**Problem 1** (3.1.1)**.** *Show that* $\operatorname{Ext}(H,G)$ *is a contravariant functor of* $H$ *for fixed* $G$ *and covariant for fixed* $H$*.*

*Proof.* Let $f : A \to B$ be an $R$-module homomorphism and let $H$ be a fixed $R$-module with a projective resolution $P$:

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "f_2"] & P_1 \arrow[r, "f_1"] & P_0 \arrow[r, "f_0"] & H \arrow[r] & 0 \end{tikzcd}$$

Then apply $\operatorname{Hom}_R(H,-)$ with both $A$ and $B$ and to $f$ for each entry in the exact sequence for the projective resolution. This gives us two chain complexes with a map at each group

$$\begin{tikzcd} \cdots & P_2^A \arrow[l] \arrow[d, "g_2^*"] & P_1^A \arrow[l, "f_2^A"'] \arrow[d, "g_1^*"] & P_0 \arrow[l, "f_1^A"'] \arrow[d, "g_0^*"] & H^A \arrow[l, "f_0^A"'] \arrow[d, "g^*"] & 0 \arrow[l] \\ \cdots & P_2^B \arrow[l] & P_1^B \arrow[l, "f_2^B"'] & P_0 \arrow[l, "f_1^B"'] & H^B \arrow[l, "f_0^B"'] & 0 \arrow[l] \end{tikzcd}$$

where $H^A := \operatorname{Hom}_R(H,A)$ and $g^* : \operatorname{Hom}_R(H,A) \to \operatorname{Hom}_R(H,B)$.

Now we will show that $g_i^*$ forms a chain map. However if we write it out we get

$$\begin{aligned} f_n^B \circ g_n^*(h) &= g_{n+1}^* \circ f_n^A(h) \\ f_n^B(g \circ h) &= g_{n+1}^*(h \circ f_n) \\ (g \circ h) \circ f_n &= g \circ (h \circ f_n) \end{aligned}$$

which are equal by associativity of function composition. Since this is a chain map this induces a homomorphism on homology which is exactly $\operatorname{Ext}(H,A) \xrightarrow{g_*} \operatorname{Ext}(H,B)$. This will preserve composition since we are sending functions to functions. Therefore $\operatorname{Ext}(H,-)$ is a covariant functor.

To show that $\operatorname{Ext}(-,G)$ is a contravariant functor we trace out the procedure gone through in class. Let $A$ and $B$ be $R$-modules with projective resolutions $P$ and $Q$ respectively. Let $f : A \to B$ be an $R$-module homomorphism. Then extend $f$ to a chain map $\alpha$ in the form

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "\partial_2^A"] \arrow[d, "\alpha_2"] & P_1 \arrow[r, "\partial_1^A"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "\partial_0^A"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r] & Q_2 \arrow[r, "\partial_2^B"] & Q_1 \arrow[r, "\partial_1^B"] & Q_0 \arrow[r, "\partial_0^B"] & B \arrow[r] & 0 \end{tikzcd}$$

Then we dualize with $\operatorname{Hom}(-,G)$ giving us two chain complexes with a chain map $\alpha^*$

$$\begin{tikzcd} \cdots & P_2^* \arrow[l, "\partial_2^A"] & P_1^* \arrow[l, "\partial_1^A"] & P_0 \arrow[l, "\partial_0^A"] & A \arrow[l] & 0 \arrow[l] \\ \cdots & Q_2 \arrow[l, "\partial_2^B"] \arrow[u, "\alpha_2^*"] & Q_1 \arrow[l, "\partial_1^B"] \arrow[u, "\alpha_1^*"] & Q_0 \arrow[l, "\partial_0^B"] \arrow[u, "\alpha_0^*"] & B \arrow[l] \arrow[u, "f^*"] & 0 \arrow[l] \end{tikzcd}$$

