**Problem 1.** *Let $G = C_2 = \langle a : a^2 = 1\rangle$, and let $V = F^2$ (where $F$ is a field). For $(\alpha, \beta) \in V$, define the action of $G$ on $V$ by $1(\alpha, \beta) = (\alpha, \beta)$ and $a(\alpha, \beta) = (\beta, \alpha)$, and extend by linearity to make $V$ into an $FG$-module. Find all $FG$-submodules of $V$.*

First note that the submodules correspond to subspaces that are invariant under the action of $FG$. In this case, due to the transposition, the only such subspaces are $\{0\}, V$, $\{(\alpha, \alpha) \in V\}$, and $\{(\alpha, \beta) \in V | \alpha + \beta = 0\}$.

**Problem 2.** *If $G = C_2 \times C_2 = \langle a, b : a^2 = b^2 = 1, ab = ba\rangle$, write the real group ring $\mathbb{R}G$ as a direct sum of $\mathbb{R}G$-submodules, each of which is 1-dimensional over $\mathbb{R}$.*

Treat this as a vector space over $\mathbb{R}$ with basis $(e_1, e_2, e_3, e_4)$ where the corresponding representation acts as

$$\varphi(a) = (e_2, e_1, e_3, e_r), \varphi(b) = (e_1, e_2, e_4, e_3)$$

Then we can express this as the direct sum of the four following 1-dimensional subspaces that are invariant under the action:

- $\{(xe_1 + xe_2)|x \in \mathbb{R}\}$
- $\{(xe_3 + xe_4)|x \in \mathbb{R}\}$
- $\{(xe_1 + ye_2)|x + y = 0\}$
- $\{(xe_3 + ye_4)|x + y = 0\}$

**Problem 3.** *Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. Define matrices $A, B, C, D$ over $\mathbb{C}$ by*

$$A = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) *Verify that each of the functions $\rho_k : G \to GL(2, \mathbb{C})$ $(k = 1, 2, 3, 4)$, given by (i) $\rho_1(a^r b^s) = A^r B^s$, (ii) $\rho_2(a^r b^s) = A^{3r}(-B)^s$, (iii) $\rho_3(a^r b^s) = (-A)^r B^s$, (iv) $\rho_4(a^r b^s) = C^r D^s$ for $0 \leq r \leq 5, 0 \leq s \leq 1$, is a representation of $G$.*

(b) *Which of the representations $\rho_k$ are faithful?*

(c) *Which of the representations are equivalent?*

(d) *Which are irreducible?*

(a) To show that the $\rho_k$ are representations we will verify that the relations of $D_{12}$ are fulfilled after applying $\rho_k$.

For $\rho_1$:

$$\rho_1(a)^6 = \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 \end{pmatrix} = I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \rho_1(b)^2$$

$$\rho_1(b)^{-1}\rho_1(a)\rho_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i\sqrt{3} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2}i\sqrt{3} + \frac{1}{2} \end{pmatrix}$$

For $\rho_2$:

$$\rho_2(a)^6 = \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{18} & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{18} \end{pmatrix} = I_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^2 = \rho_2(b)^2$$

$$\rho_2(b)^{-1}\rho_2(a)\rho_2(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} & 0 \\ 0 & \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} \end{pmatrix}$$

For $\rho_3$:

$$\rho_3(a)^6 = \begin{pmatrix} \left(-\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)^{6} & 0 \\ 0 & \left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)^{6} \end{pmatrix} = I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \rho_3(b)^2$$

$$\rho_3(b)^{-1}\rho_3(a)\rho_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i\sqrt{3}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}i\sqrt{3}-\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}i\sqrt{3}-\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}i\sqrt{3}-\frac{1}{2} \end{pmatrix}$$

For $\rho_4$:

$$\rho_4(a)^6 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}^6 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \rho_4(b)^2$$

$$\rho_4(b)^{-1}\rho_4(a)\rho_4(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

(b) We can see that both $\rho_1$ and $\rho_4$ are faithful as the order $A$ and $D$ are both 6. However $\rho_2$ and $\rho_3$ are not faithful as the order of $\rho_2(a)$ is 2 and the order of $\rho_3(a)$ is 3.

(c) To start $\rho_2$ and $\rho_3$ are not equivalent to either of the other two, or each other for that matter, due to the orders of the elements that $a$ is sent to.

However we can see that $\rho_1$ and $\rho_4$ are equivalent as $A$ and $C$ have the same eigenvalues $(e^{i*\pi/3}, e^{-i\pi/3})$ and so do matrices $B$ and $D$ $(1, -1)$.

