

*Problem 1 (8.5.7).* Provide all the details in the proof of the formula for the solution of the Dirichlet problem in a strip discussed in Section 1.3. Recall that it suffices to compute the solution at the points  $z = iy$  with  $0 < y < 1$ .

(a) Show that if  $re^{i\theta} = G(iy)$ , then

$$re^{i\theta} = i \frac{\cos \pi y}{1 + \sin \pi y}$$

This leads to two separate cases: either  $0 < y \leq 1/2$  and  $\theta = \pi/2$  or  $1/2 \leq y < 1$  and  $\theta = -\pi/2$ . In either case, show that

$$r^2 = \frac{1 - \sin \pi y}{1 + \sin \pi y} \quad \text{and} \quad P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}$$

(b) In the integral  $\frac{1}{2\pi} \int_0^\pi P_r(\theta - \varphi) \tilde{f}_0(\varphi) d\varphi$  make the change of variables  $t = F(e^{i\varphi})$ . Observe that

$$e^{i\varphi} = \frac{i - e^{\pi t}}{i + e^{\pi t}}$$

and then take the imaginary part and differentiate both sides to establish the two identities

$$\sin \varphi = \frac{1}{\cosh \pi t} \quad \text{and} \quad \frac{d\varphi}{dt} = \frac{\pi}{\cosh \pi t}$$

Hence deduce that

$$\begin{aligned} \frac{1}{2\pi} \int_0^\pi P_r(\theta - \varphi) \tilde{f}_0(\varphi) d\varphi &= \frac{1}{2\pi} \int_0^\pi \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi} \tilde{f}_0(\varphi) d\varphi \\ &= \frac{\sin \pi y}{2} \int_{-\infty}^\infty \frac{f_0(t)}{\cosh \pi t - \cos \pi y} dt \end{aligned}$$

(c) Use a similar argument to prove the formula for the integral  $\frac{1}{2\pi} \int_{-\pi}^0 P_r(\theta - \varphi) \tilde{f}_1(\varphi) d\varphi$ .

*Proof.* (a) By the definition of  $G$  we start with

$$re^{i\theta} = \frac{i - e^{i\pi y}}{i + e^{i\pi y}}$$

Replace the  $e^{i\theta}$  with sines and cosines, followed by multiplying the top and bottom by the conjugate of the bottom to get

$$\frac{i - e^{i\pi y}}{i + e^{i\pi y}} = \frac{i - \cos \pi y - i \sin \pi y}{i + \cos \pi y + i \sin \pi y} \cdot \frac{\cos \pi y - i(1 + \sin \pi y)}{\cos \pi y - i(1 + \sin \pi y)}$$

Which then simplifies to

$$i \frac{\cos \pi y}{1 + \sin \pi y}$$

In the first case listed above we lie on the positive imaginary axis, and in the second case listed above we are on the negative imaginary axis. Either way we have

$$\begin{aligned} (\pm ir) &= -r^2 = -\frac{\cos^2 \pi y}{(1 + \sin \pi y)^2} \\ r^2 &= \frac{1 - \sin^2 \pi y}{(1 + \sin \pi y)^2} \\ &= \frac{1 - \sin \pi y}{1 + \sin \pi y} \end{aligned}$$

By definition of  $P_r(\theta - \varphi)$  we have

$$P_r(\theta - \varphi) = \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2}$$

Substitute in for  $r^2$  and simplify to get

$$\begin{aligned} P_r(\theta - \varphi) &= \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} \\ &= \frac{1 - \frac{1 - \sin \pi y}{1 + \sin \pi y}}{1 - 2r \cos(\theta - \varphi) + \frac{1 - \sin \pi y}{1 + \sin \pi y}} \\ &= \frac{1 - \frac{1 - \sin \pi y}{1 + \sin \pi y}}{1 - 2r(\cos \theta \cos \varphi + \sin \theta \sin \varphi) + \frac{1 - \sin \pi y}{1 + \sin \pi y}} \cdot \frac{1 + \sin \pi y}{1 + \sin \pi y} \\ &= \frac{2 \sin \pi y}{2 - 2r(1 + \sin \pi y)(\sin \theta \sin \varphi)} \\ &= \frac{\sin \pi y}{1 - (e^{i(\theta + \pi/2)} \cos \pi y)(\sin \theta \sin \varphi)} \end{aligned}$$

