Problem 1. *Prove the Brouwer fixed point theorem for maps $f : D^n \to D^n$ by applying degree theory to the map $S^n \to S^n$ that sends both the northern and southern hemispheres of S^n to the southern hemisphere via f. [This was Brouwer's original proof.]*

Proof. Suppose that we have a map $f : D^n \to D^n$ with no fixed points. Then define a map $g : S^n \to S^n$ by identifying the northern hemisphere as the destination of f via

$$g(x) = \begin{cases} f(x) & x \in \text{northern hemisphere} \\ f(-x) & \text{otherwise} \end{cases}$$

Then g has degree 0 since it is not surjective and g has degree $(-1)^{n+1}$ since it has no fixed point from Hatcher pg. 134. This is a contradiction. $\square$

Problem 2. *Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.*

Proof. Suppose otherwise. Then f has no fixed point and as such has degree $(-1)^{2n+1} = -1$. Moreover the map $-f = f \circ A$ also has no fixed point and as such has degree $(-1)^{2n+1} = -1$. However due to how degree multiplies under composition we also have that $-\deg f = \deg -f$ which is a contradiction.

For the second portion suppose that we have a map $g : \mathbb{R}P^{2n} \to \mathbb{R}^{2n}$ and consider its lift $\widetilde{g} : S^{2n} \to S^{2n}$. Then we have the commutative diagram

$$\begin{tikzcd} S^{2n} \arrow[r, "\widetilde{g}"] & S^{2n} \arrow[d, "q"] \\ \mathbb{R}P^{2n} \arrow[u, "i"] \arrow[r, "g"] & \mathbb{R}P^{2n} \end{tikzcd}$$

where i is inclusion and q is a quotient map that agrees with the inclusion. Since $\widetilde{g}$ must respect equivalence classes we have that $\widetilde{g}(x) = \widetilde{g}(-x)$. Moreover from the previous part we have that there is an $x \in S^{2n}$ such that $\widetilde{g}(x) = \widetilde{g}(-x)$ is either x or $-x$. In either case there is a corresponding element $\bar{x} \in \mathbb{R}P^{2n}$ such that $i(\bar{x})$ is $\pm x$ and $q(\pm x) = \bar{x}$. By commutativity this implies that $g(\bar{x}) = \bar{x}$ which shows that g indeed has a fixed point. $\square$

An example of a map from $\mathbb{R}^{2n-1}$ to $\mathbb{R}P^{2n-1}$ is the map

$$(a_1, a_2, \dots, a_n, a_{n+1}) \mapsto (-a_2, a_2, \dots, -a_{n+1}, a_n)$$

where $(a_1, \dots, a_{n+1})$ is a point in $\mathbb{R}P^n$ such that not all a_i are zero. This map is well defined since the dimension is odd and will not have a fixed point.

Problem 3. *Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.*

Proof. Suppose that f had no fixed point. Then its degree would have to be $(-1)^{n+1}$ which is a contradiction. If $-f$ had no fixed point then $\deg -f = (-1)^{n+1}$ which is also a contradiction. Therefore if f has degree zero it must have a fixed point and so does $-f$.

Let $F : D^n \to \mathbb{R}^n$ be a nonzero vector field on D^n. Then define $\bar{F} : D^n \to S^{n-1}$ by normalizing

$$\bar{F}(x) = \frac{F(x)}{|F(x)|}$$

Then the restriction of $\bar{F}$ to ∂D^n is a map from S^{n-1} to S^{n-1} that has degree zero since it extends to a disc. Thus there exist points $x, y \in \partial D^n$ such that $\bar{F}(x) = x$ and $\bar{F}(y) = -y$ corresponding to a vector $F(x)$ and $F(y)$ pointing radially inward and radially outward on the boundary of D^n respectively. $\square$