TRICOLORINGS OF KNOTS

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Abstract. Empty

1. Introduction

Citation Usage EX [1, Problem 2.4.6]

2. Diagrams and Coloring

3. The Knot Group

The definition of 3-coloring is a perfectly valid definition for a knot invariant. However it is defined in terms of diagrams and as such we had to do some extra work with the Reidmeister moves to show that it was indeed an invariant. Ideally there would be an equivalent definition of coloring that does not involve diagrams. Then the fact that it is an invariant would be immediate and it may be possible to better tease out the topological property that is being captured by 3-coloring.

It turns out that there is a way to define 3-coloring in this manner and it has to do with a construction called the Knot group. Ones first instinct to get a group out of a knot might be to take the fundamental group. However since all knots are copies of S^1 the fundamental group for any knot is $\mathbb{Z}$. As such we have to take advantage of its ambient space.

Definition 3.1 (Knot Group)**.** Let K be a knot. Then the Knot group of K is the fundamental group $\pi_1(S^3 \setminus K, *)$.

This definition makes use of the embedding of the knot into S^3 and as such will not be identical for all knots. Naturally we need a way to calculate the knot group. We can do this using what is called the Wirtinger presentation [2].

Theorem 3.2 (Wirtinger)**.** *Let K be a knot and D a diagram for K. Choose an orientation for D. Then let $x_1, \ldots, x_n$ denote the arcs of D and for each crossing give a relation $r_1 \ldots, r_m$ such that $r_i \equiv (x_i x_j x_i^{-1} = x_k)$ where x_i is the top arc of the crossing and x_j is the*

Date: April 1, 2018.

FIGURE 1. Annotated Trefoil

arc underneath approaching x_k. Then the knot group of K has the presentation

$$\pi_1(S^3 \setminus K, *) \cong \langle x_1, \ldots, x_n | \rangle$$

Now we give a couple of examples.

Example 3.3. The Knot group of the unknot is

$$\langle x_1 | \rangle \cong \mathbb{Z}$$

as there are no crossings and only a single arc.

Followed by the trefoil knot.

Example 3.4. The trefoil knot has three crossings and three arcs. Label it as shown in figure 1. Then the Wirtinger presentation of the knot group of the trefoil is

$$\langle x_1, x_2, x_3 | x_1 x_2 x_1^{-1} = x_3, x_3 x_1 x_3^{-1} = x_2, x_2 x_3 x_2^{-1} = x_1 \rangle$$

However the presentation of this group can be simplified to the form

$$\langle x_1, x_1 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

The knot group gives us another invariant of knots. Unfortunately presentations of groups can be rather difficult to work with. Thus in order to make the problem more tractable we instead look at homomorphisms from the knot group to finite groups as these will be determined by where we send generators. This is how we will redefine coloring in terms of the knot group [3].

Definition 3.5 (Fox 3-coloring)**.** Let K be a knot and G the corresponding knot group. Then a Fox 3-coloring of K is a homomorphism ρ from G to the symmetries of a triangle D_6. We say that ρ is a non-trivial coloring if ρ is surjective.

Thus a knot K is Fox 3-colorable if there exists a nontrivial Fox 3-coloring.

Now a couple of examples using our prior work.

Example 3.6. The knot group of the unknot is $\mathbb{Z}$. Since $\mathbb{Z}$ has only a single generator it is not possible to create a surjective homomorphism onto D_6 as D_6 has two generators.

Example 3.7. As we calculated above the knot group for the trefoil is

$$\langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

Similarly we can write a presentation of D_6 as

$$\langle r, s | r^3 = 1, s^2 = 1, srsr = 1 \rangle$$

This gives us two non-trivial Fox 3-colorings of the trefoil. The first sends x_1 to r and x_2 to s and the other swaps which generator is sent to which generator.

As was implied by the prior examples there is a relation between Fox 3-coloring and the 3-colorability of knots. In fact it turns out that they are exactly the same.

Theorem 3.8. *A knot K has a 3-coloring if and only if there is a Fox 3-coloring of K.*

A proof of this theorem can be found within [4]. The proof that is done relates two notions of coloring using the Wirtinger presentation to marry the conditions necessary for the existence of a 3-coloring and the existence of a Fox 3-coloring.

4. n-coloring

So what are the benefits of looking at 3-coloring through the lens of Fox 3-coloring? As we mentioned before it is immediately clear that this is a knot invariant where as when we defined 3-coloring with diagrams. Another benefit is that it is more readily apparent how we can further extend Fox 3-coloring than extending 3-coloring. From the definition of 3-coloring there is some ambiguity that would need to be sorted out about precisely what rules need to change. Questions such as "How many colors are necessary?" or "What would make a crossing valid configuration?".

This is avoided with Fox 3-coloring. The only choice we made was that we are looking at homomorphisms to D_6. We could very easily have chosen any finite group and gotten the same nice structure that we had above. The reason that D_6 is the group used for Fox 3-coloring is that it is the group of symmetries of the triangle. We can define the Fox n-coloring similarly.

Definition 4.1 (Fox n-coloring)**.** Let G be the knot group of a knot K. Then an n-coloring of K is a homomorphism $\rho : G \to D_{2n}$ where ρ is called a non-trivial n-coloring if ρ is surjective.

We can also extend the notion of coloring that we started with. However it does take more work. Instead of choosing the colors that we did when originally defining 3-coloring we could just as easily have used numbers instead. Then instead of requiring that each crossing

Figure 2. Colored Crossing

either use all three colors or all the same it would be equivalent to each crossing satisfying the equation

$$2x - y - z \equiv 0 \mod 3$$

where the arcs of the crossing are labeled as in figure 2

In this form it is more apparent how we could extend 3-coloring to arbitrary n

Acknowledgements

References

[1] Kung, J.P.S., Critical problems, in *Matroid Theory* (eds. J. E. Bonin, J.G. Oxley, and B. Servatius), *Contemporary Mathematics* **197**, Amer. Math. Soc., Providence, 1996, pp. 1–127.

[2]

[3]

[4]

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