Problem 1. *Let $G = C_2 = \langle a : a^2 = 1 \rangle$, and let $V = F^2$ (where F is a field). For $(\alpha, \beta) \in V$, define the action of G on V by $1(\alpha, \beta) = (\alpha, \beta)$ and $a(\alpha, \beta) = (\beta, \alpha)$, and extend by linearity to make V into an FG-module. Find all FG-submodules of V.*

First note that the submodules correspond to subspaces that are invariant under the action of FG. In this case, due to the transposition, the only such subspaces are $\{0\}, V$, $\{(\alpha, \alpha) \in V\}$, and $\{(\alpha, \beta) \in V | \alpha + \beta = 0\}$.

Problem 2. *If $G = C_2 \times C_2 = \langle a, b : a^2 = b^2 = 1, ab = ba \rangle$, write the real group ring $\mathbb{R}G$ as a direct sum of $\mathbb{R}G$-submodules, each of which is 1-dimensional over $\mathbb{R}$.*

Treat this as a vector space over $\mathbb{R}$ with basis (e_1, e_2, e_3, e_4) where the corresponding representation acts as

$$\varphi(a) = (e_2, e_1, e_3, e_r), \varphi(b) = (e_1, e_2, e_4, e_3)$$

Then we can express this as the direct sum of the four following 1-dimensional subspaces that are invariant under the action:

- $\{(xe_1 + xe_2) | x \in \mathbb{R}\}$
- $\{(xe_3 + xe_4) | x \in \mathbb{R}\}$
- $\{(xe_1 + ye_2) | x + y = 0\}$
- $\{(xe_3 + ye_4) | x + y = 0\}$

Problem 3. *Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define matrices A, B, C, D over $\mathbb{C}$ by*

$$A = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) Verify that each of the functions $\rho_k : G \to GL(2, \mathbb{C})$ $(k = 1, 2, 3, 4)$, given by (i) $\rho_1(a^r b^s) = A^r B^s$, (ii) $\rho_2(a^r b^s) = A^{3r}(-B)^s$, (iii) $\rho_3(a^r b^s) = (-A)^r B^s$, (iv) $\rho_4(a^r b^s) = C^r D^s$ for $0 \leq r \leq 5, 0 \leq s \leq 1$, is a representation of G.

(b) Which of the representations ρ_k are faithful?

(c) Which of the representations are equivalent?

(d) Which are irreducible?

(a) To show that the ρ_k are representations we will verify that the relations of D_{12} are fulfilled after applying ρ_k.

For ρ_1:

$$\rho_1(a)^6 = \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 \end{pmatrix} = I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \rho_1(b)^2$$

$$\rho_1(b)^{-1}\rho_1(a)\rho_1(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3} + \frac{1}{2}\right)^6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}i\sqrt{3} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2}i\sqrt{3} + \frac{1}{2} \end{pmatrix}$$

For ρ_2:

$$\rho_2(a)^6 = \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{18} & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{18} \end{pmatrix} = I_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^2 = \rho_2(b)^2$$

$$\rho_2(b)^{-1}\rho_2(a)\rho_2(b) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} & 0 \\ 0 & \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \left(-\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} & 0 \\ 0 & \left(\frac{1}{2}i\sqrt{3}+\frac{1}{2}\right)^{3} \end{pmatrix}$$

For ρ_3:

$$\rho_3(a)^6 = \begin{pmatrix} \left(-\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)^{6} & 0 \\ 0 & \left(\frac{1}{2}i\sqrt{3}-\frac{1}{2}\right)^{6} \end{pmatrix} = I_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \rho_3(b)^2$$

$$\rho_3(b)^{-1}\rho_3(a)\rho_3(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}i\sqrt{3}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}i\sqrt{3}-\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}i\sqrt{3}-\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}i\sqrt{3}-\frac{1}{2} \end{pmatrix}$$

For ρ_4:

$$\rho_4(a)^6 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}^6 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \rho_4(b)^2$$

$$\rho_4(b)^{-1}\rho_4(a)\rho_4(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

(b) We can see that both ρ_1 and ρ_4 are faithful as the order A and D are both 6. However ρ_2 and ρ_3 are not faithful as the order of $\rho_2(a)$ is 2 and the order of $\rho_3(a)$ is 3.

(c) To start ρ_2 and ρ_3 are not equivalent to either of the other two, or each other for that matter, due to the orders of the elements that a is sent to.

However we can see that ρ_1 and ρ_4 are equivalent as A and C have the same eigenvalues $(e^{i*\pi/3}, e^{-i\pi/3})$ and so do matrices B and D $(1, -1)$.

