Problem 1 (13.6.6). Prove that for n odd, $n > 1$, $\Phi_{2n}(x) = \Phi_n(-x)$.

Proof. Begin with $\Phi_n(-x)$. Then we have that

$$\Phi_n(-x) = \prod_{1 \le d < n | (d,n)=1} (-x - \zeta_n^d)$$

If we pull out the negatives we get

$$\Phi_n(-x) = (-1)^{\varphi(n)} \prod_{1 \le d < n | (d,n)=1} (x - \zeta_n^{d+n/2})$$

Since $\varphi(m)$ is even for $m \ge 3$ we can safely remove it. Then we change the base of ζ_n to get

$$\Phi_n(-x) = (-1)^{\varphi(n)} \prod_{1 \le d < n | (d,n)=1} (x - \zeta_{2n}^{2d+n})$$

All of the $2d + n$ are greater than or equal to 1 and less than $2n$. Moreover as n is odd, greater than 1, and $\gcd(d, n) = 1$ we have that $\gcd(2d, n) = 1$. Since $\deg \Phi_{2n}(x) = \varphi(2n) = \varphi(n)$ and there are $\varphi(n)$ factors in the above product we must have all of the factors for $\Phi_{2n}(x)$.

Therefore

$$\Phi_n(-x) = \Phi_{2n}(x)$$

for n odd and $n > 1$. □

Problem 2 (13.6.9). Suppose A is an $n \times n$ matrix over $\mathbb{C}$ for which $A^k = I$ for some integer $k \ge 1$. Show that A can be diagonalized. Show that the matrix $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where α is an element of a field of characteristic p satisfies $A^p = I$ and cannot be diagonalized if $\alpha \neq 0$.

Proof. Let J be the Jordan normal form of A. This will exist since we are working over the complex numbers. If J is diagonalizable then A will be as well. However because we have the relation $A^k - I_n = 0$ for some $k > 1$ it follows that the minimal polynomial of A will be $x^k - 1$. However this has all distinct roots. As such the block matrices in J will have to be 1×1 since the eigenvalues are distinct. Thus J is a diagonal matrix and so is A.

For the second part first note that

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix}$$

which demonstrates that $A^p = I$ since we are in a field of characteristic p. If we calculate the minimal polynomial of A where $\alpha \neq 0$ we get $(x - 1)^2$. Since the eigenvalues are not unique it will not be diagonalizable. □

Problem 3 (13.6.10). Let φ denote the Frobenius map $x \mapsto x^p$ on the finite field $\mathbb{F}_{p^n}$. Prove that ϕ gives an isomorphism of $\mathbb{F}_{p^n}$ to itself. Prove that φ^n is the identity map and that no lower power of φ is the identity.

Proof. Since powers distribute over multiplication it is clear that φ preserves multiplication. The fact that it preserves addition follows from $\mathbb{F}_{p^n}$ being of characteristic p as:

$$(x+y)^p = \sum_0^p \binom{p}{k} x^k y^{p-k} = x^p + y^p$$

Now we must show that the map is both injective and surjective. We will start with injectivity. Suppose that $x^p = 1$. Then

$$x^p - 1^p = (x-1)^p = 0$$

Which implies that $x = 1$ since we are in a field. Since the kernel is trivial it follows that φ is injective.

For surjectivity note that $F_{p^n}^*$ is a multiplicative group of order $p^n - 1$. As such given $y \in F_{p^n}^*$ we have that $y^{p^n} = y$. It then follows that

$$\left(y^{p^{n-1}}\right)^p = \varphi\left(y^{p^{n-1}}\right) = y$$

which demonstrates that φ is surjective.

Therefore the Frobenius map φ is an isomorphism.

For the latter portion note that $\varphi^n(x) = x^{p^n}$ which is equal to x from the argument made earlier. However this cannot occur from $m < n$. If it did then we would have that $x^{p^m - 1} = x$ for all $x \in \mathbb{F}_{p^n}$. This would imply that the orders of all elements in $\mathbb{F}_{p^n}$ is at most $p^m - 1$. However this is a contradiction as the multiplicative groups for finite fields are cyclic. $\square$

Problem 4 (13.6.13). This exercise outlines a proof of Wedderburn's Theorem that a finite division ring D is a field.

