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THE SKEIN METHOD FOR THREE-MANIFOLD INVARIANTS

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ABSTRACT

A new combinatorial method of establishing the quantum $SU(2)$ 3-manifold invariants is given. Nothing but Kirby's original theorem on handle-slide moves of surgery diagrams is assumed. The method leads readily to certain techniques for the invariants' calculation.

1. Introduction

The aim of this paper is to provide the shortest and easiest proof of the existence of Witten's $SU_q(2)$ 3-manifold invariants. Whether or not this is achieved is probably a matter of opinion; maybe deeper significance emanates from approaches of greater erudition. Use is made, throughout, of the linear skein theory associated with the Kauffman bracket (and hence with the Jones polynomial); the Temperley-Lieb algebra appears as an instance of that theory. The description of 3-manifolds by means of surgery on framed links in the 3-sphere S^3 is used, together with the equivalence of such links under the moves of the Kirby calculus. It is however precisely those moves, which come naturally from the consideration of sliding handles of 4-manifolds, (not the Fenn-Rourke modification into a 'local' version) that are here used. This is made possible by a new result on the skein theory of an annulus with two specified points in its boundary; this mimics handle sliding and quickly establishes the invariants. A minor modification produces the invariants of 3-manifolds with spin structure, or with a specified cohomology class, due to Blanchet [1]. This approach combines readily with the Kauffman version of 'quantised recoupling', [5], to enable certain types of calculation to be made, and in particular the invariants for the product of a surface and a circle are here calculated from a surgery presentation.

A proof, using quantum groups, of the existence of these $SU_q(2)$ 3-manifold invariants was given first by Reshetikhin and Turaev [16]; it was amplified by Kirby and Melvin [9]. A proof using skein theory, or the Temperley-Lieb algebra, was given by the author in [12] and [13] ; a version appears also in [2]. Although the method given here is also that of skein theory, it involves no algebraic manipulation at all. As a glance at what follows confirms, the accent is on simple geometric manoeuvres, and very little knowledge is required of the reader. To achieve this lack of prerequisite certain results from elsewhere are repeated here. Conversations with L.H.Kauffman and J.D.Roberts aided the section concerned with calculations, and doubtless the pervasive aroma of Topological Quantum Field Theory exerted some influence.

2. Three-dimensional Manifolds

It is a long established fact [11] that any closed connected oriented 3-manifold is the boundary of a 4-ball B^4 to which some 2-handles have been added. Such a 4-manifold is the union of B^4 and a finite number of copies of $D^2 \times D^2$ (D^2 is the 2-dimensional disc) with the copies of $\partial D^2 \times D^2$ identified with some disjoint solid tori in S^3, the boundary of B^4. Schematically this is shown in Figure 1.

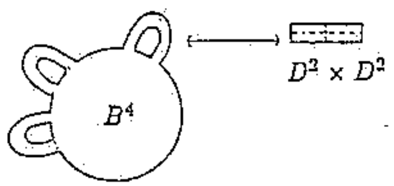


Fig.1.

Those solid tori should be thought of as a neighbourhood of a link. To effect the handle addition (that is, to make the identification) each solid torus must be parametrised to determine how many times $\partial D^2 \times D^2$ should twist when being identified with it. Thus the 4-manifold is specified by a framed link in S^3, the framing being depicted by an annulus around each component recording the amount of twisting required (see Figure 2).

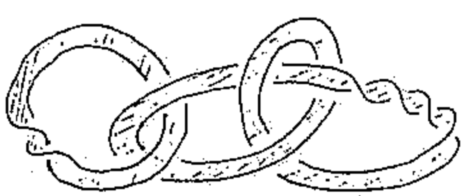


Fig. 2.

Fig. 3.

It is easier just to draw a 2-dimensional diagram of the link with the understanding that the annulus is a widening of each component *in the plane of the paper.* Extra twists of the annulus correspond to extra 'kinks' in the diagram. Figure 3 is a diagram for the framed link shown in Figure 2. Such a link diagram will be said to *represent* the 3-manifold; the 3-manifold is said to be obtained by *surgery* on S^3 along the framed link. Of course moving a framed link around in S^3 will not change things at all, and this corresponds to the equivalence of regular isotopy on representing diagrams. Regular isotopy is generated by the two (Reidemeister) moves shown in Figure 4.

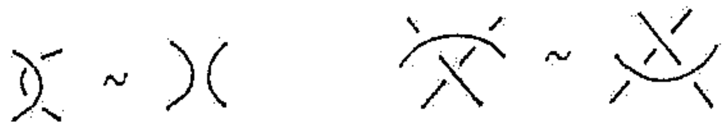


Fig. 4.

A 4-manifold of the type described above is clearly unchanged by the operation of sliding one of its handles over another one. Schematically Figure 5 shows the result of such a move on the 4-manifold of Figure 1.

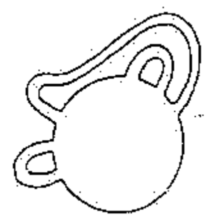


Fig. 5.

Of course this does not change the boundary 3-manifold but it does change the link diagram that represents the 3-manifold as follows. One component of the link diagram is joined by a band to a loop parallel to one of the other components (this is illustrated in Figure 6).

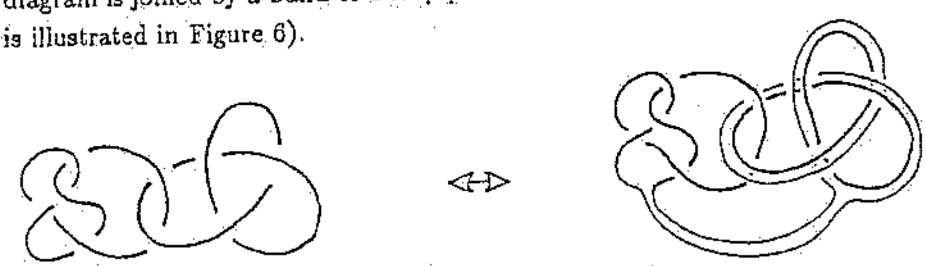


Fig. 6.

This might be thought of as dragging an arc of one component up to another and then passing it over to the far side of that component. A move of this type is called a K_2 move on the link diagram. A K_1 move consists of adding to a diagram D, or subtracting from it, a curve with precisely one crossing (see Figure 7).

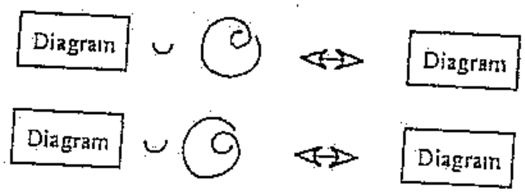


Fig. 7.

