Problem 1 (3.2.1). Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

Let α_i, β_i denote the first cohomology generators and let γ_i denote the generator for the second cohomology group for the ith tori. in the wedge sum Note that since the homology groups of the torus are free that Ext will be zero and as such the cohomology groups are the hom duals of the homology groups.

Problem 2 (3.2.2). Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is a union of n contractible open subsets, to show that the n-fold cup products of positive dimensional classes are zero.

Proof. First note that since A is contractible we get an isomorphism

$$0 \longrightarrow H^k(X, A; R) \longrightarrow H^n(X; R) \longrightarrow 0$$

Similarly we also get an isomorphism with $H^k(X, B; R)$ and $H^k(X; R)$.

Using this with the naturality of the cup product we get a commutative diagram

$$\begin{tikzcd} H^k(X, A; R) \times H^\ell(X, B; R) \arrow[r, "\smile"] \arrow[d, "\cong"] & H^{k+\ell}(X, A \cup B; R) \cong 0 \arrow[d] \\ H^k(X; R) \times H^\ell(X; R) \arrow[r, "\smile"] & H^{k+l}(X; R) \end{tikzcd}$$

However since this map factors through zero it must be the case that the cup product is zero for positive dimensions.

In the case where $X = \bigcup_i A_i$ we still have the same isomorphisms as before. As such our new diagram is

$$\begin{tikzcd} \prod_i H^{k_i}(X, A; R) \arrow[r, "\smile"] \arrow[d, "\cong"] & H^{\Sigma_i k_i}(X, \bigcup A_i; R) \cong 0 \arrow[d] \\ \prod_i H^{k_i}(X; R) \arrow[r, "\smile"] & H^{\Sigma_i k_i}(X; R) \end{tikzcd}$$

which gives us zero on the cup product for positive dimensions via the same reasoning as above. $\square$

Problem 3 (3.2.4). Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{C}P^n; \mathbb{Z}) \to H^*(\mathbb{C}P^n; \mathbb{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. [See Exercise 3 in §2.C for an example of a map without fixed points in this exceptional case.]

Proof. Recall that $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}(\alpha)/(\alpha^{n+1})$ and that $H^k(\mathbb{C}P^n; \mathbb{Z})$ is $\mathbb{Z}$ for even dimensions and zero otherwise. For $f^* : H^2(\mathbb{C}P^n; \mathbb{Z}) \to H^2(\mathbb{C}P^n; \mathbb{Z})$ this is an endomorphism on the integers. As such it must be of the form $f^*(\alpha) = c\alpha$ for some $c \in \mathbb{Z}$. Moreover since f^* is also an endomorphism on the cohomology ring it must be that $f^*(\alpha^m) = c^m\alpha^m$. Since the Ext term in the universal coefficient theorem will be zero for $\mathbb{C}P^n$ we have that f^* is the hom dual of f_*. Since multiplication maps are unaffected by hom dual the Lefschetz number of f is

$$\tau(f) = 1 + c + \cdots + c^n$$

The only possible rational roots of $\tau(f)$ are ± 1. Since all the coefficients are positive 1 is not a root. If n is even -1 cannot be as well. If n is odd and $f^*(\alpha) \neq -\alpha$ ($c \neq -1$), then $\tau(f)$ has no roots as well.

Therefore by the Lefschetz fixed point theorem the map f has a fixed point if n is even or if $f(\alpha) \neq -\alpha$. $\square$