Problem 1 (1). *For the map $\phi(x) = x\sin(x)$ of the real line to itself, what are the regular values?*

Since ϕ is a map from $\mathbb{R}$ to $\mathbb{R}$ we can acquire the critical points via the usual method from calculus. Thus the regular values the points that are not solutions to

$$\sin(x) + x\cos(x) = 0$$

Problem 2 (2). *For the map $\phi(x, y) = x^2 - y^2$ of the plane to the line, what are the regular values?*

The pushforward for ϕ will be $\phi_*(x, y) = (2x, -2y)$. Since we are mapping from 2 dimensions to 1 the regular values will be those where ϕ_* is surjective. This will be the points where neither x nor y are 0.

Problem 3 (3). *For the map $\phi(x, y) = \sin(x^2 + y^2)$ of the plane to the line, what are the regular values?*

First rewrite the function as $\phi(r) = \sin(r^2)$. Then if we take the derivative we get $\phi'(r) = 2r\cos(r^2)$. Thus the critical values are when $r = 0$ or when $r = \pm\sqrt{\frac{(2k+1)\pi}{2}}$ for $k \in \mathbb{Z}$. Translating back to x, y we have that the regular values are when the equations $x^2+y^2 = 0$ and $x^2+y^2 = \frac{(2k+1)\pi}{2}$ for $k \in \mathbb{Z}$ do not hold.

Problem 4 (5). *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve in the plane. Let K be the set of all $r \in \mathbb{R}$ such that the circle of radius r about the origin is tangent to the curve γ at some point. Show that K has empty interior in $\mathbb{R}$.*

Proof. Define $\delta(t) = |\gamma(t)|$. Then the critical points of δ are precisely the points in K. By Sard's theorem the critical points of δ are of measure zero. Thus K is of measure and cannot contain any intervals. It then follows that no point in K has an open neighborhood around it and as such the interior of K is empty. □

Problem 5 (6). *If C is a circle embedded smoothly in $\mathbb{R}^4$, show that there exists a three-dimensional hyperplane H such that the orthogonal projection of C to H is an embedding.*

Proof. Let $\gamma : C \to \mathbb{R}^4$ be a smooth embedding of C into $\mathbb{R}^4$. □