Problem 1 (3.1.3).

Proof. Let $a, b \in G$. Then $\mathrm{Inn}(ab) = \phi_{ab}$. For any $g \in G$ we have

$$\phi_{ab}(g) = abgb^{-1}a^{-1} = a\phi_b(g)a^{-1} = \phi_a \circ \phi_b(g)$$

Which implies that $\phi_{ab} = \phi_a \circ \phi_b = \mathrm{Inn}(a) \circ \mathrm{Inn}(b)$ completing the proof that Inn is a group homomorphism.

Next, for an element $a \in G$ to be in the kernel of Inn it is required to have $\phi_a(g) = g$. Then $gag^{-1} = a$ which by cancellation we get $ga = ag$ for all $g \in G$. Therefore the kernel of Inn is $Z(G)$.

Finally let $\phi_a \in \mathrm{Inn}(G)$ and $\varphi \in \mathrm{Aut}(G)$. Then the function $\varphi \circ \phi_a \circ \varphi^{-1}$ for an element $g \in G$ is

$$\varphi \circ \phi_a \circ \varphi^{-1}(g) = \varphi(a\varphi^{-1}(g)a^{-1}) = \varphi(a)\varphi \circ \varphi^{-1}(g) \circ \varphi(a^{-1}) = \varphi(a)g\varphi(a)^{-1} = \phi_{\varphi(a)}(g)$$

which shows that $\mathrm{Inn}(G)$ is closed under conjugation and is therefore a normal subgroup of $\mathrm{Aut}(G)$. $\square$

Elaborate on why this is the case. The Automorphism group for D_8 is isomorphic to D_8 The Inner Automorphism group for D_8 is isomorphic to $\mathcal{K}_4$

Problem 2 (3.4).

Proof. Let $\phi : G \to \bar{G}$ be an epimorphism with $N := \ker\phi$ and H a subgroup of G containing N. Then we will show that H is normal in G if and only if $\phi(H)$ is normal in $\bar{G}$. For the forward direction suppose that H is normal in G. Let $g' \in \bar{G}$ and $h' \in \phi(H)$. Since ϕ is an epimorphism there exist $g \in G$ and $h \in H$ such that $\phi(g) = g'$ and $\phi(h) = h'$. Then

$$g'h'g'^{-1} = \phi(g)\phi(h)\phi(g^{-1}) = \phi(ghg^{-1})$$

which is in $\phi(H)$ since H is normal. Therefore $\phi(H)$ is normal.

Now suppose that $\phi(H)$ is normal in $\bar{G}$. Let $g \in G$ and $h \in H$. Then $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) \in \phi(H)$ and since $\phi(ghg^{-1})$ lies in the image $\phi(H)$ this implies that $ghg^{-1} \in H$ and therefore H is normal.

Next consider the map $\varphi : H \mapsto \phi(H)$ on the lattice of subgroups containing N. We will show that it is a bijection by proving that it is surjective and injective. For surjectivity let $K \leq \bar{G}$. Then $\phi(K) \leq G$. Note that $\phi^{-1}(\bar{e}) \leq \phi^{-1}(K)$ which implies that $N \leq \phi^{-1}(K)$. Therefore $\phi(\phi^{-1}(K)) = K$.

For injectivity let $\phi(H) = \phi(K)$ for $H, K \leq G$ where H, K contain N. Without loss of generality let $h \in H$. Then $\phi(h) \in \phi(K)$ and $\phi(h) = \phi(k)$ for some $k \in K$. This implies that $\phi(k^{-1}h) = \bar{e}$ and it follows that $k^{-1}h \in N \leq K$. However since $k \in K$ we have that $h \in K$. Therefore $H \subset K$ and via the same reasoning $K \subset H$ and therefore $H = K$.

Therefore the map φ is a bijection.

Finally let $B \subset A$ be subgroups of G that contain N. Then define a map $C : A/B \to \phi(A)/\phi(B)$ via $C(aB) = \phi(a)\phi(B)$. To show that the map is well defined let $aB = a'B$. Then $a = a'b$ for some $b \in B$. So

$$\phi(a)\phi(B) = \phi(a'b)\phi(B) = \phi(a')\phi(b)\phi(B) = \phi(a')\phi(B)$$

which shows that the map C is well defined.

To show surjectivity let $a'\phi(B) \in \phi(A)/\phi(B)$. Then because ϕ is surjective there exists an $a \in A$ such that $\phi(a) = a'$. Then $C(aB) = a'\phi(B)$ which shows is surjective.

Finally to show injectivity let $C(aB) = C(a'B)$. Then we have

$$\phi(a)\phi(B) = \phi(a')\phi(B)$$

which implies that
$$\phi(B) = \phi(a^{-1})\phi(a')\phi(B)$$
and thus $\phi(a^{-1}a') \in N \leq B$ which implies that $aB = a'B$.

Therefore there is a bijection between A/B and $\phi(A)/\phi(B)$ and as such $|A : B| = |\phi(A) : \phi(B)|$. $\square$

Problem 3 (3.5)**.**

Proof. Let H be a normal subgroup of G where $|G : H| = p$. Let $K \leq G$ then there are two cases. If $K \leq H$ then $K \leq H$ and we are done. Otherwise suppose that $K \not\leq H$. We know that $|G/H| = p$ which implies that G/H is cyclic. There exists a $k \in K \setminus H$ such that $\langle kH \rangle = G/H$. However this means that for any $g \in G$ we have $g \in k^i H$ and therefore $g = k^i h$ where $h \in H$. Therefore $g \in HK$ and as such $G = HK$.

By the second isomorphsim theorem we have $G/H \cong HK/H \cong K/(H \cap K)$ which implies that $|K : K \cap H| = p$. $\square$

Problem 4 (3.6)**.**

Proof. 1. Let G, G' be groups. Then we will show that $G \times G'$ is a group under pointwise multiplication.

associativity: Consider elements $(a, a'), (b, b'), (c, c') \in G \times G'$. Then we have
$$\begin{aligned}((a, a') \cdot (b, b')) \cdot (c, c') &= (ab, a'b') \cdot (c, c') \\ &= ((ab)c, (a'b')c') \\ &= (a(bc), a'(b'c')) \\ &= (a, a') \cdot (bc, b'c') \\ &= (a, a') \cdot ((b, b') \cdot (c, c'))\end{aligned}$$
Which shows that the group operation is associative.

identity: Consider (e, e') made up of the identity elements of G and G' respectively. Then for $(g, g') \in G \times G'$ we have
$$(e, e') \cdot (g, g') = (eg, e'g') = (g, g') = (ge, g'e') = (g, g')(e, e')$$
which shows the existence of an identity.

inverse: Let $(g, g') \in G$. Then
$$\begin{aligned}(g, g') \cdot (g^{-1}, g'^{-1}) &= (gg^{-1}, g'g'^{-1}) \\ &= (e, e') \\ &= (g^{-1}g, g'^{-1}g') \\ &= (g^{-1}, g'^{-1})(g, g')\end{aligned}$$
Which shows that for any element we have a two sided inverse.

Therefore the Cartesian product of groups $G \times G'$ is a group under pointwise multiplication.

2. Let $M, N \trianglelefteq G$ such that $G = MN$.

$\square$

Problem 5 (3.1.17)**.**

Proof. $\square$

Problem 6 (3.1.32)**.**

Proof. $\square$