(d) We used the eigenvectors for the matrices involved in ρ_1. Moreover note that the eigenvalues are distinct and the eigenspaces are distinct. This implies that ρ_1 is irreducible. We then get that ρ_4 is irreducible by equivalence.

Similarly the eigenvalues for $-A$ are $\left[-\frac{1}{2}i\sqrt{3}-\frac{1}{2}, \frac{1}{2}i\sqrt{3}-\frac{1}{2}\right]$. Since the eigenvalues for $-A$ and B are distinct and the eigenspaces are distinct we have that ρ_3 is irreducible.

However the eigenvalues for $-B$ are $[-1, 1]$ and for A^3 they are $[-1, -1]$. Since the eigenvalues are not distinct the eigenspaces also will not. This implies that there is a non-trivial subrepresentation.

Problem 4. *Find the missing row in the following character table:*

Order of the conjugacy class	(1)	(3)	(6)	(6)	(8)
Conjugacy class	$Cl(1)$	$Cl(a)$	$Cl(b)$	$Cl(c)$	$Cl(d)$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0
χ_5					

First note that the order of the group is 24. Then using the second orthogonality relation from Dummit ($\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)}$) we know what each column with itself should be the size of the centralizer. This gives us

$$\begin{aligned} 24 - (1+1+9+9) &= 4 \\ 8 - (1+1+1+1) &= 4 \\ 4 - (1+1+1+1) &= 0 \\ 4 - (1+1+1+1) &= 0 \\ 3 - (1+1) &= 1 \end{aligned}$$

Then we can use the inner product $\langle \chi_1, \chi_5 \rangle = 0$ to get the signs correct. When we fill in our table we get

Order of the conjugacy class	(1)	(3)	(6)	(6)	(8)
Conjugacy class	$Cl(1)$	$Cl(a)$	$Cl(b)$	$Cl(c)$	$Cl(d)$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0
χ_5	2	2	0	0	-1

Problem 5. *The character table of S_3 is*

Conjugacy class	$Cl(1)$	$Cl((1\ 2))$	$Cl((1\ 2\ 3))$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let ϕ be a character such that $\phi(1) = 5, \phi((1\ 2)) = 1, \phi((1\ 2\ 3)) = 2$.

(a) Compute the inner products $\langle \phi, \chi_1 \rangle, \langle \phi, \chi_2 \rangle$, and $\langle \phi, \chi_3 \rangle$.

(b) Write the character ϕ as a linear combination of χ_1, χ_2, χ_3.

(a) If we calculate the inner products we get

$$\langle \phi, \chi_1 \rangle = 2, \langle \phi, \chi_2 \rangle = 1, \langle \phi, \chi_3 \rangle = 1$$

(b) $\phi = 2\chi_1 + \chi_2 + \chi_3$

Various bits of code that I used to help get a handle on some of this material.

```
A = Matrix(SR,[[e^(i*pi/3),0],[0,e^(-i*pi/3)]])
B = Matrix(SR,[[0,1],[1,0]])
C = Matrix(SR,[[1/2,sqrt(3)/2],[-sqrt(3)/2,1/2]])
D = Matrix(SR,[[1,0],[0,-1]])
G = DihedralGroup(6)
r,s = G.gens()

def p1(g):
    if g in [r^i for i in range(0,6)]:
        return A^([r^i for i in range(0,6)].index(g))
```

```
    elif g in [r^i*s for i in range(0,6)]:
        return A^([r^i*s for i in range(0,6)].index(g))*B

def p2(g):
    if g in [r^i for i in range(0,6)]:
        return A^(3*[r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return A^(3*[r^i*s for i in range(0,6)].index(g))*(-B)

def p3(g):
    if g in [r^i for i in range(0,6)]:
        return (-A)^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return (-A)^([r^i*s for i in range(0,6)].index(g))*B

def p4(g):
    if g in [r^i for i in range(0,6)]:
        return C^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return C^([r^i*s for i in range(0,6)].index(g))*D

def orth1(x1,x2,G):
    return 1/len(G) * sum([x1(g)*conjugate(x2(g)) for g in G])

def orth2(x,y,Cs):
    return sum([c(x)*conjugate(c(y)) for c in Cs])

S3 = SymmetricGroup(3)
cl1 = S3(()).conjugacy_class()
cl12 = S3((1,2)).conjugacy_class()
cl123 = S3((1,2,3)).conjugacy_class()

def x1(g):
    return 1

def x2(g):
    if g in cl1:
        return 1
    if g in cl12:
        return -1
    if g in cl123:
        return 1

def x3(g):
    if g in cl1:
        return 2
    if g in cl12:
        return 0
    if g in cl123:
        return -1

def phi(g):
```

```
if g in cl1:
    return 5
if g in cl12:
    return 1
if g in cl123:
    return 2
```