Problem 1. *Consider the integral*

$$\Phi_n(z) = \int_a^b \frac{x^n dx}{(x-a)^{\frac{1}{2}-\lambda}(b-x)^{\frac{1}{2}+\lambda}(x-z)}$$

where $z \notin [a,b]$ *and* $-\frac{1}{2} < \lambda < \frac{1}{2}$.

(a) Compute the integral for $n = 0$ *and* $n = 1$.

(b) Determine the domain where $\phi_n(z)$ *is holomorphic.*

Hint: Consider the contour Γ *(circle entering to go about branch points) and fix a branch of the function* $\Gamma = C_R \cup \gamma^- \cup \gamma^+$ *and* $f(\zeta) = (\zeta - a)^{\frac{1}{2}-\lambda}(b-\zeta)^{\frac{1}{2}+\lambda}$ *where* $\zeta = x + iy$

(a) Let $f(w) = (w-a)^{1/2-\lambda}(b-w)^{1/2+\lambda}$ with $w = s + iy$. Then let $\widetilde{f}(w) = f_a(w)f_b(w)$ where

$$f_a(w) = |w-a|^{1/2-\lambda}e^{i\theta_a(1/2-\lambda)}, \quad f_b(w) = |b-w|^{1/2+\lambda}e^{i\theta_b(1/2+\lambda)}$$

with $\theta_a = \arg(w-a)$ and $\theta_b = \arg(b-w)$. Using the picture I will draw here

We do the work for finding branch cuts and the like to make it all consistent. We have a, b, ∞ as branch points with a branch cut along $[a,b]$.

$$\begin{aligned}
&z = s + i0 s > b && \theta_a = 0 \theta_b = -\pi && \widetilde{f}(s+i0) = -i|\widetilde{f}(s)|e^{-i\pi\lambda} \\
&z = s + i0 a \leq s \leq b && \theta_a = 0 \theta_b = 0 && \widetilde{f}(s) = |\widetilde{f}(s)| \\
&z = s + i0 s < a && \theta_a = 0 \theta_b = -\pi && \widetilde{f}(s+i0) = i|\widetilde{f}(s)|e^{-i\pi\lambda} \\
&z = s - i0 a \leq s \leq b && \theta_a = 2\pi \theta_b = 0 && \widetilde{f}(s-i0) = -|\widetilde{f}(s)|e^{-2i\pi\lambda} \\
&z = s - i0 b < s && \theta_a = 2\pi \theta_b = \pi && \widetilde{f}(s+i0) = -i|\widetilde{f}(s)|e^{-i\pi\lambda}
\end{aligned}$$

(b) The inside of the integral is holomorphic on $\mathbb{C}\backslash x$ for fixed $x \in (a,b)$. Moreover it is continuous on $\mathbb{C} \setminus [a,b] \times (a,b)$. As such we have that $\Phi_n(z)$ is holomorphic on $\mathbb{C} \setminus [a,b]$.

Now we integrate along the suggested curve. Since the function has only a single simple pole within the curve we have

$$\begin{aligned}
\int_\Gamma \frac{w^n dw}{\widetilde{f}(w)(w-z)} &= \left(\int_{C_R} + \int_{\gamma_+} + \int_{\gamma_-} + \int_{C_{\epsilon,a-}} + \int_{C_{\epsilon,b+}} + \int_{C_{\epsilon,b+}} + \int_{C_{\epsilon,b-}} \right) \frac{w^n dw}{\widetilde{f}(w)(w-z)} \\
&= 2\pi i \operatorname{Res}_{w=z} \frac{w^n}{\widetilde{f}(w)(w-z)}
\end{aligned}$$

We ignore the two fragments of the integral along $[b, R]$ since they cancel each other out. At this point the cases for $n = 0$ and $n = 1$ diverge. When $n = 0$ it the integral along C_R and the various C_ϵs vanish since there are no terms on top. When $n = 1$ I'm not sure how to get them to vanish as they appear to grow at the same rate.

Regardless, as the pole is simple the residue is

$$2\pi i \operatorname{Res}_{w=z} \frac{w^n}{\widetilde{f}(w)(w-z)} = 2\pi i \frac{z^n}{\widetilde{f}(z)}$$

The last bit is to combine the two integrals along γ_+ and γ_-. Using the values deduced earlier we have

$$\int_{\gamma_+} \frac{w^n dw}{\widetilde{f}(w)(w-z)} + \int_{\gamma_-} \frac{w^n dw}{\widetilde{f}(w)(w-z)} = \int_a^b \frac{w^n dw}{\widetilde{f}(s+i0)(w-z)} + \int_b^a \frac{w^n dw}{\widetilde{f}(s-i0)(w-z)} = (1+e^{2\pi i\lambda}) \int_a^b \frac{s^n ds}{f(s)(s-z)}$$