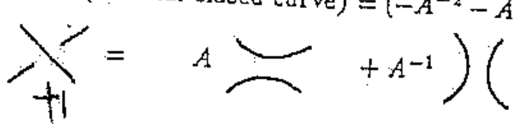
This does not change the 3-manifold though it does change the 4-manifold (by the operation of connected sum with $\pm\mathbb{C}P^2$). The theorem of Kirby [7] is that closed connected oriented 3-manifolds are equivalent if and only if any link diagrams that represent them (with respect to surgery) differ by regular isotopy and a sequence of K_1 and K_2 moves. Thus, to construct a 3-manifold invariant, it is necessary only to associate with each link diagram some algebraic concept that does not change when the diagram changes under regular isotopy or K_1 and K_2 moves. Of course any link invariant is unchanged under (regular) isotopy. It is accommodating the K_2 move that is more difficult; the K_1 move turns out to be almost a piece of administration.

3. Skein Theory

Let F be an oriented surface with a finite collection (possibly empty) of points specified in its boundary, ∂F. A *link diagram* of arcs and closed curves in F is the image of an immersion of a compact 1-manifold into F, that has as, singular set, only interior transverse double points (the crossings) to which 'over' or 'under' crossing information is associated. Further, the image of the boundary of the 1-manifold is the set of specified points in ∂F and no other point maps into ∂F. That definition is meant to contain no surprise. Two diagrams are regarded as the same if they differ by an isotopy of F relative to ∂F. The required linear skein theory of F (inspired by the Kauffman bracket [4]) is defined as follows.

Definition. *Let A be a fixed complex number. The linear skein $\mathcal{S}(F)$ of F is the vector space of formal linear sums, over $\mathbb{C}$, of link diagrams in F qotiented by the relations*

(i) $D \cup (\text{a trivial closed curve}) = (-A^{-2} - A^{2})D,$

(ii) 

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3.2 The annulus

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Here a trivial closed curve in a diagram is one that is nul-homotopic and that contains no crossing. The equation in (ii) refers to three diagrams that are identical except where shown. It is an easy excercise to show that diagrams that are regularly isotopic (that is, related by the moves of Figure 4) represent the same element of $\mathcal{S}(F)$. Although a linear skein space is associated with any oriented surface, the only surfaces needed in what follows are the plane, the annulus and the disc.

3.1 The plane

The linear skein of the plane, $\mathcal{S}(\mathbf{R}^2)$, is easily seen to be a 1-dimensional vector space with the empty diagram as a fairly natural base. ($\mathcal{S}(R^2)$ will thus be identified with $\mathbf{C}$.) This is because, by use of (ii), any link diagram in any surface can be expressed uniquely as a linear sum of diagrams with no crossing at all, and, for $\mathbf{R}^2$, it follows from (i) that those diagrams are multiples of the empty diagram. In fact this is the Kauffman bracket approach to the Jones polynomial; if D is a (zero framed) diagram of a knot K, the coordinate of $(-A^{-2}-A^2)^{-1}D$ in $\mathcal{S}(\mathbf{R}^2)$, using the empty set as base, is the Jones polynomial of K with its variable t evaluated at A^{-4}.

3.2 The annulus

The linear skein of the annulus, $\mathcal{S}(S^1 \times I)$, similarly has a base consisting of diagrams with no crossing and no nulhomotopic closed curve. Each base element is just a number of parallel curves encircling the annulus. A product of a diagram in an annulus with a diagram in another annulus can be formed by identifying together one boundary component from each annulus. This produces a third annulus containing a diagram that is the union of the two original diagrams. It is easy to see that this operation induces a well defined bilinear product on $\mathcal{S}(S^1 \times I)$ that turns it into a commutative algebra (and any choice of boundary orientation is irrelevant). Let α denote the base element that consists of one single curve encircling the annulus (Figure 8).

α

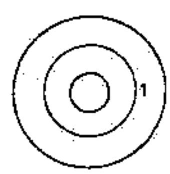


Fig. 8.

Then the base mentioned above is $\{\alpha^0, \alpha^1, \alpha^2, \ldots\}$, where α^0 denotes the empty diagram in the annulus, and $\mathcal{S}(S^1 \times I)$ is the polynomial algebra $\mathbf{C}[\alpha]$.

3.3 The Temperley-Lieb algebra

Next consider the linear skein $\mathcal{S}(D^2, 2n)$ of a disc with $2n$ points in its boundary. This has a base consisting of all diagrams with no crossing and no closed curve. There are $\frac{1}{n+1}\binom{2n}{n}$ such diagrams. Regarding the disc as a square with n points on the left edge and n on the right, a product of diagrams can be defined by juxtaposing squares. This product of diagrams extends to a well defined bilinear map that turns $\mathcal{S}(D^2, 2n)$ into an algebra, TL_n, the n^{th} Temperley-Lieb algebra. As an algebra TL_n is generated by n elements $\mathbf{1}, e_1, e_2, \ldots, e_{n-1}$ shown in Figure 9.

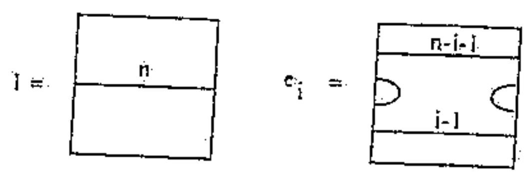


Fig. 9.

In this and later diagrams, an integer a beside an arc signifies a copies of that arc all parallel in the plane so that, for example, $\mathbf{1}$ is n parallel arcs from one side of the square to the other. Some figures will, for convenience, show the square as a rectangle!

Nothing subtle has yet occurred. It is now essential to understand the definition of the Jones-Wenzl idempotent $f^{(n)}$ of TL_n defined in [17]. In the following figures $f^{(n)}$ will be shown as a small blank square with n arcs entering and n leaving (see Figure 10); indeed, the number of such arcs is used to determine to which value of n, and hence to which Temperley-Lieb algebra, such a blank square refers.

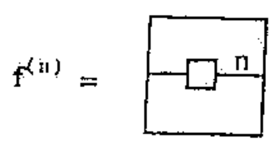


Fig. 10.

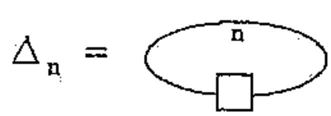


Fig. 11.

Although $f^{(n)}$ is represented by a linear sum of diagrams, it is sometimes helpful to pretend it is just one diagram! Let Δ_n be the complex number obtained by placing $f^{(n)}$ in the plane and joining the n points on the left of the square by parallel arcs to those on the right (see Figure 11). This type of definition will occur again. More precisely, Δ_n is the image of $f^{(n)}$ under the linear map $TL_n \to \mathcal{S}(\mathbb{R}^2)$ induced by mapping each diagram in the square (with $2n$ boundary points) to a planar diagram formed by the above standard joining-up process.

The element $f^{(n)}$ is defined and characterised in the following lemma.

Lemma 1. *Suppose that A^4 is not a k^{th} root of unity for $k \leq n$. Then there is a unique element $f^{(n)} \in TL_n$ such that*

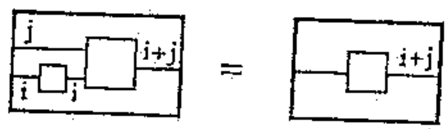
(i) $f^{(n)}e_i = 0 = e_i f^{(n)}$ *for* $0 \leq i \leq n-1$,

(ii) $(f^{(n)} - 1)$ *belongs to the algebra generated by* $\{e_1, e_2, ..., e_{n-1}\}$,

(iii) $f^{(n)} f^{(n)} = f^{(n)}$ *and*

(iv) $\Delta_n = \frac{(-1)^n(A^{2(n+1)} - A^{-2(n+1)})}{A^2 - A^{-2}}$.

