p+M_pL_p{}^2J_r+4J_pM_pL_r{}^2$.

Presentation of the system dynamics in the state space form enables the model to be analyzed by means of computer programs in order to obtain a solution. In addition, a simple form of equations allows to use the model, even in case of very complex systems, using simple matrix calculus rules.

3. CONTROL ALGORITHM

Inverted pendulum stabilization in the vertical position has been solved by using a linear-quadratic regulator (LQR). It is a regulator with feedback, that belongs to the optimal control class, where the cost criteria function is minimized. It is determined based on a set of linear differential equations describing the dynamic system and a square quality indicator describing the cost functions. A simplified block diagram showing the operation of the LQR controller for a rotary inverted pendulum is shown in Figure 2.

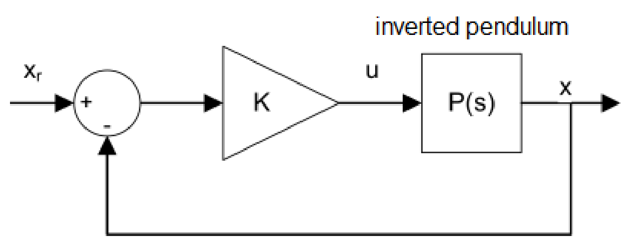


Figure 2. Block diagram of the inverted pendulum feedback model.

The LQR method is used to stabilize only linear systems, while thanks to linearization at a given point (unstable work point - pendulum rod set in vertical position) it can also be used for non-linear systems. The cost function can be interpreted as a penalty imposed on the system status and control values. By contrast, matrices Q and R, which are the weights of the status and control signal, respectively, determine how much weight of punishment is associated with the individual components. This means that these are the tuning parameters of the regulator and the properties of the control system ultimately depend on them. In practice, such a control method boils down to the proper selection of the weight matrix and which affects the cost function. The solution to the problem of linear-quadratic regulation is the vector of feedback reinforcement K. Assuming that the linear system is in equilibrium. The purpose of the control is to keep the system in the equilibrium (or set point), while maintaining the desired position of the arm, despite the occurring disturbances. It is assumed that the pendulum is to be in a vertical position and the maximum deviation from this position is to be less than 20°. To solve the problem of the LQR regulator, the following assumptions are made [10]: the system is observable and controllable [8] and state variables (θ, α) are measurable and available for feedback. The square quality indicator (cost function) used to evaluate the control law is given by:

$$J = \int_0^\infty \left[(x_r - x(t))^T Q (x_r - x(t)) + u(t)^T R u(t) dt \right. \tag{2}$$

where: x_r -reference signal, Q - a symmetrically positively semicircular state balance matrix, R - positive control matrix of the control signal, because there is only one control variable, the R matrix is simplified to the scalar value.

The method of selecting the value of elements on the main diagonals of the Q and R matrices is taken from Bryson's rule [11] and is carried out as follows:

$$Q_{ii} = \frac{1}{a_i} \ , R_{ii} = \frac{1}{b_i} \ , \ i \in \{1,2,...n\}, n \in N \, , \tag{3}$$

where: a - maximum acceptable value x_i^2,b - maximum acceptable value of u_i^2.

In this case, for the system with four state variables and one input, the following matrix weights are required:

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}, \ R = [r_1]. \tag{4}$$

The tuning of the linear-quadratic controller depends on the proper selection of the Q and R matrices, which are selected manually. Depending on what parameters are chosen, the closed loop system shows different responses. When adjusting the individual values of these matrices, their ratio should be taken into account, not their absolute value. In contrast to the individual requirements set for the control system and constraints related to the design of the control system, appropriate matrix values are adjusted to achieve satisfying adjustment effects.

LQR regulator, minimize the cost using a linear function of the state vector, which is expressed by the formula:

$$u = -Kx \, , \tag{5}$$

where: K - reinforcement matrix.

The LQR regulator's gain matrix is expressed by the following formula:

$$K = R^{-1} B^T P \, , \tag{6}$$

where: P - a matrix that is the solution to the Riccati equation.

In order to determine the optimal regulator, it is necessary to solve the continuous Riccati equation with respect to the matrix of gains, which has the form:

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \, , \tag{7}$$

In this case, the state x vector is defined as:

$$x = \left[\theta \ \alpha \ \dot{\theta} \ \dot{\alpha} \right]^T , \tag{8}$$

The reference signal is expressed by:

$$x_r = \left[\theta_r \;\; 0 \;\; 0 \;\; 0\right]^T , \tag{9}$$

where: θ_r- desired arm position angle (the reference point for the pendulum angle position is 0, i.e. the vertical position).

