Diophantine problems in rings and algebras: undecidability and reductions to rings of algebraic integers

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Abstract

We study systems of equations in different families of rings and algebras. In each such structure R we interpret by systems of equations (e-interpret) a ring of integers O of a global field. The long standing conjecture that $\mathbb Z$ is always e-interpretable in O then carries over to R, and if true it implies that the Diophantine problem in R is undecidable. The conjecture is known to be true if O has positive characteristic, i.e. if O is not a ring of algebraic integers. As a corollary we describe families of structures where the Diophantine problem is undecidable, and in other cases we conjecture that it is so. In passing we obtain that the first order theory with constants of all the aforementioned structures R is undecidable.

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1 Introduction

In this paper we study systems of equations in different families of rings and algebras. For each R in one of these families we interpret by systems of equations a ring of integers O of a global field (i.e. O is the integral closure of \mathbb{Z} or $\mathbb{F}_p[t]$ in a finite extension of \mathbb{Q} or $\mathbb{F}_p(t)$, respectively). In particular this reduces the Diophantine problem (decidability of systems of equations) in O to the same problem in R. It is known that $\mathcal{D}(O)$ is undecidable if O has positive characteristic [41], and it is conjectured to be also undecidable if otherwise O is a ring of algebraic integers [31, 6]. This leads us to conjecture that the Diophantine problem in the above structures is also undecidable. In many cases we verify that this is so.

A field K is a global field if it is either a number field (a finite extension of \mathbb{Q}) or a global function field (a finite extension of $\mathbb{F}_p(t)$, for some prime p). A ring of integers of a number field is called a ring of algebraic integers.

The Diophantine problem in a structure R (also known as Hilbert's 10th problem in R), denoted $\mathcal{D}(R)$, asks whether there exists an algorithm that, given a system of equations S with coefficients in R, determines if S has a solution in R or not. The original version of this problem was posed by Hilbert for the ring of integers \mathbb{Z} . This was solved in the negative in 1970 by Matyasevich [24] building on the work of Davis, Putnam, and Robinson [3]. Subsequently the same problem has been studied in a wide variety of rings, for instance in rings of algebraic integers, where it remains open. It is conjectured that the ring \mathbb{Z} is e-definable in all such rings [6, 31], and indeed this has been shown to be true in some partial cases, for example for rings of integers of abelian number fields [38] (see [43] for further partial results). The conjecture is known to hold provided that for each number field K there exists an elliptic curve E over K with $\mathrm{rk}E(K) = \mathrm{rk}E(\mathbb{Q}) = 1$ [32]. Later this was shown to be true assuming the Safarevich-Tate conjecture [25].

The scenario is much clearer for rings of integers of global function fields, indeed Shlapentokh [41] showed that \mathbb{Z} is e-interpretable in any such ring O, and consequently that $\mathcal{D}(O)$ is undecidable. Other results in this direction include undecidability of $\mathcal{D}(K)$ for any global function field K [8, 39], and of $\mathcal{D}(R[t])$, for R any integral domain [4, 5]. Some rings where the Diophantine problem remains open are most remarkably \mathbb{Q} (it is known however that this problem is undecidable in $\mathbb{Z}[S^{-1}]$, for S an infinite set of primes of Dirichlet density 1 [33]); the rational functions $\mathbb{C}(t)$ (even though $\mathcal{D}(\mathbb{C}(t_1, t_2))$

is undecidable [21]); and the field of Laurent series $\mathbb{F}_p((t))$. The decidability of the first order theory of $\mathbb{C}(t)$ and of $\mathbb{F}_p((t))$ remains an open problem as well, whereas a classic result of Julia Robinson states that the first order theory of \mathbb{Q} is undecidable. We refer to [34, 31, 43, 22] for further information and surveys of results in this direction.

Regarding non-commutative rings, Romankov [37] showed that $\mathcal{D}(F)$ is undecidable in several types of free rings F, which include free Lie rings, free associative or non-associative rings, and free nilpotent rings. In the same direction the second author, in collaboration with Kharlampovich, recently proved undecidability of $\mathcal{D}(A)$ in the language of rings, for A any of the following: a free associative k-algebra, a free Lie k-algebra (of rank at least 3), and many group k-algebras [15, 17]. In all these cases k is an arbitrary field. In Corollary 1.8 we obtain further results of this type. Many results regarding the first order theory of free algebras can be found in [16, 18, 19].

In this paper all rings and algebras are possibly non-associative, non-commutative, and non-unitary, unless stated otherwise. A ring (or algebra) of scalars is an associative, commutative, unitary ring (or algebra). We will always consider algebras over rings of scalars, and we fix Λ to denote such ring. Given a Λ -algebra L, we let L^2 be the Λ -module generated by all products of two elements of L. A ring is the same as a \mathbb{Z} -algebra. One of the main results of the paper is the following:

Theorem 1.1. Let R be a ring that is finitely generated as an abelian group. Suppose that R^2 is infinite. Then there exists a ring of algebraic integers O that is interpretable in R by systems of equations in the language of rings. Consequently, $\mathcal{D}(O)$ is reducible to $\mathcal{D}(R)$. On the other hand, if R^2 is finite, then $\mathcal{D}(R)$ is decidable.

In number theoretic terms, an *interpretation by systems of equations* (in short, an *e-interpretation*), is roughly a Diophantine definition up to a Diophantine equivalence relation. Here Diophantine definitions are considered by means of systems of equations, as opposed to single equations. We also convene that all systems of equations and all e-interpretations allow the use of any constant elements of the structures at hand, not necessarily in the signature. See Subsections 2.1.3, 2.1.2 and 2.3 for further comments on these matters.

Theorem 1.1 is further generalized to algebras. A Λ -algebra is called *module-finite* if it is finitely generated as a Λ -module. The language of Λ -modules \mathcal{L}_{mod} , or of Λ -algebras \mathcal{L}_{alg} , consists in the usual language of groups \mathcal{L}_{group} or of rings \mathcal{L}_{ring} , respectively, together with unary functions $\{\cdot_{\lambda} \mid \lambda \in \Lambda\}$ representing multiplication by elements of Λ (see Subsection 2.3). We write $(R; \mathcal{L})$ to indicate that a structure R is considered with a language \mathcal{L} .

Theorem 1.2. Let A be a module-finite algebra over a finitely generated ring of scalars Λ . Suppose that A^2 is infinite. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{alg})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(A; \mathcal{L}_{alg})$. If additionally Λ has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{alg})$, and $\mathcal{D}(A; \mathcal{L}_{alg})$ is undecidable. If otherwise A^2 is finite and $\mathcal{D}(A; \mathcal{L}_{mod})$ is decidable, then $\mathcal{D}(A; \mathcal{L}_{alg})$ is decidable.

If Λ is a finite field, then all the above holds after replacing $(A; \mathcal{L}_{alg})$ by $(A; \mathcal{L}_{ring})$, and $(A; \mathcal{L}_{mod})$ by $(A; \mathcal{L}_{group})$. In this case, if A^2 is finite, then $\mathcal{D}(A; \mathcal{L}_{ring})$ is decidable.

The simultaneous appearance of number fields and global function fields in our results should come as no surprise: there is a well-known analogy between these two classes of fields (see for example the preface of [45]).

Theorems 1.1 and 1.2 are further extended to a large class of finitely generated non-module-finite rings and algebras (Theorems 1.6 and 1.7). Similar statements are obtained for finite-dimensional algebras over arbitrary fields (Corollary 1.11 and Theorem 1.12). All of these will be discussed later in this introduction.

Our main results rely on the combination of two techniques: in the first we move from non-commutative algebra to commutative algebra, and in the second we move from the latter to number theory. The first reduction is achieved through the study of rings of scalars of bilinear maps between Λ -modules. This is relevant for us because much of the structure of a Λ -algebra can be "seen" in its ring multiplication operation, which is indeed a Λ -bilinear map between Λ -modules. In fact, bilinear maps also arise in a natural way in other structures, and in some cases it is possible to apply the methods presented in this paper to these, for example, in some classes of groups. This line of work will be explored further in an upcoming paper.

The approach regarding bilinear maps is described further at the end of this introduction. The reduction from commutative algebra to number theory is summarized in the following result, and also in Theorem 1.10, where algebras over arbitrary fields are considered.

Theorem 1.3. Let R be an infinite finitely generated ring of scalars. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(R; \mathcal{L}_{ring})$. Moreover:

- 1. If R is finitely generated as an abelian group, then O is a ring of algebraic integers.
- 2. If R has positive characteristic p, then O is the ring of integers of a global function field, $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable.

Some of the main results of this paper have been presented above. We now proceed to state the remaining ones, starting with the following non-unitary version of Theorem 1.3. See also Theorem 1.10 for a similar statement involving algebras.

Theorem 1.4. Let R be an infinite finitely generated associative commutative non-unitary ring. Then the conclusions of Theorem 1.3 hold for $(R; \mathcal{L}_{ring})$.

We now make an important observation.

Remark 1.5. All subsequent results are stated for Λ -algebras. These hold in particular for rings by taking $\Lambda = \mathbb{Z}$, in which case algebras become rings, modules become groups, and one has $\mathcal{L}_{alg} = \mathcal{L}_{ring}$, $\mathcal{L}_{mod} = \mathcal{L}_{group}$. The reader interested solely in rings may choose to read the whole paper with these considerations in mind (note that this does not apply for the results concerning algebras over infinite fields, which are stated at the end of this introduction and in the last section of the paper).

Theorems 1.1 and 1.2 are extended to some families of finitely generated non-module-finite rings and algebras. Let A be a Λ -algebra (or a ring) generated by a set S. If A is unitary let $\overline{S} = S \setminus \{\lambda \cdot 1 \mid \lambda \in \Lambda\}$, and otherwise let $\overline{S} = S$. In both cases let I_n be the Λ -ideal generated by all Λ -multiples of products of n elements of \overline{S} . A non-associative Λ -algebra will be called *right-normed-generated* if each I_n is generated as a Λ -module by a (possibly infinite) set of elements of the form $(s_1(s_2(\ldots(s_{k-1}s_k)\ldots)))$, with $k \geq n$ and $s_i \in \overline{S}$ for all i.

Theorem 1.6. Let A be a finitely generated algebra over a finitely generated ring of scalars Λ . Suppose that A is associative or right-normed-generated, and that $(A/I_n)^2$ is infinite for some $n \geq 1$. Then there exists a ring of integers of a global field O such that $(O; \mathcal{L}_{ring})$ is e-interpetable in $(A; \mathcal{L}_{alg})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(A; \mathcal{L}_{alg})$.

Moreover, if Λ has positive characteristic then $\mathcal{D}(A; \mathcal{L}_{alg})$ is undecidable. If $\Lambda = \mathbb{Z}$ (so A is just a ring), then O is a ring of algebraic integers. If Λ is either \mathbb{Z} or a finite field, then the whole theorem holds after replacing $(A; \mathcal{L}_{alg})$ by $(A; \mathcal{L}_{ring})$.

This theorem is proved by showing that A/I_n is e-interpretable in L. Then, since A/I_n is module-finite, it suffices to apply Theorem 1.2 and transitivity of e-interpretations.

A gradation $L = \bigoplus_{i \geq 1} L_i$ of a non-unitary algebra L will be called *simple* if L_1 generates L as an algebra. We will see that all simply graded Lie algebras are right-normed-generated. Thus Theorem 1.6 immediately yields the following:

Corollary 1.7. Let L be a finitely generated simply graded Lie Λ -algebra (for Λ as in Theorem 1.6). Suppose that $[L/I_n, L/I_n]$ is infinite for some $n \geq 1$. Then the conclusions of Theorem 1.6 hold for L.

In a free algebra F one has that $(F/I_n)^2$ is infinite for all $n \geq 2$. Hence the next result. Below Λ is as in Theorem 1.6.

Corollary 1.8. Let F be a finitely generated free associative Λ -algebra (commutative or non-commutative, and unitary or non-unitary) or a free Lie Λ -algebra of rank at least 2. Then the conclusions of Theorem 1.6 hold for F.

This complements the aforementioned results of Romankov [37] and of Kharlam-povich and Miasnikov [15, 17] regarding free algebras. We remark that in [37] it is proved (among others) that the algebras of Corollary 5.10 actually have undecidable Diophantine problem if $\Lambda = \mathbb{Z}$.

Noskov [30] proved that all finitely generated infinite rings of scalars have undecidable first order theory. Since this applies to the ring of integers of any global field, the next result follows.

Theorem 1.9. Suppose that A satisfies the hypotheses of any of the theorems and corollaries above. Then the first order theory of A in the corresponding language with constants is undecidable.

We next consider finitely generated algebras over a field. Note that an infinite field cannot be finitely generated, and so such an algebra may not fall into any of the cases studied previously. From now on k denotes an arbitrary field.

The following is analogous to Theorem 1.3. As before it relies on some results due to Shlapentokh [39, 40]. The proof is very similar to that of Theorem 1.3, with the main divergence being due to different behavior when the Krull dimension is 0. Note that only the language of rings (as opposed to the language of k-algebras) is used here.

Theorem 1.10. Let R be a nonzero finitely generated k-algebra of scalars. Suppose that R has Krull dimension at least one. Then $(\mathbb{Z}; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable.

If otherwise R has Krull dimension zero, then there exists a finite field extension K of k such that $(K; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$.

A similar result holds if R is non-unitary. In this case however one must consider $(R; \mathcal{L}_{alg})$ instead of $(R; \mathcal{L}_{ring})$.

Corollary 1.11. Let R be a finitely generated associative commutative non-unitary k-algebra such that $R^2 \neq 0$. Then the first or the second conclusion of Theorem 1.10 hold for $(R; \mathcal{L}_{alg})$.

All previous results regarding module-finite algebras over a f.g. ring of scalars have their analogue for module-finite algebras over k. As in Theorem 1.10, the e-interpretations below use only the language of rings. This contrasts with Theorems 1.1 and 1.2.

Theorem 1.12. Let R be a finitely generated k-algebra. Suppose that R satisfies one of the following:

- 1. R has finite dimension over k and $R^2 \neq 0$.
- 2. R is associative or right-normed-generated and $(R/I_n)^2 \neq 0$ for some $n \geq 1$.
- 3. R is a simply graded Lie algebra and $[R/I_n, R/I_n] \neq 0$ for some $n \geq 1$.

Then there exists a finite field extension K of k such that $(K; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(K; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(R; \mathcal{L}_{ring})$.

So for example if k is a global function field, then $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable for any R as above. The same is true if k is a field all of whose finite extensions have undecidable Diophantine problem.

From bilinear maps to commutative algebra Next, we briefly explain the approach taken in order to pass from possibly non-associative, non-commutative, and non-unitary algebras to algebras of scalars (and similarly for rings). The ideas we present here were introduced by the second author in [26], and they have been used successfully to study different first order theoretic aspects of different types of structures, including rings whose additive group is finitely generated [27], free algebras [16, 18, 19], and nilpotent groups [28, 29].

