Problem 1. *Prove that the following conditions on an R-module P are equivalent.*

(a) P is projective.

(b) P is isomorphic to a direct summand of a free R-module.

(c) If $f : M \to P$ is surjective, then there exists and R-module homomorphism $g : P \to M$ such that $f \circ g = \mathrm{id}_P$.

Proof. □

Problem 2. *Let F be a field and let $R = F \times F$. Let $e = (1, 0) \in R$ and let $P = Re$. Show that P is a projective R-module, but that P is not a free R-module.*

Proof. □

Problem 3. *Show that if R is a semisimple ring, then so is $M_n(R)$.*

Proof. □

Problem 4. *Show that if R is a semisimple ring and I is any ideal, then R/I is also semisimple.*

Proof. □

Problem 5 (7.2)**.** *Let F be a field and let*

$$R = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in F\}$$

be the ring of upper triangular matrices over F. Let $M = F^2$ and make M into a (left) R-module by matrix multiplication. Show that $End_R(M) \cong F$. Conclude that the converse of Schur's lemma is false, i.e., $End_R(M)$ can be a division ring without M being a simple R-module.

Proof. □

Problem 6 (7.4)**.** *An R-module M is said to satisfy the descending chain condition (DCC) on submodules if any strictly decreasing chain of submodules of M of finite length.*

(a) Show that if M satisfies the DCC, then any nonempty set of submodules of M contains a minimal element.

(b) Show that $\ell(M) < \infty$ if and only if M satisfies M satisfies both the ACC and DCC.

Proof. □

Problem 7 (7.5)**.** *Let*

$$R = \{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b \in \mathbb{R}; c \in \mathbb{Q}\}$$

R is a ring under matrix addition and multiplication. Show that R satisfies the ACC and DCC on left ideals but neither chain condition is valid for right ideals. Thus R is of finite length as a left R-module, but $\ell(R) = \infty$ as a right R-module.

Proof. □

Problem 8 (7.11)**.** *Let F be a field, let V be a finite-dimensional vector space over F, and let $T \in End_F(V)$. We shall say that T is semisimple if the $F[X]$-module V_T is semisimple. If $A \in M_n(F)$, we shall say that A is semisimple if the linear transformation $T_A : F^n \to F^n$ (multiplication by A) is semisimple. Let $\mathbb{F}_2$ be the field with 2 elements and let $F = \mathbb{F}_2(Y)$ be the rational function field in the indeterminate Y, and let $K = F[X]/\langle X^2 + Y\rangle$. Since $X^2 + Y \in F[X]$ is irreducible, K is a field containing F as a subfield. Now let*

$$A = C(X^2 + Y) = \begin{bmatrix} 0 & Y \\ 1 & 0 \end{bmatrix} \in M_2(F)$$

Show that A is semisimple when considered in $M_2(F)$ but A is not semisimple when considered in $M_2(K)$. Thus, semisimplicity of a matrix is not necessarily preserved when one passes to a larger field. However, prove that if L is a subfield of the complex numbers $\mathbb{C}$, then $A \in M_n(L)$ is semisimple if and only if it is also semisimple as a complex matrix.

Proof. □

Problem 9 (7.17)**.** *(a) Prove that if R is a semisimple ring and I is an ideal, then R/I is semisimple.*

(b) Show (by example) that a subring of a semisimple ring need not be semisimple.

Proof. □