Problem 1 (1.4.7)**.** *The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.*

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \leq (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mappings

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. Hint: Calculate $F \circ F$.

Proof. (a) Since $\overline{z}w \neq 1$ we can change the inequality from

$$\left|\frac{w-z}{1-\overline{w}z}\right| \leq 1$$

to

$$|w-z| \leq |1-\overline{w}z|$$

Then if we square both sides we can use the properties of the conjugate to get

$$(w-z)\overline{(w-z)} \leq (1-\overline{w}z)\overline{(1-\overline{w}z)}$$

Distribute the conjugate over the sum to get

$$(w-z)(\overline{w}-\overline{z}) \leq (1-\overline{w}z)(1-w\overline{z})$$

Multiply out to get

$$|w|^2 - z\overline{w} - w\overline{z} + |z|^2 \leq 1 - \overline{w}z - \overline{z}w + |w|^2|z|^2$$

Shuffle everything to the right and we get

$$0 \leq 1 - |w|^2|z|^2 - |w|^2 - |z|^2 = (1-|w|^2)(1-|z|^2) = (1-|w|)(1+|w|)(1-|z|)(1+|z|)$$

Since each one of these steps was invertible this inequality is equivalent to the original. At this point it is clear if either $|w|$ or $|z|$ is equal to one we have equality. Moreover when $|z| < 1$ and $|w| < 1$ we will have two negative terms and to positive terms making the inequality hold strictly.

(b) (i) If $|z| \leq 1$ then it is in the unit disc. From the above inequality $|F(z)| \leq 1$. Thus F is a map from the unit disc to itself.

For holomorphicity we multiply on the top and bottom by the conjugate of the bottom and convert it into terms of x and y to get

$$\frac{w - w^2\overline{z} - z + w|z|^2}{|1-\overline{w}z|^2} = \frac{\left(x^2+y^2\right)(a+i\,b) - (a+i\,b)(x-i\,y) + a + i\,b - x - i\,y}{(ax+by-1)^2 + (bx-ay)^2}$$

The real part is

$$\frac{ax^2 + ay^2 - ax - by + a - x}{a^2x^2 + b^2x^2 + a^2y^2 + b^2y^2 - 2\,ax - 2\,by + 1}$$

and the imaginary part is

$$\frac{bx^2 + by^2 - bx + ay + b - y}{a^2x^2 + b^2x^2 + a^2y^2 + b^2y^2 - 2\,ax - 2\,by + 1}$$

(ii) First calculate $F(0)$

$$\begin{aligned} F(0) &= \frac{w-0}{1-\overline{w}0} \\ &= \frac{w}{1} \\ &= w \end{aligned}$$

Calculating $F(w)$ we get

$$\begin{aligned} F(w) &= \frac{w-w}{1-|w|^2} \\ &= 0 \end{aligned}$$

(iii) From the second part of (a) if $|z| = 1$ then $F(z) = 1$.

(iv) To show it is a bijection we calculate $F \circ F(z)$:

$$\begin{aligned} F \circ F(z) &= \frac{w + \frac{z-w}{1-\overline{w}z}}{1 + \overline{w}\frac{z-w}{1-\overline{w}z}} \\ &= \frac{\frac{w(1-\overline{w}z)+z-w}{1-\overline{w}z}}{\frac{1-\overline{w}z+\overline{z}-\overline{w}w}{1-\overline{w}z}} \\ &= \frac{-|w|^2z + z}{1-|w|^2} \\ &= \frac{z(1-|w|^2)}{1-|w|^2} \\ &= z \end{aligned}$$

□

Problem 2 (1.4.9)**.** *Show that in polar coordinates, the Cauchy-Riemann equations take the form*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \textit{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \textit{where } z = re^{i\theta} \textit{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. Using the chain rule and relations $x = r\cos\theta$ and $y = r\sin\theta$ we get:

$$\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta \\
u_\theta &= -u_x r \sin\theta + u_y r \cos\theta \\
v_r &= v_x \cos\theta + v_y \sin\theta \\
v_\theta &= -v_x r \sin\theta + v_y r \cos\theta
\end{aligned}$$

