Problem 1 (22)**.** *Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres.*

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

(c) If X has k n-cells then $H_n(X)$ is generated by at most k elements.

$$\cdots \to \widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n) \to \widetilde{H}_k(X^n/X^{n-1}) \to \widetilde{H}_{k-1}(X^{n-1}) \to \widetilde{H}_{k-1}(X^n) \to \widetilde{H}_{k-1}(X^n/X^{n-1}) \to \cdots$$

Proof. (a) Note that $H_n(X) \cong H_n(X^n)$ since there are no simplices of higher dimension.

Suppose that X has dimension zero. Then X consists of only 0-cells and as such $C_i(X) = 0$ for $i > 0$ implying that $H_i(X) = 0$ for $i > 0$. From Hatcher we have that $H_0(X) = \bigoplus_{x \in X} \mathbb{Z}$.

Now assume that for Y with dimension $j < n$ that $H_i(Y) = 0$ for $i > j$ and that $H_j(Y)$ is free. Suppose that X has dimension n. Then from Hatcher we have the long exact sequence

$$\cdots \to \widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n) \to \widetilde{H}_k(X^n/X^{n-1}) \to \widetilde{H}_{k-1}(X^{n-1}) \to \widetilde{H}_{k-1}(X^n) \to \widetilde{H}_{k-1}(X^n/X^{n-1}) \to \cdots$$

If $k > n$ then this gives us the exact sequence

$$0 \to \widetilde{H}_k(X^n) \to 0$$

as $\widetilde{H}_k(X^{n-1})$ will be zero by induction and $\widetilde{H}_k(X^n/x^{n-1})$ will be zero since n-spheres only have nonzero reduced homology in dimension n.

Otherwise if $k = n$ we get the exact sequence

$$0 \to \widetilde{H}_n(X^n) \to \bigoplus_{n \text{ simplices}} \mathbb{Z} \to \cdots$$

The fact that $\widetilde{H}_n(X^{n-1})$ is zero by induction forces the above map to be injective. It then follows that $\widetilde{H}_n(X^n)$ is isomorphic to the subgroup of a free abelian group and as such is free as well. Moreover the number of generators is has rank equal to one fewer than the number of n-simplices (this will be used in (c)). Since reduced homology is identical to homology when the dimension is greater than zero this completes the proof.

(b) Let $n = 0$. From Hatcher we have that $H_0(X)$ is free abelian with a generator for each 0-cell.

Now assume that for dimension $j < n$ if there are no $j-1$ or $j+1$ simplices that $H_j(X)$ is free with basis in bijective correspondence with the j-cells. Then let X be a CW complex with no $n+1$ or $n-1$ simplices. Note that this implies that $X^{n+1} = X^n$ and that $X^{n-1} = X^{n-2}$ From the same long exact sequence as above we have:

$$\widetilde{H}_n(X^{n-2}) \cong 0 \to \widetilde{H}_n(X^n) \to \widetilde{H}_n(X^n/X^{n-1}) \cong \bigoplus_{n \text{ simplices}} \mathbb{Z} \to H_{n-1}(X^{n-2}) \cong 0$$

This an isomorphism between $\widetilde{H}_n(X)$ and $\bigoplus_{n \text{ simplices}} \mathbb{Z}$ and since $n > 0$ reduced homology is equivalent to homology.

Therefore if $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

(c) Consider the same long exact sequence as above. Then we have

$$\cdots \to \widetilde{H}_n(X^n) \to \widetilde{H}_n(X^{n+1}) \to \widetilde{H}_n(X^{n+1}/X^n) \cong 0$$

which makes the map from $\widetilde{H}_n(X^n)$ to $\widetilde{H}_n(X^{n+1}) \cong H_n(X)$ surjective. From (a) we have that $H_n(X^n)$ is free on at most k generators and as such $H_n(X)$ must also have at most k generators.

□

Problem 2 (26)**.** *Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ together with its limit 0. [See Example 1.25.]*

Proof. We can see that $H_1(X, A) \cong \bigoplus_1^\infty \mathbb{Z}$ from the short exact sequence

$$0 \longrightarrow H_1(X, A) \longrightarrow H_0(A) \cong \bigoplus_0^\infty \mathbb{Z} \longrightarrow H_0(X) \cong \mathbb{Z} \longrightarrow 0$$

as this short exact sequence enforces that $H_1(X, A) \oplus \mathbb{Z} \cong \bigoplus_0^\infty \mathbb{Z}$. Thus $H_1(X, A)$ has a countable number elements.

However if we examine the space X/A we can see that it is homeomorphic to the Hawaiian earring. Moreover there are a countable number of one simplices and a single zero simplex. Thus $\widetilde{H}_1(X/A)$ is not finitely generated and as such by Hatcher (49) we have that $\widetilde{H}_1(X/A)$ is uncountable.

Therefore $H_1(X, A)$ and $\widetilde{H}_1(X/A)$ are not isomorphic. □

Problem 3 (27)**.** *Let $f : (X, A) \to (Y, B)$ be a map that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.*

(a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.

(b) For the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs —there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures.]

Proof. (a) From Hatcher we have the long exact sequences with morphisms between them like so:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\
\cdots \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(Y) & \longrightarrow \cdots
\end{array}$$

Since f is a homotopy equivalence for A and X the left 2 and the right 2 f_*s are isomorphisms. Thus by the five lemma the center map f_* is an isomorphism.

(b) Suppose otherwise. Then we have a map $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that g is a homotopy equivalence. Then since $g : D^n \setminus \{0\} \to S^{n-1}$ is continuous it will force $g : D^n \to S^{n-1}$ to send 0 to S^{n-1} as well as it is continuous and therefore must preserve the limit of a sequence $\{x_n \neq 0\}_1^\infty$ that converges to 0. Since g is then a map from D^n to S^{n-1} it must be null and therefore the induced map is the zero map.

However for $k = n - 1$ we have a map

$$H_{n-1}(D^n \setminus \{0\}) \cong \mathbb{Z} \to H_n(S^{n-1}) \cong Z$$

which is both the zero map and an isomorphism since it is a homotopy equivalence. This is a contradiction.

Therefore there the inclusion map is not a homotopy equivalence of pairs for (D^n, S^{n-1}) and $(D^n, D^n \setminus \{0\})$.

□