Problem 1. *In the context of our proof of the zig-zag lemma. Prove that* $\ker(\phi_*) \subset \operatorname{Im}(\partial_*)$ *and* $\ker(\psi_*) \subset \operatorname{Im}(\phi_*)$.

Proof. □

Problem 2. *Let* $A : S^n \to S^n$ *be the antipodal map. What is* $A_* : H_n(S^n) \to H_n(S^n)$*?*

Proof. □

Problem 3. *Give a geometric description of the boundary map in the Mayer-Vietoris sequence.*

Proof. □

Problem 4. *Using the Mayer-Vietoris sequence, compute the homology of the n-Sphere,* $H_*(S^n)$.

Proof. We'll start by computing the reduced homology of S^0 and proceed by induction. The zeroth reduced homology group is the one less than the number of connected components copies of $\mathbb{Z}$. Since S^0 is two disjoint points we have that $\widetilde{H}_0(S^0) \cong \mathbb{Z}$.

Next assume that $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and is zero elsewhere. Then consider S^n as the union of a point $*$ and D^n enlarging them both slightly. Their intersection will be homotopy equivalent to S^{n-1}. Then using the Mayer-Vietoris sequence we have the long exact sequence

$$\cdots \longrightarrow H_n(S^{n-1}) \longrightarrow H_n(*) \oplus H_n(D^n) \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^{n-1}) \longrightarrow \cdots$$

All other portions of the sequence be either zero or $H_p(S^n)$ sandwiched between two zeros forcing it to be zero. Rewrite the above sequence with the portions we know and we get

$$0 \longrightarrow H_n(S^n) \longrightarrow (H_{n-1}(S^{n-1}) \cong \mathbb{Z}) \longrightarrow 0$$

Which implies that $\widetilde{H}_n(S^n) \cong \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Therefore the homology of S^n is

$$H_p(S^n) = \begin{cases} \mathbb{Z} & p = n, 0 \\ 0 & \text{else} \end{cases}$$

except for S^0 which has $H_0(S^0) = \mathbb{Z}^2$ and 0 elsewhere. □

Problem 5. *Let* $T^2 = S^1 \times S^1$ *be the torus, and* $h : S^1 \to T^2$ *an embedding of the unit circle into* T^2*. Form the space*

$$X = T^2 \cup_h D^2$$

by attaching a 2-cell D^2 *to* T^2 *via the map* h*. Compute the homology of* X*. Note that there is more than one case.*

Proof. □