Problem 1 (3.2.1). Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

Let α_i, β_i denote the first cohomology generators and let γ_i denote the generator for the second cohomology group for the ith tori in the wedge sum. Also let γ be the lone generator for $H^2(M_g)$. Note that since the homology groups of the torus are free that $\mathrm{Ext}(H_k(T^2), \mathbb{Z})$ will be zero and as such the cohomology groups are the hom duals of the homology groups. Let a_i, b_i, c_i be the dual basis elements to α_i, β_i, and γ_i respectively. The induced map $q_* : H^k(M_g) \to H^k(\bigvee_i T^2)$ will act on the basis elements of the homology groups as

$$q_*(a_i) = a_i, \quad q_*(b_9) = b_i, \quad q_*(c) = \sum_i c_i$$

where c is the lone generator of $H_2(M_g)$. By the universal coefficient theorem q^* is the hom dual of q_*. As such we can deduce the values of q^* as

$$q^*(\alpha_8) = \alpha_8, \quad q^*(\beta_i), \quad q^*(\gamma_i) = \gamma$$

Since the cohomology ring of the wedge sum is the product of the cohomology rings we can determine the cup product structure using what we know of $\prod_i H^*(T^2)$ and q^*. The cup product structure of $\prod_i H^i(T^2)$ is

$$\begin{aligned} \alpha_i \smile \beta_j &= 0 & i \neq j \\ \alpha_i \smile \beta_i &= \gamma_i \\ \beta_j \smile \alpha_j &= -\gamma_j \\ \beta_j \smile \alpha_i &= 0 & i \neq j \end{aligned}$$

Then using the quotient map we get

$$\begin{aligned} q^*(\alpha_i) \smile q^*(\beta_j) &= q^*(\alpha_i \smile \beta_j) = 0 & i \neq j \\ q^*(\alpha_i) \smile q^*(\beta_i) &= q^*(\alpha_i \smile \beta_i) = \gamma_i \\ q^*(\beta_j) \smile q^*(\alpha_j) &= q^*(\beta_j \smile \alpha_j) = -\gamma_j \\ q^*(\beta_j) \smile q^*(\alpha_i) &= q^*(\beta_j \smile \alpha_i) = 0 & i \neq j \end{aligned}$$

Which gives us the full cup product structure for $H^*(M_g)$.

Problem 2 (3.2.2). Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is a union of n contractible open subsets, to show that the n-fold cup products of positive dimensional classes are zero.

Proof. First note that since A is contractible we get an isomorphism

$$0 \longrightarrow H^k(X, A; R) \longrightarrow H^n(X; R) \longrightarrow 0$$

Similarly we also get an isomorphism with $H^k(X, B; R)$ and $H^k(X; R)$.

Using this with the naturality of the cup product we get a commutative diagram

$$\begin{tikzcd}
H^k(X,A;R) \times H^\ell(X,B;R) \arrow[r, "\smile"] \arrow[d, "\cong"] & H^{k+\ell}(X, A\cup B; R) \cong 0 \arrow[d] \\
H^k(X;R) \times H^\ell(X;R) \arrow[r, "\smile"] & H^{k+l}(X;R)
\end{tikzcd}$$

However since this map factors through zero it must be the case that the cup product is zero for positive dimensions.

In the case where $X = \bigcup_i A_i$ we still have the same isomorphisms as before. As such our new diagram is

$$\begin{tikzcd}
\prod_i H^{k_i}(X,A;R) \arrow[r, "\smile"] \arrow[d, "\cong"] & H^{\Sigma_i k_i}(X, \bigcup A_i; R) \cong 0 \arrow[d] \\
\prod_i H^{k_i}(X;R) \arrow[r, "\smile"] & H^{\Sigma_i k_i}(X;R)
\end{tikzcd}$$

which gives us zero on the cup product for positive dimensions via the same reasoning as above. $\square$

Problem 3 (3.2.4). Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{C}P^n;\mathbb{Z})$. [See Exercise 3 in §2.C for an example of a map without fixed points in this exceptional case.]

Proof. Recall that $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}(\alpha)/(\alpha^{n+1})$ and that $H^k(\mathbb{C}P^n;\mathbb{Z})$ is $\mathbb{Z}$ for even dimensions and zero otherwise. For $f^* : H^2(\mathbb{C}P^n;\mathbb{Z}) \to H^2(\mathbb{C}P^n;\mathbb{Z})$ this is an endomorphism on the integers. As such it must be of the form $f^*(\alpha) = c\alpha$ for some $c \in \mathbb{Z}$. Moreover since f^* is also an endomorphism on the cohomology ring it must be that $f^*(\alpha^m) = c^m\alpha^m$. Since the Ext term in the universal coefficient theorem will be zero for $\mathbb{C}P^n$ we have that f^* is the hom dual of f_*. Since multiplication maps are unaffected by hom dual the Lefschetz number of f is

$$\tau(f) = 1 + c + \cdots + c^n$$

The only possible rational roots, and thus integer roots, of $\tau(f)$ are ± 1. Since all the coefficients are positive 1 is not a root. If n is even -1 cannot be a root as well since there are an odd number of terms. If n is odd and $f^*(\alpha) \neq -\alpha$ (i.e. $c \neq -1$), then $\tau(f)$ will have no roots. Therefore by the Lefschetz fixed point theorem a map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if n is even or if n is odd and $f(\alpha) \neq -\alpha$. $\square$