Problem 1 (7.26)**.** *Let M and N be finitely generated R-modules over a PID R. Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors $s_1, \ldots, s_t$ (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbb{Z}} M$ is a finite group of order $\prod_{j=1}^{t} s_j^{2t-2j+1}$.*

Proof. By the structure theorem for finitely generated modules over a PID we have that

$$M \cong R^r \oplus R/(a_i) \oplus \cdots \oplus R/(a_m)$$

and

$$N \cong R^s \oplus R/(b_1) \oplus \cdots \oplus R/(b_n)$$

where $a_i | a_{i+1}$ and $b_i | b_{i+1}$ Then by Theorem 2.18 of Adkins' book the tensor $M \otimes_R N$ distributes over the direct sum. Together with $R/(a_i) \otimes_R R/(b_j) \cong R/(a_i + b_j)$ we get that

$$M \otimes_R N \cong \bigoplus_{0<i\leq m, 0<j\leq n} (R/(a_i + b_j)) \bigoplus_{0<i\leq m} \left(R/(a_i) \otimes_R R^t\right) \bigoplus_{0<j\leq n} (R^s \otimes_R R/(b_j))$$

If we let M be a finite abelian group with invariant factor decomposition

$$M \cong \bigoplus_{0<i\leq t} \mathbb{Z}_{s_i}$$

Then, from above, the tensor of M with itself is

$$M \otimes_{\mathbb{Z}} M \cong \bigoplus_{0<i,j\leq t} \mathbb{Z}_{\gcd(s_i, s_j)}$$

However since $s_i | s_j$ for $i \leq j$ we have that $\gcd(s_i, s_j)$. This will cause $2t-1$ copies of $\mathbb{Z}_{s_1}$ to appear, $2t-3$ for $\mathbb{Z}_{s_2}$, and so on until we have only a single copy of $\mathbb{Z}_{s_t}$. Thus $M \otimes_{\mathbb{Z}} M$ will have order $\prod_{j=1}^{t} s_j^{2t-2j+1}$. □

Problem 2 (7.30)**.** *(a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:*

$$K \otimes_F (F[X]/\langle f(X) \rangle \cong K[X]/\langle f(X) \rangle)$$

(b) By choosing $F, f(x)$, and K appropriately, find an example of two fields K and L containing F such that the F-algebra $K \otimes_F L$ has nilpotent elements.

Proof. (a) Define a map $\varphi : K \times F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$ via

$$\varphi(k, f) = kf$$

Now briefly verify that φ is F-middle linear

$$\begin{aligned} \varphi(ak, fb) &= akfb & &= a\varphi(k, f)b \\ \varphi(k_1 + k_2, f) &= k_1 f + k_2 f & &= \varphi(k_1, f) + \varphi(k_2, f) \\ \varphi(k, f_1 + f_2) &= kf_1 + kf_2 & &= \varphi(k, f_1) + \varphi(k, f_2) \\ \varphi(ka, f) &= kaf & &= \varphi(k, af) \end{aligned}$$

Since φ is F-middle linear it induces a map $\widetilde{\varphi} : K \otimes_F F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$. Now define a map $\psi : K[x]/\langle g(x) \rangle \to K \otimes_F F[x]/\langle g(x) \rangle$ by $\psi(kx^i) = k \otimes x^i$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses.

First $\widetilde{\varphi} \circ \psi$

$$\begin{aligned}
\widetilde{\varphi} \circ \psi(\sum a_i x^i) &= \widetilde{\varphi}(\sum a_i \otimes x^i) \\
&= \sum a_i x^i
\end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned}
\psi \circ \widetilde{\varphi}(k \otimes \sum a_i x^i) &= \psi \circ \widetilde{\varphi}(\sum k \otimes a_i x^i) \\
&= \psi(\sum \widetilde{\varphi}(k \otimes a_i x^i)) \\
&= \psi(\sum k a_i x^i) \\
&= \sum \psi(k a_i x^i) \\
&= \sum k a_i \otimes x^i \\
&= \sum k \otimes a_i x^i
\end{aligned}$$

Since $\widetilde{\varphi}$ has an inverse it is indeed an isomorphism. Which shows that $K \otimes_F F[x]/\langle g(x) \rangle$ is isomorphic to $K[x]/\langle g(x) \rangle$.

