Problem 1. *Verify explicitly that $\partial^2 = 0$.*

Proof. Consider $[v_1, \ldots, v_n]$. Then

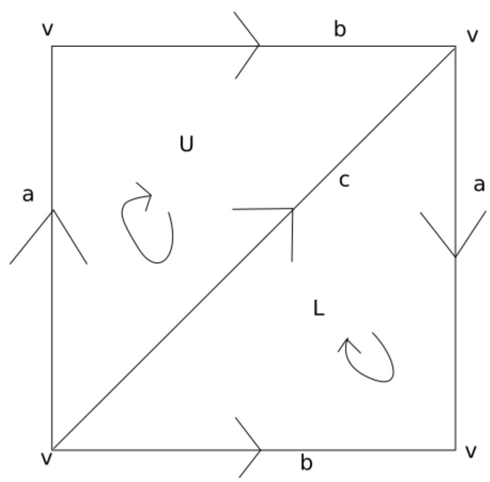
$$\partial^2([v_1, \ldots, v_n]) = \sum_{j<i} (-1)^{i+j} [v_1, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_n] + \sum_{i<j} (-1)^{i+j-1} [v_1, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_n]$$

Then if we swap i, j for the first sum and pull out a negative we get

$$\begin{aligned} \partial^2([v_1, \ldots, v_n]) &= \sum_{j<i} (-1)^{i+j} [v_1, \ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots, v_n] + \sum_{i<j} (-1)^{i+j-1} [v_1, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_n] \\ &= -\sum_{i<j} (-1)^{j+i-1} [v_1, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_n] + \sum_{i<j} (-1)^{i+j-1} [v_1, \ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots, v_n] \\ &= 0 \end{aligned}$$

Therefore $\partial^2 = 0$. $\square$

Problem 2. *Compute the simplicial homology of the Klein bottle using the Δ-complex structure, with two simplices of dimension 2.*



Proof. A:

First we'll list the images of all the various simplices.

$$\begin{aligned} \partial U &= a + b - c & & \\ \partial L &= a - b + c & & \\ \partial a &= v - v & &= 0 \\ \partial b &= v - v & &= 0 \\ \partial c &= v - v & &= 0 \\ \partial v &= 0 & & \end{aligned}$$

For $H_2(K) = \frac{\ker \partial}{\mathrm{Im} \partial}$ the image is trivial as there as there are no 3 simplices and the kernel is when $\partial(pU + qL) = (p+q)a + (p-q)b + (q-p)c$ is zero which only occurs if $p = q = 0$. Therefore $H_2(K) = 0$.

For $H_1(K)$ the kernel is the free abelian group on a, b, c and the image is generated by $(a+b-c)$ and $a - b + c$. So our group is the abelian group generated by $\langle a, b, c | a + b = c, a + c = b \rangle$. We can simplify to remove c and get $\langle a, b | 2a \rangle$. Therefore $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$.

For $H_0(K)$ the image is trivial and the kernel is the whole space. Thus $H_0(K) = \mathbb{Z}$. $\square$

Problem 3. *Show that if G is a finitely generated free abelian group and $H \subset G$ is a subgroup, then there is a basis $g_1, \ldots, g_n$ for G and integers $p_1, \ldots, p_k$ with $k \leq n$ such that each p_i divides p_{i+1}, and such that $p_1 g_1, \ldots, p_k g_k$ is a basis for H. We say that these bases for G and H are stacked. Conclude that*

$$G/H \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k} \oplus \mathbb{Z}^{n-k}$$

In particular, every finitely generated abelian group is a direct sub of cyclic groups. (Hint: You may find it helpful to use the fact that subgroups of free abelian groups are themselves free abelian.)

Proof. By the structure theorem for finitely generated modules over a PID and the fact that abelian groups are $\mathbb{Z}$-modules there exists a basis of $G = \langle g_1, \ldots, g_n \rangle$ such that $H = \langle p_1 g_1, \ldots, p_k g_k \rangle$ where $p_i | p_{i+1}$. If we then look at the quotient G/H we get

$$G/H \cong \langle g_1, \ldots, g_n | p_i g_i = 0 : 1 \leq i \leq k \rangle$$

However this implies that the subgroup generated by g_i will be $\mathbb{Z}_{p_i}$ for $i \leq k$ and $\mathbb{Z}$ otherwise. Since the group is abelian we can then conclude that

$$G/H \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k} \oplus \mathbb{Z}^{n-k}$$

□

Problem 4. *If $i : A \to X$ is the inclusion of a retract of X, show that $i_* : H_k(A) \to H_k(X)$ is a monomorphism onto a direct summand of $H_k(X)$. If A is deformation retract of X, show that i_* is an isomorphism.*

Proof. Let $i : A \to X$ be the inclusion and $r : X \to A$ the retract of X onto A. By definition $r \circ i = id_A$ which implies that the map induced on the homology $r_* \circ i_* = id_*$. Since i_* has a left inverse it must be injective. Therefore the map induced by the inclusion of a retract is injective.

Now suppose that A is a deformation retract of X. Then r is homotopic to id_X. Then we have that $i \circ r$ is homotopic to id_X which means that i is a homotopy equivalence and as such induces an isomorphism on the homology (Hatcher 2.11) of A and X. □

Problem 5. *Show that it is impossible to retract the $n-$ball B^n onto its $n-1$-sphere boundary $\partial B^n = S^{n-1}$.*

Proof. Since B^n is contractible $H_k(B^n) = 0$ for all k. If there existed a retract of B^n onto S^1 this would imply that there exists an injective function from $H_{n-1}(S^{n-1}) = \mathbb{Z}$ to the trivial group which is a contradiction.

Therefore there is no deformation retract from B^n to S^{n-1}. □

Problem 6. *Compute the simplicial homology of the Klein bottle using the Δ-complex structure, with two simplices of dimension 2, discussed in class.*

Proof. `goto A;` □