Tricolorings of Knots

Lucas Meyers

Department of Mathematics
Louisiana State University

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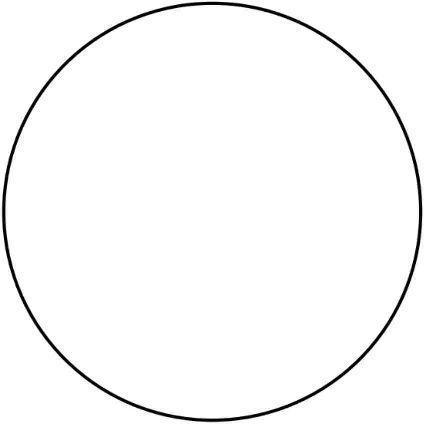
Knot Definition

- A *Knot* is a simple closed curve in S^3 $(\mathbb{R}^3 \cup \{\infty\})$.

Unknot



Trefoil

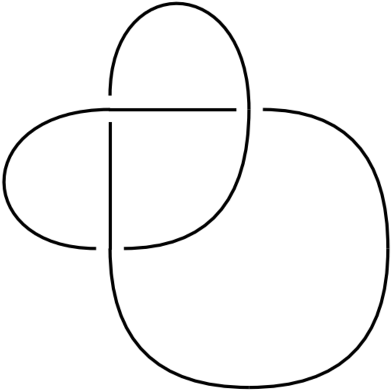
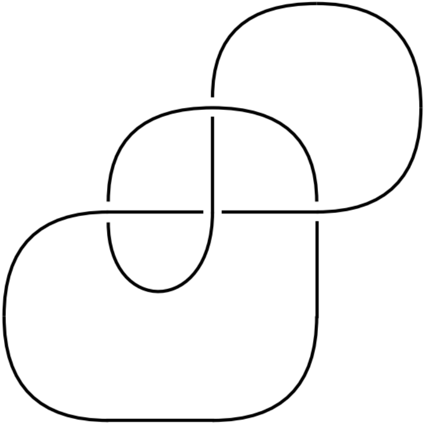


Figure-Eight



Equivalence

- All knots are homeomorphic to S^1.
- Two knots K and J are *Ambient Isotopic* if there is a homotopy $H : S^3 \times [0,1] \to S^3$ such that:
  - $H(x, 0)$ is the identity on S^3.
  - The image $H(K, 1)$ is J.
  - For each fixed $t \in [0, 1]$ the map $H(x, t)$ is a homeomorphism.

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Diagrams

- A *diagram* D of a knot K is a projection of K to a plane such that:
  - At most two points can be projected to the same point on the plane.
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  - We preserve over and under crossing information.

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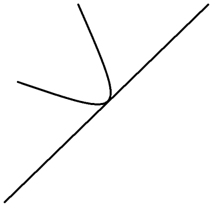

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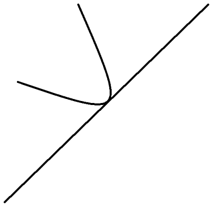

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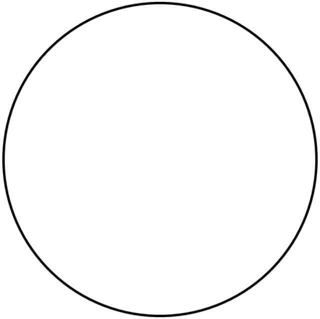
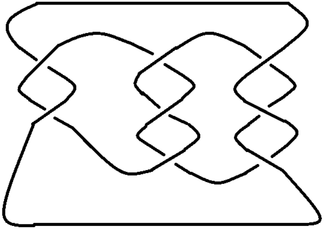
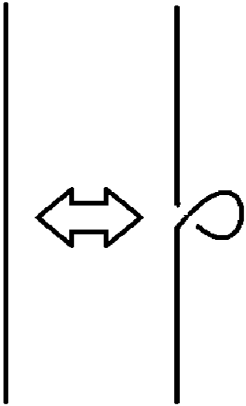




Reidmeister Moves

- **Theorem:** Two knots K and J are ambient isotopic if, and only if, their diagrams are related by a finite sequence of Reidmeister moves and planar isotopy.

