Problem 1 (28a). *Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.*

Proof. A Möbius band is homotopy equivalent to a circle, and we know the homology of the torus. Thus we have the long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(T^2 \cap M) \longrightarrow \widetilde{H}_n(T^2) \oplus \widetilde{H}_n(M) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_{n-1}(T^2 \cap M) \longrightarrow \cdots$$

The intersection $T^2 \cap M$ also has the homology of a circle giving us the exact sequence

$$\cdots \longrightarrow \widetilde{H}_2(M \cap T^2) \longrightarrow \widetilde{H}_2(T^2) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(M \cap T^2) \longrightarrow \widetilde{H}_1(M) \oplus \widetilde{H}_1(T^2) \longrightarrow$$

$$\longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_1(M \cap T^2) \longrightarrow \cdots$$

which when we substitute in we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \widetilde{H}_2(X) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{k_* - l_*} \widetilde{H}_1(X) \longrightarrow 0$$

Since the right i_*, j_* is injective the kernel is zero. Thus the image of ∂_* is 0 making the left i_*, j_* an isomorphism. Similarly the fact that the kernel of the right i_*, j_* is zero effectively gives us the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*, j_*} \mathbb{Z} \oplus \mathbb{Z}^2 \xrightarrow{k_* - l_*} \widetilde{H}_1(X) \longrightarrow 0$$

which implies that $\widetilde{H}_1(X) \cong \langle x, y, z \rangle / \langle 2x - y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.

Therefore the homology of X is

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

Problem 2 (29). *The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X into two copies of R. Also compute the relative groups $H_i(R, M_g)$.*

Proof. First note that the homology of M_g is

$$H_i(M_g) = \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z}^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Since the region R has the homotopy type of the wedge of g circles its homology is

$$H_i(R) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^g & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now we represent X as the union of two slightly thickened copies of R such that the intersection is M_g. Then using the Mayer-Vietoris sequence we get:

$$0 \longrightarrow H_3(X) \xrightarrow{\partial_*} H_2(M_g) \longrightarrow 0 \longrightarrow H_2(X) \xrightarrow{\partial_*} H_1(M_g) \xrightarrow{i_*, j_*} H_1(R) \oplus H_1(R) \xrightarrow{k_* - l_*} H_1(X) \longrightarrow 0$$

It immediately follows that $H_3(X)$ is isomorphic to $H_2(M_g) \cong \mathbb{Z}$. Moreover we know have an exact sequence which if we substitute in we get:

$$0 \longrightarrow H_2(X) \xrightarrow{\partial_*} \mathbb{Z}^{2g} \xrightarrow{i_*, j_*} \mathbb{Z}^g \oplus \mathbb{Z}^g \xrightarrow{k_* - l_*} H_1(X) \longrightarrow 0$$

Since ∂_* is injective from its location in the sequence it follows that $H_2(X) \cong \ker i_*, j_*$. Looking at the map i_*, j_* it is the inclusion of M_g into both copies of R. As such half of the meridians will get sent to zero while the others will get sent to the generators of $H_1(R)$. Thus the kernel of i_*, j_* is $\mathbb{Z}^g \cong H_1(X)$.

Next since $k_* - l_*$ is surjective we have that $H_2(X) \cong (\mathbb{Z}^g \oplus \mathbb{Z}^g)/\text{im } i_*, j_*$. The elements in the image of i_*, j_* are those of the form $(a_1, \ldots, a_g, a_1, \ldots, a_g)$. Thus the quotient will be $\mathbb{Z}^{2g}/\mathbb{Z}^g \cong \mathbb{Z}^g \cong H_2(X)$.

Therefore the homology of X is

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z}^g & i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

To calculate $H_i(R, M_g)$ we use the long exact sequence from hatcher to get

$$0 \longrightarrow H_3(R, M_g) \longrightarrow H_2(M_g) \longrightarrow 0 \longrightarrow H_2(R, M_g) \longrightarrow H_1(M_g) \longrightarrow H_1(R) \longrightarrow H_1(R, M_g) \longrightarrow 0$$

It follows immediately that $H_3(R, M_g) \cong H_2(M_g) \cong \mathbb{Z}^{2g}$. In addition we also get the exact sequence

$$0 \longrightarrow H_2(R, M_g) \xrightarrow{\partial_*} \mathbb{Z}^{2g} \xrightarrow{i_*} \mathbb{Z}^g \xrightarrow{q_*} H_1(R, M_g) \longrightarrow 0$$

The setup is the same as above. Thus $H_2(R, M_g) \cong \ker i_* \cong \mathbb{Z}^g$. In addition $H_1(R, M_g) \cong \mathbb{Z}^g/\text{im } i_*$. However unlike in the previous problem i_* is surjective. As such $H_1(R, M_g) \cong \mathbb{Z}^g/\mathbb{Z}^g \cong 0$. The zeroeth homology is zero as M_g touches the single path component of R.

Therefore the relative homology of R with respect to M_g is

$$H_i(R, M_g) = \begin{cases} \mathbb{Z} & i = 3 \\ \mathbb{Z}^g & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Problem 3 (30). *For the mapping torus T_f of a map $f : X \to X$, we constructed in Example 2.48 a long exact sequence*

$$\cdots \longrightarrow H_n(X) \xrightarrow{\text{id} - f_*} H_n(X) \longrightarrow H_n(T_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

Use this to compute the homology of the mapping tori of the following maps:

(a) *A reflection* $S^2 \to S^2$

(b) *A map* $S^2 \to S^2$ *of degree* 2.