We showed in class that $\alpha^*$ is in fact a chain map for the new complexes. As such it induces a homomorphism on homology which is $g_* : \operatorname{Ext}(B,G) \to \operatorname{Ext}(A,G)$. This shows that $\operatorname{Ext}(-,G)$ is indeed a contravariant functor as composition is preserved for the same reason as above. □

**Problem 2** (3.1.2)**.** *Show that the maps* $G \xrightarrow{n} G$ *and* $H \xrightarrow{n} H$ *multiplying each element by the integer* $n$ *induce multiplication by* $n$ *in* $\operatorname{Ext}(H,G)$*.*

*Proof.* Let $A$ be a generating set for $G$. Then we have a free resolution of $G$ of the form

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r, "i"] & F(A) \arrow[r, "f"] & G \arrow[r] & 0 \end{tikzcd}$$

where $i$ is inclusion, $f$ is the evaluation map, and $F(A)$ is the free group on $A$. Fortunately the map that multiplies by $n$ has a lift where it is also multiplication by $n$. Thus we have

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r, "i"] \arrow[d, "n"] & F(A) \arrow[r, "f"] \arrow[d, "n"] & G \arrow[r] \arrow[d, "n"] & 0 \\ 0 \arrow[r] & \ker(f) \arrow[r, "i"] & F(A) \arrow[r, "f"] & G \arrow[r] & 0 \end{tikzcd}$$

Then we dualize to get

$$\begin{tikzcd} 0 & \ker(f)^* \arrow[l] & F(A) \arrow[l, "i^*"'] & G^* \arrow[l, "f^*"'] & 0 \arrow[l] \\ 0 & \ker(f)^* \arrow[l] \arrow[u, "n^*"] & F(A)^* \arrow[l, "i^*"'] \arrow[u, "n^*"] & G^* \arrow[l, "f^*"'] \arrow[u, "n^*"] & 0 \arrow[l] \end{tikzcd}$$

Now we need to know what the $n^*$ maps are. However as it turns out if we have a map in one of the groups in the above diagram $h$. Then $n^*(h)(x) = h(nx) = nh(x)$ which implies that $n^*$ is once again the multiplication by $n$ map. Since it is the multiplication map everywhere the induced map on homology will also be the multiplication by $n$ map. Thus $\operatorname{Ext}(-, H)(\cdot n) = \cdot n$.

Similarly if we use $\operatorname{Ext}(G, -)$ we dualize first and then place the $n^*$ maps in. However since these maps are endomorphisms we get the same diagram

$$\begin{tikzcd} 0 & \ker(f)^* \arrow[l] \arrow[d, "n^*"] & F(A)^* \arrow[l, "i^*"'] \arrow[d, "n^*"] & G^* \arrow[l, "f^*"'] \arrow[d, "n^*"] & 0 \arrow[l] \\ 0 & \ker(f)^* \arrow[l] & F(A)^* \arrow[l, "i^*"'] & G^* \arrow[l, "f^*"'] & 0 \arrow[l] \end{tikzcd}$$

which must have the same then induce the same maps giving $\operatorname{Ext}(G, -)(\cdot n) = \cdot n$ $\square$

**Problem 3** (3.1.3)**.** *Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all $n$.*

We can construct a free resolution of $\mathbb{Z}_2$ of the form

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}_4 \arrow[r, "2"] & \mathbb{Z}_4 \arrow[r, "2"] & \mathbb{Z}_4 \arrow[r, "\text{mod } 2"] & \mathbb{Z}_2 \arrow[r] & 0 \end{tikzcd}$$

When we dualize with $\operatorname{Hom}(-, \mathbb{Z}_2)$ we get

$$\begin{tikzcd} \cdots & \mathbb{Z}_0 \arrow[l, "0"'] & \mathbb{Z}_2 \arrow[l, "0"'] & \mathbb{Z}_2 \arrow[l, "0"'] & \mathbb{Z}_2 \arrow[l, "0"'] & 0 \arrow[l] \end{tikzcd}$$

Which has nonzero homology groups everywhere. As such $\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all $n$.