Problem 1 (15). *For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.*

Proof. Label the maps of the exact sequence as:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

Suppose that $C = 0$. Then $\ker \beta = B$ which by exactness gives im $\alpha = B$ implying that α is surjective. On the other hand im $\gamma = 0$ which by exactness gives $\ker \delta = 0$ and as such δ is injective.

Now suppose that α is surjective and that δ is injective. Then im $\alpha = B$ which implies that $\ker \beta = B$. In addition, since $\ker \delta = 0$ by exactness we have that im $\gamma = 0$. However since the image of β is 0 by exactness $\ker \gamma = 0$. Since the kernel of γ is 0 and the image is 0 it must be that the group C is zero.

Therefore $C = 0$ if, and only if, α is surjective and δ is injective. □

Problem 2 (16). *(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.*

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path component contains at most one path-component of A.

Proof.

(a)

(b)

□

Problem 3 (18). *Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.*

Proof. □