Problem 1. *Consider the 3-sphere as the set of unit quaternions*

$$\{x + iy + jz + kw | x^2 + y^2 + z^2 + w^2 = 1\}$$

Let $\phi : U \to \mathbb{R}^3$ *be* $\phi(x + iy + jz + kw) = (y, z, w) \in \mathbb{R}^3$ *where*

$$U = \{x + iy + jz + kw \in S^3 | x > 0\}$$

Consider ϕ *as a chart. For each* $q \in S^3$*, define a chart* ψ_q *by* $\psi_q(p) = \phi(q^{-1}p)$*. Show that this set of charts is an atlas for a smooth structure on* S^3*.*

First note that every point $q \in S^3 \subset \mathbb{H}$ will be included in the domain of its corresponding chart ψ_q. Moreover since ψ_q will be a homeomorphism if ϕ is a homeomorphism as $p \mapsto q^{-1}p$ is continuous where defined. Thus all we need to show is that ϕ is a homeomorphism and that for $q, p \in S^3$ that $\psi_q \psi_p^{-1}$ is smooth.

The projection map ϕ is continuous. Define an inverse via

$$\phi^{-1}(y, z, w) = \sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw$$

This map is continuous as all of its components are continuous. Thus, since ϕ has a continuous inverse, it is indeed a homeomorphism. It then follows that ψ_q is a homeomorphism due to the reasoning above.

Now we will show that for $q, p \in S^3$ that $\psi_q \psi_p^{-1}$ is smooth where defined. Let $r = x+iy+jz+kw$ and $p = a + ib + jc + kd$. Then we expand $\psi_q \psi_p^{-1}$ as

$$\begin{aligned}
\psi_q \psi_p^{-1}(r) &= \psi_q(p\phi^{-1}(r)) \\
&= \psi_q(p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw)) \\
&= \phi(q^{-1}p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw))
\end{aligned}$$

The expression $q^{-1}p(\sqrt{1 - y^2 - z^2 - w^2} + iy + jz + kw)$ is smooth where defined and the projection ϕ is smooth. Therefore $\psi_q \psi_p^{-1}$ is smooth and therefore the ψ_qs form an atlas for $S^3 \subset \mathbb{H}$.

Problem 2. *Let* X *be a copy of the real line* $\mathbb{R}$ *and let* $\phi : X \to \mathbb{R}$ *be* $\phi(x) = x^3$*. Taking* ϕ *as a chart, this defines a smooth structure on* X*. Prove or disprove the following statements:*

1. X *is diffeomorphic to* $\mathbb{R}$*;*
2. *the identity map* $X \to \mathbb{R}$ *is a diffeomorphism;*
3. ϕ *together with the identity map comprise an atlas;*
4. *on the one-point compactification* X^+ *of* X*,* ϕ, ψ *give an atlas, where* $\psi(x) = 1/x$*, for* $x \neq 0, \infty$*, and* $\psi(\infty) = 0$*. (*ψ *is defined on* $X^+ \setminus \{0\}$*.)*

1. The smooth manifold X will be diffeomorphic to $\mathbb{R}$ by the map ϕ. To verify this we check that
$$id_{\mathbb{R}} \circ x^3 \circ x^{-1/3} = x$$
and
$$x^3 \circ x^{1/3} \circ id_{\mathbb{R}}^{-1} = x$$
are smooth. However since they are both the identity map we have that it is the case that ϕ and ϕ^{-1} are both smooth. Thus ϕ is a diffeomorphism and it follows that X and $\mathbb{R}$ are diffeomorphic.

2. The identity map will not be a diffeomorphism as

$$id_{\mathbb{R}} \circ id \circ x^{1/3} = x^{1/3}$$

   is not a smooth map and thus *id* cannot be a diffeomorphism.

3. Together they do not form an atlas as

$$id \circ x^{1/3} = x^{1/3}$$

   is not smooth at zero.

4. Yes, the maps ϕ, ψ form an atlas on X^+. To verify this note first that every point in X^+ is in the domain of ϕ or ψ, X and $X^+ \setminus \{0\}$ respectively. We also have that x^3 and $\frac{1}{x}$ are homeomorphisms on their respective domains. Finally the maps

$$\phi\psi^{-1} = x^{-3}$$

   and

$$\psi\phi^{-1} = x^{-1/3}$$

   are both smooth so long as we avoid both 0 and ∞.

   Therefore ϕ and ψ form an atlas for X^+.