Problem 1.

Proof. □

Problem 2.

Proof. Let $p : E \to B$ be a covering map where both B and E are path connected. Now suppose that the cover is normal. Then consider a loop α in B. Since E is a normal covering space for any $e \in p^{-1}(b_0)$ there is a homeomorphism such that $h(e_0) = e$ and that $p \circ h = p$. Then if we consider a lift α to $\widetilde{\alpha}$ in E it will either lift to a loop or a path uniquely. If we apply h to $\widetilde{\alpha}$ then we have $h \circ \widetilde{\alpha}$ as a loop or path based at e however since lifts are unique this is precisely the lift of α to the covering space when the basepoint is shifted. Since the cover is normal we can do this for all basepoints. Therefore all lifts of α are either loops or paths.

Otherwise suppose that given a loop in B either all of its lifts are loops or none of them are loops. Then consider $[\alpha] \in \pi_1(B, b_0)$ and $[\gamma] \in p_*(\pi_1(E, e_0))$. Then consider the lift of $\alpha * \gamma * \alpha^{-1}$. If α lifts to a loop then $\alpha * \gamma * \alpha^{-1}$ will lift to a loop which implies that $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$. Otherwise if α^{-1} lifts to a path then so does α. The path $\widetilde{\alpha}$ will go to some other point $e \in p^{-1}(b_0)$. Since γ lifts to a loop at e_0 it will also lift to a loop based at e which after concatenating the lift of α^{-1} we get back to e_0 which shows that $\alpha * \gamma * \alpha^{-1}$ lifts to a loop and therefore $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$.

Therefore a covering is normal if and only if for each loop in B, either all its lifts are loops or none of them are loops. □

Problem 3.

Proof. □

Problem 4.

Proof. Let Σ_n and Σ_m be closed surfaces of genus n and m respectively. Suppose that $m = n$. Then we can express both surfaces as a polygons with $4n$ sides glued together. However this would mean that $\chi(\Sigma_n) = \chi(\Sigma_m)$ which by Problem 1 means that they are a 1 sheeted covering spaces of each other and as such $\Sigma_n \cong \Sigma_m$.

Otherwise suppose that $\Sigma_m \cong \Sigma_n$. Then any cell complex for Σ_m would also be one for Σ_n and vice versa. Since Σ_m is of genus m it has a cell complex with $4m$ sides. Since this is also a cell complex for Σ_n we know that $n \leq m$. However by the same reasoning we can show that $m \leq n$ implying that $n = m$.

Therefore $\Sigma_n \cong \Sigma_m$ if and only if $m = n$. □

Problem 5.

Proof. Let $p : E \to B$ be a covering map with B and E both path connected. Now suppose that the group of Deck transformations acts transitively on the set of points which lie over any single point in B. Then it acts transitively on $p^{-1}(b_0)$ which implies that the cover is normal.

Otherwise suppose that E is a normal cover. Then the group of Deck transformations acts transitively on the basepoint. As B is path connected for any point $b \in B$ there is an isomorphism $\varphi : \pi_1(B, b_0) \to \pi_1(B, b)$ made by going along a path γ from b to b_0 and back. This path has a lift $\widetilde{\gamma}$ which gives rise to an isomorphism from $\pi_1(E, e_0)$ to $\pi_1(E, e)$ where $e \in p^{-1}(b)$. Then we have (E, e) as a covering space for (B, b) with the same map and then $p_*(\pi_1(E, e))$ will be normal in $\pi_1(B, b)$. Thus the group of Deck transformations will act transitively on $p^{-1}(b)$ and since this was for an arbitrary point it holds for all points $b \in B$.

Therefore a cover is normal if and only if the group of Deck transformations acts transitively on the set of points in E which lie over any single point in B. $\square$

Problem 6.

Using the Seifert van Kampen theorem we can compute that the fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is

$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *) \cong \langle a, b | a^2, b^2 \rangle$$

The subgroups for this group are:

- $\langle e \rangle$
- $\langle a \rangle$
- $\langle b \rangle$
- $\langle (ab)^n \rangle$ for all $n \in \mathbb{N}$
- $\langle a, b | a^2, b^2 \rangle$

The covering spaces, with pictures below, that correspond to having the above group of Deck transformations are:

- $\mathbb{R}P^2 \vee \mathbb{R}P^2$
- S^2 with two copies of $\mathbb{R}P^2$ wedged on at antipodal points where crossing the sphere corresponds to a loop in a.
- As the previous but with b as the crossing of the sphere.
- A wreath of an even number of spheres wedged together.
- An infinite chain of spheres wedged together at antipodal points.

Problem 7.

Proof. Let X be a path connected, locally path connected, semi-locally simply connected space. Then let $H := [\pi_1(X, *), \pi_1(X, *)]$ where $[G, G]$ denoted the commutator subgroup and let $\widetilde{X}_{ab}$ be the covering space associated with H such that $p_*(\pi_1(\widetilde{X}_{ab}, \widetilde{*})) = H$. Then since H is normal $\widetilde{X}_{ab}$ is a normal covering space and its Deck group is $\text{Deck}(\widetilde{X}_{ab}) = N(H)/H = \pi_1(X, *)/H$ which is Abelian. Moreover any quotient of $\pi_1(X, *)$ with a normal subgroup $\pi_1(X, *)/N$ that is abelian must have N containing H as a subgroup. **Add why this means $\widetilde{X}_{ab}$ covers them**.

The universal abelian cover for $S^1 \vee S^1$ is the 2d grid of real lines with a copy of $\mathbb{R}$ for each integer. As shown below. **Actually insert the picture**.

The preimage of the basepoint are each of the intersections. The Deck group for the space is $\mathbb{Z} \times \mathbb{Z}$.

For $S^1 \vee S^1 \vee S^1$ the universal abelian cover is similar however it is a 3d grid of real lines with a copy of $\mathbb{R}$ for each integer. **Insert a picture**

The preimage of the basepoint will be the intersections. The Deck group of the space is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\square$