Problem 1 (1.4.7)**.** *The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called* ***Blaschke factors****, will reappear in various applications in later chapters.*

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \leq (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mappings

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. Hint: Calculate $F \circ F$.

Proof. (a) Since $\overline{z}w \neq 1$ we can change the inequality from

$$\left|\frac{w-z}{1-\overline{w}z}\right| \leq 1$$

to

$$|w-z| \leq |1-\overline{w}z|$$

Then if we square both sides we can use the properties of the conjugate to get

$$(w-z)\overline{(w-z)} \leq (1-\overline{w}z)\overline{(1-\overline{w}z)}$$

Distribute the conjugate over the sum to get

$$(w-z)(\overline{w}-\overline{z}) \leq (1-\overline{w}z)(1-w\overline{z})$$

Multiply out to get

$$|w|^2 - z\overline{w} - w\overline{z} + |z|^2 \leq 1 - \overline{w}z - \overline{z}w + |w|^2|z|^2$$

Shuffle everything to the right and we get

$$0 \leq 1 - |w|^2|z|^2 - |w|^2 - |z|^2 = (1-|w|^2)(1-|z|^2) = (1-|w|)(1+|w|)(1-|z|)(1+|z|)$$

Since each one of these steps was invertible this inequality is equivalent to the original. At this point it is clear if either $|w|$ or $|z|$ is equal to one we have equality. Moreover when $|z| < 1$ and $|w| < 1$ we will have two negative terms and to positive terms making the inequality hold strictly.

(b) (i) If $|z| \leq 1$ then it is in the unit disc. From the above inequality $|F(z)| \leq 1$. Thus F is a map from the unit disc to itself.

For holomorphicity we know that the quotient of holomorphic functions is holomorphic. Moreover so is addition, multiplication by a constant. Since the identity map is holomorphic this means that $w - z$ and $1 - \overline{w}z$ are both holomorphic and as such $F(z) = \frac{w-z}{1-\overline{w}z}$ is holomorphic as a function of z.

(iii) From the second part of (a) if $|z| = 1$ then $F(z) = 1$.

(iv) To show it is a bijection we calculate $F \circ F(z)$:

$$
\begin{aligned}
F \circ F(z) &= \frac{w + \frac{z-w}{1-\overline{w}z}}{1 + \overline{w}\frac{z-w}{1-\overline{w}z}} \\
&= \frac{\frac{w(1-\overline{w}z)+z-w}{1-\overline{w}z}}{\frac{1-\overline{w}z+\overline{z}-\overline{w}w}{1-\overline{w}z}} \\
&= \frac{-|w|^2 z + z}{1 - |w|^2} \\
&= \frac{z(1 - |w|^2)}{1 - |w|^2} \\
&= z
\end{aligned}
$$

Since the function is its own inverse it is indeed a bijection.

□

Problem 2 (1.4.9). *Show that in polar coordinates, the Cauchy-Riemann equations take the form*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \textit{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \textit{where } z = re^{i\theta} \textit{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. Using the chain rule and relations $x = r\cos\theta$ and $y = r\sin\theta$ we get:

$$
\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta \\
u_\theta &= -u_x r \sin\theta + u_y r \cos\theta \\
v_r &= v_x \cos\theta + v_y \sin\theta \\
v_\theta &= -v_x r \sin\theta + v_y r \cos\theta
\end{aligned}
$$

At this point we have

$$\frac{1}{r}(v_\theta) = -v_x \sin\theta + v_y \cos\theta = u_y \sin\theta + u_x \cos\theta = u_r$$

and similarly

$$\frac{1}{r}u_\theta = -u_x \sin\theta + u_y \cos\theta = -v_y \sin\theta - v_x \cos\theta = -v_r$$

□

Taking the derivatives for $u = \log r$ and $v = \theta$ with respect to r and θ we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}$$
$$\frac{\partial u}{\partial \theta} = 0$$
$$\frac{\partial v}{\partial r} = 0$$
$$\frac{\partial v}{\partial \theta} = 1$$

Which fulfill the polar Cauchy-Riemann equations. Thus $\log z$ is in fact holomorphic.

Problem 3 (1.4.10)**.** *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the ***Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. First recall that

$$f_z = 1/2 f_x - i/2 f_y$$

and

$$f_{\overline{z}} = 1/2 f_x + i/2 f_y$$

Then we compute:

$$\begin{aligned} 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f &= 4\frac{\partial}{\partial z}(1/2 f_x + i/2 f_y) \\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy}) \\ &= f_{xx} + f_{yy} = \Delta f \end{aligned}$$

and similarly

$$\begin{aligned} 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} f &= 4\frac{\partial}{\partial \overline{z}}(1/2 f_x - i/2 f_y) \\ &= 4(1/4 f_{xx} + i/4 f_{yx} - i/4 f_{xy} + 1/4 f_{yy}) \\ &= f_{xx} + f_{yy} = \Delta f \end{aligned}$$

□

Problem 4 (1.4.13)**.** *Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:*

(a) $\Re(f)$ is constant;

(b) $\Im(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Proof. Separate $f = u + iv$.

