Problem 1 (7.26). *Let M and N be finitely generated R-modules over a PID R. Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors $s_1, \ldots, s_t$ (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbb{Z}} M$ is a finite group of order $\prod_{j=1}^{t} s_j^{2t-2j+1}$.*

Proof. By the structure theorem for finitely generated modules over a PID we have that

$$M \cong R^r \oplus R/(a_i) \oplus \cdots \oplus R/(a_m)$$

and

$$N \cong R^s \oplus R/(b_1) \oplus \cdots \oplus R/(b_n)$$

where $a_i | a_{i+1}$ and $b_i | b_{i+1}$ Then by Theorem 2.18 of Adkins' book the tensor $M \otimes_R N$ distributes over the direct sum. Together with $R/(a_i) \otimes_R R/(b_j) \cong R/(a_i + b_j)$ we get that

$$M \otimes_R N \cong \bigoplus_{0<i\leq m, 0<j\leq n} (R/(a_i + b_j)) \bigoplus_{0<i\leq m} \left(R/(a_i) \otimes_R R^t\right) \bigoplus_{0<j\leq n} (R^s \otimes_R R/(b_j))$$

If we let M be a finite abelian group with invariant factor decomposition

$$M \cong \bigoplus_{0<i\leq t} \mathbb{Z}_{s_i}$$

Then, from above, the tensor of M with itself is

$$M \otimes_{\mathbb{Z}} M \cong \bigoplus_{0<i,j\leq t} \mathbb{Z}_{\gcd(s_i, s_j)}$$

However since $s_i | s_j$ for $i \leq j$ we have that $\gcd(s_i, s_j)$. This will cause $2t - 1$ copies of $\mathbb{Z}_{s_1}$ to appear, $2t - 3$ for $\mathbb{Z}_{s_2}$, and so on until we have only a single copy of $\mathbb{Z}_{s_t}$. Thus $M \otimes_{\mathbb{Z}} M$ will have order $\prod_{j=1}^{t} s_j^{2t-2j+1}$. □

Problem 2 (7.30). *(a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:*

$$K \otimes_F (F[X]/\langle f(X) \rangle \cong K[X]/\langle f(X) \rangle)$$

(b) By choosing $F, f(x)$, and K appropriately, find an example of two fields K and L containing F such that the F-algebra $K \otimes_F L$ has nilpotent elements.

Proof. (a) Define a map $\varphi : K \times F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$ via

$$\varphi(k, f) = kf$$

Now briefly verify that φ is F-middle linear

$$\begin{aligned}
\varphi(ak, fb) &= akfb & &= a\varphi(k, f)b \\
\varphi(k_1 + k_2, f) &= k_1 f + k_2 f & &= \varphi(k_1, f) + \varphi(k_2, f) \\
\varphi(k, f_1 + f_2) &= kf_1 + kf_2 & &= \varphi(k, f_1) + \varphi(k, f_2) \\
\varphi(ka, f) &= kaf & &= \varphi(k, af)
\end{aligned}$$

Since φ is F-middle linear it induces a map $\widetilde{\varphi} : K \otimes_F F[x]/\langle g(x) \rangle \to K[x]/\langle g(x) \rangle$. Now define a map $\psi : K[x]/\langle g(x) \rangle \to K \otimes_F F[x]/\langle g(x) \rangle$ by $\psi(kx^i) = k \otimes x^i$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses.

First $\widetilde{\varphi} \circ \psi$

$$\begin{aligned}\widetilde{\varphi} \circ \psi(\sum a_i x^i) &= \widetilde{\varphi}(\sum a_i \otimes x^i) \\ &= \sum a_i x^i\end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned}\psi \circ \widetilde{\varphi}(k \otimes \sum a_i x^i) &= \psi \circ \widetilde{\varphi}(\sum k \otimes a_i x^i) \\ &= \psi(\sum \widetilde{\varphi}(k \otimes a_i x^i)) \\ &= \psi(\sum k a_i x^i) \\ &= \sum \psi(k a_i x^i) \\ &= \sum k a_i \otimes x^i \\ &= \sum k \otimes a_i x^i\end{aligned}$$

Since $\widetilde{\varphi}$ has an inverse it is indeed an isomorphism. Which shows that $K \otimes_F F[x]/\langle g(x)\rangle$ is isomorphic to $K[x]/\langle g(x)\rangle$.

