Problem 1 (15)**.** *For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.*

Proof. Label the maps of the exact sequence as:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

Suppose that $C = 0$. Then $\ker \beta = B$ which by exactness gives im $\alpha = B$ implying that α is surjective. On the other hand im $\gamma = 0$ which by exactness gives $\ker \delta = 0$ and as such δ is injective.

Now suppose that α is surjective and that δ is injective. Then im $\alpha = B$ which implies that $\ker \beta = B$. In addition, since $\ker \delta = 0$ by exactness we have that im $\gamma = 0$. However since the image of β is 0 by exactness $\ker \gamma = 0$. Since the kernel of γ is 0 and the image is 0 it must be that the group C is zero.

Therefore $C = 0$ if, and only if, α is surjective and δ is injective. □

Problem 2 (16)**.** *(a) Show that $H_0(X, A) = 0$ iff A meets each path-component of X.*

(b) Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path component contains at most one path-component of A.

Proof.

(a) Note that $H_0(X, A) = C_0(X, A)/\text{im } \partial_1$. It then follows that $H_0(X, A) = 0$ only when $C_0(X, A) = 0$ or im $\partial_1 = C_0(X, A)$. If $C_0(X, A) = 0$ then $A = X$ and the problem follows immediately. Otherwise it must be that $C_0(X, A) \neq 0$.

Now suppose that A meets each path component of X. Then each point $x \in X$ will have a path connecting it to A which implies that $x \in \text{im } \partial_1$. Since this holds for any point we have that im $\partial_1 = C_0(X, A)$ and as such $H_0(X, A) = 0$.

Otherwise if $H_0(X, A) = 0$ then im $\partial_1 = C_0(X, A)$ meaning that any point $x \in X$ has a one-simplex that ends at x and inside of A. This creates a path from x to A which implies that A must contact each path component of X.

(b) From Hatcher we have the long exact sequence

$$\cdots \longrightarrow H_1(A) \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\partial_*} H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0$$

First suppose that $i_* : H_1(A) \to H_1(X)$ is surjective. Then by exactness $\ker j_* = H_1(X)$ and it then follows that im $j_* = 0$ which means that $\ker \partial_* = 0$. This implies that ∂_* is injective. If the kernel of $i_* : H_0(A) \to H_0(X)$ is trivial this will imply that $H_1(X, A) = 0$ as im $\partial = 0$. However since A touches each path component of X at most once the induced map i_{*1} will be injective which gives it a trivial kernel. Thus if $i_* : H_1(A) \to H_1(X)$ is surjective and A meets each path component of X at most once we have that $H_1(X, A) = 0$.

Otherwise suppose that $H_1(X, A) = 0$. Then by exactness we have that i_{*1} is injective and as such A will meet each path component of X at most once. Also $\ker j_* = H_1(X)$ this implies that im $i_{*2} = H_1(X)$ making it surjective.

Therefore $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path component contains at most one path-component of A.

$\square$

Problem 3 (18). *Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.*

Proof. From Hatcher we have the long exact sequence

$$\cdots \longrightarrow H_1(\mathbb{R}) \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial_*} H_0(\mathbb{Q}) \xrightarrow{i_*} H_0(\mathbb{R}) \longrightarrow H_)(\mathbb{R}, \mathbb{Q}) \longrightarrow 0$$

We know the homology of $\mathbb{R}$ since it is contractible and we know that $H_0(\mathbb{R}, \mathbb{Q})$ is trivial by the previous problem. Moreover since $\mathbb{Q}$ is completely disconnected its zeroth homology will be a copy of $\mathbb{Z}$ for each rational number. This gives us the short exact sequence:

$$0 \longrightarrow H_1(\mathbb{R}, \mathbb{Q}) \xrightarrow{\partial_*} \bigoplus_{q \in \mathbb{Q}} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \longrightarrow 0$$

We can write elements of $\bigoplus_{q \in \mathbb{Q}} \mathbb{Z}$ as formal sums of rational numbers with integer coefficients $(\sum_{q \in \mathbb{Q}} a_q q)$. Then the map induced by the inclusion would be

$$i_* \left(\sum a_q q \right) = \sum a_q$$

The kernel of this map is $\{\sum a_q q | \sum a_q = 0\} \cong \operatorname{im} \partial_*$. However since ∂_* is injective by the first isomorphism theorem we have that

$$H_1(\mathbb{R}, \mathbb{Q}) \cong \{\sum a_q q | \sum a_q = 0\}$$

This group is a subgroup of a free abelian group which implies that it is also free abelian. We can write a basis as [1]

$$\langle -1 \cdot p + 1 \cdot q | q \in \mathbb{Q} \setminus \{p\} \rangle$$

$\square$

[1]Almost did this with $-1 \cdot 0 + 1 \cdot q$ but I disliked the notation