Problem 1 (3.2.3). (a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding results for maps $\mathbb{C}P^n \to \mathbb{C}P^m$?

(b) Prove the Borsuk-Ulam theorem by the following argument. Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all x. Then define $g : S^n \to S^{n-1}$ by $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$, so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Show that part (a) applies to this map.

Proof. □

Problem 2 (3.2.5). Show the ring $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ is isomorphic to $\mathbb{Z}_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ where $|\alpha| = 1$ and $|\beta| = 2$. [Use the coefficient map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ and the proof of Theorem 3.19.]

Proof. □

Problem 3 (3.2.7). Use the cup products to show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

Proof. From Hatcher the cup product structure for $\mathbb{R}P^3$ with $\mathbb{Z}_2$ coefficients is

$$H^*(\mathbb{R}P^3; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta]/(\alpha^4) \quad |\alpha| = 1$$

and the cup product structure for $\mathbb{R}P^2 \vee S^3$ with $\mathbb{Z}_2$ coefficients is

$$H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^2; \mathbb{Z}_2) \times H^*(S^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2[\beta]/(\beta^3) \times \mathbb{Z}_2[\gamma]/(\gamma^2))/(\langle 1_\beta, 1_\gamma \rangle) \quad |\beta| = 1, |\gamma| = 3$$

The latter has an element, β, which when cubed is zero. However the former has no such elements. Thus the two cohomology rings are not isomorphic and therefore $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$. □