Problem 1 (3.2.15). For a fixed coefficient field F, define the **Poincaré series** of a space X to be the formal power series $p(t) = \sum_i a_i t^i$ where a_i is the dimension of $H^i(X; F)$ as a vector space over F, assuming this dimension is finite for all i. Show that $p(X \times Y) = p(X)p(Y)$. Compute the Poincaré series for $S^n, \mathbb{R}P^n, \mathbb{R}P^\infty, \mathbb{C}P^n$, and $\mathbb{C}P^\infty$.

Proof. Since we are taking coefficients in a field F, there will be no torsion on the cohomology groups. As such we can use the Künneth formula

$$H^*(X \times Y; F) \cong H^*(X; F) \otimes_F H^*(Y; F)$$

For a fixed n this isomorphism gives

$$H^n(X \times Y; F) \cong \bigoplus_{i+j=n} H^i(X; F) \otimes H^j(Y; F)$$

Since the tensor product multiplies dimension this implies that

$$\dim(H^n(X \times Y; F)) \cong \dim\left(\bigoplus_{i+j=n} H^i(X; F) \otimes H^j(Y; F)\right) = \sum_{i+j=n} \left(\dim H^i(X; F)\right)\left(\dim H^j(Y; F)\right)$$

If we let $a_i = \dim(H^i(X; F))$ and $b_i = \dim(H^j(X; F))$ then using the above we get

$$p(X \times Y) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j t^n = \left(\sum_{i=0}^{\infty} a_i t^i\right)\left(\sum_{j=0}^{\infty} b_j t^j\right) = p(X)p(Y)$$

as desired. $\square$

Using Theorem 3.19 for the projective spaces, we get that the Poincaré polynomials for the spaces above are

$$P(S^n) = 1 + t^n$$
$$P(\mathbb{R}P^n) = \sum_0^n t^i$$
$$P(\mathbb{R}P^\infty) = \sum_0^\infty t^n$$
$$P(\mathbb{C}P^n) = \sum_0^n t^{2n}$$
$$P(\mathbb{C}P^\infty) = \sum_0^\infty t^{2n}$$

Problem 2 (3.2.16). Show that if X and Y are finite CW complexes such that $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$ contain no elements of order a power of a given prime p, then the same is true for $X \times Y$. [Apply Theorem 3.15 with coefficients in various fields.]

Proof. Suppose that $z \in H^*(X \times Y; \mathbb{Z})$ has order p^k. Let q be a prime number such that q is relatively prime to the additive order of z. This will ensure that $q(z) \neq 1$. Consider the commutative diagram

$$\begin{tikzcd} H^*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(Y;\mathbb{Z}) \arrow[r, "\times"] \arrow[d, "q_x \otimes q_y"] & H^*(X \times Y;\mathbb{Z}) \arrow[d, "q"] \\ H^*(X;\mathbb{Z}_q) \otimes_{\mathbb{Z}} H^*(Y;\mathbb{Z}_q) \arrow[r, "\times", "\cong"'] & H^*(X \times Y;\mathbb{Z}_q) \end{tikzcd}$$

where q is the quotient map. The bottom line is an isomorphism by the Kunnëth formula since finite CW complexes imply finitely generated cohomology groups and field coefficients remove torsion.

Since $\mathrm{ord}(z) = p^k$ we have that $\mathrm{ord}(q(z)) = p^j$ where $j < k$. Since the bottom groups are isomorphic and the fact that the quotient map $q_x \otimes q_y$ is surjective we can pull back $q(z)$ to $x \otimes y \in (q_x \otimes q_y)^{-1}(z)$. The order of $x \otimes y$ is $\mathrm{ord}(x \otimes y) = mp^j$. However this implies that $(x \otimes y)^m$ has order p^j which only holds if both the components of $(x \otimes y)^m$ has order dividing p^m.

Thus there is either an $x \in H^*(X;\mathbb{Z})$ or a $y \in H^*(Y;\mathbb{Z})$ that has prime power order completing the proof. □

Problem 3 (3.3.3). Show that every covering space of an orientable manifold is an orientable manifold.

Proof. Let X be an orientable manifold and $p : \widetilde{X} \to X$ be a covering space. Since X is orientable we have a function $x \mapsto \mu_x \in H_n(M|X)$ satisfying local consistency. We can then define an orientation on $\widetilde{X}$ via $\widetilde{x} \mapsto \mu_{p(\widetilde{x})}$. This will satisfy the local consistency condition since $\widetilde{X}$ is locally homeomorphic to X.

Therefore every covering space of an orientable manifold is orientable. □