From here since we know what  $\theta$  is, we can simplify to

$$P_r(\theta - \varphi) = \frac{\sin \pi y}{1 - \cos \pi y \sin \varphi}$$

(b)

(c)

□

*Problem 2* (8.5.9). Prove that the function  $u$  defined by

$$u(x, y) = \Re\left(\frac{i + z}{i - z}\right), \quad u(0, 1) = 0$$

is harmonic in the unit disc and vanishes on the boundary. Note that  $u$  is not bounded in  $\mathbb{D}$ .

*Proof.* Since  $u$  is the real part of a holomorphic function it is harmonic. If we write  $z = x + iy$  we can rewrite  $u(x, y)$  as

$$\Re\left(\frac{i + z}{i - z}\right) = \frac{1 - x^2 - y^2}{x^2 + (1 - y)^2} = \frac{1 - |z|^2}{x^2 + (1 - y)^2}$$

If  $|z| = 1$  and  $z \neq i$ , the top vanishes giving us that  $u(x, y) = 0$ . Since we decreed that  $u(0, 1) = 0$  we can conclude that  $u$  vanishes on  $\partial\mathbb{D}$ . □

*Problem 3* (8.5.16). Let

$$f(z) = \frac{i - z}{i + z} \quad \text{and} \quad f^{-1}(w) = i \frac{1 - w}{1 + w}$$

(a) Given  $\theta \in \mathbb{R}$ , find real numbers  $a, b, c, d$  so that  $ad - bc = 1$ , and so that for any  $z \in \mathbb{H}$

$$\frac{az + b}{cz + d} = f^{-1}(e^{i\theta} f(z))$$

(b) Given  $\theta \in \mathbb{R}$ , find real numbers  $a, b, c, d$  so that  $ad - bc = 1$ , and so that for any  $z \in \mathbb{H}$

$$\frac{az + b}{cz + d} = f^{-1}(\psi_\alpha(f(z)))$$

with  $\psi_\alpha$  defined in Section 2.1.

(c) Prove that if  $g$  is an automorphism of the unit disc, then there exist real numbers  $a, b, c, d$  such that  $ad - bc = 1$  and so that for any  $z \in \mathbb{H}$

$$\frac{az + b}{cz + d} = f^{-1} \circ g \circ f(z)$$

[Hint: Use parts (a) and (b)].

*Proof.* (a) The equivalent transformation is

$$\frac{\sin(\theta/2) + \cos(\theta/2)z}{\cos(\theta/2) - \sin(\theta/2)z}$$

To see this use begin  $f^{-1}(e^{i\theta}f(z))$  and simplify to get

$$\frac{(e^{i\theta} - 1) + iz(1 + e^{i\theta})}{i(1 + e^{i\theta}) - (e^{i\theta} - 1)z} =$$

(b) Similarly the equivalent transformation is

$$\frac{\Re(\alpha) - 1 + z\Im(\alpha)}{z(\Re(\alpha) + 1) - \Im(\alpha)} \cdot \frac{(1 - |\alpha|^2)^{-1/2}}{(1 - |\alpha|^2)^{-1/2}}$$

To derive this, begin with  $f^{-1} \circ \psi_\alpha \circ f(z)$  to get

$$\frac{i - i \frac{\alpha - \frac{i-z}{i+z}}{1 - \bar{\alpha} \frac{i-z}{i+z}}}{1 + \frac{\alpha - \frac{i-z}{i+z}}{1 - \bar{\alpha} \frac{i-z}{i+z}}} =$$

