Problem 1. *Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]*

Proof. Suppose that we have a map $f : D^n \to D^n$ with no fixed points. Then define a map $g : S^n \to S^n$ by identifying the northern hemisphere as the destination of f via

$$g(x) = \begin{cases} f(x) & x \in \text{northern hemisphere} \\ f(-x) & \text{otherwise} \end{cases}$$

Then g has degree 0 since it is not surjective and g has degree $(-1)^{n+1}$ since it has no fixed point from Hatcher pg. 134. This is a contradiction. $\square$

Problem 2. *Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.*

Proof. Suppose otherwise. Then f has no fixed point and as such has degree $(-1)^{2n+1} = -1$. Moreover the map $-f = f \circ A$ also has no fixed point and as such has degree $(-1)^{2n+1} = -1$. However due to how degree multiplies under composition we also have that $-\deg f = \deg -f$ which is a contradiction. $\square$

Problem 3. *Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.*

Proof. $\square$