Problem 1 (2.2.35). *Use the Mayer-Vietoris sequence to show that a nonorientable closed surface, or more generally a finite simplicial complex X for which $H_1(X)$ contains torsion, cannot be embedded as a subspace of $\mathbb{R}^3$ in such a way as to have a neighborhood homeomorphic to the mapping cylinder of some map from a closed orientable surface to X. [This assumption on a neighborhood is in fact not needed if one deduces the result from Alexander duality in 3.3]*

Proof. Let $i : X \to \mathbb{R}^{\nvDash}$ be an embedding with $N \subset \mathbb{R}^3$ such that $N \cong M_f$ where M_f is the mapping cylinder for a map $f : Y \to X$ with Y closed and orientable. Let $A = \mathbb{R}^3 \setminus i(X)$. Then $A \cap N = N \setminus i(X) \cong Y$. Note that because N retracts onto $i(X)$ that $H_n(N) = H_n(X) \oplus H_n(N, X)$. Then using Mayer-Vietrois sequence with reduced homology we get

$$0 \longrightarrow H_1(Y) \longrightarrow H_1(\mathbb{R}^3 \setminus i(X)) \oplus H_1(X) \oplus H_1(N, X) \longrightarrow 0$$

Which gives us an isomorphism. However since Y is a closed orientable surface its first homology group is isomorphic $\mathbb{Z}^{2g}$ which contains no torsion while the other group does which is a contradiction.

Therefore a finite simplicial complex with torsion cannot be embedded into $\mathbb{R}^3$ such that it has a neighborhood homeomorphic to the mapping cylinder for a map from a closed orientable surface to said complex. $\square$

Problem 2 (2.2.38). *Show that the commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[dr, "\gamma"] \arrow[dd, "k"] & & B_n \arrow[r, "i"] \arrow[dd, "j"] & C_n \arrow[dr, "\gamma"] \arrow[dd, "k"] & & B_{n-1} \arrow[r] \arrow[dd, "j"] & \cdots \\
& & A_n \arrow[ur, "\alpha_b"] \arrow[dr, "\alpha_d"] & & & A_{n-1} \arrow[ur, "\alpha_b"] \arrow[dr, "\alpha_d"] & & \\
\cdots \arrow[r] & E_{n+1} \arrow[ur, "\epsilon"] & & D_n \arrow[r, "l"] & E_n \arrow[ur, "\epsilon"] & & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

with the two sequences across the top and bottom exact, gives rise to an exact sequence

$$\cdots \to E_{n+1} \to B_n \to C_n \oplus D_n \to E_n \to B_{n-1} \to \cdots$$

where the maps are obtained from those in the previous diagram in the obvious way, except that $B_n \to C_n \oplus D_n$ has a minus sign in one coordinate.

Proof. Label the map $E_{n+1} \to B_n$, $B_n \to C_n \oplus D_n$, and $C_n \oplus D_n \to E_n$ as $\varphi, (i, -j)$, and $k + l$ respectively. There are six things to verify:

- $\ker(\varphi) \subseteq \text{im } (k + l)$: Suppose that $e \in \ker \varphi$. Then $\epsilon \circ \alpha_b(e) = 0$ and by commutativity $\beta \circ \epsilon \circ \alpha_b(e) = \epsilon \circ \alpha_d(e) = 0$. By exactness there is a $c \in C_{n+1}$ such that $\gamma(c) = \epsilon(e)$. By commutativity $\gamma(c) = \epsilon \circ k(c) = \epsilon(e)$. As such $\epsilon(e - k(c)) = 0$ and by exactness there is a $d \in D_{n+1}$ such that $l(d) = e - k(c)$. It then follows that $k + l(c + d) = k(c) + e - k(c) = e \in \text{im } (k + l)$.

- $\text{im } (k + l) \subseteq \ker(\varphi)$: Let $e \in \text{im } (k + l)$. Then there exists $c \in C_{n+1}$ and $d \in D_{n+1}$ such that $k(c) + l(d) = e$. Then $\varphi(k(c) + l(d)) = \alpha_b \circ \epsilon \circ k(c) + \alpha_b \circ \epsilon \circ l(d)$. First note that $\epsilon \circ l(d) = 0$ by exactness. By commutativity $\epsilon \circ k(c) = \gamma(c)$ which implies that $\alpha_b \circ \gamma(c) = 0$. Therefore $\varphi(e) = 0$.

