

**Problem 1.** On the 2-sphere, consider the flow

$$\theta(t, \langle x, y, z \rangle) = \langle x, y \cos(t) - z \sin(t), y \sin(t) + z \cos(t) \rangle$$

Find the vector field on  $S^2$  induced by this flow.

Define the vector field  $\xi$  on  $\mathbb{R}^3$  as  $\langle 0, -z, y \rangle$ . Then the corresponding vector field on  $S^2$  is the projection of  $\xi$  onto  $S^2$ .

**Problem 2.** Consider the vector field  $\xi(x) = x$  on  $\mathbb{R}$ . Show that  $\xi$  is the tangent field to a flow, and find the flow. (Hint: In classical notation, this vector field corresponds to the initial value problem  $dy/dt = y, y(0) = x$ .)

If we solve the differential equation  $\frac{dy}{dt} = y$  we get  $y = Ae^t$ . Solving for initial conditions we get

$$\theta(t, x) = xe^t$$

as our flow.

**Problem 3** (4). If  $X$  and  $Y$  vector fields on  $M$  then  $XY$  makes sense as an operator on smooth real valued functions on  $M$ . Show that  $[X, Y] = XY - YX$  is a vector field. (This is called the “Lie bracket” of  $X$  and  $Y$ . Sometimes it is defined with opposite sign.) Also show that  $XY$  itself is not a vector field.

*Proof.* Let  $X = a^i \frac{\partial}{\partial x_i}$  and  $Y = b^j \frac{\partial}{\partial x_j}$  be vector fields and let  $f$  be a smooth function. Then

$$XYf = a^j \frac{\partial Yf}{\partial x_j} = a^j \frac{\partial (b^i \frac{\partial f}{\partial x_i})}{\partial x_j} = a^j (\frac{\partial b^i}{\partial x_j} \frac{\partial f}{\partial x_i} + b^i \frac{\partial^2 f}{\partial x_i \partial x_j}) = a^j \frac{\partial b^i}{\partial x_j} \frac{\partial f}{\partial x_i} + a^i b^i \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and similarly

$$YXf = b^j \frac{\partial a^i}{\partial x_j} \frac{\partial f}{\partial x_i} + b^i a^i \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Then the Lie bracket of  $X$  and  $Y$  is

$$[X, Y]f = XYf - YXf = a^j \frac{\partial b^i}{\partial x_j} \frac{\partial f}{\partial x_i} + a^i b^i \frac{\partial^2 f}{\partial x_i \partial x_j} - b^j \frac{\partial a^i}{\partial x_j} \frac{\partial f}{\partial x_i} + b^i a^i \frac{\partial^2 f}{\partial x_i \partial x_j} = (a^j \frac{\partial b^i}{\partial x_j} - b^j \frac{\partial a^i}{\partial x_j}) \frac{\partial f}{\partial x_i}$$

which shows that  $[X, Y]$  is in fact a vector field.  $\square$

From the above calculation one can see that  $XY$  is not a vector field as

$$XYf = a^j \frac{\partial b^i}{\partial x_j} \frac{\partial f}{\partial x_i} + a^i b^i \frac{\partial^2 f}{\partial x_i \partial x_j}$$

which is not in the proper form to be a vector field.

**Problem 4 (5).** *Show that the Klein Bottle has an everywhere nonzero vector field. Describe the resulting flow.*

