TRICOLORINGS OF KNOTS

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Abstract. Tricoloring is a knot invariant that differentiates knots based on whether or knot their diagram can be colored in a certain way with three colors. In this paper, we introduce knots and define tricolorability in terms of both knot diagrams and the knot group. We then mention extensions of the notions of tricolorability to arbitrary n-coloring.

1. Introduction

One common pattern we see in mathematics is a concern with how certain objects fit inside other objects. Knot theory studies this sort of problem. Given a circle S^1 what different ways can we embed it into $\mathbb{R}^3$ or S^3 ($\mathbb{R}^3$ with a point at ∞)? A typical drawing of a knot can be seen in Figure 1.

Naturally, once we begin to consider knots, we are also led to consider how we might tell them apart or rather when they should be the same at all. Intuitively, the knot in Figure 1 should be considered the same as that in Figure 2 but should be different from that of trefoil seen in Figure 6. In the next section, we will see the definition of a knot, how to define their equivalence in a sensible way, followed by a method to differentiate them called tricolorability. Afterwards, we will look at an alternative definition of tricolorability that utilizes the fundamental group. Then we will introduce notions of how one may consider coloring knots with more than three colors.

2. Diagrams and Coloring

In this paper we will follow the conventions of Charles Livingston [2]. To begin, we give the definition of a knot.

Definition 2.1. A *knot* is a simple closed curve in $\mathbb{R}^3$(or S^3).

Now there are two issues that we need to consider before moving forward. The first is our notion of equivalence. Unfortunately, a homeomorphism from one knot to another is too weak a notion of equivalence

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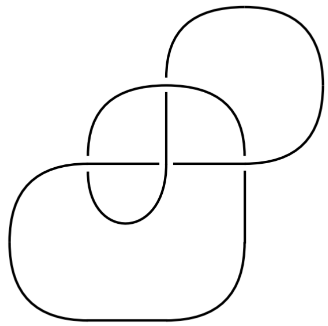

FIGURE 1. Figure-eight knot

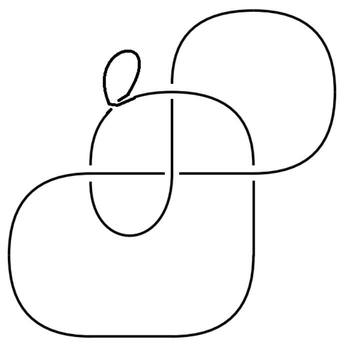

FIGURE 2. Figure-eight with a twist

as all simple closed curves are homeomorphic to S^1. Instead we will use the following:

Definition 2.2. Two knots K and J are *ambient isotopic* if there is a homotopy on the ambient space $H : S^3 \times [0,1] \to S^3$ if $H(x,0)$ is the identity and the image $H(K,1) = J$.

The second issue of initial concern is that we will be considering a subset of knots called tame. A knot is *tame* if it is ambient isotopic to a piecewise linear knot. A knot that is not tame is called *wild* and a drawing of an example of such a knot seen in Figure 3. The knot in question has a decreasing sequence of components that cannot be made linear. Wild knots have some more pathological properties and to keep things simple and consistent we will restrict ourselves to tame knots.

When we give the initial definition of tricolorability we will not work directly with the curves themselves. Instead we will deal with diagrams

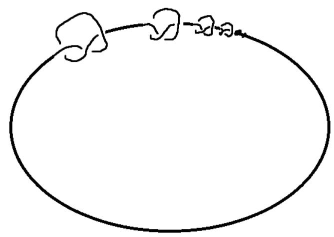

FIGURE 3. Wild knot

of the knots. All drawings of knots in this paper are examples of diagrams.

Definition 2.3. A *diagram* D for a knot K is a projection of K to a plane such that:

- At most two points are sent to the same location on the plane. These are called *crossings*.
- Given any two crossings they each have an open neighborhood that is disjoint from the other's neighborhood.
- All crossings must be either overcrossings or undercrossings.

Note that all figures in this paper are diagrams of knots.

When we project to a plane there is a loss of information that occurs. This is why we must denote all crossings as either overcrossings or undercrossings. This allows one to recover the knot in its entirety from the diagram. Otherwise it would be ambiguous what knot is recovered since we would have to make a choice of over or under for each crossing. The segments that make up the diagram are called *arcs*.

There is one other unfortunate occurrence from moving to diagrams. It is that there are infinitely many diagrams for a given knot. Moreover actually telling whether two diagrams denote the same knot is extremely difficult, though decidable, and moreover belongs to NP [4].

It turns out that there are three "moves" along with planar isotopy that we can apply to diagrams that will allow us to completely characterize when two diagrams correspond to the same knot.

