TRICOLORINGS OF KNOTS

LUCAS MEYERS

Abstract. Tricoloring is a knot invariant that differentiates knots based on whether or knot their diagram can be colored in a certain way with 3 colors. In this paper we introduce knots and define Tricolorability in terms of both knot diagrams and the knot group. We then mention extensions of the notions of tricolorability to arbitrary n-coloring.

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(1) Remove **TODO**
(2) Iron out figure/image placement before final edition. Too early now. Wait till revision is done.

1. Introduction

One common pattern we see in mathematics is a concern with how certain objects fit inside other objects. Knot theory studies this sort of problem. Given a circle S^1 what different ways can we embed it into $\mathbb{R}^3$ or $\mathbb{R}^3$ with an additional point at infinity (S^3)? A typical drawing of a knot can be seen in figure 1.

Naturally once we begin to consider knots we are also led to consider how we might tell them apart or rather when they should be the same at all. Intuitively the knot in figure 1 should be considered the same

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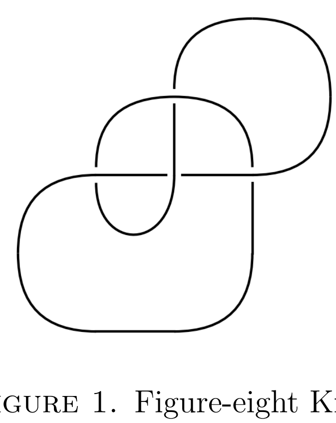

Figure 1. Figure-eight Knot

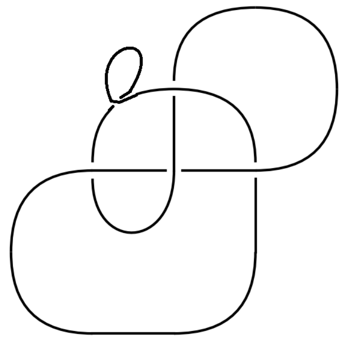

FIGURE 2. Figure-eight with a twist

as in 2 but should be different than that of trefoil seen in figure 6. In the next section we will see the definition of a knot, how to define their equivalence in a sensible way, followed by a method to differentiate them called tricolorability. Afterwards we will look at an alternative definition of tricolorability. Then we will introduce notions of how one may consider coloring knots with more than 3-colors.

2. Diagrams and Coloring

In this paper we will follow the conventions of Charles Livingston [2]. To begin we give the definition of a knot.

Definition 2.1 (Knot)**.** A Knot is a simple closed curve in $\mathbb{R}^3$(or S^3).

Now there are two issues that we need to consider before moving forward. The first is our notion of equivalence. Unfortunately a homeomorphism from one knot to another is too weak a notion of equivalence. Instead we will use an equivalence called ambient isotopy.

Definition 2.2 (Ambient Isotopy)**.** Two knots K, J are ambient isotopic if there is a homotopy on the ambient space $H : S^3 \times [0,1] \to S^3$ if $H(x,0)$ is the identity and the image $H(K,1) = J$.

Second is that we will be considering a subset of knots called tame. A knot is tame it is ambient isotopic to a piecewise linear knot. An example of a knot that is not tame is called wild and an example drawing of such can be seen in figure 3. Wild knots have some more "pathological" properties and to keep things simple and consistent.

When we give the initial definition of tricolorability we will not work directly with the curves themselves. Instead we will deal with diagrams of the knots. All drawings of knots in this paper are examples of diagrams.

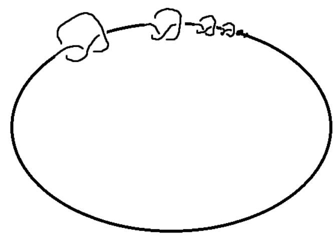

FIGURE 3. Wild Knot

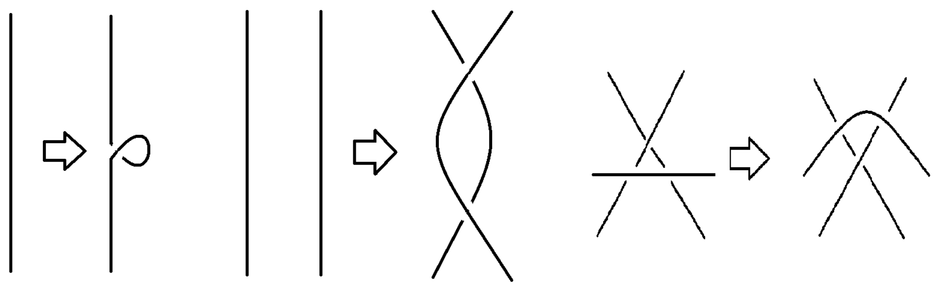

FIGURE 4. Reidmeister moves

Definition 2.3 (Diagram)**.** A diagram D for a knot K is a projection of K down to a 2-plane such that no three points are sent to the same place nor do any two points that are kinks in K.

