Problem 1 (2.7.1)**.**

Proof. First we will show that N is normal. Then given $g \in N$ for all $x \in G$ there exists an $h \in H$ such that $g = xhx^{-1}$. Let $k \in G$ then $kgk^{-1} = h \in H$ as $g \in k^{-1}Hk$. However $g \in k^{-1}xHx^{-1}k$ for any $x \in G$. It then follows that

$$h = kgk^{-1} = kk^{-1}xh'x^{-1}kk^{-1} = xh'x$$

which implies that $h \in xHx^{-1}$ for all $x \in G$ and therefore $h \in N$. Therefore N is a normal subgroup of G.

Now we will show that N is the largest normal subgroup of G contained in H. Let M be a subgroup of G such that $M \trianglelefteq G$ and $M \leq H$. Then given $g \in M$ and $x \in G$ we have $x^{-1}gx = h \in M$. However this implies that $g = xhx^{-1}$ for all x and since $M \leq H$ we have that $h \in H$ and therefore $g \in N$ and $M \leq N$.

Therefore since N is normal and normal subgroup contained in H is contained in N we have that N is the largest normal subgroup contained in H. □

Problem 2 (2.7.2)**.**

Proof. a) For reflexivity, since $H, K \leq G$ we have $exe = x \in HxK$.

For symmetry if $x \sim y$ then $x \in HyK$ which implies that $x = hyk$ for some $h \in H$ and $k \in K$. However since $H, G \leq G$ we have that $h^{-1}xk^{-1} = y$ which implies that $y \in HxK$ and therefore $y \sim x$.

For transitivity suppose that we have $x \sim y$ and $y \sim z$. Then as before we have $x = hyk$ and $y = h'zk'$. It then follows that $x = hh'zk'k$ which implies that $x \in HzK$ and therefore $x \sim z$.

Let $\bar{x}$ denote the equivalence class of x. Then if $y \in HxK$ by definition $y \sim x$ and $y \in X$. Otherwise if $y \in \bar{x}$ by definition $y \sim x$ which implies that $y \in HxK$.

Therefore $\sim$ is an equivalence relation, the equivalence classes are of the form HxK, and as such $H \setminus G/K$ forms a partition of G.

b) Suppose that $f(hW) = f(h'W)$. This implies that $h'WxK = hWxK$ and as such $hWxk_1 = hWxk_2$ for some $k_1, k_2 \in K$. Then we have $hWx = hWxk$. Using the definition of W we rewrite as

$$h'(xk_3x^{-1})x = h'xk_3 = hxk_4k = h(xk_4x^{-1})xk$$

However this implies that $h' = hxk_4kk_3^{-1}x^{-1}$ and that

$$h^{-1}h' = xk_4kk_3^{-1}x^{-1} \in xKx^{-1}, H$$

Then $h' \in hW$ and $h'W = hW$.

Therefore the function f is injective.

Now let L be a set of representatives of left cosets of W in H. Then define an equivalence relation on HxK via $h_1xK \sim h_2xK$ if $h_2^{-1}h_1 \in W$. That this is an equivalence relation follows directly from the properties of cosets. Then if we take an element of L it will map one to each equivalence class. Similarly if we had right cosets then we switch the sides to get the same property.

c) The size of L is the number of cosets of W in H. This implies that $|L| = [H : W] = \frac{|H|}{|W|}$. We also have that $|HxK| = |L| \times |K|$ since L consists of the representatives of the partition defined previously and there are $|K|$ elements in each. However we can substitute for $|L|$ to get

$$|HxK| = \frac{|H||K|}{|W| = |H \cap xKx^{-1}}$$

If we instead define W as $x^{-1}Hx \cap K$ and use R instead of L we can use the same partition and substitution to get

$$|HxK| = \frac{|H||K|}{|x^{-1}Hx \cap K|}$$

.