Proof. Note that, if $f^{(n)}$ exists, $1 - f^{(n)}$ is the identity of the algebra generated by $\{e_1, e_2, ..., e_{n-1}\}$ and so $f^{(n)}$ is certainly unique. Suppose that $A^{4k} \neq 1$ for $k \leq n+1$, so that $\Delta_k \neq 0$ for $k \leq n$. Let $f^{(1)} = 1$, and assume that $f^{(2)}, f^{(3)}, ..., f^{(n)}$ have been defined with the above properties (i), (ii), (iii) and (iv). Observe that (i) and (ii) immediately imply (iii) and that this generalises to the identity shown in Figure 12.



by (i) and (ii)

Fig.12.

Now consider the element x, say, of TL_{n-1} shown at the start of Figure 13.

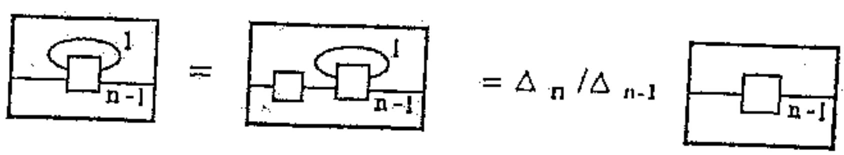


Fig.13.

The identity of Figure 12 implies that $f^{(n-1)}x = x$. But $f^{(n-1)}x$ is, by (i), just some scalar multiple of $f^{(n-1)}$ (because x is a linear sum of 1's and products of e_i's); the trick of placing squares in the plane and joining points on the left to points on the right in a standard way implies that that scalar is Δ_n/Δ_{n-1}. Now define $f^{(n+1)}$

inductively by the equation of Figure 14.

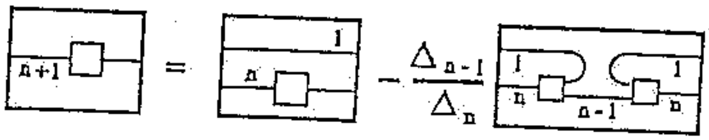


Fig.14.

Properties (i) and (ii) (and hence (iii)) for $f^{(n+1)}$ follow immediately except perhaps for the fact that $f^{(n+1)}e_n = 0$. However Figure 15 shows, using the identities of Figures 12 and 13, why that also is true.

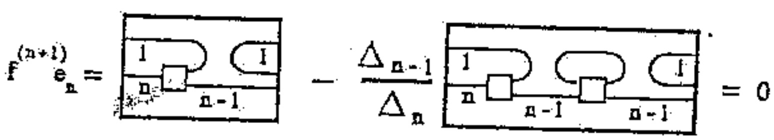


Fig.15.

Note that $f^{(0)}$ is taken to be the empty diagram and Δ_{-1} to be zero.

It remains to investigate Δ_{n+1}. Consider the operation of placing a square in an annulus and joining k points on one side to k points on the other by parallel arcs encircling the annulus. For each k that gives a linear map $TL_k \to \mathcal{S}(S^1 \times I)$. The image of $f^{(k)}$ is some polynomial $S_k(\alpha)$ in the generator α of $\mathcal{S}(S^1 \times I)$. $S_0(\alpha) = \alpha^0$ and $S_1(\alpha) = \alpha$. Inserting into the annulus, in this way, the defining relation of Figure 14 for $f^{(n+1)}$ gives the formula of Figure 16.

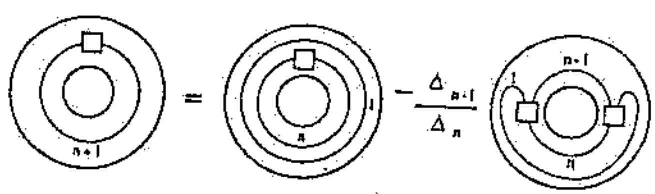


Fig.16.

However in the last diagram in Figure 16 the two small squares representing $f^{(n)}$ can be slid together to become one small square (using $f^{(n)}f^{(n)} = f^{(n)}$) and an application of the formula of Figure 13 gives

$$S_{n+1}(\alpha) = \alpha S_n(\alpha) - S_{n-1}(\alpha).$$

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This, with the above initial conditions, is the recurrence formula for the n^{th} Chebychev polynomial in α. If now the annulus is placed in the plane the ensuing linear map $\mathcal{S}(S^1 \times I) \to \mathcal{S}(\mathbf{R})$ sends α^k to $(-A^{-2} - A^2)^k$ and by definition it maps $S_k(\alpha)$ to Δ_k. Thus

$$\Delta_{n+1} = (-A^{-2} - A^2)\Delta_n - \Delta_{n-1}.$$

The induction hypothesis then shows that $\Delta_{n+1} = \frac{(-1)^{n+1}(A^{2(n+2)} - A^{-2(n+2)})}{A^2 - A^{-2}}$. □

That completes Lemma 1. The proof could have been slightly shortened by inserting the squares directly into the plane, but consideration of the annulus is important later. Attention has also been drawn to the element $S_n(\alpha) \in \mathcal{S}(S^1 \times I)$ that is $f^{(n)}$ inserted into $S^1 \times I$ with its boundary points connected up by arcs encircling the annulus.

Definition. *For a given integer r let $\omega \in \mathcal{S}(S^1 \times I)$ be defined by*

$$\omega = \sum_{n=0}^{r-2} \Delta_n S_n(\alpha).$$

3.4 The annulus with two boundary points

As a final instance of skein theory consider the linear skein of an annulus with two points specified on *one* of its boundary components, $\mathcal{S}(S^1 \times I,$ 2 points). Let $a\omega$ and $b\omega$ be the elements of $\mathcal{S}(S^1 \times I,$ 2 points) that consist of ω inserted into the annulus together with an arc, joining the two boundary points of the annulus; the arc goes 'above' ω for $a\omega$ or 'below' ω for $b\omega$ (see Figure 17). The following lemma is the new result that is the basis of this paper.

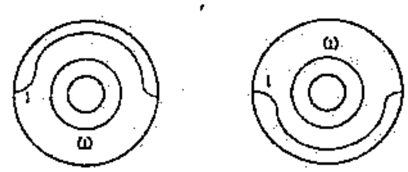
a ω  b ω

Fig.17.

Lemma 2. *In $\mathcal{S}(S^1 \times I,$ 2 points), $a\omega - b\omega$ is a linear sum of two elements each of which contains a copy of $f^{(r-1)}$. (That is, each of the two elements is the image of*

$f^{(r-1)}$ *under some map* $TL_{r-1} \to \mathcal{S}(S^1 \times I,\ 2 \text{ points})$ *formed by including a square into an annulus and joining up boundary points in some way.)*

Proof. Consider the inclusion, shown in Figure 18, of the TL_{n+1} recurrence relation of Figure 14 into the annulus.