Observe that ring multiplication \cdot of a Λ -algebra R is, by definition, a Λ -bilinear map between Λ -modules. One can try to replace Λ by a "larger" ring of scalars Δ . To do so, one needs to find a ring of scalars Δ that acts on R by Λ -module endomorphisms (thus making the additive group of R into a Δ -module), in a way that \cdot becomes a Δ -bilinear map between Δ -modules. In this case we say that Δ is a ring of scalars of the map \cdot .

These considerations apply in the exact same way if one starts with an arbitrary Λ -bilinear map $f: N \times N \to M$ between Λ -modules N and M. If f is full and non-degenerate (see Subsection 3.1) then one can define the *largest* ring of scalars of f, denoted R(f). This ring constitutes an important feature of f, and in some sense it provides an "approximation" to interpreting (in $(N, M; f; \mathcal{L}_{mod})$) multiplication of constant elements from N and M by integer variables, or alternatively by variables taking values in Λ . Another important property of R(f) is that it is interpretable in $(N, M; f; \mathcal{L}_{mod})$ by first order formulas without constants [26]. In this paper we prove that this is still true if one uses systems of equations instead (with constants).

Theorem 1.13. If f is full and non-degenerate, and if N and M are finitely generated, then both Z(Sym(f)) and the largest ring of scalars of f are e-interpretable in the two sorted structure $(N, M; f, \mathcal{L}_{mod})$.

Here Sym(f) is defined as $Sym(f) = \{\alpha \in End_{\Lambda}(A) \mid f(\alpha x, y) = f(x, \alpha y) \ \forall \ x, y \in A\}$, and Z(Sym(f)) denotes the center of Sym(f). The interest we have for Z(Sym(f)) is mostly technical. This is explained in Remark 3.11.

Idea of the proof of Theorem 1.13. There are two main observations. The first goes as follows: Both Z(Sym(f)) and R(f) can be seen as subalgebras of the algebra of Λ -endomorphisms of N, denoted $End_{\Lambda}(N)$. Let a_1, \ldots, a_k be a module generating set of N. Then each $\alpha \in End_{\Lambda}(N)$ can be identified with the tuple $(\alpha a_1, \ldots, \alpha a_k) \in N^k$, and so we can think of Z(Sym(f)) and R(f) as Λ -submodules of N^k with an extra ring multiplication operation. Using that N^k is a Notherian Λ -module, we will see that Z(Sym(f)) and R(f) as Λ -modules are e-definable in N. Moreover: since αa_i is the i-th component of α seen as a tuple, it is possible to e-interpret in N the action of any endomorphism α on any generator a_i (and thus on any constant element of N).

The second idea is to use the properties of f in order to "express" statements about endomorphisms from Z(Sym(f)) in terms of their actions on a_1, \ldots, a_k . For example, given $\alpha, \beta, \gamma \in Z(Sym(f))$, one has that $\gamma = \alpha\beta$ if and only if $f(\gamma a_i, a_j) = f(\beta a_i, \alpha a_j)$ for all $1 \leq i, j \leq k$ (this is proved using bilinearity of f and the fact that $f(\alpha\beta x, y) = f(\beta x, \alpha y)$ for all x and y). This and the considerations in the previous paragraph can be

combined to show (after some work) that multiplication in Z(Sym(f)) is e-interpretable in $(N, M; f, \mathcal{L}_{mod})$. The rest of the proof follows similarly.

In Subsection 3.3 we generalize Theorem 1.13 to the following result.

Theorem 1.14. Let $f: A \times B \to C$ be a Λ -bilinear map between finitely generated Λ -modules. Then there exists a ring of scalars Θ that is a module-finite Λ -algebra, such that $(\Theta; \mathcal{L}_{alg})$ is e-interpretable in $F = (A, B, C; f, \mathcal{L}_{mod})$. If Λ is the ring \mathbb{Z} or a field, then \mathcal{L}_{alg} and \mathcal{L}_{mod} can be replaced by \mathcal{L}_{ring} and \mathcal{L}_{group} , respectively.

As mentioned above, the ring multiplication of any module-finite Λ -algebra R is a Λ -bilinear map $\cdot: R \times R \to R$ between finitely generated Λ -modules, and $(R, R, R; \cdot, \mathcal{L}_{mod})$ is e-interpretable in $(R; \mathcal{L}_{alg})$. Applying Theorem 1.14 and transitivity of e-interpretations we manage to move from the possibly non-associative, non-commutative, and non-unitary R to an algebra of scalars.

2 Preliminaries

2.1 Model theory

2.1.1 Multi-sorted structures

A multi-sorted structure \mathcal{A} is a tuple $\mathcal{A} = (A_i; f_j, r_k, c_\ell \mid i, j, k, \ell)$, where the A_i are pairwise disjoint sets called sorts; the f_j are functions of the form $f_j : A_{\ell_1} \times \cdots \times A_{\ell_m} \to A_{\ell_{m+1}}$ for some ℓ_i ; the r_k are relations of the form $r_k : A_{s_1} \times \cdots \times A_{s_p} \to \{0,1\}$; and the c_ℓ are constants, each one belonging to some sort. The tuple $(f_j, r_k, c_\ell \mid j, k, \ell)$ is called the signature or the language of \mathcal{A} . We always assume that \mathcal{A} contains the relations "equality in A_i ", for all sorts A_i , but we do not write them in the signature. If \mathcal{A} has only one sort then \mathcal{A} is a structure in the usual sense. One can construct terms in a multi-sorted structure in an analogous way as in uniquely-sorted structures. In this case, when introducing a variable x one must specify a sort where it takes values, which we denote A_x .

A set S of generators of A is a collection of elements from different sorts such that any element from any sort can be written as a term using only constants from S and from the signature of A (and using function symbols).

Let A_1, \ldots, A_n be a collection of multi-sorted structures. We denote by (A_1, \ldots, A_n) the multi-sorted structure that is formed by all the sorts, functions, relations, and constants of each A_i . Given a function f or a relation r we use the notation (A, f) or (A, r) with analogous meaning. If two different A_i 's have the same sort, then we view one of them as a formal disjoint copy of the other.

2.1.2 Diophantine problems and reductions.

Let \mathcal{A} be a multi-sorted structure. An equation in \mathcal{A} is an expression of the form $r(\tau_1, \ldots, \tau_k)$, where r is a signature relation of \mathcal{A} (typically, the equality relation), and each τ_i is a term in \mathcal{A} (taking values in an appropriate sort) where some of its variables

may have been substituted by elements of \mathcal{A} . Such elements are called the *coefficients* (or the *constants*) of the equation. These may not be signature constants. A system of equations is a finite conjunction of equations. A *solution* to a system of equations $\wedge_i \Sigma_i(x_1, \ldots, x_n)$ on variables x_1, \ldots, x_n is a tuple $(a_1, \ldots, a_n) \in A_{x_1} \times \cdots \times A_{x_n}$ such that all equations $\Sigma_i(a_1, \ldots, a_n)$ are true in \mathcal{A} .

The *Diophantine problem* in \mathcal{A} , denoted $\mathcal{D}(\mathcal{A})$, refers to the algorithmic problem of determining if each given system of equations in \mathcal{A} (with coefficients in a fixed computable set) has a solution. Sometimes this is also called *Hilbert's tenth problem* in \mathcal{A} . An algorithm L is a solution to $\mathcal{D}(\mathcal{A})$ if, given a system of equations S in \mathcal{A} , determines whether S has a solution or not. If such an algorithm exists, then $\mathcal{D}(\mathcal{A})$ is called decidable, and, if it does not, undecidable.

An algorithmic problem P_1 is said to be *reducible* to another problem P_2 if a solution to P_2 (if it existed) could be used as a subroutine of a solution to P_1 . For example, $\mathcal{D}(\mathbb{Z})$ is undecidable, and hence $\mathcal{D}(\mathcal{A})$ is undecidable for any structure \mathcal{A} such that $\mathcal{D}(\mathbb{Z})$ is reducible to $\mathcal{D}(\mathcal{A})$.

In some cases one restricts the set of coefficients C that can be used in the input equations of the Diophantine problem of a structure. For instance, one typically takes $C = \mathbb{Z}$ when studying $\mathcal{D}(\mathbb{Q})$ (equivalently one can take $C = \{0,1\}$). In this paper we will always need that C contains certain coefficients, namely those used in a certain e-interpretation, and maybe also the preimage of some constants of the structure that is being e-interpreted. For this reason, and to simplify the exposition, we agree that C is always the whole structure, or a suitable computable subset if the structure is not countable. All structures considered in this paper are finitely generated (and thus computable), except in Subsection 6.

2.1.3 Interpretations by equations

Interpretability by equations (e-interpretability) is the analogue of the classic model-theoretic notion of interpretability by first order formulas (see [12, 23]). In e-interpretability one requires that only systems of equations with coefficients are used, instead of first order formulas. From a number theoretic viewpoint, e-interpretability is roughly Diophantine definability (by systems of equations) up to a Diophantine definable equivalence relation.

In this paper —in e-interpretations and in Diophantine problems— we consider systems of equations and not just single equations. This may contrast with some number-theoretic settings, where systems of equations are equivalent to single equations, and both notions are treated interchangeably (for example in integral domains whose field of fractions is not algebraically closed).

Let \mathcal{A} be a structure with sorts $\{A_i \mid i \in I\}$. A basic set of \mathcal{A} is a set of the form $A_{i_1} \times \cdots \times A_{i_m}$ for some m and i_j 's.

Definition 2.1. Let M be a basic set of a multi-sorted structure \mathcal{M} . A subset $A \subseteq M$ is called *definable by equations* (or e-definable) in \mathcal{M} if there exists a system of equations $\Sigma_A(x_1,\ldots,x_m,y_1,\ldots,y_k)$ on variables $(x_1,\ldots,x_m,y_1,\ldots,y_k)=(\mathbf{x},\mathbf{y})$ such that \mathbf{x} takes values in M, and such that for any tuple $\mathbf{a} \in M$, one has that $\mathbf{a} \in A$ if and only if the

system $\Sigma_A(\mathbf{a}, \mathbf{y})$ on variables \mathbf{y} has a solution in \mathcal{M} . In this case Σ_A is said to define A in \mathcal{M} .

From an algebraic geometric viewpoint, an e-definable set is a projection onto some coordinates of an affine algebraic set.

Definition 2.2. Let $\mathcal{A} = (A_1, \ldots; f, \ldots, r, \ldots, c, \ldots)$ and \mathcal{M} be two multi-sorted structures. \mathcal{A} is said to be *interpretable by equations* (or *e-interpretable*) in \mathcal{M} if for each sort A_i there exists a basic set $M(A_i)$ of \mathcal{M} , a subset $X_i \subseteq M(A_i)$, and a surjective map $\phi_i : X_i \to A_i$ such that:

- 1. X_i is e-definable in \mathcal{M} , for all i.
- 2. For each function f and each relation r in the signature of \mathcal{A} (including the equality relation of each sort), the preimage by $\phi = (\phi_1, \dots)$ of the graph of f (and of r) is e-definable in \mathcal{M} , in which case we say that f (or r) is e-interpretable in \mathcal{M} . The same terminology applies to functions and relations that are not necessarily in the signature of \mathcal{A} .

The tuple of maps $\phi = (\phi_1, ...)$ is called an *e-interpretation* of \mathcal{A} in \mathcal{M} . The coefficients appearing in the equations above are called the *coefficients of the e-interpretation*.

The next lemma illustrates a key application of e-interpretability.

Lemma 2.3. Let R be a ring, not necessarily commutative or associative. Suppose $I \leq R$ is an ideal that is e-definable as a set in R. Then $(R/I; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$.

Proof. Let $\Sigma_I(x, \mathbf{y})$ be a system of equations that e-defines I as a set in R, so that $a \in R$ belongs to I if and only if $\Sigma_I(a, \mathbf{y})$ has a solution \mathbf{y} . It suffices to check that the natural epimorphism $\pi: R \to R/I$ is an e-interpretation of R/I in R. First observe that the preimage of π is the whole R, which is e-definable in R by an empty system of equations. Regarding equality in R/I, $\pi(a_1) = \pi(a_2)$ in R/I if and only if $a_1 - a_2 \in I$, i.e. if and only if $\Sigma_I(a_1 - a_2, \mathbf{y})$ has a solution. From this it follows that the preimage of equality in R/I, $\{a_1, a_2 \in R \mid \pi(a_1) = \pi(a_2)\}$, is e-definable in R by the system of equations $\Sigma_I'(x_1, x_2, \mathbf{y})$ obtained from $\Sigma_I(x, \mathbf{y})$ after substituting each occurrence of x by $x_1 - x_2$, where x_1 and x_2 are new variables. By similar arguments the preimages of the addition and multiplication operations in R/I are e-definable in R: indeed, $\pi(a_1) + \pi(a_2) = \pi(a_3)$ if and only if $a_1 + a_2 - a_3 \in I$, and $\pi(a_1)\pi(a_2) = \pi(a_3)$ if and only if $a_1a_2 - a_3 \in I$. \square

Remark 2.4. It is clear from the proof that an analogue of Lemma 2.3 holds for other structures, such as groups with e-definable normal subgroups, modules with e-definable submodules, etc.

The next two results are fundamental. They follow from Lemma 2.7, which we present at the end of this subsection.

Proposition 2.5 (E-interpretability is transitive). If \mathcal{A} is e-interpretable in \mathcal{B} and \mathcal{B} is e-interpretable in \mathcal{M} , then \mathcal{A} is e-interpretable in \mathcal{M} .

Proposition 2.6 (Reduction of Diophantine problems). Let \mathcal{A} and \mathcal{M} be (possibly multi-sorted) structures such that \mathcal{A} is e-interpretable in \mathcal{M} . Then $\mathcal{D}(\mathcal{A})$ is reducible to $\mathcal{D}(\mathcal{M})$. As a consequence, if $\mathcal{D}(\mathcal{A})$ is undecidable, then so is $\mathcal{D}(\mathcal{M})$.

Similarly, the first order theory of A is reducible to the first order theory (with constants¹) of M, and the second is undecidable if the first is.

Both Propositions 2.5 and 2.6 are a consequence of the following lemma, which states in technical terms that if one structure is e-interpretable in the other, then one may "express" equations in the first as systems of equations in the second.

Lemma 2.7. Let $\phi = (\phi_1, \dots)$ be an e-interpretation of a multi-sorted structure $\mathcal{A} = (A_1, \dots; f, \dots, r, \dots, c, \dots)$ in another multi-sorted structure \mathcal{M} , with $\phi_i : X_i \subseteq M(A_i) \to A_i$ (see Definition 2.2). Let $\sigma(\mathbf{x}) = \sigma(x_1, \dots, x_n)$ be an arbitrary system of equations in \mathcal{A} with each variable x_i taking vaules in A_{j_i} . Then there exists a system of equations $\Sigma_{\sigma}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ in \mathcal{M} , such that each tuple of variables \mathbf{y}_i takes values in $M(A_{j_i})$, and such that a tuple $(\mathbf{b}_1, \dots, \mathbf{b}_n) \in \prod_{i=1}^n M(A_{j_i})$ is a solution to $\Sigma_{\sigma}(\mathbf{y}_1, \dots, \mathbf{y}_n)$ if and only if $(\mathbf{b}_1, \dots, \mathbf{b}_n) \in \prod_{i=1}^n X_{j_i}$ and $(\phi_{j_1}(\mathbf{b}_1), \dots, \phi_{j_n}(\mathbf{b}_n))$ is a solution to σ .

