Problem 1 (2.2.36)**.** *Show that $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ be definition. Namely, show $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ and $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$ [For the latter isomorphism the relative Mayer-Vietoris sequence yields an easy proof].*

Proof. □

Problem 2 (2.3.3)**.** *Show that if $\widetilde{h}$ is a reduced homology theory then $\widetilde{h}_n(point) = 0$ for all n. Deduce that there are suspension isomorphisms $\widetilde{h}_n(X) \cong \widetilde{h}_{n+1}(SX)$ for all n.*

Proof. First note that if we wedge the point with itself that we get a point back. Then using the wedge axiom we have $\widetilde{h}_i(\bigvee_1^2 * = *)$ is isomorphic to $\widetilde{h}_i(*) \oplus \widetilde{h}_i(*)$. The only way this could occur would be if $\widetilde{h}_i(*) \cong 0$ for all i.

For the suspension use the Mayer-Vietoris sequence along with the fact that the cone of a space is contractable to get

$$\cdots \longrightarrow \widetilde{h}_{i+1}(CX) \oplus \widetilde{h}_{i+1}(CX) \longrightarrow \widetilde{h}_{i+1}(SX) \longrightarrow \widetilde{h}_i(X) \longrightarrow \widetilde{h}_i(CX) \oplus \widetilde{h}_i(CX) \longrightarrow \cdots$$

Which is equivalent to

$$0 \longrightarrow \widetilde{h}_{i+1}(SX) \longrightarrow \widetilde{h}_i(X) \longrightarrow 0$$

This show that $\widetilde{h}_i(X) \cong \widetilde{h}_{i+1}(SX)$ for all i. □

Problem 3 (2.B.3)**.** *Let $(D, S) \subset (D^n, S^{n-1})$ be a pair of subspaces homeomorphic to (D^k, S^{k-1}), with $D \cap S^{n-1} = S$. Show that the inclusion $S^{n-1} - S \hookrightarrow D^n - D$ induces an isomorphism on homology. [Glue two copies of (D^n, D) to the two ends of $(S^{n-1} \times I, S \times I)$ to produce a k-sphere in S^n and look at the Mayer-Vietoris sequence for the complement of this k-sphere.]*

Proof. As described in the hint. Construct a $S^n \setminus S^k$ and decompose it as $A = D^n \setminus D$ and $B = S^n \setminus D$. Then $A \cap B = S^{n-1} \setminus S$. By Prop 2B.1 we have that B is null-homologous. Then from the Mayer-Vietoris sequence we have

$$\cdots \longrightarrow H_i(S^{n-1} \setminus S) \longrightarrow H_i(D^n \setminus D) \longrightarrow H_i(S^n \setminus S^k) \longrightarrow \cdots$$

Where the map between $H_i(S^{n-1} \setminus S)$ and $H_i(D^n \setminus D)$ is inclusion. Then Using 2B.1 again we know that $S^n \setminus S^k$ will have zero homology everywhere but for $i = n - k - 1$. This induces isomorphisms for $i > n - k - 1$ and $i < n - k - 2$.

Finally for $i = n - k - 1, n - k - 2$ we have □