Problem 1 (2.6.14). Suppose that f is holomorhic in an open set containing the closed unit disc, expcept for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Only need to do case where degree of pole is greater than 1.

Proof. □

Problem 2 (3.8.2). Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} dx$$

Where are the poles of $1/(1+z^4)$?

The poles of the function are at $\pm e^{i\pi/4}, \pm e^{3i\pi/4}$.

We evaluate this integral by integrating over the semicircle in the upper half plane of radius R which we will call γ_R. Then we have

$$\int_{\gamma_R} \frac{1}{1+z^4} dz = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\cap i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

However since $\frac{1}{1+z^4} \le \frac{1}{R^4-1}$ on C_R we have

$$\left| \int_{C_R} \frac{1}{1+z^4} dz \right| \le \frac{\pi R}{R^4-1}$$

which approaches 0 as R approaches infinity. This gives us that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

To calculate the residues we use the formula from the book. In this case the formula boils down to plugging in the point to $\frac{1}{1+z^4}$ where removing the factor associated with the singularity. This gives us

$$\operatorname{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{2\pi i/4} + e^{\pi i/2})(2e^{i\pi/4})} = \frac{1}{4e^{3\pi i/4}}$$

and

$$\operatorname{Res}_{z=e^{3i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{6\pi i/4} + e^{3\pi i/2})(2e^{3i\pi/4})} = \frac{1}{4e^{9\pi i/4}}$$

Add them together and we get $\frac{-i\sqrt{2}}{4}$. Multiply it by $2\pi i$ and we get $\frac{\pi}{2}$. Which gives us that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

Problem 3 (3.8.4). Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0$$

Proof. First note that $\Im(\frac{ze^{iz}}{z^2+a^2}) = \frac{z \sin z}{z^2+a^2}$. We proceed by integrating the semicircle in the upper half plane, γ_R, which we can expand as

$$\int_{\gamma_R} \frac{ze^{iz}}{z^2 + a^2} dz = \int_{C_R} \frac{ze^{iz}}{z^2 + a^2} dz + \int_{-R}^{R} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2 + a^2}$$

First we will show that $\left| \int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz \right| \to 0$ as $R \to \infty$. Start by making the substitution $z = Re^{i\theta}$. Then we get

$$\begin{aligned}
\left| \int_{C_R} \frac{ze^{iz}}{z^2 + a^2} dz \right| &= \left| \int_0^{\pi} \frac{Re^{i\theta} e^{iRe^{i\theta}}}{R^2 e^{2i\theta} + a^2} Rie^{i\theta} d\theta \right| \\
&\leq \left| \int_0^{\pi} \frac{iR^2 e^{2i\theta} e^{iRe^{i\theta}}}{R^2 - a^2} d\theta \right| \\
&\leq \left| \int_0^{\pi} \frac{R^2 e^{iRe^{i\theta}}}{R^2 - a^2} d\theta \right| \\
&\leq \frac{R^2}{R^2 - a^2} \left| \int_0^{\pi} e^{iR\cos\theta - R\sin\theta} d\theta \right| \\
&\leq \frac{R^2}{R^2 - a^2} \left| \int_0^{\pi} e^{-R\sin\theta} d\theta \right|
\end{aligned}$$

At this point we use two inequalities related to $\sin\theta$. The first is that $\sin\theta \geq 2/\pi\theta$ if $0 \leq \theta \leq \pi/2$ and the other is that $\sin\theta \geq 1 - 2/\pi\theta$ when $\pi/2 \leq \theta \leq \pi$. This gives us

$$\begin{aligned}
\frac{R^2}{R^2 - a^2} \left| \int_0^{\pi} e^{-R\sin\theta} d\theta \right| &\leq \frac{R^2}{R^2 - a^2} \left(\int_0^{\pi/2} e^{-R\pi\theta/2} d\theta + \int_{\pi/2}^{\pi} e^{-R+2/\pi R\theta} d\theta \right) \\
&= \frac{R^2}{R^2 - a^2} \left(\frac{-2e^{-R\pi/2\theta}}{\pi R} \right|_0^{\pi/2} + \frac{R^2}{R^2 - a^2} \left(\frac{\pi e^{-R+2/\pi R\theta}}{2R} \right|_{\pi/2}^{\pi}
\end{aligned}$$

At this point it is clear that the above approaches zero as R approaches infinity. Thus

$$\int_{-R}^{R} \frac{ze^{iz}}{z^2 + a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2 + a^2}$$

To calculate the residue we compute

$$\begin{aligned}
\lim_{z \to ia} (z - ia) \frac{ze^{iz}}{z^2 + a^2} &= \frac{iae^{-a}}{2ia} \\
&= \frac{e^{-a}}{2}
\end{aligned}$$

