

Problem 1 (5.6.2). Find the order of growth of the following entire functions:

(a) $p(z)$ where p is a polynomial.

(b) e^{bz^n} for $b \neq 0$.

(c) e^{e^z} .

Proof.

□

Problem 2 (5.6.6). Prove Wallis's product formula

$$\frac{\pi}{2} = \prod_{m=1}^{\infty} \frac{(2m)^2}{(2m-1)(2m+1)}$$

[Hint: Use the product formula for $\sin z$ at $z = \pi/2$.]

Proof.

□

Problem 3 (5.6.8). Prove that for every z the product below converges, and

$$\prod_{k=1}^{\infty} \cos(z/2^k) = \frac{\sin z}{z}$$

[Hint: Use the fact that $\sin 2z = 2 \sin z \cos z$.]

Proof.

□

Problem 4 (5.6.10(b)). Show that the Hadamard product for $\cos z$ is

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2} \right)$$

Proof.

□

Problem 5 (6.3.5). Use the fact that $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ to prove that

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}} = \sqrt{\pi \operatorname{sech} \pi t}$$

whenever $t \in \mathbb{R}$.

Proof.

□

Problem 6 (6.3.7). The **Beta function** is defined for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ by

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(b) Show that $B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

[Hint: For part (a), note that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds$$

and make the change of variables $s = ur, t = u(1-r)$.]

Proof. □

Problem 7 (6.3.10). An integral of the form

$$F(z) = \int_0^\infty f(t)t^{z-1} dt$$

is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t} .

(a) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)t^{z-1} dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$ and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)t^{z-1} dt = \Gamma(z) \sin\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$.

(b) Show that the second of the above is valid in the larger strip $-1 < \Re(z) < 1$, and that as a consequence, one has

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x^{3/2}} dx = \sqrt{2\pi}$$

[Hint: For the first part, consider the integral of the function $f(w) = e^{-w}w^{z-1}$ around the quarter annulus. Use the analytic continuation to prove the second part.]

Proof. □

Problem 8 (6.3.15). Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$.]

Proof. □