Definition 2.4. The *Reidmeister moves* are the moves labeled in Figure 4. They are referred to as type-I, type-II, or type-III moves respectively.

We will use, without proof, the following theorem about Reidmeister moves.

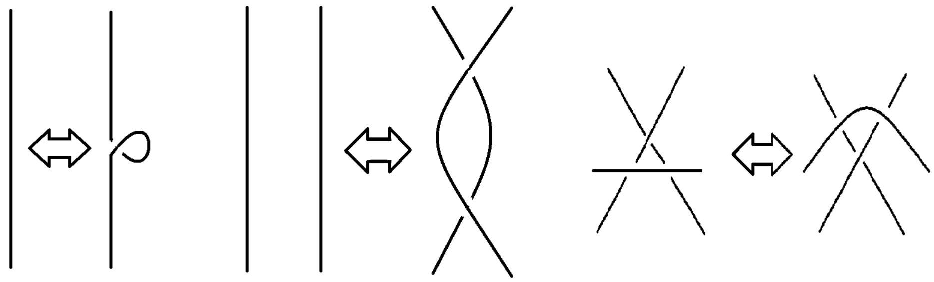

FIGURE 4. Reidmeister moves of type I, II, and III

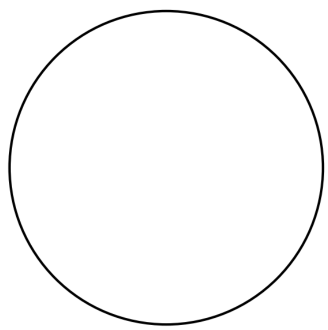

FIGURE 5. Unknot

Theorem 2.5. *Two knots K and J are ambient isotopic if and only if their diagrams are related by a finite sequence of Reidmeister moves and planar isotopy.*

The importance of Reidmeister moves and diagrams becomes apparent in defining invariants of knots. If we define an invariant on diagrams and show that it does not change under planar isotopy or the Reidmeister moves, then it will also act as an invariant on knots. This is precisely the method we will use with our initial definition of tricolorability.

There are two knots that we will return to repeatedly as they are the simplest examples of knots for which tricolorability differentiates them. The simplest knot is the *unknot* (see Figure 5), which is unique in that it has a diagram with no crossings whatsoever. The other is the *trefoil* shown in Figure 6.

With that in mind let us introduce tricolorability.

Definition 2.6. Let K be a knot with diagram D. Then a *tricoloring (3-coloring)* of K is an assignment to each arc one of three colors

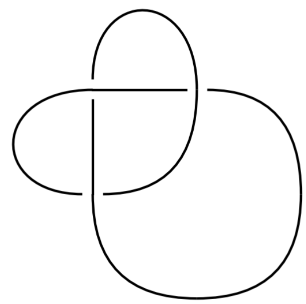

FIGURE 6. Trefoil knot

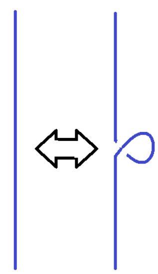

FIGURE 7. type-I

such that at each crossing either all colors appear or only single color appears. A tricoloring is called *non-trivial* if at least two colors are used.

We will use the colors red, green, and blue as our colors. Now we must show that tricoloring is in fact an invariant.

Theorem 2.7. *The existence of a non-trivial* 3*-coloring for some diagram is a knot invariant.*

Proof. Let K be a knot with diagram D. We must show that, given a coloring for D, we can color any diagram of K. First note that planar isotopy does not affect the crossings for D and, as such, does not change the 3-colorability.

Next we will have a series of pictures that demonstrate the possibilities for type-I, type-II, and type-III moves respectively.

For type-I we have there only a single case as a type-I move acts only on a single arc as seen in Figure 7.

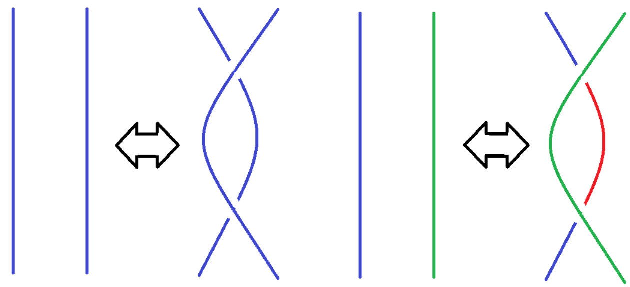

FIGURE 8. type-II

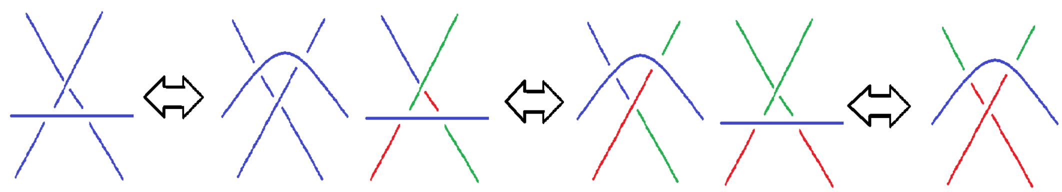

FIGURE 9. type-III

For type-II, there are two cases either both strands have the same color or they are different as seen in Figure 8.

