Problem 1 (3.1.1)**.** *Show that* $\operatorname{Ext}(H,G)$ *is a contravariant functor of* H *for fixed* G *and covariant for fixed* H.

Proof. Let $f : A \to B$ be an R-module homomorphism and let H be a fixed R-module with a projective resolution P:

$$\cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} H \longrightarrow 0$$

Then apply $\operatorname{Hom}_R(H,-)$ with both A and B and to f for each entry in the exact sequence for the projective resolution. This gives us two chain complexes with a map at each group

$$\begin{tikzcd}
\cdots & P_2^A \arrow[l] \arrow[d, "g_2^*"] & P_1^A \arrow[l, "f_2^A"'] \arrow[d, "g_1^*"] & P_0 \arrow[l, "f_1^A"'] \arrow[d, "g_0^*"] & H^A \arrow[l, "f_0^A"'] \arrow[d, "g^*"] & 0 \arrow[l] \\
\cdots & P_2^B \arrow[l] & P_1^B \arrow[l, "f_2^B"'] & P_0 \arrow[l, "f_1^B"'] & H^B \arrow[l, "f_0^B"'] & 0 \arrow[l]
\end{tikzcd}$$

where $H^A := \operatorname{Hom}_R(H,A)$ and $g^* : \operatorname{Hom}_R(H,A) \to \operatorname{Hom}_R(H,B)$.

Now we will show that g_i^* forms a chain map. However if we write it out we get

$$\begin{aligned}
f_n^B \circ g_n^*(h) &= g_{n+1}^* \circ f_n^A(h) \\
f_n^B(g \circ h) &= g_{n+1}^*(h \circ f_n) \\
(g \circ h) \circ f_n &= g \circ (h \circ f_n)
\end{aligned}$$

which are equal by associativity of function composition. Since this is a chain map this induces a homomorphism on homology which is exactly $\operatorname{Ext}(H,A) \to \operatorname{Ext}(H,B)$. This will preserve composition since we are sending functions to functions. Therefore $\operatorname{Ext}(H,-)$ is a covariant functor.

To show that $\operatorname{Ext}(-,G)$ is a contravariant functor we trace out the procedure gone through in class. Let A and B be R-modules with projective resolutions P and Q respectively. Let $f : A \to B$ be an R-module homomorphism. Then extend f to a chain map α in the form

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2^A"] \arrow[d, "\alpha_2"] & P_1 \arrow[r, "\partial_1^A"] \arrow[d, "\alpha_1"] & P_0 \arrow[r, "\partial_0^A"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2^B"] & Q_1 \arrow[r, "\partial_1^B"] & Q_0 \arrow[r, "\partial_0^B"] & B \arrow[r] & 0
\end{tikzcd}$$

Then we dualize with $\operatorname{Hom}(-,G)$ giving us two chain complexes with a chain map α^*

$$\begin{tikzcd}
\cdots & P_2^* \arrow[l, "\partial_2^A"] & P_1^* \arrow[l, "\partial_1^A"] & P_0 \arrow[l, "\partial_0^A"] & A \arrow[l] & 0 \arrow[l] \\
\cdots & Q_2 \arrow[l, "\partial_2^B"] \arrow[u, "\alpha_2^*"] & Q_1 \arrow[l, "\partial_1^B"] \arrow[u, "\alpha_1^*"] & Q_0 \arrow[l, "\partial_0^B"] \arrow[u, "\alpha_0^*"] & B \arrow[l] \arrow[u, "f^*"] & 0 \arrow[l]
\end{tikzcd}$$

We showed in class that α^* is in fact a chain map for the new complexes. As such it induces a homomorphism on homology which is $g_* : \operatorname{Ext}(B,G) \to \operatorname{Ext}(A,G)$. This shows that $\operatorname{Ext}(-,G)$ is indeed a contravariant functor as composition is preserved for the same reason as above. $\square$

Problem 2 (3.1.2)**.** *Show that the maps* $G \xrightarrow{n} G$ *and* $H \xrightarrow{n} H$ *multiplying each element by the integer* n *induce multiplication by* n *in* $\operatorname{Ext}(H,G)$.

Proof. Let A be a generating set for G. Then we have a free resolution of G of the form

$$0 \longrightarrow \ker(f) \xrightarrow{i} F(A) \xrightarrow{f} G \longrightarrow 0$$

where i is inclusion, f is the evaluation map, and $F(A)$ is the free group on A. Fortunately the map that multiplies by n has a lift where it is also multiplication by n. Thus we have

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, "i"] \arrow[d, "n"] & F(A) \arrow[r, "f"] \arrow[d, "n"] & G \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & \ker(f) \arrow[r, "i"] & F(A) \arrow[r, "f"] & G \arrow[r] & 0
\end{tikzcd}$$

Then we dualize to get

$$\begin{tikzcd}
0 & \ker(f)^* \arrow[l] & F(A) \arrow[l, "i^*"'] & G^* \arrow[l, "f^*"'] & 0 \arrow[l] \\
0 & \ker(f)^* \arrow[l] \arrow[u, "n^*"] & F(A)^* \arrow[l, "i^*"'] \arrow[u, "n^*"] & G^* \arrow[l, "f^*"'] \arrow[u, "n^*"] & 0 \arrow[l]
\end{tikzcd}$$

Now we need to know what the n^* maps are. However as it turns out if we have a map in one of the groups in the above diagram h. Then $n^*(h)(x) = h(nx) = nh(x)$ which implies that n^* is once again the multiplication by n map. Since it is the multiplication map everywhere the induced map on homology will also be the multiplication by n map. Thus $\mathrm{Ext}(-, H)(\cdot n) = \cdot n$.

Similarly if we use $\mathrm{Ext}(G, -)$ we dualize first and then place the n^* maps in. However since these maps are endomorphisms we get the same diagram

$$\begin{tikzcd}
0 & \ker(f)^* \arrow[l] \arrow[d, "n^*"] & F(A)^* \arrow[l, "i^*"'] \arrow[d, "n^*"] & G^* \arrow[l, "f^*"'] \arrow[d, "n^*"] & 0 \arrow[l] \\
0 & \ker(f)^* \arrow[l] & F(A)^* \arrow[l, "i^*"'] & G^* \arrow[l, "f^*"'] & 0 \arrow[l]
\end{tikzcd}$$

which must have the same then induce the same maps giving $\mathrm{Ext}(G, -)(\cdot n) = \cdot n$ $\square$

Problem 3 (3.1.3). *Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.*

We can construct a free resolution of $\mathbb{Z}_2$ of the form

$$\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

When we dualize with $\mathrm{Hom}(-, \mathbb{Z}_2)$ we get

$$\cdots \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \longleftarrow 0$$

Which has nonzero homology groups everywhere. As such $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2)$ is nonzero for all n.