(c) From class we know that any automorphism of  $\mathbb{D}$  is the composition of a rotation  $\rho(z)$  and a  $\psi_\alpha(z)$ . Let  $g(z) = \rho \circ \psi_\alpha(z)$ . Then

$$\begin{aligned} f^{-1} \circ g \circ f(z) &= f^{-1} \circ \rho \circ \psi_\alpha \circ f(z) \\ &= (f^{-1} \circ \rho \circ f) \circ (f^{-1} \circ \psi_\alpha \circ f)(z) \end{aligned}$$

At this point we know that both  $f^{-1} \circ \rho \circ f$  and  $f^{-1} \circ \psi_\alpha \circ f$  are of the form  $\frac{az+b}{cz+d}$  with  $ad - bc = 1$ . Moreover since composition of two transformations of the above form gives another of its kind it must be that  $g(z)$  is of the form  $\frac{az+b}{cz+d}$ . □

*Problem 4* (8.5.20). Other examples of elliptic integrals providing conformal maps from the upper half-plane to rectangles providing conformal maps from the upper half-plane to rectangles are given below.

(a) The function

$$S(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}, \quad \lambda \in \mathbb{R} \setminus \{1\}$$

maps the upper half-plane conformally to a rectangle, one of whose vertices is the image of the point at infinity.

(b) In the case  $\lambda = -1$ , the image of

$$S(z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

is a square whose side lengths are  $\frac{\Gamma^2(1/4)}{2\sqrt{2}\pi}$ .

*Proof.* (a) By proposition 8.4.1, we fall into the second case, and so the image of the upper half plane is a quadrangle with corners  $S(0)$ ,  $S(1)$ ,  $S(\lambda)$ , and  $S(\infty)$  where the angle corresponding to  $S(\infty)$  is  $\pi/2$ . In addition we also get from the proposition that the rest of the angles are  $\pi/2$  since they are each to the  $1/2$  power. So the polygon must in fact be a rectangle.

(b) From above we know that the shape is a rectangle. All that is left to confirm is that the side lengths are  $\frac{\Gamma^2(1/4)}{2\sqrt{2}\pi}$ . Starting with the side  $S(0)$  to  $S(1)$  make the substitution  $\sqrt{u} = \zeta$  to get

$$\int_0^1 \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}}$$

□

*Problem 5* (8.5.21). We consider the conformal mappings to triangles.

(a) Show that

$$S(w) = \int_0^w z^{-\beta_1} (1-z)^{-\beta_2} dz$$

with  $0 < \beta_1 < 1$ ,  $0 < \beta_2 < 1$ , and  $1 < \beta_1 + \beta_2 < 2$ , maps  $\mathbb{H}$  to a triangle whose vertices are the images of  $0, 1$ , and  $\infty$ , and with angles  $\alpha_1\pi, \alpha_2\pi$ , and  $\alpha_3\pi$ , where  $\alpha_j + \beta_j = 1$  and  $\beta_1 + \beta_2 + \beta_3 = 2$ .

(b) What happens when  $\beta_1 + \beta_2 = 1$ ?

(c) What happens when  $0 < \beta_1 + \beta_2 < 1$ ?

(d) In (a), the length of the side of the triangle opposite angle  $\alpha_j\pi$  is  $\frac{\sin(\alpha_j\pi)}{\pi} \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)$ .

*Proof.* (a) Using proposition 8.4.1, we have that the image  $S(\mathbb{H})$  is a triangle with points at  $S(0)$ ,  $S(1)$ , and  $S(\infty)$ . The angles corresponding to each side are then  $\alpha_0\pi, \alpha_1\pi, \alpha_\infty\pi$  where  $\alpha_z = 1 - \beta_z$  and  $\beta_0 + \beta_1 + \beta_\infty = 2$  from the proposition.

(b) Following in a similar fashion to the last problem we get a “degenerate” triangle that is a straight line (angle at infinity is  $\pi$ ).

(c) We get a triangle with a vertex at infinity and a vertex at  $S(0), S(1)$  with non-intersecting rays emerging from each.

