

Problem 1 (3.2.1). Assuming as known the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

Proof. □

Problem 2 (3.2.2). Using the cup product $H^k(X, A; R) \times H^\ell(X, B; R) \rightarrow H^{k+\ell}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B , then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. This applies in particular if X is a suspension. Generalize to the situation that X is a union of n contractible open subsets, to show that the n -fold cup products of positive dimensional classes are zero.

Proof. First note that since A is contractible we get an isomorphism

$$0 \longrightarrow H^k(X, A; R) \longrightarrow H^n(X; R) \longrightarrow 0$$

Similarly we also get an isomorphism with $H^k(X, B; R)$ and $H^k(X; R)$.

Using this with the naturality of the cup product we get a commutative diagram

$$\begin{array}{ccc} H^k(X, A; R) \times H^\ell(X, B; R) & \xrightarrow{\smile} & H^{k+\ell}(X, A \cup B; R) \cong 0 \\ \downarrow \cong & & \downarrow \\ H^k(X; R) \times H^\ell(X; R) & \xrightarrow{\smile} & H^{k+\ell}(X; R) \end{array}$$

However since this map factors through zero it must be the case that the cup product is zero for positive dimensions.

In the case where $X = \bigcup_i A_i$ we still have the same isomorphisms as before. As such our new diagram is

$$\begin{array}{ccc} \prod_i H^{k_i}(X, A_i; R) & \xrightarrow{\smile} & H^{\sum_i k_i}(X, \bigcup_i A_i; R) \cong 0 \\ \downarrow \cong & & \downarrow \\ \prod_i H^{k_i}(X; R) & \longrightarrow & H^{\sum_i k_i}(X; R) \end{array}$$

which gives us zero on the cup product for positive dimensions via the same reasoning as above. □

Problem 3 (3.2.4). Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$. [See Exercise 3 in §2.C for an example of a map without fixed points in this exceptional case.]

Proof. Recall that $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}(\alpha)/(\alpha^{n+1})$ and that $H^k(\mathbb{C}P^n; \mathbb{Z})$ is \mathbb{Z} for even dimensions and zero otherwise. For $f^* : H^2(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^2(\mathbb{C}P^n; \mathbb{Z})$ this is an endomorphism on the integers. As such it must be of the form $f^*(\alpha) = c\alpha$ for some $c \in \mathbb{Z}$. Moreover since f^* is also an endomorphism on the cohomology ring it must be that $f^*(\alpha^m) = c^m \alpha^m$. Since the Ext term in the universal coefficient theorem will be zero for $\mathbb{C}P^n$ we have that f^* is the hom dual of f_* . Since multiplication maps are unaffected by hom dual the Lefschetz number of f is

$$\tau(f) = 1 + c + \cdots + c^n$$

The only possible rational roots of $\tau(f)$ are ± 1 . Since all the coefficients are positive 1 is not a root. If n is even -1 cannot be as well. If n is odd and $f^*(\alpha) \neq -\alpha$ ($c \neq -1$), then $\tau(f)$ has no roots as well.

Therefore by the Lefschetz fixed point theorem the map f has a fixed point if n is even or if $f(\alpha) \neq -\alpha$. \square