There is a loss of information that occurs when projecting down to a 2-plane and we recover this by taking any intersection and denoting which strand goes above and which strand goes below. The segments that make up the diagram are called arcs.

There is one other unfortunate occurrence from moving to diagrams. It is that there are infinitely many diagrams for a given knot. Moreover actually telling whether to diagrams denote the same knot is extremely difficult although it is decidable and moreover belongs to NP [4].

It turns out that there are three "moves" along with planar isotopy that we can apply to diagrams that will remedy this.

Definition 2.4 (Reidmeister Moves)**.** The Reidmeister moves are the moves labeled in figure 4. We will refer to them as type I, type II, or type III moves respectively.

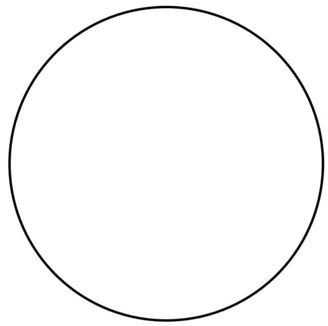

FIGURE 5. Unknot

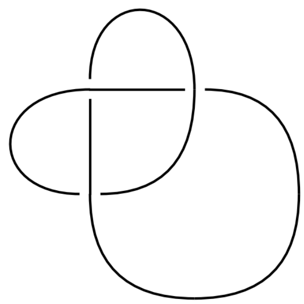

FIGURE 6. Trefoil Knot

We will use the following theorem about Reidmeister moves without proof.

Theorem 2.5. *Two knots K and J are ambient isotopic if and only if their diagrams are related by a finite sequence of Reidmeister moves and planar isotopy.*

This is where the importance of Reidmeister moves and diagrams becomes apparent in defining invariants of knots. If we define an invariant on diagrams and show that it does not change under planar isotopy or the Reidmeister moves then it will also act as an invariant on knots. This is precisely the method we will use with our initial definition of tricolorability.

Now there are two knots that we will return to over an over again in this paper as they are the simplest examples of knots for which tricolorability differentiates them. The simplest knot is the unknot 5 which is unique in that it has a diagram with no crossings whatsoever. The other is the trefoil 6.

With that in mind let us introduce tricolorability.

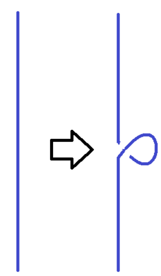

Figure 7. Type I

Definition 2.6. Let K be a knot with diagram D. Then a tricoloring (3-coloring) of K is an assignment to each arc one of three colors such that at each crossing either all colors or only single color color appear. A tricoloring is called nontrivial if at least 2 colors are used.

We will use the colors red, green, and blue as our colors. Now we must show that tricoloring is in fact an invariant.

Theorem 2.7. *The existence of a non-trivial 3-coloring for some diagram is a knot invariant.*

Proof. Let K be a knot with diagram D. We must show that given a coloring for D that we can color any diagram of K. First note that planar isotopy does not affect the crossings for D and as such does not change the 3-colorability.

Next we will have a series of pictures that demonstrate the possibilities for Type I, Type II, and Type III moves respectively.

For type I we have:

For type II there are two cases either both strands have the same color or they are different.

Finally for type III moves there are several cases based on the colors of the strands entering. We will demonstrate three such cases and the rest are similar.

Thus if a diagram D for a knot K is has a 3-coloring then any diagram for K has a 3-coloring. Therefore 3-colorability is a knot invariant. □

An immediate consequence of this is that the unknot and the trefoil knot are in fact not identical. We can see this because the unknot is not tricolorable as it only has a single arc. However the trefoil can be colored as in figure 10.