Proof. It suffices to follow step by step the proof of Theorem 5.3.2 from [12], which states that the above holds for the classical notion of interpretability by first order formulas of uniquely-sorted structures. Indeed, the result follows from the definitions after observing that σ can be rewritten as a system of equations in which all its atomic subformulas are unnested.

2.2 Rings of integers of number and function fields

An (algebraic) number field is a finite field extension of \mathbb{Q} . An (algebraic) function field F over a field k is a finite field extension of k(t). If k is a finite field then F is called a global function field. A global field is either a number field or a global function field.

Let A and B be two rings of scalars such that $A \leq B$. An element $b \in B$ is integral over A if there exists a monic polynomial from A[x] that has b as a root. The integral closure of A in B is the set C of all elements of B that are integral over A. Such set is a subring of B. An integral domain D is called integrally closed if the integral closure of D in its field of fractions is D itself.

Let K be a number field or a field of functions over k. The ring of integers of K, denoted O_K , is the integral closure of \mathbb{Z} or k[t] in K, respectively. In the first case O_K is called a ring of algebraic integers.

Rings of S-integers or holomorphy rings Our main references for this subsection are [44, 43, 10]. Throughout the end of this subsection let K be either a number field or a function field over a field k.

A (non-archimidean) valuation of K is a map $v: K \to \mathbb{Z} \cup \{\infty\}$ that satisfies all of the following:

¹The considerations made regarding the use of constants in systems of equations and Diophantine problems are made as well for first order formulas and their decidability problems (see Paragraph 3 of Subsection 2.3 or Subsection 2.1.2).

- 1. v induces a homomorphism $v: K^* \to \mathbb{Z}$ from the multiplicative group of K into the additive group of \mathbb{Z} .
- 2. $v(x+y) \ge \min\{v(x), v(y)\}\$ for all $x, y \in K$.
- 3. $v(x) = \infty$ if and only if x = 0.
- 4. $v(x) \neq 0$ for some $x \in K$.
- 5. If K is a function field, then v(c) = 0 for all c in the algebraic closure of k in K.

We will refer to a non-archimedean valuation simply as a valuation (archimidean valuations are never considered in this paper). Two valuations v_1 and v_2 are equivalent if $\{x \in K \mid v_1(x) \geq 0\} = \{x \in K \mid v_2(x) \geq 0\}$. An equivalence class of a valuation v will be called a *prime*. We identify v with its equivalence class, and we write $ord_v(x)$ instead of v(x), for any $x \in K$.

Let S be a (finite or infinite) set of primes of K. The ring of S-integers of K is defined as

$$O_{K,S} = \{x \in K \mid ord_p(x) \ge 0 \ \forall p \notin S\}.$$

In many texts $O_{K,S}$ is called a holomorphy ring when K is a function field.

Remark 2.8. Rings of S-integers include rings of integers of number and function fields. Indeed, if K is a number field, then $O_{K,\emptyset}$ is precisely the ring of algebraic integers O_K of K (this follows, for example, from B.1.15 and B.1.22 of [43]).

Suppose otherwise that K is a function field over k. The ring of integers of k(t) is $k[t] = O_{k(t),\{v_{\infty}\}}$, where v_{∞} is the prime of k(t) defined by $v_{\infty}(f/g) = deg(g) - deg(f)$ (see 3.2 of [44]). Furthermore, the ring of integers of K is $O_{K,S_{\infty}}$, where S_{∞} is the set of all primes of K that lie above v_{∞} , i.e. those primes that, when restricted to k(t), coincide with v_{∞} (see 1.2.2 and 3.2.6 of [44]). The set S_{∞} is finite (see B.1.11 of [43]), and each $p \in S_{\infty}$ is called a *pole of t*. Notice that if S is an arbitrary set of primes of K and $O_{K,S}$ contains k[t], then $S_{\infty} \subseteq S$, by definition of $O_{K,S}$ and S_{∞} .

It follows from the above remark that if K is a number field, then the field of fractions F of $O_{K,S}$ is K, for any set of primes S. Similarly, if K is a function field, then the field of fractions of $O_{K,S}$ is again K, for any set of primes S containing S_{∞} .

Proposition 2.9 (Propositions B.1.21 and B.1.27 of [43], and Corollary 3.2.8 of [44]). Let R be a subring of K. Then R is integrally closed if and only if $R = O_{F,S}$, where F is the field of fractions of R, and S is some set of primes of F.

We will also need the following result.

Lemma 2.10. Let S be a set of primes of K. Suppose that one of the following hold:

- 1. K is a number field, and $O_{K,S}$ is finitely generated as a ring.
- 2. K is a function field over k, and $O_{K,S}$ is finitely generated as a k-algebra.

Then there exists a finite subset S_0 of S such that $O_{K,S} = O_{K,S_0}$.

Proof. Let r be either \mathbb{Z} or k, depending on whether Item 1 or 2 holds. Suppose that $O_{K,S}$ is generated as a r-algebra by a finite set $A=\{a_1,\ldots,a_n\}$. Let S_0 be the subset of S consisting in all primes p such that $ord_p(a) \neq 0$ for some $a \in A$. It is well known that for any $x \in O_{K,S}$ there are only finitely many primes p of K such that $ord_p(x) \neq 0$ (see B.1.18 of [43] and 1.3.4 of [44]). Therefore S_0 is finite. By assumption, for all $x \in O_{K,S}$ there exists a polynomial $q \in r[a_1,\ldots,a_n]$ such that x=q. Using the axioms of non-archimedean valuations and the fact that $ord_p(a) = 0$ for all $p \in S \setminus S_0$ and all $a \in A$, we obtain that $ord_p(x) \geq 0$ for all $x \in O_{K,S}$ and all $p \in S \setminus S_0$ (indeed, writing $q = \sum_i r_i m_i$ for some $r_i \in r$ and some products $m_i = a_{i_1} \ldots a_{i_{j_i}}$, we have $ord_p(x) \geq \min\{ord_p(r_im_i) \mid i\} = \min\{ord_p(r_i) \mid i\} \geq 0$). Thus by definition $\mathcal{O}_{K,S} \leq \mathcal{O}_{K,S_0}$. The opposite inclusion follows immediately, since $S_0 \subseteq S$.

2.3 Notation and conventions

We would like to emphasize some relevant aspects and to fix some notation.

1. By ring or algebra of scalars we mean an associative commutative unitary ring or algebra, respectively.

Given a Λ -algebra R and a subset $S \subseteq R$, we let $\langle S \rangle_{\Lambda}$ be the Λ -submodule of R generated by a set S. We also let $R^2 = \langle xy \mid x, y \in R \rangle_{\Lambda}$.

All modules are assumed to be *left* modules. Similarly, the underlying module of an algebra is assumed to be a left module. All arguments work in the same way if we replace left for right, or left module for bimodule.

- 2. There is no restriction on the coefficients used in an e-interpretation. In particular, the coefficients may not belong to the signature. The same consideration applies for the Diophantine problem of a computable (for example countable) structure. If the structure is not computable, then the input systems of equations in the Diophantine problem must have coefficients in a fixed computable set. See Subsection 2.1.2.
- 3. The language of groups is $\mathcal{L}_{group} = (+,0)$. The language of Λ -modules is $\mathcal{L}_{mod} = (\mathcal{L}_{group}, \cdot \Lambda)$, where the $\cdot \Lambda = \{\cdot \lambda \mid \lambda \in \Lambda\}$ are unary functions representing multiplication by scalars: $\cdot_{\lambda}(x) = \lambda x$. The language of rings \mathcal{L}_{ring} is either $(+,\cdot,0)$ or $(+,\cdot,0,1)$. The language of Λ -algebras is $\mathcal{L}_{alg} = (\mathcal{L}_{ring}, \cdot \Lambda)$. The presence or lack of 1 will be clear from the context. If Λ admits a finite generating set S, then one can replace $\cdot \Lambda$ by $\cdot S = \{\cdot_{\lambda} \mid \lambda \in S\}$.

Hence, in an equation (or in a formula) over a Λ -module or Λ -algebra R, one is allowed to multiply any element of R by any constant element of Λ , but this is as far as one can involve Λ : no variable can take values in Λ , no quantification over Λ can be made, etc.

- **4.** The notion of \mathbb{Z} -module or \mathbb{Z} -algebra with the languages above is equivalent to the notion of abelian group or ring, respectively. Indeed, multiplication by scalars from \mathbb{Z} is e-interpretable using only the group addition +, since $nx = x + \cdot n + x$ for all all $n \in \mathbb{Z}$ and all element x.
- 5. Sometimes we will want to look at a Λ -algebra L as a Λ -module, or as a ring, or as a group, forgetting about the corresponding additional operations of L. We will use the notation $(L; \mathcal{L}_{mod})$, $(L; \mathcal{L}_{ring})$, $(L; \mathcal{L}_{group})$ when this is done, respectively. We will also write $(L; \mathcal{L}_{alg})$ to emphasize that L is considered with all its Λ -algebra operations. A similar terminology will be used for other structures such rings and modules.

This notation will be used extensively in expressions of the type $(L; \mathcal{L}_1)$ is e-interpretable in $(K; \mathcal{L}_2)$. This means that L with the operations of the language \mathcal{L}_1 is e-interpretable in K considered with the operations of \mathcal{L}_2 (and with constants possibly not in the signature of \mathcal{L}_2 –see Paragraph 3). In the particular case that $\mathcal{L}_1 = \mathcal{L}_2$ we will also say that L is e-interpretable in K in the language \mathcal{L}_1 (with constants).

Given some modules and algebras L_1, \ldots, L_n , we write $(L_1, \ldots, L_n; \mathcal{L}_{mod})$ instead of $((L_1; \mathcal{L}_{mod}), \ldots, (L_n; \mathcal{L}_{mod}))$. The expression $(L_1, \ldots, L_n; \mathcal{L}_{group})$ has an analogous meaning.

3 From bilinear maps to commutative rings and algebras

A brief description of the arguments used in this section can be found at the last part of the introduction.

3.1 Ring of scalars of a full non-degenerate bilinear map

Throughout this subsection, Λ denotes a (possibly infinitely generated) ring of scalars, i.e., an associative, commutative, unitary ring.

A map $f: N \times N \to M$ between Λ -modules N and M is Λ -bilinear if, for all $a \in N$, the maps $f(a,\cdot)$ and $f(\cdot,a)$ from N to M are homomorphisms of Λ -modules. We call f non-degenerate if whenever f(a,x)=0 for all $x \in N$, we have a=0, and similarly for f(x,a). The map f is full if the Λ -submodule generated by the image of f is the whole M.

The set of module endomorphisms of a Λ -module N, denoted $End_{\Lambda}(N)$, forms an associative unitary Λ -algebra once we equip it with the operations of addition and composition (henceforth called multiplication). We simply write αx instead of $\alpha(x)$, for $\alpha \in End_{\Lambda}(N)$ and $x \in N$. An action of a ring Δ on N is a ring homomorphism $\phi : \Delta \to End_{\Lambda}(N)$. Any such action ϕ endows N with a structure of Δ -module. Furthermore the action is called faithful if ϕ is an embedding.

Definition 3.1. Let $f: N \times N \to M$ be a Λ -bilinear map between Λ -modules. A ring Δ is called a *ring of scalars* of f if it is associative, commutative, and unitary, and there exist faithful actions of Δ on M and N such that $f(\alpha x, y) = f(x, \alpha y) = \alpha f(x, y)$ for all $\alpha \in \Delta$ and all $x, y \in N$.

Since the actions of a ring of scalars Δ of f on M and N are faithful, there exist ring embeddings $\Delta \hookrightarrow End_{\Lambda}(M)$ and $\Delta \hookrightarrow End_{\Lambda}(N)$. For this reason and for convenience we always assume that a ring of scalars of f is a subring of $End_{\Lambda}(N)$.

Definition 3.2. We say that Δ is the *largest* ring of scalars of f, and we denote it R(f), if for any other ring of scalars Δ' of f, one has $\Delta' \leq \Delta$ as subrings of $End_{\Lambda}(N)$.

The next result was proved by the second author in [26]. We recover its proof since we will need to examine it in the next subsection.

Theorem 3.3 ([26]). Let $f: N \times N \to M$ be a full non-degenerate Λ -bilinear map between Λ -modules. Then the largest ring of scalars R(f) of f exists and is unique.

Proof. Consider the following subsets of $End_{\Lambda}(N)$:

$$Sym(f) = \{ \alpha \in End_{\Lambda}(N) \mid f(\alpha x, y) = f(x, \alpha y) \text{ for all } x, y \in N \},$$
 (1)

$$Z(Sym(f)) = \{ \alpha \in Sym(f) \mid \alpha\beta = \beta\alpha \text{ for all } \beta \in Sym(f) \}.$$
 (2)

It is straightforward to check that both Sym(f) and Z(Sym(f)) are Λ -modules. Moreover, for all $\alpha_1, \alpha_2 \in Z(Sym(f))$ and all $x, y \in N$,

$$f(\alpha_1 \alpha_2 x, y) = f(\alpha_2 x, \alpha_1 y) = f(x, \alpha_2 \alpha_1 y) = f(x, \alpha_1 \alpha_2 y),$$

and thus $\alpha_1\alpha_2 \in Sym(f)$. Since both α_1 and α_2 commute with any element from Sym(f), so does $\alpha_1\alpha_2$. Hence, $\alpha_1\alpha_2 \in Z(Sym(f))$, and so Z(Sym(f)) is a subalgebra of $End_{\Lambda}(N)$.

Next, we show that any ring of scalars Δ of f is a subring of Z(Sym(f)). Indeed, by definition, $\Delta \subseteq Sym(f)$. To see that $\Delta \subseteq Z(Sym(f))$, let $\alpha \in \Delta$ and $\beta \in Sym(f)$. Then, for all $x, y \in N$,

$$f(\alpha \beta x, y) = \alpha f(\beta x, y) = \alpha f(x, \beta y) = f(\alpha x, \beta y) = f(\beta \alpha x, y).$$

Hence $f((\alpha\beta - \beta\alpha)x, y) = 0$ for all $x, y \in N$. Since f is non-degenerate and y is arbitrary, $(\alpha\beta - \beta\alpha)x = 0$ for all $x \in N$. It follows that $\alpha\beta = \beta\alpha$, and thus $\Delta \subseteq Z(Sym(f))$.

