Problem 1. *Consider the integral*

$$\Phi_n(z) = \int_a^b \frac{x^n dx}{(x-a)^{\frac{1}{2}-\lambda}(b-x)^{\frac{1}{2}+\lambda}(x-z)}$$

where $z \notin [a,b]$ *and* $-\frac{1}{2} < \lambda < \frac{1}{2}$.

(a) Compute the integral for $n = 0$ *and* $n = 1$.

(b) Determine the domain where $\phi_n(z)$ *is holomorphic.*

Hint: Consider the contour Γ *(circle entering to go about branch points) and fix a branch of the function* $\Gamma = C_R \cup \gamma^- \cup \gamma^+$ *and* $f(\zeta) = (\zeta - a)^{\frac{1}{2}-\lambda}(b-\zeta)^{\frac{1}{2}+\lambda}$ *where* $\zeta = x + iy$

(a) Let $f(w) = (w-a)^{1/2-\lambda}(b-w)^{1/2+\lambda}$ with $w = s + iy$. Then let $\widetilde{f}(w) = f_a(w)f_b(w)$ where

$$f_a(w) = |w-a|^{1/2-\lambda}e^{i\theta_a(1/2-\lambda)}, \quad f_b(w) = |b-w|^{1/2+\lambda}e^{i\theta_b(1/2+\lambda)}$$

with $\theta_a = \arg(w-a)$ and $\theta_b = \arg(b-w)$. Using the picture I will draw here

We do the work for finding branch cuts and the like to make it all consistent. We have a, b, ∞ as branch points with a branch cut along $[a,b]$.

$z = s + i0 s > b$	$\theta_a = 0 \ \theta_b = -\pi$	$\widetilde{f}(s+i0) = -i\lvert\widetilde{f}(s)\rvert e^{-i\pi\lambda}$
$z = s + i0 a \le s \le b$	$\theta_a = 0 \ \theta_b = 0$	$\widetilde{f}(s) = \lvert\widetilde{f}(s)\rvert$
$z = s + i0 s < a$	$\theta_a = 0 \ \theta_b = -\pi$	$\widetilde{f}(s+i0) = i\lvert\widetilde{f}(s)\rvert e^{-i\pi\lambda}$
$z = s - i0 a \le s \le b$	$\theta_a = 2\pi \ \theta_b = 0$	$\widetilde{f}(s-i0) = -\lvert\widetilde{f}(s)\rvert e^{-2i\pi\lambda}$
$z = s - i0 b < s$	$\theta_a = 2\pi \ \theta_b = \pi$	$\widetilde{f}(s+i0) = -i\lvert\widetilde{f}(s)\rvert e^{-i\pi\lambda}$

Now we integrate along the suggested curve. Since the function has only a single simple pole at z within the curve we have

$$\begin{aligned}\int_\Gamma \frac{w^n dw}{\widetilde{f}(w)(w-z)} &= \left(\int_{C_R} + \int_{\gamma_+} + \int_{\gamma_-} + \int_{C_{\epsilon,a-}} + \int_{C_{\epsilon,b+}} + \int_{C_{\epsilon,b+}} + \int_{C_{\epsilon,b-}}\right) \frac{w^n dw}{\widetilde{f}(w)(w-z)} \\ &= 2\pi i \operatorname{Res}_{w=z} \frac{w^n}{\widetilde{f}(w)(w-z)}\end{aligned}$$

We ignore the two fragments of the integral along $[b,R]$ since they cancel each other out. At this point the cases for $n = 0$ and $n = 1$ diverge. When $n = 0$ it the integral along C_R and the various C_ϵs as the top doesn't change, yet the bottom will grow at R^2 as $R \to \infty$. When $n = 1$ I'm not entirely sure how to make them vanish as once we account for arclength the top and bottom appear to grow at the same rate.

