Problem 1 (3.1.3)**.**

Proof. Let $a, b \in G$. Then $\mathrm{Inn}(ab) = \phi_{ab}$. For any $g \in G$ we have

$$\phi_{ab}(g) = abgb^{-1}a^{-1} = a\phi_b(g)a^{-1} = \phi_a \circ \phi_b(g)$$

Which implies that $\phi_{ab} = \phi_a \circ \phi_b = \mathrm{Inn}(a) \circ \mathrm{Inn}(b)$ completing the proof that Inn is a group homomorphism.

Next, for an element $a \in G$ to be in the kernel of Inn it is required to have $\phi_a(g) = g$. Then $gag^{-1} = a$ which by cancellation we get $ga = ag$ for all $g \in G$. Therefore the kernel of Inn is $Z(G)$.

Finally let $\phi_a \in \mathrm{Inn}(G)$ and $\varphi \in \mathrm{Aut}(G)$. Then the function $\varphi \circ \phi_a \circ \varphi^{-1}$ for an element $g \in G$ is

$$\varphi \circ \phi_a \circ \varphi^{-1}(g) = \varphi(a\varphi^{-1}(g)a^{-1}) = \varphi(a)\varphi \circ \varphi^{-1}(g) \circ \varphi(a^{-1}) = \varphi(a)g\varphi(a)^{-1} = \phi_{\varphi(a)}(g)$$

which shows that $\mathrm{Inn}(G)$ is closed under conjugation and is therefore a normal subgroup of $\mathrm{Aut}(G)$. $\square$

We'll start with computing the inner automorphism group of D_8. From the prior part of the problem and the second isomorphism theorem we have that $\mathrm{Inn}(D_8) \cong D_8/Z(D_8) \cong \mathcal{V}_8$.

For the automorphism group note that since isomorphisms preserve the orders of elements the choices for r and s for any element are

$$r \mapsto r, r^3 \quad s \mapsto s, sr, sr^2, sr^3, r^2$$

However since mapping s to r^2 would make the map not be surjective we can throw it out. This implies that $|\mathrm{Aut}(D_8)| \leq 8$. We also know that $|\mathrm{Inn}(D_8)| = 4$ and as such $|D_8|$ is either 4 or 8. But if we take the map

$$\varphi(r) = r, \quad \varphi(s) = sr$$

is of order 4 which means that $|\mathrm{Aut}(D_8)| = 8$.

Next define $\psi(\rho) = s\rho s$. This map is of order 2 and we will show that φ and ψ fulfill the required relations to show isomorphism with D_8. Since $\varphi^4 = id$ and $\psi^2 = id$ we already have the first two. Now we must show that $(\psi \circ \varphi)^2 = id$. Let $\eta \in D_8$ then we have

$$\begin{aligned}(\psi \circ \varphi)^2(\eta) &= ssr\varphi^2(\eta)srs \\ &= r\varphi^2(\eta)r^{-1}\end{aligned}$$

If $\eta = r^i$ then we have $rr^ir^{-1} = r^i$. Otherwise if $\eta = sr^i$ then we have

$$r\varphi^2(sr^i)r^{-1} = rsr^{i+2}r^{-1} = rsr^{i+1} = r^{-i}s = sr^i$$

which shows that $(\psi \circ \varphi)^2 = id$ completing the proof that $\mathrm{Aut}(D_8) \cong D_8$.

Problem 2 (3.4)**.**

Proof. Let $\phi : G \to \bar{G}$ be an epimorphism with $N := \ker\phi$ and H a subgroup of G containing N. Then we will show that H is normal in G if and only if $\phi(H)$ is normal in $\bar{G}$. For the forward direction suppose that H is normal in G. Let $g' \in \bar{G}$ and $h' \in \phi(H)$. Since ϕ is an epimorphism there exist $g \in G$ and $h \in H$ such that $\phi(g) = g'$ and $\phi(h) = h'$. Then

$$g'h'g'^{-1} = \phi(g)\phi(h)\phi(g^{-1}) = \phi(ghg^{-1})$$

which is in $\phi(H)$ since H is normal. Therefore $\phi(H)$ is normal.

Now suppose that $\phi(H)$ is normal in $\bar{G}$. Let $g \in G$ and $h \in H$. Then $\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) \in \phi(H)$ and since $\phi(ghg^{-1})$ lies in the image $\phi(H)$ this implies that $ghg^{-1} \in H$ and therefore H is normal.

