Problem 1 (6.1). *1. If $K \trianglelefteq G$ and G/K are solvable, then G is solvable.*

2. Prove that S_n is not solvable for $n \geq 5$.

Proof. □

Problem 2 (6.2). *A finite group G is solvable if, and only if, every composition factor of a composition series of G is cyclic of prime order.*

Proof. □

Problem 3 (6.3). *1. Let G be a group and $\phi : M(X) \to G$ a monoid homomorphism which satisfies*

$$\phi(s^{-1}) = \phi(s)^{-1} \text{ for all } s \in S$$

then for any $w \in M(X)$, $\phi(w) = \phi(r(w))$

2. Let S be a set, R a subset of $F(S)$, G a group, $\phi : S \to G$ a function and $\widetilde{\phi} : F(S) \to G$ the induced group homomorphism. If $\widetilde{\phi}(r) = e$ for all $r \in R$, then there exists a homomorphism $\overline{\phi}$ from $\langle S|R\rangle$ to G such that $\overline{\phi} \circ \pi \circ i = \phi$ where $i : S \to F(S)$ is the inclusion map, $\pi : F(S) \to \langle S|R\rangle$ is the natural surjection, and $\widetilde{\phi} : F(S) \to G$ is the homomorphism satisfying $\widetilde{\phi} \circ i = \phi$.

Proof. □

Problem 4 (6.5.1). *Using the Todd-Coxeter algorithm to determine and identify the group*

$$G = \langle x, y | x^2 = 1, y^2 = 1, xyx = yxy\rangle$$

Proof. □

Problem 5 (7.2). *1. Prove that R^X is a group under the multiplication of R.*

2. Prove that $Z(R) \cap R^X = \emptyset$.

Proof. □

Problem 6 (7.3). *1. Find the set of all zero divisors of the commutative ring $C([0,1])$ defined in example 7.3. Determine the $C([0,1])^X$.*

2. Let $D \in \mathcal{Q}$ such that the equation $x^2 = D$ has no solution $x \in \mathcal{Q}$. Prove that the set

$$\mathcal{Q}(\sqrt{D}) = \{a + b\sqrt{D} | a, b \in \mathcal{Q}\}$$

forms a field under the ordinary addition and multiplication of complex numbers.

3. *Prove $\mathbb{Z}_n$ is an integral domain if, and only if, n is prime.*

Proof. □

Problem 7 (7.4)**.** 1. *Prove that the set*

$$\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} | a, b \in \mathbb{Z}\}$$

is a subring of $\mathcal{Q}(\sqrt{D})$ and $\mathbb{Z}[\sqrt{D}]$ is an integral domain.

2. *Define the norm function $N : \mathcal{Q}(\sqrt{D}) \to \mathcal{Q}$ by*

$$N(a + b\sqrt{D}) = a^2 - Db^2$$

Prove that $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathcal{Q}(\sqrt{D})$.

3. *Show that for any $\alpha \in \mathbb{Z}[\sqrt{D}]$, α is a unit of $\mathbb{Z}[\sqrt{D}]$ if, and only if, $N(\alpha) = \pm 1$.*

Proof. □

Problem 8 (7.5)**.** *Let R be a ring. For any $a, b \in R$, if $1 - ab$ is a unit, then so is $1 - ba$.*

Proof. □

Problem 9. *Compute the commutator subgroup of S_4.*

Proof. □