Problem 1 (3.1.3)**.**

Proof. Let $a, b \in G$. Then $\operatorname{Inn}(ab) = \phi_{ab}$. For any $g \in G$ we have

$$\phi_{ab}(g) = abgb^{-1}a^{-1} = a\phi_b(g)a^{-1} = \phi_a \circ \phi_b(g)$$

Which implies that $\phi_{ab} = \phi_a \circ \phi_b = \operatorname{Inn}(a) \circ \operatorname{Inn}(b)$ completing the proof that Inn is a group homomorphism.

Next, for an element $a \in G$ to be in the kernel of Inn it is required to have $\phi_a(g) = g$. Then $gag^{-1} = a$ which by cancellation we get $ga = ag$ for all $g \in G$. Therefore the kernel of Inn is $Z(G)$.

Finally let $\phi_a \in \operatorname{Inn}(G)$ and $\varphi \in \operatorname{Aut}(G)$. Then the function $\varphi \circ \phi_a \circ \varphi^{-1}$ for an element $g \in G$ is

$$\varphi \circ \phi_a \circ \varphi^{-1}(g) = \varphi(a\varphi^{-1}(g)a^{-1}) = \varphi(a)\varphi \circ \varphi^{-1}(g) \circ \varphi(a^{-1}) = \varphi(a)g\varphi(a)^{-1} = \phi_{\varphi(a)}(g)$$

which shows that $\operatorname{Inn}(G)$ is closed under conjugation and is therefore a normal subgroup of $\operatorname{Aut}(G)$. □

Elaborate on why this is the case. The Automorphism group for D_8 is isomorphic to D_8 The Inner Automorphism group for D_8 is isomorphic to $\mathcal{K}_4$

Problem 2 (3.4)**.**

Proof. □

Problem 3 (3.5)**.**

Proof. □

Problem 4 (3.6)**.**

Proof. 1. Let G, G' be groups. Then we will show that $G \times G'$ is a group under pointwise multiplication.

associativity: Consider elements $(a, a'), (b, b'), (c, c') \in G \times G'$. Then we have

$$\begin{aligned}((a, a') \cdot (b, b')) \cdot (c, c') &= (ab, a'b') \cdot (c, c') \\ &= ((ab)c, (a'b')c') \\ &= (a(bc), a'(b'c')) \\ &= (a, a') \cdot (bc, b'c') \\ &= (a, a') \cdot ((b, b') \cdot (c, c'))\end{aligned}$$

Which shows that the group operation is associative.

identity: Consider (e, e') made up of the identity elements of G and G' respectively. Then for $(g, g') \in G \times G'$ we have

$$(e, e') \cdot (g, g') = (eg, e'g') = (g, g') = (ge, g'e') = (g, g')(e, e')$$

which shows the existence of an identity.

inverse: Let $(g, g') \in G$. Then

$$\begin{aligned}(g, g') \cdot (g^{-1}, g'^{-1}) &= (gg^{-1}, g'g'^{-1}) \\ &= (e, e') \\ &= (g^{-1}g, g'^{-1}g') \\ &= (g^{-1}, g'^{-1})(g, g')\end{aligned}$$

Which shows that for any element we have a two sided inverse.

Therefore the Cartesian product of groups $G \times G'$ is a group under pointwise multiplication.

2. Let $M, N \trianglelefteq G$ such that $G = MN$.

□

Problem 5 (3.1.17)**.**

Proof. □

Problem 6 (3.1.32)**.**

Proof. □