(b) Let F be rational functions of t with coefficients in $\mathbb{Z}_2$. Then let K be the splitting field of $x^2 - t$. Then in $K \otimes_F K$ the element $t^{1/2} \otimes 1 + 1 \otimes t^{1/2}$ is zero when squared making it an idempotent.

$\square$

Problem 3 (7.31). *Let F be a field. Show that $F[X, Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F-algebras.*

Proof. Define $\varphi : F[x] \times F[y] \to F[x] \times F[y]$ via

$$\varphi(\sum_i a_i x^i, \sum_j b_j x^j) = \sum_{i,j} a_i b_j x^i y^j$$

Since we are in a polynomial ring over a field it follows that φ is F-middle linear. As such there is a map $\widetilde{\varphi} : F[x] \otimes F[y] \to F[x, y]$ that mimics φ.

Now define $\psi : F[x, y] \to F[x] \otimes F[y]$ via $\psi(cx^i y^j) = c(x^i \otimes y^j)$ and extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses. Since both of these maps are linear we only need to verify them on either $cx^i y^j$ and $a_i x^i \otimes b_j y^j$.

First for $\widetilde{\varphi} \circ \psi$ we have

$$\begin{aligned}\widetilde{\varphi} \circ \psi(cx^i y^j) &= \widetilde{\varphi}(c(x^i \otimes y^j)) \\ &= \widetilde{\varphi}(cx^i \otimes y^j) \\ &= cx^i y^j\end{aligned}$$

Then for $\psi \circ \widetilde{\varphi}$

$$\begin{aligned}\psi \circ \widetilde{\varphi}(a_i x^i \otimes b_j y^j) &= \psi(a_i b_j x^i y^j) \\ &= a_i b_j (x^i \otimes y^j) \\ &= a_i x^i \otimes b_j y^j\end{aligned}$$

Thus $\widetilde{\varphi}$ is an isomorphism which shows that $F[x,y]$ is isomorphic to $F[x] \otimes F[y]$. □

Problem 4 (7.34)**.** *Let F be a field, V and W finite-dimensional vector spaces over F, and let $T \in End_F(V), S \in End_F(W)$.*

(a) If α is an eigenvalue of S and β is an eigenvalue of T, show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.

(b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

Proof. (a) Let u, v be eigenvectors with eigenvalues α, β for linear transformations S and T respectively. Then

$$S \otimes T(u \otimes v) = S(u) \otimes T(v) = \alpha u \otimes \beta v = \alpha\beta(u \otimes v)$$

(b) Since S and T are diagonalizable they each have a number of eigenvalues equal to the dimension of the space they act on. By the previous problem the product of two eigenvalues gives an eigenvalue which means there will be $\dim(V)\dim(W) = \dim(V \otimes W)$ eigenvalues for $S \otimes T$ making it diagonalizable.

□

Problem 5 (10.4.3)**.** *Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$ modules but are not isomorphic as $\mathbb{R}$-modules.*

Proof. The complex numbers can be given the bimodule structures $(\mathbb{R}, \mathbb{R}), (\mathbb{R}, \mathbb{C})$, and $(\mathbb{C}, \mathbb{R})$ by left and right multiplication from the correct field. As such both of the above tensors can be given be given a left $\mathbb{R}$-module structure.

To show they are not isomorphic first note that from Dummit 10.4.19 we know that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ which has rank 2 as a free $\mathbb{R}$ module. Meanwhile $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ has free rank 4 as a left $\mathbb{R}$-module (Dummit pg. 375) which implies that they cannot be isomorphic. □

Problem 6 (10.4.5)**.** *Let A be a finite abelian group of order n and let p^k be the largest power of a prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A.*

Proof. Let $Syl_p(A)$ denote the Sylow p-subgroup of A. Then by theorem 2.18 from Adkins' we have that

$$\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} A \cong \bigoplus_{q \text{ prime}} \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$$

Now we will show that $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ is zero when $q \neq p$ and is isomorphic to $Syl_p(A)$ otherwise.

