

Problem 1. Consider the 3-sphere as the set of unit quaternions

$$\{x + iy + jz + kw \mid x^2 + y^2 + z^2 + w^2 = 1\}$$

Let $\phi : U \rightarrow \mathbb{R}^3$ be $\phi(x + iy + jz + kw) = (y, z, w) \in \mathbb{R}^3$ where

$$U = \{x + iy + jz + kw \in S^3 \mid x > 0\}$$

Consider ϕ as a chart. For each $q \in S^3$, define a chart ψ_q by $\psi_q(p) = \phi(q^{-1}p)$. Show that this set of charts is an atlas for a smooth structure on S^3 .

First note that every point $q \in S^3 \subset \mathbb{H}$ will be included in the chart ψ_q . Moreover since ψ_q will be a homeomorphism if ϕ is a homeomorphism as $p \mapsto q^{-1}p$ is continuous where defined. Thus all we need to show is that ϕ is a homeomorphism and that for $q, p \in S^3$ that $\psi_q\psi_p^{-1}$ is smooth.

The projection map ϕ is continuous. Define an inverse via

$$\phi^{-1}(y, z, w) = \sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw$$

This map is continuous as all of its components are continuous. Thus, since ϕ has a continuous inverse, it is indeed a homeomorphism. It then follows that ψ_q is a homeomorphism due to the reasoning above.

Now we will show that for $q, p \in S^3$ that $\psi_q\psi_p^{-1}$ is smooth where defined. Let $r = x + iy + jz + kw$ and $p = a + ib + jc + kd$. Then we expand $\psi_q\psi_p^{-1}$ as

$$\begin{aligned} \psi_q\psi_p^{-1}(r) &= \psi_q(p\phi^{-1}(r)) \\ &= \psi_q(p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw)) \\ &= \phi(q^{-1}p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw)) \end{aligned}$$

The expression $q^{-1}p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw)$ is smooth where defined and the projection ϕ is smooth. Therefore $\psi_q\psi_p^{-1}$ is smooth and therefore the ψ_q s form a chart for $S^3 \subset \mathbb{H}$.

Problem 2. Let X be a copy of the real line \mathbb{R} and let $\phi : X \rightarrow \mathbb{R}$ be $\phi(x) = x^3$. Taking ϕ as a chart, this defines a smooth structure on X . Prove or disprove the following statements:

1. X is diffeomorphic to \mathbb{R} ;
2. the identity map $X \rightarrow \mathbb{R}$ is a diffeomorphism;
3. ϕ together with the identity map comprise an atlas;
4. on the one-point compactification X^+ of X , ϕ, ψ give an atlas, where $\psi(x) = 1/x$, for $x \neq 0, \infty$, and $\psi(\infty) = 0$. (ψ is defined on $X^+ \setminus \{0\}$.)