

Problem 1 (28a). Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

Proof. A Möbius band is homotopy equivalent to a circle, and we know the homology of the torus. Thus we have the long exact sequence

$$\cdots \longrightarrow \tilde{H}_n(T^2 \cap M) \longrightarrow \tilde{H}_n(T^2) \oplus \tilde{H}_n(M) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_{n-1}(T^2 \cap M) \longrightarrow \cdots$$

The intersection $T^2 \cap M$ also has the homology of a circle giving us the exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_2(M \cap T^2) \longrightarrow \tilde{H}_2(T^2) \longrightarrow \tilde{H}_2(X) \longrightarrow \tilde{H}_1(M \cap T^2) \longrightarrow \tilde{H}_1(M) \oplus \tilde{H}_1(T^2) \longrightarrow \\ \longrightarrow \tilde{H}_1(X) \longrightarrow \tilde{H}_1(M \cap T^2) \longrightarrow \cdots \end{aligned}$$

which when we substitute in we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \tilde{H}_2(X) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{k_* - l_*} \tilde{H}_1(X) \longrightarrow 0$$

Since the right i_*, j_* is injective the kernel is zero. Thus the image of ∂_* is 0 making the left i_*, j_* an isomorphism. Similarly the fact that the kernel of the right i_*, j_* is zero effectively gives us the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{k_* - l_*} \tilde{H}_1(X) \longrightarrow 0$$

which implies that $\tilde{H}_1(X) \cong \langle x, y, z \rangle / \langle 2x - y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.

Therefore the homology of X is

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 2 (29). The surface M_g of genus g , embedded in \mathbb{R}^3 in the standard way, bounds a compact region R . Two copies of R , glued together by the identity map between boundary surfaces M_g , form a closed 3-manifold X . Compute the homology groups of X into two copies of R . Also compute the relative groups $H_i(R, M_g)$.

Proof. First note that the homology of M_g is

$$H_i(M_g) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the region R has the homotopy type of the wedge of g circles its homology is

$$H_i(R) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^g & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now we represent X as the union of two slightly thickened copies of R such that the intersection is M_g . Then using the Mayer-Vietoris sequence we get:

$$0 \longrightarrow H_3(X) \xrightarrow{\partial_*} H_2(M_g) \longrightarrow 0 \longrightarrow H_2(X) \xrightarrow{\partial_*} H_1(M_g) \xrightarrow{i_*, j_*} H_1(R) \oplus H_1(R) \xrightarrow{k_* - l_*} H_1(X) \longrightarrow 0$$

It immediately follows that $H_3(X)$ is isomorphic to $H_2(M_g) \cong \mathbb{Z}$. Moreover we know have an exact sequence which if we substitute in we get:

$$0 \longrightarrow H_2(X) \xrightarrow{\partial_*} \mathbb{Z}^{2g} \xrightarrow{i_*, j_*} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{k_* - l_*} H_1(X) \longrightarrow 0$$

Since ∂_* is injective from its location in the sequence it follows that $H_2(X) \cong \ker i_*, j_*$. Looking at the map i_*, j_* it is the inclusion of M_g into both copies of R . As such half of the meridians will get sent to zero while the others will get sent to the generators of $H_1(R)$. Thus the kernel of i_*, j_* is $\mathbb{Z}^g \cong H_1(X)$.

Next since $k_* - l_*$ is surjective we have that $H_2(X) \cong (\mathbb{Z}^g \oplus \mathbb{Z}^g) / \text{im } i_*, j_*$. The elements in the image of i_*, j_* are those of the form $(a_1, \dots, a_g, a_1, \dots, a_g)$. Thus the quotient will be $\mathbb{Z}^{2g} / \mathbb{Z}^g \cong \mathbb{Z}^g \cong H_2(X)$.

Therefore the homology of X is

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^g & i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

To calculate $H_i(R, M_g)$ we use the long exact sequence from hatcher to get

$$0 \longrightarrow H_3(R, M_g) \longrightarrow H_2(M_g) \longrightarrow 0 \longrightarrow H_2(R, M_g) \longrightarrow H_1(M_g) \longrightarrow H_1(R) \longrightarrow H_1(R, M_g) \longrightarrow 0$$

It follows immediately that $H_3(R, M_g) \cong H_2(M_g) \cong \mathbb{Z}^{2g}$. In addition we also get the exact sequence

$$0 \longrightarrow H_2(R, M_g) \xrightarrow{\partial_*} \mathbb{Z}^{2g} \xrightarrow{i_*} \mathbb{Z}^g \xrightarrow{q_*} H_1(R, M_g) \longrightarrow 0$$

The setup is the same as above. Thus $H_2(R, M_g) \cong \ker i_* \cong \mathbb{Z}^g$. In addition $H_1(R, M_g) \cong \mathbb{Z}^g / \text{im } i_*$. However unlike in the previous problem i_* is surjective. As such $H_1(R, M_g) \cong \mathbb{Z}^g / \mathbb{Z}^g \cong 0$. The zeroth homology is zero as M_g touches the single path component of R .

Therefore the relative homology of R with respect to M_g is

$$H_i(R, M_g) = \begin{cases} \mathbb{Z} & i = 3 \\ \mathbb{Z}^g & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 3 (30). For the mapping torus T_f of a map $f : X \rightarrow X$, we constructed in Example 2.48 a long exact sequence

$$\cdots \longrightarrow H_n(X) \xrightarrow{\text{id} - f_*} H_n(X) \longrightarrow H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Use this to compute the homology of the mapping tori of the following maps:

- (a) A reflection $S^2 \rightarrow S^2$
- (b) A map $S^2 \rightarrow S^2$ of degree 2.
- (c) The map $S^1 \times S^1 \rightarrow S^1 \times S^1$ that is the identity on one factor and a reflection on the other.
- (d) The map $S^1 \times S^1 \rightarrow S^1 \times S^1$ that is a reflection on each factor.
- (e) The map $S^1 \times S^1 \rightarrow S^1 \times S^1$ that interchanges the two factors and then reflects one of the factors.

Proof. (a) Note that $H_1(T_f) = 0$ as it is between $H_1(S^2)$ and $\tilde{H}_0(S^2)$. The same holds for $n > 3$. All of the possible nonzero items are contained in the exact sequence

$$0 \longrightarrow H_3(T_f) \longrightarrow H_2(S^2) \longrightarrow H_2(S^2) \longrightarrow H_2(T_f) \longrightarrow 0$$

As before this gives us that $H_3(T_f) \cong \ker(\text{id}_* - f_*)$ and $H_2(T_f) \cong \mathbb{Z}/\text{im}(\text{id}_* - f_*)$.

If we evaluate $(\text{id}_* - f_*)(1)$ this is equal to $1 - (-1) = 2$. The kernel of this is trivial and the image is $2\mathbb{Z}$. Thus the homology of T_f is

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

- (b) Same as (a) but with a different map. If we calculate $(\text{id}_* - f_*)(1)$ this is $1 - 2 = -1$. The kernel of this map is trivial and the image is \mathbb{Z} . Thus the homology is:

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

- (c) Similar to above but $H_1(T^2)$ is not zero so we have to account for that as well. The exact sequence we get from the long exact sequence above is:

$$0$$

Probably wrong

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

- (d)

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

- (e)

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

□