Problem 1 (7.26)**.** *Let M and N be finitely generated R-modules over a PID R. Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors $s_1, \ldots, s_t$ (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbb{Z}} M$ is a finite group of order $\prod_{j=1}^{t} s_j^{2t-2j+1}$.*

Proof. By the structure theorem for finitely generated modules over a PID we have that

$$M \cong R^r \oplus R/(a_i) \oplus \cdots \oplus R/(a_m)$$

and

$$N \cong R^s \oplus R/(b_1) \oplus \cdots \oplus R/(b_n)$$

where $a_i | a_{i+1}$ and $b_i | b_{i+1}$ Then by Theorem 2.18 of Adkins' book the tensor $M \otimes_R N$ distributes over the direct sum. Together with $R/(a_i) \otimes_R R/(b_j) \cong R/(a_i + b_j)$ we get that

$$M \otimes_R N \cong \bigoplus_{0<i\leq m, 0<j\leq n} (R/(a_i + b_j)) \bigoplus_{0<i\leq m} \left(R/(a_i) \otimes_R R^t\right) \bigoplus_{0<j\leq n} (R^s \otimes_R R/(b_j))$$

If we let M be a finite abelian group with invariant factor decomposition

$$M \cong \bigoplus_{0<i\leq t} \mathbb{Z}_{s_i}$$

Then, from above, the tensor of M with itself is

$$M \otimes_{\mathbb{Z}} M \cong \bigoplus_{0<i,j\leq t} \mathbb{Z}_{\gcd(s_i, s_j)}$$

However since $s_i | s_j$ for $i \leq j$ we have that $\gcd(s_i, s_j)$. This will cause $2t-1$ copies of $\mathbb{Z}_{s_1}$ to appear, $2t-3$ for $\mathbb{Z}_{s_2}$, and so on until we have only a single copy of $\mathbb{Z}_{s_t}$. Thus $M \otimes_{\mathbb{Z}} M$ will have order $\prod_{j=1}^{t} s_j^{2t-2j+1}$. □

Problem 2 (7.30)**.** *(a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:*

$$K \otimes_F (F[X]/\langle f(X) \rangle \cong K[X]/\langle f(X) \rangle)$$

(b) By choosing $F, f(x)$, and K appropriately, find an example of two fields K and L containing F such that the F-algebra $K \otimes_F L$ has nilpotent elements.

Proof. (a) Define a map $\varphi : K \times F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$ via

$$\varphi(k, f) = kf$$

Now briefly verify that φ is F-middle linear

$$\begin{aligned}
\varphi(ak, fb) &= akfb & &= a\varphi(k, f)b \\
\varphi(k_1 + k_2, f) &= k_1 f + k_2 f & &= \varphi(k_1, f) + \varphi(k_2, f) \\
\varphi(k, f_1 + f_2) &= kf_1 + kf_2 & &= \varphi(k, f_1) + \varphi(k, f_2) \\
\varphi(ka, f) &= kaf & &= \varphi(k, af)
\end{aligned}$$

Since φ is F-middle linear it induces a map $\widetilde{\varphi} : K \otimes_F F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$. Now define a map $\psi : K[x]/\langle g(x) \rangle \to K \otimes_F F[x]/\langle g(x) \rangle$ by $\psi(kx^i) = k \otimes x^i$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses.

First $\widetilde{\varphi} \circ \psi$

$$\begin{aligned} \widetilde{\varphi} \circ \psi(\sum a_i x^i) &= \widetilde{\varphi}(\sum a_i \otimes x^i) \\ &= \sum a_i x^i \end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned} \psi \circ \widetilde{\varphi}(k \otimes \sum a_i x^i) &= \psi \circ \widetilde{\varphi}(\sum k \otimes a_i x^i) \\ &= \psi(\sum \widetilde{\varphi}(k \otimes a_i x^i)) \\ &= \psi(\sum k a_i x^i) \\ &= \sum \psi(k a_i x^i) \\ &= \sum k a_i \otimes x^i \\ &= \sum k \otimes a_i x^i \end{aligned}$$

Since $\widetilde{\varphi}$ has an inverse it is indeed an isomorphism. Which shows that $K \otimes_F F[x]/\langle g(x) \rangle$ is isomorphic to $K[x]/\langle g(x) \rangle$.

(b) Let F be rational functions of t with coefficients in $\mathbb{Z}_2$. Then let K be the splitting field of $x^2 - t$. Then in $K \otimes_F K$ the element $t^{1/2} \otimes 1 + 1 \otimes t^{1/2}$ is zero when squared making it an idempotent.

□

Problem 3 (7.31)**.** *Let F be a field. Show that $F[X, Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F-algebras.*

Proof. Define $\varphi : F[x] \times F[y] \to F[x] \times F[y]$ via

$$\varphi(\sum_i a_i x^i, \sum_j b_j x^j) = \sum_{i,j} a_i b_j x^i y^j$$

Since we are in a polynomial ring over a field it follows that φ is F-middle linear. As such there is a map $\widetilde{\varphi} : F[x] \otimes F[y] \to F[x, y]$ that mimics φ.

