Problem 1 (3.3.8). For a map $f : M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \epsilon_i$ where ϵ_i is $+1$ or -1 according to whether $f : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

Proof. Let $y \in B$ and $\{x_1, \dots, x_m\} = f^{-1}(y)$ where $x_i \in B_i$. Note that there must be finitely many B_is or the sum in the problem is not well defined. We then have the following adaption of the commutative diagram from the proof of Prop. 2.30 in Hatcher

$$\begin{tikzcd}
 & H_n(B_i, B_i \setminus x_i) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(B, B \setminus y) \arrow[d, "\cong"] \\
H_n(M, M \setminus x_i) & H_n(M, M \setminus f^{-1}(y)) \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(N, N \setminus y) \\
 & H_n(M) \arrow[ul, "\cong"] \arrow[u, "j"] \arrow[r, "f_*"] & H_n(N) \arrow[u, "\cong"']
\end{tikzcd}$$

As in the proposition the upper two arrows come from excision. The lower two isomorphisms are isomorphisms by Theorem 3.26 and come from the LES of the pair.

Since this diagram commutes and is identical to the one used to prove Prop. 2.30 except with M and N instead of S^ns we can conclude from the proof of prop. 2.30 that $\deg f = \sum_i \epsilon_i$. $\square$

Problem 2 (3.3.9). Show that a p-sheeted covering space projection $M \to N$ has degree $\pm p$, when M and N are connected closed orientable manifolds.

Proof. Since covering space projections are local homeomorphisms we can apply the previous problem to this by considering a point $y \in N$ and its p-preimages. As such it will suffice to show that the local degree of each point in $f^{-1}(y)$ agrees.

Suppose that not all of the local degrees agree for some subset of N. Then partition M into M_+ and M_- denoting the points of M where f preserves and reverses local orientation respectively. It's clear from the definition that $M_+ \cap M_- = \emptyset$. Moreover both M_+ and M_- are open as given a point $x \in M$ $f(x)$ has an open neighborhood U that is oriented and each disjoint sheet in $f^{-1}(U)$ must either have orientation preserved or reversed. Thus M_+ and M_- form a partition of M which contradicts our assumption that M was connected.

Therefore given a point $y \in N$ the local degree must be $\pm p$ and by the previous problem the degree of f is then $\pm p$. $\square$

Problem 3 (3.3.10). Show that for a degree 1 map $f : M \to N$ of connected closed orientable manifolds, the induced map $f_* : \pi_1 M \to \pi_1 N$ is surjective, hence also $f_* : H_1(M) \to H_1(N)$. [Lift f to the covering space $\widetilde{N} \to N$ corresponding to the subgroup $\operatorname{Im} f_* \subset \pi_1 N$, then consider the two cases that this covering is finite sheeted or infinite sheeted.]

Proof. Let $\widetilde{N}$ be the covering space corresponding to the subgroup $\operatorname{Im} f_* \subset \pi_1(N, *)$ with covering map p. Then we can lift f to $\widetilde{f} : M \to \widetilde{N}$. Now since degree is multiplicative under composition and covering spaces are manifolds we have that $\deg f = \deg \widetilde{f} \cdot \deg p$.

The case where $\widetilde{N}$ is an infinite sheeted cover cannot occur since as $\widetilde{N}$ would not be compact forcing the isomorphism $f_* : H_n(M) \to H_n(N)$ to factor through $H_n(\widetilde{N}) \cong 0$.

In the case where $\widetilde{f}$ has a finite number of sheets we know from the previous problem that the degree is the number of sheets. Using the same degree relation and the fact that degrees are integers the only possibilities when $\deg f = 1$ is that $\deg \widetilde{f} = \deg p = \pm 1$. However this also implies that $\widetilde{N}$ is a one-sheeted cover and as such $\widetilde{N}$ is homeomorphic to N. From this we can conclude that $\operatorname{Im} f_* = \pi_1(N, *)$ and that $f_* : \pi_1(M, *) \to \pi_1(N, *)$ must be surjective. This also implies that the induced map f_* on the 1st homology is surjective as H_1 is the abelianization of π_1. $\square$