

Problem 1 (5.6.2). Find the order of growth of the following entire functions:

(a) $p(z)$ where p is a polynomial.

(b) e^{bz^n} for $b \neq 0$.

(c) e^{e^z} .

Proof. (a) Let $n = \deg p(z)$ and $\rho > 0$. Choose $C > |a_0|$ such that $|p(z)| \leq C|z|^n$ and m so that $\rho m > n$. Then

$$m!Ce^{|z|^\rho} = m!C \sum_{k=0}^{\infty} \frac{|z|^{\rho k}}{k!} \geq C|z|^{\rho m} \geq C|z|^n \geq |p(z)|$$

Since this holds for any $\rho > 0$ we have that $\rho_{p(z)} = 0$.

(b) Using the Taylor expansion we get

$$|e^{bz^n}| \leq \left| \sum_{m=0}^{\infty} \frac{(z^n)^m}{m!} \right| \leq \sum_{m=0}^{\infty} \frac{|z^n|^m}{m!} \leq \sum_{m=0}^{\infty} \frac{|z|^{nm}}{m!} \leq e^{|z|^n}$$

which shows that $\rho_{e^{bz^n}} \leq n$. However if we choose the exponent in the definition of order to be b we get exactly $e^{bz^n} = e^{Bz^n}$. From this we can conclude that the order of e^{bz^n} is exactly n .

(c) For e^{e^z} suppose that it had order n . Then if $z = x + iy$

$$|e^{e^z}| = |e^{e^x(\cos x + i \sin x)}| = e^{e^x \cos x} < Ae^{B|z|^n}$$

Take the logarithm of both sides to get

$$e^x \cos x < C|z|^n$$

which is most assuredly not true for all $z \in \mathbb{C}$. As such e^{e^z} does not have finite order. \square

Problem 2 (5.6.6). Prove Wallis's product formula

$$\frac{\pi}{2} = \prod_{m=1}^{\infty} \frac{(2m)^2}{(2m-1)(2m+1)}$$

[Hint: Use the product formula for $\sin z$ at $z = \pi/2$.]

Proof. Plugging in $z = \pi/2$ we get

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2} \prod_{n=1}^{\infty} \frac{(2n+1)(2n-1)}{(2n)^2}$$

Which then dividing by the product gives us

$$\prod_{n=1}^{\infty} \frac{(2n)^2}{(2n+1)(2n-1)} = \frac{\pi}{2}$$

as desired. \square

Problem 3 (5.6.8). *Prove that for every z the product below converges, and*

$$\prod_{k=1}^{\infty} \cos(z/2^k) = \frac{\sin z}{z}$$

[Hint: Use the fact that $\sin 2z = 2 \sin z \cos z$.]

Proof. Start with using the suggested identity with $z/2$ to get

$$\sin z = 2 \sin(z/2) \cos(z/2)$$

Then iterate usage of the identity and divide by z to get

$$\frac{\sin z}{z} = \frac{2^N \sin(z/2^N)}{z} \prod_{k=1}^N \cos(z/2^k)$$

Use Wallis's formula on the right term to get

$$\frac{2^N \sin(z/2^N)}{z} = \frac{2^N}{z} \cdot \frac{z}{2^N} \prod_{k=1}^{\infty} \left(1 - \frac{(\frac{z}{2^N})^2}{k^2 \pi^2}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{(\frac{z}{2^N})^2}{k^2 \pi^2}\right)$$

which goes to 1 as $N \rightarrow \infty$. Which in turn gives us that

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \cos(z/2^k)$$

as desired. □

Problem 4 (5.6.10(b)). *Show that the Hadamard product for $\cos z$ is*

$$\cos \pi z = \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$$

Proof. First note that $\cos z$ has growth order 1. As such for the Hadamard product formula we get

$$\begin{aligned} \cos \pi z &= e^{P(z)} z^0 \prod_{n=-\infty}^{\infty} E_1(2z/(2n+1)) \\ &= e^{P(z)} \prod_{n=-\infty}^{\infty} \left(1 - \frac{2z}{2n+1}\right) e^{2z/(2n+1)} \\ &= e^{P(z)} \left(\prod_{n=0}^{\infty} \left(1 - \frac{2z}{2n+1}\right) e^{2z/(2n+1)}\right) \left(\prod_{n=0}^{\infty} \left(1 + \frac{2z}{2n+1}\right) e^{-2z/(2n+1)}\right) \\ &= e^{P(z)} \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right) \end{aligned}$$

Now we must show that $P(z) = 0$. Since P is at most degree 1 it will be sufficient to show P evaluates to zero at two distinct values for z . Begin with $z = 0$

$$1 = e^{P(0)} \prod_{n=0}^{\infty} 1$$

Giving that $P(0) = 0$. Then use $z = 2$ to get

$$1 = e^{P(2)} \prod_{n=0}^{\infty} \left(1 - \frac{16}{(2n+1)^2}\right)$$

The partial product for the right term is

$$\prod_{n=0}^N 1 - \frac{16}{(2n+1)^2} = \frac{(2N+3)(2N+5)}{4N^2-1}$$

Which implies that $\prod_{n=0}^{\infty} 1 - \frac{16}{(2n+1)^2} = 1$ and as such $P(2) = 0$ as well. Thus $P(z) = 0$ and we have the formula

$$\cos \pi z = \prod_{n=0}^{\infty} 1 - \frac{4z^2}{(2n+1)^2}$$

□

Problem 5 (6.3.5). *Use the fact that $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ to prove that*

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} + e^{-\pi t}}} = \sqrt{\pi \operatorname{sech} \pi t}$$

whenever $t \in \mathbb{R}$.

