Problem 1.

Proof. Let Σ, Σ' be closed orientable surfaces such that $\chi(\Sigma) = n\chi(\Sigma')$ with $n \in \mathbb{Z}_{>0}$. Since Σ, Σ' are closed and orientable we can write their Euler characteristic in terms of the genus $(2 - 2g = \chi)$ of the spaces where in this case the genus will be equivalent to the number holes in the space. This gives us

$$2 - 2g = n(2 - 2g') \Rightarrow 1 + n(g' - 1) = g$$

These numbers correspond in such a way that we can construct the covering space as a single hole in the space with lines of holes length $g' - 1$ emerging. This will give us n copies of Σ' attached at the center from a continuous deformation of Σ. Therefore Σ is an $n-$fold covering space of Σ'. $\square$

In the case where the surfaces are nonorientable both surfaces will be made of handles or crosscaps attached to a sphere. In the scenario where it is only crosscaps the Euler characteristic in terms of the genus is $\chi(\Sigma) = 2 - g$ in which case the analogous equation to the above would be $2 + n(g' - 2) = g$. So we could do a similar construction with a line of two crosscaps in the center which would combine to be a torus. However if there is a mix of crosscaps and handles we would need to know that the numbers in both surfaces correspond to ensure that we have a covering.

Problem 2.

Proof. Let $p : E \to B$ be a covering map where both B and E are path connected. Now suppose that the cover is normal. Then consider a loop α in B. Since E is a normal covering space for any $e \in p^{-1}(b_0)$ there is a homeomorphism such that $h(e_0) = e$ and that $p \circ h = p$. Then if we consider a lift α to $\widetilde{\alpha}$ in E it will either lift to a loop or a path uniquely. If we apply h to $\widetilde{\alpha}$ then we have $h \circ \widetilde{\alpha}$ as a loop or path based at e however since lifts are unique this is precisely the lift of α to the covering space when the basepoint is shifted. Since the cover is normal we can do this for all basepoints. Therefore all lifts of α are either loops or paths.

Otherwise suppose that given a loop in B either all of its lifts are loops or none of them are loops. Then consider $[\alpha] \in \pi_1(B, b_0)$ and $[\gamma] \in p_*(\pi_1(E, e_0))$. Then consider the lift of $\alpha * \gamma * \alpha^{-1}$. If α lifts to a loop then $\alpha * \gamma * \alpha^{-1}$ will lift to a loop which implies that $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$. Otherwise if α^{-1} lifts to a path then so does α. The path $\widetilde{\alpha}$ will go to some other point $e \in p^{-1}(b_0)$. Since γ lifts to a loop at e_0 it will also lift to a loop based at e which after concatenating the lift of α^{-1} we get back to e_0 which shows that $\alpha * \gamma * \alpha^{-1}$ lifts to a loop and therefore $[\alpha * \gamma * \alpha^{-1}] \in p_*(\pi_1(E, e_0))$.

Therefore a covering is normal if and only if for each loop in B, either all its lifts are loops or none of them are loops. $\square$

Problem 3.

Proof.

(a) Let B be a path connected, locally path connected, semi-locally simply connected space and let $p : E \to B$ be an $n-$sheeted covering space. Define a map $P_\gamma : p^{-1}(b_0) \to p^{-1}(b_0)$ for $[\gamma] \in \pi_1(B, b_0)$by $e_i \mapsto e_j$ if the lift $\widetilde{\gamma}$ based at e_i fulfills $\gamma(0) = e_i$ and $\gamma(1) = e_j$. The map P_γ is a bijection since $P_{\gamma^{-1}}$ will send e_j to e_i for any e_i and as such P_γ has an inverse. Therefore we can define a map $\varphi_E : \pi_1(B, b_0) \to S_n$ by $\gamma \mapsto P_\gamma$.

To show that φ_E is a homomorphism consider $[\alpha], [\beta] \in \pi_1(B, b_0)$. Suppose that $P_\alpha(e_i) = e_j$ and $P_\beta(e_j) = e_k$. Then for some e_i

$$\varphi_E(\beta) \circ \varphi_E(\alpha)(e_i) = P_\beta \circ P_\alpha(e_i) = e_k = P_{\alpha*\beta}(e_i) = \varphi_E(\alpha * \beta)$$

Therefore $\varphi_E : \pi_1(B, b_0) \to S_n$ is a homomorphism. This representation of $\pi_1(B, b_0)$ is dependent on the covering space. Therefore we can construct a map ρ from the set of equivalence classes of $n-$sheeted covering spaces and representations of $\pi_1(B, b_0)$ via $E \mapsto \varphi_E$.

We will show that ρ is a bijection by constructing its inverse. Let $\psi : \pi_1(B, b_0) \to S_n$ be a representation of $\pi_1(B, b_0)$. First let $\widetilde{E}$ be the universal cover of B viewed as homotopy classes of paths based at b_0 and consider the space $\widetilde{E} \times \{0, \ldots, n\}$. We can define the covering space E_ψ as a quotient space of $\widetilde{E}$. First define a map $h : \widetilde{E} \times \{0, \ldots, n\}$ as $h([\gamma], e_i) = \widetilde{\gamma}(1)$ where $\widetilde{\gamma}$ is the lift of γ based at e_i. Consider points e_i, e_j, and e_k such that $P_\alpha(e_i) = e_k$. Then if there is a path γ such that $h([\gamma], e_i) = e_k$ and $h([\gamma], e_j) = e_k$ then $e_i \sim e_j$. This will collapse the universal cover such that the action ψ is well defined on $X/\sim$ Therefore the quotient is a covering space that realizes the action associated with the action associated with ψ up to equivalence of covering spaces.

