Problem 1 (8.5.7). Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

(a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}$$

This leads to two separate cases: either $0 < y \leq 1/2$ and $\theta = \pi/2$ or $1/2 \leq y < 1$ and $\theta = -\pi/2$. In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \quad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}$$

(b) In the integral $\frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\widetilde{f}_0(\varphi)d\varphi$ make the change of variables $t = F(e^{i\varphi})$. Observe that

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}}$$

and then take the imaginary part and differentiate both sides to establish the two identities

$$\sin \varphi = \frac{1}{\cosh \pi t} \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}$$

Hence deduce that

$$\begin{aligned}
\frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\widetilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}\widetilde{f}_0(\varphi)d\varphi \\
&= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}dt
\end{aligned}$$

(c) Use a similar argument to prove the formula for the integral $\frac{1}{2\pi}\int_{-\pi}^0 P_r(\theta - \varphi)\widetilde{f}_1(\varphi)d\varphi$.

Proof. (a) By the definition of G we start with

$$re^{i\theta} = \frac{i - e^{i\pi y}}{i + e^{i\pi y}}$$

Replace the $e^{i\theta}$ with sines and cosines, followed by multiplying the top and bottom by the conjugate of the bottom to get

$$\frac{i - e^{i\pi y}}{i + e^{i\pi y}} = \frac{i - \cos \pi y - i \sin \pi y}{i + \cos \pi y + i \sin \pi y} \cdot \frac{\cos \pi y - i(1 + \sin \pi y)}{\cos \pi y - i(1 + \sin \pi y)}$$

Which then simplifies to

$$i\frac{\cos \pi y}{1 + \sin \pi y}$$

In the first case listed above we lie on the positive imaginary axis, and in the second case listed above we are on the negative imaginary axis. Either way we have

$$\begin{aligned}
(\pm ir) = -r^2 &= -\frac{\cos^2 \pi y}{(1 + \sin \pi y)^2} \\
r^2 &= \frac{1 - \sin^2 \pi y}{(1 + \sin \pi y)^2} \\
&= \frac{1 - \sin \pi y}{1 + \sin \pi y}
\end{aligned}$$

By definition of $P_r(\theta-\varphi)$ we have

$$P_r(\theta-\varphi) = \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}$$

Substitute in for r^2 and simplify to get

$$\begin{aligned}
P_r(\theta-\varphi) &= \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2} \\
&= \frac{1-\frac{1-\sin\pi y}{1+\sin\pi y}}{1-2r\cos(\theta-\varphi)+\frac{1-\sin\pi y}{1+\sin\pi y}} \\
&= \frac{1-\frac{1-\sin\pi y}{1+\sin\pi y}}{1-2r(\cos\theta\cos\varphi+\sin\theta\sin\varphi)+\frac{1-\sin\pi y}{1+\sin\pi y}} \cdot \frac{1+\sin\pi y}{1+\sin\pi y} \\
&= \frac{2\sin\pi y}{2-2r(1+\sin\pi y)(\sin\theta\sin\varphi)} \\
&= \frac{\sin\pi y}{1-(e^{i(\theta+\pi/2)}\cos\pi y)(\sin\theta\sin\varphi)}
\end{aligned}$$

From here since we know what θ is, we can simplify to

$$P_r(\theta-\varphi) = \frac{\sin\pi y}{1-\cos\pi y\sin\varphi}$$

(b) Since F and G are inverses we have that

$$G(t) = G(F(e^{i\varphi})) = e^{i\varphi} = \frac{i-e^{\pi t}}{i+e^{\pi t}}$$

Which by multiplying the top and bottom by the conjugate of the bottom we get

$$e^{i\varphi} = G(t) = \frac{1-e^{2\pi t}+2ie^{\pi t}}{1+e^{2\pi t}}$$

From there take the imaginary part to get

$$\sin\varphi = \frac{2e^{\pi t}}{1+e^{2\pi t}} = \operatorname{sech}\pi t$$

Taking the derivative of both sides we get

$$\cos\varphi\frac{d\varphi}{dt} = -\pi\tanh\pi t\operatorname{sech}\pi t$$

However $\cos\varphi$ is the real part of $G(t)$ which is equal to

$$\cos\varphi = \frac{1-e^{2\pi t}}{1+e^{2\pi t}} = -\tanh\pi t$$

Which cancels the $-\tanh\pi t$ on each side giving us

$$\frac{d\varphi}{dt} = \pi\operatorname{sech}\pi t$$

Now we can use the above pieces to transform the integral

$$\begin{aligned}
\frac{1}{2\pi}\int_0^\pi P_r(\theta-\varphi)\widetilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\widetilde{f}_0(\varphi)d\varphi \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi y}{1-\cos\pi y\cosh\pi t}\cdot\frac{\pi}{\cosh\pi t}dt \\
&= \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_0(t)}{\cosh\pi t-\cos\pi y}dt
\end{aligned}$$

