STOCHASTIC PROCESSES

TOPICS:

- Definitions: path, filtration, stopping time, finite dimension distribution
- Conditional expectation
- Existence of processes with given finite distribution
- Martingales
- Markov chains
- Stationary and exchangeable sequences
- Random walks
- Brownian motion
- Poisson process

DEFINITIONS

STOCHASTIC PROCESS = ANY collection of RVs

Given :

- Probability space : $(\Omega , \mathcal{A} , P)$
- Measurable space : $(S , \mathcal{B})$ → S is a set and $\mathcal{B}$ is a σ-field of subsets of S
- T : any set

A SP is ANY collection of RV, namely

$$X = \{ \overbrace{X_t}^{\text{single RV}} : t \in T \} = \text{family of RV}$$

where $\underbrace{X_t : \Omega \longrightarrow S}_{\text{definition of}}$ is a RV $\forall t \in T$

Being a RV, X_t should be measurable, which means that the set

$$\overbrace{X_t^{-1} (B) \in \mathcal{A}}^{\text{inverse image belong to } \mathcal{A}} \quad \forall B \in \mathcal{B}$$

where $X_t^{-1} (B) = \{ \omega \in \Omega : X_t (\omega) \in B \}$

Also S is said to be the STATE SPACE of the process, and the most imp. case is when $S = R$ and $\mathcal{B}$ = Borel σ-field on $\mathbb{R}^1$

T is an arbitrary set, usually called the INDEXING

SET on the PARAMETER SPACE of the process.
The natural choice is **TIME**. X_t is RV at time t

↳ T finite or countable → X DISCRETE time process
↳ T interval of $\mathbb{R}$ → X CONTINUOUS time process

A process is actually a function of 2 variables

$$X : \Omega \times t \longrightarrow S$$

and so we write $X(\omega, t) = X_t(\omega)$. For instance, fix $t \in T$, X_t is a RV, i.e. a measurable function on Ω, then we have

$$X_t(\omega) \quad \forall \omega \in \Omega$$

Fixing $\omega \in \Omega$, we obtain a function of t, namely

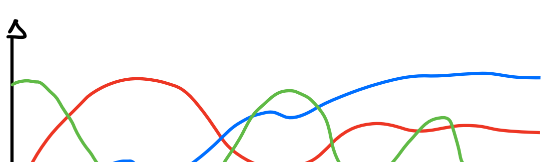
$$t \longmapsto X_t(\omega)$$ → PATH

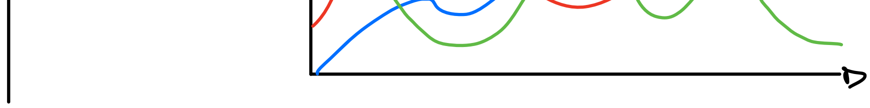
→ or trajectory

PATH = the function of t obtained by fixing $\omega \in \Omega$

EXAMPLE

If $S = \mathbb{R}$ and $T = [0, \infty)$, a PATH would be a function from $[0, \infty)$ to $\mathbb{R}$



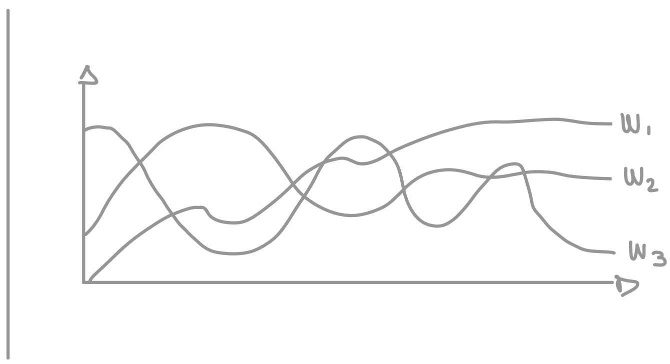


A process can be always regarded as a <u>random function</u>. It suffices to think of X as the map

$$\omega \longmapsto \text{path associated to } \omega = \underbrace{X(\omega, \cdot)}_{t \longmapsto X(\omega, t)}$$

A RV is like drawing balls from an urn, a stochastic process entail drawing FUNCTIONS!

