Problem 1. *On the 2-sphere, consider the flow*

$$\theta(t, \langle x, y, z\rangle) = \langle x, y\cos(t) - z\sin(t), y\sin(t) + z\cos(t)\rangle$$

Find the vector field on S^2 induced by this flow.

Take the derivative of θ with respect to time to get

$$\frac{\partial\theta}{\partial t}(\langle x, y, z\rangle) = \langle 0, -y\sin(t) - z\cos(t), y\cos(t) - z\sin(t)\rangle$$

Then set $t = 0$ to get the vector field

$$\xi(\langle x, y, z\rangle) = \langle 0, -z, y\rangle$$

as a vector field on S^2 corresponding to the flow θ.

Problem 2. *Consider the vector field $\xi(x) = x$ on $\mathbb{R}$. Show that ξ is the tangent field to a flow, and find the flow. (Hint: In classical notation, this vector field corresponds to the initial value problem $dy/dt = y, y(0) = x$.)*

If we solve the differential equation $\frac{dy}{dt} = y$ we get $y = Ae^t$. Solving for initial conditions we get

$$\theta(t, x) = xe^t$$

as our flow.

Problem 3 (4)**.** *If X and Y vector fields on M then XY makes sense as an operator on smooth real valued functions on M. Show that $[X, Y] = XY - YX$ is a vector field. (This is called the "Lie bracket" of X and Y. Sometimes it is defined with opposite sign.) Also show that XY itself is not a vector field.*

Proof. Let $X = a^i\frac{\partial}{\partial x_i}$ and $Y = b^i\frac{\partial}{\partial x_i}$ be vector fields and let f be a smooth function. Then

$$XYf = a^j\frac{\partial Yf}{x_j} = a^j\frac{\partial(b^i\frac{\partial f}{\partial x_i})}{\partial x_j} = a^j(\frac{\partial b^i}{\partial x_j}\frac{\partial f}{\partial x_i} + b^i\frac{\partial^2 f}{\partial x_i\partial x_j}) = a^j\frac{\partial b^i}{\partial x_j}\frac{\partial f}{\partial x_i} + a^ib^i\frac{\partial^2 f}{\partial x_i\partial x_j}$$

and similarly

$$YXf = b^j\frac{\partial a^i}{\partial x_j}\frac{\partial f}{\partial x_i} + b^ia^i\frac{\partial^2 f}{\partial x_i\partial x_j}$$

Then the Lie bracket of X and Y is

$$[X, Y]f = XYf - YXf = a^j\frac{\partial b^i}{\partial x_j}\frac{\partial f}{\partial x_i} + a^ib^i\frac{\partial^2 f}{\partial x_i\partial x_j} - b^j\frac{\partial a^i}{\partial x_j}\frac{\partial f}{\partial x_i} + b^ia^i\frac{\partial^2 f}{\partial x_i\partial x_j} = (a^j\frac{\partial b^i}{\partial x_j} - b^j\frac{\partial a^i}{\partial x_j})\frac{\partial f}{\partial x_i}$$

which shows that $[X, Y]$ is in fact a vector field. □

From the above calculation one can see that XY is not a vector field as

$$XYf = a^j\frac{\partial b^i}{\partial x_j}\frac{\partial f}{\partial x_i} + a^ib^i\frac{\partial^2 f}{\partial x_i\partial x_j}$$

which is not in the proper form to be a vector field.

Problem 4 (5). *Show that the Klein Bottle has an everywhere nonzero vector field. Describe the resulting flow.*