Problem 1 (5.6.2)**.** *Find the order of growth of the following entire functions:*

(a) $p(z)$ *where* p *is a polynomial.*

(b) e^{bz^n} *for* $b \neq 0$.

(c) e^{e^z}.

Proof. (a) Let $n = \deg p(z)$ and $\rho > 0$. Choose $C > |a_0|$ such that $|p(z)| \leq C|z|^n$ and m so that $\rho m > n$. Then

$$m!Ce^{|z|^\rho} = m!C\sum_{k=0}^{\infty}\frac{|z|^{\rho k}}{k!} \geq C|z|^{\rho m} \geq C|z|^n \geq |p(z)|$$

Since this holds for any $\rho > 0$ we have that $\rho_{p(z)} = 0$.

(b) Using the Taylor expansion we get

$$|e^{bz^n}| \leq |\sum_{m=0}^{\infty}\frac{(z^n)^m}{m!}| \leq \sum_{m=0}^{\infty}|\frac{(z^n)^m}{m!}| \leq \sum_{m=0}^{\infty}\frac{|z|^{nm}}{m!} \leq e^{b|z|^n}$$

which shows that $\rho_{e^{bz^n}} \leq n$. However if we choose the exponent in the definition of order to be b we get exactly $e^{bz^n} = e^{Bz^n}$. From this we can conclude that the order of e^{bz^n} is exactly n.

(c)

$\square$

Problem 2 (5.6.6)**.** *Prove Wallis's product formula*

$$\frac{\pi}{2} = \prod_{m=1}^{\infty}\frac{(2m)^2}{(2m-1)(2m+1)}$$

[Hint: Use the product formula for $\sin z$ *at* $z = \pi/2$*.]*

Proof. $\square$

Problem 3 (5.6.8)**.** *Prove that for every* z *the product below converges, and*

$$\prod_{k=1}^{\infty}\cos(z/2^k) = \frac{\sin z}{z}$$

[Hint: Use the fact that $\sin 2z = 2\sin z\cos z$*.]*

Proof. $\square$

Problem 4 (5.6.10(b))**.** *Show that the Hadamard product for* $\cos z$ *is*

$$\cos\pi z = \prod_{n=0}^{\infty}1 - \frac{4z^2}{(2n+1)^2}$$

Proof. □

Problem 5 (6.3.5). *Use the fact that $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ to prove that*

$$|\Gamma(1/2+it)| = \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}} = \sqrt{\pi}\,\mathrm{sech}\,\pi t$$

whenever $t \in \mathbb{R}$.

Proof. □

Problem 6 (6.3.7). *The **Beta function** is defined for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ by*

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(b) Show that $B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

[Hint: For part (a), note that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds$$

and make the change of variables $s = ur, t = u(1-r)$.]

Proof. □

Problem 7 (6.3.10). *An integral of the form*

$$F(z) = \int_0^\infty f(t) t^{z-1} dt$$

*is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t}.*

(a) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t) t^{z-1} dt = \Gamma(z)\cos(\frac{\pi z}{2})$$

for $0 < \Re(z) < 1$ and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t) t^{z-1} dt = \Gamma(z)\sin(\frac{\pi z}{2})$$

for $0 < \Re(z) < 1$.

(b) Show that the second of the above is valid in the larger strip $-1 < \Re(z) < 1$, and that as a consequence, one has

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x^{3/2}} dx = \sqrt{2\pi}$$

[Hint: For the first part, consider the integral of the function $f(w) = e^{-w}w^{z-1}$ around the quarter annulus. Use the analytic continuation to prove the second part.]

Proof. □

Problem 8 (6.3.15)**.** *Prove that for $\Re(s) > 1$,*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$.]

Proof. □