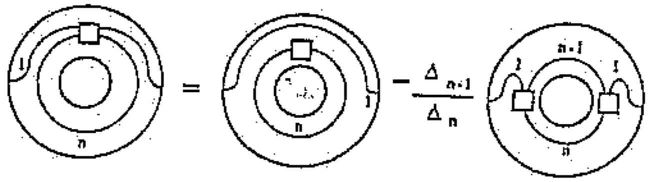


Fig.18.

The top boundary points on either side of the square are joined to the two points on the annulus boundary, the other n points on the left of the square are joined to the n on the right by parallel arcs encircling the annulus. As in the proof of Lemma 1, the two small squares in the final diagram of Figure 18 can be slid together ($f^{(n)}f^{(n)} = f^{(n)}$) to become one square and the equality can then be rearranged to become that of Figure 19.

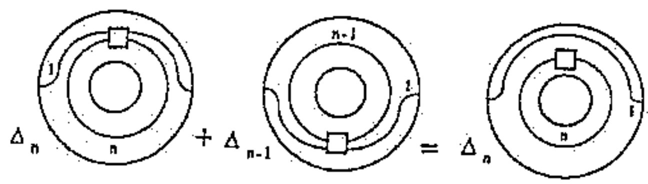


Fig.19.

Sum these equalities from $n = 0$ to $n = r-2$. The right-hand side is $a\omega$. Rotate each annulus of Figure 19 through π and sum again. The right-hand side is now $b\omega$. The left-hand sides of the formulae so obtained are almost the same; recalling that $\Delta_{-1} = 0$, the difference of these left-hand sides is the difference of the first term of Figure 19, when $n = r-2$, and its rotation; in each is a copy of $f^{(r-1)}$. □

4. Invariants

If D is a planar diagram of a link of n ordered components, D defines a multilinear map

$$< \ , \ ,\dots, \ >_D : \mathcal{S}(S^1 \times I) \times \mathcal{S}(S^1 \times I) \times \dots \times \mathcal{S}(S^1 \times I) \to \mathcal{S}(\mathbb{R}^2).$$

The map is induced (by multilinearity) by the operation of taking n diagrams in n annuli and immersing the annuli, with their diagrams, in the plane as a regular

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a scalar multiple of $f^{(r-}$
□

Corollary. *If A^4 is a pr*
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neighbourhood of the components of D. Over and under crossing information from D becomes over and under crossing information for the immersed annuli and the diagrams they contain. A simple example is shown in Figure 20.

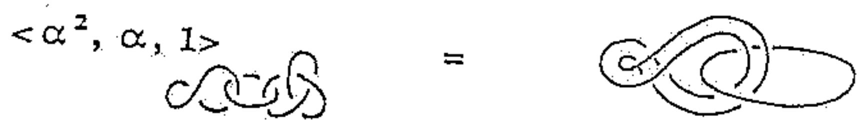

Fig.20.

Lemma 3. *Suppose that A is chosen so that A^4 is a primitive r^{th} root of unity, $r \geq 3$. Suppose that D is a planar diagram of a link of n (ordered) components and that D' is another such diagram obtained from D by a K_2 move (see Figure 6) in which a parallel of the first component of D is joined by some band to another component. Then*

$$<\omega, \;, \dots, \; >_D = <\omega, \;, \dots, \; >_{D'} .$$

Proof. It must be checked that the elements of $\mathcal{S}(\mathbb{R}^2)$, produced as described above from D and from D', with ω as the 'diagram' around the first component and any given diagrams around the others, are in fact the same element. The difference between them is the result of a sequence of moves, each consisting of moving, by regular isotopy, an arc of some component up to that labelled with ω, and changing from an immersed copy of $a\omega$ to one of $b\omega$. By Lemma 2, this difference is a linear sum of elements of $\mathcal{S}(\mathbb{R}^2)$ each containing a copy of $f^{(r-1)}$. However in $\mathcal{S}(\mathbb{R}^2)$ any element containing a copy of $f^{(r-1)}$ is *zero* if A^4 is a primitive r^{th} root of unity. That is because such an element is, for some $x \in TL_{r-1}$, the image in $\mathcal{S}(\mathbb{R}^2)$ of $f^{(r-1)}x$ under the map induced by placing the square in the plane and joining the $r-1$ points on the left to those on the right by parallel arcs. As usual, $f^{(r-1)}x$ is a scalar multiple of $f^{(r-1)}$, but $f^{(r-1)}$ maps to Δ_{r-1} which, as $A^{4r} = 1$, is zero. □

Corollary. *If A^4 is a primitive r^{th} root of unity, $r \geq 3$, and planar diagrams D and D' are related by a sequence of K_2 moves then*

$$<\omega, \omega, \dots, \omega >_D = <\omega, \omega, \dots, \omega >_{D'} .$$

4.1 The invariant for an oriented 3-manifold

In what follows U_+, U and U_- will be planar diagrams representing the unknot with framings $+1$, 0 and -1 respectively. (Recall that a diagram representing a given *framed* link is unique up to regular isotopy.) Note, at once, that the definition of ω implies that $<\omega>_U = \sum_{n=0}^{r-2} \Delta_n{}^2$. When A^4 is a primitive r^{th} root of unity, the substitution $\Delta_n = \frac{(-1)^n(A^{2(n+1)}-A^{-2(n+1)})}{A^2-A^{-2}}$, produces the formula

$$<\omega>_U = \frac{-2r}{(A^2-A^{-2})^2}.$$

It is convenient to defer the proof of the next lemma for a page or two.

Lemma 4. *Suppose $r \geq 3$.*

(i) *If A is a primitive $4r^{th}$ root of unity, then*

$$<\omega>_{U_+}<\omega>_{U_-} = <\omega>_U .$$

(ii) *If A is a primitive $2r^{th}$ root of unity and r is odd then*

$$<\omega>_{U_+}<\omega>_{U_-} = 2<\omega>_U .$$

(iii) If A is a primitive rth root r odd $<\omega>_{U_\pm} = 0$

Now comes almost at once the theorem (first proved in other forms by other authors, [18], [16]) asserting the existence of a certain 3-manifold invariant. First, note that to an oriented framed link there corresponds a *linking matrix*, namely the symmetric matrix with entries the linking numbers between the pairs of components of the link. The self linking number of a component is taken to be its linking number with a parallel copy of itself in the annulus that defines the framing. This matrix changes by congruence under K_2 moves so its numbers of positive and of negative eigenvalues do not change under K_2 moves, nor do these change if different orientations on the link's components are chosen.

