Problem 1 (11). *Show that if A is a retract of X then the map $H_n(A) \to H_n(X)$ induced by the inclusion $A \subset X$ is injective.*

Proof. Let A be a retract of X. Then there is a map such that $r : X \to A$ which when composed with inclusion we have $r \circ i = id_A$. However when we look at the induced maps on homology we see that $r_* \circ i_* = id_{A*}$. As such i_* has a left inverse and it then follows that i_* is injective. $\square$

Problem 2 (12). *Show that chain homotopy of chain maps is an equivalence relation.*

Proof. Let $f_\# \sim g_\#$ denote that $f_\#, g_\#$ are chain homotopic. Now we verify that chain homotopy is an equivalence relation.

- Let P be the trivial map. Then $\partial P + P\partial = 0 = f_\# - f_\#$. Thus $\sim$ is reflexive.
- Suppose that $f_\# \sim g_\#$ with prism P. Then for $-P$ we have that

$$(\partial - P) + (-P)\partial = -(\partial P + P\partial) = -(f_\# - g_\#) = g_\# - f_\#$$

  Thus $\sim$ is reflexive.
- Finally let $f_\# \sim g_\#$ and $g_\# \sim h_\#$ with prism P and Q respectively. Then

$$\partial(P + Q) + (P + Q)\partial = \partial P + P\partial + \partial Q + Q\partial = f_\# - g_\# + g_\# - h_\# = f_\# - h_\#$$

  Thus $\sim$ is transitive.

Therefore, since it is reflexive, symmetric, and transitive it is an equivalence relation. $\square$

Problem 3 (14). *Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine with abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?*

For this problem we'll be implicitly assuming the structure theorem of finite generated abelian groups to eliminate choices.

In order for the short exact sequence

$$0 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$

to exist we need for $\mathbb{Z}_4$ to be expressible as a quotient of $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$. However this cannot occur as there is only a single component to $\mathbb{Z}_4$ the homomorphism needs to be sent to either the $\mathbb{Z}_8$ or the $\mathbb{Z}_2$. If it is sent to the $\mathbb{Z}_8$ we can get it down to $\mathbb{Z}_2$ but $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \not\cong \mathbb{Z}_4$. Similarly if $\mathbb{Z}_4$ is sent to $\mathbb{Z}_2$ we can eliminate that component but $\mathbb{Z}_8 \not\cong \mathbb{Z}_4$. Therefore the above short exact sequence cannot exist.

For the short exact sequence

$$0 \longrightarrow \mathbb{Z}_{p^m} \longrightarrow A \longrightarrow \mathbb{Z}_{p^n} \longrightarrow 0$$

to exist we need for $\mathbb{Z}_{p^n} \cong A/\mathbb{Z}_{p^m}$. Since p is prime the only possibilities for this are $A \cong \mathbb{Z}_{p^m} \oplus \mathbb{Z}_{p^n}$ and sending $\mathbb{Z}_{p^m}$ to itself or $\mathbb{Z}_{p^{m+n}}$ and quotienting out the $\mathbb{Z}_{p^m}$.

Finally for the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z}_n \longrightarrow 0$$

to exist, as above, we need $\mathbb{Z}_n \cong B/\mathbb{Z}$. Since the first map must be injective due to its position we need to send $\mathbb{Z}$ into some copy of $\mathbb{Z}$. Thus the possibilities are $B \cong \mathbb{Z} \oplus \mathbb{Z}_n$ quotienting out the $\mathbb{Z}$ or $B \cong \mathbb{Z}$ and sending $\mathbb{Z}$ to $n\mathbb{Z}$.