Putting it all together we get, for $n = 0, 1$ that

$$\int_a^b \frac{s^n ds}{f(s)(s-z)} = \frac{2\pi i z^n}{\widetilde{f}(z)(1+e^{2\pi i\lambda})}$$

Problem 2. *Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}.$$

Proof. We start with the first integral in the case where $\xi \geq 0$. In this scenario we integrate over the lower semicircle of radius R. The integral of the outer portion C_{R^-} will approach zero by Jordan's lemma as $|\frac{a}{a^2+z^2}| \leq \frac{a}{R^2-a^2} \to 0$ as $R \to \infty$. Thus we have

$$\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

The residue is

$$2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = 2i \frac{a}{-2ia} e^{-2\pi i(-ia)|\xi|} = -e^{-2\pi a\xi}$$

Which, after swapping the bounds of integration, implies that when $\xi > 0$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}$$

However when $\xi < 0$ we instead integrate over the upper semicircle of radius R. For the same reason as above the outer portion of the integral approaches 0 as $R \to \infty$. This gives us

$$\frac{1}{\pi} \int_{-\infty}^{I} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

Similar to before if we do the residue calculation we get

$$2i \operatorname{Res}_{z=ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = 2i \frac{a}{2ia} e^{2\pi a\xi} = e^{-2\pi a\xi}$$

Therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}e^{-2\pi i x\xi}dx = e^{-2\pi a|\xi|},$$

for all $\xi \in \mathbb{R}$.

To check the other direction split the integral into two pieces. The first for positive numbers and the other the negatives.

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi = \int_{0}^{\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi - \int_{0}^{-\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi$$

Substitute $\zeta = -\xi$ into the latter to get

$$\begin{aligned}\int_{0}^{\infty} e^{-2\pi a\xi}e^{2\pi i\xi x}d\xi - \int_{0}^{-\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi &= \int_{0}^{\infty} e^{-2\pi a\xi}e^{2\pi i\xi x}d\xi + \int_{0}^{\infty} e^{-2\pi a\zeta}e^{-2\pi i\zeta x}d\zeta \\ &= \int_{0}^{\infty} e^{(-2\pi a+2\pi i x)\xi}d\xi + \int_{0}^{\infty} e^{(-2\pi a-2\pi i x)\zeta}d\zeta \\ &= \frac{1}{2\pi a-2i\pi x}+\frac{1}{2\pi a+2i\pi x} \\ &= \frac{1}{\pi}\frac{a}{a^2+x^2}\end{aligned}$$

Which completes the second part of the problem. $\square$

Problem 3. *Prove that*

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a > 0$*. Hence show that the sum equals both* $\coth \pi a$*.*

Proof. By the previous problem and the fact that $\frac{a}{a^2+x^2} \in \mathcal{F}$ the two above sums are equal by the Poisson summation formula. Now we show that this sum is equal to $\coth \pi a$.

$$\begin{aligned}\coth \pi a &= \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} \\ &= \frac{e^{\pi a}(1+e^{-2\pi a})}{e^{\pi a}(1-e^{-2\pi a})} \\ &= (1+e^{-2\pi a})\frac{1}{1-e^{-2\pi a}} \\ &= (1+e^{-2\pi a})\sum_{n=0}^{\infty} e^{-2n\pi a} \\ &= \sum_{n=0}^{\infty} e^{-2n\pi a}+\sum_{n=1}^{\infty} e^{-2n\pi a} \\ &= \sum_{n=-\infty}^{\infty} e^{-2|n|\pi a}\end{aligned}$$

$\square$

Problem 4. *(a) Let τ be fixed with $\Im(\tau) > 0$. Apply the Poisson summation formula to*

$$f(z) = (\tau + z)^{-k},$$

where k is an integer ≥ 2, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau}.$$

(b) Set $k = 2$ in the above formula to show that if $\Im(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can one conclude that the above formula hold true whenever τ is any complex number that is not an integer?

(a) Since $f \in \mathcal{F}$ when $k \geq 2$ we can use the Poisson summation formula to show the equivalence of these two series. The Fourier transform of f is of the form

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx$$

When $\xi < 0$ we calculate the above integral by integrating over the upper semicircle of radius R. The outer portion C_R will approach 0 by Jordan's lemma as $|f(R)| \leq \frac{1}{(|R|-|\tau|)^k}$ approaches zero as $R \to \infty$. Since $f(z)e^{-2\pi i x\xi}$ is holomorphic in the upper half plane the integral over the upper semicircle is also zero. This implies that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(z) e^{-2\pi i z\xi} dz = (\int_{C_R} + \int_{-R}^{R}) f(z) e^{-2\pi i z\xi} dz = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = 0$$

when $\xi < 0$.