(b) Let F be rational functions of t with coefficients in $\mathbb{Z}_2$. Then let K be the splitting field of $x^2 - t$. Then in $K \otimes_F K$ the element $t^{1/2} \otimes 1 + 1 \otimes t^{1/2}$ is zero when squared making it an idempotent.

$\square$

Problem 3 (7.31)**.** *Let F be a field. Show that $F[X,Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F-algebras.*

Proof. Define $\varphi : F[x] \times F[y] \to F[x] \times F[y]$ via

$$\varphi(\sum_i a_i x^i, \sum_j b_j x^j) = \sum_{i,j} a_i b_j x^i y^j$$

Since we are in a polynomial ring over a field it follows that φ is F-middle linear. As such there is a map $\widetilde{\varphi} : F[x] \otimes F[y] \to F[x,y]$ that mimics φ.

Now define $\psi : F[x,y] \to F[x] \otimes F[y]$ via $\psi(cx^i y^j) = c(x^i \otimes y^j)$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses. Since both of these maps are linear we only need to verify them on either $cx^i y^j$ and $a_i x^i \otimes b_j y^j$.

First for $\widetilde{\varphi} \circ \psi$ we have

$$\begin{aligned}
\widetilde{\varphi} \circ \psi(cx^i y^j) &= \widetilde{\varphi}(c(x^i \otimes y^j)) \\
&= \widetilde{\varphi}(cx^i \otimes y^j) \\
&= cx^i y^j
\end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned}\psi \circ \widetilde{\varphi}(a_i x^i \otimes b_j y^j) &= \psi(a_i b_j x^i y^j) \\ &= a_i b_j (x^i \otimes y^j) \\ &= a_i x^i \otimes b_j y^j\end{aligned}$$

Thus $\widetilde{\varphi}$ is an isomorphism which shows that $F[x, y]$ is isomorphic to $F[x] \otimes F[y]$. □

Problem 4 (7.34)**.** *Let F be a field, V and W finite-dimensional vector spaces over F, and let $T \in End_F(V), S \in End_F(W)$.*

(a) If α is an eigenvalue of S and β is an eigenvalue of T, show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.

(b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

Proof. (a) Let u, v be eigenvectors with eigenvalues α, β for linear transformations S and T respectively. Then

$$S \otimes T(u \otimes v) = S(u) \otimes T(v) = \alpha u \otimes \beta v = \alpha\beta(u \otimes v)$$

(b) Since S and T are diagonalizable they each have a number of distinct eigenvalues equal to the dimension of the space they act on. By the previous problem the product of two eigenvalues gives an eigenvalue which means there will be $\dim(V)\dim(W) = \dim(V \otimes W)$ distinct eigenvalues for $S \otimes T$ making it diagonalizable.

□

Problem 5 (10.4.3)**.** *Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$ modules but are not isomorphic as $\mathbb{R}$-modules.*

Proof. The complex numbers can be given the bimodule structures $(\mathbb{R}, \mathbb{R}), (\mathbb{R}, \mathbb{C})$, and $(\mathbb{C}, \mathbb{R})$ by left and right multiplication from the correct field. As such both of the above tensors can be given be given a left $\mathbb{R}$-module structure.

To show they are not isomorphic first note that from Dummit 10.4.19 we know that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ which has rank 2 as a free $\mathbb{R}$ module. Meanwhile $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has free rank 4 as a left $\mathbb{R}$-module (Dummit pg. 375) which implies that they cannot be isomorphic. □

Problem 6 (10.4.5)**.** *Let A be a finite abelian group of order n and let p^k be the largest power of a prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.*

Proof. Let $Syl_p(A)$ denote the Sylow p-subgroup of A. Then by theorem 2.18 from Adkins' we have that

$$\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} A \cong \bigoplus_{q \ prime} \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$$

Now we will show that $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ is zero when $q \neq p$ and is isomorphic to $Syl_p(A)$ otherwise.