(c) *The map* $S^1 \times S^1 \to S^1 \times S^1$ *that is the identity on one factor and a reflection on the other.*

(d) *The map* $S^1 \times S^1 \to S^1 \times S^1$ *that is a reflection on each factor.*

(e) *The map* $S^1 \times S^1 \to S^1 \times S^1$ *that interchanges the two factors and then reflects one of the factors.*

Proof. (a) Note that $H_1(T_f) = 0$ as it is between $H_1(S^2)$ and $\widetilde{H}_0(S^2)$. The same holds for $n > 3$. All of the possible nonzero items are contained in the exact sequence

$$0 \longrightarrow H_3(T_f) \longrightarrow H_2(S^2) \longrightarrow H_2(S^2) \longrightarrow H_2(T_f) \longrightarrow 0$$

As before this gives us that $H_3(T_f) \cong \ker(\mathrm{id}_* - f_*)$ and $H_2(T_f) \cong \mathbb{Z}/\mathrm{im}\ (\mathrm{id}_* - f_*)$.

If we evaluate $(\mathrm{id}_* - f_*)(1)$ this is equal to $1 - (-1) = 2$. The kernel of this is trivial and the image is $2\mathbb{Z}$. Thus the homology of T_f is

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

(b) Same as (a) but with a different map. If we calculate $(\mathrm{id}_* - f_*)(1)$ this is $1 - 2 = -1$. The kernel of this map is trivial and the image is $\mathbb{Z}$. Thus the homology is:

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) The exact sequence we get from the long exact sequence above is:

$$0 \longrightarrow H_3(T_f) \xrightarrow{\partial_*} H_2(T^2) \xrightarrow{\mathrm{id}_* - f_*} H_2(T^2) \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} H_1(T^2) \xrightarrow{\mathrm{id}_* - f_*} H_1(T^2) \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

If we substitute in for the known groups we get:

$$0 \longrightarrow H_3(T_f) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{\mathrm{id}_* - f_*} \mathbb{Z} \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\mathrm{id}_* - f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

We can solve for $H_3(T_f)$ and $H_1(T_f)$ in the same manner as above. The map induced on $H_2(T^2)$ be $\mathrm{id}_* - f_*$ can be seen from $(\mathrm{id}_* - f_*)(1) = 1 - (-1) = 2$. Thus the kernel is trivial making $H_3(T_f) = 0$. Similarly $H_1(T_f) \cong \mathbb{Z}^2/\mathrm{im}(id_* - f_*)$. In the left case the map applied to the generators gives $(id_* - f_*)(1,0) = (0,0)$ and $(id_* - f_*)(0,1) = 2$. This gives us that $H_1(T_f) \cong \langle x, y \rangle / \langle 2y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Lastly for $H_2(T_f)$ note that since the kernel of the right $\mathrm{id}_* - f_*$ is a copy of $\mathbb{Z}$ we have that ∂_* is surjective onto said copy of $\mathbb{Z}$. This gives us that $\mathbb{Z} \cong H_2(T_f)/(\ker \partial_*)$. However since the kernel of the left i_* is $2\mathbb{Z}$ (since the image of the left $\mathrm{id}_* - f_*$ is $2\mathbb{Z}$) it must be that $\ker \partial_* \cong \mathbb{Z}_2$. Which gives us that $H_2(T_f) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Therefore the homology of T_f is

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & i = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

(d) Once again we start with the exact sequence

$$0 \longrightarrow H_3(T_f) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z} \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

However this time the left $(id_* - f_*)(1)$ is $1 - 1 = 0$. This induces an isomorphism $H_3(T_f) \cong H_2(T^2) \cong \mathbb{Z}$. This effectively shortens our sequence to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

However as we've seen above, reflection causes $id_* - f_*$ to multiply the cycles of $H_1(T^2)$. Thus this map is injective making ∂_* the zero map. This once more induces an isomorphism with i_* between $\mathbb{Z}$ and $H_2(T_f)$.

Finally we have now shortened our exact sequence to a short exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

Which gives us that $H_1(T_f) \cong \mathbb{Z}^2/(2\mathbb{Z} \oplus 2\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Therefore the homology of T_f is

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0, 2, 3 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

(e) As before we have

$$0 \longrightarrow H_3(T_f) \xrightarrow{\partial_*} \mathbb{Z} \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z} \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

This will also cause the left $\mathrm{id}_* - f_*$ to be the zero map giving us $H_3(T_f) \cong \mathbb{Z}$. Once again shortening our sequence to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i_*} H_2(T_f) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

Note that the right instance of $\mathrm{id}_* - f_*$ is injective as $(\mathrm{id}_* - f_*)(1,0) = (0,1)$ and $(\mathrm{id}_* - f_*)(0,1) = (-1,0)$. Just as before this induces an isomorphism on $H_2(T_f) \cong \mathbb{Z}$ and giving us a short exact sequence

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\mathrm{id}_*-f_*} \mathbb{Z}^2 \xrightarrow{i_*} H_1(T_f) \longrightarrow 0$$

However from above we can see that $\mathrm{id}_* - f_*$ is an isomorphism which forces $H_1(T_f)$ to be zero. Therefore the homology of T_f is

$$H_i(T_f) = \begin{cases} \mathbb{Z} & i = 0, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

□