Standard Analysis Procedures for Field Quality Measurement of the LHC Magnets - Part I: Harmonics

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Summary

This note gives the operative definition of the standard analysis procedure to be followed in the acquisition and treatment of the raw-data from the magnetic measurement system. The basic principle of the measurement are outlined and the data treatment procedure is described up to the generation of normalized harmonics.

1. Introduction

The magnetic measurements of the LHC magnets will be based on a rotating coil system. This method delivers a measurement of the magnetic flux linked with the coil as a function of angular position. On the other hand, the main quantities of interest for field quality characterization are the harmonic coefficients of the expansion of the field. Here we wish to recall the formalism that is needed to treat the measurements from the rotating coil system in order to obtain the harmonic coefficients. Several assumptions are made, namely:

- the definition of the field errors is consistent with [1] and, as far as convenient, with the official LHC naming defined in [5];
- the right-handed reference frame of the magnet is defined with the *x* axis coincident with the zero pulse of the angular encoder (also called "index" pulse);
- a positive angle in the reference frame corresponds to a counter-clockwise rotation. This is also the positive direction taken for the encoder rotation;
- no reference is made here to the absolute orientation of the reference frame with respect to the magnet fiducials.

The expressions used here are derived mainly from Refs. [1] through [4] where the basics of the 2-D multipole expansion and its properties are discussed.

2. Magnetic field and flux definitions

2.1. Multipole expansion of the magnetic field

As generally accepted for accelerator magnets, and for use in beam optics simulation, we express the magnetic field B in the 2-D imaginary plane (x,y) using the harmonic expansion in terms of the complex variable $\mathbf{z}=x+iy$:

$$B(\mathbf{z}) = \mathbf{B}_y + i\mathbf{B}_x = \sum_{n=1}^{\infty} \mathbf{C}_n \left(\frac{\mathbf{z}}{R_{ref}} \right)^{n-1} \tag{1}$$

where the coefficients $\mathbf{C}_n$ appearing above are the complex harmonic coefficients, and R_{ref} is the reference radius (R_{ref} is presently 17 mm for LHC). The harmonic coefficients can be also explicitly written as a sum of their real and imaginary parts:

$$\mathbf{C}_n = B_n + iA_n \tag{2}$$

Uppercase notation defines the coefficients in non-normalized terms, i.e. given in [T] at the reference radius. More commonly we must refer to relative coefficients, which we will indicate with lowercase letters:

$$\mathbf{c}_n = b_n + ia_n \tag{3}$$

The normalization procedure to be adopted depends on the magnet function (i.e. the multipole *order* of the magnet) and is described in detail later, in the appropriate sections. In general the normalized coefficients are obtained for a normal magnet of order m (where $m = 1$ is a dipole) using:

$$\mathbf{c}_n = 10^4 \frac{\mathbf{C}_n}{B_m} = 10^4 \left(\frac{B_n}{B_m} + i\frac{A_n}{B_m} \right) = b_n + ia_n \tag{4}$$

where B_m is the main magnetic field expressed in a reference frame where the main skew component is zero. Note the factor 10^4 evidenced, used to produce practical relative units for the normalized coefficients. The normalized $\mathbf{c}_n$ are expressed in the form above in so called *units*.

2.2. Transformation of harmonic coefficients

Two relations among the harmonic coefficients are needed to describe a rigid translation of the reference frame in the 2-D complex plane by a vector $D\mathbf{z} = Dx + i Dy$, and a rotation of the reference frame by an angle q. They are derived from the invariance of the magnetic field:

2.2.1. Reference frame translation

If the reference frame is translated by $D\mathbf{z}$ (see Fig. 1), the harmonic coefficients $\mathbf{C}_k$ (in the original system x,y) transform into $\mathbf{C}'_n$ (in the translated system x',y') according to:

$$\mathbf{C}'_n = B'_n + iA'_n = \sum_{k=n}^{\infty} \left(\frac{(k-1)!}{(n-1)!(k-n)!} \right) \mathbf{C}_k \left(\frac{\Delta \mathbf{z}}{R_{ref}} \right)^{k-n} \tag{5}$$
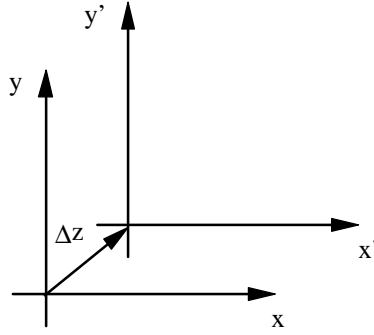
$$n \geq 1$$

Figure 1. Translation of the reference frame

2.2.2. Reference frame rotation

If the reference frame is rotated by and angle q, the harmonic coefficients $\mathbf{C}_n$ (in the original system x,y) transform into $\mathbf{C}'_n$ (in the rotated system x',y') according to:

$$\mathbf{C}'_n = B'_n + iA'_n = \mathbf{C}_n e^{inq} \tag{6}$$
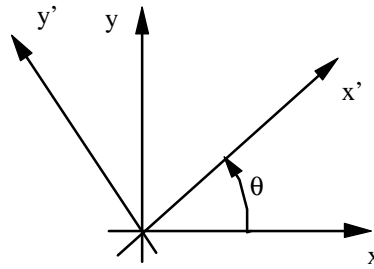

Figure 2. Rotation of the reference frame

2.3. Principle of measurement with a single turn rotating coil

The measurement by a rotating coil delivers the change of the magnetic flux linked with the rotating coil surface. As a first approximation we can imagine to model a single turn, slender coil as two filaments normal to the complex plane (see Fig. 3).
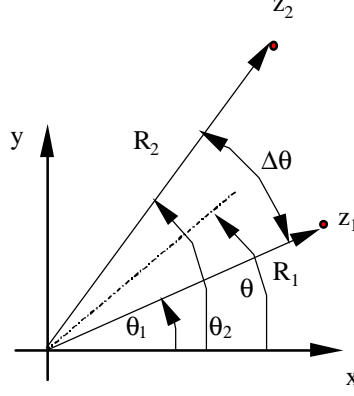
Figure 3. Filaments location in the complex plane and additional parameters used. The filaments are normal to the (x,y) plane and have length L.

The magnetic flux ψ linked by the couple of filaments of length L (along the ignorable dimension of the magnet) located at $\mathbf{z}_1$ and $\mathbf{z}_2$ in the complex plane can be calculated as:

$$\psi = L\,\mathrm{Re}\left[\int_{\mathbf{z}_1}^{\mathbf{z}_2} \mathbf{B}(\mathbf{z})d\mathbf{z}\right] \tag{7}$$

Using the multipoles expansion of the magnetic field, we obtain the following expression for the magnetic flux containing directly the harmonic coefficients:

$$\psi = L\,\mathrm{Re}\left[\sum_{n=1}^{\infty} \frac{1}{nR_{ref}^{n-1}} \mathbf{C}_n\left(\mathbf{z}_2^n - \mathbf{z}_1^n\right)\right] \tag{8}$$

