Problem 1. *Construct several examples of homotopic and non-homotopic maps.*

See attached.

Problem 2. *Show that the relation of fixed-endpoint homotopy is an equivalence relation.*

Proof. To show that the relation of fixed-endpoint homotopy ($\sim$)is an equivalence relation we will show that it is reflexive, symmetric, and transitive.

Let $f : X \to Y$ be a continuous function. Then we can construct a fixed endpoint homotopy from f to f via $H(x,t) = f(x)$ showing that $\sim$ is reflexive.

Next let $f \sim g$. Then there is a fixed endpoint homotopy $H(x,t)$ from f to g. Define $H' : X \times [0,1] \to Y$ via $H'(x,t) = H(x, 1-t)$. Then H' is a homotopy from g to f that fixes the endpoints since H did as well demonstrating that $\sim$ is reflexive.

Finally suppose that $f \sim g$ and $g \sim h$. Then there are fixed endpoint homotopies F from f to g and G from g to h. Define a new homotopy via

$$H(x,t) = \left\{ \begin{array}{cc} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{array} \right.$$

This homotopy will fix the endpoints since the two it is constructed from do.

Therefore since $\sim$ is reflexive, symmetric, and transitive it is an equivalence relation. □

Problem 3. *Construct some examples of paths which are fixed-endpoint homotopic, and some which are not.*

See attached.

Problem 4. *a) Show that any convex open subset of $\mathbb{R}^n$ is contractible.*

b) Show that a contractible space is path connected.

c) Show that if Y is contractible, then all maps

$$f : X \to Y$$

are homotopic.

d) Show that if X is contractible and Y is path-connected, then all maps

$$f : X \to Y$$

are homotopic. What happens if we remove the path-connectedness assumption?

Proof. a) Let U be a convex open subset of $\mathbb{R}^n$ and let $u \in U$. Given any point x there exists a path p_x from x to u via the definition of convexity. Then construct a homotopy from id_X to the constant map at u (c_u) via $H(x,t) = p_x(t)$.

Therefore since $id_X \sim c_u$ for U, any open convex subset of $\mathbb{R}^n$ is contractible.

b) Let X be a contractible space and let $x, y \in X$ be points in X. Since X is contractible we have a homotopy H from id_X to c_z for some point $z \in X$. Define a path from x to z via $p_x(t) = H(x,t)$ and a path from z to y via $p_y(t) = H(y,(1-t))$. Then the path $p_x * p_y$ is a path from x to y.

Therefore since a path exists between any two point in a contractible space X, X is path connected.

c) Let Y be a contractible space and $f, g : X \to Y$. Since Y is contractible there exists a homotopy H from id_Y to a constant function c_y. Define $H_f(x,t) = H(f(x),t)$ and $H_g(x,t) = H(g(x),t)$. Then define a homotopy $F : X \times [0,1] \to Y$ via

$$F(x,t) = \begin{cases} H_f(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_g(x, 2(1-t)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Therefore any two maps into a contractible space are homotopic.

d) Let X be a contractible space and Y a path-connected space. Since X is contractible there is a homotopy H from id_X to a constant map c_{x_0}. Let $f, g : X \to Y$ and p a path from $f(x_0)$ to $g(x_0)$. Then define a homotopy $F : X \times [0,1] \to Y$ from f to g via

$$F(x,t) = \begin{cases} f(H(x, 3t)) & 0 \leq t \leq \frac{1}{3} \\ p(3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ g(H(x, 3-3t)) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Therefore if X is contractible and Y is path connected any two maps $f : X \to Y$ are homotopic.

This does not hold if we remove the path connected assumption. Let $\mathbb{R}_2 = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$. Then the map $f(x) = (x, 0)$ is not homotopic to $g(x) = (x, 1)$ since there is no path between the two copies of $\mathbb{R}$.