The control strategy used to minimize the cost function is ultimately expressed in the following relationship:

$$u = K(x_r - x) = k_{p,\theta}(\theta_r - \theta) - k_{d,\theta}\dot{\theta} - k_{p,\alpha}\alpha - k_{d,\alpha}\dot{\alpha} , \tag{10}$$

where: $k_{p,\theta}$ - proportional strengthening of the arm deflection angle, $k_{d,\theta}$ - derivative of the arm deflection angle, $k_{p,\alpha}$ - proportional strengthening of the pendulum's yaw angle, $k_{d,\alpha}$ - derivative of the strengthening of the swing angle of the pendulum. The reinforcement matrix finally takes the form:

$$K = \left[-k_{p,\theta} - k_{p,\alpha} - k_{d,\theta} - k_{d,\alpha}\right]. \tag{11}$$

The final block diagram of the feedback system, taking into account the operation of the linear-quadratic regulator, for the rotary pendulum is presented in Figure 3.

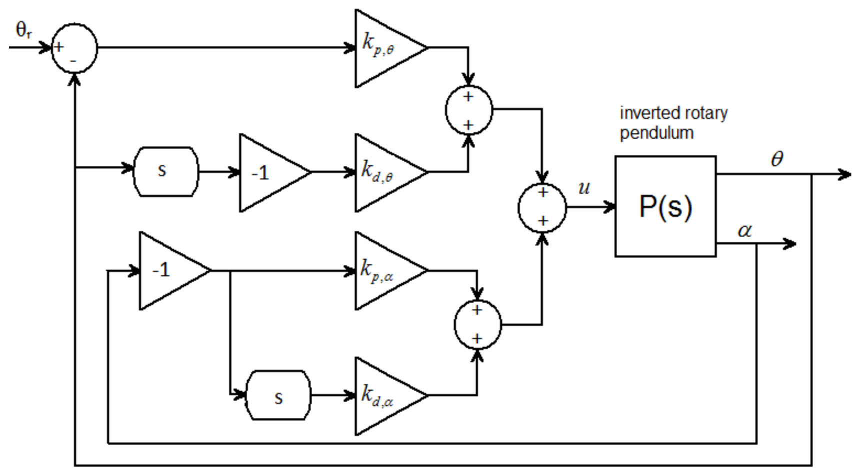


Figure 3. Actual characteristics resulting from the LQR regulator.

The description of the model using a block diagram is a simple and useful graphical method of analyzing control systems. More realistic than a mathematical model, it presents the flow of signals in the system and provides information about the relations between blocks and signals.

4. IMPLEMENTATION OF THE ALGORITHM IN THE LABVIEW ENVIRONMENT AND SIMULATION RESULTS

The algorithm of automatic regulation of the physical model of the inverted pendulum has been implemented in the LabVIEW graphic environment. The individual linear-quadratic regulation steps were carried out using the program's tools. The block diagram was created mainly on the basis of tools from the Control & Simulation tab of the function palette, which offers various algorithms and functions necessary for building and analyzing circuits and for implementing them to the equipment. The rotating pendulum rotary application has three buttons, after selecting which subroutines to open, respectively: The model of the state space, LQR controller, pendulum control, and the STOP button, terminating the application.

After substituting the parameter values of the pendulum model, in accordance with Table 1 and using the sub-scheme Model of the state space, the final description of the state space is expressed by equation (12) and used for further presentation of the results. In the Control of subroutine LQR Controller tab, the conditions for controllability and observability of the system have been calculated and confirmed.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 81{,}4315 & -45{,}6819 & -0{,}93096 \\ 0 & 122{,}004 & -43{,}9374 & -1{,}3948 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 83{,}2817 \\ 80{,}1014 \end{bmatrix} u \quad . \qquad (12)$$

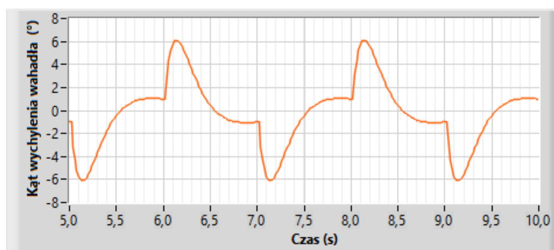
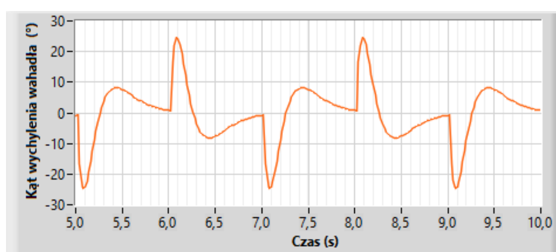
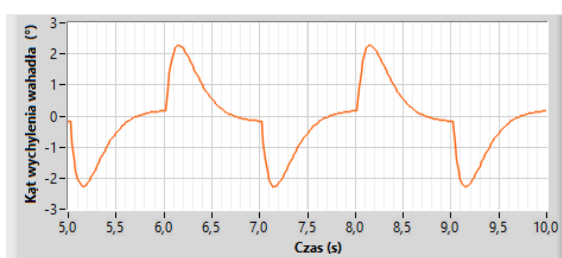
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Matrices Q and R affect the overall performance of the system. Therefore, to obtain a satisfactory design LQR regulator should be carefully selected parameters of these matrices. The control task is to balance the system in a vertical position with the following features: maximum pendulum swing angle: $|\alpha| \le 20°$, maximum control signal: $|U| \le 10V$.

