Problem 1. *Let $G = C_2 = \langle a : a^2 = 1 \rangle$, and let $V = F^2$ (where F is a field). For $(\alpha, \beta) \in V$, define the action of G on V by $1(\alpha, \beta) = (\alpha, \beta)$ and $a(\alpha, \beta) = (\beta, \alpha)$, and extend by linearity to make V into an FG-module. Find all FG-submodules of V.*

Problem 2. *If $G = C_2 \times C_2 = \langle a, b : a^2 = b^2 = 1, ab = ba \rangle$, write the real group ring $\mathbb{R}G$ as a direct sum of $\mathbb{R}G$-submodules, each of which is 1-dimensional over $\mathbb{R}$.*

Problem 3. *Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define matrices A, B, C, D over $\mathbb{C}$ by*

$$A = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{i\pi/3} \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) *Verify that each of the functions $\rho_k : G \to GL(2, \mathbb{C})$ $(k = 1, 2, 3, 4)$, given by (i) $\rho_1(a^r b^s) = A^r B^s$, (ii) $\rho_2(a^r b^s) = A^{3r}(-B)^s$, (iii) $\rho_3(a^r b^s) = (-A)^r B^s$, (iv) $\rho_4(a^r b^s) = C^r D^s$ for $0 \le r \le 5, 0 \le s \le 1$, is a representation of G.*

(b) *Which of the representations ρ_k are faithful?*

(c) *Which of the representations are equivalent?*

(d) *Which are irreducible?*

Problem 4. *Find the missing row in the following character table:*

Order of the conjugacy class	(1)	(3)	(6)	(6)	(8)
Conjugacy class	$Cl(1)$	$Cl(a)$	$Cl(b)$	$Cl(c)$	$Cl(d)$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0
χ_5					

Problem 5. *The character table of S_3 is*

Conjugacy class	$Cl(1)$	$Cl((1\ 2))$	$Cl((1\ 2\ 3))$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let ϕ be a character such that $\phi(1) = 5, \phi((1\ 2)) = 1, \phi((1\ 2\ 3)) = 2$.

(a) *Compute the inner products $\langle \phi, \chi_1 \rangle, \langle \phi, \chi_2 \rangle$, and $\langle \phi, \chi_3 \rangle$.*

(b) *Write the character ϕ as a linear combination of χ_1, χ_2, χ_3.*

Code Storage:

```
A = Matrix(SR,[[e^(i*pi/3),0],[0,e^(i*pi/3)]])
B = Matrix(SR,[[0,1],[1,0]])
C = Matrix(SR,[[1/2,sqrt(3)/2],[-sqrt(3)/2,1/2]])
D = Matrix(SR,[[1,0],[0,-1]])
G = DihedralGroup(6)
r,s = G.gens()

def p1(g):
    if g in [r^i for i in range(0,6)]:
        return A^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return A^([r^i*s for i in range(0,6)].index(g))*B

def p2(g):
    if g in [r^i for i in range(0,6)]:
        return A^(3*[r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return A^(3*[r^i*s for i in range(0,6)].index(g))*(-B)

def p3(g):
    if g in [r^i for i in range(0,6)]:
        return (-A)^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return (-A)^([r^i*s for i in range(0,6)].index(g))*B

def p4(g):
    if g in [r^i for i in range(0,6)]:
        return C^([r^i for i in range(0,6)].index(g))
    elif g in [r^i*s for i in range(0,6)]:
        return C^([r^i*s for i in range(0,6)].index(g))*D
```