Next consider the map $\varphi : H \mapsto \phi(H)$ on the lattice of subgroups containing N. We will show that it is a bijection by proving that it is surjective and injective. For surjectivity let $K \leq \bar{G}$. Then $\phi(K) \leq G$. Note that $\phi^{-1}(\bar{e}) \leq \phi^{-1}(K)$ which implies that $N \leq \phi^{-1}(K)$. Therefore $\phi(\phi^{-1}(K)) = K$.

For injectivity let $\phi(H) = \phi(K)$ for $H, K \leq G$ where H, K contain N. Without loss of generality let $h \in H$. Then $\phi(h) \in \phi(K)$ and $\phi(h) = \phi(k)$ for some $k \in K$. This implies that $\phi(k^{-1}h) = \bar{e}$ and it follows that $k^{-1}h \in N \leq K$. However since $k \in K$ we have that $h \in K$. Therefore $H \subset K$ and via the same reasoning $K \subset H$ and therefore $H = K$.

Therefore the map φ is a bijection.

Finally let $B \subset A$ be subgroups of G that contain N. Then define a map $C : A/B \to \phi(A)/\phi(B)$ via $C(aB) = \phi(a)\phi(B)$. To show that the map is well defined let $aB = a'B$. Then $a = a'b$ for some $b \in B$. So

$$\phi(a)\phi(B) = \phi(a'b)\phi(B) = \phi(a')\phi(b)\phi(B) = \phi(a')\phi(B)$$

which shows that the map C is well defined.

To show surjectivity let $a'\phi(B) \in \phi(A)/\phi(B)$. Then because ϕ is surjective there exists an $a \in A$ such that $\phi(a) = a'$. Then $C(aB) = a'\phi(B)$ which shows is surjective.

Finally to show injectivity let $C(aB) = C(a'B)$. Then we have

$$\phi(a)\phi(B) = \phi(a')\phi(B)$$

which implies that

$$\phi(B) = \phi(a^{-1})\phi(a')\phi(B)$$

and thus $\phi(a^{-1}a') \in N \leq B$ which implies that $aB = a'B$.

Therefore there is a bijection between A/B and $\phi(A)/\phi(B)$ and as such $|A : B| = |\phi(A) : \phi(B)|$. $\square$

Problem 3 (3.5)**.**

Proof. Let H be a normal subgroup of G where $|G : H| = p$. Let $K \leq G$ then there are two cases. If $K \leq H$ then $K \leq H$ and we are done. Otherwise suppose that $K \nleq H$. We know that $|G/H| = p$ which implies that G/H is cyclic. There exists a $k \in K \setminus H$ such that $\langle kH \rangle = G/H$. However this means that for any $g \in G$ we have $g \in k^iH$ and therefore $g = k^ih$ where $h \in H$. Therefore $g \in HK$ and as such $G = HK$.

By the second isomorphism theorem we have $G/H \cong HK/H \cong K/(H \cap K)$ which implies that $|K : K \cap H| = p$. $\square$

Problem 4 (3.6)**.**

Proof. 1. Let G, G' be groups. Then we will show that $G \times G'$ is a group under pointwise multiplication.

associativity: Consider elements $(a, a'), (b, b'), (c, c') \in G \times G'$. Then we have

$$\begin{aligned}((a, a') \cdot (b, b')) \cdot (c, c') &= (ab, a'b') \cdot (c, c') \\ &= ((ab)c, (a'b')c') \\ &= (a(bc), a'(b'c')) \\ &= (a, a') \cdot (bc, b'c') \\ &= (a, a') \cdot ((b, b') \cdot (c, c'))\end{aligned}$$

Which shows that the group operation is associative.

identity: Consider (e, e') made up of the identity elements of G and G' respectively. Then for $(g, g') \in G \times G'$ we have

$$(e, e') \cdot (g, g') = (eg, e'g') = (g, g') = (ge, g'e') = (g, g')(e, e')$$

which shows the existence of an identity.

inverse: Let $(g, g') \in G$. Then

$$\begin{aligned}(g, g') \cdot (g^{-1}, g'^{-1}) &= (gg^{-1}, g'g'^{-1}) \\ &= (e, e') \\ &= (g^{-1}g, g'^{-1}g') \\ &= (g^{-1}, g'^{-1})(g, g')\end{aligned}$$

Which shows that for any element we have a two sided inverse.

Therefore the Cartesian product of groups $G \times G'$ is a group under pointwise multiplication.