By what we have seen so far, Z(Sym(f)) is an associative commutative unitary algebra that acts faithfully on N. We now wish to define an action of a subring Δ of Z(Sym(f)) on M. Since f is full, for all $z \in M$ we have $z = \sum_i f(x_i, y_i)$ for some $x_i, y_i \in N$. Hence, one may try to define the following action:

$$\alpha z = \sum f(\alpha x_i, y_i) \quad \text{for} \quad \alpha \in \Delta.$$
 (3)

However, this is not necessarily well-defined, because the same $z \in M$ may have different expressions as sums of elements $f(x_i, y_i)$. With this in mind, we let Δ be the set of all $\alpha \in Z(Sym(f))$ such that

$$\sum f(\alpha x_i, y_i) = \sum f(\alpha x_i', y_i') \quad \text{whenever} \quad \sum f(x_i, y_i) = \sum f(x_i', y_i'). \tag{4}$$

Clearly, Δ is closed under addition and multiplication, and therefore it is a subring of Z(Sym(f)) with a well-defined action on M given by (3). Since the action of Δ on N is faithful, and f is a non-degenerate map, the action of Δ on M is faithful as well. It follows that Δ is a ring of scalars of f. Moreover, any ring of scalars Δ' of f satisfies (4), and thus, since $\Delta' \subseteq Z(Sym(f))$, we have $\Delta' \le \Delta$. We conclude that Δ is the unique largest ring of scalars of f.

Remark 3.4. It is clear from the proof above that R(f) is closed under multiplication by Λ . Hence, R(f) admits the structure of a Λ -algebra.

It is possible to show that the ring structure of R(f) is independent of the actions of Λ on N and M. In other words, R(f) is the same ring whether we look at f as a bilinear map between Λ -modules, or as a bilinear map between abelian groups (we do not prove this, since it will not be used or commented further in the paper). However, the second point of view may be inconvenient for e-interpretability purposes, since N and M may be finitely generated as Λ -modules, but infinitely generated as abelian groups.

3.2 E-interpreting Z(Sym(f)) and the largest ring of scalars

Throughout this subsection Λ denotes a possibly infinitely generated Noetherian ring of scalars. In this case any finitely generated Λ -module is Noetherian and finitely presented (see [7] or [11]). We refer to Subsection 2.3 for important notation and terminology conventions.

The goal of this subsection is to prove the following result.

Theorem 3.5. Let $f: N \times N \to M$ be a full non-degenerate bilinear map between finitely generated Λ -modules. Then both Z(Sym(f)) and the largest ring of scalars R(f) of f are module-finite Λ -algebras, and they are e-interpretable as Λ -algebras in $F = (N, M; f, \mathcal{L}_{mod})$. Moreover,

- 1. If Λ is either a field or the ring \mathbb{Z} , then $(Z(Sym(f)); \mathcal{L}_{ring})$ is e-interpretable in $(N, M; f, \mathcal{L}_{group})$ (i.e. multiplication by scalars is not required).
- 2. Z(Sym(f)), R(f), N, and M are all simultaneously zero, finite, or infinite.

We state some lemmas and observations before proving Theorem 3.5, starting with a useful description of $End_{\Lambda}(N)$.

Remark 3.6. Let N be a Λ -module with finite module presentation $\langle a_1,\ldots,a_m\mid \sum_i x_{j,i}a_i,\ j=1,\ldots,T\rangle_{\Lambda}$, where $x_{j,i}\in \Lambda$ for all i,j. Each element α of $End_{\Lambda}(N)$ is uniquely determined by the m-tuple $(\alpha a_1,\ldots,\alpha a_m)\in N^m$, and one has $\sum_i x_{j,i}(\alpha a_i)=0$ for all j. Conversely, any m-tuple from N^m with this property determines an element from $End_{\Lambda}(N)$. Thus $End_{\Lambda}(N)$ can be identified with the set of m-tuples $(\alpha_1,\ldots,\alpha_m)\in N^m$ that satisfy $\sum_i x_{j,i}\alpha_i=0$ for all j.

In the particular case that Λ is a field we have that N is a vector space. In particular, N is a free Λ -module, and so it admits a finite presentation with an empty set of relations. In this case, $End_{\Lambda}(N) = N^m$.

The identification of $End_{\Lambda}(N)$ with a subset of N^m is used to prove the following result.

Lemma 3.7. Let N be a finitely generated Λ -module. Then the following hold:

- 1. $(End_{\Lambda}(N); \mathcal{L}_{mod})$ is e-interpretable in $(N; \mathcal{L}_{mod})$.
- 2. Let $S_N = \{a_1, \ldots, a_m\}$ be a generating set of N, and define maps $\cdot a_i : End_{\Lambda}(N) \to N$ by $\alpha \mapsto \alpha a_i \in N$. Denote $\cdot S_N = \{\cdot a_1, \ldots, \cdot a_m\}$. Then the two-sorted structure $END_{\Lambda}(N) = (End_{\Lambda}(N), N; \cdot S_N, \mathcal{L}_{mod})$ is e-interpretable in $(N; \mathcal{L}_{mod})$.
- 3. In the particular case that Λ is a field or the ring of integers \mathbb{Z} , the previous statements are still valid after replacing \mathcal{L}_{mod} by \mathcal{L}_{group} in all structures.

Proof. As mentioned above, since Λ is a Noetherian ring of scalars, any finitely generated Λ -module is finitely presented with respect to any finite generating set. Let $\sum x_{j,i}a_i$, $j=1,\ldots,T$ be a finite set of relations of N, with $x_{j,i}\in\Lambda$ for all i,j.

Following Remark 3.6, identify each element α of $End_{\Lambda}(N)$ with the m-tuple $(\alpha_1, \ldots, \alpha_m) = (\alpha a_1, \ldots, \alpha a_m) \in N^m$. By this same remark, any m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ belongs to $End_{\Lambda}(N)$ if and only if $\sum x_{j,i}\alpha_i = 0$ for all j. This is a finite system of equations in N with variables α_i , and so $End_{\Lambda}(N)$ as a set is e-definable in $(N; \mathcal{L}_{mod})$. As observed in Remark 3.6, if Λ is a field then $End_{\Lambda}(N) = N^n$, and so the e-definition consists in an empty equation. In particular, it does not use multiplication by scalars.

The group addition of two tuples from the Λ -module $End_{\Lambda}(N)$ is obtained by component-wise addition. Hence the graph of the addition operation of $End_{\Lambda}(N)$ (which is a subset of N^{3m}) is e-definable in N. By similar reasons, so are the graphs of the equality relation of $End_{\Lambda}(N)$ and of multiplication by fixed elements of Λ (i.e. multiplication by scalars). This proves that $(End_{\Lambda}(N); \mathcal{L}_{mod})$ is e-interpretable in $(N; \mathcal{L}_{mod})$. In the case that Λ is a field, $(End_{\Lambda}(N); \mathcal{L}_{group})$ is e-interpretable in $(N; \mathcal{L}_{group})$.

It follows, of course, that the two-sorted structure $(End_{\Lambda}(N), N; \mathcal{L}_{mod})$ is e-interpretable in $(N; \mathcal{L}_{mod})$. Finally, notice that, for $\alpha \in End_{\Lambda}(N)$ and $x \in N$, the tuple

$$(\alpha, x) = (\alpha_1, \dots, \alpha_m, x) \in N^{m+1}$$

belongs to the graph of a_i if and only if $x = a_i$. In other words,

$$(y_1, \ldots, y_{m+1}) \in Graph(\cdot a_i) \subseteq N^{m+1}$$
 if and only if $y_i = y_{m+1}$,

and so the graph of a_i is e-definable in $(N; \mathcal{L}_{group})$ by the equation $y_i = y_{m+1}$. This completes the proof that $END_{\Lambda}(N)$ is e-interpretable in $(N; \mathcal{L}_{mod})$.

If Λ is a field then multiplication by scalars was not used in any equation other than when e-interpreting the scalar multiplication of $End_{\Lambda}(N)$, and if $\Lambda = \mathbb{Z}$ then a Λ -module is just a group, because $nx = x + \stackrel{n}{\cdot} \cdot + x$ for all $n \in \mathbb{Z}$. Hence, Item 3 holds. \square

Remark 3.8. It follows from Lemma 3.7 that there exists an e-interpretation ϕ of the three-sorted structure

$$F_1 = (End_{\Lambda}(N), N, M; f, \cdot S_N, \mathcal{L}_{mod})$$

in $F = (N, M; f, \mathcal{L}_{mod})$. If Λ is a field or \mathbb{Z} , then one can replace \mathcal{L}_{mod} by \mathcal{L}_{group} .

Thus by transitivity of e-interpretations (Proposition 2.5), in order to prove that $(R(f); \mathcal{L}_{alg})$ or $(Z(Sym(f)); \mathcal{L}_{alg})$ is e-interpretable in F it suffices to show that it is so in F_1 . For this one must keep in mind that an equation in F_1 can involve constants and variables from N, M, and $End_{\Lambda}(N)$; the map f; actions of endomorphisms on the a_i 's given by $\cdot S_N$; and the operations of $(N; \mathcal{L}_{mod}), (M; \mathcal{L}_{mod}),$ and $(End_{\Lambda}(N); \mathcal{L}_{mod})$ without its ring multiplication. For example, the equation $f(\alpha a_i, a_j) = f(a_i, \alpha a_j)$ on the variable α is valid in F_1 , whereas $\alpha_1\alpha_2a_i = \alpha_2\alpha_1a_i$ or $\alpha x = a_i$ is not (for variables $\alpha_1, \alpha_2, \alpha \in End_{\Lambda}(N), x \in N$). One can think of examples in which $\Lambda = N = \mathbb{Z}$ such that, if the ring multiplication of $(End_{\Lambda}(N); \mathcal{L}_{alg})$ was e-interpretable in F_1 , then multiplication of integers would be e-interpretable in the free abelian group $(\mathbb{Z}; \mathcal{L}_{group})$, contradicting the negative answer to the original Hilbert's 10th Problem.

We next prove the main result of this subsection.

Proof of Theorem 3.5. First observe that $End_{\Lambda}(N)$ is Λ -module-finite, because N^m is a Noetherian module and $End_{\Lambda}(N)$ embeds as a Λ -module into N^m , by Remark 3.6. By the same reason both R(f) and Z(Sym(f)) are Λ -module-finite as well.

Denote $F = (N, M; f, \mathcal{L}_{mod})$. We proceed to prove that $(Z(Sym(f)); \mathcal{L}_{alg})$ is e-interpretable in F. By the previous Remark 3.8, it suffices to show that $(Z(Sym(f)); \mathcal{L}_{alg})$ is e-interpretable in F_1 for some generating set $S_N = \{a_1, \ldots, a_n\}$ of N.

We start by proving that Sym(f) can be e-defined as a set in F_1 . Indeed, take any $x, y \in N$ and write $x = \sum x_i a_i$ and $y = \sum y_i a_i$ for some $x_i, y_i \in \Lambda$. Since $\alpha x = \sum x_i \alpha a_i$ for all $\alpha \in End_{\Lambda}(N)$, we have $f(\alpha x, y) = \sum x_i y_j f(\alpha a_i, a_j)$, and similarly for $f(x, \alpha y)$. It follows that

$$Sym(f) = \{ \alpha \in End_{\Lambda}(N) \mid f(\alpha a_i, a_j) = f(a_i, \alpha a_j) \text{ for all } 1 \leq i, j \leq n \}.$$

We conclude that Sym(f) is e-definable as a set in F_1 by the system

$$\bigwedge_{1 \le i,j \le n} \left[f\left(\alpha a_i, a_j\right) = f(a_i, \alpha a_j) \right] \tag{5}$$

on the single variable $\alpha \in End_{\Lambda}(N)$. Here the expressions αa_t stands for $a_t(\alpha)$ (t = 1, 2), which are terms in the structure F_1 . Observe that (5) does not use multiplication by scalars Λ .

Since the signature of F_1 contains all operations of $(End_{\Lambda}(N); \mathcal{L}_{mod})$, we have that $Sym(f) \leq End_{\Lambda}(N)$ as a Λ -module is e-interpretable in F_1 .

Next, we show that Z(Sym(f)) is e-definable as a set in F_1 . As before, this immediately implies that the Λ -module $(Z(Sym(f)); \mathcal{L}_{mod})$ is e-interpretable in F_1 . Let β_1, \ldots, β_k be a finite generating set of $(Sym(f); \mathcal{L}_{mod})$. Then, $\alpha \in Z(Sym(f))$ if and only if $\alpha \in Sym(f)$ and $\alpha\beta_t = \beta_t\alpha$ for all $t = 1, \ldots, k$. This implies that

$$f(\alpha a_i, \beta_t a_j) = f(a_i, \alpha \beta_t a_j) = f(a_i, \beta_t \alpha a_j) = f(\beta_t a_i, \alpha a_j) \quad \text{for all } i, j, t.$$
 (6)

We claim that (6) is a sufficient condition for an endomorphism $\alpha \in Sym(f)$ to belong to Z(Sym(f)). As a consequence one has that Z(Sym(f)) is definable as a set in F_1 by means of the following system of equations on the variable α :

$$\bigwedge_{\substack{t=1,\dots,k,\\1\leq i,j\leq n}} \left[f(\alpha a_i, \beta_t a_j) = f(\beta_t a_i, \alpha a_j) \right],$$
(7)

together with the system (5), which ensures that $\alpha \in Sym(f)$. Again, we have written αa_i and $\beta_t a_j$ instead of $a_i(\alpha)$ and $a_j(\beta_t)$, and similarly for the other entries of f. As before, (7) does not use multiplication by scalars Λ .

To prove the claim suppose (7) holds. Then $f(\beta_t \alpha a_i, a_j) = f(\alpha \beta_t a_i, a_j)$ for all i, j, t, and thus, for fixed i and t, $f([\beta_t, \alpha]a_i, a_j) = 0$ for all j, where $[\beta_t, \alpha] = \beta_t \alpha - \alpha \beta_t$. By bilinearity of f and the fact that a_1, \ldots, a_n generate N, we have that $f([\beta_t, \alpha]a_i, x) = 0$ for all $x \in N$ and for all i, t. Since f is non-degenerate, $[\beta_t, \alpha]a_i = 0$ for all i, t. Similar arguments as before yield $[\beta_t, \alpha]x = 0$ for all $x \in N$, and thus $[\beta_t, \alpha] = 0$ for all t. This completes the proof of the claim.

We have seen that the Λ -module $(Z(Sym(f)); \mathcal{L}_{mod})$ is e-interpretable in F_1 . By analogous reasons as above, for any triple $\gamma_1, \gamma_2, \gamma_3 \in Z(Sym(f))$ the equality $\gamma_3 = \gamma_1 \gamma_2$ holds if and only if

$$f(\gamma_3 a_i, a_j) = f(\gamma_2 a_i, \gamma_1 a_j) \quad \text{for all } 1 \le i, j \le n.$$
 (8)

Hence the ring multiplication of Z(Sym(f)) is e-interpretable in F_1 by means of (8) and appropriate systems of the form (5) and (7) (which ensure that $\gamma_i \in Z(Sym(f))$). We conclude that Z(Sym(f)) as a Λ -algebra is e-interpretable in F_1 .

