Problem 1 (5.6.2)**.** *Find the order of growth of the following entire functions:*

(a) $p(z)$ *where* p *is a polynomial.*

(b) e^{bz^n} *for* $b \neq 0$.

(c) e^{e^z}.

Proof. (a) Let $n = \deg p(z)$ and $\rho > 0$. Choose $C > |a_0|$ such that $|p(z)| \leq C|z|^n$ and m so that $\rho m > n$. Then

$$m!Ce^{|z|^\rho} = m!C\sum_{k=0}^{\infty} \frac{|z|^{\rho k}}{k!} \geq C|z|^{\rho m} \geq C|z|^n \geq |p(z)|$$

Since this holds for any $\rho > 0$ we have that $\rho_{p(z)} = 0$.

(b) Using the Taylor expansion we get

$$\left|e^{bz^n}\right| \leq \left|\sum_{m=0}^{\infty} \frac{(z^n)^m}{m!}\right| \leq \sum_{m=0}^{\infty} \left|\frac{(z^n)^m}{m!}\right| \leq \sum_{m=0}^{\infty} \frac{|z|^{nm}}{m!} \leq e^{b|z|^n}$$

which shows that $\rho_{e^{bz^n}} \leq n$. However if we choose the exponent in the definition of order to be b we get exactly $e^{bz^n} = e^{Bz^n}$. From this we can conclude that the order of e^{bz^n} is exactly n.

(c) For e^{e^z} suppose that it had order n. Then if $z = x + iy$

$$\left|e^{e^z}\right| = \left|e^{e^x(\cos x + i\sin y)}\right| = e^{e^x \cos x} < Ae^{B|z|^n}$$

Take the logarithm of both sides to get

$$e^x \cos x < C|z|^n$$

which is most assuredly not true for all $z \in \mathbb{C}$. As such e^{e^z} does not have finite order.

□

Problem 2 (5.6.6)**.** *Prove Wallis's product formula*

$$\frac{\pi}{2} = \prod_{m=1}^{\infty} \frac{(2m)^2}{(2m-1)(2m+1)}$$

[Hint: Use the product formula for $\sin z$ *at* $z = \pi/2$*.]*

Proof. Plugging in $z = \pi/2$ we get

$$1 = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2}\prod_{n=1}^{\infty} \frac{(2n+1)(2n-1)}{(2n)^2}$$

Which then dividing by the product gives us

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n+1)(2n-1)} = \frac{\pi}{2}$$

as desired. □

Problem 3 (5.6.8)**.** *Prove that for every z the product below converges, and*

$$\prod_{k=1}^{\infty} \cos\left(z/2^k\right) = \frac{\sin z}{z}$$

[Hint: Use the fact that $\sin 2z = 2 \sin z \cos z$*.]*

Proof. Start with using the suggested identity with $z/2$ to get

$$\sin z = 2 \sin(z/2) \cos(z/2)$$

Then iterate usage of the identity and divide by z to get

$$\frac{\sin z}{z} = \frac{2^N \sin\left(z/2^N\right)}{z} \prod_{k=1}^{N} \cos\left(z/2^k\right)$$

Use Wallis's formula on the right term to get

$$\frac{2^N \sin\left(z/2^N\right)}{z} = \frac{2^N}{z} \cdot \frac{z}{2^N} \prod_{k=1}^{\infty} \left(1 - \frac{\left(\frac{z}{2^N}\right)^2}{k^2\pi^2}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{\left(\frac{z}{2^N}\right)^2}{k^2\pi^2}\right)$$

which goes to 1 as $N \to \infty$. Which in turn gives us that

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \cos\left(z/2^k\right)$$

as desired. □

Problem 4 (5.6.10(b))**.** *Show that the Hadamard product for* $\cos z$ *is*

$$\cos \pi z = \prod_{n=0}^{\infty} 1 - \frac{4z^2}{(2n+1)^2}$$

Proof. First note that $\cos z$ has growth order 1. As such for the Hadamard product formula we get

$$\begin{aligned}
\cos \pi z &= e^{P(z)} z^0 \prod_{n=-\infty}^{\infty} E_1\left(2z/\left(2n+1\right)\right) \\
&= e^{P(z)} \prod_{n=-\infty}^{\infty} \left(1 - \frac{2z}{2n+1}\right) e^{2z/(2n+1)} \\
&= e^{P(z)} \left(\prod_{n=0}^{\infty} \left(1 - \frac{2z}{2n+1}\right) e^{2z/(2n+1)}\right) \left(\prod_{n=0}^{\infty} \left(1 + \frac{2z}{2n+1}\right) e^{-2z/(2n+1)}\right) \\
&= e^{P(z)} \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)
\end{aligned}$$