2. Let $M, N \trianglelefteq G$ such that $G = MN$. Define a map $\phi : G \to G/M \times G/N$ as $\phi(g) = (gM, gN)$. First we will show the map is surjective. Let $(gM, g'N) \in G/M \times G/N$. Since $MN = G$ we can rewrite $g = mn$ and $g' = m'n'$ to get

$$(mnM, m'n'N) = (Mmn, m'N) = (Mn, m'N) = (nM, m'N)$$

using normality. Then consider $\phi(m'n)$ to get

$$\phi(m'n) = (m'nM, m'nN) = (Mm'n, m'N) = (Mn, m'N) = (nM, m'N)$$

completing the proof that ϕ is surjective.

Then Kerϕ is the elements g such that $gM = M$ and $gN = N$ which consists entirely of $g \in M \cap N$. If we apply the first isomorphism theorem we get

$$G/M \cap N \cong G/M \times G/N$$

completing the proof.

□

Problem 5 (3.1.17)**.**

1. The order of $\langle r^4 \rangle$ is 2 which implies that $|D_{16}/\langle r^4 \rangle| = \frac{|D_{16}|}{|\langle r^4 \rangle|} = 8$.

2. The elements of $D_{16}/\langle r^4 \rangle$ are

$$\{\bar{r}^0, \bar{r}^1, \bar{r}^2, \bar{r}^3, \bar{s}, \bar{s}\bar{r}, \bar{s}\bar{r}^2, \bar{s}\bar{r}^3\}$$

3. The order of each element in the order listed above is

$$1, 4, 2, 4, 2, 2, 2, 2$$

4. $\overline{rs} = \bar{s}\bar{r}^3$
   $\overline{sr^2s} = \bar{r}^2$
   $\overline{s^{-1}r^{-1}sr} = \bar{r}^2$

5. The size of $\bar{H}$ is 4 and as such is of index 2 in $\bar{G}$ which is sufficient to show the normality of $\bar{H}$. Since $\bar{H}$ is of size 4 and each non-identity element, $\bar{s}, \bar{s}\bar{r}^2, \bar{r^2}$ is of order 2 it must be isomorphic to $\mathcal{V}_4$.

   The preimage of $\bar{H}$ in D_{16} is $\{e, r^2, r^4, r^6, s, sr^2, sr^4, sr^6\}$ which is isomorphic to D_8.

6. The center of $\bar{G}$ is $\bar{e}, \bar{r^2}$. Then the size of $\bar{G}/Z(\bar{G})$ is 4. Then our elements will be similar to $\{\bar{e}, \bar{r}, \bar{s}, \bar{r}\bar{s}\}$ and since each is of order 2 aside from the identity the quotient is isomorphic to $\mathcal{V}_4$.

Problem 6 (3.1.32)**.**

Proof. The proof that all subgroups of $\mathcal{Q}_8$ are normal has been copied from the last homework at the end of the problem. For the isomorphism types of the quotients we have:

- $\mathcal{Q}_8/\langle 1 \rangle \cong \mathcal{Q}_8$
- $\mathcal{Q}_8/\langle \mathcal{Q}_8 \rangle \cong \langle 1 \rangle$
- $\mathcal{Q}_8/\langle -1 \rangle \cong \mathcal{V}_4$. To see this note that the quotient group will consist of four elements each of order two. Since there are only two groups of order four and the other is cyclic it must be the Klein four group.
- $\mathcal{Q}_8/\langle i \rangle \cong \mathcal{Q}_8/\langle j \rangle \cong \mathcal{Q}_8/\langle k \rangle \cong \mathbb{Z}_2$. To see this note that the order of the quotient is two and there is only one group of order two.

Redux: The subgroups of $\mathcal{Q}_8$ are $< 1 >, < -1 >, < i >, < j >, < k >, \mathcal{Q}_8$. The trivial group and $\mathcal{Q}_8$ are both normal. For the others we have

$< -1 >$

g	$g(-1)g^{-1}$
1	-1
-1	-1
i	-1
$-i$	-1
j	-1
$-j$	-1
k	-1
$-k$	-1

$< i >$

g	$g(i)g$
1	i
-1	i
i	i
$-i$	i
j	$-i$
$-j$	$-i$
k	$-i$
$-k$	$-i$

$< j >$

g	$g(j)g$
1	j
-1	j
i	$-j$
$-i$	$-j$
j	j
$-j$	j
k	$-j$
$-k$	$-j$

$< k >$

g	$g(k)g$
1	k
-1	k
i	$-k$
$-i$	$-k$
j	$-k$
$-j$	$-k$
k	k
$-k$	k

$\square$