Problem 1.

Proof. First note that if 15° can be trisected then so can 30° as we could bisect 30°, trisect 15° and then double the resulting angle. As such it will suffice to show that we cannot trisect 15°.

A number is constructible if, and only if, both its real and imaginary parts are constructible. If 15° were constructible then so would $e^{i\cdot 10^\circ}$ as it would be the intersection of the angle and the unit circle. The real part of which is

$$\alpha = \cos 10^\circ = \frac{1}{2}\sqrt{\frac{1}{2}\left(4 + 2\cdot\left(\frac{1}{2}(1+i\sqrt{3})^{-\frac{1}{3}} + 2^{\frac{2}{3}}(1+i\sqrt{3})\right)\right)}$$

We know that a number is constructible if, and only if, we have an ascending chain of fields $\mathbb{Q} = F_0 \subset \cdots \subset F_n = \mathbb{Q}[\alpha]$ where all of the intermediate degrees are two. This enforces that the degree of the extension must be a power of 2. However for α at some point we will have to adjoin $(1+i\sqrt{3})^{-\frac{1}{3}}$ for which the extension will be of degree 3. By the tower theorem this means that $3|\mathbb{Q}[\alpha]$ but this cannot occur.

Therefore α is not constructible and it then follows that neither 15° nor 30° can be trisected. $\square$

Problem 2.

(a) Let $f(x) = x^2 - x + 1$. This polynomial has ξ as a root. Moreover it is irreducible by the rational roots theorem as ± 1 are not roots.

(b) The roots of f are ξ and $-e^{2\pi i/3} = -\xi^2$. Thus $\mathbb{Q}[\xi, -\xi^2] = \mathbb{Q}[\xi]$ is the splitting field for f.

(c) Since the degree of f is 2 it follows that $[\mathbb{Q}[\xi] : \mathbb{Q}] = 2$.

Problem 3.

(a) The roots of $f(x) = x^4 + 1$ are $r := e^{\pi i/4}, r^3, r^5$, and r^7.

(b) The roots of $g(x) = x^4 + 4$ are the same roots as above but with each multiplied by $\sqrt{2}$.

(c) The roots of $p(x) = fg(x) = (x^4+1)(x^4+4)$ are the roots of both part a and b.

(d) The roots of $q(x) = (x^4-1)(x^4+4)$ are the roots of part b as well as ± 1 and $\pm i$.

Problem 4.

(a) It will follow that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ if $f(x) = x^4 - 4x^2 + 2$ is irreducible since $f(\alpha) = 0$. However f is irreducible by Eisenstein's criterion using 2.

(b)

Problem 5.

Proof. $\square$

Problem 6.

Proof. □

Problem 7.

Proof. □