Problem 1 (3.3.2). Show that deleting a point from a manifold of dimension greater than 1 does not affect the orientability of the manifold.

Proof. Let M be an n-manifold, $n > 1$, and let y be a point in M. Without loss of generality assume that M is connected as we could apply to following reasoning to each connected component. From Hatcher Prop. 3.25 M is orientable if, and only if, the two-sheeted cover

$$\widetilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$$

has two components.

Suppose that M was orientable. Then $\widetilde{M}$ has two components. For $\widetilde{M \setminus \{y\}}$ this will be identical to $\widetilde{M}$ but missing both points covering y. However since the dimension is greater than one this will not affect the number of components. As such $\widetilde{M \setminus \{y\}}$ has two components which implies that $M \setminus \{y\}$ is orientable.

If M is not orientable, then $\widetilde{M}$ will not have two components. Removing the preimages of y in $\widetilde{M \setminus \{y\}}$ will not change the number of components from that of $\widetilde{M}$ due to the dimension. Thus $M \setminus \{y\}$ is not orientable.

Therefore, for an n-manifold with dimension greater than 1, removing a single point does not affect the orientability of the manifold. $\square$

Problem 2 (3.3.4). Given a covering space action of a group G on an orientable manifold M by orientation-preserving homeomorphisms, show that M/G is also orientable.

Proof. To start let $\mathcal{O}$ be a set of representatives for the orbits in M from G. Choose an orientation for M such that for $x \in \mathcal{O}$ if $x \mapsto \mu_x$ then $g \cdot x \mapsto \mu_{g \cdot x}$ for all $g \in G$ and we satisfy local consistency. We can do this since the cation of G is orientation preserving and for each point in the orbit have disjoint open neighborhoods.

From here we can define an orientation on M/G by using the points $x \in \mathcal{O}$ and sending them $x \mapsto \mu_x$. This is well defined since we were consistent in choice for all points in the orbit and we inherit local consistency for the same reason. Thus M/G has an orientation and as such is orientable. $\square$

Problem 3 (3.3.7). For a map $f : M \to N$ between connected, closed, orientable n-manifolds with fundamental classes $[M]$ and $[N]$, the *degree* of f is defined to be the integer d such that $f_*([M]) = d[N]$, so the sign of the degree depends on the choice of fundamental classes. Show that for any connected closed orientable n-manifold M there is a degree 1 map $M \to S^n$.

Proof. Let p be a point in a connected, closed, and oriented n-manifold M. Then p has an open neighborhood U homeomorphic to $\mathbb{R}^n$. Let q be a point in S^n. Define a map $f : M \to S^n$ by mapping U homeomorphically to $S^n \setminus \{q\} \cong \mathbb{R}^n$, preserving orientation, and mapping $M \setminus U$ to q. This map is continuous since an open set in S^n not containing q will have preimage an open set in U and a set containing q will have preimage with complement a closed neighborhood of p.

Thus f is a continuous map from M to S^n. Now we show that it has degree 1. Using local degree procedure from Hatcher pg. 136, and the fact that we mapped U homeomorphically to $S^n \setminus \{q\}$ preserving orientation, we can deduce that the local degree of f at $-q$ is 1 as $f^{-1}(-q) = \{p\}$ and the map $f_* : H_n(U, U \setminus p) \to H_n(S^n \setminus \{q\}, S^n \setminus \{q, -q\})$ is the identity map. Since p is the only point in the preimage of $-q$ the degree of f is exactly 1. $\square$