Problem 1 (22). *Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres.*

(a) If X has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free.

(b) $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.

(c) If X has k n-cells then $H_n(X)$ is generated by at most k elements.

$$\cdots \to \widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n) \to \widetilde{H}_k(X^n/X^{n-1}) \to \widetilde{H}_{k-1}(X^{n-1}) \to \widetilde{H}_{k-1}(X^n) \to \widetilde{H}_{k-1}(X^n/X^{n-1}) \to \cdots$$

Proof. (a) Suppose that X has dimension zero. Then X consists of only 0-cells and as such $C_i(X) = 0$ for $i > 0$ implying that $H_i(X) = 0$ for $i > 0$. From Hatcher we have that $H_0(X) = \bigoplus_{x\in X} \mathbb{Z}$.

Now assume that for Y with dimension $j < n$ that $H_i(Y) = 0$ for $i > j$ and that $H_j(Y)$ is free. Suppose that X has dimension n. Then from Hatcher we have the long exact sequence

$$\cdots \to \widetilde{H}_k(X^{n-1}) \to \widetilde{H}_k(X^n) \to \widetilde{H}_k(X^n/X^{n-1}) \to \widetilde{H}_{k-1}(X^{n-1}) \to \widetilde{H}_{k-1}(X^n) \to \widetilde{H}_{k-1}(X^n/X^{n-1}) \to \cdots$$

If $k > n$ then this gives us the exact sequence

$$0 \to \widetilde{H}_k(X^n) \to 0$$

as $\widetilde{H}_k(X^{n-1})$ will be zero by induction and $\widetilde{H}_k(X^n/x^{n-1})$ will be zero since n-spheres only have nonzero reduced homology in dimension n.

Otherwise if $k = n$ we get the exact sequence

$$0 \to \widetilde{H}_n(X^n) \to \bigoplus_{n \text{ simplices}} \mathbb{Z} \to \cdots$$

The fact that $\widetilde{H}_n(X^{n-1})$ is zero by induction forces the above map to be injective. It then follows that $\widetilde{H}_n(X^n)$ is isomorphic to the subgroup of a free abelian group and as such is free as well. Since reduced homology is identical to homology when the dimension is greater than zero this completes the proof.

(b) Let $n = 0$. From Hatcher we have that $H_0(X)$ is free abelian with a generator for each 0-cell.

Now assume that for dimension $j < n$ if there are no $j-1$ or $j+1$ simplices that $H_j(X)$ is free with basis in bijective correspondence with the j-cells. Then let X be a finite-dimensional CW complex with no n-simplices. Then via the same exact sequence from above we get

$$\widetilde{H}_n(X^{n-1}) \cong 0 \to \widetilde{H}_n(X^n) \to \widetilde{H}_n(X^n/X^{n-1}) \to \widetilde{H}_{n-1}(X^{n-1} \cong 0)$$

Which induces an isomorphism between the remaining groups. Since $\widetilde{H}_n(X^n/X^{n-1})$ has a generator for each n simplex it follows that $\widetilde{H}_n(X^n)$ does as well.

Next suppose that there are no $n+1$ simplices.

(c)

$\square$

Problem 2 (26)**.** *Show that $H_1(X, A)$ is not isomorphic to $\widetilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ together with its limit 0. [See Example 1.25.]*

Proof. We can see that $H_1(X, A) \cong \bigoplus_1^\infty \mathbb{Z}$ from the short exact sequence

$$0 \longrightarrow H_1(X, A) \longrightarrow H_0(A) \cong \bigoplus_0^\infty \mathbb{Z} \longrightarrow H_0(X) \cong \mathbb{Z} \longrightarrow 0$$

as this short exact sequence enforces that $H_1(X, A) \oplus \mathbb{Z} \cong \bigoplus_0^\infty \mathbb{Z}$. Thus $H_1(X, A)$ has a countable number elements.

However if we examine the space X/A we can see that it is homeomorphic to the Hawaiian earring. Moreover there are a countable number of one simplices and a single zero simplex. Thus $\widetilde{H}_1(X/A)$ is not finitely generated and as such by Hatcher (49) we have that $\widetilde{H}_1(X/A)$ is uncountable.

Therefore $H_1(X, A)$ and $\widetilde{H}_1(X/A)$ are not isomorphic. □

Problem 3 (27)**.** *Let $f : (X, A) \to (Y, B)$ be a map that both $f : X \to Y$ and the restriction $f : A \to B$ are homotopy equivalences.*

(a) Show that $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.

(b) For the inclusion $f : (D^n, S^{n-1}) \to (D^n, D^n - \{0\})$, show that f is not a homotopy equivalence of pairs —there is no $g : (D^n, D^n - \{0\}) \to (D^n, S^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs. [Observe that a homotopy equivalence of pairs $(X, A) \to (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing A and B by their closures.]

Proof. (a) From Hatcher we have the long exact sequences with morphisms between them like so:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X, A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y, B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Since f is a homotopy equivalence for A and X the left 2 and the right 2 f_*s are isomorphisms. Thus by the five lemma the center map f_* is an isomorphism.

(b) Suppose otherwise. Then we have a map $g : (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ such that g is a homotopy equivalence. Then since $g : D^n \setminus \{0\} \to S^{n-1}$ is continuous it will force $g : D^n \to S^{n-1}$ to send 0 to S^{n-1} as well as it is continuous. Since g extends is then a map from D^n that extends to a disc it is null and therefore the zero map on homology.

However for $k = n - 1$ we have a map

$$H_{n-1}(D^n \setminus \{0\}) \cong \mathbb{Z} \to H_n(S^{n-1}) \cong Z$$

which is both the zero map and an isomorphism since it is a homotopy equivalence. This is a contradiction.

Therefore there the inclusion map is not a homotopy equivalence of pairs for (D^n, S^{n-1}) and $(D^n, D^n \setminus \{0\})$.

□