Theorem 5. *Suppose that a closed oriented 3-manifold M is obtained by surgery on a framed link that is represented by a planar diagram D. Let b_+ be the number of positive eigenvalues and b_- the number negative eigenvalues of the linking matrix of this link. Suppose $r \geq 3$ and that either A is a primitive $4r^{th}$ root of unity or A is a primitive $2r^{th}$ root of unity and r is odd. Then*

$$<\omega, \omega, \ldots, \omega>_D <\omega>_{U_+}^{-b_+} <\omega>_{U_-}^{-b_-}$$

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Lemma 6. *Suppose*
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$1 \leq n \leq r-3$. When

is a well defined invariant of M.

Proof. Note that either condition on A implies that A^4 is a primitive r^{th} root of unity. As has just been explained, the above expression is invariant under K_2 moves. The last two factors make it invariant under K_1 moves, and regular isotopy of D induces regular isotopies of all the diagrams used in defining the expression. Finally note that Lemma 4 implies that neither $< \omega >_{U_-}$ nor $< \omega >_{U_+}$ is zero. □

This invariant (at least when A is a primitive $4r^{th}$ root of unity) is essentially the $SU_q(2)$ invariant of M at a 'level' coresponding to r. Observe however that if ω is replaced throughout by $\mu\omega$ where μ is a constant complex number, then another invariant is obtained. The new invariant is the old one multiplied by μ raised to the power of the first Betti number of M. Later it will be more convenient to use such a renormalisation. Perhaps it should be mentioned that it is easy to increase the apparent sophistication of this whole approach by taking A to be an indeterminant symbol rather than a complex number and by working with modules over $\mathbf{Z}[A, A^{-1}]$ rather than vector spaces, quotienting when appropriate by a cyclotomic polynomial. One can also re-phrase things in terms of the skein theory of framed links in 3-manifolds rather than link diagrams in surfaces.

The properties of ω will now be explored further in order to give a simple proof of Lemma 4. If the square, with n points specified on each of its two sides, is placed in the plane, each element of TL_n can be regarded as a linear map to $\mathbf{C}$ of the linear skein of the outside of the square. This is induced by taking a diagram inside and a diagram outside the square and regarding the union of the two as an element of $\mathcal{S}(\mathbf{R}^2) = \mathbf{C}$. (Working with S^2 rather than $\mathbf{R}^2$ would be more satisfying here as then the outside skein is another copy of TL_n.) As has already been noted, if $r \geq 3$ and A^4 is a primitive r^{th} root of unity, $f^{(r-1)}$ defines the zero map of outsides although it is not the zero element of TL_{r-1}.

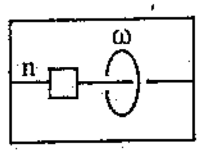

Fig.21.

Lemma 6. *Suppose $r \geq 3$ and either (i) A is a primitive $4r^{th}$ root of unity, or (ii) A is a primitive $2r^{th}$ root of unity and r is odd. The element of TL_n shown in Figure 21 (that consits of $f^{(n)}$ encirled by an ω) is the zero map of outsides if $1 \leq n \leq r-3$. When $n = (r-2)$ it is also the zero map in case (i) but is the same* *as the map given by $<\omega>_U f^{(r-2)}$ in case (ii). When $n = 0$ the element acts as multiplication by $<\omega>_U$ in both cases.*

Proof. The formula in TL_n shown in Figure 22 is an easy excercise in skein calculations, [12]; it is true whenever $f^{(n)}$ is defined.

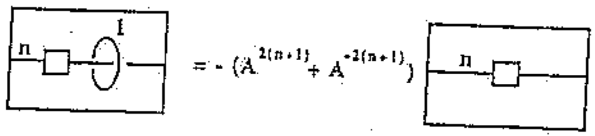

Fig.22.

Considering diagrams as maps of outsides of the square permits, by Lemma 2, the manoeuvre of sliding a curve over another one that is labelled ω. Figure 23 employs this manoeuvre and it is apparent that the map in question is zero unless $A^{-2} + A^2 = A^{-2(n+1)} + A^{2(n+1)}$. That is unless $A^{2n} = 1$ or $A^{2(n+2)} = 1$. In case (i) this does not occur for $1 \leq n \leq r-2$. In case (ii) it occurs only when $n = (r-2)$. But then placing a curve beside $f^{(r-2)}$ or encircling $f^{(r-2)}$ with it both multiply $f^{(r-2)}$ by $-A^{-2} - A^2$. Thus placing any element of $\mathcal{S}(S^1 \times I)$ (ω, for example) beside $f^{(r-2)}$ has the same effect as encircling $f^{(r-2)}$ with it. □

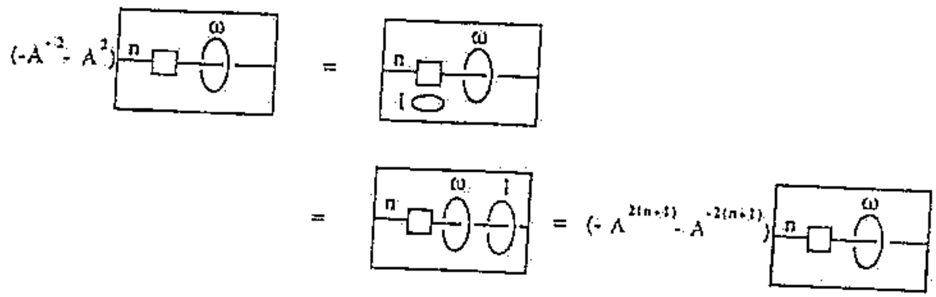

Fig.23.

Proof of Lemma 4. Figure 24 shows, using the Corollary to Lemma 3, that $<\omega>_{U_+} <\omega>_{U_-}$ is equal to a component with one crossing labelled ω simply linked with a component with no self-crossing also labelled ω.

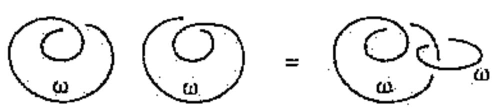

Fig.24.

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Definition. *Suppose*
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4.2 The invariant fo

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By definition the ω on the first component is $\sum_{n=0}^{r-2} \Delta_n S_n(\alpha)$ and $S_n(\alpha)$ is $f^{(n)}$ inserted into the annulus and joined around the annulus by n parallel arcs. In case (i), by Lemma 6, the linking curve labelled ω makes to be zero each term of the summation except the first (when $n = 0$). Thus $< \omega >_{U_+} < \omega >_{U_-} = < \omega >_U$. In case (ii) the $(r-2)^{th}$ term of the summation also makes a contribution, namely $< \omega >_U$ multiplied by the element of C represented by the loop with just one positive crossing annotated by $f^{(r-2)}$ (see Lemma 6). An easy skein theory exercise, [13], shows this multiplier to be $(-1)^{r-2} A^{(r-2)^2+2(r-2)} \Delta_{r-2}$ which is one when r is odd and $A^r = -1$.

□ $\langle W \rangle_U = u^{-2}$

Now choose $\mu \in \mathrm{C}$ so that $\mu^{-2} = < \omega >_{U_+} < \omega >_{U_-}$. This means that $< \mu\omega >_{U_+} = < \mu\omega >_{U_-}^{-1}$. The following renormalisation of the above invariant can then be written in terms of the signature of a linking matrix; it is this renormalisation that produces some elegant evaluations.

