Problem 1 (2.6.14). Suppose that f is holomorhic in an open set containing the closed unit disc, expcept for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Only need to do case where degree of pole is greater than 1.

Proof. □

Problem 2 (3.8.2). Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} dx$$

Where are the poles of $1/(1+z^4)$?

Proof. □

Problem 3 (3.8.4). Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0$$

Proof. □

Problem 4 (3.8.8). Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Proof. □

Problem 5 (3.8.9). Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

Hint: Use contour that goes down to 0 and up from 1.

Proof. □

Problem 6 (3.8.10). Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a$$

Hint: Integrate over upper half annulus with inner radius ϵ and outer radius R.

Proof. □

Problem 7 (3.8.13). Suppose f is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Proof. □

Problem 8 (3.9.3). If f is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$.

Proof. □