Problem 1 (2.6.14). Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Only need to do case where degree of pole is greater than 1.

Proof. We proceed by induction over the degree of the pole. The case of the simple pole was provided. Suppose that the above property holds for poles of degree k. Then if z_0 is a pole of degree $k+1$ we can construct a function

$$g(z) = (z - z_0)f(z)$$

The function g has a pole of degree k. By assumption, for the power series $g(z) = \sum_0^\infty b_n z^n$, we have that $\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = z_0$. Since power series are unique. We can write the coefficients b_n in terms of a_n.

$$f(z)(z - z_0) = \sum_0^\infty (z - z_0)a_n z^n$$

This implies that $b_n = a_{n-1} - a_n z_0$ where $a_{-1} = 0$. Now we rewrite the limit as

$$\lim_{n\to\infty} \frac{b_n}{b_{n+1}} = \lim_{n\to\infty} \frac{a_{n-1} - a_n z_0}{a_n - a_{n+1}z_0} = z_0$$

and use it to show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

To do this first we factor out a_n from the top and a_{n+1} from the bottom. This gives us

$$\lim_{n\to\infty} \frac{a_{n-1} - a_n z_0}{a_n - a_{n+1}z_0} = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \cdot \frac{\frac{a_{n-1}}{a_n} - z_0}{\frac{a_n}{a_{n+1}} - z_0}$$

We know that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}}$ exists since it is the limit of ratios for a power series. Call this limit C. Then introduce the constant $0 < \epsilon < 1$ such that

$$\frac{a_n \epsilon^n}{a_{n+1}\epsilon^{n+1}} = C(1 + \delta(n))$$

Where $1 + \delta(n)$ depends solely on ϵ and will converge to 1. Rewriting the original limit we get

$$\lim_{n\to\infty} \frac{a_n \epsilon^n}{a_{n+1}\epsilon^{n+1}} \cdot \frac{\frac{a_{n-1}\epsilon^{n-1}}{a_n \epsilon^n} - z_0}{\frac{a_n \epsilon^n}{a_{n+1}\epsilon^{n+1}} - z_0}$$

Then we have

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}\epsilon} \cdot \frac{C(1 + \delta(n)) - z_0}{C(1 + \delta(n)) - z_0}$$

The limit of the right hand term is 1. Take the limit as ϵ approaches 1 and we get

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$$

as desired. ∎

Problem 2 (3.8.2). Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} dx$$

Where are the poles of $1/(1+z^4)$?

The poles of the function are at $\pm e^{i\pi/4}, \pm e^{3i\pi/4}$.

We evaluate this integral by integrating over the semicircle in the upper half plane of radius R which we will call γ_R. Then we have

$$\int_{\gamma_R} \frac{1}{1+z^4} dz = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

However since $\frac{1}{1+z^4} \leq \frac{1}{R^4-1}$ on C_R we have

$$\left| \int_{C_R} \frac{1}{1+z^4} dz \right| \leq \frac{\pi R}{R^4 - 1}$$

which approaches 0 as R approaches infinity. This gives us that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

To calculate the residues we use the formula from the book. In this case the formula boils down to plugging in the point to $\frac{1}{1+z^4}$ where removing the factor associated with the singularity. This gives us

$$\operatorname{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{2\pi i/4} + e^{\pi i/2})(2e^{i\pi/4})} = \frac{1}{4e^{3\pi i/4}}$$

and

$$\operatorname{Res}_{z=e^{3i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{6\pi i/4} + e^{3\pi i/2})(2e^{3i\pi/4})} = \frac{1}{4e^{9\pi i/4}}$$

Add them together and we get $\frac{-i\sqrt{2}}{4}$. Multiply it by $2\pi i$ and we get $\frac{\pi}{2}$. Which gives us that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

Problem 3 (3.8.4). Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0$$

Proof. First note that $\Im(\frac{ze^{iz}}{z^2+a^2}) = \frac{z \sin z}{z^2+a^2}$ when z is on the real axis. We proceed by integrating the semicircle in the upper half plane, γ_R, which we can expand as

$$\int_{\gamma_R} \frac{ze^{iz}}{z^2+a^2} dz = \int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz + \int_{-R}^{R} \frac{ze^{iz}}{z^2+a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2+a^2}$$