At this point we have

$$\frac{1}{r}(v_\theta) = -v_x \sin\theta + v_y \cos\theta = u_y \sin\theta + u_x \cos\theta = u_r$$

and similarly

$$\frac{1}{r} u_\theta = -u_x \sin\theta + u_y \cos\theta = -v_y \sin\theta - v_x \cos\theta = -v_r$$

□

Taking the derivatives for $u = \log r$ and $v = \theta$ with respect to r and θ we get

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{1}{r} \\
\frac{\partial u}{\partial \theta} &= 0 \\
\frac{\partial v}{\partial r} &= 0 \\
\frac{\partial v}{\partial \theta} &= 1
\end{aligned}$$

Which fulfill the polar Cauchy-Riemann equations. Thus $\log z$ is in fact holomorphic.

Problem 3 (1.4.10)**.** *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the ***Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. First recall that

$$f_z = 1/2 f_x - i/2 f_y$$

and

$$f_{\overline{z}} = 1/2 f_x + i/2 f_y$$

Then we compute:

$$\begin{aligned}4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} &= 4\frac{\partial}{\partial z}(1/2 f_x + i/2 f_y)\\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy})\\ &= f_{xx} + f_{yy} = \Delta f\end{aligned}$$

and the other

$$\begin{aligned}4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} &= 4\frac{\partial}{\partial \bar{z}}(1/2 f_x - i/2 f_y)\\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy})\\ &= f_{xx} + f_{yy} = \Delta f\end{aligned}$$

□

Problem 4 (1.4.13)**.** *Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:*

(a) $\Re(f)$ is constant;

(b) $\Im(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Proof. Separate $f = u + iv$.

(a) Since u is constant $u_x = u_y = 0$. However using the Cauchy-Riemann equations we have that $v_x = v_y = 0$ as well. Since u and v are constant so is f.

(b) Since v is constant $v_x = v_y = 0$. However using the Cauchy-Riemann equations we have that $u_x = u_y = 0$ as well. Since u and v are constant so is f.

(c) Since $|f|$ is constant so is $|f|^2 = u^2 + v^2$. Take the derivative with respect to x and divide by two to get

$$uu_x + vv_x = 0$$

and for y

$$uu_y + vv_y = 0$$

Next rewrite the latter with the Cauchy-Riemann equations

$$-uv_x + vu_x = 0$$

Then multiply the first equation by u, the latter by v, and add them together to get

$$u(uu_x + vv_y)v(-uv_x + vu_x) = (u^2 + v^2)u_x = 0$$

Since $u^2 + v^2$ is a constant this implies that $u_x = v_y = 0$.

We can do the same thing by rewriting the first to get

$$(u^2 + v^2)v_x = 0$$

Which implies that $v_x = -u_y = 0$. This gives us that u and v are constant and as such so is f.

□

Problem 5 (1.4.17)**.** *Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

We prove the alternate exercise given in class.

Proof. Let $R := \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. We will show that R is the radius of convergence for the power series $\sum_0^\infty a_n(z - z_0)$.

Let $|z - z_0| < R$ and $r := |z - z_0|$

$$\left| \sum_0^\infty a_n(z - z_0)^n \right| \leq \sum_0^\infty |a_n| r^n$$

Then we apply the ratio test to the latter sum to get

$$\lim_{n\to\infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} < 1$$

which shows that the sum converges.

Now suppose that $|z - z_0| > R$. Then apply the ratio test to the power series to get

$$\lim_{n\to\infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} > 1$$

which shows that the series diverges.

Therefore R is in fact the radius of convergence. □

I suppose a way of proving the original would be to use the sequence as a power series and apply the convergence tests with Hadarmard's rule.

