Problem 1. *Consider the integral*

$$\Phi_n(z) = \int_a^b \frac{x^n dx}{(x-a)^{\frac{1}{2}-\lambda}(b-x)^{\frac{1}{2}+\lambda}(x-z)}$$

where $z \notin [a,b]$ *and* $\frac{-1}{2} < \lambda < \frac{1}{2}$.

(a) *Compute the integral for* $n = 0$ *and* $n = 1$.

(b) *Determine the domain where* $\phi_n(z)$ *is holomorphic.*

Hint: Consider the contour Γ *(circle entering to go about branch points) and fix a branch of the function* $\Gamma = C_R \cup \gamma^- \cup \gamma^+$ *and* $f(\zeta) = (\zeta - a)^{\frac{1}{2}-\lambda}(b-\zeta)^{\frac{1}{2}+\lambda}$ *where* $\zeta = x + iy$

Problem 2. *Show, by contour integration, that if* $a > 0$ *and* $\xi \in \mathbb{R}$ *then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}.$$

Proof. We start with the first integral in the case where $\xi \geq 0$. In this scenario we integrate over the lower semicircle of radius R. The integral of the outer portion C_{R^-} will approach zero by Jordan's lemma as $|\frac{a}{a^2+z^2}| \leq \frac{a}{R^2-a^2} \to 0$ as $R \to \infty$. Thus we have

$$\frac{1}{\pi}\int_{\infty}^{-\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

The residue is

$$2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = 2i \frac{a}{-2ia} e^{-2\pi i(-ia)|\xi|} = -e^{-2\pi a\xi}$$

Which, after swapping the bounds of integration, implies that when $\xi > 0$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}$$

However when $\xi < 0$ we instead integrate over the upper semicircle of radius R. For the same reason as above the outer portion of the integral approaches 0 as $R \to \infty$. This gives us

$$\frac{1}{\pi}\int_{-\infty}^{I} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

Similar to before if we do the residue calculation we get

$$2i \operatorname{Res}_{z=ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = 2i \frac{a}{2ia} e^{2\pi a\xi} = e^{-2\pi a\xi}$$

Therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|},$$

for all $\xi \in \mathbb{R}$.

To check the other direction □

Problem 3. *Prove that*

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2}=\sum_{n=-\infty}^{\infty}e^{-2\pi a|n|}$$

whenever $a>0$*. Hence show that the sum equals both* $\coth \pi a$*.*

Proof. By the previous problem and the fact that $\frac{a}{a^2+x^2}\in\mathcal{F}$ the two above sums are equal by the Poisson summation formula. Now we show that this sum is equal to $\coth \pi a$.

$$\begin{aligned}\coth \pi a &= \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}}\\ &= \frac{e^{\pi a}(1+e^{-2\pi a})}{e^{\pi a}(1-e^{-2\pi a})}\\ &= (1+e^{-2\pi a})\frac{1}{1-e^{-2\pi a}}\\ &= (1+e^{-2\pi a})\sum_{n=0}^{\infty}e^{-2n\pi a}\\ &= \sum_{n=0}^{\infty}e^{-2n\pi a}+\sum_{n=1}^{\infty}e^{-2n\pi a}\\ &= \sum_{n=-\infty}^{\infty}e^{-2|n|\pi a}\end{aligned}$$

□

Problem 4. *(a) Let* τ *be fixed with* $\Im(\tau)>0$*. Apply the Poisson summation formula to*

$$f(z)=(\tau+z)^{-k},$$

where k *is an integer* ≥ 2*, to obtain*

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^k}=\frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}m^{k-1}e^{2\pi i m\tau}.$$

(b) Set $k=2$ *in the above formula to show that if* $\Im(\tau)>0$*, then*

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\tau+n)^2}=\frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can one conclude that the above formula hold true whenever τ *is any complex number that is not an integer?*

(a) Since $f\in\mathcal{F}$ when $k\geq 2$ we can use the Poisson summation formula to show the equivalence of these two series. The Fourier transform of f is of the form

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)e^{-2\pi i x\xi}dx$$

When $\xi<0$ we calculate the above integral by integrating over the upper semicircle of radius R. The outer portion C_R will approach 0 by Jordan's lemma as $|f(R)|\leq\frac{1}{(|R|-|\tau|)^k}$

approaches zero as $R \to \infty$. Since $f(z)e^{-2\pi ix\xi}$ is holomorphic in the upper half plane the integral over the upper semicircle is also zero. This implies that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(z)e^{-2\pi iz\xi}dz = (\int_{C_R} + \int_{-R}^{R})f(z)e^{-2\pi iz\xi}dz = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx = 0$$

when $\xi < 0$.

