**Problem 1** (5.6.2). *Find the order of growth of the following entire functions:*

*(a)* $p(z)$ *where* $p$ *is a polynomial.*

*(b)* $e^{bz^n}$ *for* $b \neq 0$.

*(c)* $e^{e^z}$.

*Proof.* (a) Let $n = \deg p(z)$ and $\rho > 0$. Choose $C > |a_0|$ such that $|p(z)| \leq C|z|^n$ and $m$ so that $\rho m > n$. Then

$$m!Ce^{|z|^\rho} = m!C\sum_{k=0}^{\infty}\frac{|z|^{\rho k}}{k!} \geq C|z|^{\rho m} \geq C|z|^n \geq |p(z)|$$

Since this holds for any $\rho > 0$ we have that $\rho_{p(z)} = 0$.

(b) Using the Taylor expansion we get

$$|e^{bz^n}| \leq |\sum_{m=0}^{\infty}\frac{(z^n)^m}{m!}| \leq \sum_{m=0}^{\infty}|\frac{(z^n)^m}{m!}| \leq \sum_{m=0}^{\infty}\frac{|z|^{nm}}{m!} \leq e^{b|z|^n}$$

which shows that $\rho_{e^{bz^n}} \leq n$. However if we choose the exponent in the definition of order to be $b$ we get exactly $e^{bz^n} = e^{Bz^n}$. From this we can conclude that the order of $e^{bz^n}$ is exactly $n$.

(c) For $e^{e^z}$ suppose that it had order $n$. Then if $z = x + iy$

$$|e^{e^z}| = |e^{e^x(\cos x + i\sin y)}| = e^{e^x\cos x} < Ae^{B|z|^n}$$

Take the logarithm of both sides to get

$$e^x \cos x < C|z|^n$$

which is most assuredly not true for all $z \in \mathbb{C}$. As such $e^{e^z}$ does not have finite order.

□

**Problem 2** (5.6.6). *Prove Wallis's product formula*

$$\frac{\pi}{2} = \prod_{m=1}^{\infty}\frac{(2m)^2}{(2m-1)(2m+1)}$$

*[Hint: Use the product formula for* $\sin z$ *at* $z = \pi/2$*.]*

*Proof.* Plugging in $z = \pi/2$ we get

$$1 = \frac{\pi}{2}\prod_{n=1}^{\infty}(1 - \frac{1}{4n^2}) = \frac{\pi}{2}\prod_{n=1}^{\infty}\frac{(2n+1)(2n-1)}{(2n)^2}$$

Which then dividing by the product gives us

$$\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n+1)(2n-1)} = \frac{\pi}{2}$$

as desired. □

**Problem 3** (5.6.8). *Prove that for every $z$ the product below converges, and*

$$\prod_{k=1}^{\infty} \cos(z/2^k) = \frac{\sin z}{z}$$

*[Hint: Use the fact that* $\sin 2z = 2 \sin z \cos z$*.]*

*Proof.* Start with using the suggested identity with $z/2$ to get

$$\sin z = 2 \sin(z/2) \cos(z/2)$$

Then iterate usage of the identity and divide by $z$ to get

$$\frac{\sin z}{z} = \frac{2^N \sin(z/2^N)}{z} \prod_{k=1}^{N} \cos(z/2^k)$$

Use Wallis's formula on the right term to get

$$\frac{2^N \sin(z/2^N)}{z} = \frac{2^N}{z} \cdot \frac{z}{2^N} \prod_{k=1}^{\infty}(1 - \frac{(\frac{z}{2^N})^2}{k^2\pi^2}) = \prod_{k=1}^{\infty}(1 - \frac{(\frac{z}{2^N})^2}{k^2\pi^2})$$

which goes to 1 as $N \to \infty$. Which in turn gives us that

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \cos(z/2^k)$$

as desired. $\square$

**Problem 4** (5.6.10(b)). *Show that the Hadamard product for* $\cos z$ *is*

$$\cos \pi z = \prod_{n=0}^{\infty} 1 - \frac{4z^2}{(2n+1)^2}$$

*Proof.* First note that $\cos z$ has growth order 1. As such for the Hadamard product formula we get