During a measurement the coil rotates, and the instantaneous position is given by:

$$\begin{aligned}
\mathbf{z}_1(t) &= \mathbf{z}_1(0)e^{iq(t)} = R_1 e^{iq_1} e^{iq(t)} \\
\mathbf{z}_2(t) &= \mathbf{z}_2(0)e^{iq(t)} = R_2 e^{iq_2} e^{iq(t)}
\end{aligned} \tag{9}$$

where the time dependent position has been written using filament radii R_1 and R_2, and initial phases q_1 and q_2 in the complex plane (see Fig. 3). We can now write the following relation:

$$\begin{aligned}
\mathbf{z}_2^n - \mathbf{z}_1^n &= \left(R_2^n e^{inq_2} - R_1^n e^{inq_1}\right)e^{inq} = \\
&= \left(R_2^n e^{in\frac{\Delta q}{2}} - R_1^n e^{-in\frac{\Delta q}{2}}\right)e^{inq_0}\,e^{inq}
\end{aligned} \tag{10}$$

where we have introduced an average initial phase q_0 and a phase difference, or *opening*, Δq defined by (see Fig. 3):

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$$q_0 = \frac{q_1 + q_2}{2}$$

$$\Delta q = q_2 - q_1 \tag{11}$$

We can now put the expression Eq. (8) in the following compact form:

$$\psi(q) = L \operatorname{Re} \left[\sum_{n=1}^{\infty} \frac{\chi_n}{n R_{ref}^{n-1}} C_n e^{inq} \right] \tag{12}$$

where the complex *coil geometric factors* χ_n are related to the coil sensitivity that will be discussed later, and are defined as:

$$\chi_n = \left(R_2^n e^{in\frac{\Delta q}{2}} - R_1^n e^{-in\frac{\Delta q}{2}} \right) e^{inq_0} \tag{13}.$$

The form above is useful to describe an ideal coil that rotates rigidly around its axis, without variations of the radii and of the opening. A rigid rotation is described by a periodic change of the average phase q, while χ_n remains constant throughout the rotation.

2.4. Voltage pickup of a single turn rotating coil

The voltage seen by a single turn rotating coil is, by definition:

$$V = -\frac{\partial \psi}{\partial t} = -L \operatorname{Re} \left[\sum_{n=1}^{\infty} \frac{1}{n R_{ref}^{n-1}} \chi_n e^{inq} \left(\frac{\partial C_n}{\partial t} + in C_n \frac{\partial q}{\partial t} \right) \right] \tag{14}.$$

We see from here how a voltage is induced both by a variation of the field (time derivative of the harmonic coefficients) and by a rotation of the coil (time derivative of the angular position).

2.5. Relation between discretely sampled fluxes and harmonic coefficients

As will be explained later on, each rotating coil measurement delivers (after proper processing and normalization for the gains of the chain of amplifiers and converter) the value of the magnetic flux $\psi(q_k)$ as a function of the rotation angle q_k in a discrete series of points k for a total of N points. The sampling points are equally spaced over the $[0...2\pi]$ interval, and we indicate the sampled points in short as ψ_k. We wish to reconstruct the harmonic coefficients from this measurement. To do this we use a discrete Fourier transform (DFT) defined as follows:

$$\Psi_n = \sum_{k=1}^{N} y_k e^{-2\mathrm{p}i(n-1)\frac{(k-1)}{N}} \tag{15}$$

$$n = 1...N$$

where we have introduced the DFT complex coefficients Ψ_n. We recall that the inverse DFT (signal reconstruction) is given by:

$$y_k = \frac{1}{N} \sum_{n=1}^{N} \Psi_n e^{2\mathrm{p}i(k-1)\frac{(n-1)}{N}} \tag{16}.$$

$$k = 1...N$$

As shown in Appendix I, it is possible to establish a relation between the DFT coefficients and the field harmonic coefficients C_n. This relation, for an even number of points N, is given by:

$$C_n \approx \frac{2}{N} \frac{1}{L} \frac{n R_{ref}^{n-1}}{\chi_n} \Psi_{n+1} \tag{17}$$

$$n = 1... \frac{N}{2}$$

that is valid for the single turn idealized coil discussed up to now.

2.6. Coil sensitivity

As we have shown in the previous section, the field harmonics C_n can be obtained from the DFT coefficients of the flux using Eq. (17). In the case of an ideal coil wound with N_t turns (with negligible winding size) we can write the same expression in the following form:

$$C_n \approx \frac{2}{N} \frac{R_{ref}^{n-1}}{\kappa_n} \Psi_{n+1} \tag{18}$$

$$n = 1... \frac{N}{2}$$

where the coefficients κ_n are the complex *coil sensitivity* coefficients to the harmonic of order n. These coefficients are proportional to the coil geometric factors χ_n introduced earlier (see Eq. (13)), and are given by:

$$\kappa_n = \frac{N_t L \chi_n}{n} \tag{19}.$$

Note that the coil sensitivity is in general a complex number. Two particular cases are of importance, that of a radial and of a tangential coil. The general case, including the effect of finite winding thickness, is treated in detail in [6].

2.6.1. Perfect radial coil

For a perfect radial coil initially on the horizontal plane (zero initial phase q_0) the absolute sensitivity reduces to a real number:

$$\kappa_n^{\text{radial}} = \frac{N_t L}{n}\left(R_2^n - R_1^n\right) \tag{20}$$

where R_1 is the internal radius of the coil and R_2 is the external radius of the coil, both referred to the centers of gravity of the coil windings.

2.6.2. Perfect tangential coil

For a perfect tangential coil initially symmetric with respect to the horizontal plane (zero initial phase q_0) the sensitivity reduces to an imaginary number:

$$\kappa_n^{\text{tangential}} = \frac{2iN_t L}{n} R^n \sin\left(\frac{n\Delta q}{2}\right) \tag{21}$$

where R is the radius of the centers of gravity of the winding and Dq is the coil opening angle.

2.6.3. Series connections of coils

The expressions above apply to the case of a single coil reading the magnetic flux, what is called an *absolute* measurement. Usually in addition to an *absolute* measurement, used to determine the main field component, a *compensated* measurement is taken. This is done connecting different coils with appropriate weights (gains) so to cancel the main field component. For a set of S coils, each of sensitivity κ_n^s, connected in a compensation scheme with gains g_s, the following expression is used to obtain the total sensitivity coefficients κ_n to an harmonic n:

$$\kappa_n = \sum_{s=1}^{S} g_s \kappa_n^s \tag{22}$$

Note that to obtain the coil sensitivity of the compensated coil set we need the different sensitivities of each single coil <u>and</u> the gains used for the sum of their signals.

3.   Standard analysis of raw-data

In this section we use the relations established previously, and the results of Appendices I through IV, to describe the treatment of the raw-data generated by the rotating coil measurement. This standard procedure must be followed to generate tables of harmonic coefficients for each measurement.

3.1.   Definition of a "measurement"

Each *measurement* consists in the reading of an *absolute* and a *compensated* signals as delivered by rotating coils over a complete forward and backward rotation. The coil angular position is read by an angular encoder.

The absolute signal is obtained from the reading of a single coil (the outermost, which is in general the one with the highest sensitivity to the main field component), and is used for the determination of the main field component (or verification). The compensated (or *bucked*) signal is obtained as a combination of the signals of different coils, and is used for the determination of the field errors. In some special cases the only signal read is the absolute one (e.g. for testing and checking purposes). In this case the compensated signal is missing for the measurement analysis and the harmonics must be computed directly from the absolute signal.

3.2.   Reading of coil voltage and integration

The two voltage signals (absolute and compensated) from each coil group are sent to VME integrators, possibly through a chain of external amplifiers. A VME integrator amplifies the input voltage signal and converts the amplified voltage to a series of variable frequency pulses (voltage-to-frequency conversion). The variable frequency pulses are counted during the time interval determined by two subsequent trigger pulses from the angular encoder. The counts obtained are thus proportional to the integral of the voltage between two encoder trigger pulses. The result of each integration step, the *flux increments* $\mathrm{D}f_k$, is available on the VME bus at each angular interval. In addition the integrator, via an internal time base, provides the time interval $\mathrm{D}t_k$ between two subsequent trigger pulses from the encoder.

