Problem 1.

$$S_3 \cong \langle a, b, c | a^2, b^2, c^2, (ab)^3, (ac)^3 \rangle$$

We can identify a with (1 2), b with (2 3), and c with (1 3).

Problem 2.

Proof. Start with $\langle a, b | a^4 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$. Then the elements of this group are

$$e, a, a^2, a^3, b, ba^2, ba^3, ab, a^2b, a^3b, bab, aba, \ldots$$

However using the relations above $bab = bab^{-1} = a^{-1}$ and similarly we also have $aba = b$. In addition we have $b = b^{-1}$ and as such $ba = a^{-1}b$ which implies that $ba^i = a^{-i}b$ collapsing more elements until we are left with

$$e, a, a^2, a^3, b, ba, ba^2, ba^3$$

showing that the group is of order 8.

Now consider the symmetries of a square with vertices labeled clockwise $1, 2, 3, 4$. Then the symmetries consist of 8 permutations

$$e, (1\ 2\ 3\ 4), (1\ 4)(2\ 3), (4\ 3\ 2\ 1), (1\ 2)(3\ 4), (2\ 4), (1\ 4)(2\ 3), (1\ 3)$$

Define $\rho := (1\ 2\ 3\ 4)$ and $r := (1\ 2)(3\ 4)$. It can be seen that ρ and r generate the others. Note that since ρ is a 4-cycle and r is a product of disjoint transpositions that $\rho^4 = e$ and $r^2 = 1$. In addition if we consider $r\rho r^{-1}$ we have

$$r\rho r^{-1} = (1\ 2)(3\ 4)(1\ 2\ 3\ 4)(1\ 2)(3\ 4) = (4\ 3\ 2\ 1) = \rho^{-1}$$

Since there are eight permutations of the square and their generators fulfill the same relations we can conclude that $(a, b | a^4 = 1, b^2 = 1, bab^{-1} = a^{-1})$ is isomorphic to the group of symmetries of the square. $\square$

Problem 3.

Proof. Define a map $\varphi : G \to H$ by

$$\varphi(a) = xyx, \quad \varphi(b) = xy$$

and define φ for the rest of the elements of G by concatenation.

We will show it is a homomorphism by showing that it preserves the relation $a^2 = b^3$.

$$\begin{aligned}\varphi(a)^2 &= xyxxyx \\ &= xyxyxy \\ &= (xy)^3 \\ &= \varphi(b)^3\end{aligned}$$

proving that φ is a homomorphism.

Now we will show that $\varphi(a)$ and $\varphi(b)$ are also generators of H. This will allow us to define φ^{-1} by simply reversing the map which will show that φ is an isomorphism.

For x we have

$$y = y^{-1}x^{-1}xyx = (xy)^{-1}xyx = \varphi(b)^{-1}\varphi(a)$$

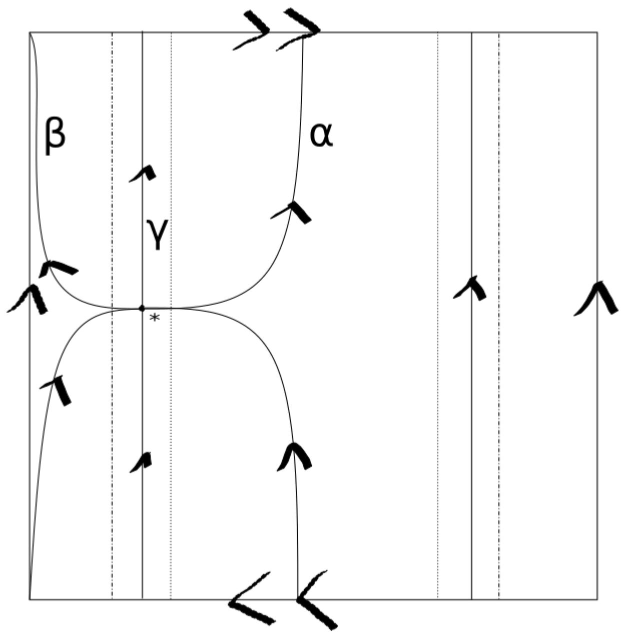
Then for y it is

$$y = yxyy^{-1}x^{-1} = xyx(xy)^{-1} = \varphi(a)\varphi(b)^{-1}$$

Since we can reach the generators of H from $\varphi(a)$ and $\varphi(b)$ they are also generators of H. Therefore there is a well-defined inverse φ^{-1} by reversing the map which shows that φ is an isomorphism.

Therefore the groups G and H are isomorphic. □

Problem 4.



Proof.

We will show that the fundamental of the Klein bottle $\pi_1(K, *)$ is isomorphic to $\langle a, b | aba^{-1}b = 1\rangle$. First as above we decompose K into U and V where U lies in the center between the dashed dotted lines and V lies on the outer edges between the dotted lines. Then U, V and $U \cap V$ can be seen to be Möbius strips. The Möbius strips are each homotopy equivalent to a circle. Therefore the fundamental groups of U, V, and $U \cap V$ are

$$\pi_1(U, *) \cong \pi_1(V, *) \cong \pi_1(U \cap V, *) \cong \pi_1(S^1, *) \cong \mathbb{Z}$$

Next we examine the generators for each of the fundamental groups. The generators for $\pi_1(U, *), \pi_1(V, *)$, and $\pi_1(U \cap V, *)$ are $[\alpha], [\beta]$, and $[\gamma]$ respectively. Take the inclusion maps $i : U \cap V \to U$ and $j : U \cap V \to V$. Then $i_*([\gamma]) = [\alpha]^2$ and similarly $j_*([\gamma]) = [\beta]^2$. We can see this because if we were to project γ and α (or β) onto a circle we would have γ go around twice as many times as α. Then by the Seifert-van Kampen Theorem we have

$$\pi_1(K, *) = \langle [\alpha], [\beta] | i_*([\gamma]) j_8([\gamma])^{-1} = 1\rangle \cong \langle [\alpha], [\beta] | [\alpha]^2[\beta]^{-2} = 1\rangle \cong \langle x, y | x^2 = y^2\rangle$$

