**Problem 1** (1.1.21)**.**

*Proof.* Let $x \in G$ have order odd $2n+1$. Then $x^{2n+1} = e$ and if we multiply by $x$ on both sides we get $x^{2n+2} = x^{2(n+1)} = x$.

Therefore if $x$ has odd order then it has $x^{2k} = x$ for some $k \in \mathbb{Z}_{>0}$. □

**Problem 2** (1.1.25)**.**

*Proof.* Suppose that $G$ is a group such that for all $x \in G$ that $x^2 = 1$. This implies that for any element $x^{-1} = x$. Then we have $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$ which shows that $G$ is commutative. □

**Problem 3** (1.1.35)**.**

*Proof.* Let $x \in G$ have order $n$. Then consider $x^m$. Using the division algorithm with $m$ and $n$ we can rewrite $m$ as $qn + d$ where $0 \leq d < n$. This implies that

$$x^m = x^{qn+d} = x^{qn}x^d = (x^n)^q x^d = e^q x^d = x^d$$

As such $x \in \{e, x^1, \dots, x^{n-1}\}$. □

**Problem 4** (1.3.6)**.**

- $(1\ 2\ 3\ 4) = (1\ 4)(1\ 3)(1\ 2)$
- $(1\ 3\ 4\ 2) = (1\ 2)(1\ 4)(1\ 3)$
- $(1\ 4\ 3\ 2) = (1\ 2)(1\ 3)(1\ 4)$
- $(4\ 3\ 2\ 1) = (4\ 1)(4\ 2)(4\ 3)$
- $(2\ 4\ 3\ 1) = (2\ 1)(2\ 3)(2\ 4)$
- $(3\ 2\ 4\ 1) = (3\ 1)(3\ 4)(3\ 2)$

**Problem 5** (1.3.9)**.**

a) $1 + 12k, 5 + 12k, 7 + 12k, 11 + 12k$ for $k \in \mathbb{Z}$.

b) $1 + 8k, 3 + 8k, 5 + 8k, 7 + 8k$ for $k \in \mathbb{Z}$.

c) $1 + 14k, 3 + 14k, 5 + 14k, 9 + 14k, 11 + 14k, 13 + 14k$ for $k \in \mathbb{Z}$.

**Problem 6** (1.3.13)**.**

*Proof.* Suppose that $g \in S_n$ is of the form $g = \prod_{i=1}^{m}(a_i\ b_i)$ where each transposition commutes. Then

$$g^2 = \left(\prod_{i=1}^{m}(a_i\ b_i)\right)^2 = \prod_{i=1}^{m}((a_i\ b_i)^2 = e) = e$$

which implies that $g$ is of order 2.

Next suppose that $g \in S_n$ is of order 2. We can write $g$ as a product of disjoint cycles $\prod_{i=1}^{m} \sigma_i$. Since the cycles are disjoint we can write $g^2$ as

$$g^2 = \left(\prod_{i=1}^{m}\sigma_i\right)^2 = \prod_{i=1}^{m}\sigma_i^2 = e$$

Then for a given $\sigma_i$ rewrite the above to

$$\prod_{i\in\{1\ldots m\}-j}\sigma_i^2 = \sigma_j^{-2}$$

However since each $\sigma_i$ is disjoint this implies that $\sigma_j^{-2} = e = \sigma_j^2$. Since this is for an arbitrary $\sigma_i$ we have that $\sigma_i^2$ for all $i$.

Therefore a permutation is of order two if and only if it is the product of disjoint $2-$cycles. $\square$

**Problem 7** (1.5.2)**.**

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| $S_3$ | $()$ | $(1\ 2)$ | $(1\ 2\ 3)$ | $(1\ 3\ 2)$ | $(1\ 3)$ | $(2\ 3)$ |
|---|---|---|---|---|---|---|
| $()$ | $()$ | $(1\ 2)$ | $(1\ 2\ 3)$ | $(1\ 3\ 2)$ | $(1\ 3)$ | $(2\ 3)$ |
| $(1\ 2)$ | $(1\ 2)$ | $()$ | $(2\ 3)$ | $(1\ 3)$ | $(1\ 3\ 2)$ | $(1\ 2\ 3)$ |
| $(1\ 2\ 3)$ | $(1\ 2\ 3)$ | $(1\ 3)$ | $(1\ 3\ 2)$ | $()$ | $(2\ 3)$ | $(1\ 2)$ |
| $(1\ 3\ 2)$ | $(1\ 3\ 2)$ | $(2\ 3)$ | $()$ | $(1\ 2\ 3)$ | $(1\ 2)$ | $(1\ 3)$ |
| $(1\ 3)$ | $(1\ 3)$ | $(1\ 2\ 3)$ | $(1\ 2)$ | $(2\ 3)$ | $()$ | $(1\ 3\ 2)$ |
| $(2\ 3)$ | $(2\ 3)$ | $(1\ 3\ 2)$ | $(1\ 3)$ | $(1\ 2)$ | $(1\ 2\ 3)$ | $()$ |

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| $D_8$ | $e$ | $r$ | $p$ | $p^2$ | $rp$ | $rp^3$ | $p^3$ | $rp^2$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $r$ | $p$ | $p^2$ | $rp$ | $rp^3$ | $p^3$ | $rp^2$ |
| $r$ | $r$ | $e$ | $rp^3$ | $rp^2$ | $p^3$ | $p$ | $rp$ | $p^2$ |
| $p$ | $p$ | $rp$ | $p^2$ | $p^3$ | $rp^2$ | $r$ | $e$ | $rp^3$ |
| $p^2$ | $p^2$ | $rp^2$ | $p^3$ | $e$ | $rp^3$ | $rp$ | $p$ | $r$ |
| $rp$ | $rp$ | $p$ | $r$ | $rp^3$ | $e$ | $p^2$ | $rp^2$ | $p^3$ |
| $rp^3$ | $rp^3$ | $p^3$ | $rp^2$ | $rp$ | $p^2$ | $e$ | $r$ | $p$ |
| $p^3$ | $p^3$ | $rp^3$ | $e$ | $p$ | $r$ | $rp^2$ | $p^2$ | $rp$ |
| $rp^2$ | $rp^2$ | $p^2$ | $rp$ | $r$ | $p$ | $p^3$ | $rp^3$ | $e$ |

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| $Q_8$ | $1$ | $i$ | $j$ | $-1$ | $-k$ | $k$ | $-1$ | $-j$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $i$ | $j$ | $-1$ | $-k$ | $k$ | $-1$ | $-j$ |
| $i$ | $i$ | $-1$ | $k$ | $-1$ | $j$ | $-j$ | $1$ | $-k$ |
| $j$ | $j$ | $-k$ | $-1$ | $-j$ | $-1$ | $i$ | $k$ | $1$ |
| $-1$ | $-1$ | $-1$ | $-j$ | $1$ | $k$ | $-k$ | $i$ | $j$ |
| $-k$ | $-k$ | $-j$ | $i$ | $k$ | $-1$ | $1$ | $j$ | $-1$ |
| $k$ | $k$ | $j$ | $-1$ | $-k$ | $1$ | $-1$ | $-j$ | $i$ |
| $-1$ | $-1$ | $1$ | $-k$ | $i$ | $-j$ | $j$ | $-1$ | $k$ |
| $-j$ | $-j$ | $k$ | $1$ | $j$ | $i$ | $-1$ | $-k$ | $-1$ |