On the other hand if $\xi > 0$, we integrate over the lower semicircle of radius R. The integral on C_{R^-} will be zero by Jordan's lemma as before. This gives us

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi ix\xi}}{(\tau+z)^k}$$

Next we calculate the residue

$$\begin{aligned}
\hat{f}(\xi) = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi ix\xi}}{(\tau+z)} &= -2\pi i \lim_{x\to-\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \frac{(\tau+x)^k e^{-2\pi ix\xi}}{(\tau+x)^k} \\
&= -2\pi i \lim_{x\to-\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} e^{-2\pi ix\xi} \\
&= -2\pi i \lim_{x\to-\tau} \frac{(-2\pi i\xi)^{k-1}}{(k-1)!} e^{-2\pi ix\xi} \\
&= \frac{(-2\pi i)^k \xi^{k-1}}{(k-1)!} e^{2\pi i\tau\xi}
\end{aligned}$$

Applying the Poisson summation formula with the above values for $\hat{f}(\xi)$ gives us that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi im\tau}.$$

(b) First we transform $\frac{\pi^2}{\sin^2(\pi\tau)}$ into a series.

$$\begin{aligned}
\frac{\pi^2}{\sin^2(\pi\tau)} &= \pi^2 \left(\frac{2i}{e^{i\pi\tau} - e^{-i\pi\tau}} \right)^2 \\
&= \frac{-4\pi^2}{e^{-i\pi\tau/2}(1-e^{2i\pi\tau})^2} \\
&= \frac{-4\pi^2}{e^{-2i\pi\tau}} \sum_{m=0}^{\infty} me^{2i\pi m\tau} \\
&= -4\pi^2 \sum_{1}^{\infty} me^{2i\pi m\tau}
\end{aligned}$$

The last step only works when $|e^{2\pi i\tau}| < 1$. However since $\Im(\tau) > 0$ this is in fact the case. Which exactly matches the series listed above when we plug in $k = 2$.

(c) This will not work if $\Im(\tau) > 0$ does not hold. The series in question may not be convergent otherwise.

Problem 5. *Compute the integral*

$$I = \int_0^\infty \frac{\log^2(x)}{x^2 + a^2} dx$$

Let Γ be the positively oriented upper half annulus of outer radius R and inner radius ϵ. The given function has a simple pole at ia inside Γ. This gives us

$$\int_\epsilon^R \frac{\ln^2(x)}{x^2 + a^2} dx + \int_{C_R} \frac{\log^2(z)}{z^2 + a^2} dz + \int_{C_\epsilon} \frac{\log^2(z)}{z^2 + a^2} dz + \int_{-R}^{-\epsilon} \frac{\log^2(z)}{z^2 + a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{\log^2(z)}{z^2 + a^2}$$

The integrals about C_R and C_ϵ both go to zero. For

$$\int_{-R}^{-\epsilon} \frac{\log^2(z)}{z^2 + a^2} dz$$

we make the substitution $z = -x$ giving us

$$\int_{-R}^{-\epsilon} \frac{\log^2(z)}{z^2 + a^2} dz = \int_\epsilon^R \frac{\log^2(-x)}{x^2 + a^2} dx$$

Which when rewriting $\log(-x)$ as $\ln(x) + i\pi$ we get

$$\int_\epsilon^R \frac{\log^2(-x)}{x^2 + a^2} dx = \int_\epsilon^R \frac{\ln^2(x) + \pi i \ln(x) - \pi^2}{x^2 + a^2} dx$$

Now rewrite the original equation as

$$2 \int_\epsilon^R \frac{\ln^2(x)}{x^2 + a^2} dx = 2\pi i \operatorname{Res}_{z=ia} \frac{\log^2(z)}{z^2 + a^2} + \pi^2 \int_\epsilon^R \frac{1}{x^2 + a^2} dx - 2\pi i \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx$$

The residue is

$$\operatorname{Res}_{z=ia} \frac{\log^2 z}{z^2 + a^2} = \frac{\log^2(ia)}{2ia} = \frac{\ln^2(a) + \pi i \ln(a) - \pi^2/4}{2ia}$$

The first integral on the right is of the derivative of $a^{-1}\arctan(x/a)$. Giving us

$$\int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{\pi}{2a}$$

The latter integral is from a previous assignment

$$\int_0^\infty \frac{\ln x}{x^2 + a^2} dx = \frac{\pi \ln a}{2a}$$

Putting this all together while letting $\epsilon \to 0$ and $R \to \infty$ we get

$$\begin{aligned} 2I &= 2\pi i \operatorname{Res}_{z=ia} \frac{\log^2(z)}{z^2 + a^2} + \pi^2 \int_\epsilon^R \frac{1}{x^2 + a^2} dx - 2\pi i \int_\epsilon^R \frac{\ln(x)}{x^2 + a^2} dx \\ &= \frac{\pi \ln^2 a}{a} + \frac{i\pi^2 \ln a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a} - \frac{i\pi^2 \ln a}{a} \\ &= \frac{\pi \ln^2 a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a} \end{aligned}$$

We can then conclude that

$$I = \int_0^\infty \frac{\log^2 x}{x^2 + a^2} dx = \frac{\pi \ln^2 a}{2a} - \frac{\pi^3}{8a} + \frac{\pi^3}{4a}$$