(a) Let Z denote the center of D. Prove that Z is a field containing $\mathbb{F}_p$ for some prime p. If $Z = \mathbb{F}_q$ prove that D has order q^n for some integer n. [D is a vector space over Z].

(b) The nonzero elements $D^\times$ of D form a multiplicative group. For any $x \in D^\times$ show that the nonzero elements of D which commute with x form a division ring which contains Z. Show that this division ring is of order q^m for some integer m and that $m < n$ if x is not an element of Z.

(c) Show that the class equation for the group $D^\times$ is

$$q^n - 1 = (q-1) + \sum_{i=1}^r \frac{q^n - 1}{|C_{D^\times}(x_i)|}$$

where $x_1, x_2, \ldots, x_r$ are representatives of the distinct conjugacy classes in $D^\times$ not contained in the center of $D^\times$. Conclude from (b) that for each i, $|C_{D^\times}(x_i)| = q^{m_i} - 1$ for some $m_i < n$.

(d) Prove that since $\frac{q^n-1}{q^{m_i}-1}$ is an integer (namely, the index $|D^\times : C_{D^\times}(x_i)|$) then m_i divides n. Conclude that $\Phi_n(x)$ divides $(x^n - 1)/(x^{m_i} - 1)$ and hence that the integer $\Phi_n(q)$ divides $(q^n - 1)/(q^{m_i} - 1)$ for $i = 1, 2, \ldots, r$.

(e) Prove that (c) and (d) imply that $\Phi_n(q) = \prod_{\zeta \text{ primitive}}(q - \zeta)$ divides $q - 1$. Prove that $|q - \zeta| > q - 1$ (complex absolute value) for any root of unity $\zeta \neq 1$ [note that 1 is the closest point on the unit circle in $\mathbb{C}$] to the point q on the real line]. Conclude that $n = 1$, i.e., that $D = Z$ is a field.

Proof. (a) Since D is a finite division ring the center Z is a commutative finite division ring. Thus Z is a field. Moreover all finite fields are of order p^n where p is a prime. If $Z \cong \mathbb{F}_{p^n}$ then this will contain $\mathbb{F}_p$ as a subfield as $\mathbb{F}_{p^n}$ is the splitting field for $x^{p^n} - x$ over $\mathbb{F}_p$.

For the second part suppose that $Z \cong \mathbb{F}_q$. Let $b_1 := 1$. Then together define $\mathcal{B}_i := \operatorname{span}\{b_1, \dots, b_i\}$ (linear combinations with coefficients in Z) and b_{i+1} to be some element in $D \setminus \mathcal{B}_i$. Since D is finite it follows that there will exist an n where $\mathcal{B}_n = D$. Now we will show that the set $\{b_1, \dots, b_n\}$ forms a basis for D. Since it already spans D we just need to show that it is linearly independent.

Suppose otherwise. Then we have a nontrivial linear combination $\sum_0^n a_i b_i = 0$ giving us that $-\frac{1}{a_n} \sum_0^{n-1} a_i b_i = b_n$. However this contradicts the assumption that $b_n \in D \setminus \mathcal{B}_{n-1}$.

Therefore $\{b_1, \dots, b_n\}$ forms a basis for D over Z and as such $|D| = q^n$ for some positive integer n.

(b) Let $x \in D^\times$ and let D_x be the set of elements that commute with x. We will show that D_x is a division ring and that $Z \subseteq D_x$. Since elements of the center commute with all elements of D it is clear that $Z \subseteq D_x$. This implies that 0 and 1 are in D_x.