When we put it all together we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx &= \Im \left(\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz \right) \\
&= \Im(2\pi i \cdot \frac{e^{-a}}{2}) \\
&= \pi e^{-a}
\end{aligned}$$

$\square$

Problem 4 (3.8.8). Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Proof. We start by rewriting $\cos\theta$ in terms of $e^{i\theta}$ which gives us

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} &= \int_0^{2\pi} \frac{d\theta}{a + b/2(e^{i\theta} + e^{-i\theta})} \\
&= 2\int_0^{2\pi} \frac{e^{i\theta}}{2ae^{i\theta} + be^{2i\theta} + b} d\theta
\end{aligned}$$

Then substitute $z = e^{i\theta}$ to get

$$\begin{aligned}
2\int_0^{2\pi} \frac{e^{i\theta}}{2ae^{i\theta} + be^{2i\theta} + b} d\theta &= \frac{2}{i}\int_{C_1} \frac{1}{bz^2 + 2az + b} dz \\
&= 4\pi \operatorname{Res} \frac{1}{bz^2 + 2az + b}
\end{aligned}$$

First we find the poles by factoring the bottom

$$\frac{1}{bz^2 + 2az + b} = \frac{1}{b(z - (-a/b + \sqrt{a^2/b^2 - 1}))(z - (-a/b - \sqrt{a^2/b^2 - 1}))}$$

The pole that occurs within the circle of radius 1 is $-a/b + \sqrt{a^2/b^2 - 1}$. We calculate the residue as

$$\begin{aligned}
\operatorname{Res}_{z=-a/b+\sqrt{a^2/b^2-1}} \frac{1}{bz^2 + 2az + b} &= \lim_{z \to -a/b+\sqrt{a^2/b^2-1}} \frac{(z - (-a/b + \sqrt{a^2/b^2 - 1}))}{b(z - (-a/b + \sqrt{a^2/b^2 - 1}))(z - (-a/b - \sqrt{a^2/b^2 - 1}))} \\
&= \frac{1}{b(-a/b + \sqrt{a^2/b^2 - 1} - (-a/b - \sqrt{a^2/b^2 - 1}))} \\
&= \frac{1}{2b\sqrt{a^2/b^2 - 1}} \\
&= \frac{1}{2\sqrt{a^2 - b^2}}
\end{aligned}$$

Which when we plug back in we get

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{4\pi}{2\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

$\square$

Problem 5 (3.8.9). Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

Hint: Use contour that goes down to 0 and up from 1.

Proof. First note that

$$\begin{aligned}
\log(1-e^{2\pi iz}) &= \log(e^{\pi iz}(-2i)(e^{\pi iz}+e^{-\pi iz})/2i) \\
&= \pi iz + \log(-2i) + \log(\sin \pi z) \\
&= \pi iz + \log(2) - i\pi/2 + \log(\sin \pi z)
\end{aligned}$$

If we integrate this with respect to z from 0 to 1 we get

$$\int_0^1 \log(1-e^{2\pi iz})dz = \pi i/2 + \log(2) - i\pi/2 + \int_0^1 \log(\sin \pi z)dz = \log(2) + \int_0^1 \log(\sin \pi z)dz$$

This will give us the desired equality if we show that $\int_0^1 \log(1-e^{2\pi iz})dz = 0$. To do this we integrate over the rectangle of width 1, height R, with two quarter circles of radius ϵ on the corners at 0 and 1 to avoid the branch points. Refer to this curve as $\gamma_{\epsilon,R}$. The entire integral will be zero as $\log(1-e^{2\cap iz})$ is holomorphic on the curve and its interior. Then we split the integral into six pieces

$$\begin{aligned}
0 = \int_{\gamma_{\epsilon,R}} \log(1-e^{2\pi iz})dz &= \int_{\epsilon}^{1-\epsilon} \log(1-e^{2\pi ix})dx && z = x \\
&+ \int_{\pi}^{\pi/2} \log(1-e^{2\pi i\epsilon e^{i\theta}})i\epsilon e^{i\theta}d\theta && z = \epsilon e^{i\theta} \\
&+ i\int_{\epsilon}^{R} \log(1-e^{2\pi i(1+it)})dt && z = 1+it \\
&+ \int_{1}^{0} \log(1-e^{2\pi i(t+iR)})dt && z = t+iR \\
&+ i\int_{R}^{\epsilon} \log(1-e^{2\pi i(it)})dt && z = it \\
&+ \int_{\pi/2}^{0} \log(1-e^{2\pi i\epsilon e^{i\theta}})d\theta && z = \epsilon e^{i\theta}
\end{aligned}$$

