**Problem 1.** *Show that $x^3 + 3x + 1$ is irreducible over $\mathbb{Q}$ and let $\theta \in \mathbb{C}$ be a root. Compute $(1+\theta)(1+\theta+\theta^2)$ and $\frac{1+\theta}{1+\theta+\theta^2}$ in $\mathbb{Q}(\theta)$.*

By the rational roots theorem if rational roots for $x^3 + 3x + 1$ exist then they must be of the form $\pm 1$. However neither of those are roots. Thus $x^3 + 3x + 1$ is irreducible over $\mathbb{Q}$.

Let $\theta \in \mathbb{C}$ be a root of $x^3 + 3x + 1$. Then for the expression $(1+\theta)(1+\theta+\theta^2)$ we have:

$$\begin{aligned}(1+\theta)(1+\theta+\theta^2) &= 1 + 2\theta + 2\theta^2 + \theta^3 \\ &= 2\theta^2 - \theta + 1 + 3\theta + \theta^3 \\ &= 2\theta^2 - \theta\end{aligned}$$

For the next expression, $\frac{1+\theta}{1+\theta+\theta^2}$, the multiplicative inverse of the bottom $1+\theta+\theta^2$ is $\frac{3}{7}\theta^2 - \frac{2}{7}\theta + \frac{8}{7}$. This can be found be multiplying $1+\theta+\theta^2$ by $(c + b\theta + c\theta^2)$ and extracting a system of linear equations. Then we have:

$$\begin{aligned}\frac{1+\theta}{1+\theta+\theta^2} &= (1+\theta)\left(\frac{3}{7}\theta^2 - \frac{2}{7}\theta + \frac{8}{7}\right) \\ &= \frac{3}{7}\theta^3 + \frac{1}{7}\theta^2 + \frac{6}{7}\theta + \frac{8}{7} \\ &= \frac{1}{7}\theta^2 - \frac{3}{7}\theta + \frac{5}{7}\end{aligned}$$

**Problem 2.** *Let $w = e^{\pi i/6}$ so that $w^{12} = 1$, but $w^k \neq 1$ for $1 \leq k < 12$. Find the minimal polynomial $m_{w,\mathbb{Q}}(x)$ and compute $[\mathbb{Q}[w] : \mathbb{Q}]$.*

Begin with the polynomial $x^{12} - 1$ which we now that $\omega$ is a root of. This factors as

$$x^{12} - 1 = (x^6 - 1)(x^6 + 1)$$

Since $\omega$ of $x^6 + 1$ and not the other we continue with it. This factors as

$$x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1)$$

As before $x^4 - x^2 + 1$ has $\omega$ as a root and the other does not. Now we will show that $x^4 - x^2 + 1$ is irreducible.

By the rational root theorem the only possible rational roots are $\pm 1$. However neither of these are roots. The only way $x^4 - x^2 + 1$ would not be irreducible is if it were the product of quadratics. Now consider $x^4 - x + 1$ as a polynomial with integer coefficients and suppose that

$$x^4 - x^2 + 1 = (ax^2 + bx + c)(dx^2 + ex + f) = adx^4 + (bd + ae)x^3 + (af + be + cd)x^2 + (bf + ce)x + fc$$

where $a, b, c, d, e \in \mathbb{Z}$. This gives us the following system of equations:

$$\begin{aligned} ad &= 1 \\ bd + ae &= 0 \\ af + dc + be &= -1 \\ bf + ce &= 0 \\ fc &= 1 \end{aligned}$$

Consider these equations as polynomials in $\mathbb{C}[a,b,c,d,e,f]$. Then consider the ideal

$$\langle ad-1, bd+ae, af+dc+be+1, bf+ce, fc-1\rangle$$

The Gröbner basis for this ideal is $\langle 1\rangle$. Since we are in an algebraically closed field there are no solutions to a set of polynomial equations when the ideal is the whole ring. Thus $x^4-x^2+1$ is not the product of quadratics and as such $x^4-x^2+1$ is irreducible.

Therefore $x^4-x^2+1$ is the minimal polynomial $m_{\omega,\mathbb{Q}}(x)$ and as such $[\mathbb{Q}[\omega]:\mathbb{Q}]=4$.

**Problem 3.** *Compute the minimal polynomial $m_{\alpha,F}(x)$ where $\alpha=\sqrt{2}+\sqrt{5}$ and $F$ is each of the following fields:*
*(a) $\mathbb{Q}$, (b) $\mathbb{Q}[\sqrt{5}]$, (c) $\mathbb{Q}[\sqrt{10}]$, (d) $\mathbb{Q}[\sqrt{15}]$.*

(a) The minimal polynomial is $x^4-14x^2+9$.

