Problem 1. *Let X denote the surface of genus two with a single boundary component. Let A denote the boundary of X. Compute the relative homology groups $H_p(X, A)$.*

Proof. The surface of genus two with a single boundary component will be a 2-holed torus with a disk removed where the boundary of the disk is A. From Hatcher we have that $H_p(X, A) \cong \widetilde{H}_p(X/A)$. However X/A is homeomorphic to the 2-holed torus. As such the relative homology will be

$$H_p(X, A) = \begin{cases} 0 & p = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & p = 1 \\ \mathbb{Z} & p = 2 \end{cases}$$

□

Problem 2. *Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but that their universal covering spaces do not.*

Proof. The first space $S^1 \times S^1$ is the torus. As such its homology groups are

$$H_p(T^2) = \begin{cases} \mathbb{Z} & p = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & p = 1 \\ 0 & \text{otherwise} \end{cases}$$

For the latter space we can use the fact that the connected sum corresponds to the direct sum of the reduced homology groups. Since for S^n the only nonzero reduced homology group is $H_n(S^n) = \mathbb{Z}$ we have

$$H_p(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & p = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & p = 1 \\ 0 & \text{otherwise} \end{cases}$$

which shows that their homology groups are isomorphic.

The universal cover of $S^1 \times S^1 \cong T^2$ is the grid $\mathbb{Z} \times \mathbb{Z}$ wherein each point is connected to its neighbors by a unit interval. The universal cover of $S^1 \vee S^1 \vee S^2$ will be the Cayley graph of $\mathbb{Z} * \mathbb{Z}$ where each point has an S^2 attached to it. The covering space of the former does not have any two cells which implies that $\widetilde{H}_2(T^2) = 0$. However since there are no three cells and each two sphere is attached at only one point $\widetilde{H}_2(S^1 \vee S^1 \vee S^2)$ will be nontrivial.

Therefore the homology of the universal covers of $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are not isomorphic even though their homologies are. □

Problem 3. *Group the following spaces into homotopy equivalence classes. Justify your answers.*

(a) Möbius band

(b) The torus

(c) $B^2 \times S^1$

(d) The torus minus one point

(e) The torus minus two points

(f) The Klein bottle minus one point

(g) $\mathbb{R}^3$ minus the z-axis

(h) $\mathbb{R}^3$ *minus the unit circle in the xy-plane,* $\{x^2 + y^2 = 0, z = 0\}$

(i) The intersection of (g) *and* (h)

(j) S^3 *minus two linked circles*

(k) S^3 *minus two unlinked circles*

There are many pictures attached showing important points as well as explanation attached below. The equivalence classes are

S^1	T^2	$S^1 \vee S^1$	$S^1 \vee S^1 \vee S^1$	$S^1 \vee (S^1 \vee S^2)$	$S^1 \vee S^2$
(a)	(b)	(d)	(e)	(k)	(h)
(c)	(i)	(f)			
(g)	(j)				

(a) For the Möbius band there is a circle running along the center with an interval attached. We can squish the interval down to a point to get a circle.

(b) The torus is a torus.

(c) Each point of the circle has a disk attached that we can squish down to a point.

(d) If we draw the torus as a square with sides identified and poke a hole in the middle we can deform the punctured disk to its boundary which is $S^1 \vee S^1$.

(e) Similar to (d) except when we puncture two holes we have to retract the part to a line between the holes.

(f) This is the same process as the torus in that when we puncture the disk we can pull it back to the boundary.

(g) We can flatten it to $\mathbb{R}^2 \setminus \{0\}$ which is homotopy equivalent to a circle.

(h) This space is almost a torus however having the z-axis left interferes with that. As such we can pull the space down to get an S^2 except with the two points on the z-axis attached. Pull them apart leaving an interval connecting them and then move the two attachment points together to get $S^1 \vee S^2$.

(i) If we take a look at a vertical slice we get an open half plane minus a point. This will be homotopy equivalent to a circle. Since there is one for each point of the circle this is a Torus.

(j) Being in S^3, as opposed to $\mathbb{R}^3$ lets us effectively take one of the circles and turn it into a line by putting it through "the point at infinity" of $\mathbb{R}^3$. If we do this with two linked circles we get a line going through a circle which is a space homeomorphic to (h).

(k) We do the same process as above except we end up with a line not going through the circle. This ends up being as shown in the picture the wedge of $\mathbb{R}^3$ with a line taken out and an $\mathbb{R}^3$ with the circle taken out. Thus giving us $S^1 \vee (S^1 \vee S^2)$.

