Problem 1 (3.1.10). For the lens space $L_m(\ell_1, \ldots, \ell_n)$ defined in Example 2.43, compute the cohomology groups using the cellular cochain complex and taking coefficients in $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_m$, and $\mathbb{Z}_p$ for p prime. Verify that the answers agree with those given by the universal coefficient theorem.

Proof. We abbreviate $L_m(\ell_1, \ldots, \ell_n)$ as L.

We know Hatcher that the cellular chain complex for a lens space $L_m(\ell_1, \ldots, l_n)$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Where there are $2n-1$ copies of $\mathbb{Z}$ in the above complex. We also know that the cellular cochain complex is isomorphic to the hom dual of the cellular cochain complex. Moreover $\operatorname{Hom}(\mathbb{Z}, A) \cong A$, determined by where the generators are sent, and $\operatorname{Hom}(\mathbb{Z}, -)(\cdot m) = \cdot m$. We know look at the cellular cochain complex for each of the listed groups in turn.

$\mathbb{Z}$: From the above information we know that cellular cochain complex for $\mathbb{Z}$ will in fact be almost exactly the same as the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Thus the cohomology groups of the lens space with integer coefficients will be isomorphic to the homology groups with a switch from k odd to k even

$$H^k(L; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, 2n-1 \\ \mathbb{Z}_m & 0 < k < 2n-1, \; k \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{Q}$: With rational coefficients the cellular cochain complex will be of the form

$$0 \longrightarrow \mathbb{Q} \xrightarrow{0} \mathbb{Q} \xrightarrow{m} \mathbb{Q} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Q} \xrightarrow{m} \mathbb{Q} \xrightarrow{0} \mathbb{Q} \longrightarrow 0$$

It's clear from this that for $k = 2n-1, 0$ that the cohomology will be $\mathbb{Q}$. The other cases are either an $\cdot m$ leaving and a 0 map entering, or vice versa. In the prior case the kernel is trivial and in the latter case the image equals the kernel. Either way the cohomology will be zero. Thus the cohomology for the lens space with rational coefficients will be

$$H^k(L; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 2n-1, 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{Z}_m$: Under $\mathbb{Z}_m$ the map that multiplies by m gives us the zero map. As such the cellular cochain complex will be

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \longrightarrow 0$$

Since the kernel is always $\mathbb{Z}_m$ and the image the identity we then have that

$$H^k(L; \mathbb{Z}_m) \cong \begin{cases} \mathbb{Z}_m & 0 \leq k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{Z}_p$: Dualizing with $\mathbb{Z}_p$ gives us the cellular cochain complex

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \longrightarrow 0$$

Then we need to consider the case where $p|m$ and when $p \nmid m$.

When $p|m$ the map $\cdot m$ becomes the zero map. As such we end up with cohomology analogous to the $\mathbb{Z}_m$ case

$$H^k(L;\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & 0 \le k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

However if $p \nmid m$. Then the map $\cdot m$ is invertible and as such an isomorphism. This makes it analogous to the $\mathbb{Q}$ case

$$H^k(L;\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p & k = 0, 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

□

Now we check our work with the universal coefficient theorem. The short exact sequence that we will be using is

$$0 \longrightarrow \text{Ext}(H_k(L), G) \longrightarrow H^k(L;G) \longrightarrow \text{Hom}(H_k(L), G) \longrightarrow 0$$

If $k = 2n-1, 0$ we get

$$0 \longrightarrow \text{Ext}(0, G) \cong 0 \longrightarrow H^k(L;G) \longrightarrow \text{Hom}(\mathbb{Z}, G) \cong \mathbb{Z} \longrightarrow 0$$

which provides $H^{2n-1}(L;G) \cong H^0(L;G) \cong G$ as expected.

Next if $0 < k < 2n-1$ and k even we start with

$$0 \longrightarrow pp\,\text{Ext}(\mathbb{Z}_m, G) \longrightarrow H^k(L;G) \longrightarrow \text{Hom}(0, G) \cong 0 \longrightarrow 0$$

Then $\text{Ext}(\mathbb{Z}_m, G) \cong H^k(L;G)$. Split into cases and we get

- $\mathbb{Z}$: $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m$.
- $\mathbb{Q}$: $\text{Ext}(\mathbb{Z}_m, \mathbb{Q}) \cong \mathbb{Q}/m\mathbb{Q} \cong 0$
- $\mathbb{Z}_m$: $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_m) \cong \mathbb{Z}_m/(m\mathbb{Z}_m \cong 0) \cong \mathbb{Z}_m$
- $\mathbb{Z}_p$: When $p|m$ we get $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) \cong \mathbb{Z}_p/(m\mathbb{Z}_p \cong 0) \cong \mathbb{Z}_p$ and when $p \nmid m$ we get $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_p) \cong \mathbb{Z}_p/(m\mathbb{Z}_p \cong \mathbb{Z}_p) \cong 0$.

