

*Problem 1* (2.6.14). Suppose that  $f$  is holomorphic in an open set containing the closed unit disc, except for a pole at  $z_0$  on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of  $f$  in the open unit disc, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = z_0.$$

Only need to do case where degree of pole is greater than 1.

*Proof.* Tried doing the sum method. Currently attempting proof by induction. □

*Problem 2* (3.8.2). Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$$

Where are the poles of  $1/(1+z^4)$ ?

The poles of the function are at  $\pm e^{i\pi/4}, \pm e^{3i\pi/4}$ .

We evaluate this integral by integrating over the semicircle in the upper half plane of radius  $R$  which we will call  $\gamma_R$ . Then we have

$$\int_{\gamma_R} \frac{1}{1+z^4} dz = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

However since  $\frac{1}{1+z^4} \leq \frac{1}{R^4-1}$  on  $C_R$  we have

$$\left| \int_{C_R} \frac{1}{1+z^4} dz \right| \leq \frac{\pi R}{R^4-1}$$

which approaches 0 as  $R$  approaches infinity. This gives us that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx + \int_{C_R} \frac{1}{1+z^4} dz = 2\pi i \operatorname{Res}_{z=e^{i\pi/4}, e^{3i\pi/4}} \frac{1}{1+z^4}$$

To calculate the residues we use the formula from the book. In this case the formula boils down to plugging in the point to  $\frac{1}{1+z^4}$  where removing the factor associated with the singularity. This gives us

$$\operatorname{Res}_{z=e^{i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{2\pi i/4} + e^{\pi i/2})(2e^{i\pi/4})} = \frac{1}{4e^{3\pi i/4}}$$

and

$$\operatorname{Res}_{z=e^{3i\pi/4}} \frac{1}{1+z^4} = \frac{1}{(e^{6\pi i/4} + e^{3\pi i/2})(2e^{3i\pi/4})} = \frac{1}{4e^{9\pi i/4}}$$

Add them together and we get  $\frac{-i\sqrt{2}}{4}$ . Multiply it by  $2\pi i$  and we get  $\frac{\pi}{2}$ . Which gives us that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

Problem 3 (3.8.4). Show that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \quad \text{for all } a > 0$$

*Proof.* First note that  $\Im(\frac{ze^{iz}}{z^2+a^2}) = \frac{z \sin z}{z^2+a^2}$ . We proceed by integrating the semicircle in the upper half plane,  $\gamma_R$ , which we can expand as

$$\int_{\gamma_R} \frac{ze^{iz}}{z^2+a^2} dz = \int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz + \int_{-R}^R \frac{ze^{iz}}{z^2+a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2+a^2}$$

First we will show that  $\left| \int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz \right| \rightarrow 0$  as  $R \rightarrow \infty$ . Start by making the substitution  $z = Re^{i\theta}$ . Then we get

$$\begin{aligned} \left| \int_{C_R} \frac{ze^{iz}}{z^2+a^2} dz \right| &= \left| \int_0^\pi \frac{Re^{i\theta} e^{iRe^{i\theta}}}{R^2 e^{2i\theta} + a^2} Rie^{i\theta} d\theta \right| \\ &\leq \left| \int_0^\pi \frac{iR^2 e^{2i\theta} e^{iRe^{i\theta}}}{R^2 - a^2} d\theta \right| \\ &\leq \left| \int_0^\pi \frac{R^2 e^{iRe^{i\theta}}}{R^2 - a^2} d\theta \right| \\ &\leq \frac{R^2}{R^2 - a^2} \left| \int_0^\pi e^{iR \cos \theta - R \sin \theta} d\theta \right| \\ &\leq \frac{R^2}{R^2 - a^2} \left| \int_0^\pi e^{-R \sin \theta} d\theta \right| \end{aligned}$$

