Problem 1 (1.6.2).

Proof. Let $\varphi : G \to H$ be an isomorphism and let $x \in G$ where $|x| = n$. Since $\varphi(x^n) = \varphi(x)^n$ it follows that $|x| \geq |\varphi(x)|$. However φ is an isomorphism which implies that φ^{-1} is also an isomorphism. Via the same reasoning this implies that $|\varphi(x)| \geq |\varphi^{-1} \circ \varphi(x) = x|$. Therefore $|x| = |\varphi(x)|$.

If φ is an isomorphism and G_n is the set of elements of order n in G then $\varphi|_{G_n}$ is a bijection and since φ preserves orders it follows that we have the same number of elements of order n in G and H for any n. $\square$

It does not hold if φ is not an isomorphism. Consider $\varphi : \mathbb{Z}/6\mathbb{Z} \to \{e\}$. Then the order of for any $\varphi(x)$ is 1.

Problem 2 (1.6.3).

Proof. Let $\varphi : G \to H$ be an isomorphism and suppose that H is Abelian. Then for $x, y \in G$ we have

$$xy = \varphi^{-1} \circ \varphi(xy) = \varphi^{-1}(\varphi(x)\varphi(y)) = \varphi^{-1}(\varphi(y)\varphi(x)) = yx$$

which implies that G is Abelian. If G is Abelian swap φ for φ^{-1} and the reasoning will be identical to above.

Therefore if $\varphi : G \to H$ is an isomorphism then G is Abelian if and only if H is Abelian. $\square$

If $\varphi : G \to H$ is a homomorphism and H is Abelian then we can show that G is Abelian if φ is injective via the proof above as there will be a well defined inverse on $\varphi(G)$.

Otherwise if G is Abelian we can show that H is Abelian if φ is surjective.

Proof. Let $x, y \in H$. Since φ is surjective there exist $a, b \in G$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Then

$$xy = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = yx$$

which implies that H is Abelian. $\square$

Problem 3 (1.6.4).

Proof. Suppose that $\varphi : \mathbb{C}^* \to \mathbb{R}^*$. Then $\phi(z) = -1$ for some $z \in \mathbb{C}$. However there is a $w \in \mathbb{C}$ such that $w^2 = z$ which implies that there exists a $y \in \mathbb{R}^*$ such that $\varphi(w) = x$ and that $x^2 = -1$ which is impossible.

Therefore $\mathbb{C}^*$ and $\mathbb{R}^*$ are not isomorphic. $\square$

Problem 4 (1.6.7).

Proof. In $\mathcal{Q}_8$ the identity has order 1, $|-1| = 2$, and $|i| = |j| = |k| = |-i| = |-j| = |-k| = 4$. However in D_8 the elements r and ρ^2 are both of order two. Since D_8 has more elements of order two than $\mathcal{Q}_8$ they cannot be isomorphic. $\square$

Problem 5 (1.6.17).

Proof. Let G be Abelian. Consider the inverse map $\varphi(g) = g^{-1}$. Then

$$\varphi(g)\varphi(h) = g^{-1}h^{-1} = h^{-1}g^{-1} = \varphi(gh)$$

and since $e = e^{-1}$ it follows that φ is a homomorphism.

Otherwise suppose that G is not Abelian. This implies that exist $g, h \in G$ such that $gh \neq hg$. Then

$$\varphi(g^{-1})\varphi(h^{-1}) = gh \neq hg = \varphi(g^{-1}h^{-1})$$

which implies that φ cannot be a homomorphism. □

Problem 6 (1.6.25)**.**

Proof. a) Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be a point in $\mathbb{R}^2$. Then rewrite $\begin{pmatrix} x \\ y \end{pmatrix}$ in polar coordinates as $\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$. Then if we multiply by the rotation matrix we get