□

Problem 3 (2.8)**.**

Proof. 1. We'll start by showing that $C_G(A)$ is a subgroup. If we have $g, h \in C_G(A)$ then $gha = gah = agh$ so it is closed under the group operation. Then if $g \in C_G(A)$ we have $ga = ag$. Multiply on the left by and right by g^{-1} and we get $ag^{-1} = g^{-1}a$. Therefore $C_G(A)$ is a subgroup.

Next consider $N_G((A))$. If we have $g, h \in N_G(A)$ then $hah^{-1} = a' \in A$. This implies that $ga'g^{-1} \in A$ and therefore $ghah^{-1}g^{-1} \in A$. Next let $g \in N_G(A)$ and $a \in A$. Then since $gAg^{-1} = A$ it follows that $g^{-1}Ag = g^{-1}gAg^{-1}g = A$. Therefore $N_G(A)$ is a subgroup.

Let $g \in C_G(A)$ then for $a \in A$ we have $gag^{-1} = agg^{-1} = a$ which implies that $g \in N_G(A)$. Therefore $C_G(A) \subset N_G(A)$.

2. Let $a \in A$ and $n \in N_G(A)$. Then $nan^{-1} \in A$ be definition of $N_G(A)$. Therefore if $A \leq G$ then $A \trianglelefteq N_G(A)$.

3. Let $z \in Z(G)$ and let $g \in G$. Then $zgz^{-1} = gzz^{-1} = g$ which implies that $Z(G)$ is a normal subgroup of G.

4. Let H be a group of index 2 in G. Then $N_G(H)$ is a subgroup of G that is normal in G. However since H is of index 2. The normalizer is either G or H. In both cases this implies that G is normal. Therefore if H is a subgroup of index 2 then it is normal.

For our counterexample S_3 take the subgroup $\langle(1\ 2)\rangle$. Then $(1\ 2\ 3)(1\ 2)(3\ 2\ 1) = (2\ 3) \notin \langle(1\ 2)\rangle$ which implies that $\langle(1\ 2)\rangle$ is not normal.

5. If n is even then the center consists of $r^{n/2}$ and the identity. Otherwise if n is odd then the center is just the identity.

6. The subgroups of $\mathcal{Q}_8$ are $< 1 >, < -1 >, < i >, < j >, < k >, \mathcal{Q}_8$. The trivial group and $\mathcal{Q}_8$ are both normal. For the others we have

$< -1 >$

g	$g(-1)g^{-1}$
1	-1
-1	-1
i	-1
$-i$	-1
j	-1
$-j$	-1
k	-1
$-k$	-1

$< i >$

g	$g(i)g$
1	i
-1	i
i	i
$-i$	i
j	$-i$
$-j$	$-i$
k	$-i$
$-k$	$-i$

$< j >$

g	$g(j)g$
1	j
-1	j
i	$-j$
$-i$	$-j$
j	j
$-j$	j
k	$-j$
$-k$	$-j$

$< k >$

g	$g(k)g$
1	k
-1	k
i	$-k$
$-i$	$-k$
j	$-k$
$-j$	$-k$
k	k
$-k$	k

□

Problem 4 (2.9.1)**.**

Proof. Since $|G| = p$ where p is prime by Lagrange's Theorem (2.12 in notes) the order of any subgroup must be either 1 or p. However the only subgroups that fulfill these criterion are either the trivial group or G itself. Therefore G cannot have any non-trivial subgroups.

Let $g \in G \setminus \{e\}$. Since there is no non-trivial subgroup the element $\langle g \rangle$ must generate the whole group. Then by definition G is cyclic. □

Problem 5 (2.9.2)**.**

Proof. Let G be a group and H a subgroup of $Z(G)$ where G/H is cyclic. Then there exists an $a \in G$ such that $\bigcup_{k \in \mathbb{Z}} a^k H = G$. This implies that for all g can be written in the form $a^k h$ where $h \in H$. Then consider $g_1 g_2$

$$g_1 g_2 = a^i h_1 a^j h_2 = h_1 a^j a^i h_2 = a^j h_2 a^i h_1 = g_2 g_1$$

which shows that G is commutative. □

Problem 6 (2.10)**.**