(c) We proceed similarly. However since $\widetilde{f}_1(\varphi) = f_1(F(e^{i\varphi}) - i)$ we instead let $t = F(e^{i\varphi} - i)$. This will cause a $+i$ to appear in the exponent which will make the calculations the same as before with a small sign change.

$$G(t+i) = e^{i\varphi} = \frac{i - e^{i\pi + \pi t}}{i + e^{\pi i + \pi t}} = \frac{1 - e^{2\pi t} - 2ie^{\pi t}}{1 + e^{2\pi t}}$$

Taking the imaginary part instead gives us

$$\sin\varphi = -\operatorname{sech}\pi t$$

and the real part

$$\cos\varphi = -\tanh\pi t$$

Which when we take the derivative again will give us

$$\cos\varphi\frac{d\varphi}{dt} = \pi\tanh\pi t\operatorname{sech}\pi t$$

This then turns into

$$\frac{d\varphi}{dt} = -\pi\operatorname{sech}\pi t$$

Now we can once again transform the integral

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta-\varphi)\widetilde{f}_1(\varphi)d\varphi &= \frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin\pi y}{1-\cos\pi y\sin\varphi}\widetilde{f}_1(\varphi)d\varphi \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\pi y}{1+\cos\pi y\operatorname{sech}\pi t}f_1(t)\cdot\frac{-\pi}{\cosh\pi t}dt \\ &= \frac{\sin\pi y}{2}\int_{-\infty}^{\infty}\frac{f_1(t)}{\cosh\pi t + \cos\pi y}dt\end{aligned}$$

□

Problem 2 (8.5.9). Prove that the function u defined by

$$u(x,y) = \Re(\frac{i+z}{i-z}), \quad u(0,1) = 0$$

is harmonic in the unit disc and vanishes on the boundary. Note that u is not bounded in $\mathbb{D}$.

Proof. Since u is the real part of a holomorphic function it is harmonic. If we write $z = x + iy$ we can rewrite $u(x,y)$ as

$$\Re(\frac{i+z}{i-z}) = \frac{1-x^2-y^2}{x^2+(1-y)^2} = \frac{1-|z|^2}{x^2+(1-y)^2}$$

If $|z| = 1$ and $z \neq i$, the top vanishes giving us that $u(x,y) = 0$. Since we decreed that $u(0,1) = 0$ we can conclude that u vanishes on $\partial\mathbb{D}$. □

Problem 3 (8.5.16). Let

$$f(z) = \frac{i-z}{i+z} \quad \text{and} \quad f^{-1}(w) = i\frac{1-w}{1+w}$$

(a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(e^{i\theta} f(z))$$

(b) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(\psi_\alpha(f(z)))$$

with ψ_a defined in Section 2.1.

(c) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z)$$

[Hint: Use parts (a) and (b)].

Proof. (a) The equivalent transformation is

$$\frac{\sin(\theta/2) + \cos(\theta/2)z}{\cos(\theta/2) - \sin(\theta/2)z}$$

To see this use begin $f^{-1}(e^{i\theta} f(z))$ and simplify to get

$$\frac{(e^{i\theta} - 1) + iz(1 + e^{i\theta})}{i(1 + e^{i\theta}) - (e^{i\theta} - 1)z} =$$

(b) Similarly the equivalent transformation is

$$\frac{\Re(\alpha) - 1 + z\Im(\alpha)}{z(\Re(\alpha) + 1) - \Im(\alpha)} \cdot \frac{(1 - |\alpha|^2)^{-1/2}}{(1 - |\alpha|^2)^{-1/2}}$$

To derive this, begin with $f^{-1} \circ \psi_\alpha \circ f(z)$ to get

$$\begin{aligned} \frac{i - i\frac{\alpha - \frac{i-z}{i+z}}{1 - \overline{\alpha}\frac{i-z}{i+z}}}{1 + \frac{\alpha - \frac{i-z}{i+z}}{1 - \overline{\alpha}\frac{i-z}{i+z}}} &= \frac{(\overline{\alpha} - 1) + iz(1 + \overline{\alpha}) + (\alpha - 1) - iz(\alpha + 1)}{i(1 - \overline{\alpha}) + z(1 + \overline{\alpha}) + i(\alpha - 1) + z(\alpha + 1)} \\ &= \frac{\overline{\alpha} - 2 + \alpha + iz\overline{\alpha} - iz\alpha}{i\alpha - i\overline{\alpha} + i2z + z\alpha + z\overline{\alpha}} \\ &= \frac{2\Re(\alpha) - 2 + 2\Im(\alpha)z}{-2\Im(\alpha) + 2z(\Re(\alpha) + 1)} \end{aligned}$$

which we can then renormalize to get

$$\frac{\Re(\alpha) - 1 + z\Im(\alpha)}{z(\Re(\alpha) + 1) - \Im(\alpha)} \cdot \frac{(|\alpha|^2 - 1)^{-1/2}}{(|\alpha|^2 - 1)^{-1/2}}$$

as desired.