(b) The minimal polynomial is $x^2-\sqrt{5}x+2$.

(c) The minimal polynomial is $x^2-(7+\sqrt{10})$.

(d) The minimal polynomial will be $x^4-14x^2+9$ as well. The fact that it is irreducible will follow from it being irreducible over $\mathbb{Q}$ if $\sqrt{15}$ is not in the span of $\{1,\alpha\}$.

**Problem 4.** *Compute the minimal polynomial $m_{\alpha,\mathbb{Q}}(x)$ where $\alpha=\sqrt{2}+\sqrt[3]{5}$.*

Consider the polynomial $f(x)=x^6-6x^4-10x^3+12x^2-60x+17$. Then $f(\alpha)=0$. Now we wish to show that $f(x)$ is irreducible. By the rational roots theorem if any rational roots exist then they will be of the form $\pm 17$ neither of which are roots. Therefore if $f(x)$ is reducible it will either be the product of two cubics or the product of a quartic and a quadratic.

Suppose that $f(x)$ was the product of two cubics. Then we would have

$$x^6-6x^4-10x^3+12x^2-60x+17=(x^3+ax^2+bx+c)(x^3+dx^2+ex+f)$$

If we multiply out the latter terms we can extract the system of equations

$$\begin{aligned}
a+d&=0\\
ad+b+e+6&=0\\
ad+be+c+f+10&=0\\
af+eb+cd-12&=0\\
bf+ec+60&=0\\
cf-17&=0
\end{aligned}$$

If we consider the ideal

$$\langle a+d,, ad+b+e, ad+be+c+f+10, af+eb+cd-12, bf+ec+60, cf-17\rangle\subset\mathbb{C}[a,b,c,d,e,f]$$

the Gröbner basis of this ideal is $\langle 1\rangle$ which implies that there are no solutions to the equation. Thus $f(x)$ cannot be the product of two cubics.

Similarly suppose that $f(x)$ was the product of a quartic and a quadratic. Then

$$x^6-6x^4-10x^3+12x^2-60x+17=(x^4+ax^3+bx^2+cx+d)(x^2+ex+f)$$

$$=x^6+(a+e)\,x^5+(ae+b+f)\,x^4+(be+af+c)\,x^3+(ce+bf+d)\,x^2+(de+cf)\,x+df$$

This gives us the system of equations

$$\begin{aligned} a + e &= 0 \\ ae + b + f + 6 &= 0 \\ be + af + c + 10 &= 0 \\ ce + bf + d - 12 &= 0 \\ de + cf + 60 &= 0 \\ df - 17 &= 0 \end{aligned}$$

As before consider the ideal

$$\langle df - 17, de + cf + 60, ce + bf + d - 12, be + af + c + 10, ae + b + f + 6 \rangle \subset \mathbb{C}[a, b, c, d, e, f]$$

The Gröbner basis for this ideal is $\langle 1 \rangle$ which implies that there are no solutions. Therefore $f(x)$ cannot be expressed as the product of a quartic and a quadratic.

Therefore $f(x)$ is irreducible and as such is in fact the minimal polynomial for $\alpha$.

**Problem 5.** *If $K$ is a field extension of the field of $F$ and $\alpha \in K$ has a minimal polynomial $f(x) \in F[x]$ of odd degree, prove that $F(\alpha) = F(\alpha^2)$. Determine whether the condition on $f(x)$ is necessary for $F(\alpha) = F(\alpha^2)$.*

*Proof.* □

**Problem 6.** *6 Let $K$ be an extension field of $F$ that is algebraic over $F$. Show that any subring $R$ of $K$ which contains $F$, i.e., $F \subseteq R \subseteq K$, is a field. Hence, prove that any subring of a finite dimensional extension field $K/F$ containing $F$ is a subfield.*

*Proof.* □

**Problem 7.** *7 Suppose that $K = F(\alpha)$ is a finite simple extension of the field $F$. Define an $F$-linear transformation $T_\alpha : K \to K$ by $T_\alpha(\beta) = \alpha\beta$ for all $\beta \in K$. Show that the minimal polynomial of $\alpha$ over $F$ is the characteristic polynomial of $T_\alpha$, that is*

$$m_{\alpha,F}(x) = det(xI - T_\alpha).$$

*Proof.* □