□

Problem 5. *Check that the fundamental group of a pointed space (X, x_0) is a group.*

Proof. To show that $\pi(X, x_0)$ is a group we need to verify that that path concatenation is associative, there is an identity, and that there are inverses. We denote the equivalence class of a loop α via $[\alpha]$

- Let $[\alpha], [\beta], [\gamma] \in \pi(X, x_0)$. Then we can define a fixed endpoint homotopy between $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ via

$$H(t,s) = \begin{cases} \alpha(\frac{4t}{s+1}) & 0 \leq t \leq \frac{s+1}{4} \\ \beta(4t - s - 1) & \frac{s+1}{4} \leq t \leq \frac{s+2}{4} \\ \gamma(\frac{4t-s-2}{2-s}) & \frac{s+2}{4} \leq t \leq 1 \end{cases}$$

  Since there is a fixed endpoint homotopy between them they lie in the same equivalence class of loops.

  Therefore the concatenation operation on loops is associative.

- Let $[\alpha] \in \pi(X, x_0)$ and let p_{x_0} be the constant loop at x_0. Consider the loop $\alpha * p_{x_0}$. We can then construct a fixed endpoint homotopy from $\alpha * p_{x_0}$ to α via

$$H(t,s) = \begin{cases} \alpha(\frac{2t}{1+s}) & 0 \leq t \leq \frac{1+s}{2} \\ p_x(\frac{2t-(1+s)}{1-s}) & \frac{1+s}{2} < t \leq 1 \end{cases}$$

  Since there is a fixed endpoint homotopy between them they lie in the same equivalence class of loops and the constant loop is a right identity.

  Therefore there exists a right identity in $\pi(X, x_0)$.

- Let $[\alpha] \in \pi(X, x_0)$. Then define $\beta(t) = \alpha(1-t)$. We can define a homotopy from $\alpha * \beta$ to p_{x_0} via
$$H(t,s) = \begin{cases} \alpha(2(1-s)t) & 0 \leq t \leq \frac{1}{2} \\ \beta(2(1-s)t + (2s-1)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$
Since there is a fixed endpoint homotopy between them they lie in the same equivalence class of loops which implies that β is a right inverse of α.

  Therefore given $[\alpha] \in \pi(X, x_0)$ there exists a right inverse.

Therefore since loop concatenation is associative, there exists an identity element, and any class of loops has an inverse we have that $\pi(X, x_0)$ is a group under the loop concatenation operation. □

Problem 6. *Show that if x_0, x_1 are in the same path component of a space X, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.*

Proof. Let p be a path from x_0 to x_1 and let $p^r(t) = p(1-t)$. Then define two functions $\varphi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\psi : \pi_1(X, x_1) \to \pi_1(X, x_0)$ as
$$\varphi([\alpha]) = [p^r * \alpha * p]$$
and
$$\psi([\alpha]) = [p * \alpha * p^r]$$
Then to show that φ is a bijection we check that they are inverses.
$$\varphi \circ \psi([\alpha]) = [p^r * p * \alpha * p^r * p] = [p^r * p] * [\alpha] * [p^r * p] = [e_{x_1}] * [\alpha] * [e_{x_1}] = [\alpha]$$
and
$$\psi \circ \varphi([\alpha]) = [p * p^r * \alpha * p * p^r] = [p * p^r] * [\alpha] * [p * p^r] = [e_{x_0}] * [\alpha] * [e_{x_0}] = [\alpha]$$
To show that it preserves the group structure we start with
$$\varphi([\alpha]) * \varphi([\beta]) = [p^r * \alpha * p] * [p^r * \beta * p] = [p^r * \alpha * p * p^r * \beta * p] = [p^r * \alpha] * [e_{x_0}] * [\beta * p]$$
At this point we have a constant map in the middle which is homotopic to the same path without. So we get
$$[p^r * \alpha] * [\beta * p] = [p^r * \alpha * \beta * p] = \varphi([\alpha * \beta] = \varphi([\alpha] * [\beta]))$$
Since φ is a bijection and preserves the group structure it is an isomorphism.

Therefore if x_1, x_1 are in the same path component of a space X, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. □