Finally, for type-III moves there are several cases based on the colors of the strands entering. We will demonstrate three such cases, given in Figure 9, and the rest are similar.

Thus if a diagram D for a knot K has a 3-coloring, then any diagram for K has a 3-coloring. Therefore 3-colorability is a knot invariant. □

An immediate consequence of this is that the unknot and the trefoil knot are in fact not identical. We can see this because the unknot is not tricolorable as it only has a single arc. However, the trefoil can be colored as in Figure 10.

3. The Knot Group

The definition of 3-coloring is a perfectly valid definition for a knot invariant. However, it is defined in terms of diagrams and, as such, we had to do some extra work with the Reidmeister moves to show

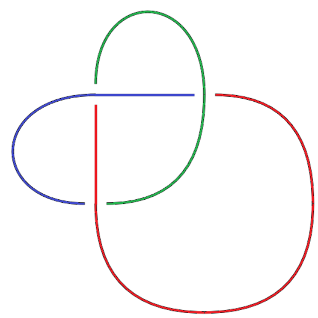

FIGURE 10. Colored trefoil

that it was indeed an invariant. Ideally, there would be an equivalent definition of coloring that does not involve diagram, but rather, only involves the knot and its embedding into S^3. Then it may be easier to tease out what topological properties are being captured by 3-coloring.

It turns out that there is a way to define 3-coloring in this manner and it has to do with a construction called the knot group. Ones first instinct to get a group out of a knot might be to take the fundamental group. However, since all knots are copies of S^1, the fundamental group for any knot is $\mathbb{Z}$. As such, we have to take advantage of its ambient space.

Definition 3.1. Let K be a knot. Then the *knot group* of K is the fundamental group $\pi_1(S^3 \setminus K, *)$.

This definition makes use of the embedding of the knot into S^3 and so will not be identical for all knots. Naturally, we need a way to calculate the knot group. We can do this using what is called the *Wirtinger presentation* [1].

Theorem 3.2 (Wirtinger)**.** *Let K be a knot and D a diagram for K. Choose an orientation for D. Let $x_1, \ldots, x_n$ denote the arcs of D and, for each crossing, give a relation $r_1 \ldots, r_m$ such that $r_i \equiv (x_i x_j x_i^{-1} = x_k)$ where, when approaching x_k is the arc leaving the crossing, the top arc is x_i and the bottom arc entering is x_j. Then the knot group of K has the presentation*

$$\pi_1(S^3 \setminus K, *) \cong \langle x_1, \ldots, x_n | r_1, \ldots, r_m \rangle$$

Now we give a couple of examples.

Example 3.3. The knot group of the unknot is

$$\langle x_1 | \rangle \cong \mathbb{Z}$$

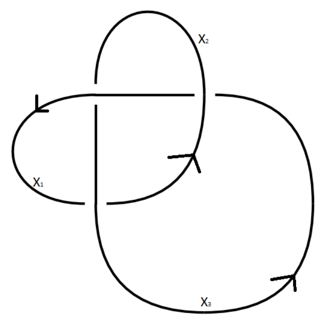

FIGURE 11. Annotated Trefoil

as there are no crossings and only a single arc.

Example 3.4. The trefoil knot has three crossings and three arcs. Label it as shown in Figure 11. Then the Wirtinger presentation of the knot group of the trefoil is

$$\langle x_1, x_2, x_3 | x_1 x_2 x_1^{-1} = x_3, x_3 x_1 x_3^{-1} = x_2, x_2 x_3 x_2^{-1} = x_1 \rangle$$

However, the presentation of this group can be simplified to the form

$$\langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

The knot group gives us another invariant of knots. Unfortunately, presentations of groups can be rather difficult to work with. Thus, to make the problem more tractable, we instead look at homomorphisms from the knot group to finite groups as these will be determined by where we send generators. This is how we will redefine coloring in terms of the knot group [5].

Definition 3.5. Let K be a knot and G the corresponding knot group. Then a *Fox* 3*-coloring* of K is a homomorphism ρ from G to the symmetry group D_6 of an equilateral triangle. We say that ρ is a *non-trivial coloring* if ρ is surjective.