The flux increments are in units of *counts* from the VFC-counter unit. Counts are proportional to the integrated voltage (Vs), and the constant of proportionality is the product of the gains in the line of amplifiers and of the transfer function of the VFC. For present integrators this constant is 500/10 (KHz/V).

The time intervals are also in units of *counts*, and are obtained from the internal reference of 1 MHz in the counter. Time counts, proportional to the time interval between two encoder pulses, are thus in units of (1/MHz), i.e. in practical terms in units of (µs).

The raw-data is considered to be the stored flux increments $\mathrm{D}f_k$ and the time intervals $\mathrm{D}t_k$ (i.e. as output from the VME integrators) for the forward and backward rotations,

in a total of N angular points per rotation (defined by the encoder trigger pulses). In addition all complementary information from the acquisition (current, gains, rotation speed, etc.) is regarded as additional raw-data. This data is stored on disk for treatment.

3.3. Non-normalized harmonics from DC measurements

This procedure is used for the standard analysis of measurements taken in DC conditions (i.e. with constant current in the measured magnet):

1. Pulse time. Compute the time of the trigger pulse from the angular encoder, for absolute and compensated signals, forward and backward rotations (Eqs. (AII.3), (AII.4))

2. Conversion. Convert the raw flux increments from integrator [counts] to [Vs], for absolute and compensated signals, forward and backward rotations (Eqs. (AII.8), (AII.9))

3. Voltage offset. Compute the voltage offset, for absolute and compensated signals, forward and backward rotations (Eq. (AII.12))

4. Drift correction. Correct the flux increments for the voltage offset, for absolute and compensated signals, forward and backward rotations (Eq. (AII.14))

5. Signal average. Average the flux increments, for absolute and compensated signals (Eq. (AII.15))

6. Magnetic flux. Integrate the flux increments, for absolute and compensated signals (Eqs. (AII.17), (AII.18))

7. Fourier transform. Frequency transform the integrated flux (DFT), for absolute and compensated signals (Eq. (AII.19))

8. Amplitude spectrum. Fold spectrum of the frequency transform, for absolute and compensated signals (Eq. (AII.20) or (AII.21))

9. Harmonics. Compute field harmonics, for absolute and compensated signals (Eq. (AII.22))

The result of this calculation are the non-normalized field harmonics in the reference frame of the rotating coil. In complex terms the coefficients obtained are:

$$\mathbf{C}_1^{abs} \qquad \mathbf{C}_1^{cmp} \qquad \text{complex harmonic, order 1}$$
$$... \qquad\qquad ...$$
$$\mathbf{C}_{Nm}^{abs} \qquad \mathbf{C}_{Nm}^{cmp} \qquad \text{complex harmonic, order } N_m$$

where N_m is the number of harmonic components retained (typically N_m=15), and we remark that two records (absolute and compensated) are generated by the procedure.

All operations on the compensated signal are possible only when the compensated signal is read-out (i.e. in a *compensated* measurement). They are skipped otherwise (i.e. in an *absolute* measurement).

3.4. Non-normalized harmonics from AC measurements

This procedure is used for the standard analysis of measurements taken in AC conditions (i.e. during ramps of current). It is identical to the procedure used for DC measurements, but skips the drift correction for the single rotations (not possible because the magnetic flux through the coil changes in time). Because of this no pulse time is needed. The drift is corrected (to first order) by the averaging of the forward and backward rotations:

1. Conversion. Convert the raw flux increments from integrator [counts] to [Vs], for absolute and compensated signals, forward and backward rotations (Eqs. (AII.8), (AII.9))

2. Signal average. Average the flux increments, for absolute and compensated signals (Eq. (AII.15))

3. Magnetic flux. Integrate the flux increments, for absolute and compensated signals (Eqs. (AII.17), (AII.18))

4. Fourier transform. Frequency transform the integrated flux (DFT), for absolute and compensated signals (Eq. (AII.19))

5. Amplitude spectrum. Fold spectrum of the frequency transform, for absolute and compensated signals (Eq. (AII.20) or (AII.21))

6. Harmonics. Compute field harmonics, for absolute and compensated signals (Eq. (AII.22))

As for the DC case, the result of this calculation are the non-normalized field harmonics in the reference frame of the rotating coil. The same structure is generated.

3.5. Feed-down correction

Feed-down correction is necessary to compute the harmonics in the magnet center. The definition of the center location is different for dipole magnets and higher order multipole magnets (see Appendix III).

3.5.1. Dipole

1. Center location. Find the center location $\Delta\mathbf{z}$ (with respect to the coil rotation axis) that cancels the normal and skew 16-pole (order $n=8$) using the compensated harmonics $\mathbf{C}_n^{cmp}$ in case of a *compensated* measurement or the absolute harmonics $\mathbf{C}_n^{abs}$ in case of an *absolute* measurement (Eqs. (AIII.1), (AIII.2))

2. <u>Feed-down correction</u>. Compute the harmonics (absolute and compensated) in the reference frame translated by $\Delta\mathbf{z}$ (Eq. (AIII.6))

3.5.2. 2*m*-pole magnet (*m*>1)

1. <u>Center location</u>. Find the center location $\Delta\mathbf{z}$ (with respect to the coil rotation axis) that cancels the normal and skew $2(m\text{-}1)$-pole using the absolute harmonics $\mathbf{C}_n^{abs}$ (Eqs. (AIII.4) or (AIII.5))

2. <u>Feed-down correction</u>. Compute the harmonics (absolute and compensated) in the reference frame translated by $\Delta\mathbf{z}$ (Eq. (AIII.6))

For both dipole and 2*m*-pole magnets the result of these calculations are the *centered* non-normalized harmonics (in a reference frame with origin in the magnet center), and the center coordinates with respect to the coil rotation axis.

3.6. Normalized harmonics

In order to obtain normalized harmonics in *units*, the non-normalized harmonics must be rotated in the reference frame of the main field first, and normalized to the main field afterwards. The procedure to be followed for a general 2*m*-pole magnet is:

1. <u>Main field module</u>. Compute from the absolute harmonics $\mathbf{C}_m^{abs}$ the main field module $|\mathbf{C}_m|$ (Eq. (AIV.1))

2. <u>Main field phase</u>. Compute the phase of the main field $\jmath_m$ from the absolute harmonics $\mathbf{C}_m^{abs}$ (Eq. (AIV.2) or (AIV.3)). The phase $\jmath_m$ is limited to the interval [-$\pi/2$ ... $\pi/2$] (Eq. (AIV.4))

3. <u>Angle</u>. Compute the direction of the main field a_m with respect to the measurement reference frame (Eq. (AIV.5))

4. <u>Rotation</u>. Compute the non-normalized harmonics (absolute and compensated) in the reference frame of the main field, rotating them by a_m (Eq. (AIV.6))

5. <u>Normalization</u>. Compute the normalized harmonics $\mathbf{c}_n$ of order n higher than the main field order m (i.e. $n > m$) (Eq. (AIV.8) or (AIV.9))

The result of this procedure are rotated harmonics (absolute and compensated), normalized to the main field for all orders higher than *m*. Note that the harmonic of order *m* has no skew (imaginary) part because of the rotation.

