Problem 1 (12)**.** *Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.*

Proof. □

Problem 2 (14)**.** *A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree. [Hints: If f is even, it factors as a composition $S^n \to \mathbb{R}P^n \to S^n$. Using the calculation of $H_n(\mathbb{R}P^n)$ in the text, show that the induced map $H_n(S^n) \to H_n(\mathbb{R}P^n)$ sends a generator to twice a generator when n is odd. It may be helpful to show that the quotient map $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$ induces an isomorphism on H_n when n is odd.]*

Proof. Let f be a map from S^n to S^n such that $f(x) = f(-x)$ for all $x \in S^n$. Since $\mathbb{R}P^n$ is a quotient space of S^n where antipodal points are identified any even map from S^n respects equivalence classes for the quotient space and as such it factors as

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "q"] & S^n \\ \mathbb{R}P^n \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

First note that $\deg q = 2$. This is because for any point $x \in \mathbb{R}P^n$ it will have two points mapping to it from S^n and it will be the identity map giving us local degrees of 1 which add up to 2. Since any even map will factor in this way any even map must have even degree.

When n is even $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ and when n is even $H_n(\mathbb{R}P^n) \cong 0$. If n is odd and f is even then $\deg f = \deg q \cdot \deg \widetilde{f}$. However since $H_n(\mathbb{R}P^n) \cong 0$ when n is even then the degree of f has to be zero as $\deg \widetilde{f} = 0$ since it is mapping out of the trivial group.

Now suppose that n is odd. There is a quotient map $r : \mathbb{R}P^n \to (\mathbb{R}P^n/\mathbb{R}P^{n-1} \equiv S^n)$. This map will have degree 1 and as such $q \circ r$ will be a map from $S^n \to S^n$ of degree two. From there if we take a map of degree k, $f_k : S^n \to S^n$ (Hatcher 2.32). Then $f_k \circ q \circ r$ will be an even map of degree $2k$. □

Problem 3 (20)**.** *For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.*

Proof. Let x_n and y_n denote the n-dimensional simplices of X and Y respectively. Then

$$\chi(X)\chi(Y) = \left(\sum_{n=0}^{\infty}(-1)^i x_i\right)\left(\sum_{m=0}^{\infty}(-1)^n\right) = \sum_{0=i=m+n}^{\infty}(-1)^i\left(\sum_{j=0}^{i} x_j y_{i-j}\right)$$

However since the number of i-simplices in $X \times Y$ is $\sum_{j=0}^{i} x_j y_{i-j}$ this demonstrates that

$$\chi(X)\chi(Y) = \chi(X \times Y)$$

□