At this point we use two inequalities related to  $\sin \theta$ . The first is that  $\sin \theta \geq 2/\pi \theta$  if  $0 \leq \theta \leq \pi/2$  and the other is that  $\sin \theta \geq 1 - 2/\pi \theta$  when  $\pi/2 \leq \theta \leq \pi$ . This gives us **Lucas you did things horribly wrong**

$$\begin{aligned} \frac{R^2}{R^2 - a^2} \left| \int_0^\pi e^{-R \sin \theta} d\theta \right| &\leq \frac{R^2}{R^2 - a^2} \left( \int_0^{\pi/2} e^{-R\pi\theta/2} d\theta + \int_{\pi/2}^\pi e^{-R+2/\pi R\theta} d\theta \right) \\ &= \frac{R^2}{R^2 - a^2} \left( \frac{-2e^{-R\pi/2\theta}}{\pi R} \Big|_0^{\pi/2} + \frac{R^2}{R^2 - a^2} \left( \frac{\pi e^{-R+2/\pi R\theta}}{2R} \Big|_{\pi/2}^\pi \right) \right) \end{aligned}$$

At this point it is clear that the above approaches zero as  $R$  approaches infinity. Thus

$$\int_{-R}^R \frac{ze^{iz}}{z^2+a^2} dz = 2\pi i \operatorname{Res}_{z=ia} \frac{ze^{iz}}{z^2+a^2}$$

To calculate the residue we compute

$$\begin{aligned} \lim_{z \rightarrow ia} (z - ia) \frac{ze^{iz}}{z^2+a^2} &= \frac{iae^{-a}}{2ia} \\ &= \frac{e^{-a}}{2} \end{aligned}$$

When we put it all together we get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx &= \Im \left( \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2} dz \right) \\ &= \Im \left( 2\pi i \cdot \frac{e^{-a}}{2} \right) \\ &= \pi e^{-a}\end{aligned}$$

□

*Problem 4 (3.8.8).* Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if  $a > |b|$  and  $a, b \in \mathbb{R}$ .

*Proof.* We start by rewriting  $\cos \theta$  in terms of  $e^{i\theta}$  which gives us

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} &= \int_0^{2\pi} \frac{d\theta}{a + b/2(e^{i\theta} + e^{-i\theta})} \\ &= 2 \int_0^{2\pi} \frac{e^{i\theta}}{2ae^{i\theta} + be^{2i\theta} + b} d\theta\end{aligned}$$

Then substitute  $z = e^{i\theta}$  to get

$$\begin{aligned}2 \int_0^{2\pi} \frac{e^{i\theta}}{2ae^{i\theta} + be^{2i\theta} + b} d\theta &= \frac{2}{i} \int_{C_1} \frac{1}{bz^2 + 2az + b} dz \\ &= 4\pi \operatorname{Res} \frac{1}{bz^2 + 2az + b}\end{aligned}$$

First we find the poles by factoring the bottom

$$\frac{1}{bz^2 + 2az + b} = \frac{1}{b(z - (-a/b + \sqrt{a^2/b^2 - 1}))(z - (-a/b - \sqrt{a^2/b^2 - 1}))}$$

The pole that occurs within the circle of radius 1 is  $-a/b + \sqrt{a^2/b^2 - 1}$ . We calculate the residue as

$$\begin{aligned}\operatorname{Res}_{z=-a/b+\sqrt{a^2/b^2-1}} \frac{1}{bz^2 + 2az + b} &= \lim_{z \rightarrow -a/b+\sqrt{a^2/b^2-1}} \frac{(z - (-a/b + \sqrt{a^2/b^2 - 1}))}{b(z - (-a/b + \sqrt{a^2/b^2 - 1}))(z - (-a/b - \sqrt{a^2/b^2 - 1}))} \\ &= \frac{1}{b(-a/b + \sqrt{a^2/b^2 - 1} - (-a/b - \sqrt{a^2/b^2 - 1}))} \\ &= \frac{1}{2b\sqrt{a^2/b^2 - 1}} \\ &= \frac{1}{2\sqrt{a^2 - b^2}}\end{aligned}$$

Which when we plug back in we get

$$\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{4\pi}{2\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

□

Problem 5 (3.8.9). Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

Hint: Use contour that goes down to 0 and up from 1.