$\mathbf{C}_1^{abs}$	$\mathbf{C}_1^{cmp}$	complex, rotated harmonic, order 1
...	...	
$\mathbf{C}_{m\text{-}1}^{abs}$	$\mathbf{C}_{m\text{-}1}^{cmp}$	complex, rotated harmonic, order *m*-1
$\mathbf{C}_m^{abs}$	$\mathbf{C}_m^{cmp}$	complex, rotated harmonic, order *m*
$\mathbf{c}_{m+1}^{abs}$	$\mathbf{c}_{m+1}^{cmp}$	complex, rotated, normalized harmonic, order *m*+1

...	...	
$c_{Nm}{}^{abs}$	$c_{Nm}{}^{cmp}$	complex, rotated, normalized harmonic, order N_m

3.7. Record of harmonics

The compensated harmonics contain in principle no component below the main field order m (as the compensation ideally cancels these components). On the other hand the absolute harmonics have a large component at the main field order m and possibly lower orders (due to feed-down) that may pollute the quality of the higher order harmonics. Therefore the standard measurement result should use the absolute harmonics up to order m, and the compensated harmonics from there on (if present). Furthermore, through the feed-down correction and the rotation in the main field reference frame some of the harmonics are canceled by definition. The information on center location and angle must be therefore added to the final harmonic record. The structure, written explicitly in terms of the real and imaginary part of the complex harmonics, is the following:

$\mathrm{D}x$	$\mathrm{D}y$	magnetic center coordinates
$B_1{}^{abs}$	$A_1{}^{abs}$	complex, rotated, absolute harmonic, order 1
...	...	
$B_{m-1}{}^{abs}$	$A_{m-1}{}^{abs}$	complex, rotated, absolute harmonic, order m-1
B_m	a_m	main field and field angle
$b_{m+1}{}^{cmp}$	$a_{m+1}{}^{cmp}$	complex, rotated, normalized, compensated harmonic, order m+1
...	...	
$b_{Nm}{}^{cmp}$	$a_{Nm}{}^{cmp}$	complex, rotated, normalized, compensated harmonic, order N_m

3.8. Measurement quality verification

A series of measurement quality checks should be performed on line with the calculation of harmonics. The aim is to exclude an evident malfunction of a mechanical or electronic component, excess electric noise, power supply instabilities and ripple:

1. Rotation speed. The rotation speed is computed according to Eq. (AII.7). A warning is raised if its variations around the average rotation speed is larger than $\pm$ 5 %

2. Amplifiers and/or integrators offset. The voltage offset is computed according to Eq. (AII.12). A warning flag is raised if the offset V_{off} is larger than 1 mV (equivalent at input)

3. Error between rotations. The flux error e_k between rotations is computed with Eq. (AII.16). A warning flag is raised when the maximum error $max\{e_k\}$ exceeds 10 % of the maximum average flux increments amplitude $max\{\mathrm{D}y_k\}$

4. <u>Electric noise</u>. The procedure is not yet specified

5. <u>Ripple</u>. The procedure is not yet specified

6. *<u>Bucking ratio</u>*. The bucking ratio b_n is computed using Eq. (AII.23) for the main harmonic and for the next lower one, i.e. *n = m-1, m*

4.   References

[1]   R. Wolf, *Field Error Definitions for LHC. Version 1*, AT-MA Internal Note 94-102, CERN, 1994.

[2]   A. Devred, M. Traveria, *Magnetic Field and Flux of Particle Accelerator Magnets in Complex Formalism*, DSM/DAPNIA/STCM Internal Note CRYOMAG/94/08, CEA-Saclay, 1994.

[3]   A. Devred, *Complex Formalism for Two Dimensional Fields*, DSM/DAPNIA/STCM Internal Note CRYOMAG/96/01, CEA-Saclay, 1996.

[4]   A. Jain, *Harmonic Coils*, presented at CERN Accelerator School on Measurements and Alignment of Accelerator and Detector Magnets, April 11-17, 1997, Anacapri, Italy

[5]   R. Wolf, *Field Error Naming Conventions for LHC Magnets*, LHC Project Document LHC-M-ES-0001.00 rev. 1.0.

[6]   L. Deniau, *Coils Calibration – Computation and Storage of Sensitivity Factors*, MTA-IN-98-021.

5.   Appendix I - Correspondence between DFT and harmonic coefficients

Any periodic function $y(q)$ of the angular position q (with period $2p$) can be expanded in Fourier series as follows:

$$y(q) = \sum_{m=-\infty}^{\infty} \mathbf{\Phi}_m e^{imq} \tag{AI.1}$$

where the complex quantities $\mathbf{\Phi}_m$ are the Fourier coefficients of the expansion, and are obtained by projection of the function $y(q)$ on the basis of the expansion:

$$\mathbf{\Phi}_m = \frac{1}{2p} \int_0^{2p} y(q) e^{-imq} dq \tag{AI.2}$$

In the integral above the period can be shifted arbitrarily along the real axis. If we take as a function $y(q)$ the magnetic flux linked with the ideal, single turn coil, given in Eq. (12):

$$y(q) = L \operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{\mathbf{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n e^{inq}\right] \tag{AI.3}$$

we have that the Fourier coefficients are:

$$\begin{aligned}
\mathbf{\Phi}_m &= \frac{1}{2p} \int_0^{2p} y(q) e^{-imq} dq \\
&= \frac{1}{2p} \int_0^{2p} L \operatorname{Re}\left[\sum_{n=1}^{\infty} \frac{\mathbf{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n e^{inq}\right] e^{-imq} dq
\end{aligned} \tag{AI.4}$$

Because the operators of integration, summation and real part are linear we are allowed to exchange their order as follows:

$$\mathbf{\Phi}_m = \frac{L}{2p} \sum_{n=1}^{\infty} \left\{ \int_0^{2p} \operatorname{Re}\left[\frac{\mathbf{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n e^{inq}\right] e^{-imq} dq \right\} \tag{AI.5}$$

We now proceed taking the real part of the term in square brackets:

$$\Phi_m = \frac{L}{2p} \sum_{n=1}^{\infty} \int_0^{2p} \left\{ \mathrm{Re}\left[\frac{\boldsymbol{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n \right] \cos(nq)\cos(mq) \right.$$

$$- i\,\mathrm{Re}\left[\frac{\boldsymbol{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n \right] \cos(nq)\sin(mq)$$

$$- \mathrm{Im}\left[\frac{\boldsymbol{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n \right] \sin(nq)\cos(mq)$$

$$\left. + i\,\mathrm{Im}\left[\frac{\boldsymbol{\chi}_n}{nR_{ref}^{n-1}} \mathbf{C}_n \right] \sin(nq)\sin(mq) \right\} dq \qquad \text{(AI.6)}$$

where we have decomposed all complex exponential functions in their harmonic functions components. We recall the following properties of the harmonic functions:

$$\int_0^{2p} \cos(nq)\cos(mq)\,dq = p\,d_{n|m|}$$

$$\int_0^{2p} \sin(nq)\sin(mq)\,dq = \begin{cases} p\,d_{n|m|} & for \quad m > 0 \\ -p\,d_{n|m|} & for \quad m < 0 \end{cases}$$

