Problem 1. *In the context of our proof of the zig-zag lemma. Prove that* $\ker(\phi_*) \subset \mathrm{Im}(\partial_*)$ *and* $\ker(\psi_*) \subset \mathrm{Im}(\phi_*)$.

Proof. The following diagram contains the necessary maps

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "\partial"'] & \vdots \arrow[d, "\partial"'] & \\
0 \arrow[r] & C_{p+1} \arrow[r, "\phi"] \arrow[d, "\partial"'] & D_{p+1} \arrow[r, "\psi"] \arrow[d, "\partial"'] & E_{p+1} \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_p \arrow[r, "\phi"] \arrow[d, "\partial"'] & D_p \arrow[r, "\psi"] \arrow[d, "\partial"'] & E_p \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{p-1} \arrow[r, "\phi"] \arrow[d, "\partial"'] & D_{p-1} \arrow[r, "\psi"] \arrow[d, "\partial"'] & E_{p-1} \arrow[r] \arrow[d, "\partial"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

with the long exact sequence from the zig-zag lemma

$$\cdots \longrightarrow H_n(C) \xrightarrow{\phi_*} H_n(D) \xrightarrow{\psi_*} H_n(E) \xrightarrow{\partial_*} H_{n-1}(C) \longrightarrow \cdots$$

First we'll show that $\ker(\phi_*) \subset \mathrm{Im}(\partial_*)$. Start with $[c] \in \ker(\phi_*)$. Then $\phi(c) = d$ where $[d] = 0$. This implies that there is a $d' \in D_{p+1}$ such that $\partial d' = d$. In addition we have $\psi \circ \partial(d) = 0$ and if we have $e := \psi(d)$ then $\partial e = 0$ by commutativity and thus $[e]$ is a class in homology. Let $e' := \psi(d')$. Then $\partial e' = e$ which implies that $[e']$ is a class in homology. Finally by commutativity we have that $\partial_*[e'] = c$ which implies that $\ker(\phi_*) \subset \mathrm{Im}(\partial_*)$.

Next we'll show that $\ker(\psi_*) \subset \mathrm{Im}(\phi_*)$. Let $[d] \in \ker(\psi_*)$. Then we know that $\partial d = 0$ and that for $e := \psi(d)$ that there exists $\partial e' := e$. Since ψ is surjective we have $\psi(d') := e'$. By commutativity we have that $\psi(d - \partial d') = 0$ so there exists an a such that $\phi(a) = d - \partial d'$ which is unique by injectivity of ϕ. In addition $\partial a = 0$ by commutativity. Then in homology we get that $\phi_*[a] = [d - \partial d'] = [d]$. Therefore $\ker(\psi_*) \subset \mathrm{Im}(\phi_*)$. □

Problem 2. *Let* $A : S^n \to S^n$ *be the antipodal map. What is* $A_* : H_n(S^n) \to H_n(S^n)$*?*

Proof. We'll start by showing that the map $r_i : S^n \to S^n$ defined by

$$r(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$$

has degree -1. Start with S^n and consider it as two D^n, called p and q, joined at their boundaries with the boundary coinciding with x_i. The only non-zero reduced homology for S^n is $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ that is generated by $p+q$. Then $r_{i*}(p+q) = -(p+q)$ since it will swap p, q and their orientations. Therefore the degree of r_i is -1.

We can express the map $A : S^n \to S^n$ as the composition $A = r_1 \circ \cdots \circ r_{n+1}$. Thus the degree of the map A is $(-1)^{n+1}$.

Therefore $A_* : H_n(S^n) \to H_n(S^n)$ is the identity map when n is odd and the inverse map when n is even. □

Problem 3. *Give a geometric description of the boundary map in the Mayer-Vietoris sequence.*

Begin with our space $X = U \cup V$ satisfying the conditions for the Mayer-Vietoris sequence. Start with a cycle $[h] \in H_n(X)$. Then we can write $h = u + v$ where u, v lie wholly in U and V respectively. Since h represents a class in homology we have $\partial x = 0$. However this implies that $\partial u + \partial v = 0$ giving us that $\partial u = -\partial v$. It then follows that $\partial u, \partial v \in U \cap V$ and thus $\partial_*[h] = [u]$.

Problem 4. *Using the Mayer-Vietoris sequence, compute the homology of the n-Sphere, $H_*(S^n)$.*

Proof. We'll start by computing the reduced homology of S^0 and proceed by induction. The zeroth reduced homology group is the one less than the number of connected components copies of $\mathbb{Z}$. Since S^0 is two disjoint points we have that $\widetilde{H}_0(S^0) \cong \mathbb{Z}$.

Next assume that $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and is zero elsewhere. Then consider S^n as the union of a point $*$ and D^n enlarging them both slightly. Their intersection will be homotopy equivalent to S^{n-1}. Then using the Mayer-Vietoris sequence we have the long exact sequence

$$\cdots \longrightarrow H_n(S^{n-1}) \longrightarrow H_n(*) \oplus H_n(D^n) \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow \cdots$$

All other portions of the sequence be either zero or $H_p(S^n)$ sandwiched between two zeros forcing it to be zero. Rewrite the above sequence with the portions we know and we get

$$0 \longrightarrow H_n(S^n) \longrightarrow (H_{n-1}(S^{n-1}) \cong \mathbb{Z}) \longrightarrow 0$$

Which implies that $\widetilde{H}_n(S^n) \cong \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Therefore the homology of S^n is

$$H_p(S^n) = \begin{cases} \mathbb{Z} & p = n, 0 \\ 0 & \text{else} \end{cases}$$

except for S^0 which has $H_0(S^0) = \mathbb{Z}^2$ and 0 otherwise. □

Problem 5. *Let $T^2 = S^1 \times S^1$ be the torus, and $h : S^1 \to T^2$ an embedding of the unit circle into T^2. Form the space*

$$X = T^2 \cup_h D^2$$

by attaching a 2-cell D^2 to T^2 via the map h. Compute the homology of X. Note that there is more than one case.

Proof. We begin by using the Mayer-Vietoris sequence to decompose X into T^2 and D^2 to get an exact sequence describing our problem. The only nonzero portions of the sequence are

$$0 \longrightarrow H_2(T^2) \oplus H_2(D^2) \longrightarrow H_2(X) \longrightarrow H_1(S^1) \longrightarrow H_1(T^2) \oplus H_1(D^2) \longrightarrow H_1(X) \longrightarrow 0$$

Then filling in for some of the groups we have the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(X) \longrightarrow 0$$

There are two cases to consider for the map h. The first is where h is homotopic to one of the generators of $H_1(T)$ and the latter is when h is null. □