(a) Since u is constant $u_x = u_y = 0$. However using the Cauchy-Riemann equations we have that $v_x = v_y = 0$ as well. Since u and v are constant so is f.

(b) Since v is constant $v_x = v_y = 0$. However using the Cauchy-Riemann equations we have that $u_x = u_y = 0$ as well. Since u and v are constant so is f.

(c) Since $|f|$ is constant so is $|f|^2 = u^2 + v^2$. Take the derivative with respect to x and divide by two to get

$$uu_x + vv_x = 0$$

and for y

$$uu_y + vv_y = 0$$

Next rewrite the latter with the Cauchy-Riemann equations

$$-uv_x + vu_x = 0$$

Then multiply the first equation by u, the latter by v, and add them together to get

$$u(uu_x + vv_y)v(-uv_x + vu_x) = (u^2 + v^2)u_x = 0$$

Since $u^2 + v^2$ is a constant this implies that $u_x = v_y = 0$.

We can do the same thing by rewriting the first to get

$$(u^2 + v^2)v_x = 0$$

Which implies that $v_x = -u_y = 0$. This gives us that u and v are constant and as such so is f.

□

Problem 5 (1.4.17). *Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

We prove the alternate exercise given in class.

Proof. Let $R := \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. We will show that R is the radius of convergence for the power series $\sum_0^\infty a_n(z - z_0)$.

Let $|z - z_0| < R$ and $r := |z - z_0|$

$$\left|\sum_0^\infty a_n(z - z_0)^n\right| \le \sum_0^\infty |a_n| r^n$$

Then we apply the ratio test to the latter sum to get

$$\lim_{n\to\infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} < 1$$

which shows that the sum converges.

Now suppose that $|z - z_0| > R$. Then apply the ratio test to the power series to get

$$\lim_{n\to\infty} \frac{|a_{n+1}|r}{|a_n|} = \frac{r}{R} > 1$$

which shows that the series diverges.

Therefore R is in fact the radius of convergence. $\square$

I suppose a way of proving the original would be to use the sequence as a power series and apply the convergence tests with Hadarmard's rule.

Problem 6 (2.6.1)**.** *Prove that*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

*These are the **Fresnel Integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.*
Hint: Integrate the function e^{-x^2} over the $\pi/4$ semicircle thing. Recall that $\int_{-\infty}^\infty e^{-x^2} = \sqrt{\pi}$.

Proof. Let γ_R denote the curve given in the supplied figure of radius R. Then parametrize the curve piecewise so we have:

$$\oint_{\gamma_R} e^{-iz^2} = \int_0^R e^{it^2}dt + \int_0^{\pi/4} iR\operatorname{cis}\theta e^{i(R\operatorname{cis}\theta)^2}d\theta + \int_0^R -\operatorname{cis}\pi/4 e^{\operatorname{cis}(\pi/4)(R-t))^2}dt$$

The integral about γ_R is zero as the function contained within is holomorphic. Now we will show that

$$\lim_{R\to\infty}\int_0^{\pi/4} iR\operatorname{cis}\theta e^{i(R\operatorname{cis}\theta)^2}d\theta = 0$$

$$\begin{aligned}
\left|\int_0^{\pi/4} iR\operatorname{cis}\theta e^{iR^2\operatorname{cis}^2\theta}d\theta\right| &\le \int_0^{\pi/4} R\left|e^{iR^2\operatorname{cis}2\theta}\right|d\theta \\
&= \int_0^{\pi/4} R\left|e^{iR^2\cos 2\theta}\right|\left|e^{-R^2\sin 2\theta}\right|d\theta \\
&\le \int_0^{\pi/4} R|\operatorname{cis}(R^2\cos 2\theta)|e^{-R^2\sin 2\theta}d\theta \\
&\le \int_0^{\pi/4} Re^{-R^2\sin 2\theta}d\theta
\end{aligned}$$

Next we use the fact that $\sin 2\theta \geq \theta$ when $0 \leq \theta \leq \pi/4$ to get

$$\begin{aligned}
\int_0^{\pi/4} Re^{-R^2 \sin 2\theta} d\theta &\leq \int_0^{\pi/4} Re^{-R^2\theta} \\
&= \left(\frac{Re^{-R^2\theta}}{-R^2} \right|_0^{\pi/4} \\
&= \frac{e^{-R^2\pi/4} - 1}{R}
\end{aligned}$$

This last term goes to zero as R approaches infinity.