We now prove Item 1. As observed in the arguments above, multiplication by scalars of F_1 was only used in order to e-interpret multiplication by scalars of Z(Sym(f)). Hence $(Z(Sym(f)); \mathcal{L}_{ring})$ is e-interpretable in $F_1 = (End_{\Lambda}(N), N, M; f, \cdot S_N, \mathcal{L}_{group})$. By Lemma 3.7 and Remark 3.8, the latter structure is e-interpretable in $(M, M; f, \mathcal{L}_{group})$, and so $(Z(Sym(f)); \mathcal{L}_{ring})$ is e-interpretable as a ring in $(M, M; f, \mathcal{L}_{group})$. This concludes the proof of Item 1.

Next we show that $(R(f); \mathcal{L}_{alg})$ is e-interpretable in $(N, M; f, \mathcal{L}_{group})$. By the previous arguments and by transitivity of e-interpretations, it suffices to prove that $(R(f); \mathcal{L}_{alg})$ is e-interpretable in $(Z(Sym(f)); \mathcal{L}_{alg})$. First recall from the proof of Theorem 3.3 that R(f) is the set of all $\alpha \in Z(Sym(f))$ such that

if
$$\sum_{i} f(x_i, y_i) = \sum_{i} f(x'_i, y'_i)$$
, then $\sum_{i} f(\alpha x_i, y_i) = \sum_{i} f(\alpha x'_i, y'_i)$.

This condition is equivalent to

if
$$\sum_{i} f(x_i, y_i) = 0$$
, then $\sum_{i} f(\alpha x_i, y_i) = 0$. (9)

We claim that $\alpha \in Z(Sym(f))$ satisfies (9) if and only if it satisfies the following condition:

if
$$\sum_{j,k} z_{j,k} f(a_j, a_k) = 0$$
 for some $z_{j,k} \in \Lambda$, then $\sum_{j,k} z_{j,k} f(\alpha a_j, a_k) = 0$. (10)

Indeed, the direct implication is immediate. Conversely, suppose that α satisfies (10), and let $\{x_i, y_i\}$ be such that $\sum_i f(x_i, y_i) = 0$. Write each x_i and y_i in terms of the generators a_1, \ldots, a_n ,

$$x_i = \sum_j x_{i,j} a_j$$
, and $y_i = \sum_k y_{i,k} a_k$, $x_{i,j}, y_{i,k} \in \Lambda$.

Since f is bilinear,

$$\sum_{i} f(x_{i}, y_{i}) = \sum_{j,k} \left(\sum_{i} x_{i,j} y_{i,k} \right) f(a_{j}, a_{k}) = 0.$$

Thus $0 = \sum_{j,k} \sum_i x_{i,j} y_{i,k} f(\alpha a_j, a_k) = \sum_j f(\alpha x_i, y_j)$. This completes the proof of the claim.

The set S of all tuples $(z_{i,j}) \in \Lambda^{n^2}$ such that $\sum_{i,j=1}^n z_{i,j} f(a_i, a_j) = 0$ forms a submodule of Λ^{n^2} , and so it admits a finite generating set, say $X = \{\mathbf{s}_i \mid i = 1, \dots, T\}$. Write $\mathbf{s}_i = (s_{i,j,k} \mid 1 \leq j, k \leq n)$. Then $\alpha \in Z(Sym(f))$ belongs to R(f) if and only if

$$\sum_{j,k} \left(\sum_{i} t_{i} s_{i,j,k} \right) f(\alpha a_{j}, a_{k}) = 0, \quad \text{for all} \quad t_{1}, \dots, t_{r} \in \Lambda.$$
 (11)

Equivalently,

$$\sum_{i} t_{i} \left(\sum_{j,k} s_{i,j,k} f(\alpha a_{j}, a_{k}) \right) = 0, \quad \text{for all} \quad t_{1}, \dots, t_{r} \in \Lambda.$$
 (12)

By making appropriate choices for the t_i 's, one sees that (12) holds if and only if each one of the expressions inside the parenthesis is 0. It follows that R(f) is e-definable as a set in $(Z(Sym(f)); \mathcal{L}_{alg})$. Consequently, R(f) is e-interpretable as a Λ -algebra in $(Z(Sym(f)); \mathcal{L}_{alg})$, since all the operations of $(R(f); \mathcal{L}_{alg})$ are already present in the signature of the latter.

Finally we prove Item 2, i.e. that Z(Sym(f)), R(f), N, and M are all simultaneously either zero, finite, or infinite. We claim that if Θ is a ring of scalars, then any finitely generated faithful Θ -module K is zero, finite, or infinite if and only if Θ is zero, finite, or infinite, respectively. Indeed, if K is finite (or zero), then $End_{\Theta}(K)$ is finite (zero) as well, because $End_{\Theta}(K)$ embeds as a Θ -module into K^n , for some n (see Remark 3.6). Since K is a faithful Θ -module, there exists an embedding $\Theta \hookrightarrow End_{\Theta}(K)$, and hence Θ is finite (zero) as well. On the other hand, if K is infinite, then, since K is finitely generated, there must exist $k \in K$ such that the set $\{\theta k \mid \theta \in \Theta\}$ is infinite. The claim follows.

Observe that both N and M are faithful R(f)-modules, and that N is also a faithful Z(Sym(f))-module. We claim that all these modules are finitely generated. Indeed, let $\Lambda_N = \{\lambda \in \Lambda \mid \lambda n = 0 \text{ for all } n \in N\}$, and define Λ_M similarly. Then N (resp. M) is a finitely generated faithful Λ/Λ_N -module (Λ/Λ_M -module). Using that f is full and non-degenerate, one can see that N is also a faithful Λ/Λ_M -module under the action

 $(\lambda + \Lambda_M)x = \lambda x$. With this action, Λ/Λ_M becomes a ring of scalars of f, and so by maximality of R(f) we have $\Lambda/\Lambda_M \leq R(f) \leq Z(Sym(f))$ as subrings of $End_{\Lambda}(N)$. Similar arguments yield $\Lambda/\Lambda_N \leq R(f) \leq Z(Sym(f))$. Hence N is finitely generated as a R(f)-module and a Z(Sym(f))-module. Similarly, M is finitely generated as a R(f)-module. This completes the proof of the claim. Item 2 follows now from such claim and from the observation in the paragraph above.

3.3 Arbitrary bilinear maps

In this subsection we keep the assumption that Λ is a *(possibly infinitely generated)* Noetherian ring of scalars. Our next goal is to generalize Theorem 3.5 to arbitrary bilinear maps.

Theorem 3.9. Let $f: A \times B \to C$ be a Λ -bilinear map between finitely generated Λ -modules. Then there exists a module-finite Λ -algebra of scalars Θ such that $(\Theta; \mathcal{L}_{alg})$ is e-interpretable in $F = (A, B, C; f, \mathcal{L}_{mod})$. Moreover,

- 1. If Λ is a field or the ring \mathbb{Z} , then $(\Theta; \mathcal{L}_{ring})$ is e-interpretable in $(A, B, C; f, \mathcal{L}_{group})$.
- 2. Θ is zero, finite or infinite if and only if both $\langle f(A,B)\rangle_{\Lambda}$ and $A/Ann_l(f)\times B/Ann_r(f)$ are zero, finite or infinite, respectively.

The proof of this result relies in constructing from f a suitable full non-degenerate bilinear map of the form $f: X \times X \to Y$, so that we can apply Theorem 3.5 to it. To this end, let the left and right *annihilators* of f be, respectively,

$$Ann_l(f) = \{ a \in A \mid f(a, y) = 0 \text{ for all } y \in B \},$$

 $Ann_r(f) = \{ b \in B \mid f(x, b) = 0 \text{ for all } x \in A \}.$

Observe that f induces a full non-degenerate Λ -bilinear map

$$f_1: A/Ann_l(f) \times B/Ann_r(f) \to \langle f(A,B) \rangle_{\Lambda},$$
 (13)

where by $\langle S \rangle_{\Lambda}$ we mean the Λ -submodule generated by S. Write $F = (A, B, C; f, \mathcal{L}_{mod})$, $A_1 = A/Ann_l(f)$, $B_1 = B/Ann_r(f)$, $C_1 = \langle f(A, B) \rangle_{\Lambda}$, and $F_1 = (A_1, B_1, C_1; f_1, \mathcal{L}_{mod})$. Note that A_1, B_1 and C_1 are finitely generated since A, B and C are Noetherian modules. If $A_1 = B_1$, then f_1 is of the desired form, and Theorem 3.5 can be applied to it. Otherwise consider the map

$$f_2: (A_1 \times B_1) \times (A_1 \times B_1) \to C_1 \times C_1$$

 $((a,b), (a',b')) \mapsto (f_1(a,b'), f_1(a',b)).$ (14)

One can easily check that f_2 is a full non-degenerate Λ -bilinear map between finitely generated Λ -modules. Denote $F_2 = (A_1 \times B_1, C_1 \times C_1; f_1, \mathcal{L}_{mod})$. Either f_1 or f_2 are of the desired form, and thus Theorem 3.5 can be applied to at least one of them. Moreover:

Lemma 3.10. Both F_1 and F_2 are e-interpretable in F. The same is true if one replaces \mathcal{L}_{mod} with \mathcal{L}_{group} in both structures.

Proof. Let $S_A = \{a_1, \ldots, a_n\}$ and $S_B = \{b_1, \ldots, b_m\}$ be generating sets of A and B, respectively. The submodules $Ann_l(f)$ and $Ann_r(f)$ are e-definable as sets in F by the systems of equations $\wedge_i f(x, b_i) = 0$ and $\wedge_i f(a_i, y) = 0$, respectively. Here x and y are variables taking values in A and in B, respectively. Any $c \in C_1 = \langle f(A, B) \rangle_{\Lambda}$ can be written as

$$c = \sum_{i,j} \lambda_{i,j} f(a_i, b_j) = \sum_j f\left(\sum_i \lambda_{i,j} a_i, b_j\right)$$
 for some $\lambda_{i,j} \in \Lambda$.

It follows that C_1 is e-definable as a set in F by the equation $z = \sum_j f(x_j, b_j)$ on variables z and $\{x_j \mid j = 1, \ldots, n\}$. These variables take values in C and in A, respectively.

The operations of $Ann_l(f)$, $Ann_r(f)$ and C_1 are e-interpretable in F because they are already present in the signature of F. Hence $(A_1; \mathcal{L}_{mod})$ and $(B_1; \mathcal{L}_{mod})$ are e-interpretable as Λ -modules in $(A; \mathcal{L}_{mod})$ and $(B; \mathcal{L}_{mod})$, respectively (see Lemma 2.3 and Remark 2.4). Moreover, from the proof of Lemma 2.3 and the fact that the e-definitions of $Ann_l(f)$ and $Ann_r(f)$ do not use multiplication by scalars, we have that $(A_1; \mathcal{L}_{group})$ and $(B_1; \mathcal{L}_{group})$ are e-interpretable in $(A; \mathcal{L}_{group})$ and $(B; \mathcal{L}_{group})$, respectively.

The preimage in F of the graph of f_1 is e-definable in F by the system of equations $z = f(x, y), z = \sum_j f(x_j, b_j)$ (the second equation ensures that z takes values in C_1). Again this equation does not use multiplication by scalars. We conclude that $F_1 = (A_1, B_1, C_1; f_1, \mathcal{L}_{mod})$ is e-interpretable in $(A, B, C; f, \mathcal{L}_{mod})$, and that the same holds if one drops multiplication by scalars in both structures.

Finally, note that $(A_1 \times B_1; \mathcal{L}_{mod})$ is e-interpretable in $(A_1, B_1; \mathcal{L}_{mod})$, and $(C_1 \times C_1; \mathcal{L}_{mod})$ is e-interpretable in $(C_1; \mathcal{L}_{mod})$ (they are basic sets of (A_1, B_1) and C_1 , and so they are defined as sets by empty systems of equations). The equations $z = f_1(x, y')$ and $z' = f_1(x', y)$ on variables x, y, x', y', z, z' e-define the graph of f_2 in F_1 . It follows that the whole two-sorted structure F_2 is e-interpretable in F_1 , and also in F by transitivity. Moreover, this is still valid if one replaces \mathcal{L}_{mod} by \mathcal{L}_{group} in the three structures F_2, F_1 and F.

Proof of Theorem 3.9. The result follows immediately after using Theorem 3.5 in order to e-interpet $(Z(Sym(f_2)); \mathcal{L}_{alg})$ or $(Z(Sym(f_1)); \mathcal{L}_{alg})$ in F_2 or F_1 , depending on whether or not $A_1 = B_1$, respectively. Items 1 and 2 are a direct consequence of Items 1 and 2 of Theorem 3.5.

Remark 3.11. In Theorem 3.9 we have e-interpreted $Z(Sym(f_2))$ in F_2 (or $Z(Sym(f_1))$ in F_1 if $A_1 = B_1$). Alternatively one can also e-interpret the largest ring of scalars $R(f_2)$ of f_2 in F_2 (similarly for f_1). This may have some advantages if one seeks to study the structure of A, B, C and f, because $R(f_2)$ is determined by "more properties" of these than $Z(Sym(f_2))$. However, when it comes to the Diophantine problem, $Z(Sym(f_2))$ is a more practical choice than $R(f_2)$, because it uses a simpler e-interpretation. For instance, as we have seen, if Λ is a field then one can drop multiplication by scalars in the e-interpretation of $(Z(Sym(f_2)); \mathcal{L}_{ring})$, whereas there is no apparent way to do the same with $(R(f_2); \mathcal{L}_{ring})$.

4 From rings of scalars to rings of integers of global fields

In the previous section we established a method for passing from bilinear maps to commutative structures. In this section we provide a reduction from the latter to rings of number theoretic flavour.

We refer to Subsection 2.3 for relevant conventions, notation, and terminology used throughout the paper. We start with two useful results.

Lemma 4.1. Let R be a module-finite algebra over a Noetherian ring of scalars A. Then $(R; \mathcal{L}_{alg})$ (in particular, $(R; \mathcal{L}_{ring})$) is e-interpretable in $(A; \mathcal{L}_{ring})$.

Proof. Let r_1, \ldots, r_n be a finite set of generators of R as an A-module, and let A^n be the direct product of n copies of A, i.e. the free A-module of rank n. Consider the natural projection $\phi: A^n \to R$ defined by $\phi(a_1, \ldots, a_n) = \sum a_i r_i$. We claim that ϕ is an e-interpretation of $(R; \mathcal{L}_{alg})$ in $(A; \mathcal{L}_{ring})$. Indeed, it is clear that ϕ is surjective and that $\phi^{-1}(R) = A^n$ is e-definable in $(A; \mathcal{L}_{alg})$. From here on, we look at r_1, \ldots, r_n indistinctly as a set of generators of R and as a natural base of A^n .