Problem 6 (2.6.1)**.** *Prove that*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

These are the ***Fresnel Integrals****. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$.*
Hint: Integrate the function e^{-x^2} over the $\pi/4$ semicircle thing. Recall that $\int_{-\infty}^\infty e^{-x^2} = \sqrt{\pi}$.

Proof. Let γ_R denote the curve given in the supplied figure of radius R. Then parametrize the curve piecewise so we have:

$$\oint_{\gamma_R} e^{-iz^2} = \int_0^R e^{it^2} dt + \int_0^{\pi/4} iR \operatorname{cis}\theta e^{i(R\operatorname{cis}\theta)^2} d\theta + \int_0^R - \operatorname{cis}\pi/4 e^{\operatorname{cis}(\pi/4)(R-t))^2} dt$$

The integral about γ_R is zero as the function contained within is holomorphic. Now we will show that

$$\lim_{R\to\infty}\int_0^{\pi/4} iR\operatorname{cis}\theta e^{i(R\operatorname{cis}\theta)^2}d\theta = 0$$

$$\begin{aligned}
\left|\int_0^{\pi/4} iR\operatorname{cis}\theta e^{iR^2\operatorname{cis}^2\theta}d\theta\right| &\leq \int_0^{\pi/4} R\left|e^{iR^2\operatorname{cis}2\theta}\right|d\theta \\
&= \int_0^{\pi/4} R\left|e^{iR^2\cos 2\theta}\right|\left|e^{-R^2\sin 2\theta}\right|d\theta \\
&\leq \int_0^{\pi/4} R|\operatorname{cis}(R^2\cos 2\theta)|e^{-R^2\sin 2\theta}d\theta \\
&\leq \int_0^{\pi/4} Re^{-R^2\sin 2\theta}d\theta
\end{aligned}$$

Next we use the fact that $\sin 2\theta \geq \theta$ when $0 \leq \theta \leq \pi/4$ to get

$$\begin{aligned}
\int_0^{\pi/4} Re^{-R^2\sin 2\theta}d\theta &\leq \int_0^{\pi/4} Re^{-R^2\theta} \\
&= \left(\frac{Re^{-R^2\theta}}{-R^2}\right|_0^{\pi/4} \\
&= \frac{e^{-R^2\pi/4}-1}{R}
\end{aligned}$$

This last term goes to zero as R approaches infinity.

Next we evaluate $\int_0^R -\operatorname{cis}(\pi/4)e^{i(R-t)^2\operatorname{cis}(\pi/2)}dt$.

$$\begin{aligned}
\int_0^R -\operatorname{cis}(\pi/4)e^{i(R-t)^2\operatorname{cis}^2(\pi/4)}dt &= \int_0^R -\operatorname{cis}(\pi/4)e^{i(R-t)^2\operatorname{cis}(\pi/2)}dt \\
&= \int_0^R -\operatorname{cis}(\pi/4)e^{-(R-t)^2)}dt
\end{aligned}$$

Then we apply the reminder in the problem giving that:

$$\int_0^R -\operatorname{cis}(\pi/4)e^{-(R-t)^2)}dt = \operatorname{cis}(\pi/4)\frac{\sqrt{\pi}}{2} = -\left(\frac{\sqrt{2\pi}}{4}+i\frac{\sqrt{2\pi}}{4}\right)$$

Which gives us

$$\lim_{R\to\infty}\int_0^R e^{it^2}dt = \lim_{R\to\infty}\int_0^R \cos t^2 + i\sin t^2 dt = \left(\frac{\sqrt{2\pi}}{4}+i\frac{\sqrt{2\pi}}{4}\right)$$

If we take the real part of the above equation we get

$$\int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

and if we take the imaginary part we get

$$\int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

$\square$

Problem 7 (2.6.11)**.** *Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.*

Proof. (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi} \Re\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right))d\varphi.$$

(b) Show that

$$\Re\left(\frac{Re^{i\gamma} - r}{Re^{i\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

Hint: For the first part, note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity. $\square$