**Problem 1** (1.1.21).

*Proof.* Let $x \in G$ have order odd $2n + 1$. Then $x^{2n+1} = e$ and if we multiply by $x$ on both sides we get $x^{2n+2} = x^{2(n+1)} = x$.

Therefore if $x$ has odd order then it $x^{2k} = x$ for some $k \in \mathbb{Z}_{>0}$. $\qquad \square$

**Problem 2** (1.1.25).

*Proof.* Suppose that $G$ is a group such that for all $x \in G$ that $x^2 = 1$. This implies that for any element $x^{-1} = x$. Then we have $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$ which shows that $G$ is commutative. $\qquad \square$

**Problem 3** (1.1.35).

*Proof.* Let $x \in G$ have order $n$. Then consider $x^m$. Using the division algorithm with $m$ and $n$ we can rewrite $m$ as $qn + d$ where $0 \le d < n$. This implies that

$$x^m = x^{qn+d} = x^{qn}x^d = (x^n)^q x^d = e^q x^d = x^d$$

As such $x \in \{e, x^1, \dots, x^{n-1}\}$. $\qquad \square$

**Problem 4** (1.3.6).

- $(1\ 2\ 3\ 4) = (1\ 4)(1\ 3)(1\ 2)$

- $(1\ 3\ 4\ 2) = (1\ 2)(1\ 4)(1\ 3)$

- $(1\ 4\ 3\ 2) = (1\ 2)(1\ 3)(1\ 4)$

- $(4\ 3\ 2\ 1) = (4\ 1)(4\ 2)(4\ 3)$

- $(2\ 4\ 3\ 1) = (2\ 1)(2\ 3)(2\ 4)$

- $(3\ 2\ 4\ 1) = (3\ 1)(3\ 4)(3\ 2)$

**Problem 5** (1.3.9).

a) $1 + 12k, 5 + 12k, 7 + 12k, 11 + 12k$ for $k \in \mathbb{Z}$.

b) $1 + 8k, 3 + 8k, 5 + 8k, 7 + 8k$ for $k \in \mathbb{Z}$.

c) $1 + 14k, 3 + 14k, 5 + 14k, 9 + 14k, 11 + 14k, 13 + 14k$ for $k \in \mathbb{Z}$.

**Problem 6** (1.3.13).

*Proof.* Suppose that $g \in S_n$ is of the form $g = \prod_{i=1}^{m}(a_i \ b_i)$ where each transposition commutes. Then

$$g^2 = \left( \prod_{i=1}^{m}(a_i \ b_i) \right)^2 = \prod_{i=1}^{m}((a_i \ b_i)^2 = e) = e$$

which implies that $g$ is of order 2.

Next suppose that $g \in S_n$ is of order 2. We can write $g$ as a product of disjoint cycles $\prod_{i=1}^{m} \sigma_i$. Since the cycles are disjoint we can write $g^2$ as

$$g^2 = \left( \prod_{i=1}^{m} \sigma_i \right)^2 = \prod_{i=1}^{m} \sigma_i^2 = e$$

Then for a given $\sigma_i$ rewrite the above to

$$\prod_{i \in \{1...m\}-j} \sigma_i^2 = \sigma_j^{-2}$$

However since each $\sigma_i$ is disjoint this implies that $\sigma_j^{-2} = e = \sigma_j^2$. Since this is for an arbitrary $\sigma_i$ we have that $\sigma_i^2$ for all $i$.

Therefore a permutation is of order two if and only if it is the product of disjoint $2-$cycles. $\square$

**Problem 7** (1.5.2).

*Proof.* $\square$