- (d) If we have one side the others follow from using law of sines. The length of the side corresponding to  $S(0)$  and  $S(1)$  has length  $|S(1) - S(0)| = |S(1)| = \left| \int_0^1 z^{-\beta_1} (1-z)^{-\beta_2} dz \right|$ .  $\square$

*Problem 6 (8.6.2).* The angle between two non-zero complex numbers  $z$  and  $w$  (taken in that order) is simply the oriented angle, in  $(-\pi, \pi]$ , that is formed between the two vectors in  $\mathbb{R}^2$  corresponding to the points  $z$  and  $w$ . This oriented angle, say  $\alpha$ , is uniquely determined by the two quantities

$$\frac{(z, w)}{|z||w|} \quad \text{and} \quad \frac{(z, -iw)}{|z||w|}$$

which are simply the cosine and sine of  $\alpha$ , respectively. Here, the notation  $(\cdot, \cdot)$  corresponds to the usual Euclidean inner product in  $\mathbb{R}^2$ , which in terms of complex numbers takes the form  $(z, w) = \Re(z\bar{w})$ .

In particular, we may now consider two smooth curves  $\gamma : [a, b] \rightarrow \mathbb{C}$  and  $\eta : [a, b] \rightarrow \mathbb{C}$ , that intersect at  $z_0$ , say  $\gamma(t_0) = \eta(t_0) = z_0$  for some  $t_0 \in (a, b)$ . If the quantities  $\gamma'(t_0)$  and  $\eta'(t_0)$  are non-zero, then they represent the tangents to the curves  $\gamma$  and  $\eta$  at the point  $z_0$ , and we say that the two curves intersect at  $z_0$  at the angle formed by the two vectors  $\gamma'(t_0)$  and  $\eta'(t_0)$ .

A holomorphic function  $f$  defined near  $z_0$  is said to **preserve angles** at  $z_0$  if for any two smooth curves  $\gamma$  and  $\eta$  intersecting at  $z_0$ , the angle formed between the curves  $\gamma$  and  $\eta$  at  $z_0$  equals the angle formed between the curves  $f \circ \gamma$  and  $f \circ \eta$  at  $f(z_0)$ . In particular we assume that the tangents to the curves  $\gamma, \eta, f \circ \gamma$ , and  $f \circ \eta$  at the point  $z_0$  and  $f(z_0)$  are all non-zero.

- (a) Prove that if  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, and  $f'(z_0) \neq 0$ , then  $f$  preserves angles at  $z_0$ . [Hint: Observe that

$$(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0)) = |f'(z_0)|^2(\gamma'(t_0), \eta'(t_0))$$

]

- (b) Conversely, prove the following: suppose  $f : \Omega \rightarrow \mathbb{C}$  is complex-valued function, that is real differentiable at  $z_0 \in \Omega$ , and  $J_f(z_0) \neq 0$ . If  $f$  preserves angles at  $z_0$ , then  $f$  is holomorphic at  $z_0$  with  $f'(z_0) \neq 0$ .

*Proof.* (a) From the problem description if the two listed quantities are preserved then so is the angle. Using the suggested inequality along with the chain rule we get

$$\begin{aligned} \frac{(f'(z_0)\gamma'(t_0), f'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} &= \frac{|f'(z_0)|^2(\gamma'(t_0), \eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} \\ &= \frac{(\gamma'(t_0), \eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|} \end{aligned}$$

And similarly

$$\begin{aligned} \frac{(f'(z_0)\gamma'(t_0), if'(z_0)\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} &= \frac{|f'(z_0)|^2(\gamma'(t_0), i\eta'(t_0))}{|f'(z_0)\gamma'(t_0)||f'(z_0)\eta'(t_0)|} \\ &= \frac{(\gamma'(t_0), i\eta'(t_0))}{|\gamma'(t_0)||\eta'(t_0)|} \end{aligned}$$

- (b) Since angles are preserved by  $f$  we can coerce the Cauchy-Riemann equations out of the two quantities above with  $z(t)$  and  $w(t) = f \circ z(t)$  and  $z(t)$  with  $\overline{w(t)}$ . Since they correlate to sine and cosine this will give us

$$(z, -iw) = 0, \quad (z, \overline{w}) = 0$$

**That's the plan anyways.**

□