Problem 1. *On an open set $U \subset \mathbb{R}^k$ show that the exterior derivative d is the only operator $d : \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying:*

(a) $d(\omega + \eta) = d\omega + d\eta$

(b) $\omega \in \Omega^p(U), \eta \in \Omega^q(U) \Rightarrow d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$

(c) $f \in \Omega^0(U) \Rightarrow df(X) = X(f)$

(d) $f \in \Omega^0(U) \Rightarrow d(df) = 0$

Deduce that d is independent of the coordinate system used to define it.

Proof. First note that due to the linearity of d (pull out a bump function) that given a point p the value of $d\omega_p$ only relies on a neighborhood of p.

Now suppose that d' was another operator that fulfilled the same properties as d. Then given $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_p} \in \Omega^p(U)$ through repeated applications of (b) we have

$$d'(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = d' f dx_{i_1} \wedge \cdots \wedge dx_{i_p} - \sum_1^p (-1)^p f dx_{i_1} \wedge \cdots \wedge d' dx_{i_j} \wedge \cdots \wedge dx_{i_p}$$

However $d' dx_{i_j} = d' d' x_{i_j} = 0$ and $d' f = df$ hold by (c), (d) and the fact that d applied to a zero-form is not affected by coordinates we have that

$$d' f dx_{i_1} \wedge \cdots \wedge dx_{i_p} - \sum_1^p (-1)^p f dx_{i_1} \wedge \cdots \wedge d' dx_{i_j} \wedge \cdots \wedge dx_{i_p} = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} = d(f dx_{i_1} \wedge \cdots \wedge dx_{i_p})$$

which shows that $d = d'$.

Since d is unique and at a point only depends on a neighborhood of said point given two neighborhoods of a point p called U, V even if they have different coordinate systems must agree on $U \cap V$. Thus it must be the case that d is independent of coordinate systems. $\square$

Problem 2. *On the unit circle S^1 in the plane, let $\theta = \arctan(y/x)$ be the usual polar coordinate. Show that $d\theta$ makes sense on S^1 and is a closed 1-form which is not exact.*

If we calculate $d\theta$ we get

$$d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = \frac{-y dx}{x^2 + y^2} + \frac{x dy}{x^2 + y^2}$$

However since we are on the unit circle the bottom term becomes 1. Thus $d\theta = -y dx + x dy$. Since θ is not continuous when $x = 0$ we cannot directly apply $d^2 = 0$ to θ. However there are only two points where θ is discontinuous. If we throw out those points we have $d^2\theta = 0$. Since $d^2\theta$ is a smooth 2-form on S^1 we cannot have nonzero points when $x = 0$ as that would violate the smoothness. Thus $d^2\theta$ is uniformly zero which shows that $d\theta$ is closed.

To see that it is not closed evaluate the integral $\int_{S^1} d\theta$. If it were closed the integral would be zero however it is 2π. Therefore θ is not exact.

Problem 3. *For $\omega \in \Omega^1(M)$, verify the special case $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$ of the invariant formula mentioned above Definition 2.4.*

Proof. Let $\omega = f dx \in \Omega^1(M)$. Then

$$\begin{aligned}
d\omega(X,Y) &= d(f dx)(X,Y) \\
&= df \wedge dx(X,Y) \\
&= df(X)dx(Y) - df(Y)dx(X) \\
&= XfYx - YfXx \\
&= (XfYx + fXYx) - (YfXx + fYXx) - f(XYx - YXx) \\
&= X(fYx) - Y(fXx) - f[X,Y]x \\
&= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])
\end{aligned}$$

which completes the proof. □