$$\begin{aligned}
\cos \pi z &= e^{P(z)} z^0 \prod_{n=-\infty}^{\infty} E_1(2z/(2n+1)) \\
&= e^{P(z)} \prod_{n=-\infty}^{\infty} (1 - \frac{2z}{2n+1})e^{2z/(2n+1)} \\
&= e^{P(z)} (\prod_{n=0}^{\infty}(1 - \frac{2z}{2n+1})e^{2z/(2n+1)})(\prod_{n=0}^{\infty}(1 + \frac{2z}{2n+1})e^{-2z/(2n+1)}) \\
&= e^{P(z)} \prod_{n=0}^{\infty}(1 - \frac{4z^2}{(2n+1)^2})
\end{aligned}$$

Now we must show that $P(z) = 0$. Since $P$ is at most degree 1 it will be sufficient to show $P$ evaluates to zero at two distinct values for $z$. Begin with $z = 0$

$$1 = e^{P(0)} \prod_{n=0}^{\infty} 1$$

Giving that $P(0) = 0$. Then use $z = 2$ to get

$$1 = e^{P(2)} \prod_{n=0}^{\infty} (1 - \frac{16}{(2n+1)^2})$$

The partial product for the right term is

$$\prod_{n=0}^{N} 1 - \frac{16}{(2n+1)^2} = \frac{(2N+3)(2N+5)}{4N^2 - 1}$$

Which implies that $\prod_{n=0}^{\infty} 1 - \frac{16}{(2n+1)^2} = 1$ and as such $P(2) = 0$ as well. Thus $P(z) = 0$ and we have the formula

$$\cos \pi z = \prod_{n=0}^{\infty} 1 - \frac{4z^2}{(2n+1)^2}$$

□

**Problem 5** (6.3.5)**.** *Use the fact that* $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$ *to prove that*

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}} = \sqrt{\pi \operatorname{sech} \pi t}$$

*whenever* $t \in \mathbb{R}$.

*Proof.* Use the above identity with $s = 1/2 + it$ to get

$$\begin{aligned}
\Gamma(1/2 + it)\Gamma(1/2 - it) &= \frac{2\pi}{\sin \pi s} \\
&= \frac{2\pi i}{e^{i\pi/2 - \pi t} - e^{-i\pi/2 + \pi t}} \\
&= \frac{2\pi i}{i(e^{\pi t} - e^{-\pi t})} \\
&= \frac{2\pi}{e^{\pi t} - e^{-\pi t}}
\end{aligned}$$

Then using the fact that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ and that $s\bar{s} = |s|^2$ we get

$$|\Gamma(1/2 + it)|^2 = \frac{2\pi}{e^{\pi t} - e^{-\pi t}}$$

Which leads us to the desired identity

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} - e^{-\pi t}}}$$

□

**Problem 6** (6.3.7)**.** *The* ***Beta function*** *is defined for* $\Re(\alpha) > 0$ *and* $\Re(\beta) > 0$ *by*

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

*(a) Prove that* $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

*(b) Show that* $B(\alpha,\beta)=\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du$.

*[Hint: For part (a), note that*

$$\Gamma(\alpha)\Gamma(\beta)=\int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-t-s}dtds$$

*and make the change of variables* $s=ur, t=u(1-r)$*.]*

*Proof.* (a) Begin with $\Gamma(\alpha)\Gamma(\beta)$ and make the above suggested substitution to get

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta)&=(\int_0^\infty e^{-t}t^{\alpha-1}dt)(\int_0^\infty e^{-s}s^{\beta-1}ds)\\&=\int_0^\infty\int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-(t+s)}dtds\\&=\int_0^1\int_0^\infty(u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-u}(-u)dudr\\&=\int_0^1\int_0^\infty -u^{\alpha+\beta-1}e^{-u}(1-r)^{\alpha-1}r^{\beta-1}dudr\\&=\int_0^1(\int_0^\infty -u^{\alpha+\beta-1}e^{-u}du)(1-r)^{\alpha-1}r^{\beta-1}dr\\&=\int_0^1\Gamma(\alpha+\beta)(1-r)^{\alpha-1}r^{\beta-1}dr\\&=\Gamma(\alpha+\beta)\int_0^1(1-r)^{\alpha-1}r^{\beta-1}dr\\&=\Gamma(\alpha+\beta)B(\alpha,\beta)\end{aligned}$$

