**Problem 1** (4.2).

1. Let $G$ be a finite group and $H$ a subgroup of index $n$. Define $N := \bigcap_{x \in G} xHx^{-1}$ which we know is a normal subgroup of $G$ contained in $H$ by a prior problem. Now let $G/N$ act on $G/H$ by $gN \cdot g'H \mapsto gg'H$. To see this is well defined let $g'H \in G/H$. Then

$$gN \cdot g'H = gg'NH = gg'H$$

However this action is equivalent to a homomorphism $\varphi : G/N \to S_{|G/H|=n}$ which by the first isomorphism theorem implies that $G/N$ is isomorphic to some subgroup of $S_n$ and as such $|G/N| = |G : N| \big| n!$ completing the proof.

2. Let $G$ be a finite group where $p$ is the smallest prime factor of $|G| = n$. Let $H$ be a subgroup of $G$ with index $p$. Then by problem 4.2.1 there exists a subgroup $N \trianglelefteq G$ such that $N \leq H$ and $|G : N| \big| p!$. However $|G : N|$ cannot be less than $p$ because if it were then with $|G| = |N||G : N|$ we would have $|G|$ divisible by a smaller prime. On the other hand $|G : N|$ cannot be larger than $p$. If it were then $pm \big| |G|$ where $m$ is a product of numbers smaller than $p$ again contradicting that $p$ is the smallest prime that divides $|G|$.

   Thus $|G : N| = p$ which via Lagrange's Theorem gives us that $|H| = |N|$. However since $N \leq H$ it must be the case that $N = H$.

   Therefore $H$ is a normal subgroup.

3. Let $G$ be a group and $H$ a subgroup of index 2. Then there are only two cosets for $H$ which are $H, gH$ for some $g \in G \setminus H$. However since there are only two this implies that $gH = Hg$. Since this holds for all cosets of $H$ we have that $H$ is normal.

   Therefore any subgroup of index 2 is normal.

4. Let $N$ be a normal subgroup and $K$ a conjugacy class $K$ with some representative $k \in K$. If $K \cap N = \phi$ then we're done. Otherwise suppose that $K \cap N \neq \phi$. Then there is some $\alpha \in K \cap N$. Then $\alpha = gkg^{-1}$ for some $g \in G$. This implies that $g^{-1}\alpha g = k$ however since $\alpha \in N$ so is $g^{-1}\alpha g = k$. Therefore $k \in N$ and as such $K \subset N$.

**Problem 2** (4.3).

*Proof.* □

**Problem 3** (4.4.1).

*Proof.* □

**Problem 4** (4.4.2).

*Proof.* □

**Problem 5** (4.4.3).

*Proof.* □