(d)

**Problem 4.** *Find the missing row in the following character table:*

| *Order of the conjugacy class* | (1) | (3) | (6) | (6) | (8) |
|---|---|---|---|---|---|
| *Conjugacy class* | $Cl(1)$ | $Cl(a)$ | $Cl(b)$ | $Cl(c)$ | $Cl(d)$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_3$ | 3 | $-1$ | 1 | $-1$ | 0 |
| $\chi_4$ | 3 | $-1$ | $-1$ | 1 | 0 |
| $\chi_5$ | | | | | |

First note that the order of the group is 24. Then using the second orthogonality relation from Dummit ($\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)}$) we know what each column with itself should be the size of the centralizer. This gives us

$$\begin{aligned}
24 - (1+1+9+9) &= 4 \\
8 - (1+1+1+1) &= 4 \\
4 - (1+1+1+1) &= 0 \\
4 - (1+1+1+1) &= 0 \\
3 - (1+1) &= 1
\end{aligned}$$

Then when we fill in our table we get

| Order of the conjugacy class<br>Conjugacy class | (1)<br>$Cl(1)$ | (3)<br>$Cl(a)$ | (6)<br>$Cl(b)$ | (6)<br>$Cl(c)$ | (8)<br>$Cl(d)$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_3$ | 3 | $-1$ | 1 | $-1$ | 0 |
| $\chi_4$ | 3 | $-1$ | $-1$ | 1 | 0 |
| $\chi_5$ | 2 | 2 | 0 | 0 | 1 |

**Problem 5.** *The character table of $S_3$ is*

| *Conjugacy class* | $Cl(1)$ | $Cl((1\ 2))$ | $Cl((1\ 2\ 3))$ |
|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 |
| $\chi_3$ | 2 | 0 | $-1$ |

*Let $\phi$ be a character such that $\phi(1) = 5, \phi((1\ 2)) = 1, \phi((1\ 2\ 3)) = 2$.*

*(a) Compute the inner products $\langle \phi, \chi_1 \rangle, \langle \phi, \chi_2 \rangle$, and $\langle \phi, \chi_3 \rangle$.*

*(b) Write the character $\phi$ as a linear combination of $\chi_1, \chi_2, \chi_3$.*

(a) If we calculate the inner products we get

$$\langle \phi, \chi_1 \rangle = 2, \langle \phi, \chi_2 \rangle = 1, \langle \phi, \chi_3 \rangle = 1$$

(b) $\phi = 2\chi_1 + \chi_2 + \chi_3$

Various bits of code that I used to help get a handle on some of this material.

```
A = Matrix(SR,[[e^(i*pi/3),0],[0,e^(-i*pi/3)]])
B = Matrix(SR,[[0,1],[1,0]])
C = Matrix(SR,[[1/2,sqrt(3)/2],[-sqrt(3)/2,1/2]])
D = Matrix(SR,[[1,0],[0,-1]])
G = DihedralGroup(6)
r,s = G.gens()

def p1(g):
    if g in [r^i for i in range(0,6)]:
        return A^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
```

```
        return A^([r^i*s for i in range(0,6)].index(g))*B

def p2(g):
    if g in [r^i for i in range(0,6)]:
        return A^(3*[r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return A^(3*[r^i*s for i in range(0,6)].index(g))*(-B)

def p3(g):
    if g in [r^i for i in range(0,6)]:
        return (-A)^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return (-A)^([r^i*s for i in range(0,6)].index(g))*B

def p4(g):
    if g in [r^i for i in range(0,6)]:
        return C^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return C^([r^i*s for i in range(0,6)].index(g))*D

def orth1(x1,x2,G):
    return 1/len(G) * sum([x1(g)*conjugate(x2(g)) for g in G])

def orth2(x,y,Cs):
    return sum([c(x)*conjugate(c(y)) for c in Cs])

S3 = SymmetricGroup(3)
cl1 = S3(()).conjugacy_class()
cl12 = S3((1,2)).conjugacy_class()
cl123 = S3((1,2,3)).conjugacy_class()

def x1(g):
    return 1

def x2(g):
    if g in cl1:
        return 1
    if g in cl12:
        return -1
    if g in cl123:
        return 1

def x3(g):
    if g in cl1:
        return 2
    if g in cl12:
        return 0
    if g in cl123:
        return -1

def phi(g):
    if g in cl1:
```

```
    return 5
if g in cl12:
    return 1
if g in cl123:
    return 2
```