Next we evaluate $\int_0^R -\operatorname{cis}(\pi/4) e^{i(R-t)^2 \operatorname{cis}(\pi/2)} dt$.

$$\begin{aligned}
\int_0^R -\operatorname{cis}(\pi/4) e^{i(R-t)^2 \operatorname{cis}^2(\pi/4)} dt &= \int_0^R -\operatorname{cis}(\pi/4) e^{i(R-t)^2 \operatorname{cis}(\pi/2)} dt \\
&= \int_0^R -\operatorname{cis}(\pi/4) e^{-(R-t)^2)} dt
\end{aligned}$$

Then we apply the reminder in the problem giving that:

$$\int_0^R -\operatorname{cis}(\pi/4) e^{-(R-t)^2)} dt = \operatorname{cis}(\pi/4) \frac{\sqrt{\pi}}{2} = -\left(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4} \right)$$

Which gives us

$$\lim_{R\to\infty} \int_0^R e^{it^2} dt = \lim_{R\to\infty} \int_0^R \cos t^2 + i \sin t^2 dt = \left(\frac{\sqrt{2\pi}}{4} + i\frac{\sqrt{2\pi}}{4} \right)$$

If we take the real part of the above equation we get

$$\int_0^\infty \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}$$

and if we take the imaginary part we get

$$\int_0^\infty \sin(x^2) dx = \frac{\sqrt{2\pi}}{4}$$

$\square$

Problem 7 (2.6.11). *Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.*

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \Re \left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z} \right) d\varphi.$$

(b) *Show that*

$$\Re\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

Hint: For the first part, note that if $w = R^2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.

Proof. (a) First we write the Cauchy integral formula and subtract the quantity $0 = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta-\frac{R^2}{\overline{z}}}d\zeta$ to get

$$f(z) = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta)}{\zeta-\frac{R^2}{\overline{z}}}d\zeta$$

We do a substitution $\zeta = Re^{i\varphi}$ and get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left(\frac{Re^{i\varphi}}{Re^{i\varphi}-z} - \frac{Re^{i\varphi}}{Re^{i\varphi}-\frac{R^2}{\overline{z}}}\right)d\varphi$$

Now we want to show that

$$\left(\frac{Re^{i\varphi}}{Re^{i\varphi}-z} - \frac{Re^{i\varphi}}{Re^{i\varphi}-\frac{R^2}{\overline{z}}}\right) = \Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)$$

First note that

$$\frac{Re^{i\varphi}}{Re^{i\varphi}-\frac{R^2}{\overline{z}}}\cdot\frac{\overline{z}Re^{-i\varphi}\frac{1}{R}}{\overline{z}Re^{-i\varphi}\frac{1}{R}} = \frac{-\overline{z}}{\overline{\zeta}-\overline{z}}$$

Then if we sub back in for ζ in the prior equation we get

$$\begin{aligned}\frac{Re^{i\varphi}}{Re^{i\varphi}-z} - \frac{Re^{i\varphi}}{Re^{i\varphi}-\frac{R^2}{\overline{z}}} &= \frac{\zeta}{\zeta-z} + \frac{\overline{z}}{\overline{\zeta}-\overline{z}}\\ &= \frac{\zeta(\overline{\zeta}-\overline{z})+\overline{z}(\zeta-z)}{|\zeta-z|^2}\\ &= \frac{|\zeta|^2-|z|^2}{|\zeta-z|^2}\end{aligned}$$

Now we'll start working from the other end. Splitting $\zeta = a+ib$ and $z = x+iy$ we get

$$\begin{aligned}\Re\left(\frac{\zeta+z}{\zeta-z}\right) &= \Re\left(\frac{(a+x)+i(b+y)}{(a-x)+i(b-y)}\cdot\frac{(a-x)-i(b-y)}{(a-x)-i(b-y)}\right)\\ &= \frac{a^2+b^2-x^2-y^2}{(a-x)^2+(b-y)^2}\\ &= \frac{|\zeta|^2-|z|^2}{|\zeta-z|^2}\end{aligned}$$

Which completes the proof.

(b) First we rewrite $\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}$. Then we multiply by the conjugate on the bottom and take the real part.

$$\begin{aligned}\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r} &= \frac{R\cos\gamma+iR\sin\gamma+r}{R\cos\gamma+iR\sin\gamma-r}\cdot\frac{(R\cos\gamma-r)-iR\sin\gamma}{(R\cos\gamma-r)-iR\sin\gamma}\\ &= \frac{(R\cos\gamma+iR\sin\gamma+r)((R\cos\gamma-r)-i\sin\gamma)}{(R\cos\gamma-r)^2+R^2\sin^2\gamma}\end{aligned}$$

Then we take the real part to get

$$\begin{aligned}
\Re\left(\frac{(R\cos\gamma + iR\sin\gamma + r)((R\cos\gamma - r) - i\sin\gamma)}{(R\cos\gamma - r)^2 + R^2\sin^2\gamma}\right) &= \frac{R^2\cos^2\gamma + R^2\sin^2\gamma - r^2}{R^2\cos^2\gamma - 2Rr\cos\gamma + r^2 + R^2\sin^2\gamma} \\
&= \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}
\end{aligned}$$

□