Problem 1 (1). *Show that a second countable Hausdorff space X with a functional structure F is an n-manifold if, and only if, every point in X has a neighborhood U such that there are functions $f_1, \ldots, f_n \in F(U)$ such that: a real valued function on g on U is in $F(U)$ if, and only if there exists a smooth function $h(x_1, \ldots, x_n)$ of n real variables $\ni g(p) = h(f_1(p), \ldots, f_n(p))$ for all $p \in U$.*

Proof. □

Problem 2 (3). *Show that a map $f : M \to N$ between smooth manifolds, with functional structures F_M and F_N, is smooth in the sense of definition 2.5 if, and only if, it is smooth in the sense of definition 2.4.*

2.3: A morphism of functionally structured spaces

$$(X, F_X) \to (Y, F_Y)$$

is a map $\phi : X \to Y$ such that the composition $f \mapsto f \circ \phi$ carries $F_Y(U)$ into $F_X(\phi^{-1}(U))$.

2.4: A morphism is a smooth map if it is a morphism of functionally structured spaces for smooth manifolds as functionally structured spaces.

2.5: A map $f : M \to N$ between two smooth manifolds is said to be smooth if, for any charts ϕ on M and ψ on N, the function $\psi \circ f \circ \phi^{-1}$ is smooth where it is defined.

Proof. Let ϕ, ψ be charts for M, N respectively and let $f : M \to N$ be a map that is smooth in the sense that if $g \in F_N(V)$ for $V \subset N$ open then $g \circ f \in F_M(f^{-1}(V))$. Consider $\psi \circ f \circ \phi^{-1}$. Since both ϕ, ψ are diffeomorphisms from their domain to their image they and their inverses will be smooth. It then follows that if V is the domain of ψ that $\psi \in F_N(V)$. By our assumption we then have that $\psi \circ f \in F_M(f^{-1}(V))$ implying that $\psi \circ f$ is smooth. Since smoothness of functions is preserved by composition we have that $\psi \circ f \circ \phi^{-1}$ is smooth.

Now suppose that $f : M \to N$ was smooth in the sense that $\psi \circ f \circ \phi^{-1}$ is smooth wherever it is defined. Let $V \subset N$ and let $V_\alpha := V \cap U_\alpha$ where $\psi_\alpha : U_\alpha \to \mathbb{R}$ is a chart of N. Then because of the union property of charts it will suffice to show that the definition of smoothness holds for one such V_α. Let $g \in F_N(V_\alpha)$. Then $g \circ f \in F_M(f^{-1}(V_\alpha))$. □

Problem 3 (4). *Let X be the graph of a real valued function $\theta(x) = |x|$ of a real variable x. Define a functional structure on X by taking $f \in F(U)$ if, and only if, f is the restriction to U of a C^∞ function on some open set V in the plane with $U = V \cap X$. Show that X with this structure is not diffeomorphic to the real line with the usual C^∞ structure.*

Proof. □