(c) From class we know that any automorphism of $\mathbb{D}$ is the composition of a rotation $\rho(z)$ and a $\psi_\alpha(z)$. Let $g(z) = \rho \circ \psi_\alpha(z)$. Then

$$\begin{aligned} f^{-1} \circ g \circ f(z) &= f^{-1} \circ \rho \circ \psi_\alpha \circ f(z) \\ &= (f^{-1} \circ \rho \circ f) \circ (f^{-1} \circ \psi_\alpha \circ f)(z) \end{aligned}$$

At this point we know that both $f^{-1} \circ \rho \circ f$ and $f^{-1} \circ \psi_\alpha \circ f$ are of the form $\frac{az+b}{cz+d}$ with $ad - bc = 1$. Moreover since composition of two transformations of the above form gives another of its kind it must be that $g(z)$ is of the form $\frac{az+b}{cz+d}$.

$\square$

Problem 4 (8.5.20). Other examples of elliptic integrals providing conformal maps form the upper half-plane to rectangles providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$S(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}, \qquad \lambda \in \mathbb{R} \setminus \{1\}$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda = -1$, the image of

$$S(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

Proof. (a) By proposition 8.4.1, we fall into the second case, and so the image of the upper half plane is a quadrangle with corners $S(0), S(1), S(\lambda)$, and $S(\infty)$ where the angle corresponding to $S(\infty)$ is $\pi/2$. In addition we also get from the proposition that the rest of the angles are $\pi/2$ since they are each to the 1/2 power. So the polygon must in fact be a rectangle.

(b) From above we know that the shape is a rectangle. All that is left to confirm is that the side lengths are $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$. Starting with the side $S(0)$ to $S(1)$ make the substitution $\sqrt{u} = \zeta$ to get

$$\int_0^1 \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}} = \frac{1}{2}\int_0^1 u^{-3/4}(1-u)^{-1/2}du = \frac{B(1/4,1/2)}{2}$$

From the last homework we have that

$$\frac{B(1/4,1/2)}{2} = \frac{\Gamma(1/4)\Gamma(1/2)}{2\Gamma(3/4)} = \frac{\Gamma(1/4)\sqrt{\pi}}{2\frac{\sqrt{2}\pi}{\Gamma(1/4)}} = \frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$$

To get another side we do $|S(-1) - 0| = |S(-1)|$ instead doing the substitution $-\sqrt{u} = \zeta$ to get

$$\left|\int_0^{-1} \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}\right| = \left|\frac{1}{2}\int_0^1 \frac{du}{u^{3/4}(u-1)^{1/2}}\right| = \frac{B(1/4,1/2)}{2} = \frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$$

Since we have a rectangle and two non-opposite sides are equal in length the polygon must be a square.

$\square$

Problem 5 (8.5.21). We consider the conformal mappings to triangles.

(a) Show that

$$S(w) = \int_0^w z^{-\beta_1}(1-z)^{-\beta_2}dz$$

with $0 < \beta_1 < 1$, $0 < \beta_2 <$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of $0, 1$, and ∞, and with angles $\alpha_1\pi, \alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is $\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$.

Proof. (a) Using proposition 8.4.1, we have that the image $S(\mathbb{H})$ is a is a triangle with points at $S(0), S(1)$, and $S(\infty)$. The angles corresponding to each side are then $\alpha_0\pi, \alpha_1\pi, \alpha_\infty\pi$ where $\alpha_z = 1 - \beta_z$ and $\beta_0 + \beta_1 + \beta_\infty = 2$ from the proposition.

(b) Following in a similar fashion to the last problem we get a "degenerate" triangle that is a straight line (angle at infinity is π).

(c) We get a triangle with a vertex at infinity and a vertex at $S(0), S(1)$ with non-intersecting rays emerging from each.