We now check that the preimage of the equality relation on R is e-definable in A. Since A is Noetherian, R is finitely presented as an A-module. Let this presentation be $\langle r_1, \ldots, r_n \mid w_1, \ldots, w_k \rangle_A$, where for all $i = 1, \ldots, k$ we have $w_i = \sum a_{i,j}r_j$ for some $a_{i,j} \in A$ $(j = 1, \ldots, n)$. If $x_i, y_i \in A$ $(i = 1, \ldots, n)$, then $\sum x_i r_i = \sum y_i r_i$ holds in R if and only if $\sum x_i r_i = \sum y_i r_i + \sum z_j w_j$ holds in A^n . Expanding $w_j = \sum a_{i,j}r_i$, and collecting summands with the same r_i , we conclude that

$$\sum_{i} x_{i} r_{i} = \sum_{i} y_{i} r_{i} \quad \text{in } R \quad \text{if and only if} \quad \bigwedge_{i} \left(x_{i} - y_{i} = \sum_{j} z_{j} a_{i,j} \right) \quad \text{in } A. \tag{15}$$

The right-hand side is a finite system of equations on variables $\{x_i, y_i, z_i \mid i\}$. This defines in A the preimage of equality in R.

Next we prove that the preimage of the graph of the ring multiplication in R is edefinable in $(A; \mathcal{L}_{alg})$. For each $1 \leq i, j, k \leq n$ let $b_{j,k,i} \in A$ be such that $r_j r_k = \sum b_{j,k,i} r_i$. Take any three elements x, y, z of R, and let x_j, y_k, z_i be elements of A such that $x = \sum x_j r_j$, $y = \sum y_k r_k$, and $z = \sum z_i r_i$. Then the equality in R

$$xy = \left(\sum x_j r_j\right) \left(\sum y_k r_k\right) = \sum z_i r_i = z$$

is equivalent to

$$\sum_{i} \left(\sum_{j,k} x_j y_k b_{j,k,i} \right) r_i = \sum_{i} z_i r_i.$$
 (16)

Using (15) we see that (16) can be written as a system of equations on variables $\{x_j, y_k, z_i\}$ and some new variables $\{w_\ell\}$.

Finally, e-interpretability of the remaining operations of R (addition and mulitplication by scalars) follows in a similar way using (15) and the expressions $\sum x_i r_i + \sum y_i r_i = \sum (x_i + y_i)r_i$, and $a(\sum x_i r_i) = \sum (ax_i)r_i$, for $a, x_i, y_i \in A$.

We will also need the following:

Lemma 4.2. Let I be a finitely generated ideal of a ring of scalars A. Then I as a set is e-definable in A, and $(A/I; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$.

Proof. Let a_1, \ldots, a_n be a generating set of I. Then the equation $x = \sum x_i r_i$ on variables $\{x\} \cup \{x_i \mid i\}$ e-defines I in A as a set. Lemma 2.3 now implies that $(A/I; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$.

4.1 Subrings of global fields

The goal of this subsection is to prove the result below. See Subsection 2.2 for an explanation of the terminology used here.

Theorem 4.3. Let R be a finitely generated integral domain whose field of fractions K is a global field. Then the ring of integers of K is e-interpretable in R in the language of rings (with constants). Furthermore, if K has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable.

The proof will be obtained essentially by putting together some already known results, which we state below. Most of them can be found in Shlapentokh's book [43]. We will follow its notation and terminology closely. The next statement was originally proved by Shlapentokh in [41] (here we follow a different reference of the same author).

Theorem 4.4 (10.6.2 and 2.1.2 and 2.1.10 of [43]). Suppose that K is a global function field of characteristic p, and let O_K be its ring of integers. Then there exists a transcendental element $t \in K$ such that $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(O; \mathcal{L}_{ring})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is undecidable.

Proof. Following the terminology of [43], Proposition 10.6.2 states that $\mathbb{F}_p[t]$ is Diophantinegenerated over O_K , while 2.1.2 and 2.1.10 of the same reference yield that in this case this is equivalent to $\mathbb{F}_p[t]$ being e-definable in $(O_K; \mathcal{L}_{ring})$ as a set (in the proof of Corollary 4.6 we will comment further on this equivalence). Now $\mathcal{D}(O; \mathcal{L}_{ring})$ is undecidable because $\mathcal{D}(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(O; \mathcal{L}_{ring})$, and because $\mathcal{D}(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is undecidable by [5].

We shall also use the fact, stated below, that integrality at a prime (and thus at finitely many) is e-definable in a global field. For global function fields this was proved by Shlapentokh in [42]. For number fields this is implicit in the work of Robinson [35, 36], and it was written down explicitly by Kim and Roush in [20]. See also [9] or Theorems 4.2.4 and 4.3.4 of [43] for a unified account of these results.

Theorem 4.5 (E-definability of integrality at a prime for global fields). Let K be a global field and let p be a prime of K. Then the ring of S-integers $O_{K,S} = \{x \in K \mid ord_p(x) \geq 0\}$ is e-definable as a set in K. Here S is the set of all primes of K but p.

Since $O_{K,S}$ is a subring of K, it follows that $O_{K,S}$ is e-interpretable as a ring in K (following the notation of the above Theorem 4.5). As a consequence we have:

Corollary 4.6. Let O_K be the ring of integers of a global field K. Let S be a finite set of primes of K. In the case that K is a finite extension of $\mathbb{F}_p(t)$, assume further that S contains all the poles of t, i.e. that $S_{\infty} \subseteq S$ (see Remark 2.8). Then $(O_K; \mathcal{L}_{ring})$ is e-interpretable in $(O_{K,S}; \mathcal{L}_{ring})$.

This result is stated in Subsections 2.3.2 and 2.3.3 of [43] for number fields and global function fields, respectively, though we could not find full direct proofs. In any case, it follows quickly from several results of the cited reference, as we see now.

Proof. We simply connect some results from Shlapentokh's book [43], while following its terminology and numeration. Unless stated otherwise all results that have 3 digits in the numeration are from this reference. The author uses the notion of Diophantine-generation (see Definition B.1.20) instead of e-definition or e-interpretation. In 2.1.2 and 2.1.10 it is proved that if R_1 and R_2 are two subrings of a global field such that $R_1 \leq R_2$, then R_1 is Dioph-generated over R_2 if and only if R_1 as a set is is e-definable in $(R_2; \mathcal{L}_{ring})$, in which case $(R_1; \mathcal{L}_{ring})$ is e-interpretable in $(R_2; \mathcal{L}_{ring})$. During this proof we will always have the case $R_1 \leq R_2$, and thus for our purposes there is no difference between Diophantine-generation and e-definability. Moreover, both of them imply e-interpretability as a ring.

Denote the complement of a set of primes W by W^c (in the set of all non-archimedean primes). Observe that $O_{K,S^c} = \cap_{p \in S} O_{K,p^c}$. Since S is finite, this intersection is finite, and by the previous Theorem 4.5 we obtain that O_{K,S^c} is e-definable (in particular, Dioph-generated) in K. Lemma 2.2.2 states that the field of fractions of a domain D is Dioph-generated over D, provided that the set of non-zero elements of D is e-definable in D. In Proposition 2.2.4 it is proved that $O_{K,W}$ satisfies this last condition for any set of primes W. Therefore K is Dioph-generated over $O_{K,S}$, and by transitivity of Dioph-generation (Theorem 2.1.15), O_{K,S^c} is Dioph-generated over $O_{K,S}$. In 2.1.19 it is seen that the intersection of finitely many subrings of an integral domain R, each Dioph-generated over R, is still Dioph-generated over R. Hence, if K is a number field, then $O_K = O_{K,\emptyset} = O_{K,S^c} \cap O_{K,S}$ is Dioph-generated over $O_{K,S}$. Since $O_K \leq O_{K,S}$, we have then that $O_K \in C_{Fing}$ is e-interpretable in $O_{K,S}$: C_{Fing} .

Suppose otherwise that K is a global function field. Recall that in this case $O_K = O_{K,S_{\infty}}$ (see Remark 2.8). By the same argument as before, $O_K = O_{K,S_{\infty}} = O_{K,(S \setminus S_{\infty})^c} \cap O_{K,S}$, and so O_K is Dioph-generated over $O_{K,S}$ (recall that S_{∞} is finite and thus $S \setminus S_{\infty}$ is finite as well). Since $O_K \leq O_{K,S}$ because $S_{\infty} \subseteq S$, we obtain again that $(O_K; \mathcal{L}_{ring})$ is e-interpretable in $(O_{K,S}; \mathcal{L}_{ring})$.

We will also need the following auxiliary result.

Lemma 4.7 (Corollary 4.6.5 of [13]). Let D be a finitely generated integral domain. Then the integral closure \overline{D} of D in a finite extension of the field of fractions of D is module-finite over D (i.e. \overline{D} is finitely generated as a D-module).

We can now prove the main theorem of this subsection.

Proof of Theorem 4.3. By Lemma 4.7, the integral closure \overline{R} of R in K is module-finite over R. Then Lemma 4.1 implies that $(\overline{R}; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$. Since \overline{R} is module-finite over R and R is f.g., \overline{R} is f.g. as well. Then Proposition 2.9 and Lemma 2.10 yield that $\overline{R} = O_{K,S}$ for some global field K and some finite set S of primes of K. By Theorem 4.6, the ring of integers O_K of K is e-interpretable in $O_{K,S}$ in the language of rings, and so by transitivity O_K is e-interpretable in R in the same language. The last part of the theorem is a consequence of Theorem 4.4.

Remark 4.8. Say that the rank of a ring is the maximum number of \mathbb{Z} -linearly independent elements of the ring. Following the notation of Theorem 4.3, suppose that K is a number field and that R has finite rank n. We claim that R and O_K have the same rank. Indeed, since R has characteristic zero, R is freely generated as a \mathbb{Z} -module by n elements. It follows that $|K:\mathbb{Q}|=n$, because K is the field of fractions of R. On the other hand, it is well known that the rank of O_K coincides with the degree $|K:\mathbb{Q}|$, and hence the claim is proved.

4.2 Rings of scalars

Recall that by a ring (algebra) of scalars we mean an associative, commutative, unitary ring (algebra). In this subsection we prove the following:

Theorem 4.9. Let A be an infinite finitely generated ring of scalars. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(A; \mathcal{L}_{ring})$. Moreover:

- 1. If A is finitely generated as an abelian group, then O is a ring of algebraic integers.
- 2. If A has positive characteristic, then O is the ring of integers of a global function field, $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$ for some prime p, and $\mathcal{D}(A; \mathcal{L}_{ring})$ is undecidable.

This theorem is proved by reducing it to the case when A is a subring of a global field, and then applying the main theorem of the previous subsection. To make such a reduction we closely follow Noskov's arguments from [30], where he proved that any infinite finitely generated ring of scalars has undecidable first order theory.

Proof. Throughout the proof we will use implicitly the facts that e-interpretability is transitive, and that the quotient by any ideal of a Noetherian ring R is e-interpretable in R, by Lemma 4.2.

Suppose first that A is not an integral domain, and let N be the nilradical of A, i.e. the ideal formed by all nilpotent elements of N. Equivalently, N is the intersection of all minimal prime nonzero ideals of A. There are finitely many such ideals q_1, \ldots, q_n in a Noetherian ring (see Theorem 87 of [14]). Since A is not an integral domain, we have $n \geq 1$, because A contains at least one nonzero maximal ideal. As argued in 3.2 of [30] there exists i such that A/q_i is infinite. Moreover A/q_i is a finitely generated integral domain. Note also that if A is finitely generated as an abelian group, then so is A/q_i . Similarly, if A has positive characteristic, then so does A/q_i .

In views of the previous paragraph, assume that A is an infinite finitely generated integral domain. The Krull dimension of A is the largest integer k for which there exists a proper ascending chain of prime ideals $p_0 < p_1 < \ldots < p_k < A$. Such k is finite under our assumptions (see Section 8.2.1 of [7]). It is not possible that k = 0, for in this case A would be a finitely generated Artinian domain, and thus a finitely generated field (see Proposition 8.30 of [2]), a contradiction because such a field is necessarily finite. Hence $k \geq 1$. We may assume that k = 1, since if $k \geq 2$ then A/p_{k-1} is a finitely generated integral domain of Krull dimension 1. Moreover A/p_{k-1} is finitely generated as an abelian group if A is, and it has positive characteristic if A does.

In 1.3, 1.4, and 2.2 of [30] it is proved that if an infinite finitely generated integral domain R has Krull dimension 1, then one of the following hold:

- 1. R is a subring of a number field (if R has characteristic 0).
- 2. There exists a prime number p and a transcendental element $t \in R$ such that $\mathbb{F}_p[t] \leq R$ and R is integral over $\mathbb{F}_p[t]$. This follows by Noether normalization assuming that R has positive characteristic p, in which case R is a finitely generated \mathbb{F}_p -algebra. In particular, R is a subring of a global function field.

It follows that the field of fractions K of A_1 is a global field. Since A_1 is finitely generated, Theorem 4.3 implies that the ring of integers O_K of K is e-interpretable in A_1 in the language of rings (with constants). By transitivity, $(O_K; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$, and $\mathcal{D}(O_K; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(A; \mathcal{L}_{ring})$ by Proposition 2.6.

If A is finitely generated as an abelian group, then it has characteristic 0, because it is infinite. Hence, A_1 is a subring of a number field, and O_K is a ring of algebraic integers. If otherwise A has characteristic p > 0, then Theorem 4.3 and transitivity of e-interpretations yield that $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$, and that $\mathcal{D}(A; \mathcal{L}_{ring})$ is undecidable.

Remark 4.10. The ring O_K of Theorem 4.9 has rank at most the rank of R (recall that the rank of a ring is its maximum number of \mathbb{Z} -linearly independent elements). This follows by Remark 4.8 and by the proof of the previous Theorem 4.9, since the rank of a ring does not increase after taking a quotient.

5 Rings and algebras over finitely generated rings of scalars

The following lemma is a combination of the results obtained so far. It constitutes the main "general tool" presented in this paper (see also Lemma 6.4). Throuhgout the rest of the section we will explore its consequences. In a subsequent paper we plan to apply it further to the area of group theory. We refer again to Subsection 2.3 for important notation and terminology conventions.

Lemma 5.1. Let Λ be a ring of scalars, let $f: A \times B \to C$ be a Λ -bilinear map between finitely generated Λ -modules, and write $C_1 = \langle Im(f) \rangle_{\Lambda}$. Suppose that $(A, B, C; f, \mathcal{L}_{mod})$ is e-interpretable in some structure \mathcal{M} . Then there exists a module-finite Λ -algebra of

scalars R such that $(R; \mathcal{L}_{alg})$ is e-interpretable in \mathcal{M} . The ring R is 0, finite, or infinite if and only if C_1 is 0, finite, or infinite, respectively.

Furthermore, if C_1 is infinite and Λ is finitely generated, then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$ (hence in \mathcal{M}), and additionally in this case:

- 1. If Λ has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in \mathcal{M} , and $\mathcal{D}(\mathcal{M})$ is undeciable.
- 2. If $\Lambda = \mathbb{Z}$ then O is a ring of algebraic integers.