Next we show that D_x is closed under addition, multiplication, and inverses. Let $r, s \in D_x$. Then for addition we have

$$(r + s)x = rx + sx = xr + xs = x(r + s)$$

For multiplication we have

$$(rs)x = r(sx) = r(xs) = (rx)s = (xr)s = x(rs)$$

Finally for inverses since we have $rx = xr$ for $r \in D_x$ if we multiply on the left and right by r^{-1} it follows that

$$xr^{-1} = r^{-1}x$$

implying that $r^{-1} \in D_x$ whenever $r \in D_x$.

Therefore D_x is a finite division ring. Moreover it fulfills the same hypothesis as in part (a) so it must be of size q^m for some m. If $x \notin Z$ then there is some element $y \in D$ such that $yx \neq xy$. Then $y \notin D_x$. However since both D and D_x are vector spaces over Z with size q^n and q^m respectively it must be that $n > m$ since D_x is a strict subset of D.

(c) Let $D^\times$ act on itself by conjugation. Then the class equation for $D^\times$ will be

$$|D^\times| = |Z| + \sum_1^r \frac{|D^\times|}{|C_{D^\times(x_i)}|}$$

If we substitute in for known values we get that

$$q^n - 1 = (q - 1) + \sum_1^r \frac{q^n - 1}{|C_{D^\times}(x_i)}$$

Note that the stabilizer for each x_i will in fact be D_{x_i} from part (c). As such the order will be $q^{m_i} - 1$ where $m_i < n$ as $x_i \notin Z$.

(d) From class we have the formula

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

Applying that formula here we can rewrite the quotient as

$$\frac{q^n - 1}{q^{m_i} - 1} = \frac{\prod_{d|n} \Phi_d(q)}{\prod_{d'|m_i} \Phi_{d'}(q)} \in \mathbb{Z}$$

However since this is an integer we must have that each term $\Phi_{d'}(q)$ divides evenly into the above which implies that $d'|n$. Since $m_i|m_i$ we have that $m_i|n$.

Moreover if we replace q with x the result still holds. It then follows since $\Phi_n(x)|(x^n - 1)$ as $m_i < n$ so it will not be canceled. Thus $\Phi_n(q)|\frac{q^n-1}{q^{m_i}-1}$.

(e) From (d) we have that $\Phi_n(q)$ divides $q^n - 1$ and that it divides $\frac{q^n-1}{q^{m_i}-1}$ for each i. Thus from the class equation in part (c) it must be the case that $\Phi_n(q)$ also divides $q - 1$.

Now we will show that $|q - \zeta| > q - 1$ if $\zeta \neq 1$. If we evaluate $|q - \zeta|^2$ we get

$$|q - \zeta|^2 = (q - \cos\theta)^2 + (-\sin\theta)^2 = q^2 + 1 - 2q\cos\theta$$

Then if we compare this to $(q - 1)^2$ we see that

$$q^2 + 1 - 2q\cos\theta \geq q^2 - 1 - 2q$$

and since $\zeta \neq 1$ we have that $\cos\theta < 1$ which makes this inequality strict. Thus $n = 1$ which implies that $Z = D$ and that D is in fact a field.

□

Problem 5 (14.1.4). Prove that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic.

Proof. Suppose that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ were isomorphic. Then there would be an isomorphism $\varphi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$. Let $\varphi(\sqrt{2}) = a + b\sqrt{3}$. Then we have that

$$\varphi(2) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = 2$$

it then follows that $(a + b\sqrt{3})^2 = 2$. However by expanding we get

$$a^2 + 3b^2 + 2ab\sqrt{3} = 2$$

which implies that either a or b is zero since we are in a field. If $b = 0$ then $a^2 = 2$ which implies that $\sqrt{2} \in \mathbb{Q}[\sqrt{3}]$ which is a contradiction. On the other hand if $a = 0$ then $b^2 = 2/3$ which implies that $\sqrt{3}b = \sqrt{2}$. Then $\sqrt{2/3} \in \mathbb{Q}[\sqrt{3}]$ once again which is a contradiction.

Therefore the fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic. □

Problem 6 (14.2.4). Let p be a prime. Determine the elements of the Galois group of $x^p - 2$.

