

Problem 1. In the context of our proof of the zig-zag lemma. Prove that $\ker(\phi_*) \subset \text{Im}(\partial_*)$ and $\ker(\psi_*) \subset \text{Im}(\phi_*)$.

Proof. The following diagram contains the necessary maps

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
0 & \longrightarrow & C_{p+1} & \xrightarrow{\phi} & D_{p+1} & \xrightarrow{\psi} & E_{p+1} \longrightarrow 0 \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
0 & \longrightarrow & C_p & \xrightarrow{\phi} & D_p & \xrightarrow{\psi} & E_p \longrightarrow 0 \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
0 & \longrightarrow & C_{p-1} & \xrightarrow{\phi} & D_{p-1} & \xrightarrow{\psi} & E_{p-1} \longrightarrow 0 \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

with the long exact sequence from the zig-zag lemma

$$\cdots \longrightarrow H_n(C) \xrightarrow{\phi_*} H_n(D) \xrightarrow{\psi_*} H_n(E) \xrightarrow{\partial_*} H_{n-1}(C) \longrightarrow \cdots$$

First we'll show that $\ker(\phi_*) \subset \text{Im}(\partial_*)$. Start with $[c] \in \ker(\phi_*)$. Then $\phi(c) = d$ where $[d] = 0$. This implies that there is a $d' \in D_{p+1}$ such that $\partial d' = d$. In addition we have $\psi \circ \partial(d) = 0$ and if we have $e := \psi(d)$ then $\partial e = 0$ by commutativity and thus $[e]$ is a class in homology. Let $e' := \psi(d')$. Then $\partial e' = e$ which implies that $[e']$ is a class in homology. Finally by commutativity we have that $\partial_*[e'] = c$ which implies that $\ker(\phi_*) \subset \text{Im}(\partial_*)$.

Next we'll show that $\ker(\psi_*) \subset \text{Im}(\phi_*)$. Let $[d] \in \ker(\psi_*)$. Then we know that $\partial d = 0$ and that for $e := \psi(d)$ that there exists $\partial e' := e$. Since ψ is surjective we have $\psi(d') := e'$. By commutativity we have that $\psi(d - \partial d') = 0$ so there exists an c such that $\phi(c) = d - \partial d'$ which is unique by injectivity of ϕ . In addition $\partial c = 0$ by commutativity. Then in homology we get that $\phi_*[c] = [d - \partial d'] = [d]$. Therefore $\ker(\psi_*) \subset \text{Im}(\phi_*)$. \square

Problem 2. Let $A : S^n \rightarrow S^n$ be the antipodal map. What is $A_* : H_n(S^n) \rightarrow H_n(S^n)$?

Proof. We'll start by showing that the map $r_i : S^n \rightarrow S^n$ defined by

$$r(x_1, \dots, x_i, \dots, x_{n+1}) = (x_1, \dots, -x_i, \dots, x_{n+1})$$

has degree -1 . Start with S^n and consider it as two D^n , called p and q , joined at their boundaries with the boundary coinciding with x_i . The only non-zero reduced homology for S^n is $\tilde{H}_n(S^n) \cong \mathbb{Z}$ that is generated by $p + q$. Then $r_{i*}(p + q) = -(p + q)$ since it will swap p, q and their orientations. Therefore the degree of r_i is -1 .

We can express the map $A : S^n \rightarrow S^n$ as the composition $A = r_1 \circ \cdots \circ r_{n+1}$. Thus the degree of the map A is $(-1)^{n+1}$.

Therefore $A_* : H_n(S^n) \rightarrow H_n(S^n)$ is the identity map when n is odd and the inverse map when n is even. \square

Problem 3. Give a geometric description of the boundary map in the Mayer-Vietoris sequence.

Begin with our space $X = U \cup V$ satisfying the conditions for the Mayer-Vietoris sequence. Start with a cycle $[h] \in H_n(X)$. Then we can write $h = u + v$ where u, v lie wholly in U and V respectively. Since h represents a class in homology we have $\partial h = 0$. However this implies that $\partial u + \partial v = 0$ giving us that $\partial u = -\partial v$. It then follows that $\partial u, \partial v \in U \cap V$ and thus $\partial_*[h] = [\partial u]$.

Problem 4. Using the Mayer-Vietoris sequence, compute the homology of the n -Sphere, $H_*(S^n)$.

Proof. We'll start by computing the reduced homology of S^0 and proceed by induction. The zeroth reduced homology group is the one less than the number of connected components copies of \mathbb{Z} . Since S^0 is two disjoint points we have that $\tilde{H}_0(S^0) \cong \mathbb{Z}$.

