Problem 1. *Prove that the following conditions on an R-module P are equivalent.*

(a) P *is projective.*

(b) P *is isomorphic to a direct summand of a free R-module.*

(c) If $f : M \to P$ *is surjective, then there exists and R-module homomorphism* $g : P \to M$ *such that* $f \circ g = \mathrm{id}_P$.

Proof. First we will show that (a) implies (c). Let P be a projective module and $f : M \to P$ be surjective. Then the identity map id_P fulfills the projective conditions and there exists a map $g : P \to M$ such that $f \circ g = \mathrm{id}_P$.

Next we will show that (c) implies (b). Let P be a module that fulfills property (c). Then let F be the free group with generators the elements of P and Q the free group with generators the relations of P. This creates a short exact sequence inclusion and projection of the form

$$0 \longrightarrow Q \xrightarrow{i} F \xrightarrow{\pi} P \longrightarrow 0$$

However condition (c) states that this sequence splits. As such $F \cong Q \oplus P$ by the splitting lemma which implies that condition (b) holds.

Finally we show that (b) implies (a). Let $F = P \oplus Q$ where F is free and let $\pi : F \to P$ be the projection map. Then let $f : P \to N$ and $g : M \to N$ where g is surjective.

Let x be a generator of F and n_x as $f \circ \pi(x) \in N$. By surjectivity of g there exists $m_x \in M$ such that $g(m_x) = n_x$. By the universal property of free modules there exists a unique map $\bar{f} : F \to M$ such that $g \circ \bar{f}(x) = f \circ \pi(x)$. Now define $\tilde{f} : P \to M$ by $\tilde{f}(p) = \bar{f}(p + 0_q)$. Then

$$g \circ \tilde{f}(p) = f \circ \pi(p + 0_q) = f(p)$$

which completes the proof. □

Problem 2. *Let F be a field and let $R = F \times F$. Let $e = (1, 0) \in R$ and let $P = Re$. Show that P is a projective R-module, but that P is not a free R-module.*

Note that R is a free module over itself. Since $P \oplus R(0,1) \cong R$ we have that P is projective by problem 1. However P is not free as if we let $x, y \in F \setminus \{0\}$ then $(0, y) \cdot (x, 0) = 0$ even though neither element is zero.

Problem 3. *Show that if R is a semisimple ring, then so is $M_n(R)$.*

Proof. By Theorem 7.1.28 from Adkins' book a ring is semisimple if and only if every R-module is semisimple. Since $M_n(R)$ is an R-module it is semisimple. □

Problem 4. *Show that if R is a semisimple ring and I is any ideal, then R/I is also semisimple.*

Proof. Suppose that $R = \bigoplus_\alpha M_\alpha$ is a semisimple ring and let $\pi : R \to R/I$ be the projection map. Then $\pi(M_\alpha)$ is an ideal of R/I which implies that $R/I \cong \bigoplus_\alpha \pi(M_\alpha)$. Moreover since the preimages of ideals are ideals the components $\pi(M_\alpha)$ must also be simple otherwise it would violate the simplicity of the M_αs.

Therefore since the $\pi(M_\alpha)$s are simple and ideals of R/I we have a decomposition of R/I into simple submodules as a R/I module. It then follows that R/I is a semisimple ring. □

Problem 5 (7.2). *Let F be a field and let*

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in F \right\}$$

be the ring of upper triangular matrices over F. Let $M = F^2$ and make M into a (left) R-module by matrix multiplication. Show that $End_R(M) \cong F$. Conclude that the converse of Schur's lemma is false, i.e., $End_R(M)$ can be a division ring without M being a simple R-module.

Proof. Define a map $\psi : F \to End_R(M)$ by $\Psi(x) = \varphi_x$ where φ_x is defined as

$$\varphi_x \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} xd \\ xe \end{pmatrix}$$

Since the maps φ_x are equivalent to scalar multiplication by field elements it is clear from the definition that φ_x are in fact endomorphisms and respect the module structure. Now we will show that ψ is indeed an isomorphism.

The fact that the kernel of ψ is trivial follows from the fact that we are multiplying by field elements and as such there are no zero divisors. However to show that ψ is surjective we must utilize the fact that the endomorphisms play nice with the module structure.

First note that for $\phi \in End_R(M)$ we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \phi \begin{pmatrix} d \\ e \end{pmatrix} = \phi \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix} \right)$$

If we split ϕ into two pieces like so

$$\phi \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \phi_1(d, e) \\ \phi_2(d, e) \end{pmatrix}$$

The module structure gives us two equations

$$a\phi_1(d, e) + b\phi_2(d, e) = \phi_1(ad + be, ce)$$

and

$$c\phi_2(d, e) = \phi_2(ad + be, ce)$$

Note that we could define a new endomorphism by swapping ϕ_1 and ϕ_2. As such any statement we make about one applies to the other.

First if we let $a = 1, b = c = 0$ in the latter equation this gives us that $\phi_2(d, 0) = 0$. Thus ϕ_1 and ϕ_2 are zero whenever the right coordinate are zero. Next let $a = 1, b = -1, c = 0$ in the first equation and we get that $\phi_1(d, e) = \phi_2(d, e)$. Finally if we set $b = 0, c = 1$ and let $a \in F$ we get that we can pull constants out of the right term. Similarly setting $a = 1, b = 0$ and letting $c \in F$ we get that we can pull constants out of the right term. This implies that ϕ_1 and ϕ_2 are equivalent to multiplying by $\phi(1, 1) = x$. Thus the map $\phi = \varphi_x$ and implying that ψ is surjective.