$$\int_0^{2p} \cos(nq)\sin(mq)\,dq = 0 \qquad \text{(AI.7)}$$

$$\int_0^{2p} \sin(nq)\cos(mq)\,dq = 0$$

where d_{ij} is the Kronecker delta function. In its definition we have taken into account the fact that while n is non-zero and always positive, m can span the positive and negative integer sets. Using the properties (AI.6) into Eq. (AI.5) we finally obtain that:

$$\Phi_m = \begin{cases} \dfrac{Lp}{2p}\left(\mathrm{Re}\left[\dfrac{\boldsymbol{\chi}_m}{mR_{ref}^{m-1}} \mathbf{C}_m \right] + i\,\mathrm{Im}\left[\dfrac{\boldsymbol{\chi}_m}{mR_{ref}^{m-1}} \mathbf{C}_m \right] \right) & for \quad m > 0 \\[4mm] \dfrac{Lp}{2p}\left(\mathrm{Re}\left[\dfrac{\boldsymbol{\chi}_{|m|}}{|m|R_{ref}^{|m|-1}} \mathbf{C}_{|m|} \right] - i\,\mathrm{Im}\left[\dfrac{\boldsymbol{\chi}_{|m|}}{|m|R_{ref}^{|m|-1}} \mathbf{C}_{|m|} \right] \right) & for \quad m < 0 \end{cases} \qquad \text{(AI.8)}$$

that leads finally to the following result:

$$\Phi_m = \begin{cases} \dfrac{L}{2}\dfrac{\boldsymbol{\chi}_m}{mR_{ref}^{m-1}} \mathbf{C}_m & for \quad m > 0 \\[4mm] \dfrac{L}{2}\dfrac{\boldsymbol{\chi}_{|m|}}{|m|R_{ref}^{|m|-1}} \mathbf{C}_{|m|}^* & for \quad m < 0 \end{cases} \qquad \text{(AI.9)}$$

where the star superscript stands for the complex conjugate operator. Both Eqs. (AI.8) and (AI.9) show clearly that the spectrum of Fourier coefficients of the real function $y(q)$ has symmetric real part and anti-symmetric imaginary part. As common practice in signal analysis theory, half of the spectrum amplitude is contained in the *positive* frequencies semi-axis (positive values of *m*), and the other half is in the *negative* frequencies semi-axis (negative values of *m*). Only one half of the spectrum is sufficient to describe the expansion completely, and we can therefore take:

$$\mathbf{C}_n = \frac{2}{L} \frac{nR_{ref}^{n-1}}{\chi_n} \mathbf{\Phi}_n$$

$$n = 1...\infty$$

(AI.10)

The considerations above are valid for a continuous, periodic signal $y(q)$. In our case however we are dealing with a regularly spaced, discrete sample of this periodic signal. It is therefore necessary to introduce a further relation, namely that on the Fourier series expansion of a periodically sampled periodic signal. It is common practice to describe the operation of sampling of a signal as the product of the continuous signal $y(q)$) and a regular array of *delta* functions as follows:

$$S[y(q)] = \sum_{k=-\infty}^{\infty} y(q)d(q - k\Delta q) = \sum_{n=-\infty}^{\infty} y_k d(q - k\Delta q)$$

(AI.11)

where $S[y(q)]$ stands for the sampling operator on the signal, k is the sampling index, y_k is the signal at the sampling point $k Dq$, and Dq is the sampling interval defined as:

$$\Delta q = \frac{2p}{N}$$

(AI.12)

where N is the number of samples taken, that we assume for the moment to be even. Note that the sampled distribution $S[y(q)]$ above is <u>not</u> identical to the series of values of the signal at the sampling points. Even from a dimensional point of view the sampled signal differs from the continuous signal by the dimensions of the inverse of the sampling interval. Equation (AI.11) is in fact only useful to represent an approximation of the continuous signal through its average value during the sampling interval, as we have:

$$\int_{q+\left(k-\frac{1}{2}\right)\Delta q}^{q+\left(k+\frac{1}{2}\right)\Delta q} y(q)dq = \overline{y}\,\Delta q \approx y_k \Delta q = \Delta q \int_{q+\left(k-\frac{1}{2}\right)\Delta q}^{q+\left(k+\frac{1}{2}\right)\Delta q} y(q)d(q - k\Delta q)dq$$

(AI.13)

where the over-bar denotes the average value in the integration interval. Note that, as we said before, this representation involves forcibly an approximation of the continuous signal. In particular we can interpret the discrete sampling as an approximation of the function in an average sense during the sampling interval, or also:

$$\lim_{\Delta q \to 0} \left\{ \int_{-\infty}^{\infty} y(q) dq - \Delta q \int_{-\infty}^{\infty} S[y(q)] dq \right\} = 0 \qquad (AI.14)$$

which is equivalent in turn to:

$$y(q) \approx \Delta q S[y(q)] \qquad (AI.15).$$

In analogy to what we have done for the continuous signal, we can compute the Fourier series coefficients Σ_m of the approximate sampled signal using the definition Eq. (AI.2) and the approximation Eq. (AI.15):

$$\Sigma_m = \frac{\Delta q}{2p} \int_{-\frac{\Delta q}{2}}^{2p - \frac{\Delta q}{2}} S[y(q)] e^{-imq} dq$$

$$= \frac{1}{N} \int_{-\frac{\Delta q}{2}}^{2p - \frac{\Delta q}{2}} \sum_{k=-\infty}^{\infty} y(q) d(q - k\Delta q) e^{-imq} dq \qquad (AI.16)$$

where we have shifted the extremes of integration by half a sampling interval, so that the sampled signals that fall inside the integration domain are those with a sampling index $0 \le k \le N\text{-}1$. Because of linearity of the operators of summation and integration, we exchange the operators as done previously, and we integrate obtaining:

$$\Sigma_m = \frac{1}{N} \sum_{k=0}^{N-1} y_k e^{-2pim\frac{k}{N}} \qquad (AI.17).$$

Equation (AI.17) is indeed identical to the definition of the discrete Fourier transform (DFT) as given in Eq. (15) apart for a shift of the extremes of summation and the factor $1/N$ appearing in front of the summation. Therefore we can state that the DFT coefficients Ψ_m of an array of samples y_k are identical (apart for a proportionality factor) to the Fourier coefficients Σ_m of the approximation of the continuous signal given by Eq. (AI.15), or:

$$\Sigma_m = \frac{1}{N} \Psi_{m+1} \qquad (AI.18).$$

On the other hand the Fourier coefficients Σ_m of the sampled signal approximation can be considered as an approximation of the Fourier coefficients Φ_m of the continuous signal $y(q)$. Therefore we can write that:

$$\Phi_m \approx \Sigma_m \qquad (AI.19).$$

Finally using the result of Eqs. (AI.10). (AI.18) and (AI.19) we can write that :

$$\mathbf{C}_n \approx \frac{2}{N} \frac{1}{L} \frac{n R_{ref}^{n-1}}{\boldsymbol{\chi}_n} \boldsymbol{\Psi}_{n+1}$$

$$n = 1 ... \frac{N}{2}$$
(AI.20)

which gives the desired relation between the harmonic coefficients $\mathbf{C}_n$ and the DFT coefficients $\boldsymbol{\Psi}_{n+1}$ In the case of an *even* number of samples N. The case of an *odd* number is obtained readily by a simple change of the upper boundary of the index n:

$$\mathbf{C}_n \approx \frac{2}{N} \frac{1}{L} \frac{n R_{ref}^{n-1}}{\boldsymbol{\chi}_n} \boldsymbol{\Psi}_{n+1}$$

$$n = 1 ... \frac{N-1}{2}$$
(AI.21)