EXAMPLE



$\Omega = \{\omega_1, \omega_2, \omega_3\}$

You pick an object from Ω
PROCESS: result of drawing

EQUALITY OF THE PROCESS

Two process X and Y can be equal on different in several senses

1) $X \sim Y \iff (X_{t_1}, \dots X_{t_n}) \sim (Y_{t_1}, \dots Y_{t_n}) \quad \forall n \geq 1 \quad \forall t_1, \dots t_n \in T$

2) X equivalent to $Y \iff P(X_t \neq Y_t) = 0 \quad \forall t \in T$

3) X indistinguible from Y provided

$$\Updownarrow$$

$$X_t(\omega) = Y_t(\omega) \quad \forall t \in T \quad \forall \omega \in A$$

where $A \in \mathcal{A}$ is such that $P(A) = 1$

then

$$(3) \;\substack{\longrightarrow \\ \not\longleftarrow}\; (2) \;\substack{\longrightarrow \\ \not\longleftarrow}\; (1)$$

EXAMPLE

$X = Y$ according to (2) but $X \neq Y$ acc. to (3)

Let V be a RV s.t. $V \geq 0$ and $P(V = v) = 0$ $\forall v \geq 0$

↳ for instance $V = |Z|$ where $Z \sim N(0, 1)$

Define $X(t, \omega) = 0 \quad \forall t \geq 0 \quad \forall \omega \in \Omega$

$$Y(t, \omega) \begin{cases} 1 & \text{if } t = V(\omega) \\ 0 & \text{otherwise} \end{cases}$$

Then X and Y are not indistinguible, in fact for $t = V(\omega)$, we get

$$Y(\omega, t) = 1 \neq 0 = X(t, \omega)$$

However

$$P(X_t \neq Y_t) = P(Y_t = 0) = P(V = t) = 0$$

hence X and Y are equivalent

STOPPING TIMES

Let $T = \{0, 1, 2, \dots\}$. A **FILTRATION** is an increasing sequence of sub-σ-fields of $\mathcal{A}$, that is

$$\mathcal{F} \subset \mathcal{F} \subset \mathcal{F} \subset \dots \subset \mathcal{A}$$

A **STOPPING TIME** is a map which takes

$$T : \Omega \longrightarrow \{+\infty, 0, 1, 2, \dots\}$$
$$\text{s.t. } \{T = n\} \in \mathcal{F}_n \quad \forall n \geq 0$$

In general a σ-field $\mathcal{G} \subset \mathcal{A}$ may be used to describe our state of information. It suffices to assume that an event A is known to be true or false $\forall A \in \mathcal{G}$

EXAMPLE

$\mathcal{G} = \{\emptyset, \Omega\} \to$ our info is null
$\mathcal{G} = \{\emptyset, \Omega, A, A^{-1}\} \to$ we know if event A happened
$\mathcal{G} = \mathcal{A} \to$ we have info for all events

A normal interpretation of filtration is time.
Over time we increase information

$$\underset{\text{info time 0}}{\mathcal{F}_0} \subset \underset{\text{info time 1}}{\mathcal{F}_1} \subset \underset{\text{info time 2}}{\mathcal{F}_2} \subset \dots \subset \mathcal{A}$$

The stopping time T should be regarded as the 1st time when something happens

$T = n \to$ happened at time n
$T = +\infty \to$ not happened

EXAMPLE

(X_n) sequence of real RV $\qquad A \in \mathcal{B}(\mathbb{R})$

$$T = \inf\{n : X_n \in A\} = \text{first time } n \text{ s.t. } X_n \in A$$

$$\{T = n\} = \{X_j \notin A \ \forall j<n, X_n = A\}$$

$X_1, X_2, \dots X_{n-1} \notin A, \ X_n \in A$

$$\{T = +\infty\} = \{X_n \notin A \ \forall n\}$$

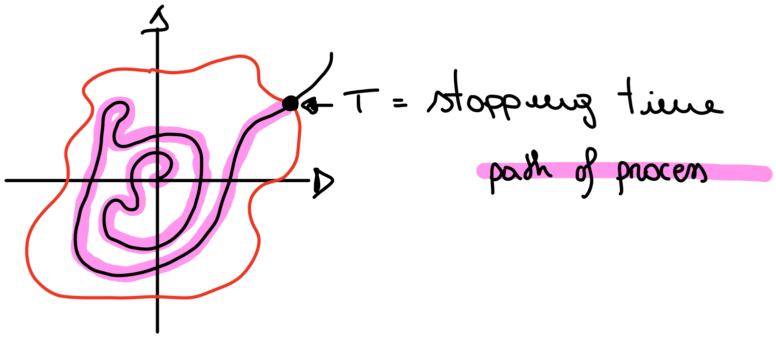
$\inf \emptyset \overset{\text{def}}{=} +\infty$

EXAMPLE interpretation of $\{T=n\} \in \mathcal{F}_n \ \forall n$

$T = n \longleftrightarrow$ I stop playing at time n
$\hookrightarrow$ depend only on info available at time n
Namely the event $\{T = n\} \in \mathcal{F}_n$

