Problem 1 (12). *Show that the quotient map $S^1 \times S^1 \to S^2$ collapsing the subspace $S^1 \vee S^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2. On the other hand, show via covering spaces that any map $S^2 \to S^1 \times S^1$ is nullhomotopic.*

Proof. Consider the long exact sequence in reduced homology

$$\cdots \longrightarrow \widetilde{H}_2(S^1 \vee S^1) \longrightarrow \widetilde{H}_2(T^2) \longrightarrow \widetilde{H}_2(S^2) \longrightarrow \widetilde{H}_1(S^1 \vee S^1) \longrightarrow \widetilde{H}_1(T^2) \longrightarrow \cdots$$

First note that $H_2(S^1 \vee S^1) = 0$ since there are no two cells which forces the map induced by the quotient to be injective. In addition the map from $\widetilde{H}_1(S^1 \vee S^1)$ to $\widetilde{H}_1(T^2)$ is an isomorphism. Thus the kernel of the boundary map is zero making the map induced by the quotient surjective.

Therefore the map induced by the quotient is an isomorphism between nonzero groups and as such cannot be null.

Let f be a map from $S^2 \to T^2$. Then define $\bar{f} : I^2 \to T^2$ by the usual quotient of I^2 to S^2 by identifying the boundary. However $\bar{f}$ is a homotopy and as such it has a lift $\widetilde{f} : I^2 \to \mathbb{R}^2$. However since $\bar{f}(\partial I^2)$ is a single point and the preimage of any point under the covering map is discrete it must be that $\widetilde{f}(\partial I^2)$ also is a single point. Thus $\widetilde{f}$ is can be identified with a map from S^2 to $\mathbb{R}^2$ such that $\widetilde{f}$ composed with the covering map is f. However since f factors through a contractible space it follows that f is null. $\square$

Problem 2 (14). *A map $f : S^n \to S^n$ satisfying $f(x) = f(-x)$ for all x is called an even map. Show that an even map $S^n \to S^n$ must have even degree, and that the degree must in fact be zero when n is even. When n is odd, show that there exist even maps of any given even degree. [Hints: If f is even, it factors as a composition $S^n \to \mathbb{R}P^n \to S^n$. Using the calculation of $H_n(\mathbb{R}P^n)$ in the text, show that the induced map $H_n(S^n) \to H_n(\mathbb{R}P^n)$ sends a generator to twice a generator when n is odd. It may be helpful to show that the quotient map $\mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1}$ induces an isomorphism on H_n when n is odd.]*

Proof. Let f be a map from S^n to S^n such that $f(x) = f(-x)$ for all $x \in S^n$. Since $\mathbb{R}P^n$ is a quotient space of S^n where antipodal points are identified any even map from S^n respects equivalence classes for the quotient space and as such it factors as

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "q"] & S^n \\ \mathbb{R}P^n \arrow[ur, "\widetilde{f}"'] & \end{tikzcd}$$

First note that $\deg q = 2$. This is because for any point $x \in \mathbb{R}P^n$ it will have two points mapping to it from S^n and it will be the identity map giving us local degrees of 1 which add up to 2. Since any even map will factor in this way any even map must have even degree.

When n is even $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ and when n is even $H_n(\mathbb{R}P^n) \cong 0$. If n is odd and f is even then $\deg f = \deg q \cdot \deg \widetilde{f}$. However since $H_n(\mathbb{R}P^n) \cong 0$ when n is even then the degree of f has to be zero as $\deg \widetilde{f} = 0$ since it is mapping out of the trivial group.

Now suppose that n is odd. There is a quotient map $r : \mathbb{R}P^n \to (\mathbb{R}P^n/\mathbb{R}P^{n-1} \equiv S^n)$. This map will have degree 1 and as such $q \circ r$ will be a map from $S^n \to S^n$ of degree two. From there if we take a map of degree k, $f_k : S^n \to S^n$ (Hatcher 2.32). Then $f_k \circ q \circ r$ will be an even map of degree $2k$. $\square$

Problem 3 (20). *For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.*

Proof. Let x_n and y_n denote the n-dimensional simplices of X and Y respectively. Then

$$\chi(X)\chi(Y) = \left(\sum_{n=0}^{\infty}(-1)^i x_i\right)\left(\sum_{m=0}^{\infty}(-1)^n\right) = \sum_{0=i=m+n}^{\infty}(-1)^i\left(\sum_{j=0}^{i} x_j y_{i-j}\right)$$

However since the number of i-simplices in $X \times Y$ is $\sum_{j=0}^{i} x_j y_{i-j}$ this demonstrates that

$$\chi(X)\chi(Y) = \chi(X \times Y)$$

$\square$