Problem 1 (3.2.15). For a fixed coefficient field F, define the **Poincaré series** of a space X to be the formal power series $p(t) = \sum_i a_i t^i$ where a_i is the dimension of $H^i(X; F)$ as a vector space over F, assuming this dimension is finite for all i. Show that $p(X \times Y) = p(X)p(Y)$. Compute the Poincaré series for $S^n, \mathbb{R}P^n, \mathbb{R}P^\infty, \mathbb{C}P^n$, and $\mathbb{C}P^\infty$.

Proof. Since we are taking coefficients in a field F, there will be no torsion on the cohomology groups. As such we can use the Künneth formula

$$H^*(X \times Y; F) \cong H^*(X; F) \otimes_F H^*(Y; F)$$

For a fixed n this isomorphism gives

$$H^n(X \times Y; F) \cong \bigoplus_{i+j=n} H^i(X; F) \otimes H^j(Y; F)$$

Since the tensor product multiplies dimension this implies that

$$\dim(H^n(X \times Y; F)) \cong \dim\left(\bigoplus_{i+j=n} H^i(X; F) \otimes H^j(Y; F)\right) = \sum_{i+j=n} \left(\dim H^i(X; F)\right)\left(\dim H^j(Y; F)\right)$$

If we let $a_i = \dim(H^i(X; F))$ and $b_i = \dim(H^j(X; F))$ then using the above we get

$$p(X \times Y) = \sum_{n=0}^{\infty} \sum_{i+j=n} a_i b_j t^n = \left(\sum_{i=0}^{\infty} a_i t^i\right)\left(\sum_{j=0}^{\infty} b_j t^j\right) = p(X)p(Y)$$

as desired. □

Using Theorem 3.19 for the projective spaces, we get that the Poincaré polynomials for the spaces above are

$$p(S^n) = 1 + t^n$$
$$p(\mathbb{R}P^n) = \sum_0^n t^i$$
$$p(\mathbb{R}P^\infty) = \sum_0^\infty t^n$$
$$p(\mathbb{C}P^n) = \sum_0^n t^{2n}$$
$$p(\mathbb{C}P^\infty) = \sum_0^\infty t^{2n}$$

Problem 2 (3.2.16). Show that if X and Y are finite CW complexes such that $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$ contain no elements of order a power of a given prime p, then the same is true for $X \times Y$. [Apply Theorem 3.15 with coefficients in various fields.]

Proof (Revised): Given a finitely generated abelian group G define $\mathrm{rk}_p(G)$ to be the number of occurences of the prime p in the primary decomposition $G \cong \mathbb{Z}^r \bigoplus_{p,k} \mathbb{Z}_{p^k}$.

From the Universal Coefficient theorem, if the cohomology groups are finitely generated, we get

$$\begin{aligned} H^k(X;\mathbb{Z}) &\cong \mathrm{Tor}(H_{k-1}(X)) \oplus \mathbb{Z}^{\mathrm{rk}(H_k(X))} \\ H^k(Y;\mathbb{Z}) &\cong \mathrm{Tor}(H_{k-1}(Y)) \oplus \mathbb{Z}^{\mathrm{rk}(H_k(Y))} \\ H^k(X\times Y;\mathbb{Z}) &\cong \mathrm{Tor}(H_{k-1}(X\times Y)) \oplus \mathbb{Z}^{\mathrm{rk}(H_k(X\times Y))} \end{aligned}$$

The values for Hom and Ext with $\mathbb{Q}$ and $\mathbb{Z}_p$ are

$$\begin{array}{ll} \mathrm{Ext}(H_{k-1},\mathbb{Q}) \cong 0 & \mathrm{Hom}(H_i,\mathbb{Q}) \cong \mathbb{Q}^{\mathrm{rk}(H_k)} \\ \mathrm{Ext}(H_{k-1},\mathbb{Z}_p) \cong \mathbb{Z}_p^{\mathrm{rk}_p(H_{k-1})} & \mathrm{Hom}(H_i,\mathbb{Q}) \cong \mathbb{Q}^{\mathrm{rk}(H_k)+\mathrm{rk}_p(H_k)} \end{array}$$