Type I



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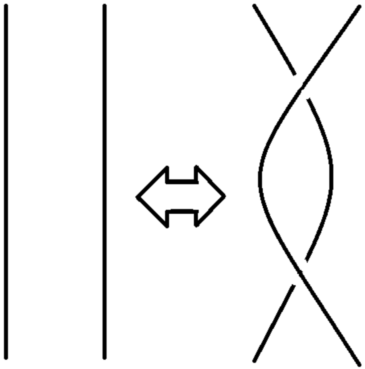

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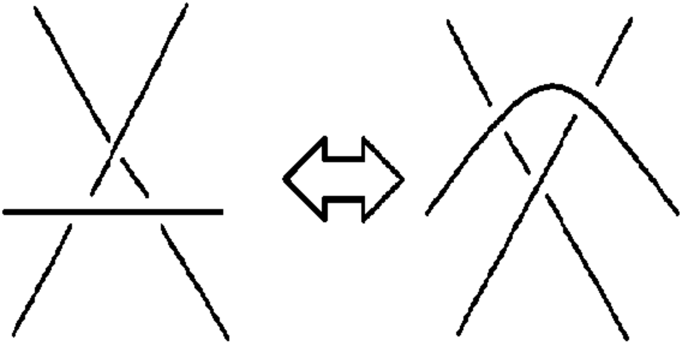

Tricoloring

- A *tricoloring* of a diagram is an assignment of one of three colors to each arc such that for all crossings:
  - Only a single color is present.
  - All three colors are present.
- A tricoloring is *nontrivial* if at least two colors are used.
- **Theorem:** The existence of a non-trivial tricoloring is a knot invariant. Such a knot is called tricolorable (or 3-colorable).

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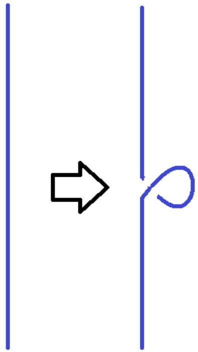

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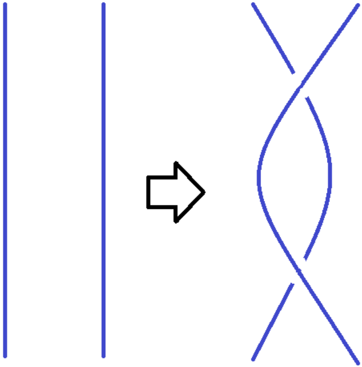

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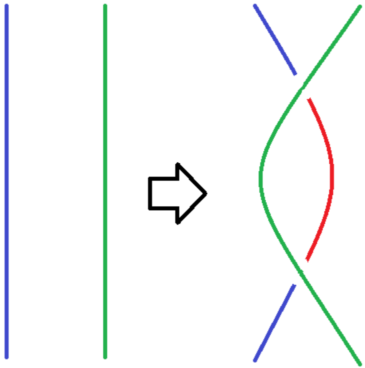

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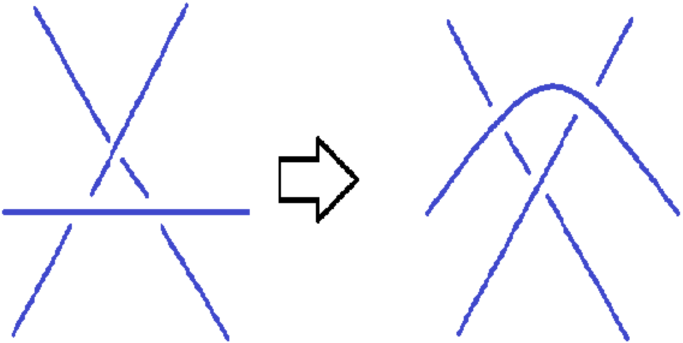

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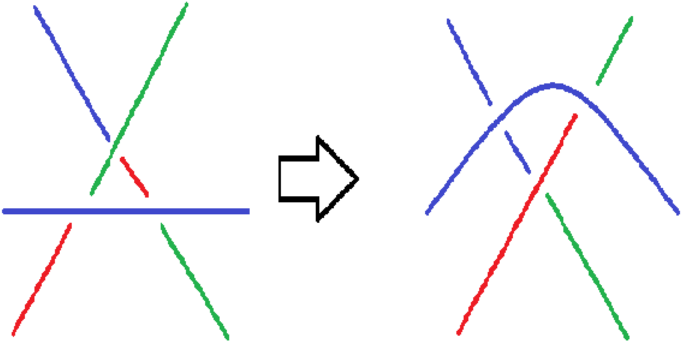

