

Problem 1.

Proof. First note that if 15° can be trisected then so can 30° as we could bisect 30° , trisect 15° and then double the resulting angle. As such it will suffice to show that we cannot trisect 15° .

A number is constructible if, and only if, both its real and imaginary parts are constructible. If 15° were constructible then so would $e^{i \cdot 10^\circ}$ as it would be the intersection of the angle and the unit circle. The real part of which is

$$\alpha = \cos 10^\circ = \frac{1}{2} \sqrt{\frac{1}{2} \left(4 + 2 \cdot \left(\frac{1}{2} (1 + i\sqrt{3})^{-\frac{1}{3}} + 2^{\frac{2}{3}} (1 + i\sqrt{3}) \right) \right)}$$

We know that a number is constructible if, and only if, we have an ascending chain of fields $\mathbb{Q} = F_0 \subset \cdots \subset F_n = \mathbb{Q}[\alpha]$ where all of the intermediate degrees are two. This enforces that the degree of the extension must be a power of 2. However for α at some point we will have to adjoin $(1 + i\sqrt{3})^{-\frac{1}{3}}$ for which the extension will be of degree 3. By the tower theorem this means that $3|\mathbb{Q}[\alpha]$ but this cannot occur.

Therefore α is not constructible and it then follows that neither 15° nor 30° can be trisected. \square

Problem 2.

- (a) Let $f(x) = x^2 - x + 1$. This polynomial has ξ as a root. Moreover it is irreducible by the rational roots theorem as ± 1 are not roots.
- (b) The roots of f are ξ and $-e^{2\pi i/3} = -\xi^2$. Thus $\mathbb{Q}[\xi, -\xi^2] = \mathbb{Q}[\xi]$ is the splitting field for f .
- (c) Since the degree of f is 2 it follows that $[\mathbb{Q}[\xi] : \mathbb{Q}] = 2$.

Problem 3.

- (a) The roots of $f(x) = x^4 + 1$ are
- (b) The roots of $g(x) = x^4 + 4$ are
- (c) The roots of $p(x) = fg(x) = (x^4 + 1)(x^4 + 4)$ are
- (d) The roots of $q(x) = (x^4 - 1)(x^4 + 4)$ are

Problem 4.

- (a) It will follow that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ if $f(x) = x^4 - 4x^2 + 2$ is irreducible since $f(\alpha) = 0$. It has no rational roots by the rational root theorem however we must now show that it cannot be the product of two irreducible quadratics.
- (b)

Problem 5.

Proof.

□

Problem 6.

Proof.

□

Problem 7.

Proof.

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