Problem 1 (1.1.21).

Proof. Let $x \in G$ have order odd $2n+1$. Then $x^{2n+1} = e$ and if we multiply by x on both sides we get $x^{2n+2} = x^{2(n+1)} = x$.

Therefore if x has odd order then it $x^{2k} = x$ for some $k \in \mathbb{Z}_{>0}$. □

Problem 2 (1.1.25).

Proof. Suppose that G is a group such that for all $x \in G$ that $x^2 = 1$. This implies that for any element $x^{-1} = x$. Then we have $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$ which shows that G is commutative. □

Problem 3 (1.1.35).

Proof. Let $x \in G$ have order n. Then consider x^m. Using the division algorithm with m and n we can rewrite m as $qn + d$ where $0 \leq d < n$. This implies that

$$x^m = x^{qn+d} = x^{qn}x^d = (x^n)^q x^d = e^q x^d = x^d$$

As such $x \in \{e, x^1, \dots, x^{n-1}\}$. □

Problem 4 (1.3.6).

- $(1\ 2\ 3\ 4) = (1\ 4)(1\ 3)(1\ 2)$
- $(1\ 3\ 4\ 2) = (1\ 2)(1\ 4)(1\ 3)$
- $(1\ 4\ 3\ 2) = (1\ 2)(1\ 3)(1\ 4)$
- $(4\ 3\ 2\ 1) = (4\ 1)(4\ 2)(4\ 3)$
- $(2\ 4\ 3\ 1) = (2\ 1)(2\ 3)(2\ 4)$
- $(3\ 2\ 4\ 1) = (3\ 1)(3\ 4)(3\ 2)$

Problem 5 (1.3.9).

a) $1 + 12k, 5 + 12k, 7 + 12k, 11 + 12k$ for $k \in \mathbb{Z}$.

b) $1 + 8k, 3 + 8k, 5 + 8k, 7 + 8k$ for $k \in \mathbb{Z}$.

c) $1 + 14k, 3 + 14k, 5 + 14k, 9 + 14k, 11 + 14k, 13 + 14k$ for $k \in \mathbb{Z}$.

Problem 6 (1.3.13).

Proof. Suppose that $g \in S_n$ is of the form $g = \prod_{i=1}^{m}(a_i\ b_i)$ where each transposition commutes. Then

$$g^2 = \left(\prod_{i=1}^{m}(a_i\ b_i)\right)^2 = \prod_{i=1}^{m}((a_i\ b_i)^2 = e) = e$$

which implies that g is of order 2.

Next suppose that $g \in S_n$ is of order 2. We can write g as a product of disjoint cycles $\prod_{i=1}^{m}\sigma_i$. Since the cycles are disjoint we can write g^2 as

$$g^2 = \left(\prod_{i=1}^{m}\sigma_i\right)^2 = \prod_{i=1}^{m}\sigma_i^2 = e$$

Then for a given σ_i rewrite the above to

$$\prod_{i \in \{1 \ldots m\} - j} \sigma_i^2 = \sigma_j^{-2}$$

However since each σ_i is disjoint this implies that $\sigma_j^{-2} = e = \sigma_j^2$. Since this is for an arbitrary σ_i we have that σ_i^2 for all i.

Therefore a permutation is of order two if and only if it is the product of disjoint $2-$cycles. □

Problem 7 (1.5.2)**.**

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S_3	()	(1 2)	(1 2 3)	(1 3 2)	(1 3)	(2 3)
()	()	(1 2)	(1 2 3)	(1 3 2)	(1 3)	(2 3)
(1 2)	(1 2)	()	(2 3)	(1 3)	(1 3 2)	(1 2 3)
(1 2 3)	(1 2 3)	(1 3)	(1 3 2)	()	(2 3)	(1 2)
(1 3 2)	(1 3 2)	(2 3)	()	(1 2 3)	(1 2)	(1 3)
(1 3)	(1 3)	(1 2 3)	(1 2)	(2 3)	()	(1 3 2)
(2 3)	(2 3)	(1 3 2)	(1 3)	(1 2)	(1 2 3)	()

-

D_8	e	r	p	p^2	rp	rp^3	p^3	rp^2
e	e	r	p	p^2	rp	rp^3	p^3	rp^2
r	r	e	rp^3	rp^2	p^3	p	rp	p^2
p	p	rp	p^2	p^3	rp^2	r	e	rp^3
p^2	p^2	rp^2	p^3	e	rp^3	rp	p	r
rp	rp	p	r	rp^3	e	p^2	rp^2	p^3
rp^3	rp^3	p^3	rp^2	rp	p^2	e	r	p
p^3	p^3	rp^3	e	p	r	rp^2	p^2	rp
rp^2	rp^2	p^2	rp	r	p	p^3	rp^3	e

-

Q_8	1	i	j	-1	$-k$	k	-1	$-j$
1	1	i	j	-1	$-k$	k	-1	$-j$
i	i	-1	k	-1	j	$-j$	1	$-k$
j	j	$-k$	-1	$-j$	-1	i	k	1
-1	-1	-1	$-j$	1	k	$-k$	i	j
$-k$	$-k$	$-j$	i	k	-1	1	j	-1
k	k	j	-1	$-k$	1	-1	$-j$	i
-1	-1	1	$-k$	i	$-j$	j	-1	k
$-j$	$-j$	k	1	j	i	-1	$-k$	-1