First we will show that $\left|\int_{C_R} \frac{ze^{iz}}{z^2+a^2}dz\right| \to 0$ as $R \to \infty$. Start by making the substitution $z = Re^{i\theta}$. Then we get

$$\begin{aligned}
\left|\int_{C_R} \frac{ze^{iz}}{z^2+a^2}dz\right| &= \left|\int_0^\pi \frac{Re^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2}Rie^{i\theta}d\theta\right| \\
&\leq \left|\int_0^\pi \frac{iR^2e^{2i\theta}e^{iRe^{i\theta}}}{R^2-a^2}d\theta\right| \\
&\leq \left|\int_0^\pi \frac{R^2e^{iRe^{i\theta}}}{R^2-a^2}d\theta\right| \\
&\leq \frac{R^2}{R^2-a^2}\left|\int_0^\pi e^{iR\cos\theta - R\sin\theta}d\theta\right| \\
&\leq \frac{R^2}{R^2-a^2}\left|\int_0^\pi e^{-R\sin\theta}d\theta\right|
\end{aligned}$$

Next note that $\sin\theta$ is symmetric about $\pi/2$. Together with the inequality $\sin\theta \geq 2/\pi\theta$ for $0 \leq \theta \leq \pi/2$ we get

$$\begin{aligned}
\frac{R^2}{R^2-a^2}\left|\int_0^\pi e^{-R\sin\theta}d\theta\right| &\leq \frac{2R^2}{R^2-a^2}\left(\int_0^{\pi/2} e^{-R\pi\theta/2}d\theta\right) \\
&= \frac{2R^2}{R^2-a^2}\left(\frac{-2e^{-R\pi/2\theta}}{\pi R}\Bigg|_0^{\pi/2}\right.
\end{aligned}$$

At this point it is clear that the above approaches zero as R approaches infinity. Thus

$$\int_{-R}^R \frac{ze^{iz}}{z^2+a^2}dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2+a^2}$$

To calculate the residue we compute

$$\begin{aligned}
\lim_{z\to ia}(z-ia)\frac{ze^{iz}}{z^2+a^2} &= \frac{iae^{-a}}{2ia} \\
&= \frac{e^{-a}}{2}
\end{aligned}$$

When we put it all together we get

$$\begin{aligned}
\int_{-\infty}^\infty \frac{x\sin x}{x^2+a^2}dx &= \Im\left(\int_{-\infty}^\infty \frac{ze^{iz}}{z^2+a^2}dz\right) \\
&= \Im(2\pi i \cdot \frac{e^{-a}}{2}) \\
&= \pi e^{-a}
\end{aligned}$$

$\square$

Problem 4 (3.8.8). Prove that

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Proof. We start by rewriting $\cos\theta$ in terms of $e^{i\theta}$ which gives us

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} &= \int_0^{2\pi}\frac{d\theta}{a+b/2(e^{i\theta}+e^{-i\theta})}\\ &= 2\int_0^{2\pi}\frac{e^{i\theta}}{2ae^{i\theta}+be^{2i\theta}+b}d\theta\end{aligned}$$

Then substitute $z=e^{i\theta}$ to get

$$\begin{aligned}2\int_0^{2\pi}\frac{e^{i\theta}}{2ae^{i\theta}+be^{2i\theta}+b}d\theta &= \frac{2}{i}\int_{C_1}\frac{1}{bz^2+2az+b}dz\\ &= 4\pi\,\mathrm{Res}\,\frac{1}{bz^2+2az+b}\end{aligned}$$

First we find the poles by factoring the bottom

$$\frac{1}{bz^2+2az+b} = \frac{1}{b(z-(-a/b+\sqrt{a^2/b^2-1}))(z-(-a/b-\sqrt{a^2/b^2-1}))}$$