*Proof.* First note that

$$\begin{aligned} \log(1 - e^{2\pi iz}) &= \log(e^{\pi iz}(-2i)(e^{\pi iz} + e^{-\pi iz})/2i) \\ &= \pi iz + \log(-2i) + \log(\sin \pi z) \\ &= \pi iz + \log(2) - i\pi/2 + \log(\sin \pi z) \end{aligned}$$

If we integrate this with respect to  $z$  from 0 to 1 we get

$$\int_0^1 \log(1 - e^{2\pi iz}) dz = \pi i/2 + \log(2) - i\pi/2 + \int_0^1 \log(\sin \pi z) dz = \log(2) + \int_0^1 \log(\sin \pi z) dz$$

This will give us the desired equality if we show that  $\int_0^1 \log(1 - e^{2\pi iz}) dz = 0$ . To do this we integrate over the rectangle of width 1, height  $R$ , with two quarter circles of radius  $\epsilon$  on the corners at 0 and 1 to avoid the branch points. Refer to this curve as  $\gamma_{\epsilon, R}$ . The entire integral will be zero as  $\log(1 - e^{2\pi iz})$  is holomorphic on the curve and its interior. Then we split the integral into six pieces

$$\begin{aligned} 0 &= \int_{\gamma_{\epsilon, R}} \log(1 - e^{2\pi iz}) dz = \int_{\epsilon}^{1-\epsilon} \log(1 - e^{2\pi ix}) dx & z = x \\ &+ \int_{\pi}^{\pi/2} \log(1 - e^{2\pi i(1+\epsilon e^{i\theta})}) i\epsilon e^{i\theta} d\theta & z = 1 + \epsilon e^{i\theta} \\ &+ i \int_{\epsilon}^R \log(1 - e^{2\pi i(1+it)}) dt & z = 1 + it \\ &+ \int_1^0 \log(1 - e^{2\pi i(t+iR)}) dt & z = t + iR \\ &+ i \int_R^{\epsilon} \log(1 - e^{2\pi i(it)}) dt & z = it \\ &+ \int_{\pi/2}^0 \log(1 - e^{2\pi i\epsilon e^{i\theta}}) d\theta & z = \epsilon e^{i\theta} \end{aligned}$$

First note that for the two vertical portions of  $\gamma_{\epsilon, R}$ , the third and fifth, that

$$\log(1 - e^{2\pi i(1+it)}) = \log(1 - e^{2\pi i(it)})$$

as  $e^{2\pi i} = 1$ . Since the only difference then is that the limits of integration are swapped these two cancel each other out. What is left to show is that

$$\left| \int_1^0 \log(1 - e^{2\pi i(t+iR)}) dt \right|, \quad \left| \int_{\pi}^{\pi/2} \log(1 - e^{2\pi i\epsilon e^{i\theta}}) i\epsilon e^{i\theta} d\theta \right|, \quad \left| \int_{\pi/2}^0 \log(1 - e^{2\pi i\epsilon e^{i\theta}}) d\theta \right|$$

all approach zero as  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$ .

Starting with the first we have

$$\begin{aligned}
\left| \int_0^1 \log(1 - e^{2\pi i(t+iR)}) dt \right| &\leq \int_0^1 |\log(1 - e^{-2\pi R} e^{2\pi it})| dt \\
&\leq \int_0^1 \log|1 - e^{-2\pi R} e^{2\pi it}| + |i \operatorname{Arg}(1 - e^{-2\pi R} e^{2\pi it})| dt \\
&\leq \int_0^1 \log(|1| + |e^{-2\pi R} e^{2\pi it}|) + |i \operatorname{Arg}(1 - e^{-2\pi R} e^{2\pi it})| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + |i \operatorname{Arg}(1 - e^{-2\pi R} e^{2\pi it})| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + |\operatorname{Arg}(1 - e^{-2\pi R} e^{2\pi it})| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + \left| \arctan\left(\frac{\Im(1 - e^{-2\pi R} e^{2\pi it})}{\Re(1 - e^{-2\pi R} e^{2\pi it})}\right) \right| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + \left| \arctan\left(\frac{e^{-2\pi R} \sin(2\pi t)}{1 - e^{-2\pi R} \cos(2\pi t)}\right) \right| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + \left| \frac{e^{-2\pi R} \sin(2\pi t)}{1 - e^{-2\pi R} \cos(2\pi t)} \right| dt \\
&\leq \int_0^1 \log(1 + e^{-2\pi R}) + \left| \frac{e^{-2\pi R}}{1 - e^{-2\pi R}} \right| dt \\
&= \log(1 + e^{-2\pi R}) + \left| \frac{e^{-2\pi R}}{1 - e^{-2\pi R}} \right|
\end{aligned}$$

which goes to zero as  $R \rightarrow \infty$ .

