Problem 1. *Verify explicitly that $\partial^2 = 0$.*

Proof. Consider $[v_1, \dots, v_n]$. Then

$$\partial^2([v_1,\dots,v_n]) = \sum_{j<i}(-1)^{i+j}[v_1,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n] + \sum_{i<j}(-1)^{i+j-1}[v_1,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n]$$

Then if we swap i, j for the first sum and pull out a negative we get

$$\begin{aligned}\partial^2([v_1,\dots,v_n]) &= \sum_{j<i}(-1)^{i+j}[v_1,\dots,\hat{v_j},\dots,\hat{v_i},\dots,v_n] + \sum_{i<j}(-1)^{i+j-1}[v_1,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n] \\ &= -\sum_{i<j}(-1)^{j+i-1}[v_1,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n] + \sum_{i<j}(-1)^{i+j-1}[v_1,\dots,\hat{v_i},\dots,\hat{v_j},\dots,v_n] \\ &= 0\end{aligned}$$

Therefore $\partial^2 = 0$. □

Problem 2. *Compute the simplicial homology of the Klein bottle using the Δ-complex structure, with two simplices of dimension 2.*

Proof. **Lucas insert pretty picture hither.**

First we'll list the images of all the various simplices.

$$\begin{aligned}\partial U &= a + b - c \\ \partial L &= a - b + c \\ \partial a &= v - v && = 0 \\ \partial b &= v - v && = 0 \\ \partial c &= v - v && = 0 \\ \partial v &= 0\end{aligned}$$

For $H_2(K) = \frac{\ker \partial}{\mathrm{Im} \partial}$ the image is trivial as there as there are no 3 simplices and the kernel is when $\partial(pU + qL) = (p+q)a + (p-q)b + (q-p)c$ is zero which only occurs if $p = q = 0$. Therefore $H_2(K) = 0$.

For $H_1(K)$ the kernel is the free abelian group on a, b, c and the image is generated by $(a+b-c)$ and $a - b + c$. So our group is the abelian group generated by $\langle a, b, c | a + b = c, a + c = b\rangle$. We can simplify to remove c and get $\langle a, b | 2a\rangle$. Therefore $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$.

For $H_0(K)$ the image is trivial and the kernel is the whole space. Thus $H_0(K) = \mathbb{Z}$. □

Problem 3. *Show that if G is a finitely generated free abelian group and $H \subset G$ is a subgroup, then there is a basis $g_1, \dots, g_n$ for G and integers $p_1, \dots, p_k$ with $k \le n$ such that each p_i divides p_{i+1}, and such that $p_1 g_1, \dots, p_k g_k$ is a basis for H. We say that these bases for G and H are stacked. Conclude that*

$$G/H \cong \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k} \oplus \mathbb{Z}^{n-k}$$

In particular, every finitely generated abelian group is a direct sub of cyclic groups. (Hint: You may find it helpful to use the fact that subgroups of free abelian groups are themselves free abelian.)

Proof. □

Problem 4. *If $i : A \to X$ is the inclusion of a retract of X, show that $i_* : H_k(A) \to H_k(X)$ is a monomorphism onto a direct summand of $H_k(X)$. If A is deformation retract of X, show that i_* is an isomorphism.*

Proof. Let $i : A \to X$ be the inclusion and $r : X \to A$ the retract of X onto A. By definition $r \circ i = id_A$ which implies that the map induced on the homology $r_* \circ i_* = id_*$. Since i_* has a left inverse it must be injective. Therefore the map induced by the inclusion of a retract is injective.

Now suppose that A is a deformation retract of X. Then r is homotopic to id_X. Then we have that $i \circ r$ is homotopic to id_X which means that i is a homotopy equivalence and as such induces an isomorphism on the homology (Hatcher 2.11) of A and X. □

Problem 5. *Show that it is impossible to retract the $n-$ball B^n onto its $n-1$-sphere boundary $\partial B^n = S^{n-1}$.*

Proof. Since B^n is contractible $H_k(B^n) = 0$ for all k. If there existed a deformation retract of B^n onto S^1 this would imply that there exists an injective function from $H_{n-1}(S^{n-1}) = \mathbb{Z}$ to the trivial group which is a contradiction.

Therefore there is no deformation retract from B^n to S^{n-1}. □

Problem 6. *Compute the simplicial homology of the Klein bottle using the Δ-complex structure, with two simplices of dimension 2, discussed in class.*

Proof. □