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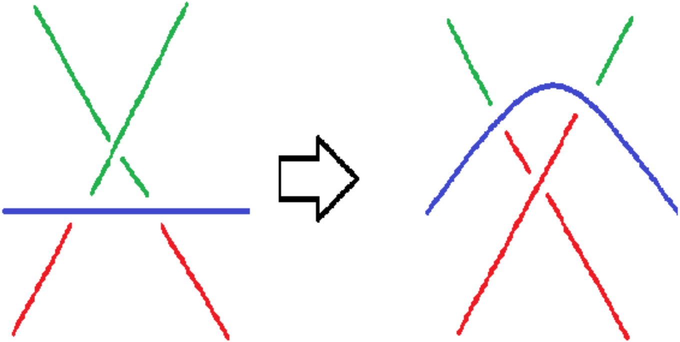
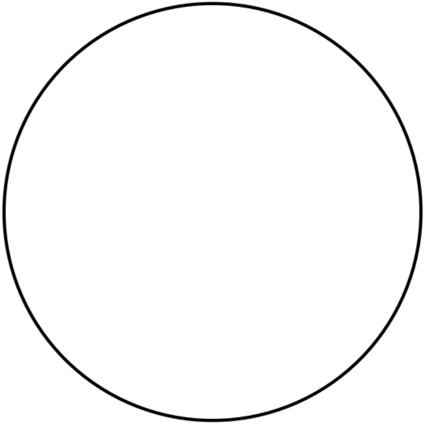


Uncolorability of the Unknot



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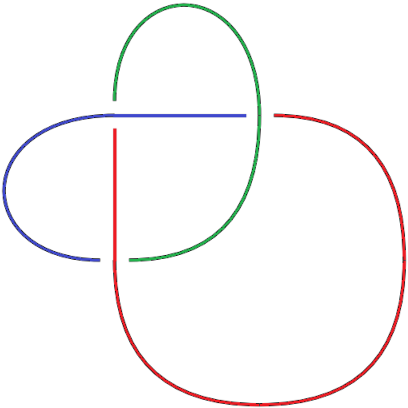
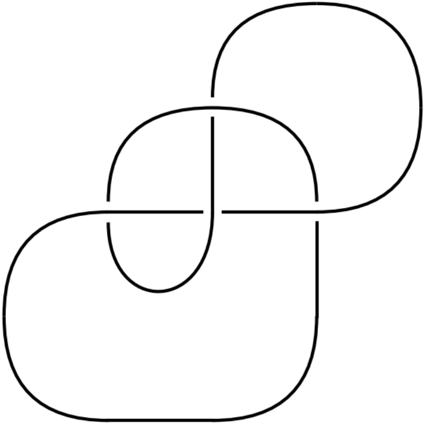


Uncolorability of the Figure-Eight Knot



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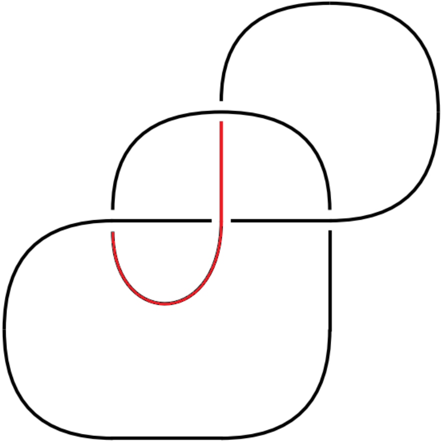

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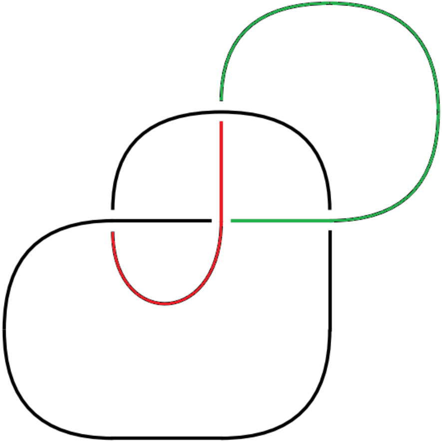

