Problem 1.

Proof. Let X be a space such that all paths having the same endpoints are fixed-endpoint homotopic. Then given a loop α based as some point x_0 this loop is homotopic to the constant map at x_0 which implies that $\pi_1(X, x_0)$ is trivial for all X. Therefore X is simply connected.

Now suppose that X is a simply connected space. Let α, β be loops from x_0 to x_1. For a fixed $t' \in [0, 1]$ using path connectedness we can construct a loop $l_{t'}$ such that $l_{t'}(0) = l_{t'}(1) = \alpha(t')$, $l_{t'}(\frac{1}{2}) = \beta(t)$ and that the function $p(t) = l_t(\frac{1}{2})$ is a path. If we had two loops that separated t

Since X is simply connected there is a homotopy $L_{t'}$ from $l_{t'}$ to the constant map at $\alpha(t')$. Then we can construct a homotopy H from β to α as

$$H(t, s) = L_t(\frac{1}{2}, s)$$

which will be continuous because of the continuity of the $L_{t'}$s and that we choose $l_{t'}$ such that as we vary over t there is a path.

Therefore a space X is simply connected if and only if all paths with the same endpoints are fixed endpoint homotopic. $\square$

Problem 2.

Proof. Let $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$. For a continuous map h we have h_* defined as $h_*([\gamma]) = [h \circ \gamma]$. Then if we consider $(g \circ f)_*$:

$$(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_* \circ f_*([\gamma])$$

Therefore $(g \circ f)_* = g_* \circ f_*$. $\square$

Problem 3.

Proof. $\square$

Problem 4.

Proof. $\square$

Problem 5.

Proof. Let B be simply connected, E path connected, and $p : E \to B$ a covering map. By the Theorem from class since E is path connected there exists a surjective map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$. However since B is simply connected the fundamental group is trivial which implies that $p^{-1}(b_0)$ has only one element. Since B is simply connected it is also connected so it follows that $p^{-1}(b)$ contains only a single element for all $b \in B$. As there is only one copy of B in E this implies that $p|_E = p$ and as such p is a homeomorphism.

Therefore if B is simply connected, then any covering map for which E is path connected is a homeomorphism. $\square$

Problem 6.

Proof. Let $h : (X, x_0) \to (Y, y_0)$ be an inessential map and let $[\gamma] \in \pi(X, x_0)$. Since h is inessential it follows that there is a homotopy H that sends h to the constant map at y_0. If we take $h_*([\gamma]) = [h \circ \gamma]$ then we can create a homotopy $H'(t, s) = H(\gamma(t), s)$ that will take γ to the constant map which implies that d $h_*([\gamma]) = [y_0]$. Since this happens for an arbitrary loop in $\pi_1(X, x_0)$ it follows that the map h_* is trivial.

Therefore if $h : (X, x_0) \to (Y, y_0)$ is inessential then the induced homomorphism h_* is trivial. $\square$