

Problem 1 (1.4.7). *The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.*

(a) Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left| \frac{w - z}{1 - \bar{w}z} \right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Hint: Why can one assume that z is real? It then suffices to prove that

$$(r - w)(r - \bar{w}) \leq (1 - rw)(1 - r\bar{w})$$

with equality for appropriate r and $|w|$.

(b) Prove that for a fixed w in the unit disc \mathbb{D} , the mappings

$$F : z \mapsto \frac{w - z}{1 - \bar{w}z}$$

satisfies the following conditions:

- (i) F maps the unit disc to itself (that is, $F : \mathbb{D} \rightarrow \mathbb{D}$), and is holomorphic.
- (ii) F interchanges 0 and w , namely $F(0) = w$ and $F(w) = 0$.
- (iii) $|F(z)| = 1$ if $|z| = 1$.
- (iv) $F : \mathbb{D} \rightarrow \mathbb{D}$ is bijective. *Hint: Calculate $F \circ F$.*

Proof. (a) Since $\bar{z}w \neq 1$ we can change the inequality from

$$\left| \frac{w - z}{1 - \bar{w}z} \right| \leq 1$$

to

$$|w - z| \leq |1 - \bar{w}z|$$

Then if we square both sides we can use the properties of the conjugate to get

$$(w - z)(\overline{w - z}) \leq (1 - \bar{w}z)(\overline{1 - \bar{w}z})$$

Distribute the conjugate over the sum to get

$$(w - z)(\bar{w} - \bar{z}) \leq (1 - \bar{w}z)(1 - w\bar{z})$$

Multiply out to get

$$|w|^2 - z\bar{w} - w\bar{z} + |z|^2 \leq 1 - \bar{w}z - \bar{z}w + |w|^2|z|^2$$

Shuffle everything to the right and we get

$$0 \leq 1 - |w|^2|z|^2 - |w|^2 - |z|^2 = (1 - |w|^2)(1 - |z|^2) = (1 - |w|)(1 + |w|)(1 - |z|)(1 + |z|)$$

Since each one of these steps was invertible this inequality is equivalent to the original. At this point it is clear if either $|w|$ or $|z|$ is equal to one we have equality. Moreover when $|z| < 1$ and $|w| < 1$ we will have two negative terms and two positive terms making the inequality hold strictly.

- (b) (i) If $|z| \leq 1$ then it is in the unit disc. From the above inequality $|F(z)| \leq 1$. Thus F is a map from the unit disc to itself.

For holomorphicity we **FINISH THIS**.

- (ii) First calculate $F(0)$

$$\begin{aligned} F(0) &= \frac{w-0}{1-\overline{w}0} \\ &= \frac{w}{1} \\ &= w \end{aligned}$$

Calculating $F(w)$ we get

$$\begin{aligned} F(w) &= \frac{w-w}{1-|w|^2} \\ &= 0 \end{aligned}$$

- (iii) From the second part of (a) if $|z| = 1$ then $F(z) = 1$.

- (iv) To show it is a bijection we calculate $F \circ F(z)$:

$$\begin{aligned} F \circ F(z) &= \frac{w + \frac{z-w}{1-\overline{w}z}}{1 + \overline{w} \frac{z-w}{1-\overline{w}z}} \\ &= \frac{\frac{w(1-\overline{w}z) + z-w}{1-\overline{w}z}}{\frac{1-\overline{w}z + \overline{z}-\overline{w}w}{1-\overline{w}z}} \\ &= \frac{-|w|^2 z + z}{1-|w|^2} \\ &= \frac{z(1-|w|^2)}{1-|w|^2} \\ &= z \end{aligned}$$

□

Problem 2 (1.4.9). Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. Using the chain rule and relations $x = r \cos \theta$ and $y = r \sin \theta$ we get:

$$\begin{aligned} u_r &= u_x \cos \theta + u_y \sin \theta \\ u_\theta &= -u_x r \sin \theta + u_y r \cos \theta \\ v_r &= v_x \cos \theta + v_y \sin \theta \\ v_\theta &= -v_x r \sin \theta + v_y r \cos \theta \end{aligned}$$

At this point we have

$$\frac{1}{r}(v_\theta) = -v_x \sin \theta + v_y \cos \theta = u_y \sin \theta + u_x \cos \theta = u_r$$

and similarly

$$\frac{1}{r}u_\theta = -u_x \sin \theta + u_y \cos \theta = -v_y \sin \theta - v_x \cos \theta = -v_r$$

□

Taking the derivatives for $u = \log r$ and $v = \theta$ with respect to r and θ we get

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{1}{r} \\ \frac{\partial u}{\partial \theta} &= 0 \\ \frac{\partial v}{\partial r} &= 0 \\ \frac{\partial v}{\partial \theta} &= 1\end{aligned}$$

Which fulfill the polar Cauchy-Riemann equations. Thus $\log z$ is in fact holomorphic.