EXAMPLE

$S = \mathbb{R}^2, \quad X_0 = (0,0)$



FINITE DIMENSIONAL DISTRIBUTIONS

Let X_t the process indexed by T (indexing process). $\forall n \geq 1$ $\forall t_1, \dots t_n \in T$ we have an n-dimensional random vector

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n})$$

The distribution of such RV $(X_{t_1} \dots X_{t_n})$ $\forall n \geq 1$ $\forall t_1 \dots t_n \in T$ are called finite dimensional distributions

EXAMPLE

$T = \{1, 2\}$ = a pair of RV

$$\left.\begin{array}{l} X_1 \sim \text{Binomial} \\ X_2 \sim \text{Poisson} \\ (X_1, X_2) \sim N \end{array}\right\}$$

cannot be as if the joint is N, also the marginals should be

In our appication we choose the finite dimensional distribution and we look for a process having such a finite dimensional distribution. But, as the previous example shows, such a process does not exist. However, there are some theorems, called consistency theorems which provide conditions on the f.d.d. under which the process with such finite dimensional distribution exists.

CONDITIONAL EXPECTATIONS

In order to define conditional expectation 3 things are required:

- $(\Omega, \mathcal{A}, P)$ prob. space
- $\mathcal{G} \subset \mathcal{A}$ sub σ-field of $\mathcal{A}$
- X real RV s.t. $E|X| < +\infty$

DEFINITION

By definition, a conditional expectation of X given $\mathcal{G}$ is ANY real RV $V: \Omega \longrightarrow \mathbb{R}$ s.t.

I) $E|V| < +\infty$

II) V is $\mathcal{G}$-measurable

III) $E[\mathbb{1}_A X] = E[\mathbb{1}_A V] \quad \forall A \in \mathcal{G}$

Remarks :

- On III) since $\Omega \in \mathcal{G}$, we obtain
$$E X = E[X \mathbb{1}_\Omega] = E[V \mathbb{1}_\Omega] = E V$$

- A RV $V : \Omega \longrightarrow \mathbb{R}$ is said to be $\mathcal{G}$-measurable if $V^{-1} \in \mathcal{G}$ $\forall B \in \mathcal{B}(\mathbb{R})$. This is a strong condition as $\mathcal{G} \subset \mathcal{A}$.
In other terms, V is a RV not only in the big prob space $(\Omega, \mathcal{A}, P)$, but also in the small prob. space $(\Omega, \mathcal{G}, P)$.
For instance if $\mathcal{G} = \{\emptyset, \Omega\}$, then if X is $\mathcal{G}$-measurable, it must be that
$$X^{-1}(B) = \Omega \quad \text{or} \quad X^{-1}(B) = \emptyset \qquad \forall B \in \mathcal{B}(\mathbb{R})$$
Hence, if $\mathcal{G} = \{\emptyset, \Omega\}$ the only $\mathcal{G}$-measur. RV are the constants.
At the opposite extremes, if $\mathcal{G} = \mathcal{A}$ every RV is $\mathcal{G}$-measurable.

EXAMPLE

Fix $A \in \mathcal{A}$ and define $\mathcal{G} = \{\emptyset, \Omega, A, A^c\}$ and

$$X = 3\,\mathbb{1}_A - 2\,\mathbb{1}_{A^c} = \begin{cases} 3 & \text{on } A \\ -2 & \text{on } A^c \end{cases}$$

$$X^{-1}(B) = \begin{cases} \Omega & \text{if } -2, 3 \in B \\ \emptyset & \text{if } -2, 3 \notin B \\ A & \text{if } 3 \in B,\ -2 \notin B \\ A^c & \text{if } 3 \notin B,\ -2 \in B \end{cases}$$

But under info $\mathcal{G}$ we know whether A is True or false, and thus X becomes a constant

IMPORTANT THEOREM

A conditional expectation V always exists and it is almost surely unique, namely, if V_1 and V_2 are both conditional expectation, then

$$P(V_1 \neq V_2) = 0$$

From now on to denote the conditional exp of X given $\mathcal{G}$, we adopt notation $V = E[X \mid \mathcal{G}]$

Interpretation: $E[X \mid \mathcal{G}]$ = our prediction of X under the info $\mathcal{G}$

$E[X]$ = our pred. of X without info

The requirement that $E[X \mid \mathcal{G}]$ should be $\mathcal{G}$-measurable is now clear since any prediction of X under the info $\mathcal{G}$ should be something which only depends on the info $\mathcal{G}$