The pole that occurs within the circle of radius 1 is $-a/b+\sqrt{a^2/b^2-1}$. We calculate the residue as

$$\begin{aligned}\mathrm{Res}_{z=-a/b+\sqrt{a^2/b^2-1}}\frac{1}{bz^2+2az+b} &= \lim_{z\to -a/b+\sqrt{a^2/b^2-1}}\frac{(z-(-a/b+\sqrt{a^2/b^2-1}))}{b(z-(-a/b+\sqrt{a^2/b^2-1}))(z-(-a/b-\sqrt{a^2/b^2-1}))}\\ &= \frac{1}{b(-a/b+\sqrt{a^2/b^2-1}-(-a/b-\sqrt{a^2/b^2-1}))}\\ &= \frac{1}{2b\sqrt{a^2/b^2-1}}\\ &= \frac{1}{2\sqrt{a^2-b^2}}\end{aligned}$$

Which when we plug back in we get

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \frac{4\pi}{2\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

□

Problem 5 (3.8.9). Show that

$$\int_0^1 \log(\sin\pi x)dx = -\log 2$$

Hint: Use contour that goes down to 0 and up from 1.

Proof. First note that

$$\begin{aligned}\log(1-e^{2\pi iz}) &= \log(e^{\pi iz}(-2i)(e^{\pi iz}+e^{-\pi iz})/2i)\\ &= \pi iz+\log(-2i)+\log(\sin\pi z)\\ &= \pi iz+\log(2)-i\pi/2+\log(\sin\pi z)\end{aligned}$$

If we integrate this with respect to z from 0 to 1 we get

$$\int_0^1 \log(1-e^{2\pi iz})dz = \pi i/2 + \log(2) - i\pi/2 + \int_0^1 \log(\sin \pi z)dz = \log(2) + \int_0^1 \log(\sin \pi z)dz$$

This will give us the desired equality if we show that $\int_0^1 \log(1-e^{2\pi iz})dz = 0$. To do this we integrate over the rectangle of width 1, height R, with two quarter circles of radius ϵ on the corners at 0 and 1 to avoid the branch points. Refer to this curve as $\gamma_{\epsilon,R}$. The entire integral will be zero as $\log(1-e^{2\pi iz})$ is holomorphic on the curve and its interior. Then we split the integral into six pieces

$$\begin{aligned}
0 = \int_{\gamma_{\epsilon,R}} \log(1-e^{2\pi iz})dz &= \int_\epsilon^{1-\epsilon} \log(1-e^{2\pi ix})dx && z = x \\
&+ \int_\pi^{\pi/2} \log(1-e^{2\pi i(1+\epsilon e^{i\theta})}) i\epsilon e^{i\theta} d\theta && z = 1+\epsilon e^{i\theta} \\
&+ i\int_\epsilon^R \log(1-e^{2\pi i(1+it)})dt && z = 1+it \\
&+ \int_1^0 \log(1-e^{2\pi i(t+iR)})dt && z = t+iR \\
&+ i\int_R^\epsilon \log(1-e^{2\pi i(it)})dt && z = it \\
&+ \int_{\pi/2}^0 \log(1-e^{2\pi i\epsilon e^{i\theta}})d\theta && z = \epsilon e^{i\theta}
\end{aligned}$$

First note that for the two vertical portions of $\gamma_{\epsilon,R}$, the third and fifth, that

$$\log(1-e^{2\pi i(1+it)}) = \log(1-e^{2\pi i(it)})$$

as $e^{2\pi i} = 1$. Since the only difference then is that the limits of integration are swapped these two cancel each other out. What is left to show is that

$$\left|\int_1^0 \log(1-e^{2\pi i(t+iR)})dt\right|, \quad \left|\int_\pi^{\pi/2} \log(1-e^{2\pi i\epsilon e^{i\theta}}) i\epsilon e^{i\theta} d\theta\right|, \quad \left|\int_{\pi/2}^0 \log(1-e^{2\pi i\epsilon e^{i\theta}})d\theta\right|$$

all approach zero as $\epsilon \to 0$ and $R \to \infty$.

