Problem 1 (7.14). *1. Let R be a commutative ring with $1 \neq 0$ and $I_1, \dots, I_n$ pairwise comaximal ideals of R. Prove that*

$$(R/(I_1 \dots I_n))^X \cong (R/I_1)^X \times \dots \times (R/I_n)^X$$

as groups.

2. Let m, n be relatively prime positive integers. Prove that

$$(\mathbb{Z}_{mn})^X \cong (\mathbb{Z}_m)^X \times (\mathbb{Z}_n)^X$$

as groups.

3. Solve the system of congruences:

$$\begin{aligned} x &\equiv 2 \mod 9 \\ x &\equiv 3 \mod 5 \\ x &\equiv 1 \mod 7 \\ x &\equiv 5 \mod 11 \end{aligned}$$

Proof. 1.

2.

3. $x \cong 533 \mod 3465$

□

Problem 2 (8.1). *Prove that the division algorithm holds for any polynomial ring over a field.*

Proof. □

Problem 3 (8.2). *1. Prove that $a|b$ iff $b \in (a)$ iff $(b) \subseteq (a)$.*

2. If $a|b$ and $a|c$, prove that $a|(bx + cy)$ for all $x, y \in R$.

3. Suppose $b \neq 0$. If $a|b$ and $b|c$, then $a|c$.

4. If d is a greatest common divisor of a, b then du is also a greatest common divisor of a, b for any unit u of R.

Proof. □

Problem 4 (8.3). *An element p in an integral domain R is prime if, and only if, $p|ab$ implies $p|a$ or $p|b$ for any $a, b \in R$.*

Proof. □

Problem 5 (8.4)**.** *Let R be a UFD and $a, b \in R \setminus \{0\}$. Then a, b has a greatest common divisor in R. If a, b are relatively prime and $a|bc$ for some $c \in R$, then $a|c$.*

Proof. □

Problem 6 (G1)**.** *Let H be a normal subgroup of a group G, and let K be a subgroup of H.*

1. *Give an example of this situation where K is not a normal subgroup of G.*
2. *Prove that if the normal subgroup H is cyclic, then K is normal in G.*

Proof.
1. Consider S_5 and A_5. We know that $A_5 \trianglelefteq S_5$ however the subgroup $\langle(1\ 2\ 3)\rangle$ is not normal in S_5.

□

Problem 7 (G2)**.** *Prove that every finite group of order at least three has a nontrivial automorphism.*

Proof. □

Problem 8 (R1)**.** *Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} | a, b \in \mathbb{Z}\}$.*

1. *Why is R an integral domain?*
2. *What are the units in R?*
3. *Is the element 2 irreducible in R?*
4. *If $x, y \in R$, and $2|xy$, does it follow that 2 divides either x or y? Justify your answer.*

Proof. □

Problem 9 (R2)**.**
1. *Give an example of an integral domain with exactly 9 elements.*
2. *Is there an integral domain with exactly 10 elements? Justify your answer.*

Proof.
1. $\mathbb{Z}_3[\sqrt{2}]$
2.

□

Problem 10 (R3)**.** *Let*

$$F = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} | a, b \in \mathbb{Q} \right\}$$

1. *Prove that F is a field under the usual matrix operations of addition and multiplication.*

2. Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{2})$.

Proof. $\square$