Therefore the equivalences classes of $n-$sheeted covering spaces are in bijection with representations of $\pi_1(B, b_0)$ in S_n

(b) Suppose that $\widetilde{X}$ is path connected. Then consider the path from α in $\widetilde{X}$ such that $\alpha(0) = e_i$ and $\alpha(1) = e_j$. Then $p \circ \alpha$ is a loop that such that the action will permute e_i and e_j. Since we can do this for any such i and j the representation is transitive.

Otherwise suppose that the representation $\varphi : \pi_1(B, b_0) \to S_n$ is transitive. Then given $e_i \in p^{-1}(b_0)$ there is a path to $e_j \in p^{-1}(b_0)$. The map from the fundamental group based at b_0 to some $b \in B$ induces an isomorphism of fundamental groups so the action will remain transitive. As such by the same reasoning we can always go between preimages of a point. However since B is path connected if we lift a path from b to b' this will lift to a path between two specific preimages giving us a way to construct paths between any two points of E.

Therefore $\widetilde{X}$ is path connected if and only if the representation is transitive.

□

Problem 4.

Proof. Let Σ_n and Σ_m be closed surfaces of genus n and m respectively. Suppose that $m = n$. Then we can express both surfaces as a polygons with $4n$ sides glued together. However this would mean that $\chi(\Sigma_n) = \chi(\Sigma_m)$ which by Problem 1 means that they are a 1 sheeted covering spaces of each other and as such $\Sigma_n \cong \Sigma_m$.

Otherwise suppose that $\Sigma_m \cong \Sigma_n$. Then any cell complex for Σ_m would also be one for Σ_n and vice versa. Since Σ_m is of genus m it has a cell complex with $4m$ sides. Since this is also a cell complex for Σ_n we know that $n \leq m$. However by the same reasoning we can show that $m \leq n$ implying that $n = m$.

Therefore $\Sigma_n \cong \Sigma_m$ if and only if $m = n$. □

Problem 5.

Proof. Let $p : E \to B$ be a covering map with B and E both path connected. Now suppose that the group of Deck transformations acts transitively on the set of points which lie over any single point in B. Then it acts transitively on $p^{-1}(b_0)$ which implies that the cover is normal.

Otherwise suppose that E is a normal cover. Then the group of Deck transformations acts transitively on the basepoint. As B is path connected for any point $b \in B$ there is an isomorphism $\varphi : \pi_1(B, b_0) \to \pi_1(B, b)$ made by going along a path γ from b to b_0 and back. This path has a lift $\widetilde{\gamma}$ which gives rise to an isomorphism from $\pi_1(E, e_0)$ to $\pi_1(E, e)$ where $e \in p^{-1}(b)$. Then we have (E, e) as a covering space for (B, b) with the same map and then $p_*(\pi_1(E, e))$ will be normal in $\pi_1(B, b)$. Thus the group of Deck transformations will act transitively on $p^{-1}(b)$ and since this was for an arbitrary point it holds for all points $b \in B$.

Therefore a cover is normal if and only if the group of Deck transformations acts transitively on the set of points in E which lie over any single point in B. $\square$

Problem 6.

Using the Seifert van Kampen theorem we can compute that the fundamental group of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is

$$\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *) \cong \langle a, b | a^2, b^2 \rangle$$

The subgroups for this group are:

- $\langle e \rangle$
- $\langle a \rangle$
- $\langle b \rangle$
- $\langle (ab)^n \rangle$ for all $n \in \mathbb{N}$
- $\langle a, b | a^2, b^2 \rangle$

The covering spaces, with pictures attached, that correspond to having the above group of Deck transformations are:

- $\mathbb{R}P^2 \vee \mathbb{R}P^2$ which projects to $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2, *)$
- $\mathbb{R}P^2$ where b lifts to a constant loop. This corresponds to the subgroup $\langle a \rangle$
- $\mathbb{R}P^2$ where a lifts to a constant loop. This corresponds to the subgroup $\langle b \rangle$
- A wreath of $2n$ spheres wedged together at distinct points. This projects to the subgroup $\langle (ab)^n \rangle$
- An infinite chain of spheres wedged together at antipodal points. This projects to the subgroup $\{e\}$.

Since we have a covering space for each subgroup of $\langle a, b | a^2, b^2 \rangle$ and covering spaces that project to the same subgroup are equivalent we can conclude that these characterize all covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

Problem 7.

Proof. Let X be a path connected, locally path connected, semi-locally simply connected space. Then let $H := [\pi_1(X, *), \pi_1(X, *)]$ where $[G, G]$ denoted the commutator subgroup and let $\widetilde{X}_{ab}$ be the covering space associated with H such that $p_*(\pi_1(\widetilde{X}_{ab}, \widetilde{*})) = H$. Then since H is normal $\widetilde{X}_{ab}$ is a normal covering space and its Deck group is $\text{Deck}(\widetilde{X}_{ab}) = N(H)/H = \pi_1(X, *)/H$ which is Abelian. Moreover any quotient of $\pi_1(X, *)$ with a normal subgroup $\pi_1(X, *)/N$ that is abelian must have N containing H as a subgroup. Thus $\widetilde{X}_{ab}$ must cover the covering space corresponding to N.

The universal abelian cover for $S^1 \vee S^1$ is the 2d grid of real lines with a copy of $\mathbb{R}$ for each integer.

The preimage of the basepoint are each of the intersections. The Deck group for the space is $\mathbb{Z} \times \mathbb{Z}$.

For $S^1 \vee S^1 \vee S^1$ the universal abelian cover is similar however it is a 3d grid of real lines with a copy of $\mathbb{R}$ for each integer. Both pictures are attached.

The preimage of the basepoint will be the intersections. The Deck group of the space is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\square$