Starting with the first we have

$$\begin{aligned}
\left|\int_0^1 \log(1-e^{2\pi i(t+iR)})dt\right| &\leq \int_0^1 \left|\log(1-e^{-2\pi R}e^{2\pi it})\right| dt \\
&\leq \int_0^1 \log|1-e^{-2\pi R}e^{2\pi it}| + |i\,\mathrm{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(|1|+|e^{-2\pi R}e^{2\pi it}|) + |i\,\mathrm{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |i\,\mathrm{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\,\mathrm{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\arctan(\frac{\Im(1-e^{-2\pi R}e^{2\pi it})}{\Re(1-e^{-2\pi R}e^{2\pi it})})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\arctan(\frac{e^{-2\pi R}\sin(2\pi t)}{1-e^{-2\pi R}\cos(2\pi t)})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}\sin(2\pi t)}{1-e^{-2\pi R}\cos(2\pi t)}|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}}{1-e^{-2\pi R}}|dt \\
&= \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}}{1-e^{-2\pi R}}|
\end{aligned}$$

which goes to zero as $R \to \infty$.

Next we show that $|\int_{\pi/2}^0 \log(1-e^{2\pi i\epsilon e^{i\theta}})d\theta| \to 0$ as $\epsilon \to 0$.

$$\begin{aligned}
|\int_{\pi/2}^0 \log(1-e^{2\pi i\epsilon e^{i\theta}})\epsilon i e^i\theta d\theta| &\leq \int_0^{\pi/2} |\log(1-e^{2\pi i\epsilon e^{i\theta}})\epsilon i e^{i\theta}|d\theta \\
&\leq \epsilon\int_0^{\pi/2} |\log(1-e^{2\pi i\epsilon e^{i\theta}})|d\theta \\
&\leq \epsilon\int_0^{\pi/2} |\log|1-e^{2\pi i\epsilon e^{i\theta}}| + i\,\mathrm{Arg}(1-e^{2\pi i\epsilon e^{i\theta}})|d\theta \\
&\leq \epsilon\int_0^{\pi/2} \log(1+|e^{2\pi i\epsilon e^{i\theta}}|) + |\,\mathrm{Arg}(1-e^{2\pi i\epsilon e^{i\theta}})|d\theta \\
&\leq \epsilon\int_0^{\pi/2} \log(1+|e^{2\pi i\epsilon e^{i\theta}}|) + \pi d\theta \\
&\leq \epsilon\int_0^{\pi/2} \log(1+e^{-2\pi\epsilon\sin\theta}) + \pi d\theta \\
&\leq \epsilon\int_0^{\pi/2} \log(1+e^{2\pi\epsilon}) + \pi d\theta \\
&= \epsilon\cdot\frac{\pi}{2}\log(1+e^{2\pi\epsilon}) + \epsilon\frac{\pi^2}{2}
\end{aligned}$$

Which at this point clearly goes to zero as $\epsilon \to 0$. The last quarter-circle integral is identical once we note that

$$\int_\pi^{\pi/2} \log(1-e^{2\pi i(1+\epsilon e^{i\theta})})i\epsilon e^{i\theta}d\theta = \int_\pi^{\pi/2} \log(1-e^{2\pi i(\epsilon e^{i\theta})})i\epsilon e^{i\theta}d\theta$$

We can then bound this integral the same as the prior. Since the arc length of the quarter-circle is the same we can safely conclude that this integral also approaches zero as $\epsilon \to 0$.

Therefore, since every portion of $\int_{\gamma_{\epsilon,R}} \log(1-e^{2\pi i z})$ is zero aside from $\int_{-\epsilon}^{\epsilon} \log(\sin(\pi x))dx$, we can conclude that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

□

Problem 6 (3.8.10). Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2} dx = \frac{\pi}{2a} \log a$$

Hint: Integrate over upper half annulus with inner radius ϵ and outer radius R.

Proof. Let $\gamma_{\epsilon,R}$ denote the curve around half annulus in the upper half plane with inner radius ϵ and outer radius R. The function $\frac{\log x}{x^2+a^2}$ has a pole at ia. Then the integral over $\gamma_{\epsilon,R}$ is

$$\begin{aligned}
\int_{\gamma_{\epsilon,R}} \frac{\log z}{z^2+a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{\log z}{z^2+a^2} \\
&= \int_{C_R} \frac{\log z}{z^2+a^2} dz \\
&+ \int_{C_\epsilon} \frac{\log z}{z^2+a^2} dz \\
&+ \int_\epsilon^R \frac{\log x}{x^2+a^2} dx \\
&- \int_R^\epsilon \frac{\log(-x)}{x^2+a^2} dx
\end{aligned}$$

First we show that the integral on C_R approaches 0 as $R \to \infty$.

$$\begin{aligned}
|\int_{C_R} \frac{\log z}{z^2+a^2} dz| &\le \int_{C_R} |\frac{\log z}{z^2+a^2}| dz \\
&\le \int_{C_R} \frac{|\log R + i\pi}{R^2-a^2} dz \\
&\le \frac{\pi R(\log R + i\pi)}{R^2-a^2}
\end{aligned}$$

This approaches zero as R approaches infinity.