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} = \begin{pmatrix} r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ r\sin\theta\cos\phi + r\sin\phi\cos\theta \end{pmatrix} = \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix}$$

which corresponds to a rotation of $\begin{pmatrix} x \\ y \end{pmatrix}$ counter-clockwise by θ as the new angle is $\theta + \phi$.

b) **Show φ is a homomorphism.**

c) Suppose that we have two $a, b \in D_{2n}$ such that $\varphi(a) = \varphi(b)$. Since any element of D_{2n} can be written in the form $s^i r^m$ it follows that

$$\varphi(a) = \varphi(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^i \begin{pmatrix} \cos\frac{2m\pi}{n} & -\sin\frac{2m\pi}{n} \\ \sin\frac{2m\pi}{n} & \cos\frac{2m\pi}{n} \end{pmatrix}$$

where $i = 0, 1$ since s is of order two and $0 \leq m < n$ as r is of order n.

□

Problem 7 (1.6.26)**.**

Proof. Since $\mathcal{Q}_8$ is finite we can show that $\varphi : \mathcal{Q}_8 \to \mathrm{GL}_2(\mathbb{C})$ by calculating the value of φ for each $q \in \mathcal{Q}_8$.

$$
\begin{aligned}
\varphi(1) &= \varphi(i^4) &&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\varphi(-1) &= \varphi(i^2) &&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\varphi(i) &= \varphi(i) &&= \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \\
\varphi(-i) &= \varphi(i^3) &&= \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \\
\varphi(j) &= \varphi(j) &&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
\varphi(-j) &= \varphi(j^3) &&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\varphi(k) &= \varphi(ij) &&= \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \\
\varphi(-k) &= \varphi(ji) &&= \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}
\end{aligned}
$$

Since each element of $\mathcal{Q}_8$ maps to a distinct element of $\mathrm{GL}_2(\mathbb{C})$ the homomorphism φ is injective. □

Problem 8 (2.1.3)**.**

Proof.

a)

	1	r^2	s	sr^2
1	1	r^2	s	sr^2
r^2	r^2	1	sr^2	s
s	s	sr^2	1	r^2
sr^2	sr^2	s	r^2	1

b)

	1	r^2	sr	sr^3
1	1	r^2	sr	sr^3
r^2	r^2	1	sr^3	sr
sr	sr	sr^3	1	r^2
sr^3	sr^3	sr	r^2	1

□

Problem 9 (2.1.10(a))**.**

Proof. Let $H, K \leq G$. Then we will show that $H \cap K \leq G$.

- Since $H, K \leq G$ it follows that $e \in H$ and $e \in K$ which implies that $e \in H \cap K$
- Let $g \in H \cap K$. Then $g \in H$ and $g \in K$ which implies that $g^{-1} \in H$ and $g^{-1} \in K$. It then follows that $g^{-1} \in H \cap K$.
- Now let $g, h \in H \cap K$. Then $g, h \in H$ and $g, h \in K$ which implies that $gh \in H$ and $gh \in K$. It then follows that $gh \in H \cap K$.

Therefore if $H, K \leq G$ then $H \cap K \leq G$. □

Problem 10 (2.3.1)**.**

The subgroups of the form $\langle x \rangle$ of $\mathbb{Z}_{45}$ are $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle 3 \rangle$, $\langle 5 \rangle$, $\langle 9 \rangle$, and $\langle 15 \rangle$. **Containment drawn below**.

Problem 11 (2.3.3)**.**

The generators will be the elements of order 48 which will consist of the elements that are relatively prime to 48. These are

$$1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47$$

Problem 12 (2.3.12(a))**.**

Proof. A group is cyclic if it can be generated by a single element. There are a total of four elements in $Z_2 \times Z_2$. These are $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$. The order of $(0, 0)$ is 1, and the order of the rest of the elements is 2. However since the size of the group is 4 it follows that none of the elements could generate the group as the size of the group generated is at most two.

Therefore the group $Z_2 \times Z_2$ is not cyclic. □