Problem 1. *Prove that the following conditions on an R-module P are equivalent.*

(a) *P is projective.*

(b) *P is isomorphic to a direct summand of a free R-module.*

(c) *If $f : M \to P$ is surjective, then there exists and R-module homomorphism $g : P \to M$ such that $f \circ g = \mathrm{id}_P$.*

Proof. The splitting lemma states that if we have a short exact sequence of modules

$$0 \longrightarrow Q \xrightarrow{q} F \xrightarrow{f} P \longrightarrow 0$$

then the following conditions are equivalent:

- $F \cong Q \oplus P$
- There exists a map $g : P \to F$ such that $gf = \mathrm{id}_P$
- There exists a map $h : Q \to F$ such that $hq = \mathrm{id}_Q$

Let P be projective module, F the free group with generators the elements of P, and finally let Q be the free group generated by the relations of P. Then we get a short exact sequence as above with f projection and q inclusion. □

Problem 2. *Let F be a field and let $R = F \times F$. Let $e = (1, 0) \in R$ and let $P = Re$. Show that P is a projective R-module, but that P is not a free R-module.*

Note that R is a free module over itself. Since $P \oplus R(0, 1) \cong R$ we have that P is projective by problem 1. However P is not free as if we let $x, y \in F \setminus \{0\}$ then $(0, y) \cdot (x, 0) = 0$ even though neither element is zero.

Problem 3. *Show that if R is a semisimple ring, then so is $M_n(R)$.*

Proof. By Theorem 7.1.28 from Adkins' book a ring is semisimple if and only if every R-module is semisimple. Since $M_n(R)$ is an R-module it is semisimple. □

Problem 4. *Show that if R is a semisimple ring and I is any ideal, then R/I is also semisimple.*

Proof. By Theorem 7.1.28 from Adkins' book a ring is semisimple if and only if every R-module is semisimple. Since R/I is an R-module it is semisimple. □

Problem 5 (7.2)**.** *Let F be a field and let*

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in F \right\}$$

be the ring of upper triangular matrices over F. Let $M = F^2$ and make M into a (left) R-module by matrix multiplication. Show that $End_R(M) \cong F$. Conclude that the converse of Schur's lemma is false, i.e., $End_R(M)$ can be a division ring without M being a simple R-module.

Proof. Define a map $\psi : F \to End_R(M)$ by $\Psi(x) = \varphi_x$ where φ_x is defined as

$$\varphi_x \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} xd \\ xe \end{pmatrix}$$

Since the maps φ_x are equivalent to scalar multiplication by field elements it is clear from the definition that φ_x are in fact endomorphisms and respect the module structure. Now we will show that ψ is indeed an isomorphism.

The fact that the kernel of ψ is trivial follows from the fact that we are multiplying by field elements and as such there are no zero divisors. However to show that ψ is surjective we must utilize the fact that the endomorphisms play nice with the module structure.

First note that for $\phi \in End_R(M)$ we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \phi \begin{pmatrix} d \\ e \end{pmatrix} = \phi \left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d \\ e \end{pmatrix} \right)$$

If we split ϕ into two pieces like so

$$\phi \begin{pmatrix} d \\ e \end{pmatrix} = \begin{pmatrix} \phi_1(d,e) \\ \phi_2(d,e) \end{pmatrix}$$

The module structure gives us two equations

$$a\phi_1(d,e) + b\phi_2(d,e) = \phi_1(ad + be, ce)$$

and

$$c\phi_2(d,e) = \phi_2(ad + be, ce)$$

Note that we could define a new endomorphism by swapping ϕ_1 and ϕ_2. As such any statement we make about one applies to the other.

First if we let $a = 1, b = c = 0$ in the latter equation this gives us that $\phi_2(d, 0) = 0$. Thus ϕ_1 and ϕ_2 are zero whenever the right coordinate are zero. Next let $a = 1, b = -1, c = 0$ in the first equation and we get that $\phi_1(d,e) = \phi_2(d,e)$. Finally if we set $b = 0, c = 1$ and let $a \in F$ we get that we can pull constants out of the right term. Similarly setting $a = 1, b = 0$ and letting $c \in F$ we get that we can pull constants out of the right term. This implies that ϕ_1 and ϕ_2 are equivalent to multiplying by $\phi(1,1) = x$. Thus the map $\phi = \varphi_x$ and implying that ψ is surjective.

