Problem 1.

Proof. Let X be a space such that all paths having the same endpoints are fixed-endpoint homotopic. Then given a loop α based as some point x_0 this loop is homotopic to the constant map at x_0 which implies that $\pi_1(X, x_0)$ is trivial for all X. Therefore X is simply connected.

Now suppose that X is simply connected and let α, β be paths from the point x_0 to x_1. Consider the path $\alpha * \beta^{-1} * \beta$. Since $\alpha * \beta^{-1}$ is a loop based at x_0 it is homotopic to the constant map at x_0. This implies that $\alpha * \beta^{-1} * \beta$ is homotopic to $c_{x_0} * \beta \sim \beta$. Now if we look at $\beta^{-1} * \beta$ this is homotopic to the constant path based at x_1 and by the same reasoning $\alpha * \beta^{-1} * \beta \sim \alpha$. By transitivity this implies that $\alpha \sim \beta$.

Therefore a space X is simply connected if and only if all paths are fixed endpoint homotopic. □

Problem 2.

Proof. Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$. For a continuous map h we have h_* defined as $h_*([\gamma]) = [h \circ \gamma]$. Then if we consider $(g \circ f)_*$:

$$(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_* \circ f_*([\gamma])$$

Therefore $(g \circ f)_* = g_* \circ f_*$. □

Problem 3.

Proof. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Let $F : [0,1]^2 \to B$ be continuous with $F(0,0) = b_0$. For each point $b \in B$ let U_b be an open neighborhood of b such that the preimage under p is the union of disjoint open sets homeomorphic to U_b. Then $\{F^{-1}(U_b)\}$ is an open cover of I^2. Let $\{F^{-1}(U_\beta)\}$ be a finite subcover of $\{F^{-1}(U_b)\}$. Choose a U_β that contains b_0. Then for $p^{-1}(U_\beta) = \coprod V_\alpha$ choose the $V_{\alpha'}$ containing e_0 and define $\tilde{F} : F^{-1}(U_\beta) \to E$ as equivalent to $F|_{F^{-1}(U_\beta)}$ as $V_{\alpha'}$ is homeomorphic to U_β.

Then for each γ such that $F^{-1}(U_\gamma) \cap F^{-1}(U_\beta) \neq \phi$ pick the $V_{\gamma'}$ from $p^{-1}(U_\gamma) = \coprod_\gamma V_\gamma$ such that $V_{\gamma'} \cap V_{\alpha'} \neq \phi$. Define $\tilde{F} : F^{-1}(U_\gamma) \to E$ in the same way as before such that the $\tilde{F}$s agree on the common components of their image. Then repeat this process for all U_β in our finite subcover. Since we have a finite number of $\tilde{F}$s and they are defined to be equal on the common pieces on their image we can create $\tilde{F} : I^2 \to E$ using the pasting lemma such that $p \circ \tilde{F} = F$. As there was only one choice to make at any given point of the construction this lift is unique.

If F is a fixed endpoint homotopy then $F|_{[0,1]\times[0]}$ and $F|_{[0,1]\times[1]}$ are constant paths that lift to constant paths in E. If we define $\tilde{F}$ as above then the restrictions applied to $\tilde{F}$ will also create constant paths since $p \circ \tilde{F} = F$ and there is no way to jump to another element of $p^{-1}(b_0)$.

Therefore if $p : E \to B$ is a covering map and $F : [0,1]^2 \to B$ is a homotopy then there exists a lift $\tilde{F} : [0,1]^2 \to E$ where if F is fixed endpoint homotopic then so is $\tilde{F}$. □

Problem 4.

Proof. Let E be a pointed space, B a connected pointed space, $p : E \to B$ a covering map, and let $|p^{-1}(b)| = n$. Define $X_n := \{b \in B | |p^{-1}(b)| = n\}$. Let x be a point in X_n and let U_x be an open neighborhood of x. Then the preimage $p^{-1}(U_x)$ is $\coprod_{k=1}^n V_k$ which implies that all other points in

U_x are in X_n. It then follows that $U_x \subset X_n$. Since we did this for an arbitrary point of X_n this implies that all points of X_n are interior and as such X_n is open. Then we can express X as

$$X = \bigcup_n X_n$$

where $X_i \cap X_j = \phi$ if $i \neq j$. If there were two or more values for which X_n was nonempty this would form a partition of B contradicting the assumption that the space is connected.

Therefore if $p : E \to B$ is a covering map, B is connected and if there exists a $b \in B$ such that $|p^{-1}(b)| = n$ then $|p^{-1}(b')| = n$ for all $b' \in B$. □

Problem 5.

Proof. Let B be simply connected, E path connected, and $p : E \to B$ a covering map. By the Theorem from class since E is path connected there exists a surjective map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$. However since B is simply connected the fundamental group is trivial which implies that $p^{-1}(b_0)$ has only one element. Since B is simply connected it is also connected so it follows that $p^{-1}(b)$ contains only a single element for all $b \in B$. As there is only one copy of B in E this implies that $p|_E = p$ and as such p is a homeomorphism.

Therefore if B is simply connected, then any covering map for which E is path connected is a homeomorphism. □

Problem 6.

Proof. Let $h : (X, x_0) \to (Y, y_0)$ be an inessential map and let $[\gamma] \in \pi(X, x_0)$. Since h is inessential it follows that there is a homotopy H that sends h to the constant map at y_0. If we take $h_*([\gamma]) = [h \circ \gamma]$ then we can create a homotopy $H'(t, s) = H(\gamma(t), s)$ that will take γ to the constant map which implies that d $h_*([\gamma]) = [y_0]$. Since this happens for an arbitrary loop in $\pi_1(X, x_0)$ it follows that the map h_* is trivial.

Therefore if $h : (X, x_0) \to (Y, y_0)$ is inessential then the induced homomorphism h_* is trivial. □