Next we show that the integral along C_ϵ approaches 0 as $\epsilon \to 0$.

$$\begin{aligned}
|\int_{C_\epsilon} \frac{\log z}{z^2+a^2} dz| &\le \int_{C_\epsilon} |\frac{\log z}{z^2+a^2}| dz \\
&\le \int_{C_\epsilon} |\log \epsilon + i\pi| dz \\
&\le \int_{C_\epsilon} \log \epsilon + \pi dz \\
&\le \pi\epsilon(\log \epsilon + \pi)
\end{aligned}$$

which also approaches 0 as ϵ approaches 0.

The residue calculation is

$$2\pi i \operatorname{Res}_{z=ia} \frac{\log z}{z^2+a^2} = 2\pi i \frac{\log ia}{2ia} = \frac{\pi \log a}{a} + \frac{i\pi^2}{2a}$$

Note that for the last integral, since we are on the principle branch we have

$$-\int_R^\epsilon \frac{\log(-x)}{x^2+a^2}dx = \int_\epsilon^R \frac{\log x}{x^2+a^2}dx + \int_\epsilon^R \frac{i\pi}{x^2+a^2}dx$$

Which gives us that

$$2\int_0^\infty \frac{\log x}{x^2+a^2}dx + i\pi \int_0^\infty \frac{1}{x^2+a^2}dx = \frac{\pi \log a}{a} + \frac{i\pi^2}{2a}$$

Since we know that $i\pi \int_0^\infty \frac{1}{x^2+a^2}dx = \frac{i\pi^2}{2a}$ we get

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi \log a}{2a}$$

as desired. □

Problem 7 (3.8.13). Suppose f is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z-z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Proof. We show this by proving the contrapositive. Suppose z_0 is a singularity that is not removable and let $\epsilon > 0$ and $A \in \mathbb{R}_{>0}$. Then it is either a pole of order k or essential.

If z_0 is a pole of order k then we can write f as $f(z) = \sum_{n=0}^{k} \frac{a_{-n}}{(z-z_0)^n} \cdot g(z)$ where g is holomorphic on $D_r(z_0)$. Then there exists a constant B such that $B|(z-z_0)^{-k}| < |f(z)|$ when z is sufficiently close to z_0. In addition when z is sufficiently close to $A|z-z_0|^{-1+\epsilon} \leq B|z-z_0|^{-k}$ which then implies that $A|z-z_0|^{-1+\epsilon} \leq |f(z)|$ when sufficiently close.

On the other hand if k is an essential singularity we have a Laurent series in $D_r(z_0) \setminus \{z_0\}$ of the form

$$f(z) = \sum_0^\infty b_n(z-z_0)^{-n} + \sum_0^\infty a_n(z-z_0)^n$$

If we cut off the series for the negative powers at the first nonzero coefficient we get

$$g(z) = b_k(z-z_0)^k + \sum_0^\infty a_n(z-z_0)^n$$

where $B|g| < |f|$ sufficiently close to z_0 for some nonzero constant B. Since g has a pole of order k at z_0 this then reduces to the case where we have a pole of order k where we shrink the radius of the disk such that $|g| < |f|$ holds within. □

Problem 8 (3.9.3). If f is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$.

Proof. Since f has a pole of order k we can write f as

$$f(z) = (z - z_0)^k g(z)$$

where g is holomorphic. Moreover since g is holomorphic it is equal to its power series $g(z) = \sum_0^\infty a_n(z - z_0)^n$. Then we distribute to get

$$f(z) = \sum_{n=0}^{k-1} \frac{a_n}{(z - z_0)^{k-n}} + \sum_0^\infty a_{n+k}(z - z_0)^n$$

completing the proof. □