(d) If we have one side the others follow from using law of sines. The length of the side corresponding to $S(0)$ and $S(1)$ has length $|S(1) - S(0)| = |S(1)| = |\int_0^1 z^{-\beta_1}(1-z)^{-\beta_2}dz|$. However this is exactly $B(1-\beta_1, 1-\beta_2)$. Using that the relation between the beta function and the gamma from the prior homework along with the property that $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$ we get

$$B(1-\beta_1, 1-\beta_2) = \frac{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(\beta_\infty)} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(1-\alpha_\infty)} = \frac{\sin(\alpha_\infty\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_\infty)$$

The length of the other sides then follows immediately from the application of law of sines. □

Problem 6 (8.6.2). The angle between two non-zero complex numbers z and w (taken in that order) is simply the oriented angle, in $(-\pi, \pi]$, that is formed between the two vectors in $\mathbb{R}^2$ corresponding to the points z and w. This oriented angle, say α, is uniquely determined by the two quantities

$$\frac{(z, w)}{|z||w|} \quad \text{and} \quad \frac{(z, -iw)}{|z||w|}$$

which are simply the cosine and sine of α, respectively. Here, the notation $(\cdot, \cdot)$ corresponds to the usual Euclidean inner product in $\mathbb{R}^2$, which in terms of complex numbers takes the form $(z, w) = \Re(z\overline{w})$.

In particular, we may now consider two smooth curves $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [a, b] \to \mathbb{C}$, that intersect at z_0, say $\gamma(t_0) = \eta(t_0) = z_0$ for some $t_0 \in (a, b)$. If the quantities $\gamma'(t_0)$ and $\eta'(t_0)$ are non-zero, then they represent the tangents to the curves γ and η at the point z_0, and we say that the two curves intersect at z_0 at the angle formed by the two vectors $\gamma'(t_0)$ and $\eta'(t_0)$.

A holomorphic function f defined near z_0 is said to **preserve angles** at z_0 if for any two smooth curves γ and η intersecting at z_0, the angle formed between the curves γ and η at z_0 equals the angle formed between the curves $f \circ \gamma$ and $f \circ \eta$ at $f(z_0)$. In particular we assume that the tangents to the curves $\gamma, \eta, f \circ \gamma$, and $f \circ \eta$ at the point z_0 and $f(z_0)$ are all non-zero.

(a) Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'(z_0) \neq 0$, then f preserves angles at z_0. [Hint: Observe that

$$(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0)) = |f'(z_0)|^2(\gamma'(t_0), \eta'(t_0)))$$

]

(b) Conversely, prove the following: suppose $f : \Omega \to \mathbb{C}$ is complex-valued function, that is real differentiable at $z_0 \in \Omega$, and $J_f(z_0) \neq 0$. If f preserves angles at z_0, then f is holomorphic at z_0 with $f'(z_0) \neq 0$.

Proof. (a) From the problem description if the two listed quantities are preserved then so is the angle. Using the suggested inequality along with the chain rule we get

$$\begin{aligned}\frac{(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} &= \frac{|f'(z_0)|^2(\gamma'(t_0), \eta'(t_0)))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} \\ &= \frac{(\gamma'(t_0), \eta'(t_0)))}{|\gamma'(t_0)||\eta'(t_0)|}\end{aligned}$$

And similarly

$$\begin{aligned}\frac{(f'(z_0)\gamma'(t_0), if'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} &= \frac{|f'(z_0)|^2(\gamma'(t_0), i\eta'(t_0)))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} \\ &= \frac{(\gamma'(t_0), i\eta'(t_0)))}{|\gamma'(t_0)||\eta'(t_0)|}\end{aligned}$$

(b) Let $z(t) = x(t)+iy(t)$ be a smooth curve in the complex plane and $f(x, y) = u(x, y)+iv(x, y)$ a complex valued function fulfilling the conditions above. Define $w(t) = f \circ z(t)$. Then take the derivative of w to get

$$\begin{aligned}w'(t) &= u_x x' + u_y y' + iv_x x' + iv_y y' \\ &= f_x x' + f_y y' \\ &= \frac{1}{2}(2f_x x' + 2f_y y' + if_y x' - if_y x' + if_x y' - if_x y') \\ &= \frac{1}{2}(f_x - if_y)(x' + iy') + \frac{1}{2}(f_x + if_y)(x' - iy') \\ &= \frac{1}{2}(f_x - if_y)z' + \frac{1}{2}(f_x + if_y)\overline{z'}\end{aligned}$$

With the right side containing one form of the Cauchy-Riemann equations.

Next consider

$$\frac{w'}{z'} = \frac{1}{2}(f_x - if_y) + \frac{1}{2}(f_x + if_y)\frac{\overline{z'}}{z'}$$

Since f preserves angles it stands to reason that as we vary t that the argument of the expression will not change. If we write $z(t) = re^{i\theta}$ then we have $\frac{\overline{z'}}{z} = re^{-2\theta}$. As such we can only actually get a constant argument in the above expression if $(f_x + if_y) = 0$. However this is equivalent to f being holomorphic.

□