As a final remark, note that in the form given by Eq. (15) the DFT coefficients are in an array ranging from 1 to N, containing as first term the DC component (irrelevant for our purposes, as it correspond to the arbitrary constant in the definition of the magnetic flux), then the ordered series of the positive frequencies and finally the swapped series of negative frequencies. The ordering is as follows, for even or odd values of N (the only difference being the presence or absence of the Nyquist frequency at the mid location):

	N even		*N* odd
$\boldsymbol{\Psi}_1$	DC component	$\boldsymbol{\Psi}_1$	DC component
$\boldsymbol{\Psi}_2$	1st harmonic	$\boldsymbol{\Psi}_2$	1st harmonic
$\boldsymbol{\Psi}_3$	2nd harmonic	$\boldsymbol{\Psi}_3$	2nd harmonic
.	.	.	.
.	.	.	.
$\boldsymbol{\Psi}_{N/2}$	$(N/2-1)$th harmonic	.	.
$\boldsymbol{\Psi}_{N/2+1}$	Nyquist harmonic	$\boldsymbol{\Psi}_{(N-1)/2+1}$	$(N-1)/2$th harmonic
$\boldsymbol{\Psi}_{N/2+2}$	$-(N/2-1)$th harmonic	$\boldsymbol{\Psi}_{(N-1)/2+2}$	$-(N-1)/2$th harmonic
.	.	.	.
.	.	.	.
$\boldsymbol{\Psi}_{N-1}$	-2nd harmonic	$\boldsymbol{\Psi}_{N-1}$	-2nd harmonic
$\boldsymbol{\Psi}_N$	-1st harmonic	$\boldsymbol{\Psi}_N$	-1st harmonic

6. Appendix II - Calculation methods

This section defines formally the operations necessary to the treatment of the data obtained during a coil rotation in order to compute the harmonic coefficients of the magnetic field, and additional quantities useful to control the measurement quality. In general this appendix does not make reference to the rotation direction or the cabling connection of the coil(s). This is done on purpose, as the treatment described here is the same, independently on the nature of the flux source. The data delivered by the acquisition system is considered to be a set of N time interval counts Dt_k and N flux increment counts Dj_k, as described in the main body of the note. The quantities are given in increasing angle order, starting at the first *zero* (or *index*) pulse of the encoder. The angular trigger pulses are nominally equally spaced.

6.1. Pulse angular position

The encoder pulses triggering the counter, after the initial *zero* (or *index*) pulse at nominal angle 0, correspond to the angular positions:

$$q_1 = 0 \qquad \text{(rad)} \qquad \text{(AII.1)}$$

$$q_{k+1} = 2p\,\frac{k}{N} \qquad \text{(rad)} \qquad \text{(AII.2)}$$

$$k = 1 \dots N$$

where we note that $q_{N+1} = q_1$.

6.2. Pulse time

The time of arrival of each trigger pulse from the encoder is computed from the time increments by a running sum:

$$t_1 = 0 \qquad \text{(s)} \qquad \text{(AII.3)}$$

$$t_{k+1} = t_k + \Delta t_k = t_k + \frac{\Delta t_k}{f} \qquad \text{(s)} \qquad \text{(AII.4)}$$

$$k = 1 \dots N$$

where f is the frequency of the reference clock for the integrator counters, and for present integrators:

$$f = 10^6 \qquad \text{(Hz)} \qquad \text{(AII.5)}.$$

Note that the initial time is set to 0 (all times are referred to the start of the rotation, first zero pulse from the encoder), and that the $N+1$ points correspond to the arrival time of the encoder trigger pulses (N pulses per turn, with a zero at the beginning and at the end of a rotation).

6.3. Mid-time

The mid time is the average time between two subsequent encoder pulses. It is used as the time stamp for all quantities referred to the interval (integrator count, time step, coil voltage, rotation speed). The mid-time of an interval is computed as:

$$t_{k+1/2} = \frac{t_k + t_{k+1}}{2} \qquad \text{(s)} \qquad \text{(AII.6)}$$

$k = 1 \dots N.$

6.4. Rotation speed

During the rotation the speed of the coil rotation is not necessarily constant. The *instantaneous* rotation speed is given by:

$$v_k = \frac{2p}{N\Delta t_k} \qquad \text{(rad/s)} \qquad \text{(AII.7)}$$

$k = 1 \dots N.$

6.5. Conversion of integrator count into coil flux

The integrator counts Dj_k must be converted to physical units of magnetic flux by using the conversion factor for the VFC and counter unit, in addition to the gains of the amplifier chain. This conversion is given by:

$$\Delta y_k = \frac{\Delta j_k}{K} \qquad \text{(Vs)} \qquad \text{(AII.8)}$$

$k = 1 \dots N$

where K is the conversion factor determined by the gain of the acquisition line and the VFC-counter transfer function. Explicitly:

$$K = VFC\, G_{int}\, G_{amp} \qquad \text{(counts/V s)} \qquad \text{(AII.9)}$$

where G_{int} and G_{amp} are respectively the integrator and amplifier gain in the acquisition chain. The factor *VFC* comes from the voltage-to-frequency conversion and counting procedure. For the present integrators that have a standard of frequency at 500 kHz and an input voltage range of 0...10 V we have:

$$VFC = 50000 \qquad \text{(counts/V s)} \qquad \text{(AII.10)}.$$

6.6. Coil voltage

The average voltage V_k picked-up by the rotating coil during an angular step Dq is calculated directly from the fluxes increments and the time interval as delivered by the integrators:

$$V_k = -\frac{\Delta y_k}{\Delta t_k} \qquad \text{(V)} \qquad \text{(AII.11)}$$

$k = 1 \dots N.$

6.7. Offset voltage correction

6.7.1. Average offset calculation

The integrator reading of the coil voltage can be affected by an offset caused by the cable connections and amplifier stages. It is possible to correct this offset assuming that the voltage offset does not change during a measurement. The average voltage offset is given by:

$$V_{off} = -\frac{\sum_{k=1}^{N} \Delta y_k}{t_{N+1}} \qquad \text{(V)} \qquad \text{(AII.12)}.$$

Note that this calculation assumes that the initial and final fluxes are the same in the rotating coil. <u>This is true only for a constant field in time</u> (and the same initial and final angular position). Hence the average voltage offset can be computed from the measurements only in the *steady state* current case.

6.7.2. Offset correction on the coil voltage

The correction is done by simple subtraction of the offset:

$$V_k = V_k - V_{off} \qquad \text{(V)} \qquad \text{(AII.13)}$$

$k = 1 \dots N.$

6.7.3. Offset correction on the flux increments

To correct the offset voltage on the flux increments we need to subtract the average offset integrated over the time between two subsequent angular pulses. The voltage offset results in a drift in the flux (voltage integral). The correction is given by:

$$\Delta y_k = \Delta y_k + V_{off} \Delta t_k \qquad \text{(Vs)} \qquad \text{(AII.14)}$$

$k = 1 \dots N.$

Combining Eq. (AII.12) and (AII.14) we could obtain an alternative relation:

$$\Delta y_k = \Delta y_k - \frac{\Delta t_k}{t_{N+1}} \sum_{k=1}^{N} \Delta y_k \qquad \text{(Vs)}$$

$k = 1 \dots N.$

that only involves flux increments.

