Problem 1. *Consider the real valued function $f(x,y,z) = (2-(x^2+y^2)^{1/2})^2+z^2$ on $\mathbb{R}^3\setminus\{(0,0,z)\}$. Show that 1 is a regular value of f. Identify the manifold $M = f^{-1}(1)$.*

First we calculate the partials

$$\frac{\partial f}{\partial x} = x\left(2 - \frac{4}{(x^2+y^2)^{1/2}}\right), \quad \frac{\partial f}{\partial y} = y\left(2 - \frac{4}{(x^2+y^2)^{1/2}}\right), \quad \frac{\partial f}{\partial z} = 2z$$

The critical points of f are those for which the rank of f_* is zero. This requires that $z = 0$. So if 1 were to be a critical value it would have to occur when $(x^2+y^2)^{1/2}$ is either 1 or 3. However neither of these would force all the partial derivatives to be zero. As such 1 is a regular value of f.

The manifold is the torus T^2.

Problem 2. *Show that the manifold M of Problem 1 is transverse to the surface*

$$N = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2 = 4\}$$

Identify the manifold $M \cap N$.

The points of intersection are

$$N \cap M = \{(x,y,z) | z = \pm 1, x^2+y^2 = 4\}$$

Given any point $p \in \mathbb{R}^3$ the tangent space $T_p(\mathbb{R}^3)$ is equivalent to $\mathbb{R}^3$. Define the function $h_+ : T^2 \to \mathbb{R}$ as the height of a point $p \in T^2$ with respect the given embedding. We can write this as

$$h_+(x,y) = \sqrt{-\left(\sqrt{x^2+y^2}-2\right)^2+1}$$

By taking the partial derivatives with respect to x and y we get

$$\frac{\partial f}{\partial x} = -\frac{x\left(\sqrt{x^2+y^2}-2\right)}{\sqrt{x^2+y^2}\sqrt{-\left(\sqrt{x^2+y^2}-2\right)^2+1}}, \quad \frac{\partial f}{\partial y} = -\frac{y\left(\sqrt{x^2+y^2}-2\right)}{\sqrt{x^2+y^2}\sqrt{-\left(\sqrt{x^2+y^2}-2\right)^2+1}}$$

which are both zero when $x^2+y^2 = 4$. Thus the height is not changing on that circle and it then follows that the tangent plane for such a point is parallel to the xy-plane. However since N is a cylinder in the z direction the tangent plane at any point will include the z direction. As such at any point of the intersection of M, N for $z > 0$ we have that $TM + TN = \mathbb{R}^3$. Identical reasoning follows for $z < 0$. Thus the manifolds N and M are transverse.

The manifold is two disjoint copies of S^1.

Problem 3. *Show that the manifold M of Problem 1 is not transverse to the surface*

$$N = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2 = 1\}$$

Is $M \cap N$ a manifold?

The points of intersection are

$$N \cap M = \{(x,y,z) = z = 0, x^2+y^2 = 1\}$$

Using the partials we calculated in problem one. The normal vector for any point $(x,y,0) \in T^2$ along the intersection will be $\langle -2x, -2y, 0\rangle$. Similarly for N the normal vector will be $\langle 2x, 2y, 0\rangle$. However the planes defined by these normal vectors will be parallel and as such their sum cannot be $\mathbb{R}^3$. Thus the manifolds M and N are not transverse.

It is a manifold and it is a single copy of S^1.