Regardless, as the pole is simple the residue is

$$2\pi i \operatorname{Res}_{w=z} \frac{w^n}{\widetilde{f}(w)(w-z)} = 2\pi i \frac{z^n}{\widetilde{f}(z)}$$

The last bit is to combine the two integrals along γ_+ and γ_-. Using the values deduced earlier we have

$$\int_{\gamma_+} \frac{w^n dw}{\widetilde{f}(w)(w-z)} + \int_{\gamma_-} \frac{w^n dw}{\widetilde{f}(w)(w-z)} = \int_a^b \frac{w^n dw}{\widetilde{f}(s+i0)(w-z)} + \int_b^a \frac{w^n dw}{\widetilde{f}(s-i0)(w-z)} = (1+e^{2\pi i\lambda}) \int_a^b \frac{s^n ds}{f(s)(s-z)}$$

Putting it all together we get, for $n = 0, 1$ that

$$\int_a^b \frac{s^n ds}{f(s)(s-z)} = \frac{2\pi i z^n}{\widetilde{f}(z)(1+e^{2\pi i\lambda})}$$

(b) The inside of the integral is holomorphic on $\mathbb{C} \backslash x$ for fixed $x \in (a,b)$. Moreover it is continuous on $\mathbb{C} \setminus [a,b] \times (a,b)$. As such we have that $\Phi_n(z)$ is holomorphic on $\mathbb{C} \setminus [a,b]$.

Problem 2. *Show, by contour integration, that if $a > 0$ and $\xi \in \mathbb{R}$ then*

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|},$$

and check that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}.$$

Proof. We start with the first integral in the case where $\xi \geq 0$. In this scenario we integrate over the lower semicircle of radius R. The integral of the outer portion C_{R^-} will approach zero by Jordan's lemma as $|\frac{a}{a^2+z^2}| \leq \frac{a}{R^2-a^2} \to 0$ as $R \to \infty$. Thus we have

$$\frac{1}{\pi} \int_{\infty}^{-\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

The residue is

$$2i \operatorname{Res}_{z=-ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} = 2i \frac{a}{-2ia} e^{-2\pi i(-ia)|\xi|} = -e^{-2\pi a\xi}$$

Which, after swapping the bounds of integration, implies that when $\xi \geq 0$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = e^{-2\pi a|\xi|}$$

However when $\xi < 0$ we instead integrate over the upper semicircle of radius R. For the same reason as above the outer portion of the integral approaches 0 as $R \to \infty$. This gives us

$$\frac{1}{\pi} \int_{-\infty}^{I} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} dx = 2i \operatorname{Res}_{z=ia} \frac{a}{a^2+z^2} e^{-2\pi i z\xi}$$

Similar to before if we do the residue calculation we get

$$2i\,\mathrm{Res}_{z=ia}\,\frac{a}{a^2+z^2}e^{-2\pi iz\xi} = 2i\frac{a}{2ia}e^{2\pi a\xi} = e^{-2\pi a\xi}$$

Therefore

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a}{a^2+x^2}e^{-2\pi ix\xi}dx = e^{-2\pi a|\xi|},$$

for all $\xi \in \mathbb{R}$.

To check the other direction split the integral into two pieces. The first for positive numbers and the other the negatives.

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi = \int_0^{\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi - \int_0^{-\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi$$

Substitute $\zeta = -\xi$ into the latter to get

$$\begin{aligned}
\int_0^{\infty} e^{-2\pi a\xi}e^{2\pi i\xi x}d\xi - \int_0^{-\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi &= \int_0^{\infty} e^{-2\pi a\xi}e^{2\pi i\xi x}d\xi + \int_0^{\infty} e^{-2\pi a\zeta}e^{-2\pi i\zeta x}d\zeta \\
&= \int_0^{\infty} e^{(-2\pi a+2\pi ix)\xi}d\xi + \int_0^{\infty} e^{(-2\pi a-2\pi ix)\zeta}d\zeta \\
&= \frac{1}{2\pi a-2i\pi x} + \frac{1}{2\pi a+2i\pi x} \\
&= \frac{1}{\pi}\frac{a}{a^2+x^2}
\end{aligned}$$

Which completes the second part of the problem. □

Problem 3. *Prove that*

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}$$

whenever $a > 0$. *Hence show that the sum equals both* $\coth \pi a$.

