Problem 1 (3.2.3). (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding results for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

(b) Prove the Borsuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Show that part (a) applies to this map.

Proof. (a) Suppose that $f : \mathbb{R}P^n \to \mathbb{R}P^m$ was non-trivial for the first cohomology. The cup product structure for $\mathbb{R}P^k$ with $\mathbb{Z}_2$ coefficients is $\mathbb{Z}_2[\alpha]/(\alpha^{k+1})$. As such f induces a map on the cohomology rings

$$f^* : \mathbb{Z}_2[\alpha]/(\alpha^{m+1}) \to \mathbb{Z}_2[\beta]/(\beta^{n+1})$$

wherein $f^*(\alpha) = \beta$ since the map is nontrivial in first cohomology. However $\alpha^{m+1} = 0$ which would imply that $f(\alpha^{m+1}) = \beta^{m+1} = 0$. However this is a contradiction since $n > m$.

(b) Let $g : S^n \to S^{n-1}$ be the map as defined above. Then this map induces a map $g' : S^n \to \mathbb{R}P^{n-1}$ since $g(x) = -g(-x)$. Then g' induces a map $g'' : \mathbb{R}P^n \to \mathbb{R}P^{n-1}$ as now $g'(x) = g'(-x)$. On the cellular cochain complex $\mathbb{Z}_2$ coefficients the map g'' induces the identity map since it also does on the cellular chain complex (since g will send a path between antipodal points on S^n to a path between antipodal points on S^{n-1}). However since all the coboundary maps are zero the induces map on the cellular cochains will induce the same map on the cohomology. However since $(g'')^*$ is not trivial this contradicts part (a).

□

Problem 2 (3.2.5). Show the ring $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ is isomorphic to $\mathbb{Z}_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ where $|\alpha| = 1$ and $|\beta| = 2$. [Use the coefficient map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ and the proof of Theorem 3.19.]

Proof. Using the cellular cochain complex for $\mathbb{R}P^\infty$ with $\mathbb{Z}_{2k}$ coefficients

$$0 \longrightarrow \mathbb{Z}_{2k} \xrightarrow{0} \mathbb{Z}_{2k} \xrightarrow{2} \mathbb{Z}_{2k} \xrightarrow{0} \cdots$$

we get the cohomology of $\mathbb{R}P^\infty$ with $\mathbb{Z}_{2k}$ coefficients is

$$H^i(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \cong \begin{cases} \mathbb{Z}_{2k} & i = 0 \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

Take the following commutative diagrams with the cellular cochains with $\mathbb{Z}_{2k}$ and $\mathbb{Z}_2$ coefficients with the $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ running between them.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_{2k} \arrow[r, "0"] \arrow[d] & \mathbb{Z}_{2k} \arrow[r, "2"] \arrow[d] & \mathbb{Z}_{2k} \arrow[r, "0"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 \arrow[r, "2"] & \mathbb{Z}_2 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

Similarly to the diagram on Hatcher pg. 222, this map will be injective on even dimension. Since $H^2(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \cong \mathbb{Z}_{2k}/2\mathbb{Z}_{2k}$ when we send the generator for the cohomology ring in dimension 1, α, it'll get sent to $\alpha^2 = k\beta$. Together this gives us that

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \cong \mathbb{Z}_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$$

□

Since $2\beta = 0$ this effectively means that β is redundant whenever k is odd, but necessary if k is even.

Problem 3 (3.2.7). Use the cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

Proof. From Hatcher the cup product structure for $\mathbb{R}P^3$ with $\mathbb{Z}_2$ coefficients is

$$H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^4) \quad |\alpha| = 1$$

and the cup product structure for $\mathbb{R}P^2 \vee S^3$ with $\mathbb{Z}_2$ coefficients is

$$H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^2; \mathbb{Z}_2) \times H^*(S^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2[\beta]/(\beta^3) \times \mathbb{Z}_2[\gamma]/(\gamma^2))/(\langle 1_\beta, 1_\gamma \rangle) \quad |\beta| = 1, |\gamma| = 3$$

The latter has an element, β, which when cubed is zero. However the former has no such elements. Thus the two cohomology rings are not isomorphic and therefore $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$. $\square$