Next assume that $\tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and is zero elsewhere. Then consider S^n as the union of a point $*$ and D^n enlarging them both slightly. Their intersection will be homotopy equivalent to S^{n-1} . Then using the Mayer-Vietoris sequence we have the long exact sequence

$$\cdots \longrightarrow H_n(S^{n-1}) \longrightarrow H_n(*) \oplus H_n(D^n) \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow \cdots$$

All other portions of the sequence be either zero or $H_p(S^n)$ sandwiched between two zeros forcing it to be zero. Rewrite the above sequence with the portions we know and we get

$$0 \longrightarrow H_n(S^n) \longrightarrow (H_{n-1}(S^{n-1}) \cong \mathbb{Z}) \longrightarrow 0$$

Which implies that $\tilde{H}_n(S^n) \cong \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Therefore the homology of S^n is

$$H_p(S^n) = \begin{cases} \mathbb{Z} & p = n, 0 \\ 0 & \text{else} \end{cases}$$

except for S^0 which has $H_0(S^0) = \mathbb{Z}^2$ and 0 otherwise. □

Problem 5. Let $T^2 = S^1 \times S^1$ be the torus, and $h : S^1 \rightarrow T^2$ an embedding of the unit circle into T^2 . Form the space

$$X = T^2 \cup_h D^2$$

by attaching a 2-cell D^2 to T^2 via the map h . Compute the homology of X . Note that there is more than one case.

Proof. We begin by using the Mayer-Vietoris sequence to decompose X into T^2 and D^2 to get an exact sequence describing our problem. The only nonzero portions of the sequence are

$$0 \longrightarrow H_2(T^2) \oplus H_2(D^2) \xrightarrow{k_* - l_*} H_2(X) \xrightarrow{\partial_*} H_1(S^1) \xrightarrow{i_* \oplus j_*} H_1(T^2) \oplus H_1(D^2) \xrightarrow{k_* - l_*} H_1(X) \longrightarrow 0$$

Then filling in for some of the groups we have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(X) \longrightarrow 0$$

For $H_0(X)$ there is a single connected component which implies that $H_0(X) \cong \mathbb{Z}$. Otherwise for the higher homotopy groups there are two cases to consider for the map h . The first is where h is homotopic to one of the generators of $H_1(T)$ and the latter is when h is null.

Suppose that $h \sim (m, n) \in H_1(T^2)$. Then the kernel of $i_* \oplus j_*$ is 0 as $i_* \oplus j_*(1) = (m, n)$. This implies that the image of ∂_* is 0 giving us that $k_* - l_*$ is surjective from exactness. Also from exactness the map is injective which means that $H_2(X) \cong H_2(T^2) \oplus H_2(D^2) \cong \mathbb{Z}$.

For $H_1(X)$ note that the image of $i_* \oplus j_*$ is $k(m, n)$ which by exactness is the kernel of $k_* - l_*$. We know that $k_* - l_*$ is surjective by exactness. It then follows from the first isomorphism theorem that

$$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} / \ker(k_* - l_*) \cong \mathbb{Z} \oplus \mathbb{Z} / \langle (m, n) \rangle \cong \mathbb{Z}_m \oplus \mathbb{Z}_n$$

Now we consider the case where h is null. If h is null then the kernel of $i_* \oplus j_*$ is $H_1(S^1)$. Then $\text{Im}(\partial_*) = H_1(S^1)$ which implies that ∂_* is surjective. Using the first isomorphism theorem we have that $H_2(X) / \ker \partial_* \cong H_1(S^1) \cong \mathbb{Z}$. However $\ker \partial_* = \text{Im}(k_* - l_*) \cong \mathbb{Z}$ as $H_2(T^2) \oplus H_2(D^2) \cong \mathbb{Z}$ and $k_* - l_*$ is injective by exactness. Thus $H_2(X) / \mathbb{Z} \cong \mathbb{Z}$ which implies that $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

For $H_1(X)$, by the first isomorphism theorem we have that $H_1(T^2) \oplus H_1(D^2) / \ker(k_* - l_*) \cong H_1(X)$. However since $\ker(k_* - l_*) = \text{Im}(i_* \oplus j_*) \cong 0$ we have that $\mathbb{Z} \oplus \mathbb{Z} / 0 \cong \mathbb{Z} \oplus \mathbb{Z} \cong H_1(X)$.

Therefore if $h \sim (m, n) \in H_1(T^2)$ the nonzero homology groups are

$$\begin{aligned} H_0(X) &\cong \mathbb{Z} \\ H_1(X) &\cong \mathbb{Z}_m \oplus \mathbb{Z}_n \\ H_2(X) &\cong \mathbb{Z} \end{aligned}$$

and if h is null

$$\begin{aligned} H_0(X) &\cong \mathbb{Z} \\ H_1(X) &\cong \mathbb{Z} \oplus \mathbb{Z} \\ H_2(X) &\cong \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

□