Problem 1. *Give an example of fields $F \subset E \subset K$ such that K is a root extension of F but E is not a root extension of F. (Hint: Look at $K = \mathbb{Q}(\zeta)$ where ζ is the primitive 7th root of unity).*

Proof. $\square$

Problem 2. *Let $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{3})$, $K = \mathbb{Q}(\sqrt{\sqrt{3}+1})$. Show that E/F and K/E are Galois extensions, but that K/F is not Galois. Find the minimal polynomial of $\sqrt{\sqrt{3}+1}$ and find its Galois group.*

Proof. $\square$

Problem 3. *Find a root extension of $\mathbb{Q}$ containing the splitting fields of each of the following polynomials.*

(a) $x^4 + 1$

(b) $x^4 + 3x^2 + 1$

(c) $x^5 + 4x^3 + x$

(d) $(x^3 - 2)(x^7 - 5)$

Proof. $\square$

Problem 4. *Give an example of a polynomial $\mathbb{Q}[x]$ which is solvable by radicals, but whose splitting field is not a root extension of $\mathbb{Q}$.*

Proof. $\square$

Problem 5. *For r a positive integer, define $f_r(x) \in \mathbb{Q}[x]$ by*

$$f_r(x) = (x^2 + 4)x(x^2 - 4)(x^2 - 16) \cdots (x^2 - 4r^2)$$

(a) *Give a (rough) sketch of the graph of $f_r(x)$*

(b) *Show that if k is an odd integer, then $|f_r(k)| \geq 5$.*

(c) *Show that $g_r(x) = f_r(x) - 2$ is irreducible over $\mathbb{Q}$ and determine its Galois group when $2r + 3 = p$ is prime.*

Proof. $\square$