Problem 1 (2). *If the curve $\phi : \mathbb{R} \to \mathbb{R}^n$ is an embedding then show that $\phi_*(d/dt)$ coincides with the classical notion of the tangent vector to the curve ϕ under the identification of the tangent space to a euclidean space with the euclidean space.*

Proof. The traditional definition of a tangent vector in Euclidean space for $\phi(t) = (\phi_1(t), \ldots, \phi_n(t))$ would be

$$\phi'(t) = (\phi_1'(t), \ldots, \phi_n'(t))$$

However if we identify the tangent space as euclidean space then the pushforward $\phi_* : \mathbb{R} \to \mathbb{R}^n$ will be

$$\phi_* \left(\frac{d}{dt} \right) = \frac{d\phi}{dt} = (\phi_1', \ldots, \phi_n')$$

which shows that the two notions agree. □

Problem 2 (3). *For a smooth function f defined on a neighborhood of a point $p \in \mathbb{R}^n$, the gradient $\nabla f = \mathrm{grad} f$ of f is the vector*

$$\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

For a vector $v \in \mathbb{R}^n$ show that the directional derivative D_v, denoted by D_{γ_v} where $\gamma_v(t) = p + tv$, satisfies the equation

$$D_v f = \langle \nabla f, v \rangle$$

the standard inner product of ∇f with v in $\mathbb{R}^n$.

Proof. First note that by definition that

$$\langle \nabla f, v \rangle = \frac{\partial f}{\partial x_i} v^i$$

Likewise for the other term we have

$$D_{\gamma_v} f = \frac{d}{dt} f(\gamma_v(t))|_{t=0} = \frac{\partial f}{\partial x_i} \frac{d\gamma_v}{dt}|_{t=0} = \frac{\partial f}{x_i} v^i$$

which completes the proof. □

Problem 3 (4). *If $M^m \subset \mathbb{R}^n$ is a smoothly embedded manifold and f is a smooth real valued function defined on a neighborhood of $p \in M^m$ in $\mathbb{R}^n$ and which is constant on M, show that ∇f is perpendicular to $T_p(M)$ at p.*

Proof. Since our manifold is smoothly embedded in $\mathbb{R}^n$ we can treat the tangent space as a subspace of $\mathbb{R}^n$. Let $(v_1, \ldots, v_m)$ be a basis for $T_p(M)$. Then each of the $\nabla f(p) \cdot v_i$ will be zero as f is constant on M and v_i is a tangent vector for M. Since $\nabla f(p)$ is perpendicular to all vectors in the basis for $T_p(M)$ it is perpendicular to $T_p(M)$ itself. □