If Λ is \mathbb{Z} or a field, then the whole lemma holds after replacing $(A, B, C; f, \mathcal{L}_{mod})$ by $(A, B, C; f, \mathcal{L}_{group})$ and $(R; \mathcal{L}_{alg})$ by $(R; \mathcal{L}_{ring})$, i.e. multiplication by scalars is not required.

Proof. By Theorem 3.9, there exists a module-finite Λ -algebra of scalars R such that $(R; \mathcal{L}_{alg})$ is e-interpretable in $(A, B, C; f, \mathcal{L}_{mod})$, and so in $(R; \mathcal{L}_{alg})$ by transitivity of e-interpretations. The statement regarding the cardinality of R follows from Item 2 of Theorem 3.9.

Suppose that C_1 is infinite and that Λ is finitely generated. Then R is infinite and finitely generated as a ring. Hence, by Theorem 4.9 there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and thus in \mathcal{M} .

If Λ has positive characteristic, then so does R, because it is a unitary algebra over Λ . Hence $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in \mathcal{M} , by Item 2 of Theorem 4.9 and by transitivity. If $\Lambda = \mathbb{Z}$, then R is finitely generated as an abelian group, and so O is a ring of algebraic integers by Item 1 of Theorem 4.9.

If Λ is \mathbb{Z} or a field, then $(R; \mathcal{L}_{ring})$ is e-interpretable in $(A, B, C; f, \mathcal{L}_{group})$, by Item 1 of Theorem 3.9. Hence if the latter is e-interpretable in \mathcal{M} , then the lemma holds after replacing \mathcal{L}_{alg} by \mathcal{L}_{ring} and \mathcal{L}_{mod} by \mathcal{L}_{group} , due to the previous considerations and due to transitivity of e-interpretations.

5.1 Module-finite rings and algebras

Throughout this subsection Λ denotes a finitely generated ring of scalars.

The following is one of the main results of the paper. A similar theorem will be obtained in Subsection 6, where we study the case when Λ is an arbitrary field (possibly infinitely generated). The case $\Lambda = \mathbb{Z}$ will be considered separately afterwards.

Theorem 5.2. Let R be a (possibly non-associative, non-commutative, and non-unitary) module-finite algebra over a finitely generated ring of scalars Λ . Suppose that R^2 is infinite. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{alg})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(R; \mathcal{L}_{alg})$. Moreover:

- 1. If Λ has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{alg})$, and $\mathcal{D}(R; \mathcal{L}_{alg})$ is undecidable.
- 2. If R^2 is finite and $\mathcal{D}(R;\mathcal{L}_{mod})$ is decidable, then $\mathcal{D}(R;\mathcal{L}_{alg})$ is decidable.

If Λ is a finite field, then all the above holds after replacing $(R; \mathcal{L}_{alg})$ by $(R; \mathcal{L}_{ring})$.

Proof. The ring multiplication operation \cdot of R induces a Λ -bilinear map between finitely generated Λ -modules $\cdot: R \times R \to R$, with $\langle Im(\cdot) \rangle_{\Lambda} = R^2$. Since the three-sorted structure $(R, R, R; \cdot, \mathcal{L}_{mod})$ is e-interpretable in $(R; \mathcal{L}_{alg})$, the theorem (except Item 2) follows from Lemma 5.1.

We now prove Item 2. Let Σ be a system of equations in the Λ -algebra R. By adding new variables and equations, we may assume that Σ consists of 1) equations in the Λ -module $(R; \mathcal{L}_{mod})$ (i.e. Λ -linear equations), together with 2) equations of the form $x_1 = x_2 \cdot x_3$, where each x_i is a variable that appears exactly once among all the equations of the second type. We say that x_2 is a variable of type left, and that x_3 is a variable of type right.

Note that $Ann_l(\cdot)$ and $Ann_r(\cdot)$ are finite index submodules of R, by Item 2 of Theorem 3.9. Take a_1,\ldots,a_s and b_1,\ldots,b_t to be full systems of coset representatives of $R/Ann_l(\cdot)$ and $A/Ann_r(\cdot)$, respectively. Let also S_R be a finite generating set of R. For each variable x of type left, choose a coset representative $a_{i(x)} \in \{a_1,\ldots,a_s\}$, and introduce a new variable x'. Then replace each occurrence of x in Σ by $a_{i(x)} + x'$. We wish x' to take values in $Ann_l(\cdot)$. Clearly, the system of equations $\wedge_{r \in S_R} zr = 0$ on the variable z e-defines $Ann_l(\cdot)$ in R. Hence, for each new variable x', add $\wedge_{r \in S_R} x'r = 0$ to Σ in order to ensure that x' takes values in $Ann_l(\cdot)$. Notice that this is a system of equations in the Λ -module $(R; \mathcal{L}_{mod})$. Proceed in an analogous way with each variable y of type right. Let Σ' be the resulting system of equations. Since there are finitely many coset representatives, the number of all possible resulting systems Σ' is finite. Let Σ_1,\ldots,Σ_m be all of them. It is clear that Σ has a solution if and only if Σ_i has a solution for some i.

We now prove that it is possible to decide algorithmically if each one of the Σ_i has a solution or not, in which case our proof is concluded. Indeed, each Σ_i consists in some equations in the Λ -module $(R; \mathcal{L}_{mod})$, together with some equations of the form $x = (a_i + x')(b_j + y')$, where x' and y' are bound to take values in $Ann_l(\cdot)$ and in $Ann_l(\cdot)$, respectively. Hence each equation $x = (a_i + x')(b_j + y')$ can be replaced with $x = a_i b_j$, which is an equation in $(R; \mathcal{L}_{mod})$. Thus Σ_i is equivalent to a system of Λ -linear equations, which is decidable by hypothesis.

The following is essentially a particular case of the previous Theorem 5.2. It is stated separately due to its independent interest.

Theorem 5.3. Let A be a ring (possibly non-associative, non-commutative, and non-unitary). Assume that A is finitely generated as an abelian group, and that A^2 is infinite. Then there exists a ring of algebraic integers O such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(A, \mathcal{L}_{ring})$. If otherwise A^2 is finite, then $\mathcal{D}(R; \mathcal{L}_{ring})$ is decidable.

Proof. The first part of the theorem follows in the same way as Theorem 5.2, taking $\Lambda = \mathbb{Z}$ and observing that here O is a ring of algebraic integers by Item 2 of Lemma 5.1. The last part follows by Item 2 of Theorem 5.2 (note that $(A; \mathcal{L}_{mod})$ is just a finitely generated abelian group, and so $\mathcal{D}(A; \mathcal{L}_{mod})$ is decidable).

5.2 Finitely generated rings and algebras satisfying an infiniteness condition

In this subsection Λ denotes a possibly infinitely generated ring of scalars.

We next apply the previous Theorems 5.2 and 5.3 to certain classes of possibly non-module-finite finitely generated rings and algebras R. The approach consists in e-defining an ideal I_n in R that contains "enough" products of at least n elements of R (for example, the ideal generated by all such products), so that the quotient R/I_n is infinite and module-finite. Then it suffices to apply the results from Subsection 5.1, together with transitivity of e-interpretations. This approach has two limitations:

- 1. I_n can be difficult to e-define if R is non-associative (hence Definition 5.5).
- 2. R/I_n may be finite. For instance, if R is unitary one cannot simply take I_n to be the ideal generated by all products of n elements of R, since then $I_n = R$ (hence the next definition).

Definition 5.4. Any unitary Λ -algebra R admits a generating set of the form $\{1\} \cup T$ for some set T such that $T \cap \{\lambda \cdot 1 \mid \lambda \in \Lambda\} = \emptyset$. We let $I_n(T)$, or simply I_n , be the ideal generated by all all products of n elements of T ($n \geq 1$). If R is non-unitary, then I_n denotes the ideal generated by all Λ -multiples of all products of n elements of R. In both cases I_n is a Λ -subalgebra of R.

Throughout the rest of the section R denotes a finitely generated Λ -algebra, possibly non-module finite, non-associative, non-commutative, and non-unitary. We fix a finite set $T = \{a_1, \ldots, a_m\}$ as in Definition 5.4, and we define the ideals I_n accordingly.

Definition 5.5. Suppose that R is non-associative. Then R is called *right-norm-generated* if, for all $n \geq 1$, I_n is generated as a Λ -module by a (possibly infinite) subset of $\{(a_{i_1}(a_{i_2}(\ldots(a_{i_{k-1}}a_{i_k})\ldots))) \mid k \geq n, \ 1 \leq i_1,\ldots,i_k \leq m\}.$

Later we will see that many graded Lie algebras are right-normed-generated.

Lemma 5.6. Suppose that R is associative or right-normed-generated, and let $n \geq 1$. Then $(R/I_n; \mathcal{L}_{ring})$ and $(R/I_n; \mathcal{L}_{alg})$ are e-interpretable in $(R; \mathcal{L}_{ring})$ and $(R; \mathcal{L}_{alg})$, respectively. Moreover, R/I_n is a module-finite Λ -algebra.

Proof. Let $T = \{a_1, \ldots, a_m\}$, and assume first that R is right-normed-generated. Then each element of I_n is a finite sum of elements of the form

$$\lambda(a_{i_1}(a_{i_2}(\dots(a_{i_{k-1}}a_{i_k})\dots))), \quad \lambda \in \Lambda, \quad k \ge n. \tag{17}$$

Hence each element as in (17) can be written in the form $(a_{i_1}(\ldots(a_{i_{n-1}}y)\ldots))$ for some $y \in R$ in the non-unitary case, and in the form $(a_{i_1}(\ldots(a_{i_{n-1}}(a_{i_n}y))\ldots))$ in the unitary case. Consequently, if R is non-unitary, then I_n is e-definable as a set in $(R; \mathcal{L}_{ring})$ by the equation

$$x = \sum_{1 \le i_1, \dots, i_{n-1} \le m} (a_{i_1}(\dots(a_{i_{n-1}}y_{i_1, \dots, i_{n-1}})\dots))$$
(18)

on variables x and $\{y_{i_1,\dots,i_{n-1}}\}$ (the e-definition is in $(R; \mathcal{L}_{ring})$ because it makes no use of multiplication by scalars Λ). Observe however that if R is unitary then a solution to (18) may yield $x \in I_{n-1}$, for example if each $y_{i_1,\dots,i_{n-1}}$ is a Λ -multiple of 1. This can be solved by taking the equation

$$x = \sum_{1 \le i_1, \dots, i_n \le m} (a_{i_1}(\dots(a_{i_{n-1}}(a_{i_n}y_{i_1, \dots, i_n}))))$$
 (19)

By our previous considerations, (19) e-defines I_n as a set in $(R; \mathcal{L}_{ring})$ when R is unitary. Thus in both cases $(R/I_n; \mathcal{L}_{ring})$ and $(R/I_n; \mathcal{L}_{alg})$ are e-interpretable in $(R; \mathcal{L}_{ring})$ and $(R; \mathcal{L}_{alg})$ by Lemma 4.2.

If R is associative, then any element $x \in I_n$ is also a sum of elements of the form (17), and the proof follows as in the non-associative case.

Finally, note that R/I_n is generated as a Λ -module by the projection of all products of less than n elements of T, together with $1 + I_n$ if R is unitary. It follows that R/I_n is module-finite.

We now state the main result of this subsection. The ideals I_n are defined with respect to any set T satisfying the condition of Definition 5.4.

Theorem 5.7. Let R be a finitely generated Λ -algebra (possibly non-module-finite, non-associative, non-commutative and non-unitary). Suppose that Λ is finitely generated, that R is associative or right-normed-generated, and that $(R/I_n)^2$ is infinite for some $n \geq 1$. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{alg})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(R; \mathcal{L}_{alg})$. Moreover:

- 1. If Λ has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{alg})$, and $\mathcal{D}(R; \mathcal{L}_{alg})$ is undecidable.
- 2. If R is a ring (i.e. $\Lambda = \mathbb{Z}$) then O is a ring of algebraic integers.

If Λ is \mathbb{Z} or a finite field then all the above holds after replacing $(R; \mathcal{L}_{alg})$ by $(R; \mathcal{L}_{ring})$.

Proof. By Lemma 5.6, R/I_n is a module-finite Λ -algebra that is e-interpretable as an algebra in $(R; \mathcal{L}_{alg})$. The same result states that R/I_n is e-interpretable as a ring in $(R; \mathcal{L}_{ring})$. By hypothesis, we can take n so that $(R/I_n)^2$ is infinite. Now the result follows by Theorems 5.2 and 5.3 applied to R/I_n , and by transitivity of e-interpretations.

We next apply the previous theorem to a class of non-associative algebras. Recall that a Λ -algebra L is called graded (over a semigroup (I, +)) if there exists a Λ -module direct-sum decomposition $L = \bigoplus_{i \in I} L_i$ such that $L_i L_j \subseteq L_{i+j}$ for all $i, j \in I$. In this case, each L_i is called an homogeneous component of L, and the elements from L_n are said to be homogeneous of degree n. We say that L is simply graded if L_1 generates A as a Λ -algebra, assuming that L is graded over $\mathbb{N} = \{1, 2, \dots\}$.

The following result is a generalization of Lemma 3.3 of [1]. The proof is essentially the same.

Lemma 5.8. Any simply graded Lie Λ -algebra is right-normed-generated.

Proof. Let $L = \bigoplus_{n \geq 1} L_n$ be a gradation of L over $\mathbb{N} = \{1, 2, \dots\}$ (note that L is non-unitary), and let $T = \{a_1, \dots\}$ be a generating set of L_1 as a Λ -module. Then T generates L as a Λ -algebra, and moreover the degree of a product of k elements from T is k. We prove that the set $S = \{[a_{i_1}, [\dots [a_{i_{k-1}}, a_{i_k}] \dots]] \mid a_{i_j} \in T, k \geq 1\}$ generates L as a Λ -module. It suffices to see that any homogeneous element $w \in L$ is a Λ -linear combination of elements from S. This follows by induction on the degree of w and by the identity [[x, y], z] = [x, [y, z]] - [y, [x, z]] (see Lemma 3.3 of [1]).

Now let $n \geq 1$ and $x \in I_n(T) \subseteq \bigoplus_{i \geq n} L_i$ (see Definition 5.4 for an explanation of the terminology $I_n(T)$). By the previous claim, x is a Λ -linear combination of elements from S, and so $x = \sum_{s \in S} \lambda_s s$ for some $\lambda_s \in \Lambda$, almost all 0. Using the gradation of L one obtains that $\lambda_s = 0$ for all s of degree less than n, i.e. for all $s = [a_{i_1}, [\dots [a_{i_{k-1}}, a_{i_k}] \dots]]$ with k < n. In conclusion, I_n is generated as a L-module by all elements of S of degree at least n.