Definition. *Suppose $r \geq 3$ and that either A is a primitive $4r^{th}$ root of unity or A is a primitive $2r^{th}$ root of unity and r is odd. Let M be a closed oriented 3-manifold. Define the invariant $\mathcal{I}_A(M)$ by*

$$\mathcal{I}_A(M) = < \mu\omega, \mu\omega, ..., \mu\omega >_D < \mu\omega >_{U_-}^{\sigma} < \mu\omega >_U^{-1}$$

where σ is the signature of the linking matrix of a link diagram D that is a surgery diagram for M.

As easy examples, consider the manifolds S^3 and $S^1 \times S^2$. The empty diagram represents S^3 so $\mathcal{I}_A(S^3) = < \mu\omega >_U^{-1}$. This is $(A^2 - A^{-2})(-2r)^{-\frac{1}{2}}$ in the first of the above cases and $(A^2 - A^{-2})(-r)^{-\frac{1}{2}}$ in the second. The diagram U represents $S^1 \times S^2$ so $\mathcal{I}_A(S^1 \times S^2) = 1$. $\checkmark$

4.2 The invariant for a framed link in a 3-manifold

Suppose L is a framed ordered link of $\#L$ components in a closed oriented 3-manifold M. As before, M has a planar surgery diagram D, but now extra components may be added (linking in with D in possibly a complicated way) to form a link diagram $D \cup E$ with E representing the framed link L. Any two such diagrams representing the pair (M, L) differ only by (i) regular isotopy, (ii) K_1 and K_2 moves on D that avoid E and (iii) K_2 moves in which a component of E is joined by a band to a parallel of a component of D (the result being a component of the 'new E'). The proof of Theorem 5 extends at once to show that $\mathcal{I}_A(M, L)$, as defined below, is an invariant of the pair (M, L).

Definition. *Suppose $r \geq 3$ and that either A is a primitive $4r^{th}$ root of unity or A is a primitive $2r^{th}$ root of unity and r is odd. Let L be a framed link with ordered components in a closed oriented 3-manifold M and suppose the link diagram $D \cup E$ represents (M, L). Define $\mathcal{I}_A(M, L)$ to be the linear form on $\mathcal{S}(S^1 \times I)^{\otimes \# L}$ given by*

$$\mathcal{I}_A(M, L) = < \mu\omega, \mu\omega, \ldots, \mu\omega, \ , \ , \ldots, \ >_{D \cup E} < \mu\omega >_{U_-}^{\sigma} < \mu\omega >_{U}^{-1},$$

where σ is the signature of the linking matrix of D and the blanks occur in the positions corresponding to the components of E.

Note that if a complex number is required instead of a form, it might be considered natural to insert α into each blank in the expression for $\mathcal{I}_A(M, L)$.

4.3 Invariants for oriented spin 3-manifolds

This section shows how the invariants of Blanchet, [1], for 3-manifolds with spin structure or specified cohomology class, can be rapidly established using the preceding techniques; [1] contains much interpretation not repeated here. Consider again Lemma 2 and the definition of $\omega \in \mathcal{S}(S^1 \times I)$.

Definition. *For a given integer r let ω_0 and ω_1 be the elements of $\mathcal{S}(S^1 \times I)$ defined by*

$$\omega_0 = \sum_{\substack{n=0 \\ n \text{ even}}}^{r-2} \Delta_n S_n(\alpha), \quad \omega_1 = \sum_{\substack{n=0 \\ n \text{ odd}}}^{r-2} \Delta_n S_n(\alpha).$$

Of course, $\omega = \omega_0 + \omega_1$, and the two elements of $\mathcal{S}(S^1 \times I,\ 2 \text{ points})$ shown in Figure 17 can be changed to $a\omega_0$ or $a\omega_1$, $b\omega_0$ or $b\omega_1$ by replacing ω with ω_0 or ω_1. Then, a 'spin' version of Lemma 2 is as follows.

Lemma 2*. *In $\mathcal{S}(S^1 \times I,\ 2 \text{ points})$, each of $a\omega_0 - b\omega_1$ and $a\omega_1 - b\omega_0$ is a multiple of an element containing a copy of $f^{(r-1)}$.*

A proof of this is an immediate adaptation of a proof of Lemma 2. This result means that if D is a planar diagram of a link and D' is D with the second component banded to a parallel of the first, then $< \omega_0, \alpha, \ldots, \ >_D = < \omega_1, \alpha, \ldots, \ >_{D'}$, provided A^4 is a primitive r^{th} root of unity. Note that the fact that any $S_{2n}(\alpha)$ is an even polynomial in α and $S_{2n+1}(\alpha)$ is an odd polynomial implies that ω_0 is an even polynomial and ω_1 is odd. Thus, in the above circumstance, $<\omega_i, \omega_j, \ldots, \quad >_D = <\omega_{i+j}, \omega_j, \ldots, >_{D'}$, where $i+j$ is interpreted modulo 2.

Spin structures are defined elsewhere; an account is in [8]. Here, just note that if a 3-manifold M is obtained by surgery on a framed link L, a spin structure on M specifies a sublink K of L such that $K.X = X.X$ for every sublink X of L, where the dot denotes linking number modulo 2. If $L_1, L_2, \ldots, L_n$ are the components of L define $c(i) = 1$ if $L_i \in K$ and $c(i) = 0$ otherwise. Suppose W is the 4-manifold obtained by adding 2-handles to the 4-ball along L and let $x_i \in H_2(W; \mathbf{Z}_2)$ be the element represented by the core of the i^{th} handle together with the cone on L_i in the 4-ball. Then, coresponding to K is the class $\sum_{i=1}^{n} c(i)x_i$. It is clear how this class changes if the handle on L_j is slid over that on L_i, namely, $c(i)$ changes to $c(i)+c(j)$ modulo 2 and the other $c(k)$'s are unchanged. Hence, if D is a diagram representing L and A^4 is a primitive r^{th} root of unity,

$$<\omega_{c(1)}, \omega_{c(2)}, \ldots, \omega_{c(n)}>_D$$

is not changed by handle-sliding (that is K_2) moves, as slidings change the allocation of ω_0's and ω_1's in exactly the same way that they change the $c(i)$'s.

The element $\sum_{i=1}^{n} c(i)x_i \in H_2(W; \mathbf{Z}_2)$ is an obstruction to extending the spin structure of $M = \partial W$ over the whole of W. The spin structure does not extend over the extra 2-handle of a K_1 move so, if L_{n+1} is introduced by such a move, $c(n+1) = 1$. Now direct calculation in [1] shows that $<\omega_1>_{U_+}<\omega_1>_{U_-}$ is non-zero if A is a primitive $4r^{th}$ root of unity, r not congruent to 2 modulo 4. Thus, with the usual notation,

$$<\omega_{c(1)}, \omega_{c(2)}, \ldots, \omega_{c(n)}>_D <\omega_1>_{U_+}^{-b_+}<\omega_1>_{U_-}^{-b_-}$$

is an invariant of the spin manifold M.

