Problem 1 (28a). *Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.*

Proof. A Möbius band is homotopy equivalent to a circle, and we know the homology of the torus. Thus we have the long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(T^2 \cap M) \longrightarrow \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(M) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_{n-1}(T^2 \cap M) \longrightarrow \cdots$$

The intersection $T^2 \cap M$ also has the homology of a circle giving us the exact sequence

$$\cdots \longrightarrow \widetilde{H}_2(M \cap T^2) \longrightarrow \widetilde{H}_2(T^2) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(M \cap T^2) \longrightarrow \widetilde{H}_1(M) \oplus \widetilde{H}_1(T^2) \longrightarrow$$

$$\longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_1(M \cap T^2) \longrightarrow \cdots$$

which when we substitute in we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \widetilde{H}_2(X) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{k_* - l_*} \widetilde{H}_1(X) \longrightarrow 0$$

Since the right i_*, j_* is injective the kernel is zero. Thus the image of ∂_* is 0 making the left i_*, j_* an isomorphism. Similarly the fact that the kernel of the right i_*, j_* is zero effectively gives us the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{k_* - l_*} \widetilde{H}_1(X) \longrightarrow 0$$

which implies that $\widetilde{H}_1(X) \cong \langle x, y, z \rangle / \langle 2x - y \rangle \cong \langle x, z \rangle \cong \mathbb{Z}^2$.

Therefore the homology of X is

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Problem 2 (29). *The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X into two copies of R. Also compute the relative groups $H_i(R, M_g)$.*

Proof. $\square$

Problem 3 (30). *For the mapping torus T_f of a map $f : X \to X$, we constructed in Example 2.48 a long exact sequence*

$$\cdots \longrightarrow H_n(X) \xrightarrow{\text{id} - f_*} H_n(X) \longrightarrow H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Use this to compute the homology of the mapping tori of the following maps:

(a) A reflection $S^2 \to S^2$

(b) A map $S^2 \to S^2$ of degree 2.

(c) The map $S^1 \times S^1 \to S^1 \times S^1$ that is the identity on one factor and a reflection on the other.

(d) The map $S^1 \times S^1 \to S^1 \times S^1$ that is a reflection on each factor.

(e) The map $S^1 \times S^1 \to S^1 \times S^1$ that interchanges the two factors and then reflects one of the factors.

Proof. □