Problem 4. *Compute the homology of the CW complex obtained from the cube* $I \times I \times I$ *by identifying opposite faces after a* $1/4$ *twist.*

Proof. The picture above is a CW complex X. The boundary maps will be

$$\begin{aligned}
\partial I &= E - E + F - F + G = 0 \\
\partial E &= a + b + c + d \\
\partial F &= a - b - c + d \\
\partial G &= a + b - c - d \\
\partial a &= v - u \\
\partial b &= u - v \\
\partial c &= v - u \\
\partial d &= u - v \\
\partial u &= 0 \\
\partial v &= 0
\end{aligned}$$

We'll start with $H_3(X)$. Since there are no 4-cells we have that $\mathrm{Im}(\partial_4) = 0$ and since $\partial I = 0$ it must be that $H_3(X) \cong \mathbb{Z}$.

Next for $H_2(X)$ the image of ∂_3 is zero. As such $H_2(X) \cong \ker \partial_2$. If we look at ∂_2 as a matrix it will be

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

If we reduce this to Smith normal form we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

for which the kernel is zero. Therefore $H_2(X) = 0$.

Next we will compute $H_1(X)$. Using the matrix above we have that $\mathrm{Im}\partial_2 = \langle a, 2b, 2c\rangle$. Since $\partial_1 a = \partial_1 c = -\partial_1 b = -\partial_1 d$ the kernel of ∂_1 will linear combinations where the number of a, cs and the number of b, ds are equal. If we mod this out by the image of ∂_2 we have $a = 0$, d is left alone, $2b = 0$, and $2c = 0$. However since d and c have to agree with the as and bs we are left with $\langle b, c | 2b = 0, 2c = 0 \rangle$. This gives us that $H_1(X) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Finally $H_0(X) \cong \mathbb{Z}$ as there is one connected component.

Thus the homology groups for this space are

$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0, 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & p = 1 \\ 0 & p = 2 \end{cases}$$

□

Problem 5. *A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all $x \in S^n$ is called and even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree. [Hint: First show that if f is even, then it necessarily factors as a composition $S^n \to \mathbb{R}P^n \to S^n$.]*

Proof. Let f be a map from S^n to S^n such that $f(x) = f(-x)$ for all $x \in S^n$. Since $\mathbb{R}P^n$ is a quotient space of S^n where antipodal points are identified any even map from S^n respects equivalence classes for the quotient space and as such it factors as

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "q"] & S^n \\ \mathbb{R}P^n \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

First note that $\deg q = 2$. This is because for any point $x \in \mathbb{R}P^n$ it will have two points mapping to it from S^n and it will be the identity map giving us local degrees of 1 which add up to 2. Since any even map will factor in this way any even map must have even degree.

When n is even $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ and when n is even $H_n(\mathbb{R}P^n) \cong 0$. If n is odd and f is even then $\deg f = \deg q \cdot \deg \widetilde{f}$. However since $H_n(\mathbb{R}P^n) \cong 0$ when n is even then the degree of f has to be zero as $\deg \widetilde{f} = 0$ since it is mapping out of the trivial group.

Now suppose that n is odd. There is a quotient map $r : \mathbb{R}P^n \to (\mathbb{R}P^n / \mathbb{R}P^{n-1} \equiv S^n)$. This map will have degree 1 and as such $q \circ r$ will be a map from $S^n \to S^n$ of degree two. From there if we take a map of degree k, $f_k : S^n \to S^n$ (Hatcher 2.32). Then $f_k \circ q \circ r$ will be an even map of degree $2k$. □

Problem 6. *Let $\mathcal{C} = (C_i, \partial)$ be a chain complex over $\mathbb{R}$ with only finitely many $C_i \neq 0$. Show that the following methods for computing Euler characteristic yield the same answer:*

$$\chi(\mathcal{C}) = \sum_i (-1)^i \mathrm{rk} C_i$$

and

$$\chi(\mathcal{C}) = \sum_i (-1)^i \mathrm{rk} H_i(\mathcal{C})$$

The same is true with $\mathbb{Z}$ coefficients, but requires a tiny bit more thought.

Proof. Note that if any of the C_is are infinite dimensional then the sum is infinite and the equality will hold.

Otherwise suppose that all of the C_is are of finite dimension and start with the sum using homology

$$\chi(\mathcal{C}) = \sum_i (-1)^i (\mathrm{rk} H_i(\mathcal{C}))$$

Since $H_i(\mathcal{C}) = \ker \partial_i / \mathrm{Im}\partial_{i+1}$ the sum is equivalent to

$$\sum_i (-1)^i (\mathrm{rk} \ker \partial_i - \mathrm{Im}\partial_{i+1})$$

As there are only finitely many C_i that are nonzero we can rearrange the terms as we desire to get

$$\sum_i (-1)^i (\mathrm{rk} \ker \partial_i + \mathrm{Im}\partial_i)$$

by shifting all of the images up one term. By the rank nullity theorem we then have

$$\sum_i (-1)^i \mathrm{rk} C_i$$

which completes the proof. $\square$