Problem 1 (3.1.6a). *Directly from the definitions, compute the simplicial cohomology groups of $S^1 \times S^1$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients, using the Δ-complex structure given in §2.1.*

Using the given structure we have for our chain complex

$$0 \longrightarrow C_2 = \langle U, L \rangle \longrightarrow C_1 = \langle a, b, c \rangle \longrightarrow C_0 = \langle v \rangle \longrightarrow 0$$

The value of the boundary map on each of these simplices with $\mathbb{Z}_2$ coefficients

$$\begin{aligned} \partial U &= a + b - c \\ \partial L &= a + b - c \\ \partial a &= 0 \\ \partial b &= 0 \\ \partial c &= 0 \\ \partial v &= 0 \end{aligned}$$

When we dualize the complex we get

$$0 \xleftarrow{\delta_2} C^2 \xleftarrow{\delta_1} C^1 \xleftarrow{\delta_0} C^0 \longleftarrow 0$$

For each group in the dual complex there are $2^{|C_n|}$ maps determined by where they send the generators. The maps that generate each are

$$\begin{aligned} C^0 &= \left\langle \begin{pmatrix} v \\ 1 \end{pmatrix} \right\rangle \\ C^1 &= \left\langle \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \\ C^2 &= \left\langle \begin{pmatrix} U & L \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} U & L \\ 0 & 1 \end{pmatrix} \right\rangle \end{aligned}$$

For $H^0(T^2; \mathbb{Z}_2)$ we need $\ker \delta_0$. However since ∂ is the zero map out of C_1 so will be δ_0. As such $H^0(T^2; \mathbb{Z}_2) \cong \mathbb{Z}_3$.

For $H^1(T^2; \mathbb{Z}_2)$ the image of δ_0 is trivial. However for $\ker \delta_1$ this will be exactly the maps that send two generators to 1 and the zero map. As such we have

$$\ker \delta_1 = \left\langle \begin{pmatrix} a & b & c \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 1 & 0 & 1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} a & b & c \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b & c \\ 1 & 1 & 0 \end{pmatrix} \right\rangle$$

Which gives us that $H^1(T^2; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Finally for $H^2(T^2; \mathbb{Z}_2)$ the kernel of δ_2 is everything. As such the only thing we need to determine is the image of δ_1.

$$\begin{aligned} \operatorname{Im} \delta_2 &= \left\langle \delta_2 \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \end{pmatrix}, \delta_2 \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \end{pmatrix}, \delta_2 \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} a & b & c \\ 1 & 0 & 0 \end{pmatrix} \circ \partial, \begin{pmatrix} a & b & c \\ 0 & 1 & 0 \end{pmatrix} \circ \partial, \begin{pmatrix} a & b & c \\ 0 & 0 & 1 \end{pmatrix} \circ \partial \right\rangle \\ &= \left\langle \begin{pmatrix} U & L \\ 1 & 1 \end{pmatrix} \right\rangle \end{aligned}$$

Which gives us

$$H^2(T^2;\mathbb{Z}_2) \cong \left\langle \begin{pmatrix} U & L \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} U & L \\ 1 & 0 \end{pmatrix} \right\rangle / \left\langle \begin{pmatrix} U & L \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} U & L \\ 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}_2$$

All other cohomology groups are zero since there are no simplices of dimension higher than two.

Problem 2 (3.1.8a). *Compute $H^i(S^n;G)$ by induction on n in two ways: using the long exact sequence of a pair, and using the Mayer-Vietoris sequence.*

Proof. First note that $H^0(S^0;G) \cong G \oplus G$ and in particular $\widetilde{H}^0(S^0;G) \cong G$ with $\widetilde{H}^k(S^0;G) = 0$ for all $k > 0$.

We start by using the long exact sequence of the pair in relative cohomology using $(D^n, \partial D^n = S^{n-1})$, so $D^n/\partial D^n = S^n$, slightly thickening the boundary so we have a proper excisive couple. Assume that $\widetilde{H}^{n-1}(S^{n-1};G) \cong G$ and that $\widetilde{H}^k(S^{n-1};G) = 0$ for $k \neq n-1$. Now we deduce the reduced cohomology of the n-sphere.

