Problem 1 (1.4.7). *The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.*

(a) *Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that*

$$\left|\frac{w-z}{1-\bar{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\bar{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Hint: Wy can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\bar{w}) \leq (1-rw)(1-r\bar{w})$$

with equality for appropriate r and $|w|$.

(b) *Prove that for a fixed w in the unit disc $\mathbb{D}$, the mappings*

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

satisfies the following conditions:

(i) *F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.*

(ii) *F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.*

(iii) *$|F(z)| = 1$ if $|z| = 1$.*

(iv) *$F : \mathbb{D} \to \mathbb{D}$ is bijective. Hint: Calculate $F \circ F$.*

Proof. □

Problem 2 (1.4.9). *Show that in polar coordinates, the Cauchy-Riemann equations take the form*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Proof. □

Problem 3 (1.4.10). *Show that*

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \Delta,$$

*where Δ is the **Laplacian***

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Proof. □

Problem 4 (1.4.13). *Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:*

(a) $\Re(f)$ is constant;

(b) $\Im(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Proof. □

Problem 5 (1.4.17). *Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

Proof. □

Problem 6 (2.6.1). *Prove that*

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

These are the ***Fresnel Integrals****. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty}\int_0^R$.*
Hint: Integrate the function e^{-x^2} over the $\pi/4$ semicircle thing. Recall that $\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}$.

Proof. □

Problem 7 (2.6.11). *Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.*

Proof. (a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi}\Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right))d\varphi.$$

(b) Show that

$$\Re\left(\frac{Re^{i\gamma} - r}{Re^{i\gamma} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\gamma + r^2}.$$

Hint: For the first part, note that if $w = R^2/\bar{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity. $\square$

Problem 8 (2.6.14)**.** *Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if*

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$$

Proof. $\square$