Now we must show that $P(z) = 0$. Since P is at most degree 1 it will be sufficient to show P evaluates to zero at two distinct values for z. Begin with $z = 0$

$$1 = e^{P(0)} \prod_{n=0}^{\infty} 1$$

Giving that $P(0) = 0$. Then use $z = 2$ to get

$$1 = e^{P(2)} \prod_{n=0}^{\infty} \left(1 - \frac{16}{(2n+1)^2}\right)$$

The partial product for the right term is

$$\prod_{n=0}^{N} 1 - \frac{16}{(2n+1)^2} = \frac{(2N+3)(2N+5)}{4N^2 - 1}$$

Which implies that $\prod_{n=0}^{\infty} 1 - \frac{16}{(2n+1)^2} = 1$ and as such $P(2) = 0$ as well. Thus $P(z) = 0$ and we have the formula

$$\cos \pi z = \prod_{n=0}^{\infty} 1 - \frac{4z^2}{(2n+1)^2}$$

□

Problem 5 (6.3.5)**.** *Use the fact that* $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$ *to prove that*

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}} = \sqrt{\pi}\,\mathrm{sech}\,\pi t$$

whenever $t \in \mathbb{R}$*.*

Proof. Use the above identity with $s = 1/2 + it$ to get

$$\begin{aligned}
\Gamma(1/2 + it)\Gamma(1/2 - it) &= \frac{2\pi}{\sin \pi s} \\
&= \frac{2\pi i}{e^{i\pi/2 - \pi t} - e^{-i\pi/2 + \pi t}} \\
&= \frac{2\pi i}{i(e^{\pi t} - e^{-\pi t})} \\
&= \frac{2\pi}{e^{\pi t} - e^{-\pi t}}
\end{aligned}$$

Then using the fact that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ and that $s\bar{s} = |s|^2$ we get

$$|\Gamma(1/2 + it)|^2 = \frac{2\pi}{e^{\pi t} - e^{-\pi t}}$$

Which leads us to the desired identity

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} - e^{-\pi t}}}$$

□

Problem 6 (6.3.7)**.** *The* ***Beta function*** *is defined for* $\Re(\alpha) > 0$ *and* $\Re(\beta) > 0$ *by*

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) *Prove that* $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(b) *Show that* $B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

[Hint: For part (a), note that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds$$

and make the change of variables $s = ur, t = u(1-r)$*.]*

Proof. (a) Begin with $\Gamma(\alpha)\Gamma(\beta)$ and make the above suggested substitution to get

$$\begin{aligned}
\Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty e^{-t} t^{\alpha-1} dt\right)\left(\int_0^\infty e^{-s} s^{\beta-1} ds\right) \\
&= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(t+s)} dt ds \\
&= \int_0^1 \int_0^\infty (u(1-r))^{\alpha-1} (ur)^{\beta-1} e^{-u} (-u) du dr \\
&= \int_0^1 \int_0^\infty -u^{\alpha+\beta-1} e^{-u} (1-r)^{\alpha-1} r^{\beta-1} du dr \\
&= \int_0^1 \left(\int_0^\infty -u^{\alpha+\beta-1} e^{-u} du\right) (1-r)^{\alpha-1} r^{\beta-1} dr \\
&= \int_0^1 \Gamma(\alpha+\beta)(1-r)^{\alpha-1} r^{\beta-1} dr \\
&= \Gamma(\alpha+\beta) \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr \\
&= \Gamma(\alpha+\beta) B(\alpha,\beta)
\end{aligned}$$

Which when we divide through gives us

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Begin with the integral side of the problem and make the substitution $\frac{1-t}{t} = u$ to get

$$\begin{aligned}
\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du &= \int_0^\infty \left(\frac{u}{1+u}\right)^{\alpha-1} (1+u)^{-\beta-1} du \\
&= \int_1^0 (1-t)^{\alpha-1} \left(t^{-1}\right)^{-\beta-1} \left(-t^{-2}\right) dt \\
&= \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt \\
&= B(\alpha,\beta)
\end{aligned}$$

□

Problem 7 (6.3.10)**.** *An integral of the form*

$$F(z) = \int_0^\infty f(t) t^{z-1} dt$$

*is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t}.*

(a) *Prove that*

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)\, t^{z-1} dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$ and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)\, t^{z-1} dt = \Gamma(z) \sin\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$.

(b) *Show that the second of the above is valid in the larger strip $-1 < \Re(z) < 1$, and that as a consequence, one has*

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x^{3/2}} dx = \sqrt{2\pi}$$

[Hint: For the first part, consider the integral of the function $f(w) = e^{-w} w^{z-1}$ around the quarter annulus. Use the analytic continuation to prove the second part.]

