Problem 1 (2.1.17). *(a) Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.*

(b) Compute the groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus two with A and B the circles shown (A is circle around join of Tori and B is around tube of right tori). [What are X/A and X/B.]

Proof. (a) Beginning with S^2 use the long exact sequence for homology to get

$$0 \to H_2(S^2) \cong \mathbb{Z} \to H_2(S^2, A) \to 0 \to H_1(S^2) \cong 0 \to H_1(S^2, A) \to H_0(A) \cong \mathbb{Z}^{|A|} \to H_0(S^2) \cong \mathbb{Z} \to 0$$

From this we can deduce that

$$H_i(S^2, A) \cong \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^{|A|-1} & i = 1 \\ 0 & \text{else} \end{cases}$$

Similarly for T^2 we have the long exact sequence

$$0 \to H_2(T^2) \cong \mathbb{Z} \to H_2(T^2, A) \to 0 \to H_1(T^2) \cong \mathbb{Z}^2 \to H_1(T^2, A) \to H_0(A) \cong \mathbb{Z}^{|A|} \to H_0(T^2) \cong \mathbb{Z} \to 0$$

Which gives us the relative homology

$$H_i(T^2, A) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^{|A|+2} & i = 1 \\ 0 & \text{else} \end{cases}$$

(b) Starting with A and after substituting for known groups we have the long exact sequence

$$0 \to \mathbb{Z} \to H_2(\Sigma_2, A) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2, A) \to \mathbb{Z} \to \mathbb{Z} \to 0$$

However the last map $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism since it is the inclusion of one connected component to another. So we can shorten our exact sequence to

$$0 \to \mathbb{Z} \to H_2(\Sigma_2, A) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2, A) \to 0$$

Finally note that the map from $\mathbb{Z}$ to $\mathbb{Z}^4$ is going to be the zero map. As such this will force two shorter exact sequences

$$0 \to \mathbb{Z} \to H_2(\Sigma_2, A) \to \mathbb{Z} \to 0, 0 \to \mathbb{Z}^4 \to H_1(\Sigma_2, A) \to 0$$

which give us

$$H_i(\Sigma_2, A) = \begin{cases} \mathbb{Z}^2 & i = 2 \\ \mathbb{Z}^4 & i = 1 \\ 0 & \text{else} \end{cases}$$

The case for B is similar. We get down to the same exact sequence

$$0 \to \mathbb{Z} \to H_2(\Sigma_2, B) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2, B) \to 0$$

However in this case the map $\mathbb{Z} \to \mathbb{Z}^4$ sends 1 to one of the four generators. Since this is injective the map will induce an isomorphism for the second homology and will induce the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2, B) \to 0$$

giving us the homology

$$H_i(\Sigma_2, B) = \begin{cases} \mathbb{Z} & i = 2 \\ \mathbb{Z}^3 & i = 1 \\ 0\text{else} \end{cases}$$

□

Problem 2 (2.2.40). *From the long exact sequence of homology groups associated to the short exact sequence of chain complexes* $0 \to C_i(X) \xrightarrow{n} C_i(X) \xrightarrow{\varphi} C_i(X;\mathbb{Z}_n) \to 0$ *deduce immediately that there are short exact sequences*

$$0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}_n) \to n - Torsion(H_{i-1}(X)) \to 0$$

where $n-Torsion(G)$ *is the kernel of the map* $G \xrightarrow{n} G$, $g \mapsto ng$. *Use this to show that* $\widetilde{H}_i(X;\mathbb{Z}_p) = 0$ *for all* i *and all primes* p *iff* $\widetilde{H}_i(X)$ *is a vector space over* $\mathbb{Q}$ *for all* i.

Proof. Since we have a short exact sequence of chain complexes we can form a long exact sequence in homology, using the zig-zag lemma, of the form

$$\cdots \xrightarrow{\partial_*} H_i(X) \xrightarrow{n_*} H_i(X) \xrightarrow{\varphi_*} H_i(X;\mathbb{Z}_n) \longrightarrow \cdots$$

From this we can make a short exact sequence

$$0 \to H_i(X)/nH_i(X) \to H_i(X;\mathbb{Z}_n) \to n - Torsion(H_{i-1}(X)) \to 0$$

which will be exact since the first map is quotienting out by the kernel of φ_* and the latter map is surjective since $n - Torsion$ is defined as the kernel of n_*.

Suppose that $\widetilde{H}(X;\mathbb{Z}_p) = 0$ for all primes p. This then forces the other two groups for the short exact sequence to be zero. It then follows that for all n there is no torsion in $\widetilde{H}_i(X)$ and that $H_i(X)/nH_i(X) = 0$ for all i and n. Then we can define an action of the rationals on $\widetilde{H}_i(X)$ via $(\frac{p}{q}) \cdot g = h$ where h is the unique solution to $p \cdot g = q \cdot h$. This will never be zero since there is no torsion in $\widetilde{H}_i(X)$ and the fact that a solution always exists comes from $H_i(X)/nH_i(X) = 0$. Thus $\widetilde{H}_i(X)$ is a $\mathbb{Q}$ vector space for all i.