Therefore ψ is in fact an isomorphism and as such $F \cong End_R(M)$. Since M is not simple but it's endomorphisms form a field this is a counterexample to the converse of Schur's lemma. □

Problem 6 (7.4). *An R-module M is said to satisfy the descending chain condition (DCC) on submodules if any strictly decreasing chain of submodules of M of finite length.*

(a) *Show that if M satisfies the DCC, then any nonempty set of submodules of M contains a minimal element.*

(b) Show that $\ell(M) < \infty$ if and only if M satisfies M satisfies both the ACC and DCC.

Proof. (a) Let $\mathcal{M}$ be the set of submodules of M partially ordered by reverse inclusion. Since M satisfies DCC every chain in $\mathcal{M}$ will have a maximal element and as such by Zorn's lemma there is a maximal element of $\mathcal{M}$ that corresponds to a minimal submodule.

(b) Suppose that $\ell(M) < \infty$. Then given a series of submodules $\{N\}$ we can refine it to a composition series $\{N'\}$ which will be of finite length. Since the length of $\{N\}$ is less than that of $\{N'\}$ it must also have finite length and as such has a minimal point and a maximal point. Since this holds for any series it follows that M satisfies both ACC and DCC.

Otherwise suppose that $\ell(M) = \infty$ and let $\{N\}$ be a composition series. Since all composition series have the same length this series must be infinite and as such M breaks either ACC or DCC.

□

Problem 7 (7.5). *Let*

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b \in \mathbb{R}; c \in \mathbb{Q} \right\}$$

R is a ring under matrix addition and multiplication. Show that R satisfies the ACC and DCC on left ideals but neither chain condition is valid for right ideals. Thus R is of finite length as a left R-module, but $\ell(R) = \infty$ as a right R-module.

Starting with R as a left R-module we can construct a composition series as

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \subset \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \subset \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

the quotients of which are isomorphic to $\mathbb{R}, \mathbb{R}$, and $\mathbb{Q}$ respectively. Since we have a composition series of finite length it must be that $\ell(R) < \infty$ as a left module and from the prior problem satisfies ACC and DCC.

However for R as a right module start with the submodule that has $a = c = 0$ and $b \in \mathbb{Q}$. Multiplication looks like

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & cf \\ 0 & 0 \end{bmatrix}$$

Since $f \in \mathbb{Q}$ it will be closed under the action of R. Then create an ascending series of submodules by adjoining roots of primes $\mathbb{Q}[\sqrt{2}, \sqrt{3}, \ldots]$. Since there are infinitely many primes this chain is infinite and as such $\ell(R) = \infty$ as a right R-module.

For the descending series let $\mathcal{B}$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Then do the same construction as above but instead each time take away an element of $\mathcal{B}$. This will create an infinite descending series.

Problem 8 (7.11). *Let F be a field, let V be a finite-dimensional vector space over F, and let $T \in End_F(V)$. We shall say that T is semisimple if the $F[X]$-module V_T is semisimple. If $A \in M_n(F)$, we shall say that A is semisimple if the linear transformation $T_A : F^n \to F^n$ (multiplication by A) is semisimple. Let $\mathbb{F}_2$ be the field with 2 elements and let $F = \mathbb{F}_2(Y)$ be the rational function field in the indeterminate Y, and let $K = F[X]/\langle X^2 + Y \rangle$. Since $X^2 + Y \in F[X]$ is irreducible, K is a field containing F as a subfield. Now let*

$$A = C(X^2 + Y) = \begin{bmatrix} 0 & Y \\ 1 & 0 \end{bmatrix} \in M_2(F)$$

Show that A is semisimple when considered in $M_2(F)$ but A is not semisimple when considered in $M_2(K)$. Thus, semisimplicity of a matrix is not necessarily preserved when one passes to a larger field. However, prove that if L is a subfield of the complex numbers $\mathbb{C}$, then $A \in M_n(L)$ is semisimple if and only if it is also semisimple as a complex matrix.

From Adkins' book the linear transformation will be semisimple as described above if the minimal polynomial is the product of irreducible factors. The minimal polynomial for A is $X^2 + Y$ which is irreducible for F but not irreducible for K.

Proof. Let $\mathbb{L}$ be a subfield of $\mathbb{C}$ and $A \in M_n(\mathbb{L})$. Then as above the matrix A is semisimple only when its minimal polynomial is the product of distinct irreducible factors. If A is irreducible as a complex matrix then it will definitely be irreducible as a matrix over $\mathbb{L}$. On the other hand if the minimal polynomial of A is the product of distinct irreducible factors over $\mathbb{L}$ then it will factor into linear terms when we move to $\mathbb{C}$ as $\mathbb{C}$ is a perfect field.

Therefore a matrix is semisimple on a subfield of the complex numbers if and only if it is semisimple over the complex numbers themselves. □

Problem 9 (7.17)**.** *(a) Prove that if R is a semisimple ring and I is an ideal, then R/I is semisimple.*

(b) Show (by example) that a subring of a semisimple ring need not be semisimple.

Proof. (a) See problem 4.

(b) The rationals are simple, and as such semisimple, since they are a field. However the integers are a subring of the rationals and the integers are not semisimple.

□