6.8. Filtering

The fluxes increments Dy_k may be filtered to remove noise components. The procedures to be applied are not yet defined. In any case, filtering should:

* preserve the amplitude spectrum for low harmonic components within 0.5×10^{-4} (at least down to order 15);
* preserve the phase of the main field harmonic within 0.1 (mrad);
* preserve the phase of the other harmonics within 1 %.

6.9. Average flux increments

As described in the main body, the flux increments Dy_k are measured for a forward and a backward rotation. To improve rejection to linear systematic errors, and indeed to allow for a proper treatment of measurements during ramps, the two rotations must be averaged. If we indicate with superscripts "+" and "-" the forward and backward rotations respectively, and we remember that a backward rotation causes a sign inversion in the coil voltage, we obtain average flux increments as:

$$\Delta y_k = \frac{\Delta y_k^+ - \Delta y_k^-}{2} \qquad \text{(Vs)} \qquad \text{(AII.15)}$$

$k = 1 \dots N.$

6.10. Flux increments error

A good indicator of systematic errors in a measurement is the difference between a forward and backward rotation. This difference is computed as:

$$e_k = \Delta y_k^+ + \Delta y_k^- \qquad \text{(Vs)} \qquad \text{(AII.16)}$$

$k = 1 \dots N.$

6.11. Integration of fluxes increments

The flux increments Dy_k are proportional to the magnetic flux change over an angular step (integral of coil voltage). To obtain the actual value of the flux y_k for the points in a rotation it is necessary to perform a running sum of the increments. The procedure is:

$$y_1 = 0 \qquad \text{(Vs)} \qquad \text{(AII.17)}$$

$$y_{k+1} = y_k + \Delta y_k \qquad \text{(Vs)} \qquad \text{(AII.18)}$$
$$k = 1 \ldots N.$$

The above running sum gives *N+1* values for the magnetic flux as a function of angular position. The first value, a constant set arbitrarily to zero, corresponds to the first *zero* pulse of the encoder (at the angle q_l=0). The last value is for the angular position at exactly one turn (2p) from the starting one (the second *zero* pulse from the encoder). The last point is therefore not necessary for a Fourier analysis. Note that after proper drift correction it should be equal to the first point, i.e. nominally:

$$y_{N+1} = y_1 = 0.$$

6.12. Fourier transform

A DFT must performed on the values of the fluxes y_k at *N* points to compute the harmonic coefficients. The *N* points are chosen such that:

$$0 \leq k \leq N\text{-}1.$$

The DFT coefficients are computed as defined by Eq. (15), or:

$$\Psi_n = \sum_{k=1}^{N} y_k e^{-2pi(n-1)\frac{(k-1)}{N}} \qquad \text{(Vs)} \qquad \text{(AII.19)}$$
$$n = 1 \ldots N.$$

The result of this calculation is the set of harmonic coefficients for the fluxes Ψ_n.

6.13. Spectrum folding and normalization

The DFT of the real flux signal is by definition a function with even real part and odd imaginary part. The amplitudes of the real and imaginary parts are equally distributed in the positive and negative frequency spectrum. To obtain the total amplitude in a reference frame as defined in [1], avoiding the *negative frequency* part of the spectrum, we must fold the spectrum about its midpoint. In addition the resulting folded coefficients must be normalized by the number of points (see Eq. (17) and Appendix 1). This is achieved for an *even* number of points $N=2K$; $K \in Z^+$:

$$\Xi_n = \frac{\Psi_{n+1} + \Psi^*_{N-n+1}}{N} \qquad \text{(Vs)} \qquad \text{(AII.20)}$$
$$n = 1 \ldots N/2\text{-}1,$$

and for an *odd* number of points $N=2K+1$; $K \in Z^+$:

$$\Xi_n = \frac{\Psi_{n+1} + \Psi^*_{N-n+1}}{N} \qquad \text{(Vs)} \qquad \text{(AII.21)}$$
$$n = 1 \ldots (N\text{-}1)/2.$$

The star superscript stands for the complex conjugate operator. Note that the resulting folded spectrum does not contain the DC component (first coefficient), which is related to the arbitrary constant that we have set for the starting value of the flux in the integration procedure, nor the Nyquist frequency component (midpoint coefficient).

6.14. Field harmonics

The folded and normalized Fourier coefficients for the fluxes Ξ_n are processed to obtain the non-normalized harmonic coefficients $\mathbf{C}_n$ in accordance to Eq. (18):

$$\mathbf{C}_n \approx \frac{R_{ref}^{n-1}}{\kappa_n} \Xi_n \qquad \text{(T)} \qquad \text{(AII.22)}$$

$n = 1 \dots N_m,$

where the κ_n are the coil calibration coefficients, defined up to order N_m.

6.15. Bucking ratio

The complex bucking ratio $\mathbf{b}_n$ is the ratio of the harmonic n in the absolute and compensated measurement. It is computed as:

$$\beta_n = \frac{\Psi_n^{abs}}{\Psi_n^{cmp}} \qquad \text{(-)} \qquad \text{(AII.23)}$$

where the superscript indicates absolute (*abs*) or compensated (*cmp*) measurements. The module of $\mathbf{b}_n$ gives a measurement of the goodness of the compensation of harmonic n. The phase gives a measurement of the angular error between coils.

7. Appendix III - Centering and feed-down correction

The measuring coil is in general not centered with respect to the magnetic center of the magnet under measurement. This results in *feed-down* from higher harmonics onto lower harmonics as evidenced by Eq. (5). To provide a consistent basis for the analysis, all measurements are referred, through the feed-down removal procedure, to the magnetic center. The absolute position of the magnetic center along the magnet is not supposed to be known in the reference frame of the magnet fiducials. The determination of this absolute position is a separate task, to be solved by means of an axis-localization device. As no position measurement is used in the feed-down removal, all information must be provided by the magnetic field measurement. The feed-down removal procedure is broken into two parts: a center localization and a feed-down correction. While the feed-down correction is rather universal and independent on the type of magnet being measured, the center localization depends strongly on the type of magnet.

7.1. Center localization

The center localization procedure is based on canceling non-allowed harmonics in the measured spectrum. We define here the center coordinate $\Delta\mathbf{z}$ with respect to the coil rotation axis, as shown in Fig. AIII.1. With this definition a physical translation of the magnet by the distance $-\Delta\mathbf{z}$ is such that the magnet axis is coincident with the rotating coil axis. The same effect (see next) is obtained by a reference frame translation by the distance $\Delta\mathbf{z}$.
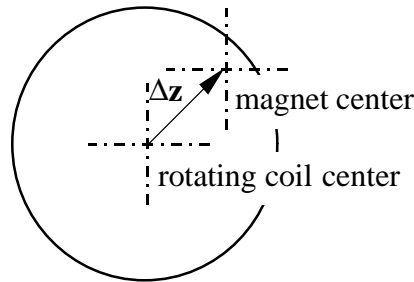

Figure AIII-1. Definition of the magnetic center coordinates

While center localization is easy and reliable for magnets of order higher than 1 (quadrupole and corrector magnets), in the case of a dipole it can be a delicate matter.