Now suppose that $\widetilde{H}_i(X)$ is a $\mathbb{Q}$ vector space. Then there is no torsion in $\widetilde{H}_i(X)$ as otherwise it wouldn't be a vector space. Moreover since the action of $\mathbb{Q}$ is well defined we have that $H_i(X)/nH_i(X)$ will be zero for all n. Since there is no torsion and the former group is zero it follows that the group $H_i(X;\mathbb{Z}_n)$ will be zero since it is in the middle of the short exact sequence. □

Problem 3 (2.2.43(a)). *Show that a chain complex of a free abelian groups* C_n *splits as a direct sum of subcomplexes* $0 \to L_{n+1} \to K_n \to 0$ *with at most two nonzero terms. [Show that the short exact sequence* $0 \to \ker\partial \to C_n \to$ *im* $\partial \to 0$ *splits and take* $K_n = \ker\partial$*.]*

Proof. First note that the short exact sequence

$$0 \longrightarrow \ker\partial_n \longrightarrow C_n \longrightarrow \text{im } \partial_{n+1} \longrightarrow 0$$

splits into

$$0 \longrightarrow \ker\partial_n \longrightarrow \ker\partial_n \oplus \text{im } \partial_{n+1} \longrightarrow \text{im } \partial_{n+1} \longrightarrow 0$$

since im ∂_{n+1} is free. Let $K_n := \ker\partial_n$ and let $L_n :=$ im ∂_{n+1}. The sequence $0 \to L_{n+1} \to K_n \to 0$ is a chain complex as $\partial^2 = 0$. It then follows that we can express the chain complex

$$\cdots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow \cdots$$

as a direct sum of the form

$$0 \longrightarrow L_{n-1} \longrightarrow \cdots$$

$$0 \longrightarrow L_n \longrightarrow K_{n-1} \longrightarrow 0$$

$$0 \longrightarrow L_{n+1} \longrightarrow K_n \longrightarrow 0$$

$$\cdots \longrightarrow K_{n+1} \longrightarrow 0$$

□

Problem 4 (2.3.4). *Show that the wedge axiom for homology theories follows from the other axioms in the case of finite wedge sums.*

Proof. From Hatcher we have that the Mayer-Vietoris sequence holds, even without the wedge axiom. If we calculate the wedge of two spaces X_1 and X_2, then we can deduce their homology using Mayer-Vietoris sequence along with the homology of the point being null to get

$$0 \longrightarrow h_i(X) \oplus h_i(Y) \longrightarrow h_i(X_1 \vee X_2) \longrightarrow 0$$

Now assume that the wedge axiom holds when we wedge n spaces together. As before if we calculate the homology of $\bigvee_1^{n+1} X_i$ we get

$$0 \longrightarrow h_i(\bigvee_1^n X_i) \oplus h_i(X_{n+1}) \cong \bigoplus_1^n h_i(X_i) \longrightarrow h_i(\bigvee_1^{n+1} X_i) \longrightarrow 0$$

Therefore the wedge axiom follows from the other homology axioms when restricted to finite wedge sums. □

Problem 5 (2.B.5). *Let S be an embedded k-sphere in S^n for which there exists a disk $D^n \subset S^n$ intersecting S in the disk $D^k \subset D^n$ defined by the first k coordinates of D^n. Let $D^{n-k} \subset D^n$ be the disk defined by the last $n-k$ coordinates, with boundary sphere S^{n-k-1}. Show that the inclusions $S^{n-k-1} \hookrightarrow S^n \setminus S$ induces an isomorphism on homology groups.*

Proof. □

Problem 6 (2.B.10). *Use the transfer sequence for the covering $S^\infty \to \mathbb{R}P^\infty$ to compute $H_n(\mathbb{R}P^\infty; \mathbb{Z}_2)$.*

Proof. First recall that the infinite sphere S^∞ has zero homology except for zeroth homology. It then follows if we apply the transfer sequence for $i > 1$

$$0 \longrightarrow H_i(\mathbb{R}P^\infty; \mathbb{Z}_2) \longrightarrow H_{i-1}(\mathbb{R}P^\infty; \mathbb{Z}_2) \longrightarrow 0$$

we see that $H_i(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong H_{i-1}(\mathbb{R}P^\infty;\mathbb{Z}_2)$ for $i > 1$. We know that $H_0(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2$ since there only a single connected component. As such all we need to figure out is $H_1(\mathbb{R}P^\infty;\mathbb{Z}_2)$ and this will determine the rest.

Using the transfer sequence at the tail end we get

$$0 \longrightarrow H_i(\mathbb{R}P^\infty;\mathbb{Z}_2) \xrightarrow{\partial_*} H_0(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2 \xrightarrow{\tau_*} H_0(S^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2 \xrightarrow{p_*} H_0(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2 \longrightarrow 0$$

First note that p_* will be an isomorphism as it a surjective map between finite groups. This makes τ_* the zero map making ∂_* surjective. However ∂_* is already injective by exactness. So ∂_* is an isomorphism and thus $H_1(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$. Using our isomorphisms for the higher homology groups we get

$$H_i(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2$$

for all $i \geq 0$. $\square$