Proof. (a) Let $\gamma_{\epsilon,R}$ be the quarter annulus in the upper right quarter plane with inner radius ϵ and outer radius R. Since there are no singularities within for $f(w)$, the value of the integral is zero. Thus we have

$$\int_{\gamma_{\epsilon,R}} e^{-w} w^{z-1} dw = 0 = \int_\epsilon^R e^{-t} t^{z-1} dt + \int_\epsilon^R e^{-it} t^{z-1} i dt + \int_{C_\epsilon} e^{-w} w^{z-1} dw + \int_{C_R} e^{-w} w^{z-1} dw$$

The leftmost term in the sum is $\Gamma(z)$, the two integrals about C_ϵ and C_R will both approach zero as $\epsilon \to 0$ and $R \to \infty$. This gives us

$$\begin{aligned} 0 &= \Gamma(z) + i^z \int_0^\infty e^{-it} t^{z-1} dt = \Gamma(z) + i^z \left(\int_0^\infty \cos(t)\, t^{z-1} dt - i \int_0^\infty \sin(t)\, t^{z-1} dt \right) \\ &= \Gamma(z) + i^z \left(\mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z)\right) \end{aligned}$$

Rearrange to get

$$\mathcal{M}(\cos)(z) - i\mathcal{M}(\sin)(z) = \Gamma(z)\, i^{-z} = \Gamma(z)\cos(\pi z/2) - i\Gamma(z)\sin(\pi z/2)$$

If z is real then by taking the real and imaginary parts of the above equality we get

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)\, t^{z-1} dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right)$$

and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)\, t^{z-1} dt = \Gamma(z) \sin\left(\frac{\pi z}{2}\right)$$

For $\mathcal{M}(\cos)(z)$ the original integral exists in the strip defined by $0 < \Re(z) < 1$. Since the integral agrees with $\Gamma(z)\cos(\pi z/2)$ on interval $(0,1)$ and $\Gamma(z)\cos(\pi z/2)$ is analytic in said strip it must be that $\mathcal{M}(\cos)(z)$ continues analytically to the whole strip with values $\Gamma(z)\cos(\pi z/2)$. The same reasoning applies to $\mathcal{M}(\sin)(z)$.

(b) Using that $\Gamma(z) = (z-1)\Gamma(z-1)$ and that $\cos(z) = \sin(z + \pi/2)$ we can continue $\mathcal{M}(\sin)(z)$ to the larger strip by

$$\mathcal{M}(\sin)(z-1) = \Gamma(z-1)\sin(\pi/2(z-1)) = \frac{\Gamma(z)\cos(\pi z/2)}{z-1}$$

Now if we plug-in $-1/2$ in for z we get

$$\mathcal{M}(\sin)(-1/2) = \int_0^\infty \frac{\sin(t)}{t^{3/2}}dt = \frac{\Gamma(1/2)\cos(-\pi/4)}{-1/2} = \frac{2\sqrt{\pi}}{\sqrt{2}} = \sqrt{2\pi}$$

Using $z = 0$ we have

$$\begin{aligned}
\mathcal{M}(\sin)(0) = \int_0^\infty \frac{\sin(t)}{t}dt &= \lim_{z\to 0}\Gamma(z)\sin(\pi z/2) \\
&= \lim_{z\to 0}\Gamma(z+1)\frac{\sin(\pi z/2)}{z} \\
&= \left(\lim_{z\to 0}\Gamma(z+1)\right)\left(\lim_{z\to 0}\frac{\sin(\pi z/2)}{z}\right) \\
&= 1\lim_{z\to 0}\frac{\sin(\pi z/2)}{z} \\
&= \lim_{z\to 0}\frac{\pi/2\cos(\pi z/2)}{1} \\
&= \pi/2
\end{aligned}$$

□

Problem 8 (6.3.15)**.** *Prove that for* $\Re(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx$$

[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$*.]*

Proof. Start with the right side of the equation and use the suggested series with a substitution $t = nx$ to get

$$\begin{aligned}
\frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx &= \frac{1}{\Gamma(s)}\int_0^\infty \sum_{n=1}^\infty x^{s-1}e^{-nx}dx \\
&= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty \int_0^\infty e^{-nx}dx \\
&= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty \int_0^\infty n^{-s}t^{s-1}e^{-t}dt \\
&= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty n^{-s}\int_0^\infty t^{s-1}e^{-t}dt \\
&= \frac{1}{\Gamma(s)}\sum_{n=1}^\infty n^{-s}\Gamma(s) \\
&= \sum_{n=1}^\infty n^{-s} \\
&= \zeta(s)
\end{aligned}$$

□