Problem 3 (1.4.10). *Show that*

$$4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} = 4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} = \Delta,$$

where Δ is the **Laplacian**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. First recall that

$$f_z = 1/2 f_x - i/2 f_y$$

and

$$f_{\bar{z}} = 1/2 f_x + i/2 f_y$$

Then we compute:

$$\begin{aligned}4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} &= 4 \frac{\partial}{\partial z} (1/2 f_x + i/2 f_y) \\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy}) \\ &= f_{xx} + f_{yy} = \Delta f\end{aligned}$$

and the other

$$\begin{aligned}4 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} &= 4 \frac{\partial}{\partial \bar{z}} (1/2 f_x - i/2 f_y) \\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy}) \\ &= f_{xx} + f_{yy} = \Delta f\end{aligned}$$

□

Problem 4 (1.4.13). Suppose that f is holomorphic in an open set Ω . Prove that in any one of the following cases:

- (a) $\Re(f)$ is constant;
- (b) $\Im(f)$ is constant;
- (c) $|f|$ is constant;

one can conclude that f is constant.

Proof. Separate $f = u + iv$.

- (a) Since u is constant $u_x = u_y = 0$. However using the Cauchy-Riemann equations we have that $v_x = v_y = 0$ as well. Since u and v are constant so is f .
- (b) Since v is constant $v_x = v_y = 0$. However using the Cauchy-Riemann equations we have that $u_x = u_y = 0$ as well. Since u and v are constant so is f .
- (c) Since $|f|$ is constant so is $|f|^2 = u^2 + v^2$. Take the derivative with respect to x and divide by two to get

$$uu_x + vv_x = 0$$

and for y

$$uu_y + vv_y = 0$$

Next rewrite the latter with the Cauchy-Riemann equations

$$-uv_x + vu_x = 0$$

Then multiply the first equation by u , the latter by v , and add them together to get

$$u(uu_x + vv_y)v(-uv_x + vu_x) = (u^2 + v^2)u_x = 0$$

Since $u^2 + v^2$ is a constant this implies that $u_x = v_y = 0$.

We can do the same thing by rewriting the first to get

$$(u^2 + v^2)v_x = 0$$

Which implies that $v_x = -u_y = 0$. This gives us that u and v are constant and as such so is f .

□

Problem 5 (1.4.17). Show that if $\{a_n\}_{n=0}^\infty$ is a sequence of non-zero complex numbers such that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n \rightarrow \infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

We prove the alternate exercise given in class.

Proof. Let $R := \lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|}$. We will show that R is the radius of convergence for the power series $\sum_0^\infty a_n(z - z_0)^n$.

Let $|z - z_0| < R$ and $r := |z - z_0|$

$$\left| \sum_0^\infty a_n(z - z_0)^n \right| \leq \sum_0^\infty |a_n| r^n$$

Then we apply the ratio test to the latter sum to get

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} < 1$$

which shows that the sum converges.

Now suppose that $|z - z_0| > R$. Then apply the ratio test to the power series to get

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} > 1$$

which shows that the series diverges.

Therefore R is in fact the radius of convergence. □

I suppose a way of proving the original would be to use the sequence as a power series and apply the convergence tests with Hadarmard's rule.

Problem 6 (2.6.1). *Prove that*

$$\int_0^\infty \sin(x^2) dx = \int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

*These are the **Fresnel Integrals**. Here, \int_0^∞ is interpreted as $\lim_{R \rightarrow \infty} \int_0^R$.*

Hint: Integrate the function e^{-x^2} over the $\pi/4$ semicircle thing. Recall that $\int_{-\infty}^\infty e^{-x^2} = \sqrt{\pi}$.

Proof. □

Problem 7 (2.6.11). *Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0 .*

Proof. (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi} \Re\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right)) d\varphi.$$

(b) Show that

$$\Re\left(\frac{Re^{i\gamma} - r}{Re^{i\gamma} + r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr \cos \gamma + r^2}.$$

Hint: For the first part, note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity. □