

Problem 1.*Proof.*

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Problem 2.

Proof. Let $p : E \rightarrow B$ be a covering map where both B and E are path connected. Now suppose that the cover is normal. Then consider a loop α in B . Since E is a normal covering space for any $e \in p^{-1}(b_0)$ there is a homeomorphism such that $h(e_0) = e$ and that $p \circ h = p$. Then if we consider a lift α to $\tilde{\alpha}$ in E it will either lift to a loop or a path uniquely. If we apply h to $\tilde{\alpha}$ then we have $h \circ \tilde{\alpha}$ as a loop or path based at e however since lifts are unique this is precisely the lift of α to the covering space when the basepoint is shifted. Since the cover is normal we can do this for all basepoints. Therefore all lifts of α are either loops or paths.

Otherwise suppose that given a loop in B either all of its lifts are loops or none of them are loops. Then consider $[\alpha] \in \pi_1(B, b_0)$ and $[\gamma] \in p_*(\pi_1(E, e_0))$. Then consider the lift of $\alpha * \gamma * \alpha^{-1}$. If α lifts to a loop then $\alpha * \gamma * \alpha^{-1}$ will lift to a loop which implies that $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$. Otherwise if α^{-1} lifts to a path then so does α . The path $\tilde{\alpha}$ will go to some other point $e \in p^{-1}(b_0)$. Since γ lifts to a loop at e_0 it will also lift to a loop based at e which after concatenating the lift of α^{-1} we get back to e_0 which shows that $\alpha * \gamma * \alpha^{-1}$ lifts to a loop and therefore $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$.

Therefore a covering is normal if and only if for each loop in B , either all its lifts are loops or none of them are loops. □

Problem 3.*Proof.*

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Problem 4.

Proof. Let Σ_n and Σ_m be closed surfaces of genus n and m respectively. Suppose that $m = n$. Then we can express both surfaces as a polygons with $4n$ sides glued together. However this would mean that $\chi(\Sigma_n) = \chi(\Sigma_m)$ which by Problem 1 means that they are a 1 sheeted covering spaces of each other and as such $\Sigma_n \cong \Sigma_m$.

Otherwise suppose that $\Sigma_m \cong \Sigma_n$. Then any cell complex for Σ_m would also be one for Σ_n and vice versa. Since Σ_m is of genus m it has a cell complex with $4m$ sides. Since this is also a cell complex for Σ_n we know that $n \leq m$. However by the same reasoning we can show that $m \leq n$ implying that $n = m$.

Therefore $\Sigma_n \cong \Sigma_m$ if and only if $m = n$. □

Problem 5.

Proof. Let $p : E \rightarrow B$ be a covering map with B and E both path connected. Now suppose that the group of Deck transformations acts transitively on the set of points which lie over any single point in B . Then it acts transitively on $p^{-1}(b_0)$ which implies that the cover is normal.

Otherwise suppose that E is a normal cover. Then the group of Deck transformations acts transitively on the basepoint. As B is path connected for any point $b \in B$ there is an isomorphism $\varphi : \pi_1(B, b_0) \rightarrow \pi_1(B, b)$ made by going along a path γ from b to b_0 and back. This path has a

lift $\tilde{\gamma}$ which gives rise to an isomorphism from $\pi_1(E, e_0)$ to $\pi_1(E, e)$ where $e \in p^{-1}(b)$. Then we have (E, e) as a covering space for (B, b) with the same map and then $p_*(\pi_1(E, e))$ will be normal in $\pi_1(B, b)$. Thus the group of Deck transformations will act transitively on $p^{-1}(b)$ and since this was for an arbitrary point it holds for all points $b \in B$.

Therefore a cover is normal if and only if the group of Deck transformations acts transitively on the set of points in E which lie over any single point in B . \square

Problem 6.

Using the Seifert van Kampen theorem we can compute that the fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is

$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *) \cong \langle a, b | a^2, b^2 \rangle$$

The subgroups for this group are:

- $\langle e \rangle$
- $\langle a \rangle$
- $\langle b \rangle$
- $\langle (ab)^n \rangle$ for all $n \in \mathbb{N}$
- $\langle a, b | a^2, b^2 \rangle$

The covering spaces, with pictures below, that correspond to having the above group of Deck transformations are:

- $\mathbb{R}P^2 \vee \mathbb{R}P^2$
- $\mathbb{R}P^2$ where b lifts to a constant loop.
- $\mathbb{R}P^2$ where a lifts to a constant loop.
- A wreath of an even number of spheres wedged together.
- An infinite chain of spheres wedged together at antipodal points.

Problem 7.

Proof. Let X be a path connected, locally path connected, semi-locally simply connected space. Then let $H := [\pi_1(X, *), \pi_1(X, *)]$ where $[G, G]$ denoted the commutator subgroup and let \tilde{X}_{ab} be the covering space associated with H such that $p_*(\pi_1(\tilde{X}_{ab}, \tilde{*})) = H$. Then since H is normal \tilde{X}_{ab} is a normal covering space and its Deck group is $\text{Deck}(\tilde{X}_{ab}) = N(H)/H = \pi_1(X, *)/H$ which is Abelian. Moreover any quotient of $\pi_1(X, *)$ with a normal subgroup $\pi_1(X, *)/N$ that is abelian must have N containing H as a subgroup. **Add why this means \tilde{X}_{ab} covers them.**

The universal abelian cover for $S^1 \vee S^1$ is the 2d grid of real lines with a copy of \mathbb{R} for each integer. As shown below. **Actually insert the picture.**

The preimage of the basepoint are each of the intersections. The Deck group for the space is $\mathbb{Z} \times \mathbb{Z}$.

For $S^1 \vee S^1 \vee S^1$ the universal abelian cover is similar however it is a 3d grid of real lines with a copy of \mathbb{R} for each integer. **Insert a picture**

The preimage of the basepoint will be the intersections. The Deck group of the space is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. \square