**Check logarithm inequalities. Might not work for  $\epsilon$ .** Next we show that  $|\int_{\pi/2}^0 \log(1 - e^{2\pi i \epsilon e^{i\theta}}) d\theta| \rightarrow 0$  as  $\epsilon \rightarrow 0$ .

$$\begin{aligned}
\left| \int_{\pi/2}^0 \log(1 - e^{2\pi i \epsilon e^{i\theta}}) \epsilon i e^{i\theta} d\theta \right| &\leq \int_0^{\pi/2} |\log(1 - e^{2\pi i \epsilon e^{i\theta}}) \epsilon i e^{i\theta}| d\theta \\
&\leq \epsilon \int_0^{\pi/2} |\log(1 - e^{2\pi i \epsilon e^{i\theta}})| d\theta \\
&\leq \epsilon \int_0^{\pi/2} |\log|1 - e^{2\pi i \epsilon e^{i\theta}}| + i \operatorname{Arg}(1 - e^{2\pi i \epsilon e^{i\theta}})| d\theta \\
&\leq \epsilon \int_0^{\pi/2} \log(1 + |e^{2\pi i \epsilon e^{i\theta}}|) + |\operatorname{Arg}(1 - e^{2\pi i \epsilon e^{i\theta}})| d\theta \\
&\leq \epsilon \int_0^{\pi/2} \log(1 + |e^{2\pi i \epsilon e^{i\theta}}|) + \pi d\theta \\
&\leq \epsilon \int_0^{\pi/2} \log(1 + e^{-2\pi \epsilon \sin \theta}) + \pi d\theta \\
&\leq \epsilon \int_0^{\pi/2} \log(1 + e^{2\pi \epsilon}) + \pi d\theta \\
&= \epsilon \cdot \frac{\pi}{2} \log(1 + e^{2\pi \epsilon})
\end{aligned}$$

Which at this point clearly goes to zero as  $\epsilon \rightarrow 0$ . The last quarter-circle integral is identical once

we note that

$$\int_{\pi}^{\pi/2} \log(1 - e^{2\pi i(1+\epsilon e^{i\theta})}) i\epsilon e^{i\theta} d\theta = \int_{\pi}^{\pi/2} \log(1 - e^{2\pi i(\epsilon e^{i\theta})}) i\epsilon e^{i\theta} d\theta$$

We can then bound this integral the same as the prior. Since the arc length of the quarter-circle is the same we can safely conclude that this integral also approaches zero as  $\epsilon \rightarrow 0$ .

Therefore, since every portion of  $\int_{\gamma_{\epsilon,R}} \log(1 - e^{2\pi iz})$  is zero aside from  $\int_{-\epsilon}^{\epsilon} \log(\sin(\pi x)) dx$ , we can conclude that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2$$

□

*Problem 6 (3.8.10).* Show that if  $a > 0$ , then

$$\int_0^{\infty} \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a$$

Hint: Integrate over upper half annulus with inner radius  $\epsilon$  and outer radius  $R$ .

*Proof.* Let  $\gamma_{\epsilon,R}$  denote the curve around half annulus in the upper half plane with inner radius  $\epsilon$  and outer radius  $R$ . The function  $\frac{\log z}{z^2 + a^2}$  has a pole at  $ia$ . Then the integral over  $\gamma_{\epsilon,R}$  is

$$\begin{aligned} \int_{\gamma_{\epsilon,R}} \frac{\log z}{z^2 + a^2} dz &= 2\pi i \operatorname{Res}_{z=ia} \frac{\log z}{z^2 + a^2} \\ &= \int_{C_R} \frac{\log z}{z^2 + a^2} dz \\ &\quad + \int_{C_{\epsilon}} \frac{\log z}{z^2 + a^2} dz \\ &\quad + \int_{\epsilon}^R \frac{\log x}{x^2 + a^2} dx \\ &\quad - \int_R^{\epsilon} \frac{\log -x}{x^2 + a^2} dx \end{aligned}$$

First we show that the integral on  $C_R$  approaches 0 as  $R \rightarrow \infty$ .

$$\left| \int_{C_R} \frac{\log z}{z^2 + a^2} dz \right| \leq \pi R \cdot \frac{R + i\pi}{R^2 - a^2}$$

This approaches zero as  $R$  approaches infinity.