Now we will show that $\langle x, y | x^2 = y^2\rangle \cong \langle a, b | aba^{-1}b = 1\rangle$. First note that we can rewrite $aba^{-1}b = 1$ as $bab = a$ via

$$aba^{-1} = b^{-1} \Rightarrow ab = b^{-1}a \Rightarrow bab = a$$

Define a map φ such that

$$\varphi(x) = ba, \quad \varphi(y) = a$$

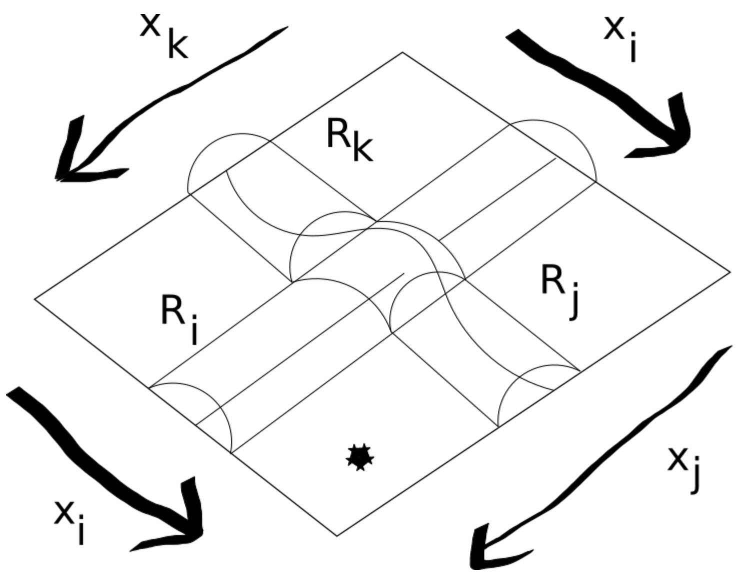
where we define φ on other elements by concatenation. Then we can show that φ is a homomorphism by showing it preserves the relation $x^2 = y^2$ via

$$\begin{aligned}\varphi(x)^2 &= baba \\ &= a^2 \\ &= \varphi(y)^2\end{aligned}$$

Next we will show that we can reach both generators a, b which from the same reasoning as problem 3 will show that φ is an isomorphism. We can express a immediately as $\varphi(y) = a$. For b we have $\varphi(x)\varphi(y)^{-1} = baa^{-1} = a$. It then follows that φ is an isomorphism.

Therefore the fundamental group of the Klein bottle has presentation $\langle a, b | aba^{-1}b = 1\rangle$. □

Problem 5.

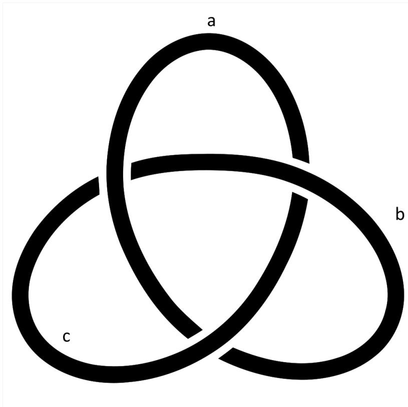


Proof. a) First we look at the space T with the Rs attached. Going along T and passing through R_n to return to the basepoint is a generator which we will call x_n. For each arc α_i there will be a corresponding generator x_i. Given a crossing we can frame it as in the drawing above. Then if we go along the loop x_i with the orientation listed followed by x_j, x_i^{-1}, and then x_k^{-1} we can pull the loop back to the constant loop obtaining the relation $x_i x_j x_i^{-1} x_k^{-1} = 1$. This can be rewritten as $x_i x_j x_i^{-1} = x_k$. Then the presentation of $\pi_1(\mathbb{R}^3 \setminus K) \cong \langle x_1, \ldots, x_m | x_i x_j x_i^{-1} = x_k$ for each crossing$\rangle$ where m is the number of crossings.

b) Since the presentation will consist solely of relations of the form $x_i x_j x_i^{-1} = x_k$ the Abelianization of the group would reduce all such relations to the form $x_i x_i^{-1} x_j = x_j x_k$. However since K is a knot each distinct arc will be related to each other either directly or through transitivity. This means that all generators will be equivalent leaving us with $\mathbb{Z}$ as the Abelianization of $\pi_1(\mathbb{R}^3 \setminus K)$.

□

Problem 6.



Proof.
Using Problem 5 we can deduce that the presentation of the group is

$$\langle a, b, c | aba^{-1} = c, bcb^{-1} = a, cac^{-1} = b \rangle$$

However we can simplify this presentation by plugging the first relation into the latter two to get

$$\begin{aligned} bcb^{-1} &= a \\ baba^{-1}b^{-1} &= a \\ bab &= aba \end{aligned}$$

and similarly

$$\begin{aligned} cac^{-1} &= b \\ aba^{-1}aab^{-1}a^{-1} &= b \\ abab^{-1}a^{-1} &= b \\ aba &= bab \end{aligned}$$

Since we get the same relation and have removed all instances of the generator c we can safely trim down our original presentation to $\langle a, b | aba = bab \rangle$.

Therefore the fundamental group of the trefoil knot is isomorphic to $\langle x, y | xyx = yxy \rangle$. Moreover this group is not Abelian as $xy = yxyx^{-1}$. It then follows that the trefoil knot is not equivalent to the unknot as the fundamental group of the unknot is Abelian. □