Let's start when $p \neq q$. Then let q^l be the highest power of q that divides n. As p and q are relatively prime there exist α and β such that $\alpha p^k + \beta q^l = 1$. Then given $x \otimes a \in \mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ we have

$$x \otimes a = x(\alpha p^k + \beta q^l) \otimes a = (x\alpha p^k + x\beta q^l) \otimes a = x \otimes q^l a = x \otimes 0 = x \otimes p^k 0 = xp^k \otimes 0 = 0 \otimes 0$$

which shows that every element of $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_q(A)$ is trivial making the group itself trivial.

The fact that $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_p(A) \cong Syl_p(A)$ follows from both constituents have size p^k. As such the inner action of $\mathbb{Z}$ is equivalent to the corresponding action by $\mathbb{Z}_{p^k}$. Thus $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} Syl_p(A)$ is isomorphic to $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}_{p^k}} Syl_p(A)$ which is then isomorphic to $Syl_p(A)$.

Therefore $\mathbb{Z}_{p^k} \otimes_{\mathbb{Z}} A$ is isomorphic to the Sylow p-subgroup of A. □

Problem 7 (10.4.10)**.** *Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \dots, e_n$.*

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_i$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \dots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where the n_i are merely assumed to be R-linearly independent then it is not necessarily true that all the m_i are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$, and the element $1 \otimes 2$.]

Proof. (a) Let $m \otimes n \in M \otimes_R N$. Then since N is free we can rewrite $m \otimes n$ as

$$m \otimes n = m \otimes \left(\sum_1^n a_i e_i \right) = \sum_i^n m \otimes a_i e_i = \sum_1^n a_i m \otimes e_i$$

Since this process is entirely reversible if another element had the same decomposition then it would be equal to the original value. As such we know that $\sum_1^n 0 \otimes e_i = 0$ and by uniqueness it follows that this is the only way of writing zero.

(b) Let $R = \mathbb{Z}, n = 1, M = \mathbb{Z}_2$, and $N = \mathbb{Z}$. Then $1 \otimes 2 = 1 \cdot 2 \otimes 1 = 0 \otimes 1 = 0$ which fulfills the conditions lain out above.

□

Problem 8 (10.4.24)**.** *Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring: $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.*

Proof. Let $\varphi : \mathbb{Z}[i] \times \mathbb{R} \to \mathbb{C}$ be defined via $\varphi(k, x) = kx$. Since it is effectively the same map that we defined before we can see that it is $\mathbb{Z}$-middle linear. As such there is an induced map $\widetilde{\varphi} : \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{C}$ where $\widetilde{\varphi}(k \otimes x) = kx$. Define $\psi : \mathbb{C} \to \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R}$ via $\psi(x) = 1 \otimes x$ and $\psi(ix) = i \otimes x$ when $x \in \mathbb{R}$ then extending linearly. We will now show that $\widetilde{\varphi}$ and ψ are inverses.

First for $\psi \circ \widetilde{\varphi}$ we have

$$\begin{aligned}
\psi \circ \widetilde{\varphi}((a + bi) \otimes x) &= \psi(ax + bxi) \\
&= \psi(ax) + \psi(bxi) \\
&= 1 \otimes ax + i \otimes bx \\
&= a \otimes x + bi \otimes x \\
&= (a + bi) \otimes x
\end{aligned}$$

Then for $\widetilde{\varphi} \circ \psi$

$$\begin{aligned}\widetilde{\varphi} \circ \psi(u + iv) &= \widetilde{\varphi}(1 \otimes u + i \otimes v) \\ &= \widetilde{\varphi}(1 \otimes u) + \widetilde{\varphi}(i \otimes v) \\ &= u + iv\end{aligned}$$

Thus ψ and $\widetilde{\varphi}$ are inverses which makes $\widetilde{\varphi}$ an isomorphism which shows that $\mathbb{C}$ is isomorphic to $\mathbb{Z}[i] \otimes \mathbb{R}$. $\square$

Problem 9 (10.4.27). *(a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).*

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1 \epsilon_2 = 0, \epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called orthogonal idempotents in A). Deduce that A is isomorphic as a ring to the direct product of two principle ideals: $A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\varphi : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \bar{z_2})$, where $\bar{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

