

Problem 1 (11). Show that if A is a retract of X then the map $H_n(A) \rightarrow H_n(X)$ induced by the inclusion $A \subset X$ is injective.

Proof. Let A be a retract of X . Then there is a map such that $r : X \rightarrow A$ which when composed with inclusion we have $r \circ i = id_A$. However when we look at the induced maps on homology we see that $r_* \circ i_* = id_{A*}$. As such i_* has a left inverse and it then follows that i_* is injective. \square

Problem 2 (12). Show that chain homotopy of chain maps is an equivalence relation.

Proof. Let $f_{\#} \sim g_{\#}$ denote that $f_{\#}, g_{\#}$ are chain homotopic. Now we verify that chain homotopy is an equivalence relation.

- Let P be the trivial map. Then $\partial P + P\partial = 0 = f_{\#} - f_{\#}$. Thus \sim is reflexive.
- Suppose that $f_{\#} \sim g_{\#}$ with prism P . Then for $-P$ we have that

$$(\partial - P) + (-P)\partial = -(\partial P + P\partial) = -(f_{\#} - g_{\#}) = g_{\#} - f_{\#}$$

Thus \sim is reflexive.

- Finally let $f_{\#} \sim g_{\#}$ and $g_{\#} \sim h_{\#}$ with prism P and Q respectively. Then

$$\partial(P + Q) + (P + Q)\partial = \partial P + P\partial + \partial Q + Q\partial = f_{\#} - g_{\#} + g_{\#} - h_{\#} = f_{\#} - h_{\#}$$

Thus \sim is transitive.

Therefore, since it is reflexive, symmetric, and transitive it is an equivalence relation. \square

Problem 3 (14). Determine whether there exists a short exact sequence $0 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow 0$. More generally, determine with abelian groups A fit into a short exact sequence $0 \rightarrow \mathbb{Z}_{p^m} \rightarrow A \rightarrow \mathbb{Z}_{p^n} \rightarrow 0$ with p prime. What about the case of short exact sequences $0 \rightarrow \mathbb{Z} \rightarrow A \rightarrow \mathbb{Z}_n \rightarrow 0$?

For this problem we'll be implicitly assuming the structure theorem of finite generated abelian groups to eliminate choices.

In order for the short exact sequence

$$0 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$

to exist we need for \mathbb{Z}_4 to be expressible as a quotient of $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and \mathbb{Z}_4 . Define the first map to send $1 \in \mathbb{Z}_4$ to $(2, 1) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$ followed by mapping $(2, 1)$ to 1. Then $\mathbb{Z}_8 \oplus \mathbb{Z}_2 / \mathbb{Z}_4 \cong \mathbb{Z}_4$ which allows for the above sequence to be exact.

For the short exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^m} \longrightarrow A \longrightarrow \mathbb{Z}_{p^n} \longrightarrow 0$$

to exist we need for $\mathbb{Z}_{p^n} \cong A / \mathbb{Z}_{p^m}$. Since p is prime the only possibilities for this are $A \cong \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ and sending \mathbb{Z}_{p^m} to itself or $\mathbb{Z}_{p^{m+n}}$ and quotienting out the \mathbb{Z}_{p^m} .

Finally for the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

to exist, as above, we need $\mathbb{Z}_n \cong B / \mathbb{Z}$. Since the first map must be injective due to its position we need to send \mathbb{Z} into some copy of \mathbb{Z} . Thus the possibilities are $B \cong \mathbb{Z} \oplus \mathbb{Z}_n$ quotienting out the \mathbb{Z} or $B \cong \mathbb{Z}$ and sending \mathbb{Z} to $n\mathbb{Z}$.