Problem 1 (2). *If the curve $\phi : \mathbb{R} \to \mathbb{R}^n$ is an embedding then show that $\phi_*(d/dt)$ coincides with the classical notion of the tangent vector to the curve ϕ under the identification of the tangent space to a euclidean space with the euclidean space.*

Proof. The traditional definition of a tangent vector in Euclidean space for $\phi(t) = (\phi_1(t), \dots, \phi_n(t))$ would be

$$\phi'(t) = (\phi_1'(t), \dots, \phi_n'(t))$$

However if we identify the tangent space as euclidean space then the pushforward $\phi_* : \mathbb{R} \to \mathbb{R}^n$ will be

$$\phi_* \left(\frac{d}{dt} \right) = \frac{d\phi}{dt} = (\phi_1', \dots, \phi_n')$$

which shows that the two notions agree. □

Problem 2 (3). *For a smooth function f defined on a neighborhood of a point $p \in \mathbb{R}^n$, the gradient $\nabla f = \text{grad} f$ of f is the vector*

$$\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$$

For a vector $v \in \mathbb{R}^n$ show that the directional derivative D_v, denoted by D_{γ_v} where $\gamma_v(t) = p + tv$, satisfies the equation

$$D_v f = \langle \nabla f, v \rangle$$

the standard inner product of ∇f with v in $\mathbb{R}^n$.

Proof. First note that by definition that

$$\langle \nabla f, v \rangle = \frac{\partial f}{\partial x_i} v^i$$

Likewise for the other term we have

$$D_{\gamma_v} f = \frac{d}{dt} f(\gamma_v(t))|_{t=0} = \frac{\partial f}{\partial x_i} \frac{d\gamma_v}{dt}|_{t=0} = \frac{\partial f}{x_i} v^i$$

which completes the proof. □

Problem 3 (4). *If $M^m \subset \mathbb{R}^n$ is a smoothly embedded manifold and f is a smooth real valued function defined on a neighborhood of $p \in M^m$ in $\mathbb{R}^n$ and which is constant on M, show that ∇f is perpendicular to $T_p(M)$ at p.*

Proof. Since we are working in a neighborhood of $p \in M^m$ we can work in local coordinates. Then the gradient of f will be $\nabla f = \langle \frac{\partial f}{x_1}, \dots, \frac{\partial f}{x_n} \rangle$. Moreover we have that $\nabla f(x) = 0$ for $x \in M^m$ since f is constant on M^m.

Let $\sum a^i \frac{\partial}{\partial x_i}$ be in $T_p(M^m)$. Then we have that

$$\nabla f(x) \cdot \sum a_i \frac{\partial}{x_i} = 0$$

which implies that ∇f is normal to $T_p(M^m)$. □