Proof. (a) Let $e_1 = 1 \otimes 1, e_2 = 1 \otimes i, e_3 = i \otimes 1, e_4 = i \otimes i$.. Then multiply out to get

$$aa'e_1^2 + a'be_1e_2 + ab'e_1e_2 + bb'e_2^2 + a'ce_1e_3 + ac'e_1e_3 + b'ce_2e_3 + bc'e_2e_3 + cc'e_3^2 + a'de_1e_4 + ad'e_1e_4$$

$$+ b'de_2e_4 + bd'e_2e_4 + c'de_3e_4 + cd'e_3e_4 + dd'e_4^2$$

Evaluating the multiplication gets us

$$aa'e_1 + a'be_2 + ab'e_2 + bb'(-e_1) + a'ce_3 + ac'e_3 + b'ce_4 + bc'e_4 + cc'(-e_1) + a'de_4 + ad'e_4 + b'd(-e_3) + bd'(-e_3)$$

$$c'd(-e_2) + cd'(-e_2) + dd'e_1$$

Which we can then simplify to

$$(aa' - bb' - cc' + dd')e_1 + (a'b + ab' - c'd - cd')e_2 + (a'c + ac' - b'd - bd')e_3 + (b'c + bc' + a'd + ad')e_4$$

(b) First adding

$$\epsilon_1 + \epsilon_2 = \frac{1}{2}(1 \otimes 1 + i \otimes i) + \frac{1}{2}(1 \otimes 1 - i \otimes i) = \frac{2}{2}(1 \otimes 1) = 1 \otimes 1 = 1$$

Then multiplying

$$\epsilon_1 \epsilon_2 = \frac{1}{2}(1 \otimes 1 + i \otimes i)\frac{1}{2}(1 \otimes 1 - i \otimes i) = \frac{1}{4}(1 \otimes 1 + (-1 \otimes -1)) = \frac{1}{4}(1 \otimes 1 - 1 \otimes 1) = 0$$

Squaring ϵ_1 gets you

$$\epsilon_1^2 = \frac{1}{4}(1 \otimes 1 + 1 \otimes 1 + i \otimes i + i \otimes i) = \frac{1}{2}(1 \otimes 1 + i \otimes i) = \epsilon_1$$

Similarly for ϵ_2

$$\epsilon_2^2 = \frac{1}{4}(1 \otimes 1 + 1 \otimes 1 - i \otimes i - i \otimes i) = \frac{1}{2}(1 \otimes 1 - i \otimes i) = \epsilon_2$$

Then from exercise 7.6.1 from Dummit we have that $A \cong A\epsilon_1 \times A\epsilon_2$.

(c) Since the space $\mathbb{C} \times \mathbb{C}$ can be treated as a vector space over $\mathbb{R}$. This along with the fact that we can pull real constants out of the complex conjugate implies that φ is $\mathbb{R}$-bilinear.

(d) First we calculate $\Phi(\epsilon_1)$ directly

$$\Phi(\epsilon_1) = \Phi(\frac{1}{2}(1 \otimes 1 + i \otimes i)) = \frac{1}{2}(\Phi(1 \otimes 1) + \Phi(i \otimes i)) = \frac{1}{2}((1,1) + (-1,1)) = (0,1)$$

Similarly for $\Phi(\epsilon_2)$ we have

$$\Phi(\epsilon_1) = \Phi(\frac{1}{2}(1 \otimes 1 - i \otimes i)) = \frac{1}{2}(\Phi(1 \otimes 1) - \Phi(i \otimes i)) = \frac{1}{2}((1,1) - (-1,1)) = (1,0)$$

Next we show that Φ is $\mathbb{C}$-linear.

$$\Phi(w(z_1 \otimes z_2)) = \Phi(wz_1 \otimes z_2) = (wz_1z_2, wz_1\bar{z_2}) = w(z_1z_2, z_1\bar{z_2}) = w\Phi(z_1 \otimes z_2)$$

Since Φ is $\mathbb{C}$-linear and we have elements that map to $(1,0)$ and $(0,1)$ it must be surjective as this allows us to reach any element using linearity.

To see that Φ is injective note that if $\Phi(z_1 \otimes z_2) = 0$ then either z_1 or z_2 must be zero since $\mathbb{C}$ is a field. Then

$$z_1 \otimes 0 = 0(z_1 \otimes 0) = 0 \otimes 0 = 0$$

The case for $z_1 = 0$ follows similarly. As such the kernel of Φ is trivial which implies that Φ is in fact an isomorphism.

$\square$