Problem 1 (1). *For the map $\phi(x) = x\sin(x)$ of the real line to itself, what are the regular values?*

Since ϕ is a map from $\mathbb{R}$ to $\mathbb{R}$ we can acquire the critical points via the usual method from calculus. Thus the regular values the points that are not solutions to

$$\sin(x) + x\cos(x) = 0$$

Problem 2 (2). *For the map $\phi(x,y) = x^2 - y^2$ of the plane to the line, what are the regular values?*

The pushforward for ϕ will be $\phi_*(x,y) = (2x, -2y)$. Since we are mapping from 2 dimensions to 1 the regular values will be those where ϕ_* is surjective. This will be the points where neither x nor y are 0.

Problem 3 (3). *For the map $\phi(x,y) = \sin(x^2+y^2)$ of the plane to the line, what are the regular values?*

First rewrite the function as $\phi(r) = \sin(r^2)$. Then if we take the derivative we get $\phi'(r) = 2r\cos(r^2)$. Thus the critical values are when $r = 0$ or when $r = \pm\sqrt{\frac{(2k+1)\pi}{2}}$ for $k \in \mathbb{Z}$. Translating back to x, y we have that the regular values are when the equations $x^2+y^2 = 0$ and $x^2+y^2 = \frac{(2k+1)\pi}{2}$ for $k \in \mathbb{Z}$ do not hold.

Problem 4 (5). *Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a smooth curve in the plane. Let K be the set of all $r \in \mathbb{R}$ such that the circle of radius r about the origin is tangent to the curve γ at some point. Show that K has empty interior in $\mathbb{R}$.*

Proof. Define $\delta(t) = |\gamma(t)|$. The function δ is smooth so long as γ avoids the origin. However since γ is smooth this can only happen either a finite number of times or if γ constant at the origin. In the latter case $K = \emptyset$.

In the other case however $\mathbb{R} \setminus \{t \in \mathbb{R} | \gamma(t)\}$ is open and as such a manifold. Thus we have that δ is smooth and we can apply Sard's theorem. Then the critical points of δ are precisely the points in K without the removed points above. By Sard's theorem the critical points of δ are of measure zero. Thus K is the union of a finite set and a measure zero set and as such is of measure zero and cannot contain any intervals. It then follows that no point in K has an open neighborhood around it and as such the interior of K is empty. □

Problem 5 (6). *If C is a circle embedded smoothly in $\mathbb{R}^4$, show that there exists a three-dimensional hyperplane H such that the orthogonal projection of C to H is an embedding.*

Proof. Let $\gamma : C \to \mathbb{R}^4$ be a smooth embedding of C into $\mathbb{R}^4$. There are two obstacles to finding a hyperplane for which the projection of γ onto H is an embedding. The first is that we need to preserve injectivity of the function itself and the second is that we need to ensure that it remains an immersion.

To deal with the latter define a function

$$f(t) : S^1 \to S^3$$

via $f(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$. This function will give us the unit tangent vector to γ at time t. Moreover it is well defined and smooth since γ is an embedding and as such there will be no point wherein $|\gamma'(t)| = 0$.

Next define $g(s,t) : S^1 \times s^1 \setminus \{(t,t)|t \in S^1\} \to S^3$ as

$$G(s,t) = \frac{\gamma(s) - \gamma(t)}{|\gamma(s) - \gamma(t)|}$$

This function will also be well defined and smooth since γ is an embedding and due to us explicitly removing points where $s = t$.

Since translating a 3-dimensional hyperplane would not affect the projection of γ we can specify each hyperplane H by its unit normal vector n which is an element of S^3.

When projecting γ onto a hyperplane with normal vector n we break the immersion property if $f(t) = n$ for any t. Similarly we break injectivity if $g(s,t) = n$ for any s, t in the domain of g. However since S^3 is a 3-manifold, S^1 is a 1-manifold, $S^1 \times s^1 \setminus \{(t,t)|t \in S^1\}$ is a 2-manifold, and the fact that Sard's theorem implies that space filling curves are not smooth it must be the case that there is a vector $n \in S^3$ such that n is not in the image of either f or g. It then follows that the projection of γ onto the hyperplane specified by n is an embedding.

Therefore if C is a smoothly embedded circle in $\mathbb{R}^4$ there exists a 3-dimensional hyperplane H such that the projection of C to H is an embedding. $\square$