

*Problem 1 (3.3.1).* Show that there exist nonorientable 1-dimensional manifolds if the Hausdorff condition is dropped from the definition of a manifold.

*Proof.* We construct a tear drop space shown below as the quotient

$$(0, 1] \times \{0, 1\} / \sim$$

where  $(x, 0) \sim (x, 1)$  if  $x < \frac{1}{2}$  and  $(1, 0) \sim (1, 1)$ . Each point in this space has a neighborhood homeomorphic to  $\mathbb{R}$  since we had  $x < \frac{1}{2}$  instead of  $x \leq \frac{1}{2}$  in the relation. The left chunk is homeomorphic to an open interval, as is the right chunk. The two points  $(1/2, 0)$  and  $(1/2, 1)$  also have open neighborhoods homeomorphic to  $\mathbb{R}$  although they cannot be separated.

The picture below demonstrates the nonorientability of the space.

□

*Problem 2 (3.3.11).* If  $M_g$  denotes the closed orientable surface of genus  $g$ , show that degree 1 maps  $M_g \rightarrow M_h$  exist iff  $g \geq h$ .

*Proof.* Suppose that  $g \geq h$ . Decompose  $M_g$  as  $M_h \# M_{g-h}$  where  $\#$  is the connected sum. Then map the  $M_h$  component of  $M_g$  with an orientation preserving homeomorphism to  $M_h$  and map  $M_{g-h}$  to a point. This will be a degree 1 map by local degree of any point aside from the one that  $M_{g-h}$  is sent to.

On the other hand suppose that  $g < h$  and there existed a degree 1 map  $M_g \rightarrow M_h$ . Then by one of the problems from the last homework we would have a surjective map on the first homologies  $\mathbb{Z}^{2g} \rightarrow \mathbb{Z}^{2h}$ . However this is a contradiction since  $g < h$ .

Therefore a degree 1 map only exists if  $g \geq h$ .

□

*Problem 3 (3.3.16).* Show that  $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$  for all  $\alpha \in C_k(X; R)$ ,  $\varphi \in C^l(X; R)$ , and  $\psi \in C^m(X; R)$ . Deduce that cap product makes  $H_*(X; R)$  a right  $H^*(X; R)$ -module.

*Proof.*

□

*Problem 4 (3.3.17).* Show that a direct limit of exact sequences is exact. More generally show that homology commutes with direct limits: If  $\{C_\alpha, f_{\alpha\beta}\}$  is a directed system of chain complexes, with the maps  $f_{\alpha\beta} : C_\alpha \rightarrow C_\beta$  chain maps, then  $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$ .

*Proof.*

□

*Problem 5 (3.3.20).* Show that  $H_c^0(X; G) = 0$  if  $X$  is path-connected and noncompact.

*Proof.* A 0-cochain is a cocycle if it is constant on each path component. So if  $\varphi \in \ker \delta : \Delta_c^0(X; G) \rightarrow \Delta_c^1(X; G)$  then it must be a constant function since  $X$  is path connected. However since  $\varphi$  has compact support and  $X$  is non-compact it must be that  $\varphi \equiv 0$ . Since  $\Delta_c^0(X; G) \cong H_c^0(X; G)$  it follows that  $H_c^0(X; G) \cong 0$ .  $\square$

*Problem 6* (3.3.25). Show that if a closed orientable manifold  $M$  of dimension  $2k$  has  $H_{k-1}(M; \mathbb{Z})$  torsion-free, then  $H_k(M; \mathbb{Z})$  is also torsion-free.

*Proof.* By Poincaré duality we have that  $H_k(M; \mathbb{Z}) \cong H^k(M; \mathbb{Z})$ . By 3.3 from Hatcher  $H^k$  gets its free component from  $H_k(M; \mathbb{Z})$  and its torsion component from  $H_{k-1}(M; \mathbb{Z})$ . Since  $H_{k-1}(M; \mathbb{Z})$  is torsion free, so is  $H_k(M; \mathbb{Z})$ .  $\square$

*Problem 7* (3.3.32). Show that a compact manifold does not retract onto its boundary.

*Proof.* Let  $M$  be a compact  $n$ -manifold with boundary  $\partial M$ . Let us also assume that  $M$  is connected. Suppose that there does exist a retract of  $M$  onto  $\partial M$ . This would imply that the inclusion map on homology,  $i_*$ , is injective. The LES of the pair in homology with  $\mathbb{Z}_2$  coefficients gives us

$$0 \longrightarrow H_n(M; \mathbb{Z}_2) \longrightarrow H_n(M, \partial M; \mathbb{Z}_2) \xrightarrow{\partial_*} H_{n-1}(\partial M; \mathbb{Z}_2) \xrightarrow{i_*} H_{n-1}(M; \mathbb{Z}_2) \longrightarrow \dots$$

Since  $i_*$  is injective this implies that  $\partial_*$  must be the zero map. By Lefschetz duality  $H_n(M; \mathbb{Z}_2) \cong H^0(M, \partial M; \mathbb{Z}_2) \cong 0$ . This implies that  $H_n(M, \partial M; \mathbb{Z}_2) \cong 0$ . However, also by Lefschetz duality, we have  $H_n(M, \partial M; \mathbb{Z}_2) \cong H^0(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ . Which gives us a contradiction.

If  $M$  is not connected then apply the above proof to a connected component with nonzero boundary to get the same contradiction.  $\square$

*Problem 8* (3.3.33). Show that if  $M$  is a compact contractible  $n$ -manifold then  $\partial M$  is a homology  $(n-1)$ -sphere.

*Proof.* Since  $M$  is contractible  $H_i(M) \cong 0$  for  $i > 0$ . The the LES of the pair in homology gives an isomorphism

$$0 \longrightarrow H_i(M, \partial M) \longrightarrow H_{i-1}(\partial M) \longrightarrow 0$$

By Lefschetz duality  $H_i(M, \partial M) \cong H^{n-i}(M)$ . These two isomorphisms together along with the fact that  $H^0(M) \cong \mathbb{Z}$  gives us that

$$H_i(\partial M) \cong \begin{cases} \mathbb{Z} & i = n-1, 0 \\ 0 & \text{otherwise} \end{cases}$$

Which makes  $\partial M$  a homology  $(n-1)$ -sphere.  $\square$