

Problem 1 (7.26). Let M and N be finitely generated R -modules over a PID R . Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors s_1, \dots, s_t (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbb{Z}} M$ is a finite group of order $\prod_{j=1}^t s_j^{2t-2j+1}$.

Proof. By the structure theorem for finitely generated modules over a PID we have that

$$M \cong R^r \oplus R/(a_1) \oplus \cdots \oplus R/(a_m)$$

and

$$N \cong R^s \oplus R/(b_1) \oplus \cdots \oplus R/(b_n)$$

where $a_i | a_{i+1}$ and $b_i | b_{i+1}$. Then by Theorem 2.18 of Adkins' book the tensor $M \otimes_R N$ distributes over the direct sum. Together with $R/(a_i) \otimes_R R/(b_j) \cong R/(a_i + b_j)$ we get that

$$M \otimes_R N \cong \bigoplus_{0 < i \leq m, 0 < j \leq n} (R/(a_i + b_j)) \bigoplus_{0 < i \leq m} (R/(a_i) \otimes_R R^t) \bigoplus_{0 < j \leq n} (R^s \otimes_R R/(b_j))$$

If we let M be a finite abelian group with invariant factor decomposition

$$M \cong \bigoplus_{0 < i \leq t} \mathbb{Z}_{s_i}$$

Then, from above, the tensor of M with itself is

$$M \otimes_{\mathbb{Z}} M \cong \bigoplus_{0 < i, j \leq t} \mathbb{Z}_{\gcd(s_i, s_j)}$$

However since $s_i | s_j$ for $i \leq j$ we have that $\gcd(s_i, s_j) = s_i$. This will cause $2t - 1$ copies of \mathbb{Z}_{s_1} to appear, $2t - 3$ for \mathbb{Z}_{s_2} , and so on until we have only a single copy of \mathbb{Z}_{s_t} . Thus $M \otimes_{\mathbb{Z}} M$ will have order $\prod_{j=1}^t s_j^{2t-2j+1}$. \square

Problem 2 (7.30). (a) Let F be a field and K a field containing F . If $f(X) \in F[X]$, show that there is an isomorphism of K -algebras:

$$K \otimes_F (F[X]/\langle f(X) \rangle) \cong K[X]/\langle f(X) \rangle$$

(b) By choosing $F, f(x)$, and K appropriately, find an example of two fields K and L containing F such that the F -algebra $K \otimes_F L$ has nilpotent elements.

Proof. (a) Define a map $\varphi : K \times F[X]/\langle g(x) \rangle \rightarrow K[X]/\langle g(x) \rangle$ via

$$\varphi(k, f) = kf$$

Now briefly verify that φ is F -middle linear

$$\begin{aligned} \varphi(ak, fb) &= akfb &= a\varphi(k, f)b \\ \varphi(k_1 + k_2, f) &= k_1f + k_2f &= \varphi(k_1, f) + \varphi(k_2, f) \\ \varphi(k, f_1 + f_2) &= kf_1 + kf_2 &= \varphi(k, f_1) + \varphi(k, f_2) \\ \varphi(ka, f) &= kaf &= \varphi(k, af) \end{aligned}$$

Since φ is F -middle linear it induces a map $\tilde{\varphi} : K \otimes_F F[X]/\langle g(x) \rangle \rightarrow K[X]/\langle g(x) \rangle$. Now define a map $\psi : K[X]/\langle g(x) \rangle \rightarrow K \otimes_F F[X]/\langle g(x) \rangle$ by $\psi(kx^i) = k \otimes x^i$ and extending linearly. We will now show that $\tilde{\varphi}$ and ψ are inverses.

First $\tilde{\varphi} \circ \psi$

$$\begin{aligned}\tilde{\varphi} \circ \psi(\sum a_i x^i) &= \tilde{\varphi}(\sum a_i \otimes x^i) \\ &= \sum a_i x^i\end{aligned}$$

Then for $\psi \circ \tilde{\varphi}$

$$\begin{aligned}\psi \circ \tilde{\varphi}(k \otimes \sum a_i x^i) &= \psi \circ \tilde{\varphi}(\sum k \otimes a_i x^i) \\ &= \psi(\sum \tilde{\varphi}(k \otimes a_i x^i)) \\ &= \psi(\sum k a_i x^i) \\ &= \sum \psi(k a_i x^i) \\ &= \sum k a_i \otimes x^i \\ &= \sum k \otimes a_i x^i\end{aligned}$$

Since $\tilde{\varphi}$ has an inverse it is indeed an isomorphism. Which shows that $K \otimes_F F[x]/\langle g(x) \rangle$ is isomorphic to $K[x]/\langle g(x) \rangle$.

