Problem 1 (7.6). *1. Let F be a non-trivial field and $F[[x]]$ the set of all formal power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_i \in F$. Prove that $F[[x]]$ is an integral domain under the following addition and multiplication:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{i+j=n} a_i b_j\right) x^n$$

Prove that $f(x)$ is a unit if and only if the constant term of $f(x)$ is non-zero.

2. *Let R be a ring and S a subring of R. Prove that $M_n(S)$ is a subring of $M_n(R)$ for any integer $n \geq 1$.*

3. *Let R be a commutative ring and G a finite group.*
   (a) *Prove that g is a unit of $R[G]$ for any $g \in G$.*
   (b) *Prove or disprove that $G = R[G]^X$.*
   (c) *If S is a subring of R, then $S[G]$ is a subring of $R[G]$.*

4. *Let R be a commutative ring and G be a finite group*
   (a) *Let $\Lambda = \sum_{g \in G} g$. Prove that Λ is in the center of $R[G]$.*
   (b) *Let K be a conjugacy class in G. Prove that $k = \sum_{g \in K} g$ is in the center of $R[G]$.*
   (c) *Let $K_1, \ldots, K_r$ be the conjugacy classes of G and $k_i = \sum_{g \in K_i} g$ for $i = 1, \ldots, r$. Prove that x is in the center of $R[G]$ if, and only if, $x = \sum_{i=1}^{r} a_i k_i$ for some $a_i \in R$.*

Proof. 1.

2.

3. (a) Let $g \in G$. Then it has an inverse $g^{-1} \in G$ for which both $g, g^{-1} \in R[G]$. Thus we have $gg^{-1} = e = 1 \in R[G]$ which shows that g is a unit of $R[G]$.
   (b)
   (c)

4. (a)
   (b)
   (c)

□

Problem 2 (7.7). *For any nonzero integers a, b, prove that $(a, b) = (\gcd(a, b)), (a) \cap (b) = (lcm(a, b))$ and that $(a)(b) = (ab)$.*

Proof. □

Problem 3 (7.8). *Let G be a finite group and R a commutative ring. Show that the map $\epsilon : R[G] \to R$ given by*

$$\epsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$$

is a surjective ring homomorphism and $\ker \epsilon$ is the ideal generated by the set $\{g - e | g \in G\}$.

Proof. □

Problem 4 (7.10). 1. *Prove that $x^2 = 0$ or 1 for all $x \in \mathbb{Z}_4$*

2. *Prove that the equation $x^2 + y^2 = 3z^2$ has no nontrivial integer solution.*

Proof. 1. For each case we have

- $0^2 \equiv 0 \mod 4$
- $1^2 \equiv 1 \mod 4$
- $2^2 \equiv 0 \mod 4$
- $3^2 \equiv 1 \mod 4$

Therefore the polynomial $x^2 = 0$ or 1 for all $x \in \mathbb{Z}_4$.

2.

□

Problem 5 (7.11). *Let D be a square-free integer and I the ideal $(x^2 - D)$ of $\mathbb{Q}[x]$. Prove that*

$$\mathbb{Q}[x]/I \cong \mathbb{Q}(\sqrt{D})$$

as rings. Find all the ideals of $\mathbb{Q}[x]$ containing I.

Proof. □