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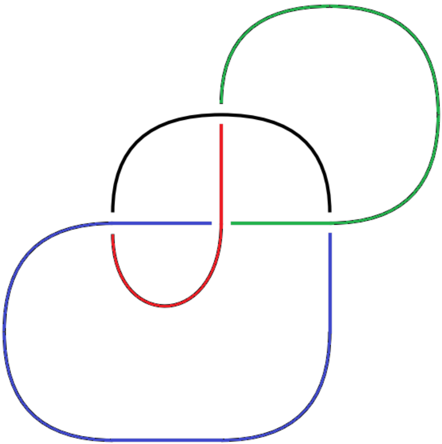

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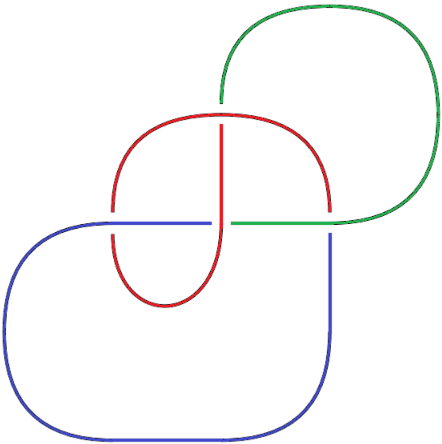

Knot Group

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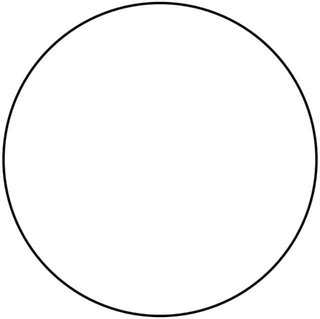

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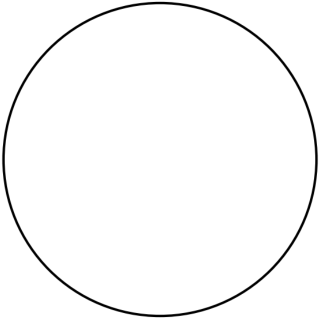

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Wirtinger Presentation

- For other knots we use a tool called the *Wirtinger Presentation*.
- Let K be a knot and D a diagram for K. Choose an orientation of D. Let $x_1, \ldots, x_n$ denote the arcs of D and, for each crossing, give a relation $r_1, \ldots, r_m$ such that $r_i \equiv (x_i x_j x_i^{-1} = x_k)$ where, when approaching x_k is the arc leaving the crossing, the top arc is x_i and the bottom arc entering is x_j.
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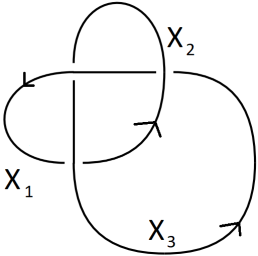

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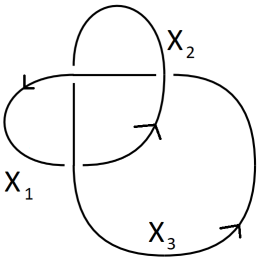

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Fox 3-Colorings of the Unknot and Trefoil

- There is no nontrivial Fox 3-coloring of the unknot as D_6 has two generators and the knot group of the unknot, $\mathbb{Z}$, only has one.
- We can give a nontrivial Fox 3-coloring of the trefoil, $\langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$, using the presentation of D_6 as

$$\langle r, s | r^3 = s^2 = 1, srsr = 1 \rangle$$

by mapping x_1 to r and x_2 to s or vice versa.
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Equivalence of Tricoloring and Fox 3-Coloring

- **Theorem:** A Knot K has a 3-coloring if, and only if, there is a Fox 3-coloring of K.

Consequences

- Since for any two groups G and H the homomorphisms $\hom(G, H)$ form a group this implies that colorings of knots also form a group.
- Unlike with tricolorability it more obvious how to extend Fox 3-colorings to an arbitrary number of colors.
  - A *Fox n-coloring* of a knot K is a homomorphism from the knot group of K to D_{2n}.
- Given any group G we can define a *Fox G-coloring* as a homomorphism from the knot group of K to the group G.
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Extending Tricolorability

- We can also use the fact that Fox 3-colorings and tricolorability are equivalent to guide us in extending the notion of *n*-colorings to diagrams as well.
- Instead of assigning colors to each arc we use elements of $\mathbb{Z}_n$.
- Then a diagram has a nontrivial coloring if at least two numbers are used and for each crossing, if we label the overcrossing arc x and the two undercrossing arcs as y and z, the following equation is satisfied

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