Problem 1 (1). *Show that a second countable Hausdorff space X with a functional structure F is an n-manifold if, and only if, every point in X has a neighborhood U such that there are functions $f_1, \ldots, f_n \in F(U)$ such that: a real valued function on g on U is in $F(U)$ if, and only if, there exists a smooth function $h(x_1, \ldots, x_n)$ of n real variables $\ni g(p) = h(f_1(p), \ldots, f_n(p))$ for all $p \in U$.*

Proof. Let X be a second countable Hausdorff space with a functional structure F. Suppose that X is an n-manifold and let $x \in X$ be a point with $U \subset X$ an open neighborhood of X. Then by definition we have an algebra $F(U)$. Since $F(U)$ is an algebra we have a basis $(f_1, \ldots, f_n)$. Let $g(p) = \sum c_i f_i(p)$ for $p \in U$. Define $h = (x_1, \ldots, x_n) = \sum c_i x_i$. Then $g(p) = h(f_1(p), \ldots, f_n(p))$ for all $p \in U$. If g were not in $F(U)$ then it would not be smooth. As such it could not be of the form $h(f_1(p), \ldots, f_n(p))$ as if it were it would be smooth. Therefore if X is an n-manifold then $g \in F(U)$ if, and only if, there is a smooth h of n real variables such that $g(p) = h(f_1(p), \ldots, f_n(p))$.

Now suppose that given a real valued g on U that $g \in F(U)$ if, and only if, there is a smooth function h such that $g(p) = h(f_1(p), \ldots, f_n(p))$ for all $p \in U$. Let $x \in X$ and let U be the corresponding neighborhood of x where the above holds. Define $\phi : X \to V \subset \mathbb{R}^n$ via

$$\phi(p) = (f_1(p), \ldots, f_n(p))$$

This map is continuous since each f_i is continuous. Moreover if $h \in C^\infty(V)$ then

$$h \circ \phi = h(f_1, \ldots, f_n)$$

which places $h \circ \phi \in F_X(U = \phi^{-1}(V))$ by our assumption. Thus ϕ is a morphism of functional structures from $(X, F_X(U))$ to (V, C^∞).

In order to construct the inverse we will need to apply the inverse function theorem.

Use a modification of ϕ to get the inverse with the inverse function theorem and compose to get a well defined inverse Give a bit on why this must cause also be a morphism of functional structures.

Thus ϕ is a bijection. Now consider $g \in F(U)$.

Then

$$g \circ \phi^{-1}(x) = h(f_1 \circ \phi^{-1}(x), \ldots, f_n \circ \phi^{-1}(x)) \in F(V)$$

Which shows that ϕ is in fact an isomorphism on functionally structured spaces and as such X is locally isomorphic to $(\mathbb{R}^n, C^\infty)$ and thus a smooth manifold. □

Problem 2 (3). *Show that a map $f : M \to N$ between smooth manifolds, with functional structures F_M and F_N, is smooth in the sense of definition 2.5 if, and only if, it is smooth in the sense of definition 2.4.*

Proof. First note that given a chart $\phi : U \to \mathbb{R}^n$ that ϕ is a diffeomorphism from U to $\phi(U)$ with the smooth structure on $\phi(U)$ being generated by the identity map id_U. To verify this we have that $id_U \circ \phi \circ \phi^{-1} = id_U$ is smooth since the id_U is smooth. Checking that ϕ^{-1} is smooth is similar as we check $\phi \circ \phi^{-1} \circ id_U = id_U$ which is smooth.

Let ϕ, ψ be charts for M, N respectively and let $f : M \to N$ be a map that is smooth in the sense that if $g \in F_N(V)$ for $V \subset N$ open then $g \circ f \in F_M(f^{-1}(V))$. Consider $\psi \circ f \circ \phi^{-1}$. Since both ϕ, ψ are diffeomorphisms from their domain to their image they and their inverses will be smooth. It then follows that if V is the domain of ψ that $\psi \in F_N(V)$. By our assumption we then have that $\psi \circ f \in F_M(f^{-1}(V))$ implying that $\psi \circ f$ is smooth. Since smoothness of functions is preserved by composition we have that $\psi \circ f \circ \phi^{-1}$ is smooth.

Now suppose that $f : M \to N$ was smooth in the sense that $\psi \circ f \circ \phi^{-1}$ is smooth wherever it is defined. Due to the union property of functional structures it will suffice to show that $g \in F_N(V)$ implies $g \circ f \in F_M(f^{-1}(V))$ for a V sufficiently small such that both V and f^{-1} fall in the domains of single charts ψ, ϕ. Let $g \in F_N(V)$. Then g is smooth which implies that $g \circ \psi^{-1}$ is smooth. By our assumption $\psi \circ f \circ \phi^{-1}$ is smooth. Then if we compose we get that

$$(g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ \phi = g \circ f$$

is smooth. Thus $g \circ f \in F_M(f^{-1}(V))$.

Therefore the two definitions for smooth maps are equivalent. $\square$

Problem 3 (4). *Let X be the graph of a real valued function $\theta(x) = |x|$ of a real variable x. Define a functional structure on X by taking $f \in F(U)$ if, and only if, f is the restriction to U of a C^∞ function on some open set V in the plane with $U = V \cap X$. Show that X with this structure is not diffeomorphic to the real line with the usual C^∞ structure.*

Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be $x + y$ rotated counterclockwise $\pi/4$ so that the former x, y axes line up with U. Then $f|_U \equiv |r|$ for $r \in U$ which implies that $|\cdot| \in F(X)$ however this function is not smooth on the real line. Thus X and $\mathbb{R}$ with the usual structure are not diffeomorphic.