- $\ker(i,-j) \subseteq \text{im } \varphi$: Let $b \in \ker(i,-j)$. Then we have that $i(b) = j(b) = 0$. By exactness there is an $a \in A_n$ such that $\alpha_b(a) = b$. From commutativity $\alpha_d(a) = j \circ \alpha_b(a) = 0$. It then follows from exactness that there is a $c \in C_{n+1}$ such that $k(c) = a$. Finally by commutativity $varphi(k(c)) = b$

- $\text{im } \varphi \subseteq \ker(i,-j)$: Let $b \in \text{im } \varphi$. Then there exists an $e \in E_{n+1}$ such that $\varphi(e) = \alpha_b \circ \epsilon(e) = b$. Apply $i, -j$ to get $i \circ \alpha_b \circ \epsilon(e) - j \circ \alpha_b \circ \epsilon(e)$. The left term is zero by exactness and the right term is transformed to $\alpha_d \circ \epsilon(e)$ which is also zero by exactness. Therefore $i, -j(b) = 0$.

- $\ker(k+l) \subseteq \text{im } (i,-j)$: Let $c + d \in \ker(k+l)$. Then $k(c) + l(d) = 0$ which implies that $k(c) = -l(d)$. By exactness we have that $\gamma(c) = -\epsilon \circ l(d) = 0$. Thus there exists a $b \in B_n$ such that $i(b) = c$. Moreover $l(d + j(b)) = l(d) - l(d) = 0$ which by exactness gives us an $a \in A_n$ such that $\alpha_d(a) = d + j(b)$. Then $j(\alpha_b(a) - j(b)) = d + j(b) - j(b) = d$ by commutativity. Then if note that $i(\alpha_b(a) - b) = -c$ we see that $\alpha_b(a) = 0$ which implies that $j(-b) = d$ showing that $i, -j(b) = c + d$.

- $\text{im } (i,-j) \subseteq \ker(k+l)$: Let $c, d \in \text{im } (i,-j)$. Then there exists a $b \in B_n$ such that $i, -j(b) = c + d$. Note that by commutativity $l \circ j(-b) = i \circ k(-b) = l(d)$. Thus $k(c) + l(d) = k \circ i(b) + k \circ i(-b) = k \circ i(b - b) = 0$.

Therefore since all of the above conditions hold the sequence it commutative. □

Problem 3 (2.B.1). *Compute $H_i(S^n \setminus X)$ when X is he subspace of S^n homeomorphic to $S^k \vee S^\ell$ or to $S^k \coprod S^\ell$.*

Let $A := S^n \setminus S^k$ and $B := S^n \setminus S^\ell$. Then $A \cap B = S^n \setminus X$. Now we split into two cases. First if we are doing the wedge of S^k and S^ℓ then $A \cup B = \mathbb{R}^n$. Using the Mayer-Vietoris sequence and lemma $2.B.1$ we get

$$0 \longrightarrow H_i(S^n \setminus X) \longrightarrow H_i(S^k) \oplus H_i(S^\ell) \longrightarrow 0$$

for $i > 0$. Thus if $k \neq \ell$

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z} & i = n - k - 1, n - \ell - 1 \\ 0 & \text{else} \end{cases}$$

Otherwise if $k = \ell$ then

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z}^2 & i = n - k - 1 \\ 0 & \text{else} \end{cases}$$

The disjoint union case is similar. If $k, \ell > 0$ then it is the same the only difference being the portion for $i = n$ where

$$0 \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^n \setminus X) \longrightarrow 0$$

Then if $k \neq \ell$

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z} & i = n - k - 1, n - \ell - 1, n - 1 \\ 0 & \text{else} \end{cases}$$

Otherwise if $k = \ell$ then

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z} & i = n - 1 \\ \mathbb{Z}^2 & i = n - k - 1 \\ 0 & \text{else} \end{cases}$$

On the other hand if either k or ℓ is zero then for n we have

$$0 \longrightarrow H_n(S^n) \longrightarrow H_{n-1}(S^n \setminus X) \longrightarrow H_{n-1}(A) \oplus H_{n-1}(B) \longrightarrow 0$$

Then if $k = \ell = 0$ we have

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z}^3 & i = n-1 \\ 0 & \text{else} \end{cases}$$

If just $k = 0$ we have

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z}^2 & i = n-1 \\ \mathbb{Z} & i = n-\ell-1 \\ 0 & \text{else} \end{cases}$$

Finally if just $\ell = 0$ we have

$$H_i(S^n \setminus X) = \begin{cases} \mathbb{Z}^2 & i = n-1 \\ \mathbb{Z} & i = n-k-1 \\ 0 & \text{else} \end{cases}$$