Problem 1. *Consider the integral*

$$\Phi_n(z) = \int_a^b \frac{x^n dx}{(x-a)^{\frac{1}{2}-\lambda}(b-x)^{\frac{1}{2}+\lambda}(x-z)}$$

where $z \notin [a,b]$ *and* $\frac{-1}{2} < \lambda < \frac{1}{2}$.

(a) *Compute the integral for* $n = 0$ *and* $n = 1$.

(b) *Determine the domain where* $\phi_n(z)$ *is holomorphic.*

Hint: Consider the contour Γ *and fix a branch of the function* $\Gamma = C_R \cup \gamma^- \cup \gamma^+$ *and* $f(\zeta) = (\zeta - a)^{\frac{1}{2}-\lambda}(b-\zeta)^{\frac{1}{2}+\lambda}$ *where* $\zeta = x + iy$

Proof. □

Problem 2. *Show, by contour integration, that if* $a > 0$ *and* $\xi \in \mathbb{R}$ *then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}.$$

Proof. □

Problem 3. *Prove that*

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a > 0$. *Hence show that the sum equals both* $\coth \pi a$.

Proof. □

Problem 4. (a) *Let* τ *be fixed with* $Im(\tau) > 0$. *Apply the Poisson summation formula to*

$$f(z) = (\tau + z)^{-k},$$

where k *is an integer* ≥ 2, *to obtain*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m \tau}.$$

(b) *Set* $k = 2$ *in the above formula to show that if* $Im(\tau) > 0$, *then*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) *Can one conclude that the above formula hold true whenever τ is any complex number that is not an integer?*

Proof. $\square$

Problem 5. *Compute the integral*

$$I = \int_0^\infty \frac{\log^2(x)}{x^2 + a^2} dx$$

Proof. $\square$