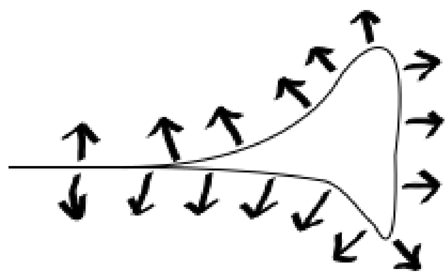
Problem 1 (3.3.1). Show that there exist nonorientable 1-dimensional manifolds if the Hausdorff condition is dropped form the definition of a manifold.

Proof. We construct a tear drop space shown below as the quotient

$$(0,1] \times \{0,1\}/\sim$$

where $(x,0) \sim (x,1)$ if $x < \frac{1}{2}$ and $(1,0) \sim (1,1)$. Each point in this space has a neighborhood homeomorphic to $\mathbb{R}$ since we had $x < \frac{1}{2}$ instead of $x \leq \frac{1}{2}$ in the relation. The left chunk is homeomorphic to an open interval, as is the right chunk. The two points $(1/2, 0)$ and $(1/2, 1)$ also have open neighborhoods homeomorphic to $\mathbb{R}$ although they cannot be separated.

The picture below demonstrates the nonorientability of the space.



□

Problem 2 (3.3.11). If M_g denotes the closed orientable surface of genus g, show that degree 1 maps $M_g \to M_h$ exist iff $g \geq h$.

Proof. Suppose that $g \geq h$. Decompose M_g as $M_h \# M_{g-h}$ where $\#$ is the connected sum. Then map the M_h component of M_g with an orientation preserving homeomorphism to M_h and map M_{g-h} to a point. This will be a degree 1 map by local degree of any point aside from the one that M_{g-h} is sent to.

On the other hand suppose that $g < h$ and there existed a degree 1 map $M_g \to M_h$. Then by one of the problems from the last homework we would have a surjective map on the first homologies $\mathbb{Z}^{2g} \to \mathbb{Z}^{2h}$. However this is a contradiction since $g < h$.

Therefore a degree 1 map only exists if $g \geq h$. □

Problem 3 (3.3.16). Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$ for all $\alpha \in C_k(X; R)$, $\varphi \in C^l(X; R)$, and $\psi \in C^m(X; R)$. Deduce that cap product makes $H_*(X; R)$ a right $H^*(X; R)$-module.

Proof. □

Problem 4 (3.3.17). Show that a direct limit of exact sequences is exact. More generally show that homology commutes with direct limits: If $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes, with the maps $f_{\alpha\beta} : C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Proof. □

Problem 5 (3.3.20). Show that $H_c^0(X;G) = 0$ if X is path-connected and noncompact.

Proof. A 0-cochain is a cocycle if it is constant on each path component. So if $\varphi \in \ker\delta : \Delta_c^0(X;G) \to \Delta_c^1(X;G)$ then it must be a constant function since X is path connected. However since φ has compact support and X is non-compact it must be that $\varphi \equiv 0$. Since $\Delta_c^0(X;G) \cong H_c^0(X;G)$ it follows that $H_c^0(X;G) \cong 0$. □

Problem 6 (3.3.25). Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsion-free, then $H_k(M;\mathbb{Z})$ is also torsion-free.

Proof. By Poincaré duality we have that $H_k(M;\mathbb{Z}) \cong H^k(M;\mathbb{Z})$. By 3.3 from Hatcher H^k gets its free component from $H_k(M;\mathbb{Z})$ and its torsion component from $H_{k-1}(M;\mathbb{Z})$. Since $H_{k-1}(M;\mathbb{Z})$ is torsion free, $H_k(M;\mathbb{Z})$ is torsion free as well. □

Problem 7 (3.3.32). Show that a compact manifold does not retract onto its boundary.

Proof. Let M be a compact n-manifold with boundary ∂M. Let us also assume that M is connected. Suppose that there does exist a retract of M onto ∂M. This would imply that the inclusion map on homology, i_*, is injective. The LES of the pair in homology with $\mathbb{Z}_2$ coefficients gives us

$$0 \longrightarrow H_n(M;\mathbb{Z}_2) \longrightarrow H_n(M,\partial M;\mathbb{Z}_2) \xrightarrow{\partial_*} H_{n-1}(\partial M;\mathbb{Z}_2) \xrightarrow{i_*} H_{n-1}(M;\mathbb{Z}_2) \longrightarrow \cdots$$

Since i_* is injective this implies that ∂_* must be the zero map. By Lefschetz duality $H_n(M;\mathbb{Z}_2) \cong H^0(M,\partial M;\mathbb{Z}_2) \cong 0$. This implies that $H_n(M,\partial M;\mathbb{Z}_2) \cong 0$. However, also by Lefschetz duality, we have $H_n(M,\partial M;\mathbb{Z}_2) \cong H^0(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Which gives us a contradiction.

If M is not connected then apply the above proof to a connected component with nonzero boundary to get the same contradiction. □

Problem 8 (3.3.33). Show that if M is a compact contractible n-manifold then ∂M is a homology $(n-1)$-sphere.

Proof. Since M is contractible $H_i(M) \cong 0$ for $i > 0$. The the LES of the pair in homology gives an isomorphism

$$0 \longrightarrow H_i(M,\partial M) \longrightarrow H_{i-1}(\partial M) \longrightarrow 0$$

By Lefschetz duality $H_i(M,\partial M) \cong H^{n-i}(M)$. These two isomorphisms together along with the fact that $H^0(M) \cong \mathbb{Z}$ gives us that

$$H_i(\partial M) \cong \begin{cases} \mathbb{Z} & i = n-1, 0 \\ 0 & \text{otherwise} \end{cases}$$

Which makes ∂M a homology $(n-1)$-sphere. □