

**Problem 1 (4.2).**

1. Let  $G$  be a finite group and  $H$  a subgroup of index  $n$ . Define  $N := \bigcap_{x \in G} xHx^{-1}$  which we know is a normal subgroup of  $G$  contained in  $H$  by a prior problem. Now let  $G/N$  act on  $G/H$  by  $gN \cdot g'H \mapsto gg'H$ . To see this is well defined let  $g'H \in G/H$ . Then

$$gN \cdot g'H = gg'NH = gg'H$$

However this action is equivalent to a homomorphism  $\varphi : G/N \rightarrow S_{|G/H|=n}$  which by the first isomorphism theorem implies that  $G/N$  is isomorphic to some subgroup of  $S_n$  and as such  $|G/N| = |G : N| \mid n!$  completing the proof.

2. Let  $G$  be a finite group where  $p$  is the smallest prime factor of  $|G| = n$ . Let  $H$  be a subgroup of  $G$  with index  $p$ . Then by problem 4.2.1 there exists a subgroup  $N \leq G$  such that  $N \leq H$  and  $|G : N| \mid p!$ . However  $|G : N|$  cannot be less than  $p$  because if it were then with  $|G| = |N||G : N|$  we would have  $|G|$  divisible by a smaller prime. On the other hand  $|G : N|$  cannot be larger than  $p$ . If it were then  $pm \mid |G|$  where  $m$  is a product of numbers smaller than  $p$  again contradicting that  $p$  is the smallest prime that divides  $|G|$ .

Thus  $|G : N| = p$  which via Lagrange's Theorem gives us that  $|H| = |N|$ . However since  $N \leq H$  it must be the case that  $N = H$ .

Therefore  $H$  is a normal subgroup.

3. Let  $G$  be a group and  $H$  a subgroup of index 2. Then there are only two cosets for  $H$  which are  $H, gH$  for some  $g \in G \setminus H$ . However since there are only two this implies that  $gH = Hg$ . Since this holds for all cosets of  $H$  we have that  $H$  is normal.

Therefore any subgroup of index 2 is normal.

4. Let  $N$  be a normal subgroup and  $K$  a conjugacy class  $K$  with some representative  $k \in K$ . If  $K \cap N = \phi$  then we're done. Otherwise suppose that  $K \cap N \neq \phi$ . Then there is some  $\alpha \in K \cap N$ . Then  $\alpha = gkg^{-1}$  for some  $g \in G$ . This implies that  $g^{-1}\alpha g = k$  however since  $\alpha \in N$  so is  $g^{-1}\alpha g = k$ . Therefore  $k \in N$  and as such  $K \subset N$ .

**Problem 2 (4.3).**

1. The conjugacy classes of  $\mathcal{Q}_8$  are

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$$

The conjugacy classes for  $D_8$  are

$$\{e\}, \{s\}, \{r, r^3\}, \{r^2\}$$

2. a) Let  $c = (c_1 \dots c_t) \in S_n$  and  $\sigma \in S_n$ . Then let  $k = \sigma(c_i)$  for some  $i$ . Then

$$\sigma c \sigma^{-1}(k) = \sigma c \sigma^{-1}(\sigma(c_i)) = \sigma c(c_i) = \sigma(c_{i+1})$$

Next suppose that  $k \neq \sigma(c_i)$  for any  $i$ . Then

$$\sigma c \sigma^{-1}(k) = \sigma c(\sigma^{-1}(k)) = \sigma(\sigma^{-1}(k)) = k$$

Which implies that  $\sigma c \sigma^{-1} = (\sigma(c_1) \dots \sigma(c_t))$ .

- b) Given a  $t$ -cycle  $c$  we can send it to any other  $t$ -cycle by conjugation. This means that the conjugacy class for  $c$  is the permutations of  $t$  numbers selected from 1 through  $n$  up to rotation. The permutations number  $\frac{n!}{(n-t)!}$  and we account for rotation by  $\frac{n!}{(n-t)!t}$ .

Given an orbit we have the formula  $|\mathcal{O}| = \frac{|G|}{|\text{Stab}|}$ . If we act on  $S_n$  by conjugation then the stabilizer for a  $t$ -cycle is the centralizer. Using the formula for any given  $t$ -cycle  $c$  we have

$$\frac{n!}{(n-t)!t} = \frac{n!}{|C_{S_n}(c)|}$$

which implies that  $|C_{S_n}(c)| = (n-t)!t$ .

- c) The conjugacy classes of  $A_5$  are

- (a)  $()$  of order 1.
- (b)  $(a\ b)(c\ d)$  of order 15.
- (c)  $(a\ b\ c)$  of order 20.
- (d) The 5-cycle conjugacy class containing  $(1\ 2\ 3\ 4\ 5)$  of order 12.
- (e) The 5-cycle conjugacy class containing  $(2\ 1\ 3\ 4\ 5)$  of order 12.

Since normal subgroups are the unions of conjugacy classes in order for  $A_5$  to have a nontrivial normal subgroup we would require for us to be able to add a combinations of conjugacy classes to get a number that divides 60. However there is no combination that does that. Therefore  $A_5$  has no normal subgroups and as such is simple.

### Problem 3 (4.4.1).

*Proof.* Let  $G$  be a group of order  $11^2 13^2$ . Then let  $P$  be a Sylow 11-subgroup and  $Q$  a Sylow 13-subgroup. Then  $n_{11}(G) \equiv 1 \pmod{11}$  and  $n_{13}(G) \equiv 1 \pmod{13}$ . However the only number that work for  $n_{11}(G)$  and  $n_{13}(G)$  is 1 since  $n_p \mid |G : P|$ . As such the groups  $P, Q$  are the only subgroups of their size. However Sylow  $p$ -subgroups are closed under conjugation and as such this implies that  $P$  and  $Q$  are normal. This means that the product  $PQ$  is well defined. The order of  $PQ$  is  $|PQ| = \frac{|P||Q|}{|P \cap Q|}$ . However the intersection  $P \cap Q = \{e\}$  as the order of any element of  $P$  or  $Q$  divides the order of the subgroup and the orders of  $P$  and  $Q$  are relatively prime. As such  $|PQ| = |P||Q| = 11^2 13^2 = |G|$  and so  $PQ = G$ . However since  $P, Q$  are order of a prime squared they are Abelian and the normality implies  $PQ = QP$  which shows that  $G$  is Abelian.

Therefore any group of order  $11^2 13^2$  is Abelian.  $\square$

### Problem 4 (4.4.2).

*Proof.* Let  $|G| = 77 = 7 \cdot 11$ . Then there are subgroups with  $|H| = 7$  and  $|K| = 11$ . Since they are of prime order they are cyclic with generators  $h$  and  $k$  respectively. However since 7 and 11 are relatively prime  $hk$  has order  $7 \cdot 11 = 77$  which implies that  $G$  is cyclic.  $\square$

### Problem 5 (4.4.3).

*Proof.* Since  $30 = 2 \cdot 3 \cdot 5$  there are subgroups of size 3 and 5 which we'll call  $H$  and  $K$  respectively. However since  $H$  and  $K$  are of prime order they are cyclic and as such  $\langle h \rangle = H$  and  $\langle k \rangle = K$ . Since the orders of  $h, k$  are relatively prime the order of  $hk$  is fifteen.  $\square$