Proof. First note that the roots of $x^p - 2$ are $\sqrt[p]{2}\zeta_p^k$ for $0 \leq k < p$, that the splitting field of $x^p - 2$ is $\mathbb{Q}[\sqrt[p]{2}, \zeta_p]$, and that $x^p - 2$ is separable. This implies that $|\mathrm{Gal}(x^p - 2)| = [\mathbb{Q}(\sqrt[p]{2}, \zeta) : \mathbb{Q}]$ which is equal to $p(p - 1)$ since the minimal polynomial for $\sqrt[p]{2}$ is $x^p - 2$ and the minimal polynomial for ζ_p is $\Phi_p(x)$ with degree $p - 1$.

From the Fundamental theorem of Galois theory we have that the Galois groups for both $\mathbb{Q}[\sqrt[p]{2}]$ and $\mathbb{Q}[\zeta_p]$ correspond to subgroups for $\text{Gal}(\mathbb{Q}[\sqrt[p]{2},\zeta])$. The Galois group for $\mathbb{Q}[\zeta_p]$ will be isomorphic to $\mathbb{Z}_p$ as it is generated by the isomorphism σ which maps $\zeta_p^k \mapsto \zeta_p^{k+1}$. The Galois group for $\mathbb{Q}[\sqrt[p]{2}]$ will be given by isomorphisms τ_k which takes $\sqrt[p]{2}$ to $\sqrt[p]{2}^k$. Note that $\tau_j \circ \tau_k = \tau_{jk}$ giving us that the Galois group for $\mathbb{Q}[\sqrt[p]{2}]$ is isomorphic to $\mathbb{Z}_p^*$.

If we conjugate σ^i by τ_k we get

$$\tau_k^{-1} \circ \sigma \circ \tau_k(\sqrt[p]{2}\zeta^j) = \tau_k^{-1}(\sqrt[p]{2}\zeta^{jk+1}) = \sqrt[p]{2}\zeta^{j+k^{-1}}$$

which implies that $\tau_k^{-1} \circ \sigma \circ \tau_k$ is in $\langle\sigma\rangle$. Thus $\langle\sigma\rangle$ is a normal subgroup and we there is a subgroup of $\text{Gal}(x^p-2)$ that is isomorphic to the semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ which happens to have the same order as $\text{Gal}(x^p-2)$.

Therefore the Galois group of $x^p - 2$ is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p^*$. □

Problem 7 (14.2.5). Prove that the Galois group of x^p-2 for p a prime is isomorphic to the group, G, of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{F}_p, a \neq 0$.

Proof. Define a map $\varphi : \mathbb{F}_p \rtimes \mathbb{F}_p^* \to G$ from the Galois group of $x^p - 2$ to the above group of matrices via

$$\varphi(b,a) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

and define the inverse by

$$\varphi^{-1}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = (b,a)$$

We can see that φ is a homomorphism since multiplication in the matrix group is of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} xz & xw+y \\ 0 & 1 \end{pmatrix}$$

which precisely mirrors the multiplication in the semi-direct product.

Therefore the Galois group of $x^p - 2$ is isomorphic to the group G. □

Problem 8 (14.2.14). Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e., is a Galois extension of degree 4 with cyclic Galois group.

Proof. For the sake of brevity let $\alpha := \sqrt{2+\sqrt{2}}$. We know from a prior homework that the degree of $\mathbb{Q}(\alpha)$ is 4. We also know that minimal polynomial for α is $x^4 - 4x^2 + 2$ whose roots are $\pm\sqrt{2 \pm \sqrt{2}}$. Thus the Galois group for this field must be of size 4. Define $\sigma : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$ by its action on the roots

$$\alpha \mapsto \alpha^3 - \alpha, \quad -\alpha \mapsto -\alpha^3 + \alpha, \quad \alpha^3 - \alpha \mapsto -\alpha, \quad -\alpha^3 + \alpha \mapsto \alpha$$

This is of order 4. As such the Galois group must be isomorphic to $\mathbb{Z}_4$. □