Similarly, a given element of $H^1(M; \mathbf{Z}_2)$ coresponds to an element of $H_2(W; \mathbf{Z}_2)$. If this is written as $\sum_{i=1}^{n} c(i)x_i$ (the $\{x_i\}$ being the base mentioned above), then the $c(i)$'s behave as before under handle sliding moves. This time, an additional $(n+1)$'st handle, introduced by a K_1 move, makes no contribution and so $c(n+1) = 0$. But, see [1], if A is a primitive $4r^{th}$ root of unity, r not divisible by four, $<\omega_0>_{U_+}<\omega_0>_{U_-} \neq 0$. Hence, for these values of A,

$$<\omega_{c(1)}, \omega_{c(2)}, \ldots, \omega_{c(n)}>_D <\omega_0>_{U_+}^{-b_+}<\omega_0>_{U_-}^{-b_-}$$

is an invariant of M together with the specified cohomology class.

5. Calculations for the Product of a Surface with S^1

What follows in this section includes a combinatorial version of some of the calculations of Kirillov and Reshetikhin [10]. To begin with, a result concerning a specific calculation in $\mathcal{S}(\mathbf{R}^2)$ is required. Figure 25 shows copies of $f^{(x+y)}$, $f^{(y+z)}$ and $f^{(z+x)}$ inserted into the plane with sets of parallel arcs joining them together.

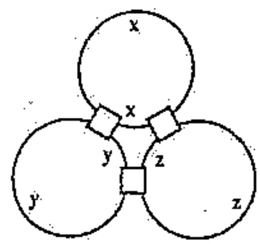


Fig.25.

Definition. *The complex number $\theta(x+y, y+z, z+x)$ is the element of $\mathcal{S}(\mathbf{R}^2)$ shown in Figure 25.*

It is shown in [14], and in [6] and [15], by a slightly intricate induction argument (which is not difficult once the answer is known) that

$$\theta(x+y, y+z, z+x) = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!},$$

where $\Delta_n! = \Delta_n\Delta_{n-1}\ldots\Delta_0$ and $\Delta_{-1} = 1$.

In everything that follows, A will be chosen so that certainly A^4 is a primitive r^{th} root of unity. Then it is immediate from the above formula that $\theta(x+y, y+z, z+x) \neq 0$, if $x+y+z \leq r-2$, and $\theta(x+y, y+z, z+x) = 0$, if $x+y+z = r-1$. $\Delta_{r-1} = 0$

Now, in the following figures, a triad of lines labelled a, b, c coming together in a black dot will be used as shorthand for the map of outsides defined by a disc with $a+b+c$ points on its boundary containing the element (of the skein of such a pointed disc) shown in Figure 26.

Fig.26.

There, $a = y + z$, $b = z + x$ and $c = x + y$. This triad was introduced by Kauffman [5]. A triple (a, b, c) of integers is said to be admissible if such a triad exists and is not the zero map. Note that the presence of the $f^{(a)}$, $f^{(b)}$ and $f^{(c)}$ inside the triad means that the only outside diagram, that might conceivably give a non-zero evaluation with this triad, is that which produces Figure 25. The evaluation is then $\theta(a, b, c)$ and the above fomula determines when that is zero. Thus admissibility is as follows.

Definition. *A triple of non-negative integers (a, b, c) is admissible if $a + b + c$ is even, $a + b + c \leq 2(r - 2)$ and $a \leq b + c$, $b \leq c + a$ and $c \leq a + b$.*

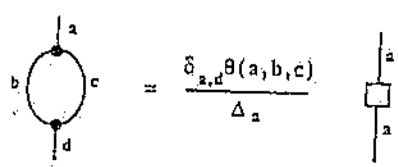


Fig.27.

A useful identity is that shown in Figure 27. The Kronecker delta function occurs because if, say, $a > d$ then the copy of $f^{(a)}$ in the top triad must, in an expansion of the remainder of the diagram in diagrams with no crossing, always abut some 'e_i', as in each such diagram, some curve leaving the top must return to the top. When $a = d$ the left diagram must be some scalar multiple of $f^{(a)}$, and the multiplier is readily found by joining, in the plane, points at the top of the diagram to points at the bottom.

As remarked in [5] (see also [14]), for fixed a, b, c and d, the space of all maps of outsides, spanned by diagrams consisting of Figure 29 with any diagram placed in the inner circle, has a base consisting of all double triads, as shown in Figure 28, with j varying such that (a, b, j) and (j, c, d) are admissible. The quantum $6j$-symbols are the change of base matrix of the formula in Figure 30.

Fig.28.

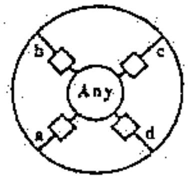


Fig. 29.

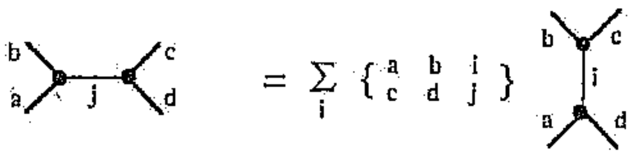


Fig.30.

In Figure 31 is a picture of a strands with an $f^{(a)}$ inserted beside b strands with an $f^{(b)}$. As a map of outsides that must, by the above result be expressible as a linear sum of basis elements. The coefficients of the linear sum are determined by adjoining an (a,b,c)-triad and using Figure 27.

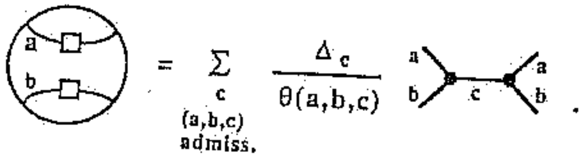
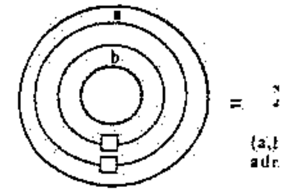


Fig.31.

Lemma 7. *The two elements $S_a(\alpha)S_b(\alpha)$ and $\sum_c S_c(\alpha)$, the summation being over all c with (a,b,c) admissible, of $\mathcal{S}(S^1 \times I)$ are equal as maps of outsides. (That is, for any planar diagram D, $< \ , \ , \dots, \ >_D$ does not change if one of the variables is changed from one of these elements to the other.)*



Proof. The proof is
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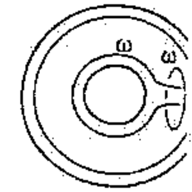
Figure 33 shows an

$\beta =$ 

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Theorem 8. *Let F_g be*
primitive $4r^{th}$ root of un

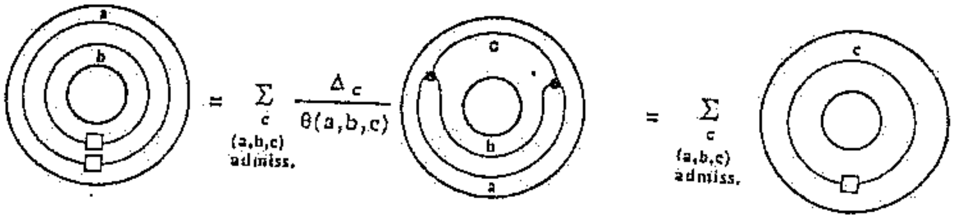


Fig.32.