For general k we have

$$\cdots \longrightarrow \widetilde{H}^k(S^n;G) \longrightarrow \widetilde{H}^k(D^n;G) \longrightarrow \widetilde{H}^k(S^{n-1};G) \longrightarrow \cdots$$

For $k \neq n$ this sequence will look like

$$\widetilde{H}^{k-1}(S^{n-1};G) = 0 \longrightarrow \widetilde{H}^k(S^n;G) \longrightarrow 0 = \widetilde{H}^k(D^n;G)$$

forcing $\widetilde{H}^k(S^n;G) = 0$ for $k \neq n$. However when $k = n$ we have

$$\widetilde{H}^{n-1}(D^n;G) = 0 \longrightarrow \widetilde{H}^{n-1}(S^{n-1};G) \cong G \longrightarrow \widetilde{H}^n(S^n;G) \longrightarrow 0 = \widetilde{H}^n(D^n;G)$$

Which gives an isomorphism $\widetilde{H}^{n-1}(S^{n-1};G) \cong \widetilde{H}^n(S^n;G) \cong G$.

Next we'll prove the same fact using the Mayer-Vietoris sequence. Break up S^n as two copies of D^n with intersection S^{n-1}. As before assume that $\widetilde{H}^{n-1}(S^{n-1};G) \cong G$ and $\widetilde{H}^k(S^{n-1};G) \cong 0$ for $n \neq k$.

The Mayer-Vietoris sequence with the above decomposition will be

$$\cdots \longrightarrow \widetilde{H}^k(S^n;G) \longrightarrow \widetilde{H}^k(D^n;G) \oplus \widetilde{H}^k(D^n;G) \longrightarrow \widetilde{H}^k(S^{n-1};G) \longrightarrow \cdots$$

Similarly when $n \neq k$ we get

$$\widetilde{H}^{k-1}(S^{n-1};G) = 0 \longrightarrow \widetilde{H}^k(S^n;G) \longrightarrow 0 = \widetilde{H}^k(D^n;G) \oplus \widetilde{H}^k(D^n;G)$$

Once again forcing $\widetilde{H}^k(S^n;G) = 0$ for $n \neq k$. However if $n = k$ we instead get

$$\widetilde{H}^{n-1}(D^n;G) \oplus \widetilde{H}^{n-1}(D^n;G) = 0 \longrightarrow \widetilde{H}^{n-1}(S^{n-1};G) \cong G \longrightarrow \widetilde{H}^n(S^n;G) \longrightarrow 0 = \widetilde{H}^n(D^n;G) \oplus \widetilde{H}^n(D^n;G)$$

This gives an isomorphism $\widetilde{H}^{n-1}(S^{n-1};G) \cong \widetilde{H}^n(S^n;G) \cong G$.

From both the above arguments we have determined the reduced cohomology of the n-sphere and from this we can see that

$$H^k(S^n;G) \cong \begin{cases} G & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$\square$

Problem 3 (3.1.9)**.** *Show that if $f : S^n \to S^n$ has degree d then $f^* : H^n(S^n; G) \to H^n(S^n; G)$ is multiplication by d.*

Proof. Let $f : S^n \to S^n$ be a map of degree d. From Hatcher page 196 we have a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(S^n), G) \arrow[r] & H^n(S^n; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(S^n), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(S^n), G) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(S^n; G) \arrow[r, "h"] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(S^n), G) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

Note however that since $H_{n-1}(S^n) = 0$ that we also have $\mathrm{Ext}(H_{n-1}(S^n), G) = 0$. This implies that h is an isomorphism. Moreover we know that f_* on the right is multiplication by d as f has degree d. Since the hom dual of a multiplication map is the same map back we get

$$\begin{tikzcd}
0 \arrow[r] & H^n(S^n; G) \arrow[r, "h"] & \mathrm{Hom}(H_n(S^n), G) \arrow[r] & 0 \\
0 \arrow[r] & H^n(S^n; G) \arrow[r, "h"] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(S^n), G) \arrow[r] \arrow[u, "\cdot d"] & 0
\end{tikzcd}$$

as our new commutative diagram. However at this point it is clear that f^* must also be multiplication by d.

Therefore if a map has degree d then the induced map on cohomology is multiplication by d. □