Proof. Use the above identity with $s = 1/2 + it$ to get

$$\begin{aligned} \Gamma(1/2 + it)\Gamma(1/2 - it) &= \frac{2\pi}{\sin \pi s} \\ &= \frac{2\pi i}{e^{i\pi/2 - \pi t} - e^{-i\pi/2 + \pi t}} \\ &= \frac{2\pi i}{i(e^{\pi t} - e^{-\pi t})} \\ &= \frac{2\pi}{e^{\pi t} - e^{-\pi t}} \end{aligned}$$

Then using the fact that $\Gamma(\bar{s}) = \overline{\Gamma(s)}$ and that $s\bar{s} = |s|^2$ we get

$$|\Gamma(1/2 + it)|^2 = \frac{2\pi}{e^{\pi t} - e^{-\pi t}}$$

Which leads us to the desired identity

$$|\Gamma(1/2 + it)| = \sqrt{\frac{2\pi}{e^{\pi t} - e^{-\pi t}}}$$

□

Problem 6 (6.3.7). *The **Beta function** is defined for $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ by*

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt$$

(a) Prove that $B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(b) Show that $B(\alpha, \beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du$.

[Hint: For part (a), note that

$$\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-t-s} dt ds$$

and make the change of variables $s = ur, t = u(1-r)$.]

Proof. (a) Begin with $\Gamma(\alpha)\Gamma(\beta)$ and make the above suggested substitution to get

$$\begin{aligned} \Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty e^{-t} t^{\alpha-1} dt \right) \left(\int_0^\infty e^{-s} s^{\beta-1} ds \right) \\ &= \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} e^{-(t+s)} dt ds \\ &= \int_0^1 \int_0^\infty (u(1-r))^{\alpha-1} (ur)^{\beta-1} e^{-u} (-u) du dr \\ &= \int_0^1 \int_0^\infty -u^{\alpha+\beta-1} e^{-u} (1-r)^{\alpha-1} r^{\beta-1} du dr \\ &= \int_0^1 \left(\int_0^\infty -u^{\alpha+\beta-1} e^{-u} du \right) (1-r)^{\alpha-1} r^{\beta-1} dr \\ &= \int_0^1 \Gamma(\alpha+\beta) (1-r)^{\alpha-1} r^{\beta-1} dr \\ &= \Gamma(\alpha+\beta) \int_0^1 (1-r)^{\alpha-1} r^{\beta-1} dr \\ &= \Gamma(\alpha+\beta) B(\alpha, \beta) \end{aligned}$$

Which when we divide through gives us

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(b) Begin with the integral side of the problem and make the substitution $\frac{1-t}{t} = u$ to get

$$\begin{aligned} \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du &= \int_0^\infty \left(\frac{u}{1+u} \right)^{\alpha-1} (1+u)^{-\beta-1} du \\ &= \int_1^0 (1-t)^{\alpha-1} (t^{-1})^{-\beta-1} (-t^{-2}) dt \\ &= \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt \\ &= B(\alpha, \beta) \end{aligned}$$

□

Problem 7 (6.3.10). An integral of the form

$$F(z) = \int_0^\infty f(t) t^{z-1} dt$$

is called a **Mellin transform**, and we shall write $\mathcal{M}(f)(z) = F(z)$. For example, the gamma function is the Mellin transform of the function e^{-t} .

(a) Prove that

$$\mathcal{M}(\cos)(z) = \int_0^\infty \cos(t)t^{z-1}dt = \Gamma(z) \cos\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$ and

$$\mathcal{M}(\sin)(z) = \int_0^\infty \sin(t)t^{z-1}dt = \Gamma(z) \sin\left(\frac{\pi z}{2}\right)$$

for $0 < \Re(z) < 1$.

(b) Show that the second of the above is valid in the larger strip $-1 < \Re(z) < 1$, and that as a consequence, one has

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x^{3/2}} dx = \sqrt{2\pi}$$

[Hint: For the first part, consider the integral of the function $f(w) = e^{-w}w^{z-1}$ around the quarter annulus. Use the analytic continuation to prove the second part.]

Proof. □

Problem 8 (6.3.15). Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

[Hint: Write $1/(e^x - 1) = \sum_{n=1}^\infty e^{-nx}$.]

Proof. Start with the right side of the equation and use the suggested series with a substitution $t = nx$ to get

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty x^{s-1} e^{-nx} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty e^{-nx} dx \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty \int_0^\infty n^{-s} t^{s-1} e^{-t} dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty n^{-s} \int_0^\infty t^{s-1} e^{-t} dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=1}^\infty n^{-s} \Gamma(s) \\ &= \sum_{n=1}^\infty n^{-s} \\ &= \zeta(s) \end{aligned}$$

□