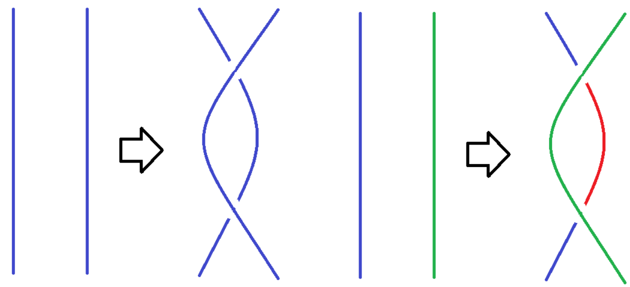

FIGURE 8. Type II

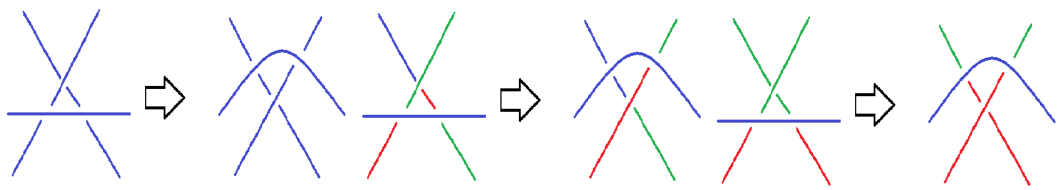

FIGURE 9. Type III

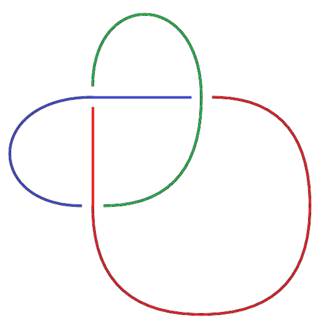

FIGURE 10. Colored trefoil

3. The Knot Group

The definition of 3-coloring is a perfectly valid definition for a knot invariant. However it is defined in terms of diagrams and as such we had to do some extra work with the Reidmeister moves to show that it was indeed an invariant. Ideally there would be an equivalent definition of coloring that does not involve diagrams. Then the fact that it is an invariant would be immediate and it may be possible to better tease out the topological property that is being captured by 3-coloring.

It turns out that there is a way to define 3-coloring in this manner and it has to do with a construction called the Knot group. Ones first instinct to get a group out of a knot might be to take the fundamental group. However since all knots are copies of S^1 the fundamental group for any knot is $\mathbb{Z}$. As such we have to take advantage of its ambient space.

Definition 3.1 (Knot Group)**.** Let K be a knot. Then the Knot group of K is the fundamental group $\pi_1(S^3 \setminus K, *)$.

This definition makes use of the embedding of the knot into S^3 and as such will not be identical for all knots. Naturally we need a way to calculate the knot group. We can do this using what is called the Wirtinger presentation [1].

Theorem 3.2 (Wirtinger)**.** *Let K be a knot and D a diagram for K. Choose an orientation for D. Then let $x_1, \ldots, x_n$ denote the arcs of D and for each crossing give a relation $r_1 \ldots, r_m$ such that $r_i \equiv (x_i x_j x_i^{-1} = x_k)$ where x_i is the top arc of the crossing and x_j is the arc underneath approaching x_k. Then the knot group of K has the presentation*

$$\pi_1(S^3 \setminus K, *) \cong \langle x_1, \ldots, x_n | \rangle$$

Now we give a couple of examples.

Example 3.3. The Knot group of the unknot is

$$\langle x_1 | \rangle \cong \mathbb{Z}$$

as there are no crossings and only a single arc.

Followed by the trefoil knot.

Example 3.4. The trefoil knot has three crossings and three arcs. Label it as shown in figure 11. Then the Wirtinger presentation of the knot group of the trefoil is

$$\langle x_1, x_2, x_3 | x_1 x_2 x_1^{-1} = x_3, x_3 x_1 x_3^{-1} = x_2, x_2 x_3 x_2^{-1} = x_1 \rangle$$

However the presentation of this group can be simplified to the form

$$\langle x_1, x_1 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

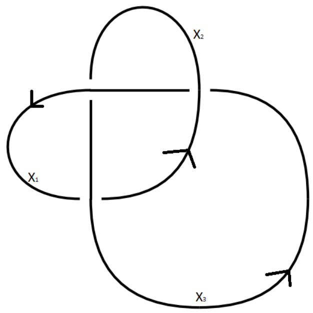

FIGURE 11. Annotated Trefoil

The knot group gives us another invariant of knots. Unfortunately presentations of groups can be rather difficult to work with. Thus to make the problem more tractable we instead look at homomorphisms from the knot group to finite groups as these will be determined by where we send generators. This is how we will redefine coloring in terms of the knot group [5].