First note that for the two vertical portions of $\gamma_{\epsilon,R}$, the third and fifth, that

$$\log(1-e^{2\pi i(1+it)}) = \log(1-e^{2\pi i(it)})$$

as $e^{2\pi i} = 1$. Since the only difference then is that the limits of integration are swapped these two cancel eachother out. What is left to show is that

$$\left|\int_1^0 \log(1-e^{2\pi i(t+iR)})dt\right|, \quad \left|\int_{\pi}^{\pi/2} \log(1-e^{2\pi i\epsilon e^{i\theta}})i\epsilon e^{i\theta}d\theta\right|, \quad \left|\int_{\pi/2}^{0} \log(1-e^{2\pi i\epsilon e^{i\theta}})d\theta\right|$$

all approach zero as $\epsilon \to 0$ and $R \to \infty$.

Starting with the first we have

$$
\begin{aligned}
\left|\int_0^1 \log(1-e^{2\pi i(t+iR)})dt\right| &\leq \int_0^1 \left|\log(1-e^{-2\pi R}e^{2\pi it})\right| dt \\
&\leq \int_0^1 \log|1-e^{-2\pi R}e^{2\pi it}| + |i\operatorname{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(|1|+|e^{-2\pi R}e^{2\pi it}|) + |i\operatorname{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |i\operatorname{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\operatorname{Arg}(1-e^{-2\pi R}e^{2\pi it})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\arctan(\frac{\Im(1-e^{-2\pi R}e^{2\pi it})}{\Re(1-e^{-2\pi R}e^{2\pi it})})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\arctan(\frac{e^{-2\pi R}\sin(2\pi t)}{1-e^{2\pi R}\cos(2\pi t)})|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}\sin(2\pi t)}{1-e^{2\pi R}\cos(2\pi t)}|dt \\
&\leq \int_0^1 \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}}{1-e^{2\pi R}}|dt \\
&= \log(1+e^{-2\pi R}) + |\frac{e^{-2\pi R}}{1-e^{2\pi R}}|
\end{aligned}
$$

which goes to zero as $R \to \infty$. □

Problem 6 (3.8.10). Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx = \frac{\pi}{2a}\log a$$

Hint: Integrate over upper half annulus with inner radius ϵ and outer radius R.

Proof. □

Problem 7 (3.8.13). Suppose f is holomorphic in a punctured disc $D_r(z_0) \setminus \{z_0\}$. Suppose also that

$$|f(z)| \leq A|z-z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$, and all z near z_0. Show that the singularity of f at z_0 is removable.

Proof. We show this by proving the contrapositive. Suppose z_0 is a singularity that is not removable and let $\epsilon > 0$ and $A \in \mathbb{R}_{>0}$. Then it is either a pole of order k or essential.

If z_0 is a pole of order k then we can write f as $f(z) = \sum_{n=0}^{k} \frac{a_{-n}}{(z-z_0)^n} \cdot g(z)$ where g is holomorphic on $D_r(z_0)$. Then there exists a constant B such that $B|(z-z_0)^{-k}| < |f(z)|$ when z is sufficiently close to z_0. In addition when z is sufficiently close to z_0 we that $|f(z)| \geq B|z-z_0|^{-k} \geq A|z-z_0|^{-1+\epsilon}$ as $k \geq 1$.

On the other hand if k is an essential singularity we have a Laurant series in $D_r(z_0) \setminus \{z_0\}$ of the form

$$f(z) = \sum_0^\infty b_n(z - z_0)^{-n} + \sum_0^\infty a_n(z - z_0)^n$$

If we cut off the series for the negative powers at the first nonzero coefficient we get

$$g(z) = b_k(z - z_0)^k + \sum_0^\infty a_n(z - z_0)^n$$

where $|g| < |f|$ sufficiently close to z_0. Since g has a pole of order k at z_0 this then reduces to the case where we have a pole of order k where we shrink the radius of the disk such that $|g| < |f|$ holds within. $\square$

Problem 8 (3.9.3). If f is holomorphic in the deleted neighborhood $\{0 < |z - z_0| < r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{(z - z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z - z_0| < r\}$.

Proof. Since f has a pole of order k we can write f as

$$f(z) = (z - z_0)^k g(z)$$

where g is holomorphic. Moreover since g is holomorphic it is equal to its power series $g(z) = \sum_0^\infty a_n(z - z_0)^n$. Then we distribute to get

$$f(z) = \sum_{n=0}^{k-1} \frac{a_n}{(z - z_0)^{k-n}} + \sum_0^\infty a_{n+k}(z - z_0)^n$$

completing the proof. $\square$