On the other hand if $\xi > 0$, we integrate over the lower semicircle of radius R. The integral on C_{R^-} will be zero by Jordan's lemma as before. This gives us

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi i x\xi}}{(\tau+z)^k}$$

Next we calculate the residue

$$\begin{aligned}
\hat{f}(\xi) = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi i x\xi}}{(\tau+z)} &= -2\pi i \lim_{x\to-\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \frac{(\tau+x)^k e^{-2\pi i x\xi}}{(\tau+x)^k} \\
&= -2\pi i \lim_{x\to-\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} e^{-2\pi i x\xi} \\
&= -2\pi i \lim_{x\to-\tau} \frac{(-2\pi i\xi)^{k-1}}{(k-1)!} e^{-2\pi i x\xi} \\
&= \frac{(-2\pi i)^k \xi^{k-1}}{(k-1)!} e^{2\pi i \tau\xi}
\end{aligned}$$

Applying the Poisson summation formula with the above values for $\hat{f}(\xi)$ gives us that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau}.$$

(b) First we transform $\frac{\pi^2}{\sin^2(\pi\tau)}$ into a series.

$$\begin{aligned}
\frac{\pi^2}{\sin^2(\pi\tau)} &= \pi^2\left(\frac{2i}{e^{i\pi\tau}-e^{-i\pi\tau}}\right)^2 \\
&= \frac{-4\pi^2}{e^{-i\pi\tau/2}(1-e^{2i\pi\tau})^2} \\
&= \frac{-4\pi^2}{e^{-2i\pi\tau}}\sum_{m=0}^{\infty} me^{2i\pi m\tau} \\
&= -4\pi^2\sum_{1}^{\infty} me^{2i\pi m\tau}
\end{aligned}$$

The last step only works when $|e^{2\pi i\tau}| < 1$. However since $\Im(\tau) > 0$ this is in fact the case. Which exactly matches the series listed above when we plug in $k = 2$.

(c) This will not work if $\Im(\tau) > 0$ does not hold. The series in question may not be convergent otherwise.

Problem 5. *Compute the integral*

$$I = \int_0^\infty \frac{\log^2(x)}{x^2+a^2}dx$$

Let Γ be the positively oriented upper half annulus of outer radius R and inner radius ϵ. The given function has a simple pole at ia inside Γ. This gives us

$$\int_\epsilon^R \frac{\ln^2(x)}{x^2+a^2}dx + \int_{C_R}\frac{\log^2(z)}{z^2+a^2}dz + \int_{C_\epsilon}\frac{\log^2(z)}{z^2+a^2}dz + \int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz = 2\pi i\,\mathrm{Res}_{z=ia}\frac{\log^2(z)}{z^2+a^2}$$

The integrals about C_R and C_ϵ both go to zero. For

$$\int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz$$

we make the substitution $z = -x$ giving us

$$\int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz = \int_\epsilon^R \frac{\log^2(-x)}{x^2+a^2}dx$$

Which when rewriting $\log(-x)$ as $\ln(x) + i\pi$ we get

$$\int_\epsilon^R \frac{\log^2(-x)}{x^2+a^2}dx = \int_\epsilon^R \frac{\ln^2(x)+\pi i\ln(x)-\pi^2}{x^2+a^2}dx$$

Now rewrite the original equation as

$$2\int_\epsilon^R \frac{\ln^2(x)}{x^2+a^2}dx = 2\pi i\,\mathrm{Res}_{z=ia}\frac{\log^2(z)}{z^2+a^2} + \pi^2\int_\epsilon^R \frac{1}{x^2+a^2}dx - 2\pi i\int_\epsilon^R \frac{\ln(x)}{x^2+a^2}dx$$

The residue is

$$\mathrm{Res}_{z=ia}\frac{\log^2 z}{z^2+a^2} = \frac{\log^2(ia)}{2ia} = \frac{\ln^2(a)+\pi i\ln(a)-\pi^2/4}{2ia}$$

The first integral on the right is of the derivative of $a^{-1}\arctan(x/a)$. Giving us

$$\int_0^\infty \frac{1}{a^2+x^2}dx = \frac{\pi}{2a}$$

The latter integral is from a previous assignment

$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}$$

Putting this all together while letting $\epsilon \to 0$ and $R \to \infty$ we get

$$\begin{aligned}
2I &= 2\pi i \operatorname{Res}_{z=ia} \frac{\log^2(z)}{z^2+a^2} + \pi^2 \int_\epsilon^R \frac{1}{x^2+a^2}dx - 2\pi i \int_\epsilon^R \frac{\ln(x)}{x^2+a^2}dx \\
&= \frac{\pi \ln^2 a}{a} + \frac{i\pi^2 \ln a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a} - \frac{i\pi^2 \ln a}{a} \\
&= \frac{\pi \ln^2 a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a}
\end{aligned}$$

We can then conclude that

$$I = \int_0^\infty \frac{\log^2 x}{x^2+a^2}dx = \frac{\pi \ln^2 a}{2a} - \frac{\pi^3}{8a} + \frac{\pi^3}{4a}$$