Therefore ψ is in fact an isomorphism and as such $F \cong End_R(M)$. Since M is not simple but it's endomorphisms form a field this is a counterexample to the converse of Schur's lemma. □

Problem 6 (7.4)**.** *An R-module M is said to satisfy the descending chain condition (DCC) on submodules if any strictly decreasing chain of submodules of M of finite length.*

(a) Show that if M satisfies the DCC, then any nonempty set of submodules of M contains a minimal element.

(b) Show that $\ell(M) < \infty$ if and only if M satisfies M satisfies both the ACC and DCC.

Proof. (a) Let $\mathcal{M}$ be the set of submodules of M partially ordered by reverse inclusion. Since M satisfies DCC every chain in $\mathcal{M}$ will have a maximal element and as such by Zorn's lemma there is a maximal element of $\mathcal{M}$ that corresponds to a minimal submodule.

(b) Suppose that $\ell(M) < \infty$. Then given a series of submodules $\{N\}$ we can refine it to a composition series $\{N'\}$ which will be of finite length. Since the length of $\{N\}$ is less than that of $\{N'\}$ it must also have finite length and as such has a minimal point and a maximal point. Since this holds for any series it follows that M satisfies both ACC and DCC.

Otherwise suppose that $\ell(M) = \infty$ and let $\{N\}$ be a composition series. Since all composition series have the same length this series must be infinite and as such M breaks either ACC or DCC.

□

Problem 7 (7.5)**.** *Let*

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b \in \mathbb{R}; c \in \mathbb{Q} \right\}$$

R is a ring under matrix addition and multiplication. Show that R satisfies the ACC and DCC on left ideals but neither chain condition is valid for right ideals. Thus R is of finite length as a left R-module, but $\ell(R) = \infty$ as a right R-module.

Starting with R as a left R-module we can construct a composition series as

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \subset \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \subset \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

the quotients of which are isomorphic to $\mathbb{R}, \mathbb{R}$, and $\mathbb{Q}$ respectively. Since we have a composition series of finite length it must be that $\ell(R) < \infty$ as a left module and from the prior problem satisfies ACC and DCC.

However for R as a right module start with the submodule that has $a = c = 0$ and $b \in \mathbb{Q}$. Multiplication looks like

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & cf \\ 0 & 0 \end{bmatrix}$$

Since $f \in \mathbb{Q}$ it will be closed under the action of R. Then create an ascending series of submodules by adjoining roots of primes $\mathbb{Q}[\sqrt{2}, \sqrt{3}, \ldots]$. Since there are infinitely many primes this chain is infinite and as such $\ell(R) = \infty$ as a right R-module.

Problem 8 (7.11)**.** *Let F be a field, let V be a finite-dimensional vector space over F, and let $T \in End_F(V)$. We shall say that T is semisimple if the $F[X]$-module V_T is semisimple. If $A \in M_n(F)$, we shall say that A is semisimple if the linear transformation $T_A : F^n \to F^n$ (multiplication by A) is semisimple. Let $\mathbb{F}_2$ be the field with 2 elements and let $F = \mathbb{F}_2(Y)$ be the rational function field in the indeterminate Y, and let $K = F[X]/\langle X^2 + Y \rangle$. Since $X^2 + Y \in F[X]$ is irreducible, K is a field containing F as a subfield. Now let*

$$A = C(X^2 + Y) = \begin{bmatrix} 0 & Y \\ 1 & 0 \end{bmatrix} \in M_2(F)$$

Show that A is semisimple when considered in $M_2(F)$ but A is not semisimple when considered in $M_2(K)$. Thus, semisimplicity of a matrix is not necessarily preserved when one passes to a larger field. However, prove that if L is a subfield of the complex numbers $\mathbb{C}$, then $A \in M_n(L)$ is semisimple if and only if it is also semisimple as a complex matrix.

Proof. □

Problem 9 (7.17)**.** *(a) Prove that if R is a semisimple ring and I is an ideal, then R/I is semisimple.*

(b) *Show (by example) that a subring of a semisimple ring need not be semisimple.*

Proof. (a) By Theorem 7.1.28 from Adkins' book a ring is semisimple if and only if every R-module is semisimple. Since R/I is an R-module it is semisimple.

(b) The rationals are simple, and as such semisimple, since they are a field. However the integers are a subring of the rationals and the integers are not semisimple.

$\square$