Proof. By the previous problem and the fact that $\frac{a}{a^2+x^2} \in \mathcal{F}$ the two above sums are equal by the Poisson summation formula. Now we show that this sum is equal to $\coth \pi a$.

$$\begin{aligned}
\coth \pi a &= \frac{e^{\pi a}+e^{-\pi a}}{e^{\pi a}-e^{-\pi a}} \\
&= \frac{e^{\pi a}(1+e^{-2\pi a})}{e^{\pi a}(1-e^{-2\pi a})} \\
&= (1+e^{-2\pi a})\frac{1}{1-e^{-2\pi a}} \\
&= (1+e^{-2\pi a})\sum_{n=0}^{\infty} e^{-2n\pi a} \\
&= \sum_{n=0}^{\infty} e^{-2n\pi a} + \sum_{n=1}^{\infty} e^{-2n\pi a} \\
&= \sum_{n=-\infty}^{\infty} e^{-2|n|\pi a}
\end{aligned}$$

□

Problem 4. *(a) Let τ be fixed with $\Im(\tau) > 0$. Apply the Poisson summation formula to*

$$f(z) = (\tau + z)^{-k},$$

where k is an integer ≥ 2, to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m\tau}.$$

(b) Set $k = 2$ in the above formula to show that if $\Im(\tau) > 0$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}.$$

(c) Can one conclude that the above formula hold true whenever τ is any complex number that is not an integer?

(a) Since $f \in \mathcal{F}$ when $k \geq 2$ we can use the Poisson summation formula to show the equivalence of these two series. The Fourier transform of f is of the form

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx$$

When $\xi < 0$ we calculate the above integral by integrating over the upper semicircle of radius R. The outer portion C_R will approach 0 by Jordan's lemma as $|f(R)| \leq \frac{1}{(|R|-|\tau|)^k}$ approaches zero as $R \to \infty$. Since $f(z)e^{-2\pi i x\xi}$ is holomorphic in the upper half plane the integral over the upper semicircle is also zero. This implies that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(z) e^{-2\pi i z\xi} dz = (\int_{C_R} + \int_{-R}^{R}) f(z) e^{-2\pi i z\xi} dz = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = 0$$

when $\xi < 0$.

On the other hand if $\xi > 0$, we integrate over the lower semicircle of radius R. The integral on C_{R^-} will be zero by Jordan's lemma as before. This gives us

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi i x\xi}}{(\tau+z)^k}$$

Next we calculate the residue

$$\begin{aligned}
\hat{f}(\xi) = -2\pi i \operatorname{Res}_{x=-\tau} \frac{e^{-2\pi i x\xi}}{(\tau+z)} &= -2\pi i \lim_{x \to -\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} \frac{(\tau+x)^k e^{-2\pi i x\xi}}{(\tau+x)^k} \\
&= -2\pi i \lim_{x \to -\tau} \frac{1}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} e^{-2\pi i x\xi} \\
&= -2\pi i \lim_{x \to -\tau} \frac{(-2\pi i\xi)^{k-1}}{(k-1)!} e^{-2\pi i x\xi} \\
&= \frac{(-2\pi i)^k \xi^{k-1}}{(k-1)!} e^{2\pi i \tau\xi}
\end{aligned}$$

Applying the Poisson summation formula with the above values for $\hat{f}(\xi)$ gives us that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m\tau}.$$

(b) First we transform $\frac{\pi^2}{\sin^2(\pi\tau)}$ into a series.

$$\begin{aligned}
\frac{\pi^2}{\sin^2(\pi\tau)} &= \pi^2\left(\frac{2i}{e^{i\pi\tau}-e^{-i\pi\tau}}\right)^2 \\
&= \frac{-4\pi^2}{e^{-i\pi\tau/2}(1-e^{2i\pi\tau})^2} \\
&= \frac{-4\pi^2}{e^{-2i\pi\tau}}\sum_{m=0}^{\infty} me^{2i\pi m\tau} \\
&= -4\pi^2\sum_{1}^{\infty} me^{2i\pi m\tau}
\end{aligned}$$

The last step only works when $|e^{2\pi i\tau}| < 1$. However since $\Im(\tau) > 0$ this is in fact the case. Which exactly matches the series listed above when we plug in $k = 2$.