In order to provide the above requirements, a square signal generator with an amplitude of 40° and a frequency of 0.5 Hz was used.

Initially, the parameters of the weight matrices recorded in table 2 were selected, for which the values of the gain matrix were respectively: K=[-1, 14,15, -1,53, 1,93], K=[-5,28, 28,24 -3,10 3,57], K=[-0,32 11,08 -1,21 1,46] and three different response characteristics were obtained (pendulum swing angle versus time), also shown in Table 2

Table 2. Parameters of the weight matrix and response characteristics

No.	Q and R parameters	Response characteristics (pendulum swing angle versus time)
1	$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0{,}1 & 0 \\ 0 & 0 & 0 & 0{,}1 \end{bmatrix}$ $R = [1]$	
2	$Q = \begin{bmatrix} 50 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0{,}1 & 0 \\ 0 & 0 & 0 & 0{,}1 \end{bmatrix}$ $R = [1]$	
3	$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0{,}1 & 0 \\ 0 & 0 & 0 & 0{,}1 \end{bmatrix}$ $R = [10]$	

The simulations were carried out for carefully selected values of the R and Q weight matrices:

$$Q = \begin{bmatrix} 28 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 \\ 0 & 0 & 0{,}01 & 0 \\ 0 & 0 & 0 & 0{,}01 \end{bmatrix} \quad R = [1]. \tag{13}$$

for which the feedback gain vector K, minimizing the cost function, is given as: K=[-5,29 28,08 -3,10 3,58].

The characteristics shown in Figure 4 were recorded in the Pendulum Control subroutine for the pendulum physical model being started. In this case, the control of the swing of the pendulum from a freely hanging position was not included.

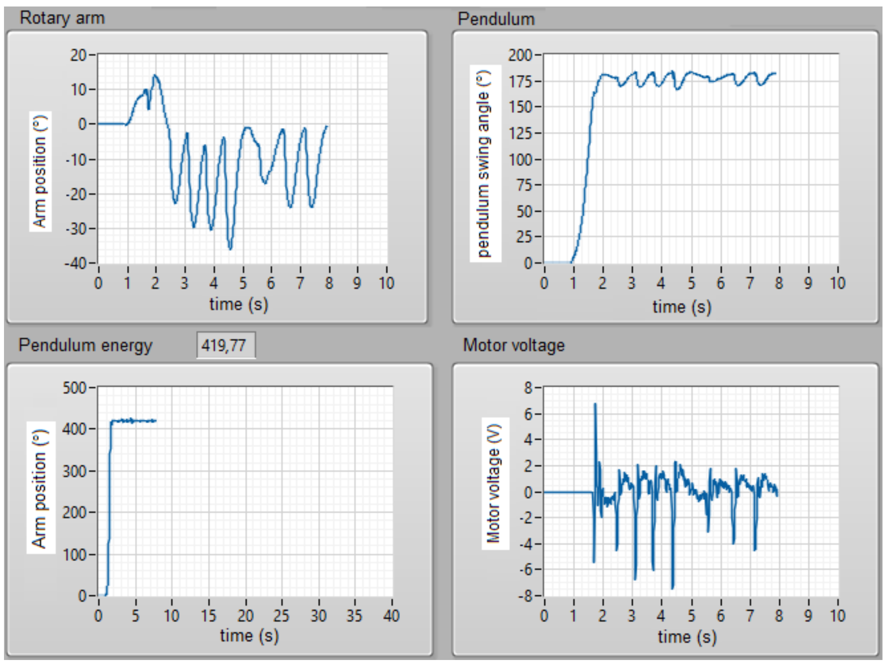


Figure 4. Actual characteristics resulting from the LQR regulator.

Initially, the pendulum was in a freely hanging position, then it was manually lifted up and precipitated from the equilibrium. From the obtained results, it can be noticed that the LQR controller not only maintains the pendulum in a vertical position, but also maintains stability in this position. The voltage on the motor and the swing angle of the pendulum in accordance with the assumptions do not exceed the maximum values.

5. CONCLUSIONS

The article presents a control system, which was aimed at pivoting the pendulum from a freely hanging, stable position to an unstable upper position, and using the LQR regulator to balance this position. The algorithm has been implemented in the LabVIEW environment, which results in an application for automatic control of the pendulum, as well as for didactic purposes. In addition, the correctness of its operation was checked by conducting a series of computer simulations based on a derived mathematical model. In addition, waveforms representing the operation of the physical model of the pendulum were recorded. In addition, the appropriate controller parameters were selected for the correct operation of the system.

In the developed project, in addition to the LQR algorithm, alternative algorithms may be used, such as eg PID controller, polarity shifting or sliding control, which were not considered. The developed application can be extended by adding the function of the above algorithms in order to compare the operation of individual controllers.

The results of the conducted simulation tests can be used, among other things, in the practical determination of the controller settings of the robot or machine as part of an industrial production line.

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