Problem 1 (2.2.36)**.** *Show that $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for all i and n, where $H_i = 0$ for $i < 0$ be definition. Namely, show $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$ and $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$ [For the latter isomorphism the relative Mayer-Vietoris sequence yields an easy proof].*

Proof. First we show that $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$. Swap out $H_i(X)$ for $H_i(X \times \{x_0\})$. Then apply the long exact sequence for the pair to get

$$\cdots \to H_{i+1}(X \times S^n, X \times \{x_0\}) \to H_i(X \times \{x_0\}) \to H_i(X \times S^n) \to H_i(X \times S^n, X \times \{x_0\}) \to H_{i-1}(X \times \{x_0\}) \to \cdots$$

Since $X \times \{x_0\}$ is a retract of $X \times S^n$ we have that the sequence splits. Thus by the splitting lemma we have that $H_i(X \times S^n) \cong H_i(X) \oplus H_i(X \times S^n, X \times \{x_0\})$.

For the next isomorphism $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$ apply the relative Mayer-Vietoris sequence with $A = X \times S^n \setminus \{x_0\}$, $B = X \times S^n \setminus \{x_1\}$ ($x_1 \neq x_0$), and $C, D = X \times \{x_0\}$. Then we have

$$\cdots \to H_i(X \times S^n \setminus \{x_0\}, X \times \{x_0\}) \oplus H_i(X \times S^n \setminus \{x_0\}, X \times \{x_1\}) \to H_i(X \times S^n, X \times \{x_0\})$$

$$\to H_{i-1}(X \times S^{n-1}, X \times \{x_0\}) \to H_{i-1}(X \times S^n \setminus \{x_0\}, X \times \{x_0\}) \oplus H_{i-1}(X \times S^n \setminus \{x_0\}, X \times \{x_1\}) \to \cdots$$

However the terms on the left and the right are all zero. As such this induces an isomorphism between $H_i(X \times S^n, X \times \{x_0\}) \cong H_{i-1}(X \times S^{n-1}, X \times \{x_0\})$.

Finally apply the second isomorphism n times to the first isomorphism to get $H_i(X \times S^n) \cong H_i(X \times \{x_0\}) \oplus H_{i-n}(X \times S^0, X \times \{x_0\})$. However the relative homology will make one of the two copies of X in $X \times S^0$ null. As such $H_{i-n}(X \times S^0, X \times \{x_0\}) \cong H_{i-n}(X)$.

Therefore $H_i(X \times S^n) \cong H_i(X) \oplus H_{i-n}(X)$ for all i, n where we assume $H_i(X) = 0$ for $i < 0$. □

Problem 2 (2.3.3)**.** *Show that if $\widetilde{h}$ is a reduced homology theory then $\widetilde{h}_n(point) = 0$ for all n. Deduce that there are suspension isomorphisms $\widetilde{h}_n(X) \cong \widetilde{h}_{n+1}(SX)$ for all n.*

Proof. First note that if we wedge the point with itself that we get a point back. Then using the wedge axiom we have $\widetilde{h}_i(\bigvee_1^2 * = *)$ is isomorphic to $\widetilde{h}_i(*) \oplus \widetilde{h}_i(*)$. The only way this could occur would be if $\widetilde{h}_i(*) \cong 0$ for all i.

For the suspension use the Mayer-Vietoris sequence along with the fact that the cone of a space is contractible to get

$$\cdots \longrightarrow \widetilde{h}_{i+1}(CX) \oplus \widetilde{h}_{i+1}(CX) \longrightarrow \widetilde{h}_{i+1}(SX) \longrightarrow \widetilde{h}_i(X) \longrightarrow \widetilde{h}_i(CX) \oplus \widetilde{h}_i(CX) \longrightarrow \cdots$$

Which is equivalent to

$$0 \longrightarrow \widetilde{h}_{i+1}(SX) \longrightarrow \widetilde{h}_i(X) \longrightarrow 0$$

This show that $\widetilde{h}_i(X) \cong \widetilde{h}_{i+1}(SX)$ for all i. □

Problem 3 (2.B.3)**.** *Let $(D, S) \subset (D^n, S^{n-1})$ be a pair of subspaces homeomorphic to (D^k, S^{k-1}), with $D \cap S^{n-1} = S$. Show that the inclusion $S^{n-1} - S \hookrightarrow D^n - D$ induces an isomorphism on homology. [Glue two copies of (D^n, D) to the two ends of $(S^{n-1} \times I, S \times I)$ to produce a k-sphere in S^n and look at the Mayer-Vietoris sequence for the complement of this k-sphere.]*

Proof. Using the construction listed in the problem decompose $S^n \setminus S^k$ as $A \cup B$ where $A = (S^{n-1} \times I, S \times I)$ filled in with $D^n \setminus D$ on one side and B as the filling on the other. Then using the fact that taking the Cartesian product with the interval does not change the homology and applying the Mayer-Vietoris sequence we get:

$$\cdots \longrightarrow H_i(S^{n-1} \setminus S) \longrightarrow H_i(D^n \setminus D) \oplus H_i(D^n \setminus D) \longrightarrow H_i(S^n \setminus S^k) \longrightarrow \cdots$$

However $H_i(S^n \setminus S^k) = 0$ unless $i = n-k-1$. Likewise since $S \cong S^{k-1}$ as a subspace we have that $H_i(S^{n-1} \setminus S) \cong 0$ unless $i = (n-1)-(k-1)-1 = n-k-1$. Similarly $H_i(D^n \setminus D) = 0$ unless $i = n-k-1$. They will all be $\mathbb{Z}$ otherwise. Thus we get the short exact sequence

$$\cdots \longrightarrow H_{n-k-1}(S^{n-1} \setminus S) \xrightarrow{i_*, i_*} H_{n-k-1}(D^n \setminus D) \oplus H_{n-k-1}(D^n \setminus D) \xrightarrow{k_* - k_*} H_{n-k-1}(S^n \setminus S^k) \longrightarrow \cdots$$

From this we can see that the kernel of $k_* - k_*$ is $\{(x,x) \in \mathbb{Z}^2\}$. However since $(1,1) \in \ker(k_* - k_*)$ from exactness it must be that $i_*(1) = 1$ which shows that i_* is an isomorphism. Since i_* is the zero map between groups that are zero elsewhere this shows that the map induced by inclusion is indeed an isomorphism for all i. $\square$