7.1.1. Dipole center localization

In the case of a dipole we define the dipole axis as the line along which higher order, non-allowed harmonics of order 8 and/or 10 are zero. These harmonics should be close to zero by symmetry, and are high enough to be little influenced by fabrication

errors. Using Eq. (5) we write therefore that the dipole center, computed zeroing the 16-pole is located at $\Delta\mathbf{z}$ given by the solution of:

$$\sum_{k=8}^{15}\left(\frac{(k-1)!}{(n-1)!(k-n)!}\right)\mathbf{C}_k\left(\frac{\Delta\mathbf{z}}{R_{ref}}\right)^{k-n}=0 \qquad\qquad\text{(AIII.1).}$$

Note that the l.h.s. of Eq. (AIII.1) is a polynomial of 7th degree, and has therefore in general 7 complex roots. The center is defined then by the root that minimizes the following cost function:

$$f(\Delta\mathbf{z})=\sum_{k=4}^{7}\frac{\left|\mathbf{C}'_{2k}\right|}{\left|\mathbf{C}_{2k+1}\right|} \qquad\qquad\text{(AIII.2)}$$

where the primed harmonics are intended as computed in the translated (centered) reference frame. Equation (AIII.2) is a weighted sum of non-allowed harmonics in the centered reference frame. Alternatively the definition of the center can be based on zeroing the 20-pole. In this case the center is given by the value of $\Delta\mathbf{z}$ that is a solution of:

$$\sum_{k=10}^{15}\left(\frac{(k-1)!}{(n-1)!(k-n)!}\right)\mathbf{C}_k\left(\frac{\Delta\mathbf{z}}{R_{ref}}\right)^{k-n}=0 \qquad\qquad\text{(AIII.3)}$$

and minimizes the same cost function as given by Eq. (AIII.2). For the center localization the compensated harmonics should be used (if available).

A difficulty for the definition of the center is represented by non-allowed multipoles appearing at low fields (non-uniform magnetization and current distribution effects) or during ramps (non-uniform inter-strand resistance distribution). As a consequence, all center localization <u>must</u> be performed at moderate to high field (> 2 T) and in steady state.

7.1.2. 2m-pole center localization ($m > 1$)

For the localization of the center of a general 2m-pole magnet the feed-down from the main field component (a 2m-pole) to the following harmonic, the 2(m-1)-pole, is used. The axis is the line along which the 2(m-1)-pole is zero. The definition of the center location, based on the measured harmonic coefficients, is the following:

$$\Delta\mathbf{z}=-\frac{R_{ref}}{m-1}\frac{\mathbf{C}_{m-1}}{\mathbf{C}_m} \qquad\qquad\text{(m)}\qquad\text{(AIII.4)}$$

that can be written explicitly in terms of the x and y coordinates separating the real and imaginary part:

$$\Delta x = -\frac{R_{ref}}{m-1}\frac{B_m B_{m-1} + A_m A_{m-1}}{B_m^2 + A_m^2}$$

$$\Delta y = -\frac{R_{ref}}{m-1}\frac{B_m A_{m-1} - A_m B_{m-1}}{B_m^2 + A_m^2}$$

(m)   (AIII.5)

The expression above neglects contributions of order higher than linear in the off-centering, and is justified by the fact that the dominant field component is large compared to the field errors of higher order.

7.2. Feed-down correction

Once the coil center coordinates Dx and Dy in the reference frame of the magnetic axis are determined, it is possible to correct the measured harmonics, both absolute and compensated, using Eq. (5)

$$\mathbf{C}'_n = B'_n + iA'_n = \sum_{k=n}^{N_m}\left(\frac{(k-1)!}{(n-1)!(k-n)!}\right)\mathbf{C}_k\left(\frac{\Delta \mathbf{z}}{R_{ref}}\right)^{k-n}$$

$$n = 1...N_m$$

(T)   (AIII.6)

where we remember that the complex center coordinate $\Delta \mathbf{z}$ is given by:

$$\Delta \mathbf{z} = \Delta x + i\Delta y$$

found previously.

8.   Appendix IV - Normalized harmonic coefficients

The normalization of the harmonic coefficients depends on the main harmonic measured. Here follows the general definition of the normalization procedure for a magnet of order m (a $2m$-pole).

8.1.   Main field module

The main field module $/\mathbf{C}_m/$ is computed from the normal and skew components of $\mathbf{C}_m$ as taken from the absolute measurements:

$$\left|\mathbf{C}_m\right| = \sqrt{B_m^2 + A_m^2} \qquad \text{(T)} \qquad \text{(AIV.1)}$$

8.2.   Main field phase φ_m

The field phase φ_m is defined in the interval $[-\pi/2 ... \pi/2]$ by the relation:

$$e^{i\varphi_m} = \frac{B_m - iA_m}{\left|\mathbf{C}_m\right|} \qquad \text{(-)} \qquad \text{(AIV.2)}$$

which can be explicitly written using real variables only as:

$$\cos\left(\varphi_m\right) = \frac{B_m}{\left|\mathbf{C}_m\right|}$$
$$\sin\left(\varphi_m\right) = -\frac{A_m}{\left|\mathbf{C}_m\right|} \qquad \text{(-)} \qquad \text{(AIV.3)}$$

Note that both expressions above are needed to determine correctly the trigonometric quadrant. The angle φ_m is the oriented angle formed by the main field and the y axis (see Fig. 4).
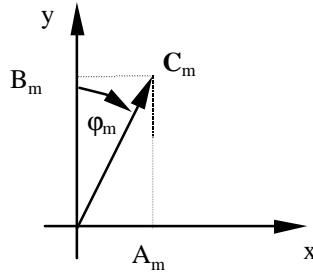

Figure 4. Definition of the main field phase

To obtain a univoque definition of the main field direction (see next section) the phase φ_m must be limited after calculation to the interval $[-\pi/2 ... \pi/2]$. This is done adding the fixed angle π in case that the phase falls outside the specified range:

if $j_m > \pi/2$ $j_m = j_m - \pi$

if $j_m < -\pi/2$ $j_m = j_m + \pi$ (rad)   (AIV.4)

8.3.   *Main field direction* α_m

The field direction a_m is given by:

$$a_m = \frac{j_m}{m}$$ (rad)   (AIV.5)

With this definition a physical rotation of the magnet by the angle $-a_m$ is such that the magnetic field becomes purely normal. The same effect (see next section) is obtained by a reference frame rotation by the angle a_m.

8.4.   *Rotation of non-normalized harmonic coefficients*

The non-normalized harmonic coefficients (absolute and compensated) are rotated by the angle a_m to bring them in the reference frame of the main field using Eq. (6):

$$\mathbf{C}'_n = B'_n + iA'_n = \mathbf{C}_n e^{ina_m}$$ (T)   (AIV.6)

After the rotation the absolute component m (the main field) will necessarily be only normal, and the skew component m will be zero. The normal component will have the same magnitude as the main field module. The sign will be positive or negative depending on the fact that the phase has been taken as computed with Eq. (AIV.2), or limited in the interval [$-\pi/2 ... \pi/2$] with Eq. (AIV.4).

8.2.   *Main field derivative*

The main field derivative g_m is computed from the normal component of the main field after rotation B_m as follows:

$$g_m = \frac{B_m}{R_{ref}^{m-1}}$$ (T/m)   (AIV.7)

8.5.   *Normalization*

The non-normalized and rotated coefficients are normalised by referring them to the main field. A multiplying factor of 10^4 is used to transform them in practical *units*:

$$c_n = 10^4 \frac{\mathbf{C}_n}{B_m}$$ (units)   (AIV.8)

$$n = m + 1 ... N_m$$

The normalization above is equivalent to:

$$b_n = 10^4 \frac{B_n}{B_m} \quad ; \quad a_n = 10^4 \frac{A_n}{B_m} \qquad \text{(units)} \qquad \text{(AIV.9)}.$$

$$n = m + 1 \ldots N_m$$