(c) This will not work if $\Im(\tau) > 0$ does not hold. The series in question may not be convergent otherwise.

Problem 5. *Compute the integral*

$$I = \int_0^\infty \frac{\log^2(x)}{x^2+a^2}dx$$

Let Γ be the positively oriented upper half annulus of outer radius R and inner radius ϵ. The given function has a simple pole at ia inside Γ. This gives us

$$\int_\epsilon^R \frac{\ln^2(x)}{x^2+a^2}dx + \int_{C_R}\frac{\log^2(z)}{z^2+a^2}dz + \int_{C_\epsilon}\frac{\log^2(z)}{z^2+a^2}dz + \int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz = 2\pi i\operatorname{Res}_{z=ia}\frac{\log^2(z)}{z^2+a^2}$$

The integrals about C_R and C_ϵ both go to zero. For

$$\int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz$$

we make the substitution $z = -x$ giving us

$$\int_{-R}^{-\epsilon}\frac{\log^2(z)}{z^2+a^2}dz = \int_\epsilon^R \frac{\log^2(-x)}{x^2+a^2}dx$$

Which when rewriting $\log(-x)$ as $\ln(x) + i\pi$ we get

$$\int_\epsilon^R \frac{\log^2(-x)}{x^2+a^2}dx = \int_\epsilon^R \frac{\ln^2(x)+\pi i\ln(x)-\pi^2}{x^2+a^2}dx$$

Now rewrite the original equation as

$$2\int_\epsilon^R \frac{\ln^2(x)}{x^2+a^2}dx = 2\pi i\operatorname{Res}_{z=ia}\frac{\log^2(z)}{z^2+a^2} + \pi^2\int_\epsilon^R\frac{1}{x^2+a^2}dx - 2\pi i\int_\epsilon^R\frac{\ln(x)}{x^2+a^2}dx$$

The residue is

$$\operatorname{Res}_{z=ia}\frac{\log^2 z}{z^2+a^2} = \frac{\log^2(ia)}{2ia} = \frac{\ln^2(a)+\pi i\ln(a)-\pi^2/4}{2ia}$$

The first integral on the right is of the derivative of $a^{-1}\arctan(x/a)$. Giving us

$$\int_0^\infty \frac{1}{a^2+x^2}dx = \frac{\pi}{2a}$$

The latter integral is from a previous assignment

$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi \ln a}{2a}$$

Putting this all together while letting $\epsilon \to 0$ and $R \to \infty$ we get

$$\begin{aligned}
2I &= 2\pi i \operatorname{Res}_{z=ia} \frac{\log^2(z)}{z^2+a^2} + \pi^2 \int_\epsilon^R \frac{1}{x^2+a^2}dx - 2\pi i \int_\epsilon^R \frac{\ln(x)}{x^2+a^2}dx \\
&= \frac{\pi \ln^2 a}{a} + \frac{i\pi^2 \ln a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a} - \frac{i\pi^2 \ln a}{a} \\
&= \frac{\pi \ln^2 a}{a} - \frac{\pi^3}{4a} + \frac{\pi^3}{2a}
\end{aligned}$$

We can then conclude that

$$I = \int_0^\infty \frac{\log^2 x}{x^2+a^2}dx = \frac{\pi \ln^2 a}{2a} - \frac{\pi^3}{8a} + \frac{\pi^3}{4a}$$