- (b) Let F be rational functions of t with coefficients in \mathbb{Z}_2 . Then let K be the splitting field of $x^2 - t$. Then in $K \otimes_F K$ the element $t^{1/2} \otimes 1 + 1 \otimes t^{1/2}$ is zero when squared making it an idempotent.

□

Problem 3 (7.31). Let F be a field. Show that $F[X, Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F -algebras.

Proof. Define $\varphi : F[x] \times F[y] \rightarrow F[x] \otimes F[y]$ via

$$\varphi(\sum_i a_i x^i, \sum_j b_j y^j) = \sum_{i,j} a_i b_j x^i y^j$$

Since we are in a polynomial ring over a field it follows that φ is F -middle linear. As such there is a map $\tilde{\varphi} : F[x] \otimes F[y] \rightarrow F[x, y]$ that mimics φ .

Now define $\psi : F[x, y] \rightarrow F[x] \otimes F[y]$ via $\psi(cx^i y^j) = c(x^i \otimes y^j)$ and extending linearly. We will now show that $\tilde{\varphi}$ and ψ are inverses. Since both of these maps are linear we only need to verify them on either $cx^i y^j$ and $a_i x^i \otimes b_j y^j$.

First for $\tilde{\varphi} \circ \psi$ we have

$$\begin{aligned}\tilde{\varphi} \circ \psi(cx^i y^j) &= \tilde{\varphi}(c(x^i \otimes y^j)) \\ &= \tilde{\varphi}(cx^i \otimes y^j) \\ &= cx^i y^j\end{aligned}$$

Then for $\psi \circ \tilde{\varphi}$

$$\begin{aligned}\psi \circ \tilde{\varphi}(a_i x^i \otimes b_j y^j) &= \psi(a_i b_j x^i y^j) \\ &= a_i b_j (x^i \otimes y^j) \\ &= a_i x^i \otimes b_j y^j\end{aligned}$$

Thus $\tilde{\varphi}$ is an isomorphism which shows that $F[x, y]$ is isomorphic to $F[x] \otimes F[y]$. \square

Problem 4 (7.34). Let F be a field, V and W finite-dimensional vector spaces over F , and let $T \in \text{End}_F(V)$, $S \in \text{End}(W)$.

- (a) If α is an eigenvalue of S and β is an eigenvalue of T , show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.
- (b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

Proof. \square

Problem 5 (10.4.3). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left \mathbb{R} modules but are not isomorphic as \mathbb{R} -modules.

Proof. \square

Problem 6 (10.4.5). Let A be a finite abelian group of order n and let p^k be the largest power of a prime p dividing n . Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p -subgroup of A .

Proof. \square

Problem 7 (10.4.10). Suppose R is commutative and $N \cong R^n$ is a free R -module of rank n with R -module basis e_1, \dots, e_n .

- (a) For any nonzero R -module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \dots, n$.
- (b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R -linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}$, $n = 1$, $M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

Proof. \square

Problem 8 (10.4.24). Prove that the extension of scalars from \mathbb{Z} to the Gaussian integers $\mathbb{Z}[i]$ of the ring \mathbb{R} is isomorphic to \mathbb{C} as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

Proof.

□

Problem 9 (10.4.27). (a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1 \epsilon_2 = 0$, $\epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called orthogonal idempotents in A). Deduce that A is isomorphic as a ring to the direct product of two principle ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\varphi : \mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \bar{z}_2)$, where \bar{z}_2 denotes the complex conjugate of z_2 , is an \mathbb{R} -bilinear map.

(d) Let Φ be the \mathbb{R} -module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is \mathbb{C} -linear, where the action of \mathbb{C} is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a \mathbb{C} -algebra isomorphism.

Proof.

□