Problem 1 (4.2)**.**

1. Let G be a finite group and H a subgroup of index n. Define $N := \bigcap_{x \in G} xHx^{-1}$ which we know is a normal subgroup of G contained in H by a prior problem. Now let G/N act on G/H by $gN \cdot g'H \mapsto gg'H$. To see this is well defined let $g'H \in G/H$. Then

$$gN \cdot g'H = gg'NH = gg'H$$

However this action is equivalent to a homomorphism $\varphi : G/N \to S_{|G/H|=n}$ which by the first isomorphism theorem implies that G/N is isomorphic to some subgroup of S_n and as such $|G/N| = |G : N| \big| n!$ completing the proof.

2. Let G be a finite group where p is the smallest prime factor of $|G| = n$. Let H be a subgroup of G with index p. Then by problem 4.2.1 there exists a subgroup $N \trianglelefteq G$ such that $N \leq H$ and $|G : N| \big| p!$. However $|G : N|$ cannot be less than p because if it were then with $|G| = |N||G : N|$ we would have $|G|$ divisible by a smaller prime. On the other hand $|G : N|$ cannot be larger than p. If it were then $pm \big| |G|$ where m is a product of numbers smaller than p again contradicting that p is the smallest prime that divides $|G|$.

   Thus $|G : N| = p$ which via Lagrange's Theorem gives us that $|H| = |N|$. However since $N \leq H$ it must be the case that $N = H$.

   Therefore H is a normal subgroup.

3. Let G be a group and H a subgroup of index 2. Then there are only two cosets for H which are H, gH for some $g \in G \setminus H$. However since there are only two this implies that $gH = Hg$. Since this holds for all cosets of H we have that H is normal.

   Therefore any subgroup of index 2 is normal.

4. Let N be a normal subgroup and K a conjugacy class K with some representative $k \in K$. If $K \cap N = \phi$ then we're done. Otherwise suppose that $K \cap N \neq \phi$. Then there is some $\alpha \in K \cap N$. Then $\alpha = gkg^{-1}$ for some $g \in G$. This implies that $g^{-1}\alpha g = k$ however since $\alpha \in N$ so is $g^{-1}\alpha g = k$. Therefore $k \in N$ and as such $K \subset N$.

Problem 2 (4.3)**.**

1. The conjugacy classes of $\mathcal{Q}_8$ are

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$$

   The conjugacy classes for D_8 are

$$\{e\}, \{s\}, \{r, r^3\}, \{r^2\}$$

2. a) Let $c = (c_1 \ \ldots \ c_t) \in S_n$ and $\sigma \in S_n$. Then let $k = \sigma(c_i)$ for some i. Then

$$\sigma c \sigma^{-1}(k) = \sigma c \sigma^{-1}(\sigma(c_i)) = \sigma c(c_i) = \sigma(c_{i+1})$$

   Next suppose that $k \neq \sigma(c_i)$ for any i. Then

$$\sigma c \sigma^{-1}(k) = \sigma c(\sigma^{-1}(k)) = \sigma(\sigma^{-1}(k)) = k$$

   Which implies that $\sigma c \sigma^{-1} = (\sigma(c_1) \ \ldots \ \sigma(c_t))$.

b) Given a t-cycle c we can send it to any other t-cycle by conjugation. This means that the conjugacy class for c is the permutations of t numbers selected from 1 through n up to rotation. The permutations number $\frac{n!}{(n-t)!}$ and we account for rotation by $\frac{n!}{(n-t)!t}$. Given an orbit we have the formula $|\mathcal{O}| = \frac{|G|}{|\text{Stab}|}$. If we act on S_n by conjugation then the stabilizer for a t-cycle is the centralizer. Using the formula for any given t-cycle c we have

$$\frac{n!}{(n-t)!t} = \frac{n!}{|C_{S_n}(c)|}$$

which implies that $|C_{S_n}(c)| = (n-t)!t$.

c) The conjugacy classes of A_5 are

(a) () of order 1.

(b) $(a\ b)(c\ d)$ of order 15.

(c) $(a\ b\ c)$ of order 20.

(d) The 5-cycle conjugacy class containing (1 2 3 4 5) of order 12.

(e) The 5-cycle conjugacy class containing (2 1 3 4 5) of order 12.

Since normal subgroups are the unions of conjugacy classes in order for A_5 to have a nontrivial normal subgroup we would require for us to be able to add a combinations of conjugacy classes to get a number that divides 60. However there is no combination that does that. Therefore A_5 has no normal subgroups and as such is simple.

Problem 3 (4.4.1)**.**

Proof. Let G be a group of order $11^2 13^2$. Then let P be a Sylow 11-subgroup and Q a Sylow 13-subgroup. Then $n_{11}(G) \equiv 1 \mod 11$ and $n_{13}(G) \equiv 1 \mod 13$. However the only number that work for $n_{11}(G)$ and $n_{13}(G)$ is 1 since $n_p \big| |G : P|$. As such the groups P, Q are the only subgroups of their size. However Sylow p-subgroups are closed under conjugation and as such this implies that P and Q are normal. This means that the product PQ is well defined. The order of PQ is $|PQ| = \frac{|P||Q|}{|P \cap Q|}$. However the intersection $P \cap Q = \{e\}$ as the order of any element of P or Q divides the order of the subgroup and the orders of P and Q are relatively prime. As such $|PQ| = |P||Q| = 11^2 13^2 = |G|$ and so $PQ = G$. However since P, Q are order of a prime squared they are Abelian and the normality implies $PQ = QP$ which shows that G is Abelian.

Therefore any group of order $11^2 13^2$ is Abelian. $\square$

Problem 4 (4.4.2)**.**

Proof. Let $|G| = 77 = 7 \cdot 11$. Then there are subgroups with $|H| = 7$ and $|K| = 11$. Since they are of prime order they are cyclic with generators h and k respectively. However since 7 and 11 are relatively prime hk has order $7 \cdot 11 = 77$ which implies that G is cyclic. $\square$

Problem 5 (4.4.3)**.**

Proof. Since $30 = 2 \cdot 3 \cdot 5$ there are subgroups of size 3 and 5 which we'll call H and K respectively. However since H and K are of prime order they are cyclic and as such $\langle h \rangle = H$ and $\langle k \rangle = K$. Since the orders of h, k are relatively prime the order of hk is fifteen. $\square$