Homework I

Solutions *(consisting of a pdf file containing the analytical derivations and the codes used to solve Exercise 3 and possibly Exercise 4)* have to be uploaded on the course website by h 23:59 of Wednesday November 25, 2020 under the name Homework1.

Problem 1. Consider unitary o-d network flows on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 1. First assume that the links have capacities

$$C_1 = C_4 = 3\,, \qquad C_2 = C_3 = C_5 = 2\,.$$

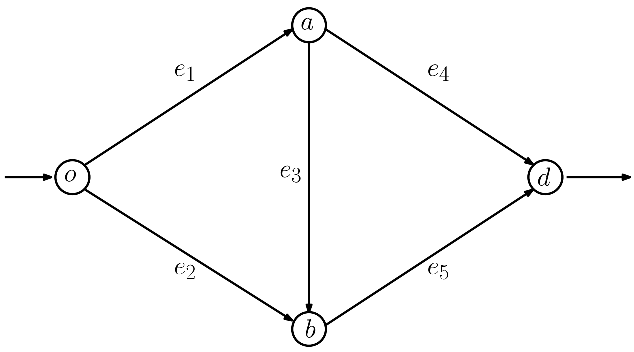


Figure 1

(a) What is infimum of the total capacity that needs to be removed for no feasible unitary flows from o to d to exist?

(b) Where should 2 units of additional capacity be allocated in order to maximize the feasible throughput from o to d? Now, assume the link capacities are all infinite and let the delay functions on the links be given by

$$d_1(x) = d_5(x) = x + 1\,, \qquad d_3(x) = 1\,, \qquad d_2(x) = d_4(x) = 5x + 1\,.$$

(c) Compute the user optimum (i.e., the Wardrop equilibrium) flow vector.

(d) Compute the social optimum flow vector, i.e., the flow vector that minimizes the average delay from o to d.

(e) Compute the price of anarchy.

(f) Find a vector of tolls on the links that reduce the price of anarchy to 1.

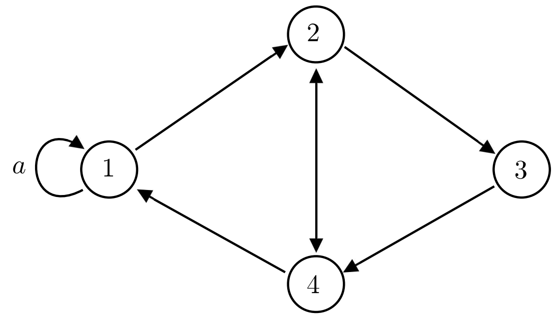


Figure 2: Graph for Problem 2. All link weights are unitary except for the selfloop at node 1, which has weight $a \geq 0$.

Problem 2. Let $\mathcal{G}$ be the graph displayed in Figure 2.

(a) Determine its weight matrix W, normalized weight matrix P, and Laplacian matrix L.

(b) Consider the French-DeGroot opinion dynamics on $\mathcal{G}$:

$$x(t+1) = Px(t) .$$

(c) For which values of $a \geq 0$ does the opinion profile $x(t)$ converge to some limit as $t \to +\infty$ for every initial condition $x(0)$? Motivate your answer.

(d) For $a = 0$, determine, if it exists, the limit opinion profile $\lim_{t \to +\infty} x(t)$ when the initial opinions are

$$x_1(0) = -1 , \qquad x_2(0) = 1 , \qquad x_3(0) = -1 , \qquad x_4(0) = 1 .$$

(e) Determine, if it exists, the minimum value of $a \geq 0$ such that $\lim_{t \to +\infty} x_1(t) \leq 0$ with the same initial conditions as in (d).

(f) For initial condition such that $x_i(0)$, for all $i = 1, 2, 3, 4$, are independent and identically distributed random variables with expected value 0 and variance 1, determine, if it exists, the value of $a \geq 0$ that minimizes the variance of $\lim_{t \to +\infty} x_1(t)$

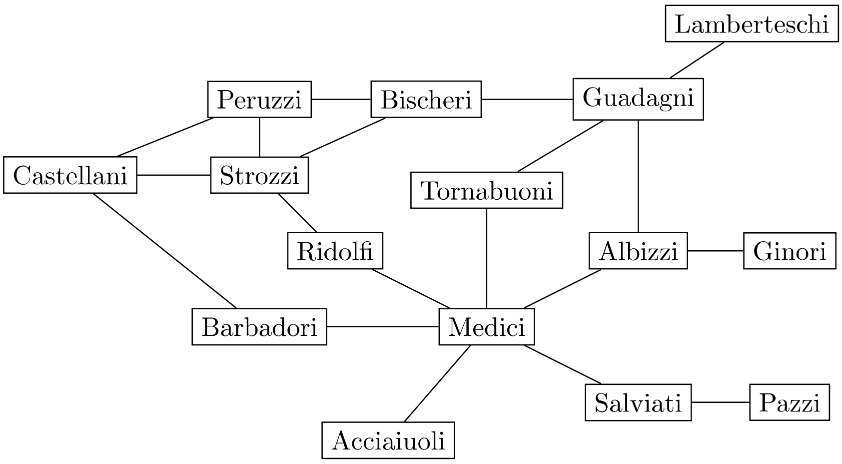


Figure 3: Intermarriages in Florence during the 15th century.

Problem 3. Consider the network of Figure 3 illustrating the marriage-relationships between some of the most influential families in 14th century Florence.

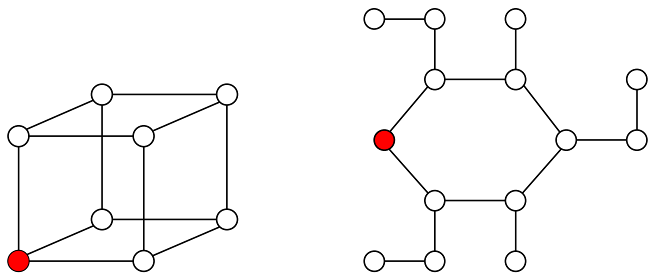
(a) Compute analytically the limit of the distributed averaging dynamics on the graph when the initial states are equal to +1 for the Medici, −1 for the Strozzi, and 0 for all other nodes;

(b) Write down a Matlab or Python code to simulate the averaging dynamics with stubborn node set $\mathcal{S} = \{\text{Medici}, \text{Strozzi}\}$ and opinions $u_{\text{Medici}} = 1$ and $u_{\text{Strozzi}} = -1$; plot the trajectories of the different states; deduce the equilibrium state vector;

(c) Compute analytically the equilibrium vector of the averaging dynamics with stubborn node set $\mathcal{S} = \{\text{Medici}, \text{Strozzi}, \text{Castellani}, \text{Guadagni}\}$ and opinions $u_{\text{Medici}} = 1$ and $u_{\text{Strozzi}} = u_{\text{Castellani}} = u_{\text{Guadagni}} = -1$.

(d) Write down a Matlab or Pythoncode for the iterative distributed computation of the PageRank centrality in the network with $\beta = 0.15$ and uniform input.

Problem 4. Consider the two simple graphs below where the red node is to be interpreted as a stubborn node 0 with opinion $x_0 = 0$.



Find the position s for a second stubborn node with opinion $x_s = 1$ in such a way that, given x the asymptotic opinion profile relative to the averaging dynamics model, the quantity

$$H(s) = \frac{1}{n} \sum_{i \in \mathcal{V}} x_i$$

is maximized.

You can either solve this problem in an analytical way or rather through a numerical simulation. In this last case you have to post the code used for the simulation.