Definition 3.5 (Fox 3-coloring)**.** Let K be a knot and G the corresponding knot group. Then a Fox 3-coloring of K is a homomorphism ρ from G to the symmetries of a triangle D_6. We say that ρ is a non-trivial coloring if ρ is surjective.

Thus a knot K is Fox 3-colorable if there exists a nontrivial Fox 3-coloring.

Now a couple of examples using our prior work.

Example 3.6. The knot group of the unknot is $\mathbb{Z}$. Since $\mathbb{Z}$ has only a single generator it is not possible to create a surjective homomorphism onto D_6 as D_6 has two generators.

Example 3.7. As we calculated above the knot group for the trefoil is

$$\langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle$$

Similarly we can write a presentation of D_6 as

$$\langle r, s | r^3 = 1, s^2 = 1, srsr = 1 \rangle$$

This gives us two non-trivial Fox 3-colorings of the trefoil. The first sends x_1 to r and x_2 to s and the other swaps which generator is sent to which generator.

As was implied by the prior examples there is a relation between Fox 3-coloring and the 3-colorability of knots. In fact it turns out that they are exactly the same.

Theorem 3.8. *A knot K has a 3-coloring if and only if there is a Fox 3-coloring of K.*

A proof of this theorem can be found within [3]. The proof that is done relates two notions of coloring using the Wirtinger presentation to marry the conditions necessary for the existence of a 3-coloring and the existence of a Fox 3-coloring.

4. n-coloring

So what are the benefits of looking at 3-coloring through the lens of Fox 3-coloring? As we mentioned before it is immediately clear that this is a knot invariant where as when we defined 3-coloring with diagrams. Another benefit is that it is more readily apparent how we can further extend Fox 3-coloring than extending 3-coloring. From the definition of 3-coloring there is some ambiguity that would need to be sorted out about precisely what rules need to change. Questions such as "How many colors are necessary?" or "What would make a crossing valid configuration?".

This is avoided with Fox 3-coloring. The only choice we made was that we are looking at homomorphisms to D_6. We could very easily have chosen any finite group and gotten the same nice structure that we had above. The reason that D_6 is the group used for Fox 3-coloring is that it is the group of symmetries of the triangle. We can define the Fox n-coloring similarly.

Definition 4.1 (Fox n-coloring)**.** Let G be the knot group of a knot K. Then an n-coloring of K is a homomorphism $\rho : G \to D_{2n}$ where ρ is called a non-trivial n-coloring if ρ is surjective.

We can also extend the notion of coloring that we started with. However it does take more work. Instead of choosing the colors that we did when originally defining 3-coloring we could just as easily have used numbers instead. Then instead of requiring that each crossing either use all three colors or all the same it would be equivalent to each crossing satisfying the equation

$$2x - y - z \equiv 0 \mod 3$$

where the arcs of the crossing are labeled as in figure 12.

In this form it is more apparent how we could extend 3-coloring to arbitrary n by simply replacing the modulus. Going back to our initial example of a knot, the figure eight knot, we have an example of a knot that is not 3-colorable but is 5-colorable as shown in figure 13.

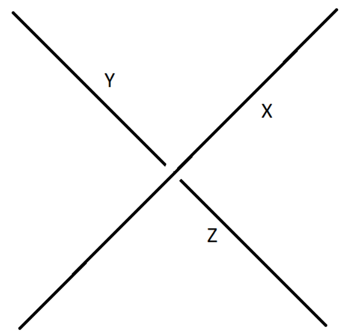

Figure 12. Colored Crossing

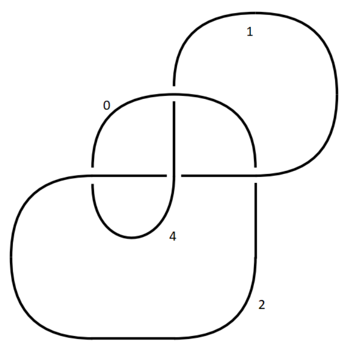

Figure 13. Figure-eight 5-coloring

Acknowledgments

References

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Mathematics Department, Louisiana State University, Baton Rouge, Louisiana
E-mail address: `lmeye22@lsu.edu`