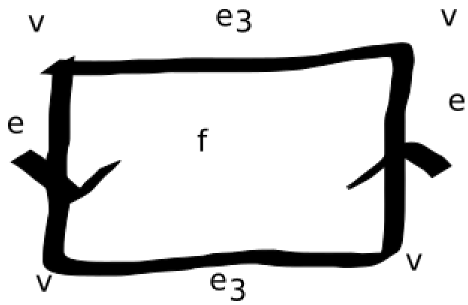
Problem 1 (1). *What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the ordering of vertices?*

If we take a look at the following identification for X



We can see that it is a Möbius strip. As further evidence of we can compute the homology of the space.

Since we are identifying edges e_1 and e_2 this also induces the identification of v_1, v_2, and v_3. We'll call them e and v respectively. Then:

$$\partial f = 2e + e_3$$
$$\partial e = v - v = 0$$
$$\partial e_3 = v - v = 0$$

Then for $H_2(X)$ we have im $\partial_3 = 0$ and $\ker \partial_2 = 0$. Thus $H_2(X) \cong 0$.
For H_1 we have im $\partial_2 = \langle 2e + e_3 \rangle$ and $\ker \partial_1 = \langle e, e_3 \rangle$. Then we have

$$H_1(X) \cong \langle e, e_3 \rangle / \langle 2e + e_3 \rangle \cong \langle e, e_3 | 2e = -e_3 \rangle \cong \langle e \rangle \cong \mathbb{Z}$$

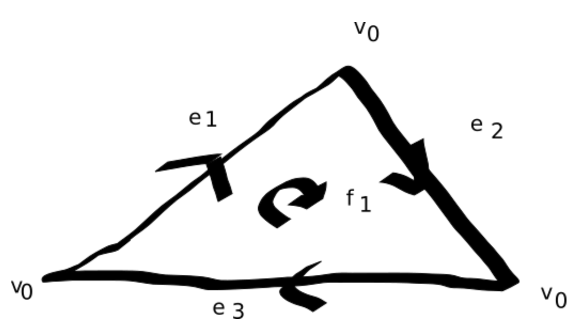
Finally $H_0(X) = \mathbb{Z}$ as there is only a single connected component.
Therefore the homology for X is

$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0, 1 \\ 0 & p \geq 2 \end{cases}$$

Since H_2 is trivial that implies that the surface X is nonorientable. Moreover the first homology group is $\mathbb{Z}$ which matches the fundamental group of the Möbius strip.

Problem 2 (4). *Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.*

Use the picture below to guide the boundary map for X.



The boundary map for our components will be:

$$\begin{aligned}\partial f &= e_1 + e_2 + e_2 \\ \partial e_1 &= v - v = 0 \\ \partial e_2 &= v - v = 0 \\ \partial e_3 &= v - v = 0\end{aligned}$$

For $H_2(X)$ we have im $\partial_3 = 0$ and $\ker \partial_2 = 0$. Thus $H_2(X) \cong 0$.
For $H_1(X)$ we have im $\partial_2 = \langle e_1 + e_2 + e_3 \rangle$ and $\ker \partial_1 = \langle e_1, e_2, e_3 \rangle$. Thus

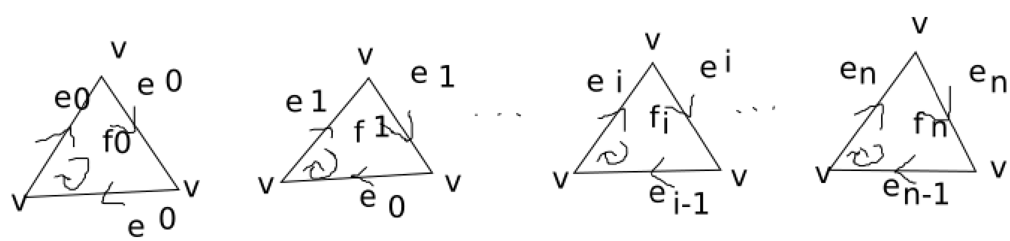
$$H_1(X) = \langle e_1, e_2, e_3 \rangle / \langle e_1 + e_2 + e_3 \rangle \cong \langle e_1, e_2, e_3 | e_1 + e_2 = -e_3 \rangle \cong \langle e_1 + e_2 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$$

Finally there is only a single connected component so $H_0(X) \cong \mathbb{Z}$.
Therefore the homology for X is

$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & p = 1 \\ 0 & p \geq 2 \end{cases}$$

Problem 3 (6)**.** *Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplicies $\Delta_0^2, \cdots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge of $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.*

We'll use the following diagram for the space X to define the homology.



The values for the boundary maps are

$$\begin{aligned}\partial e_i &= v_0 - v_0 = 0 \\ \partial f_0 &= 3e_0 \\ \partial f_i &= 2e_i + e_{i-1} & 1 \leq i \leq n\end{aligned}$$

For $H_2(X)$ we have that im $\partial_3 = 0$. For $\ker \partial_2$ start with the matrix

$$\begin{pmatrix} 3 & 1 & 0 & \cdots & 0 \\ 0 & 2 & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ & \vdots & & 2 & 1 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix}$$

Then the smith normal form of the matrix is

$$\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & & 1 & 0 \\ 0 & \cdots & & 0 & 3 \cdot 2^{n-1} \end{pmatrix}$$

The null space for this matrix is trivial. Thus $\ker \partial_2 = 0$ and thus $H_2(X) \cong 0$.

Then for $H_1(X)$ we have that $\ker \partial_1 = \langle e_i \rangle$ and from the above matrix we know that $\operatorname{im} \partial_2 = \langle e_1, \ldots, e_{n-1}, 3 \cdot 2^{n-1} e_n \rangle$ giving us that $H_1(X) \cong \mathbb{Z}_{3 \cdot 2^{n-1}}$.

Finally $H_0(X) \cong \mathbb{Z}$ as there is only one connected component.

Therefore the homology of X is

$$H_p(X) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathbb{Z}_{3 \cdot 2^{n-1}} & p = 1 \\ 0 & p \geq 2 \end{cases}$$