Proof. The proof is shown in Figure 32 where Figure 31 is first applied at the top of the diagram, then Figure 27 at the bottom. □

Figure 33 shows an element of $\mathcal{S}(S^1 \times I)$ that will be denoted β.

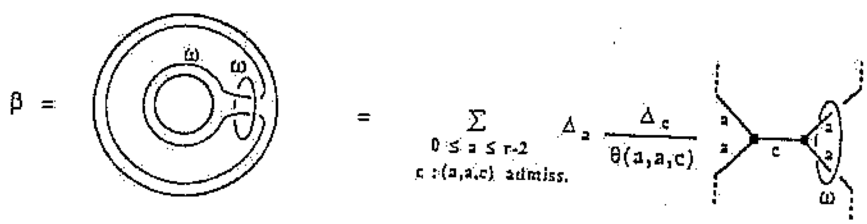


Fig.33.

Regarding β as a map of outsides, expanding one of the ω's as $\sum_{a=0}^{r-2} \Delta_a S_a(\alpha)$ and using Figure 31, a summation expression for β is obtained; this is also depicted in Figure 33 (there, the dotted lines are to be understood to encircle the annulus). But, by Lemma 6, if A is a primitive $4r^{th}$ root of unity the only non-zero contribution to that expression is when $c = 0$. Thus as maps

$$\beta = \sum_{a=0}^{r-2} < \omega >_U (S_a(\alpha))^2.$$

This is also true, by Lemma 6, when A is a primitive $2r^{th}$ root of unity and r is odd, because no triple of the form $(a, a, r-2)$ is admissible. This will be used in the proof of the next result.

Theorem 8. *Let F_g be the closed orientable surface of genus g and let A be a primitive $4r^{th}$ root of unity, $r \geq 3$. Then $\mathcal{I}_A(S^1 \times F_g)$ is an integer, namely the* *number of ways of labelling the diagram of Figure 34 with integers a_i, $0 \leq a_i \leq r-2$ so that the three labels at any node form an admissible triple.*

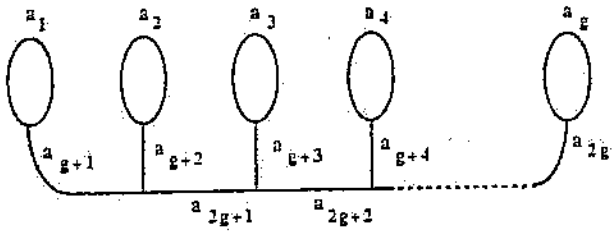


Fig.34.

Proof. $S^1 \times F_g$ is obtained by surgery on a link that consists of g copies of the Borromean rings connect-summed together on one component, each component having the zero framing. A diagram D for such a link is obtained by taking g copies of an annulus containing the link of Figure 33, threading an unknotted closed curve through the annuli and then taking the resultant link of $2g+1$ components. Then $< \omega, \omega, ..., \omega >_D = < \omega, \beta^g >_H$ where H is the diagram of the simple Hopf link of two curves and two crossings. Thus, as the signature of the linking matrix of this link is zero (and using $< \mu\omega >_U = \mu^{-1}$),

$$\mathcal{I}_A(S^1 \times F_g) = \mu^{2g+2} < \omega, \beta^g >_H .$$

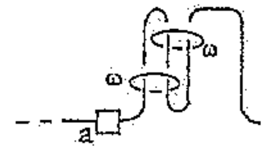
Now $\beta = \sum_{a=0}^{r-2} \mu^{-2}(S_a(\alpha))^2$, so β^g can be expressed as a sum of the $S_n(\alpha)$'s by Lemma 7. Then, by Lemma 6, $< \omega, \beta^g >_H$ is $\mu^{-2-2g} N$ where N is the number of times $S_0(\alpha)$ appears in the expansion (by Lemma 7) for $(\sum_{a=0}^{r-2}(S_a(\alpha))^2)^g$ as a sum of the $S_n(\alpha)$'s. A moment's reflection should equate N with the number of admissible labellings of the diagram of Figure 34. □

As explained in [3], [14], this result is to be expected, as this N is the dimension of the vector space associated to F_g by the relevant Topological Quantum Field Theory. Another evaluation (attributed to Verlinde) of this number can easily be given in the following way.

Theorem 9. *let A be a primitive $4r^{th}$ root of unity, $r \geq 3$.* $\mathcal{I}_A(S^1 \times F_g) = (-2r)^{g-1} \sum_{a=0}^{r-2} (A^{2(a+1)} - A^{-2(a+1)})^{2-2g}$.

Proof. The sum of Borromean rings that is a surgery diagram for $S^1 \times F_g$ is the diagram on the left of Figure 34 (ignore the annotation) repeated (along the dotted lines) g times in a circle to give a diagram D. Now insert $S_a(\alpha)$ along the last 'circle' component and ω along all the other components. Figure 35 shows the

result of performing twice th



Thus

$< \omega, \omega, ..., \omega, S_a($

So,

$\mathcal{I}_A(S^1 \times F_g) = \mu$

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Note that the same proo primitive $2r^{th}$ root of unity, r one obtains

$\mathcal{I}_A(S^1 \times F_g) = 2^{-g}$

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result of performing twice the manoeuvre first shown in Figure 33.

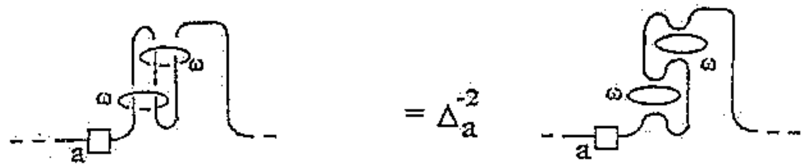

Fig.35.

Thus

$$< \omega, \omega, ..., \omega, S_a(\alpha) >_D = (\Delta_a^{-2} < \omega >_U^2)^g \Delta_a = \Delta_a^{1-2g}\mu^{-4g}.$$

So,

$$\mathcal{I}_A(S^1 \times F_g) = \mu^{2g+2} < \omega, \omega, ..., \omega >_D = \mu^{2-2g} \sum_{a=0}^{r-2} \Delta_a^{2-2g},$$

and substitution of the formulae for μ^2 and Δ_a gives the stated formula. □

Note that the same proofs give versions of Theorems 8 and 9 when A be a primitive $2r^{th}$ root of unity, r being odd. With N being the same integer as before, one obtains

$$\mathcal{I}_A(S^1 \times F_g) = 2^{-g} N = \frac{1}{2}(-r)^{g-1} \sum_{a=0}^{r-2} (A^{2(a+1)} - A^{-2(a+1)})^{2-2g}.$$

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Introduction

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