Problem 1 (7.26)**.** *Let M and N be finitely generated R-modules over a PID R. Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors $s_1, \ldots, s_t$ (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbb{Z}} M$ is a finite group of order $\prod_{j=1}^{t} s_j^{2t-2j+1}$.*

Proof. □

Problem 2 (7.30)**.** *(a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:*

$$K \otimes_F (F[X]/\langle f(X)\rangle \cong K[X]/\langle f(X)\rangle)$$

(b) By choosing $F, f(x)$, and K appropriately, find an example of two fields K and L containing F such that the F-algebra $K \otimes_F L$ has nilpotent elements.

Proof. □

Problem 3 (7.31)**.** *Let F be a field. Show that $F[X,Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F-algebras.*

Proof. □

Problem 4 (7.34)**.** *Let F be a field, V and W finite-dimensional vector spaces over F, and let $T \in End_F(V), SS \in End(W)$.*

(a) If α is an eigenvalue of S and β is an eigenvalue of T, show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.

(b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

Proof. □

Problem 5 (10.4.3)**.** *Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$ modules but are not isomorphic as $\mathbb{R}$-modules.*

Proof. □

Problem 6 (10.4.5)**.** *Let A be a finite abelian group of order n and let p^k be the largest power of a prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.*

Proof. □

Problem 7 (10.4.10)**.** *Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.*

(a) *For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^{n} m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^{n} m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.*

(b) *Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]*

Proof. □

Problem 8 (10.4.24)**.** *Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.*

Proof. □

Problem 9 (10.4.27)**.** (a) *Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).*

(b) *Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1\epsilon_2 = 0, \epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called orthogonal idempotents in A). Deduce that A is isomorphic as a ring to the direct product of two principle ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).*

(c) *Prove that the map $\varphi : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \bar{z}_2)$, where $\bar{z}_2$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.*

(d) *Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.*

Proof. □