Using the universal coefficient theorem once more we get

$$\begin{array}{ll} H^k(X;\mathbb{Q}) \cong \mathbb{Q}^{\mathrm{rk}(H_i(X))} & H^k(X;\mathbb{Z}_p) \cong \mathbb{Z}_p^{\mathrm{rk}(H_k(X))+\mathrm{rk}_p(H_k(X))+\mathrm{rk}_p(H_{k-1}(X))} \\ H^k(Y;\mathbb{Q}) \cong \mathbb{Q}^{\mathrm{rk}(H_i(Y))} & H^k(Y;\mathbb{Z}_p) \cong \mathbb{Z}_p^{\mathrm{rk}(H_k(Y))+\mathrm{rk}_p(H_k(Y))+\mathrm{rk}_p(H_{k-1}(Y))} \\ H^k(X\times Y;\mathbb{Q}) \cong \mathbb{Q}^{\mathrm{rk}(H_i(X\times Y))} & H^k(X\times Y;\mathbb{Z}_p) \cong \mathbb{Z}_p^{\mathrm{rk}(H_k(X\times Y))+\mathrm{rk}_p(H_k(X\times Y))+\mathrm{rk}_p(H_{k-1}(X\times Y))} \end{array}$$

Since X, Y are finite CW complexes and we are working with field coefficients we can use the Kunnëth theorem to get

$$\begin{aligned} H^*(X\times Y;\mathbb{Q}) &\cong H^*(X;\mathbb{Q}) \otimes_{\mathbb{Z}} H^*(Y;\mathbb{Q}) \\ H^k(X\times Y;\mathbb{Q}) &\cong \bigoplus_{i+j=k} H^i(X;\mathbb{Q}) \otimes H^j(Y;\mathbb{Q}) \\ \mathbb{Q}^{\mathrm{rk}(H_k(X\times Y))} &\cong \bigoplus_{i+j=k} \mathbb{Q}^{\mathrm{rk}(H_i(X))\,\mathrm{rk}(H_j(X))} \end{aligned}$$

Which gives the equality

$$\mathrm{rk}(H_k(X\times Y)) = \sum_{i+j=k} \mathrm{rk}(H_i(X))\,\mathrm{rk}(H_j(X))$$

If we repeat the process with $\mathbb{Z}_p$ coefficients we get the equality

$$\begin{aligned} \mathrm{rk}(H_i(X\times Y)) + \mathrm{rk}_p(H_i(X\times Y)) + \mathrm{rk}_p(H_{i-1}(X\times Y)) = \sum_{i+j=k} &(\mathrm{rk}(H_k(X)) + \mathrm{rk}_p(H_k(X)) + \mathrm{rk}_p(H_{k-1}(X))) \\ &\cdot(\mathrm{rk}(H_k(Y)) + \mathrm{rk}_p(H_k(Y)) + \mathrm{rk}_p(H_{k-1}(Y))) \end{aligned}$$

Now we begin the proof proper. Assume that $H^*(X;\mathbb{Z})$ and $H^*(Y;\mathbb{Z})$ have no elements of order p^k. Since the torsion of the kth cohomology group comes the torsion of the $(k-1)$ homology group it is clear that for a space Z that the cohomology ring has no prime power order elements if, and only if, $\mathrm{rk}_p(H_k(Z)) = 0$ for all k. As such our assumption is equivalent to $\mathrm{rk}_p(H_k(X)) = \mathrm{rk}_p(H_k(Y)) = 0$ for all k.

We proceed to show that $\mathrm{rk}_p(H_k(X\times Y)) = 0$ by induction. We start with the inductive case. Using the equality from $\mathbb{Z}_p$ coefficients and placing zeros where appropriate we get

$$\mathrm{rk}(H_i(X\times Y)) + \mathrm{rk}_p(H_i(X\times Y)) = \sum_{i+j=k} \mathrm{rk}(H_k(X)\,\mathrm{rk}(H_k(Y))$$

Then subtracting the equality from the $\mathbb{Q}$ coefficients to cancel the first terms we get

$$\mathrm{rk}_p(H_k(X\times Y))) = 0$$

which completes the inductive case.

If $k = 0$ we still have

$$\mathrm{rk}(H_0(X \times Y)) + \mathrm{rk}_p(H_0(X \times Y)) + \mathrm{rk}_p(H_{-1}(X \times Y)) = \mathrm{rk}(H_0(X))\,\mathrm{rk}(H_0(Y))$$

However since $H_{-1}(X \times Y) \cong 0$ we get that

$$\mathrm{rk}_p(H_0(X \times Y)) = 0$$

This completes the proof that $\mathrm{rk}_P(H_k(X \times Y)) = 0$ for all k and as such we have that $H^*(X \times Y; \mathbb{Z})$ has no elements of power of p order. □

Problem 3 (3.3.3). Show that every covering space of an orientable manifold is an orientable manifold.

Proof. Let X be an orientable manifold and $p : \widetilde{X} \to X$ be a covering space. Since X is orientable we have a function $x \mapsto \mu_x \in H_n(M|X)$ satisfying local consistency. We can then define an orientation on $\widetilde{X}$ via $\widetilde{x} \mapsto \mu_{p(\widetilde{x})}$. This will satisfy the local consistency condition since $\widetilde{X}$ is locally homeomorphic to X.

Therefore every covering space of an orientable manifold is orientable. □