Problem 1 (8.5.7). Provide all the details n the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points $z = iy$ with $0 < y < 1$.

(a) Show that if $re^{i\theta} = G(iy)$, then

$$re^{i\theta} = i\frac{\cos \pi y}{1 + \sin \pi y}$$

This leads to two separate cases: either $0 < y \leq 1/2$ and $\theta = \pi/2$ or $1/2 \leq y < 1$ and $\theta = -\pi/2$. In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \quad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}$$

(b) In the integral $\frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\widetilde{f}_0(\varphi)d\varphi$ make the change of variables $t = F(e^{i\varphi})$. Observe that

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}}$$

and then take the imaginary part and differentiate both sides to establish the two identities

$$\sin \varphi = \frac{1}{\cosh \pi t} \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}$$

Hence deduce that

$$\begin{aligned}\frac{1}{2\pi}\int_0^\pi P_r(\theta - \varphi)\widetilde{f}_0(\varphi)d\varphi &= \frac{1}{2\pi}\int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}\widetilde{f}_0(\varphi)d\varphi \\ &= \frac{\sin \pi y}{2}\int_{-\infty}^{\infty} \frac{f_0(t)}{\cosh \pi t - \cos \pi y}dt\end{aligned}$$

(c) Use a similar argument to prove the formula for the integral $\frac{1}{2\pi}\int_{-\pi}^0 P_r(\theta - \varphi)\widetilde{f}_1(\varphi)d\varphi$.

Proof. □

Problem 2 (8.5.9). Prove that the function u defined by

$$u(x, y) = \Re(\frac{i + z}{i - z}), \quad u(0, 1) = 0$$

is harmonic in the unit disc and vanishes on the boundary. Note that u is not bounded in $\mathbb{D}$.

Proof. □

Problem 3 (8.5.16). Let

$$f(z) = \frac{i - z}{i + z} \quad \text{and} \quad f^{-1}(w) = i\frac{1 - w}{1 + w}$$

(a) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az + b}{cz + d} = f^{-1}(e^{i\theta}f(z))$$

with ψ_a defined in Section 2.1.

(b) Given $\theta \in \mathbb{R}$, find real numbers a, b, c, d so that $ad - bc = 1$, and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1}(\psi_\alpha(f(z)))$$

with ψ_a defined in Section 2.1.

(c) Prove that if g is an automorphism of the unit disc, then there exist real numbers a, b, c, d such that $ad - bc = 1$ and so that for any $z \in \mathbb{H}$

$$\frac{az+b}{cz+d} = f^{-1} \circ g \circ f(z)$$

[Hint: Use parts (a) and (b)].

Proof. □

Problem 4 (8.5.20). Other examples of elliptic integrals providing conformal maps form the upper half-plane to rectangles providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}, \qquad \lambda \in \mathbb{R} \setminus \{1\}$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case $\lambda = -1$, the image of

$$\int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

Proof. □

Problem 5 (8.5.21). We consider the conformal mappings to triangles.

(a) Show that

$$\int_0^z z^{-\beta_1}(1-z)^{-\beta_2} dz$$

with $0 < \beta_1 < 1$, $0 < \beta_2 <$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of $0, 1$, and ∞, and with angles $\alpha_1\pi, \alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

(b) What happens when $\beta_1 + \beta_2 = 1$?

(c) What happens when $0 < \beta_1 + \beta_2 < 1$?

(d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is $\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)$.

Proof. □

Problem 6 (8.6.2). The angle between two non-zero complex numbers z and w (taken in that order) is simply the oriented angle, in $(-\pi, \pi]$, that is formed between the two vectors in $\mathbb{R}^2$ corresponding to the points z and w. This oriented angle, say α, is uniquely determined by the two quantities

$$\frac{(z,w)}{|z||w|} \quad \text{and} \quad \frac{(z,-iw)}{|z||w|}$$

which are simply the cosine and sine of α, respectively. Here, the notation $(\cdot,\cdot)$ corresponds to the usual Euclidean inner product in $\mathbb{R}^2$, which in terms of complex numbers takes the form $(z,w) = \Re(z\overline{w})$.

In particular, we may now consider two smooth curves $\gamma : [a,b] \to \mathbb{C}$ and $\eta : [a,b] \to \mathbb{C}$, that intersect at z_0, say $\gamma(t_0) = \eta(t_0) = z_0$ for some $t_0 \in (a,b)$. If the quantities $\gamma'(t_0)$ and $\eta'(t_0)$ are non-zero, then they represent the tangents to the curves γ and η at the point z_0, and we say that the two curves intersect at z_0 at the angle formed by the two vectors $\gamma'(t_0)$ and $\eta'(t_0)$.

A holomorphic function f defined near z_0 is said to **preserve angles** at z_0 if for any two smooth curves γ and η intersecting at z_0, the angle formed between the curves γ and η at z_0 equals the angle formed between the curves $f \circ \gamma$a nd $f \circ \eta$ at $f(z_0)$. In particular we assume that the tangents to the curves $\gamma, \eta, f \circ \gamma$, and $f \circ \eta$ at the point z_0 and $f(z_0)$ are all non-zero.

(a) Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'(z_0) \neq 0$, then f preserves angles at z_0. [Hint: Observe that

$$(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0)) = |f'(z_0|^2(\gamma'(t_0), \eta'(t_0)))$$

]

(b) Conversely, prove the following: suppose $f : \Omega \to \mathbb{C}$ is complex-valued function, that is real differentiable at $z_0 \in \Omega$, and $J_f(z_0) \neq 0$. If f preserves angles at z_0, then f is holomorphic at z_0 with $f'(z_0) \neq 0$.

Proof. □