Which when we divide through gives us

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Begin with the integral side of the problem and make the substitution $\frac{1-t}{t}=u$ to get

$$\begin{aligned}\int_0^\infty\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}du&=\int_0^\infty(\frac{u}{1+u})^{\alpha-1}(1+u)^{-\beta-1}du\\&=\int_1^0(1-t)^{\alpha-1}(t^{-1})^{-\beta-1}(-t^{-2})dt\\&=\int_0^1(1-t)^{\alpha-1}t^{\beta-1}dt\\&=B(\alpha,\beta)\end{aligned}$$

□

**Problem 7** (6.3.10)**.** *An integral of the form*

$$F(z)=\int_0^\infty f(t)t^{z-1}dt$$

*is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function $e^{-t}$.*

*(a) Prove that*

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)t^{z-1}dt = \Gamma(z)\cos(\frac{\pi z}{2})$$

*for $0 < \Re(z) < 1$ and*

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)t^{z-1}dt = \Gamma(z)\sin(\frac{\pi z}{2})$$

*for $0 < \Re(z) < 1$.*

*(b) Show that the second of the above is valid in the larger strip $-1 < \Re(z) < 1$, and that as a consequence, one has*

$$\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x^{3/2}}dx = \sqrt{2\pi}$$

*[Hint: For the first part, consider the integral of the function $f(w) = e^{-w}w^{z-1}$ around the quarter annulus. Use the analytic continuation to prove the second part.]*

*Proof.* (a) Let $\gamma_{\epsilon,R}$ be the quarter annulus in the upper right quarter plane with inner radius $\epsilon$ and outer radius $R$. Since there are no singularities within for $f(w)$, the value of the integral is zero. Thus we have

$$\int_{\gamma_{\epsilon,R}} e^{-w}w^{z-1}dw = 0 = \int_\epsilon^R e^{-t}t^{z-1}dt + \int_\epsilon^R e^{-it}t^{z-1}idt + \int_{C_\epsilon} e^{-w}w^{z-1}dw + \int_{C_R} e^{-w}w^{z-1}dw$$

The leftmost term in the sum is $\Gamma(z)$, the two integrals about $C_\epsilon$ and $C_R$ will both approach zero as $\epsilon \to 0$ and $R \to \infty$. This gives us

$$0 = \Gamma(z)+i^z\int_0^\infty e^{-it}t^{z-1}dt = \Gamma(z)+i^z(\int_0^\infty \cos(t)t^{z-1}dt - i\int_0^\infty \sin(t)t^{z-1}dt) = \Gamma(z)+i^z(\mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z))$$

Rearrange to get

$$\mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z) = \Gamma(z)i^{-z} = \Gamma(z)\cos(\pi z/2) - i\Gamma(z)\sin(\pi z/2)$$

**Something happens**. This then gives us that

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)t^{z-1}dt = \Gamma(z)\cos(\frac{\pi z}{2})$$

and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)t^{z-1}dt = \Gamma(z)\sin(\frac{\pi z}{2})$$

(b)

□

**Problem 8** (6.3.15)**.** *Prove that for $\Re(s) > 1$,*

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$$

*[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$.]*

*Proof.* Start with the right side of the equation and use the suggested series with a substitution $t = nx$ to get

$$\begin{aligned}
\frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty x^{s-1} e^{-nx} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty e^{-nx} dx \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty n^{-s} t^{s-1} e^{-t} dt \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty n^{-s} \int_0^\infty t^{s-1} e^{-t} dt \\
&= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty n^{-s} \Gamma(s) \\
&= \sum_{n=1}^\infty n^{-s} \\
&= \zeta(s)
\end{aligned}$$

$\square$