Now define $\psi : F[x, y] \to F[x] \otimes F[y]$ via $\psi(cx^i y^j) = c(x^i \otimes y^j)$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses. Since both of these maps are linear we only need to verify them on either $cx^i y^j$ and $a_i x^i \otimes b_j y^j$.

First for $\widetilde{\varphi} \circ \psi$ we have

$$\begin{aligned} \widetilde{\varphi} \circ \psi(cx^i y^j) &= \widetilde{\varphi}(c(x^i \otimes y^j)) \\ &= \widetilde{\varphi}(cx^i \otimes y^j) \\ &= cx^i y^j \end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned}\psi \circ \widetilde{\varphi}(a_i x^i \otimes b_j y^j) &= \psi(a_i b_j x^i y^j)\\ &= a_i b_j (x^i \otimes y^j)\\ &= a_i x^i \otimes b_j y^j\end{aligned}$$

Thus $\widetilde{\varphi}$ is an isomorphism which shows that $F[x, y]$ is isomorphic to $F[x] \otimes F[y]$. □

Problem 4 (7.34)**.** *Let F be a field, V and W finite-dimensional vector spaces over F, and let $T \in End_F(V), S \in End(W)$.*

(a) If α is an eigenvalue of S and β is an eigenvalue of T, show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.

(b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

Proof. □

Problem 5 (10.4.3)**.** *Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$ modules but are not isomorphic as $\mathbb{R}$-modules.*

Proof. The complex numbers can be given the bimodule structures $(\mathbb{R}, \mathbb{R}), (\mathbb{R}, \mathbb{C})$, and $(\mathbb{C}, \mathbb{R})$ by left and right multiplication from the correct field. As such both of the above tensors can be given be given a left $\mathbb{R}$-module structure.

To show they are not isomorphic consider that in $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ we have $1 \otimes 1 = i^4 \otimes i^4 = i \otimes -i$. However in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ $1 \otimes 1$ has an equivalence class that is entirely real since only real numbers can travel between the sides of the tensor. As such the two cannot be isomorphic. □

Problem 6 (10.4.5)**.** *Let A be a finite abelian group of order n and let p^k be the largest power of a prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.*

Proof. Let $Syl_p(A)$ denote the Sylow p-subgroup of A. Then by theorem 2.18 from Adkins' we have that

$$\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} A \cong \bigoplus_{q \text{ prime}} \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$$

Now we will show that $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ is zero when $q \neq p$ and is isomorphic to $Syl_p(A)$ otherwise. □

Problem 7 (10.4.10)**.** *Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.*

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

(b) *Show that if* $\sum m_i \otimes n_i = 0$ *in* $M \otimes N$ *where the* n_i *are merely assumed to be* R*-linearly independent then it is not necessarily true that all the* m_i *are 0. [Consider* $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$*, and the element* $1 \otimes 2$*.]*

Proof. □

Problem 8 (10.4.24). *Prove that the extension of scalars from* $\mathbb{Z}$ *to the Gaussian integers* $\mathbb{Z}[i]$ *of the ring* $\mathbb{R}$ *is isomorphic to* $\mathbb{C}$ *as a ring:* $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ *as rings.*

Proof. □

Problem 9 (10.4.27). (a) *Write down a formula for the multiplication of two elements* $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ *and* $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ *in the example* $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ *following proposition 21 (where* $1 = 1 \otimes 1$ *is the identity of* A*).*

(b) *Let* $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ *and* $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$*. Show that* $\epsilon_1 \epsilon_2 = 0, \epsilon_1 + \epsilon_2 = 1$*, and* $\epsilon_j^2 = \epsilon_j$ *for* $j = 1, 2$ *(*ϵ_1 *and* ϵ_2 *are called orthogonal idempotents in* A*). Deduce that* A *is isomorphic as a ring to the direct product of two principle ideals:* $A \cong A\epsilon_1 \times A\epsilon_2$ *(cf. Exercise 1, Section 7.6).*

(c) *Prove that the map* $\varphi : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ *by* $\varphi(z_1, z_2) = (z_1 z_2, z_1 \bar{z_2})$*, where* $\bar{z_2}$ *denotes the complex conjugate of* z_2*, is an* $\mathbb{R}$*-bilinear map.*

(d) *Let* Φ *be the* $\mathbb{R}$*-module homomorphism from* A *to* $\mathbb{C} \times \mathbb{C}$ *obtained from* φ *in (c). Show that* $\Phi(\epsilon_1) = (0, 1)$ *and* $\Phi(\epsilon_2) = (1, 0)$*. Show also that* Φ *is* $\mathbb{C}$*-linear, where the action of* $\mathbb{C}$ *is on the left tensor factor in* A *and on both factors in* $\mathbb{C} \times \mathbb{C}$*. Deduce that* Φ *is surjective. Show that* Φ *is a* $\mathbb{C}$*-algebra isomorphism.*

Proof. □