Finally if $0 < k < 2n-1$ and k odd we start with

$$0 \longrightarrow \text{Ext}(0, G) \cong 0 \longrightarrow H^k(L;G) \longrightarrow \text{Hom}(\mathbb{Z}_m, G) \cong 0 \longrightarrow 0$$

giving $H^k(L;G) \cong \text{Hom}(\mathbb{Z}_m, G)$. Split into the same cases as above to get

- $\mathbb{Z}$: $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}) \cong 0$
- $\mathbb{Q}$: $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}) \cong 0$
- $\mathbb{Z}_m$: $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_m) \cong \mathbb{Z}_m$
- $\mathbb{Z}_p$: As above if $p|m$ we get $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_p) \cong \mathbb{Z}_p$ and otherwise if $p \nmid m$ then $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_p) \cong 0$

Problem 2 (3.1.11). Let X be a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m.

(a) Show that the quotient map $X \to X/S^n = S^{n+1}$ induces the trivial map on $\widetilde{H}_i(-;\mathbb{Z})$ for all i, but not on $H^{n+1}(-;\mathbb{Z})$. Deduce that the splitting in the universal coefficient theorem for cohomology cannot be natural.

(b) Show that the inclusion $S^n \hookrightarrow X$ induces the trivial map on $\widetilde{H}^i(-;\mathbb{Z})$ for all i, but not on $H_n(-;\mathbb{Z})$.

Proof. (a) We can deduce that the map induced by $q : X \to S^{n+1}$ is the trivial on homology since $\widetilde{H}_k(X) \cong 0$ for $k \neq n$ implies that it can only be the zero map and for $k = n$ we have $q_* : \mathbb{Z}_m \to (H_n(S^{n+1}) = 0)$.

Using the universal coefficient theorem we can calculate the cohomology of X with $\mathbb{Z}$ coefficients. It is clear that $\widetilde{H}^k(X;\mathbb{Z}) \cong 0$ when $k \neq n+1$ as both Hom and Ext will be zero in the universal coefficient theorem. However when $k = n+1$

$$0 \longrightarrow \mathrm{Ext}(H_n(X) \cong \mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m \longrightarrow H^{n+1}(X) \longrightarrow \mathrm{Hom}(H_{n+1}(X) \cong 0, \mathbb{Z}) \cong 0 \longrightarrow 0$$

Giving us that $H^{n+1}(X) \cong \mathbb{Z}_m$. From the same reasoning as before we know that the induced map $q^* : \widetilde{H}^k(S^{n+1}) \to \widetilde{H}^k(X;\mathbb{Z})$ will be the zero map for $k \neq n+1$. When $k = n+1$ if we look at $q_\#$ on the chain level it will send the e^{n+1} cell of X to the e^{n+1} cell for S^{n+1}. As such when we look at the induced map $q^* : \widetilde{H}^{n+1}(S^{n+1}) = \mathbb{Z} \to \widetilde{H}^{n+1}(X) = \mathbb{Z}_m$ it will send the generator of the former to the generator of the latter. As such the map q^* is not trivial on cohomology when $k = n$.

Using the commutative diagram from pg. 196 of Hatcher we have

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_n(X),\mathbb{Z}) \arrow[r] & H^{n+1}(X) \arrow[r] & \mathrm{Hom}(H_{n+1}(X),\mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_n(S^{n+1}),\mathbb{Z}) \arrow[r] \arrow[u, "(q_*)^*"] & H^{n+1}(S^{n+1}) \arrow[r] \arrow[u, "q^*"] & \mathrm{Hom}(H_{n+1}(X),\mathbb{Z}) \arrow[r] \arrow[u, "(q_*)^*"] & 0
\end{tikzcd}$$

Substituting in we get

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_m \arrow[r] & \mathbb{Z}_m \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u, "0"] & \mathbb{Z} \arrow[r] \arrow[u, "q^*"] & \mathbb{Z} \arrow[r] \arrow[u, "0"] & 0
\end{tikzcd}$$

From splitting we get a map from $0 \to \mathbb{Z}_m$ and a map $\mathbb{Z} \to \mathbb{Z}$ going back. If the splitting was natural this would give us commutativity in

$$\begin{tikzcd}
\mathbb{Z}_m & 0 \arrow[l] \\
\mathbb{Z} \arrow[u, "q^*"] & \mathbb{Z} \arrow[l] \arrow[u, "0"]
\end{tikzcd}$$

However this is not the case. Thus the splitting for the universal coefficient theorem is not natural.

(b) For the same reason as q_* being trivial on reduced homology above, i^* will be trivial on reduced cohomology since there are no groups that are nonzero for the same index.

However when it comes to reduced homology when $k = n$ the induced map i_* will map the generator of $\widetilde{H}_n(S^n) = \mathbb{Z}$ to the generator of the reduced homology of X which is also a copy of S^n. As such $i_*(k) = k \mod m$ on the nth homology group.

□