Problem 1 (13.6.6). Prove that for n odd, $n > 1$, $\Phi_{2n}(x) = \Phi_n(-x)$.

Proof. Begin with $\Phi_n(-x)$. Then we have that

$$\Phi_n(-x) = \prod_{1 \le d < n | (d,n)=1} (-x - \zeta_n^d)$$

If we pull out the negatives we get

$$\Phi_n(-x) = (-1)^{\varphi(n)} \prod_{1 \le d < n | (d,n)=1} (x - \zeta_n^{d+n/2})$$

Since $\varphi(m)$ is even for $m \ge 3$ we can safely remove it. Then we change the base of ζ_n to get

$$\Phi_n(-x) = (-1)^{\varphi(n)} \prod_{1 \le d < n | (d,n)=1} (x - \zeta_{2n}^{2d+n})$$

All of the $2d + n$ are greater than or equal to 1 and less than $2n$. Moreover as n is odd, greater than 1, and $\gcd(d, n) = 1$ we have that $\gcd(2d, n) = 1$. Since $\deg \Phi_{2n}(x) = \varphi(2n) = \varphi(n)$ and there are $\varphi(n)$ factors in the above product we must have all of the factors for $\Phi_{2n}(x)$.

Therefore

$$\Phi_n(-x) = \Phi_{2n}(x)$$

for n odd and $n > 1$. ☐

Problem 2 (13.6.9). Suppose A is an $n \times n$ matrix over $\mathbb{C}$ for which $A^k = I$ for some integer $k \ge 1$. Show that A can be diagonalized. Show that the matrix $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ where α is an element of a field of characteristic p satisfies $A^p = I$ and cannot be diagonalized if $\alpha \neq 0$.

Proof. Let J be the Jordan normal form of A. This will exist since we are working over the complex numbers. If J is diagonalizable then A will be as well. However because we have the relation $A^k - I_n = 0$ for some $k > 1$ it follows that the characteristic polynomial of A will be $x^k - 1$. However this has all distinct roots. As such the block matrices in J will have to be 1×1 since the eigenvalues are distinct. Thus J is a diagonal matrix and so is A.

For the second part first note that

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix}$$

which demonstrates that $A^p = I$ since we are in a field of characteristic p. If we calculate the characteristic polynomial of A where $\alpha \neq 0$ we get $(x-1)^2$. Since the eigenvalues are not unique it will not be diagonalizable. ☐

Problem 3 (13.6.10). Let φ denote the Frobenius map $x \mapsto x^p$ on the finite field $\mathbb{F}_{p^n}$. Prove that ϕ gives an isomorphism of $\mathbb{F}_{p^n}$ to itself. Prove that φ^n is the identity map and that no lower power of φ is the identity.

Proof. Since powers distribute over multiplication it is clear that φ preserves multiplication. The fact that it preserves addition follows from $\mathbb{F}_{p^n}$ being of characteristic p as:

$$(x+y)^p = \sum_0^p \binom{p}{k} x^k y^{p-k} = x^p + y^p$$

Now we must show that the map is both injective and surjective. We will start with injectivity. Suppose that $x^p = 1$. Then

$$x^p - 1^p = (x-1)^p = 0$$

Which implies that $x = 1$ since we are in a field. Since the kernel is trivial it follows that φ is injective.

For surjectivity note that $F^*_{p^n}$ is a multiplicative group of order $p^n - 1$. As such given $y \in F^*_{p^n}$ we have that $y^{p^n} = y$. It then follows that

$$\left(y^{p^{n-1}}\right)^p = \varphi\left(y^{p^{n-1}}\right) = y$$

which demonstrates that φ is surjective.

Therefore the Frobenius map φ is an isomorphism.

For the latter portion note that $\varphi^n(x) = x^{p^n}$ which is equal to x from the argument made earlier. However this cannot occur from $m < n$. If it did then we would have that $x^{p^m - 1} = x$ for all $x \in \mathbb{F}_{p^n}$. This would imply that the orders of all elements in $\mathbb{F}_{p^n}$ is at most $p^m - 1$. However this is a contradiction as the multiplicative groups for finite fields are cyclic. □

Problem 4 (13.6.13). This exercise outlines a proof of Wedderburn's Theorem that a finite division ring D is a field.