Thus a knot K is *Fox* 3*-colorable* if there exists a nontrivial Fox 3-coloring.

Now a couple of examples using our prior work.

Example 3.6. Recall that the knot group of the unknot is $\mathbb{Z}$. Since $\mathbb{Z}$ has only a single generator, it is not possible to create a surjective homomorphism onto D_6 as D_6 has two generators.

Example 3.7. As we calculated above, the knot group for the trefoil is

$$\langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle.$$

Similarly, we can write a presentation of D_6 as

$$\langle r, s | r^3 = 1, s^2 = 1, srsr = 1\rangle.$$

This gives us two non-trivial Fox 3-colorings of the trefoil. The first sends x_1 to r and x_2 to s and the other swaps which generator is sent to which generator.

As was implied by the prior examples, there is a relation between Fox 3-coloring and the 3-colorability of knots. In fact, it turns out that they are the same.

Theorem 3.8. *A knot K has a 3-coloring if and only if there is a Fox 3-coloring of K.*

A proof of this theorem can be found in [3]. The proof there relates two notions of coloring using the Wirtinger presentation to marry the conditions necessary for the existence of a 3-coloring and the existence of a Fox 3-coloring.

4. n-coloring

What are the benefits of looking at 3-coloring through the lens of Fox 3-coloring? Unlike with coloring diagrams, it is immediate that Fox 3-coloring is a knot invariant as the fundamental group is invariant under homotopy equivalence. So Fox 3-coloring must be invariant under ambient isotopy as well. Another benefit is that it is more readily apparent how we can further extend Fox 3-coloring than extending 3-coloring. From the definition of 3-coloring, there is some ambiguity that would need to be sorted out about precisely what rules need to change. For example, one of our conditions was that either all colors appear at a crossing or none. Since a crossing has at most three arcs entering it would not be possible to have all colors meet if we were attempting to color with more than three.

This is avoided with Fox 3-coloring. The only choice we made was that we are looking at homomorphisms to D_6. We could very easily have chosen any finite group and gotten the same nice structure that we had above. The reason that D_6 is the group used for Fox 3-coloring is that it is the group of symmetries of the triangle. We can define the Fox n-coloring similarly. Let D_{2n} be the symmetry group of a regular n-gon.

Definition 4.1. Let G be the knot group of a knot K. Then a *Fox n-coloring* of K is a homomorphism $\rho : G \to D_{2n}$ where ρ is called a *non-trivial n-coloring* if ρ is surjective.

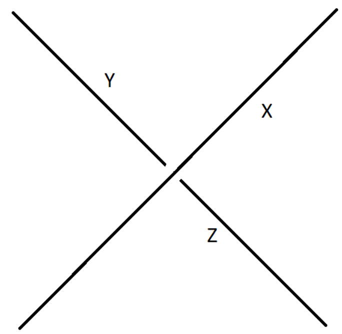

FIGURE 12. Colored Crossing

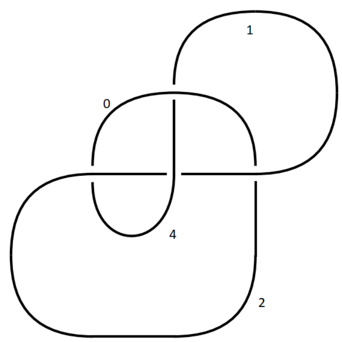

FIGURE 13. Figure-eight 5-coloring

We can also extend the notion of coloring that we started with. However, it does take more effort. Instead of choosing the colors that we did when originally defining 3-coloring, we could just as easily have used numbers instead. Then, instead of requiring that each crossing either have all three colors appear or only a single color appear, would be equivalent to each crossing satisfying the equation

$$2x - y - z \equiv 0 \mod 3$$

where the arcs of the crossing are labeled as in Figure 12.

In this form, it is more apparent how we could extend 3-coloring to arbitrary n by simply replacing the modulus. Going back to our initial example of a knot, the figure-eight knot, we have an example of a knot that is not 3-colorable but is 5-colorable as shown in Figure 13.

References

[1] Hatcher, A., *Algebraic Topology,* Cambridge University Press, Cambridge, 2001.

[2] Livingston, C., *Knot Theory,* Mathematical Association of America, Washington DC, 1996.

[3] Medwid, M., *Generalized P-Colorings of Knots,* Bowling Green University, 2014.

[4] Poonen, Bjorn. Undecidable Problems: a Sampler, in *Interpreting GDel*, pp. 211241., doi:10.1017/cbo9780511756306.015.

[5] Fort, M. K., and Fox R.H., *A Quick Trip Through Knot Theory*, in Topology of 3-Manifolds and Related Topics, Dover Publications, Mineola, 2010.

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