Let's start when $p \neq q$. Then let q^l be the highest power of q that divides n. As p and q are relatively prime there exist α and β such that $\alpha p^k + \beta q^l = 1$. Then given $x \otimes a \in \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ we have

$$x \otimes a = x(\alpha p^k + \beta q^l) \otimes a = (x\alpha p^k + x\beta q^l) \otimes a = x \otimes q^l a = x \otimes 0 = x \otimes p^k 0 = xp^k \otimes 0 = 0 \otimes 0$$

which shows that every element of $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ is trivial making the group itself trivial.

The fact that $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_p(A) \cong Syl_p(A)$ follows from both constituents have size p^k. As such the inner action of $\mathbb{Z}$ is equivalent to the corresponding action by $\mathbb{Z}_{p^k}$. Thus $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_p(A)$ is isomorphic to $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}_{p^k}} Syl_p(A)$ which is then isomorphic to $Syl_p(A)$.

Therefore $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A. □

Problem 7 (10.4.10)**.** *Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.*

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

Proof. (a)

(b) Let $R = \mathbb{Z}, n = 1, M = \mathbb{Z}_2$, and $N = \mathbb{Z}$. Then $1 \otimes 2 = 1 \cdot 2 \otimes 1 = 0 \otimes 1 = 0$ which fulfills the conditions lain out above.

□

Problem 8 (10.4.24)**.** *Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.*

Proof. Let $\varphi : \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C}$ be defined via $\varphi(k, x) = kx$. Since it is effectively the same map that we defined before we can see that it is $\mathbb{Z}$-middle linear. As such there is an induced map $\widetilde{\varphi} : \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}$ where $\widetilde{\varphi}(k \otimes x) = kx$. Define $\psi : \mathbb{C} \to \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ via $\psi(x) = 1 \otimes x$ and $\psi(ix) = i \otimes x$ when $x \in \mathbb{R}$ then extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses.

First for $\psi \circ \widetilde{\varphi}$ we have

$$\begin{aligned} \psi \circ \widetilde{\varphi}((a + bi) \otimes x) &= \psi(ax + bxi) \\ &= \psi(ax) + \psi(bxi) \\ &= 1 \otimes ax + i \otimes bx \\ &= a \otimes x + bi \otimes x \\ &= (a + bi) \otimes x \end{aligned}$$

Then for $\widetilde{\varphi} \circ \psi$

$$\begin{aligned} \widetilde{\varphi} \circ \psi(u + iv) &= \widetilde{\varphi}(1 \otimes u + i \otimes v) \\ &= \widetilde{\varphi}(1 \otimes u) + \widetilde{\varphi}(i \otimes v) \\ &= u + iv \end{aligned}$$

Thus ψ and $\widetilde{\varphi}$ are inverses which makes $\widetilde{\varphi}$ an isomorphism which shows that $\mathbb{C}$ is isomorphic to $\mathbb{Z}[i] \otimes \mathbb{R}$. □

Problem 9 (10.4.27). *(a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).*

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1 \epsilon_2 = 0, \epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called orthogonal idempotents in A). Deduce that A is isomorphic as a ring to the direct product of two principle ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\varphi : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \bar{z_2})$, where $\bar{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

Proof. (a) Multiply it out to get

$$aa'e_1^2 + a'be_1e_2 + ab'e_1e_2 + bb'e_2^2 + a'ce_1e_3 + ac'e_1e_3 + b'ce_2e_3 + bc'e_2e_3 + cc'e_3^2 + a'de_1e_4 + ad'e_1e_4 + b'de_2e_4 + bd'e_2e_4 + c'de_3e_4$$

(b)

(c)

(d)

$\square$