Next we show that the integral along  $C_{\epsilon}$  approaches 0 as  $\epsilon \rightarrow 0$ .

$$\left| \int_{C_{\epsilon}} \frac{\log z}{z^2 + a^2} dz \right| \leq$$

The residue calculation is

$$2\pi i \operatorname{Res}_{z=ia} \frac{\log z}{z^2 + a^2} = 2\pi i \frac{\log ia}{2ia} = \pi \log a + \frac{i\pi^2}{2}$$

Note that for the last integral, since we are on the principle branch we have

$$-\int_R^\epsilon \frac{\log -x}{x^2 + a^2} dx = \int_\epsilon^R \frac{\log x}{x^2 + a^2} dx + \int_\epsilon^R \frac{i\pi}{x^2 + a^2} dx$$

Which gives us that

$$2 \int_0^\infty \frac{\log x}{x^2 + a^2} dx + i\pi \int_0^\infty \frac{1}{x^2 + a^2} dx = \pi \log a + \frac{i\pi^2}{2}$$

Since we know that  $i\pi \int_0^\infty \frac{1}{x^2 + a^2} dx = \frac{i\pi^2}{2}$  we get

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi \log a}{2}$$

as desired. □

*Problem 7* (3.8.13). Suppose  $f$  is holomorphic in a punctured disc  $D_r(z_0) \setminus \{z_0\}$ . Suppose also that

$$|f(z)| \leq A|z - z_0|^{-1+\epsilon}$$

for some  $\epsilon > 0$ , and all  $z$  near  $z_0$ . Show that the singularity of  $f$  at  $z_0$  is removable.

*Proof.* We show this by proving the contrapositive. Suppose  $z_0$  is a singularity that is not removable and let  $\epsilon > 0$  and  $A \in \mathbb{R}_{>0}$ . Then it is either a pole of order  $k$  or essential.

If  $z_0$  is a pole of order  $k$  then we can write  $f$  as  $f(z) = \sum_{n=0}^k \frac{a_{-n}}{(z-z_0)^n} \cdot g(z)$  where  $g$  is holomorphic on  $D_r(z_0)$ . Then there exists a constant  $B$  such that  $B|(z-z_0)^{-k}| < |f(z)|$  when  $z$  is sufficiently close to  $z_0$ . In addition when  $z$  is sufficiently close to  $z_0$  we that  $|f(z)| \geq B|z-z_0|^{-k} \geq A|z-z_0|^{-1+\epsilon}$  as  $k \geq 1$ .

On the other hand if  $k$  is an essential singularity we have a Laurant series in  $D_r(z_0) \setminus \{z_0\}$  of the form

$$f(z) = \sum_0^\infty b_n(z-z_0)^{-n} + \sum_0^\infty a_n(z-z_0)^n$$

If we cut off the series for the negative powers at the first nonzero coefficient we get

$$g(z) = b_k(z-z_0)^k + \sum_0^\infty a_n(z-z_0)^n$$

where  $|g| < |f|$  sufficiently close to  $z_0$ . Since  $g$  has a pole of order  $k$  at  $z_0$  this then reduces to the case where we have a pole of order  $k$  where we shrink the radius of the disk such that  $|g| < |f|$  holds within. □

*Problem 8* (3.9.3). If  $f$  is holomorphic in the deleted neighborhood  $\{0 < |z - z_0| < r\}$  and has a pole of order  $k$  at  $z_0$ , then we can write

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{(z-z_0)} + g(z)$$

where  $g$  is holomorphic in the disc  $\{|z - z_0| < r\}$ .

*Proof.* Since  $f$  has a pole of order  $k$  we can write  $f$  as

$$f(z) = (z - z_0)^k g(z)$$

where  $g$  is holomorphic. Moreover since  $g$  is holomorphic it is equal to its power series  $g(z) = \sum_0^\infty a_n (z - z_0)^n$ . Then we distribute to get

$$f(z) = \sum_{n=0}^{k-1} \frac{a_n}{(z - z_0)^{k-n}} + \sum_0^\infty a_{n+k} (z - z_0)^n$$

completing the proof. □