Problem 1 (12.1).

1. *Let R be a ring and M and R-module. Prove that $r0 = 0$ for $r \in R$. If R has the identity 1, then $(-1)x = -x$ for $x \in M$.*

2. *Let R be a ring and M, N, L be R-modules. Prove:*

   (a) $\hom_R(M, N)$ *is an abelian group under addition*

   $$(\phi + \psi)(m) = \phi(m) + \psi(m)$$

   If R is commutative, $\hom(M, N)$ *is an R-module with the R-action given by*

   $$(r\phi)(m) = r\phi(m)$$

   (b) *If $\phi \in \hom_R(M, N)$ and $\psi \in \hom_R(N, L)$, then $\psi \circ \phi \in \hom_R(M, L)$.*

   (c) $\hom_R(M, M)$ *is a ring with identity with composition as multiplication.*

3. *Prove that $\hom_{\mathbb{Z}}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_d$ where $d = \gcd(m, n)$.*

Proof. □

Problem 2 (12.2). *Let A, B be submodules of an R-module M. Prove that $A + B$ and $A \cap B$ are submodules of M. Moreover, the equality*

$$A \cap (B + C) = B + (A \cap C)$$

holds for all R-submodules C if $B \subseteq A$.

Proof. □

Problem 3 (12.4). *Let M be an R-module.*

1. *For any submodules $N_1, \ldots, N_n$ of M, their sum $N_1 + \cdots + N_n$ is the smallest submodule of M which contains $N_1 \cup \cdots \cup N_n$.*

2. *For any subset A of M, RA is the smallest submodule of M which contains A.*

Proof. □

Problem 4 (12.5). *Show that $\mathbb{Z}_{p^e}$, regarded as a $\mathbb{Z}$-module is not a direct sum of any two non-zero submodules, where p is a prime and $e > 0$. Does it hold for $\mathbb{Z}$? Does it hold for $\mathbb{Z}_{12}$?*

Proof. □

Problem 5 (12.7). *Let R be a PID and p a prime in R.*

1. *If M is a finitely generated p-primary R-module, then M/pM is an $R/(p)$-module with the R-action given by* (12.6)*. Moreover, show that the mapping ϕ defined in* (12.7) *is a $R/(p)$-module map.*

2. *Let $\phi : M_1 \to M_2$ be an isomorphism finitely generated p-primary R-modules. Prove that $\phi|_{pM_1} : pM_1 \to pM_2$ is an isomorphism of R-module. Show that the map $\bar{\phi} : M_1/pM_1 \to M_2/pM_2$ defined by*
$$\bar{\phi}(m + pM_1) = \phi(m) + pM_2$$
is an isomorphism of $R/(p)$-vector spaces.

Proof. □

Problem 6 (12.8)**.** 1. *Find the Smith normal form of the integer matrix*

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 3 \end{bmatrix}$$

2. *Determine the invariant factor decomposition of $\mathbb{Z}^3/K$ where K is generated by $f_1(2, 1, -3)$ and $f_2 = (1, -1, 2)$.*

Proof. □

Problem 7 (12.9)**.** 1. *Find a basic for the submodule K of $\mathbb{Q}[x]^3$ generated by*

$$f_1 = (2x - 1, x, x^2 + 3), \quad f_2 = (x, x, x^2), \quad f_3 = (x + 1, 2x, 2x^2 - 3)$$

2. *Find the invariant factors and elementary divisors of the $\mathbb{Q}[x]$-module $\mathbb{Q}[x]^3/K$.*

Proof. □

Problem 8 (12.11)**.** *Let F be a field and V an n-dimensional vector space over F with an ordered basis $\mathcal{B}$.*

1. *Let T be a linear operator on V. For any ordered basis $\mathcal{B}'$ of V, the matrices $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ are similar over F. Conversely, if $A \in M_n(F)$ is similar to $[T]_{\mathcal{B}}$ over F, there exists a basis $\mathcal{B}'$ such that $[T]_{\mathcal{B}'} = A$.*

Proof. □

Problem 9 (12.13)**.** *Find the rational canonical form of the matrix*

$$A = \begin{bmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{bmatrix} \in M_3(\mathbb{Q})$$

Consider $A \in M_3(\mathbb{C})$ and find the Jordan canonical form of A.

Proof. □