Problem 1. *Show that* $x^3 + 3x + 1$ *is irreducible over* $\mathbb{Q}$ *and let* $\theta \in \mathbb{C}$ *be a root. Compute* $(1+\theta)(1+\theta+\theta^2)$ *and* $\frac{1+\theta}{1+\theta+\theta^2}$ *in* $\mathbb{Q}(\theta)$.

Proof. $\square$

Problem 2. *Let* $w = e^{\pi i/6}$ *so that* $w^{12} = 1$, *but* $w^k \neq 1$ *for* $1 \leq k < 12$. *Find the minimal polynomial* $m_{w,\mathbb{Q}}(x)$ *and compute* $[\mathbb{Q}[w] : \mathbb{Q}]$.

Proof. $\square$

Problem 3. *Compute the minimal polynomial* $m_{\alpha,F}(x)$ *where* $\alpha = \sqrt{2} + \sqrt{5}$ *and* F *is each of the following fields:*
(a) $\mathbb{Q}$, *(b)* $\mathbb{Q}[\sqrt{5}]$, *(c)* $\mathbb{Q}[\sqrt{10}]$, *(d)* $\mathbb{Q}[\sqrt{15}]$.

Proof. $\square$

Problem 4. *Compute the minimal polynomial* $m_{\alpha,\mathbb{Q}}(x)$ *where* $\alpha = \sqrt{2} + \sqrt[3]{5}$.

Proof. $\square$

Problem 5. *If* K *is a field extension of the field of* F *and* $\alpha \in K$ *has a minimal polynomial* $f(x) \in F[x]$ *of odd degree, prove that* $F(\alpha) = F(\alpha^2)$. *Determine whether the condition on* $f(x)$ *is necessary for* $F(\alpha) = F(\alpha^2)$.

Proof. $\square$

Problem 6. *6 Let* K *be an extension field of* F *that is algebraic over* F. *Show that any subring* R *of* K *which contains* F, *i.e.,* $F \subseteq R \subseteq K$, *is a field. Hence, prove that any subring of a finite dimensional extension field* K/F *containing* F *is a subfield.*

Proof. $\square$

Problem 7. *7 Suppose that* $K = F(\alpha)$ *is a finite simple extension of the field* F. *Define an* F*-linear transformation* $T_\alpha : K \to K$ *by* $T_\alpha(\beta) = \alpha\beta$ *for all* $\beta \in K$. *Show that the minimal polynomial of* α *over* F *is the characteristic polynomial of* T_α, *that is*

$$m_{\alpha,F}(x) = det(xI - T_\alpha).$$

Proof. $\square$