

Problem 1. Consider the real valued function $f(x, y, z) = (2 - (x^2 + y^2)^{1/2})^2 + z^2$ on $\mathbb{R}^3 \setminus \{(0, 0, z)\}$. Show that 1 is a regular value of f . Identify the manifold $M = f^{-1}(1)$.

First we calculate the partials

$$\frac{\partial f}{\partial x} = x \left(2 - \frac{4}{(x^2 + y^2)^{1/2}} \right), \quad \frac{\partial f}{\partial y} = y \left(2 - \frac{4}{(x^2 + y^2)^{1/2}} \right), \quad \frac{\partial f}{\partial z} = 2z$$

The critical points of f are those for which the rank of f_* is zero. This requires that $z = 0$. So if 1 were to be a critical value it would have to occur when $(x^2 + y^2)^{1/2}$ is either 1 or 3. However neither of these would force all the partial derivatives to be zero. As such 1 is a regular value of f .

The manifold is the torus T^2 .

Problem 2. Show that the manifold M of Problem 1 is transverse to the surface

$$N = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 4\}$$

Identify the manifold $M \cap N$.

The points of intersection are

$$N \cap M = \{(x, y, z) | z = \pm 1, x^2 + y^2 = 4\}$$

Take the normal vector to M at one of the points of intersection using the partial derivatives above to get $\langle 0, 0, \pm 2 \rangle$ where the positive is the top circle and the bottom circle is the negative. Similarly the normal vector to N is $\langle 2x, 2y, 0 \rangle$. Since both x and y cannot be zero on the intersection and the normal vectors for each surface are not parallel we have that for $p \in M \cap N$ that $T_p M + T_p N = T_p \mathbb{R}^3$ which shows that M and N are transverse.

The manifold is two disjoint copies of S^1 .

Problem 3. Show that the manifold M of Problem 1 is not transverse to the surface

$$N = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$$

Is $M \cap N$ a manifold?

The points of intersection are

$$N \cap M = \{(x, y, z) = z = 0, x^2 + y^2 = 1\}$$

Using the partials we calculated in problem one. The normal vector for any point $(x, y, 0) \in T^2$ along the intersection will be $\langle -2x, -2y, 0 \rangle$. Similarly for N the normal vector will be $\langle 2x, 2y, 0 \rangle$. However the planes defined by these normal vectors will be parallel and as such their sum cannot be \mathbb{R}^3 . Thus the manifolds M and N are not transverse.

It is a manifold and it is a single copy of S^1 .