Problem 1 (1). *Show that a second countable Hausdorff space X with a functional structure F is an n-manifold if, and only if, every point in X has a neighborhood U such that there are functions $f_1, \dots, f_n \in F(U)$ such that: a real valued function on g on U is in $F(U)$ if, and only if there exists a smooth function $h(x_1, \dots, x_n)$ of n real variables $\ni g(p) = h(f_1(p), \dots, f_n(p))$ for all $p \in U$.*

Proof. ☐

Problem 2 (3). *Show that a map $f : M \to N$ between smooth manifolds, with functional structures F_M and F_N, is smooth in the sense of definition 2.5 if, and only if, it is smooth in the sense of definition 2.4.*

2.3: A morphism of functionally structured spaces

$$(X, F_X) \to (Y, F_Y)$$

is a map $\phi : X \to Y$ such that the composition $f \mapsto f \circ \phi$ carries $F_Y(U)$ into $F_X(\phi^{-1}(U))$.

2.4: A morphism is a smooth map if it is a morphism of functionally structured spaces for smooth manifolds as functionally structured spaces.

2.5: A map $f : M \to N$ between two smooth manifolds is said to be smooth if, for any charts ϕ on M and ψ on N, the function $\psi \circ f \circ \phi^{-1}$ is smooth where it is defined.

Proof. Let ϕ, ψ be charts for M, N respectively and let $f : M \to N$ be a map that is smooth in the sense that if $g \in F_N(V)$ for $V \subset N$ open then $g \circ f \in F_M(f^{-1}(V))$. Then consider $\psi \circ f \circ \phi^{-1}$. ☐

Problem 3 (4). *Let X be the graph of a real valued function $\theta(x) = |x|$ of a real variable x. Define a functional structure on X by taking $f \in F(U)$ if, and only if, f is the restriction to U of a C^∞ function on some open set V in the plane with $U = V \cap X$. Show that X with this structure is not diffeomorphic to the real line with the usual C^∞ structure.*

Proof. ☐