(a) Let Z denote the center of D. Prove that Z is a field containing $\mathbb{F}_p$ for some prime p. If $Z = \mathbb{F}_q$ prove that D has order q^n for some integer n. [D is a vector space over Z].

(b) The nonzero elements $D^\times$ of D form a multiplicative group. For any $x \in D^\times$ show that the nonzero elements of D which commute with x form a division ring which contains Z. Show that this division ring is of order q^m for some integer m and that $m < n$ if x is not an element of Z.

(c) Show that the class equation for the group $D^\times$ is

$$q^n - 1 = (q-1) + \sum_{i=1}^{r} \frac{q^n - 1}{|C_{D^\times}(x_i)|}$$

where $x_1, x_2, \dots, x_r$ are representatives of the distinct conjugacy classes in $D^\times$ not contained in the center of $D^\times$. Conclude from (b) that for each i, $|C_{D^\times}(x_i)| = q^{m_i} - 1$ for some $m_i < n$.

(d) Prove that since $\frac{q^n-1}{q^{m_i}-1}$ is an integer (namely, the index $|D^\times : C_{D^\times}(x_i)|$) then m_i divides n. Conclude that $\Phi_n(x)$ divides $(x^n - 1)/(x^{m_i} - 1)$ and hence that the integer $\Phi_n(q)$ divides $(q^n - 1)/(q^{m_i} - 1)$ for $i = 1, 2, \dots, r$.

(e) Prove that (c) and (d)e imply that $\Phi(q) = \prod_{\zeta \text{ primitive}}(q - \zeta)$ divides $q - 1$. Prove that $|q - \zeta| > q - 1$ (complex absolute value) for any root of unity $\zeta \neq 1$ [note that 1 is the closest point on the unit circle in $\mathbb{C}$] to the point q on the real line]. Conclude that $n = 1$, i.e., that $D = Z$ is a field.

Proof. $\square$

Problem 5 (14.1.4). Prove that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic.

Proof. Suppose that $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ were isomorphic. Then there would be an isomorphism $\varphi : \mathbb{Q}[\sqrt{2}] \to \mathbb{Q}[\sqrt{3}]$. Let $\varphi(\sqrt{2}) = a + b\sqrt{3}$. Then we have that

$$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 2$$

it then follows that $(a + b\sqrt{3})^2 = 2$. However by expanding we get

$$a^2 + 3b^2 + 2ab\sqrt{3} = 2$$

which implies that either a or b is zero since we are in a field. If $b = 0$ then $a^2 = 2$ which implies that $\sqrt{2} \in \mathbb{Q}[\sqrt{3}]$ which is a contradiction. On the other hand if $a = 0$ then $b^2 = 2/3$ which implies that $\sqrt{3}b = \sqrt{2}$. Then $\sqrt{2/3} \in \mathbb{Q}[\sqrt{3}]$ once again which is a contradiction.

Therefore the fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are not isomorphic. $\square$

Problem 6 (14.2.4). Let p be a prime. Determine the elements of the Galois group of $x^p - 2$.

Proof. $\square$

Problem 7 (14.2.5). Prove that the Galois group of $x^p - 2$ for p a prime is isomorphic to the group of matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a, b \in \mathbb{F}_p, a \neq 0$.

Proof. $\square$

Problem 8 (14.2.14). Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic field, i.e., is a Galois extension of degree 4 with cyclic Galois group.

Proof. For the sake of brevity let $\alpha := \sqrt{2+\sqrt{2}}$. We know from a prior homework that the degree of $\mathbb{Q}(\alpha)$ is 4. We also know that minimal polynomial for α is $x^4 - 4x^2 + 2$ whose roots are $\pm\sqrt{2 \pm \sqrt{2}}$. Thus the Galois group for this field must be of size 4. Define $\sigma : \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$ by its action on the roots

$$\alpha \mapsto \alpha^3 - \alpha, \quad -\alpha \mapsto -\alpha^3 + \alpha, \quad \alpha^3 - \alpha \mapsto -\alpha, \quad -\alpha^3 - \alpha \mapsto \alpha$$

Probably want to justify why this is a homomorphism at all

This is of order 4. As such the Galois group must be isomorphic to $\mathbb{Z}_4$. $\square$