The next two corollaries follow immediately from Theorem 5.7 and Lemma 5.8. As usual, $[R/I_n, R/I_n]$ denotes the Λ -module generated by $\{[x, y] \mid x, y \in R/I_n\}$.

Corollary 5.9. Let L be a finitely generated simply graded Lie Λ -algebra. Assume that $[R/I_n, R/I_n]$ is infinite for some $n \geq 1$, and that Λ is finitely generated. Then the conclusions of Theorem 5.7 hold for L.

Corollary 5.10. Let F be a free associative Λ -algebra (possibly non-commutative and non-unitary) or a free Lie algebra of rank at least 2, with Λ finitely generated. Then the conclusions of Theorem 5.7 hold for F.

Proof. In both cases $(F/I_n)^2$ is infinite for all $n \geq 3$. Moreover, any free Lie algebra is simply graded. Thus the result follows from Theorem 5.7 and the previous Corollary 5.9.

Corollary 5.10 complements Romankov's [37], and Kharlampovich and Miasnikov's [15, 17] papers on free algebras. In the first reference it is proved that $\mathcal{D}(F; \mathcal{L}_{ring})$ is undecidable for many types of free rings F. In particular it is proved that the algebras of Corollary 5.10 actually have undecidable Diophantine problem if $\Lambda = \mathbb{Z}$. In the references [15, 17] it is proved that $\mathcal{D}(F; \mathcal{L}_{ring})$ is undecidable if Λ is an arbitrary field, and F is a free associative non-commutative unitary algebra, or a free Lie algebra of rank at least 3. Note that an infinite field is necessarily infinitely generated, and so our Corollary 5.10 does not cover many of the cases considered in [15, 17].

5.3 Finitely generated associative commutative non-unitary rings and algebras

In this subsection we study non-unitary rings and algebras of scalars.

Lemma 5.11. Let A be a finitely generated associative commutative non-unitary algebra over a (possibly infinitely generated) ring of scalars Θ . Then the following exist:

- 1. A ring of scalars Λ and a module-finite Λ -algebra of scalars B such that $(B; \mathcal{L}_{alg})$ is e-interpretable in $(A; \mathcal{L}_{alg})$.
- 2. A finitely generated Θ -algebra of scalars C such that $(C; \mathcal{L}_{alg})$ is e-interpetable in $(A; \mathcal{L}_{alg})$.

Additionally, C is module-finite if A is module-finite, and if A^2 is zero, nonzero or infinite, then both B and C are zero, nonzero or infinite as well, respectively.

Proof. The set $\Lambda = \Theta + A = \{\theta + a \mid \theta \in \Theta, a \in A\}$ is a ring of scalars under the obvious operations of addition and multiplication. Moreover, Λ acts naturally by endomorphisms on A, and with this action A is a Λ -algebra. During this proof we write A_{Λ} and A_{Θ} to refer to A seen as a Λ -algebra or as a Θ -algebra, respectively (we will proceed similarly with other algebras below). The operation of multiplication by a given scalar $\theta + a \in \Lambda$ is e-interpreted in $(A_{\Theta}; \mathcal{L}_{alg})$ by the equation $y = \theta x + ax$. Thus, $(A_{\Lambda}; \mathcal{L}_{alg})$ is e-interpretable in $(A_{\Theta}; \mathcal{L}_{alg})$. Suppose that A_{Θ} is generated as a Θ -algebra by n elements a_1, \ldots, a_n . Then A_{Λ} is generated as a Λ -module by these same elements, since for all $x \in A$ there exists $y_1, \ldots, y_n \in A \leq \Lambda$ such that $x = \sum_i y_i a_i$. In particular, A is a module-finite Λ -algebra.

The ring multiplication of A_{Λ} is a Λ -bilinear map between finitely generated Λ -modules $\cdot: A \times A \to A$. Moreover, $(A_{\Lambda}, A_{\Lambda}, A_{\Lambda}; \cdot, \mathcal{L}_{mod})$ is e-interpretable in $(A_{\Lambda}; \mathcal{L}_{alg})$, which in turn is e-interpretable in $(A_{\Theta}; \mathcal{L}_{alg})$. Hence, by the first part of Lemma 5.1, there exists a module-finite Λ -algebra of scalars D_{Λ} such that $(D_{\Lambda}; \mathcal{L}_{alg})$ is e-interpretable in $(A_{\Theta}; \mathcal{L}_{alg})$. Since $\Theta \leq \Lambda$, also D is a Θ -algebra. Clearly, $(D_{\Theta}; \mathcal{L}_{alg})$ is e-interpretable in $(D_{\Lambda}; \mathcal{L}_{alg})$, and so D_{Θ} is e-interpretable in A_{Θ} in the language of Θ -algebras (with constants).

Let S_D be a finite set of generators of D_{Λ} as a Λ -module. Identify each a_i with the element $a_i \cdot 1$ of D. Write $S_A = \{a_1, \ldots, a_n\}$. Then D_{Θ} is generated as a Θ -algebra by the finite set $S_D \cup S_A$. Now taking $B = D_{\Lambda}$ and $C = D_{\Theta}$ completes the proof of Items 1 and 2.

A similar argument as above shows that if A_{Θ} is generated as a Θ -module by a finite set, say S_A , then D_{Θ} is generated as a Θ -module by $S_D \cup S_A$, thus in this case both A_{Θ} and D_{Θ} are module-finite. Finally, if $(A_{\Theta})^2$ is zero, nonzero or infinite, then $(A_{\Lambda})^2 = \langle Im(\cdot) \rangle_{\Lambda}$ is nonzero or infinite as well, respectively, and hence the three B, C and D are also respectively zero, nonzero or infinite, by Lemma 5.1.

Below we obtain a non-unitary version of Theorem 4.9. We convene that the *characteristic* of a non-unitary ring A is defined as the maximum positive integer n such that nx = 0 for all $x \in R$.

Theorem 5.12. Let A be a finitely generated associative commutative non-unitary ring, with A^2 infinite. Then the conclusions of Theorem 4.9 hold for $(A; \mathcal{L}_{ring})$.

Proof. Let n be the characteristic of A, and write $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ (if n = 0 let $\mathbb{Z}_n = \mathbb{Z}$). Then A is a faithful finitely generated \mathbb{Z}_n -algebra. Let C be the infinite finitely generated \mathbb{Z}_n -algebra given by Lemma 5.11. By this same result, C as a \mathbb{Z}_n -algebra (and thus as

a ring) is e-interpretable in A as a \mathbb{Z}_n -algebra. The latter is e-interpretable in $(A; \mathcal{L}_{ring})$ because multiplication by any scalar $s \in \mathbb{Z}_n$ is e-interpretable in $(A; \mathcal{L}_{group})$ by the equation y = x + . $\cdot .$ $\cdot .$ Hence $(C; \mathcal{L}_{ring})$ is e-interpretable in $(A; \mathcal{L}_{ring})$. Note that C is finitely generated as a ring because it is finitely generated as a \mathbb{Z}_n -algebra. So by Theorem 4.9 applied to C there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(C; \mathcal{L}_{ring})$, and also in $(A; \mathcal{L}_{ring})$ by transitivity. By Lemma 5.11 again, if A is \mathbb{Z}_n -module-finite then so is C. It follows then that if A is finitely generated as an abelian group then so is C. In this case Item 1 of Theorem 4.9 yields that O is a ring of algebraic integers. Finally, if A has positive characteristic (i.e. n > 0) then C has positive characteristic as well, being a \mathbb{Z}_n -algebra. Hence by Item 2 of Theorem 4.9 there exists a prime p such that $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(C; \mathcal{L}_{ring})$, and so in $(A; \mathcal{L}_{ring})$. This makes $\mathcal{D}(A; \mathcal{L}_{ring})$ undecidable in this case. \square

Lemma 5.11 can be used further to study non-unitary Θ -algebras of scalars, for certain Θ . We note that, unlike in several previous results, here we cannot replace \mathcal{L}_{alg} by \mathcal{L}_{ring} in case Θ is a field. Note also that the case $\Theta = \mathbb{Z}$ has already been treated in the previous Theorem 5.12.

Theorem 5.13. Let L be a finitely generated associative commutative non-unitary algebra over a finitely generated ring of scalars Θ , with L^2 infinite. Then there exists a ring of integers O of a global field such that $(O; \mathcal{L}_{ring})$ is e-interpretable in $(L; \mathcal{L}_{alg})$, and $\mathcal{D}(O; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(L; \mathcal{L}_{alg})$. Moreover, if Θ has positive characteristic p, then $(\mathbb{F}_p[t]; \mathcal{L}_{ring})$ is e-interpretable in $(L; \mathcal{L}_{alg})$, and $\mathcal{D}(L; \mathcal{L}_{alg})$ is undecidable.

Proof. Let B be the module-finite Λ -algebra of scalars given by Item 1 of Lemma 5.11, where $\Lambda = \Theta + B$. Suppose that L^2 is infinite. By this same lemma, B is infinite, and since B is unitary we have that $B^2 = B$ is infinite as well. Note further that if Θ has positive characteristic then so does Λ . The result now follows by transitivity of e-interpretations and by Theorem 5.2 applied to B.

5.4 Undecidability of first order theories

The first order theory T (or elementary theory) of a structure M over a language \mathcal{L} is the set of all first order sentences over \mathcal{L} that are true in M. One says that T is decidable if there exists an algorithm that, given a sentence ϕ over \mathcal{L} , determines if ϕ is true in M or not, i.e. if ϕ belongs to T. If such an algorithm does not exist then T is said to be undecidable.

Noskov proved in [30] that the first order theory of an infinite finitely generated ring of scalars is undecidable in the language of rings with constants. In particular this is true for the ring of integers of any global field. Thus using transitivity of e-interpretations and Proposition 2.6 we immediately obtain the following:

Theorem 5.14. Let R be a ring or an algebra satisfying the hypotheses of one of the Theorems 4.9, 5.2, 5.3, 5.7, 5.12, or Corollary 5.9. Let \mathcal{L} denote \mathcal{L}_{ring} if R is a ring or an algebra over a field. Otherwise let $\mathcal{L} = \mathcal{L}_{alg}$. Then the first order theory of R in the language \mathcal{L} (with constants) is undecidable.

The same statement holds if R satisfies the hypotheses of Theorem 5.13, i.e. if R is a finitely generated associative commutative non-unitary algebra over a finitely generated ring of scalars Λ , with R^2 infinite. In this case however we need to take \mathcal{L} to be the language of Λ -algebras, regardless of whether Λ is a field or not.

In particular, Theorem 5.14 holds for finitely generated associative commutative non-unitary rings R with R^2 infinite (by Theorem 5.12). This result is close to Noskov's original one, but to our knowledge there is no apparent way to prove it with "direct" methods from commutative algebra.

6 Finitely generated algebras over infinite fields

Throughout this subsection k denotes an arbitrary field.

We next study finitely generated algebras over k. Say R is one such algebra. A separate treatment is required, because an infinite field is necessarily infinitely generated as a ring. Hence the general tool (Lemma 5.1) from the previous section cannot be applied to R. More precisely, the k-algebra of scalars L obtained from Theorem 3.9 by considering the k-bilinear map $\cdot: R \times R \to R$ is not finitely generated as a ring, and so the results of Section 5 cannot be applied to L.

Except in Corollary 6.3, all e-interpretations and reductions of this section are done in the language of rings with constants: the full expressiveness of the language of algebras is never required. This contrasts with our previous results obtained for algebras over arbitrary f.g. rings of scalars. Note also that the algebras in this section are no longer countable, and thus Diophantine problems must be considered using coefficients in a fixed computable subset (see Subsections 2.3 or 2.1.2).

We next state the analogue of Theorem 4.9 for algebras over fields. Recall that an algebra of scalars is an associative commutative unitary algebra, and that by function field over k we refer to a finite field extension of k(t) (see Subsection 2.2). The Krull dimension of a ring R is the largest integer n for which there exists a strictly ascending chain of prime ideals $p_0 \leq \cdots \leq p_n$ in R. We refer to Section 8 of [7] for further information on this notion.

Theorem 6.1. Let R be a nonzero finitely generated k-algebra of scalars. Suppose that R has Krull dimension at least one. Then $(\mathbb{Z}; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable.

If otherwise R has Krull dimension zero, then there exists a finite field extension K of k such that $(K; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$.

To prove this theorem we shall need the following result due to Shlapentokh. Its proof (besides a trivial last step that we prove below) can be found in Theorems 3.1 and 4.1 of [39], and in Theorem 3.1 and Remark 3.1 of [40].

Theorem 6.2 ([39, 40]). Let S be a nonempty finite set of primes of a function field K. Then $(\mathbb{Z}; \mathcal{L}_{ring})$ is e-interpretable in $(O_{K,S}; \mathcal{L}_{ring})$.

Proof. In Theorem 3.1 and Remark 3.1 of [40] it is shown that if K has characteristic 0 then $(\mathbb{Z}; \mathcal{L}_{ring})$ is e-interpretable in $(O_{K,S}; \mathcal{L}_{ring})$. In Theorems 3.1 and 4.1 of [39] it is

proved that if otherwise K has positive characteristic p, then $(\mathbb{Z}; +, |_p, |)$ is e-interpretable in $(O_{K,S}; \mathcal{L}_{ring})^2$. The former is the so called arithmetic with addition and localized divisibility. Here | is the relation of divisibility (x|y) iff $\exists z \ y = xz$, and $|_p$ is the relation of localized divisibility: $x|_p y$ iff there exists $n \in \mathbb{N}$ such that $y = \pm xp^n$. In Section 4 of [5], Denef shows that $(\mathbb{Z}; +, |_p, |)$ is e-interpretable in $(\mathbb{Z}; \mathcal{L}_{ring})$ (using only 0 and 1 as coefficients), and so the result follows by transitivity of e-interpretations.

We next prove Theorem 6.1. Several ideas are parallel to arguments used in Subsections 4.1 and 4.2, with the main difference being that the structures at hand behave differently when the Krull dimension is 0. Indeed, a f.g. integral domain R of Krull dimension 0 has to be necessarily a finite field, while if R is a nontrivial f.g. k-algebra of Krull dimension 0, then R is a finite field extension of k.

Proof of Theorem 5.7. Note that any f.g. k-algebra A is Noetherian, because k is a Noetherian ring. Thus $(A/I; \mathcal{L}_{alg})$ is e-interpretable in $(A; \mathcal{L}_{alg})$ for any ideal I, by Lemma 4.2. Throughout the proof we will use implicitly this fact, as well as transitivity of e-interpretations.

Suppose first that R is not a domain, and let $p_0 \leq \cdots \leq p_d$ be a maximal ascending chain of prime ideals of R. Then R/p_0 is a domain with the same Krull dimension as R. Moreover R/p_0 is nonzero (since by definition a prime ideal is proper).

In views of the previous paragraph, asssume for the rest of the proof that R is an integral domain, and let F be the field of fractions of R. Since R is a finitely generated algebra over a field k, the Krull dimension d of R coincides with the transcendence degree of F over k (see Section 8.2.1 of [7]), and by Noether normalization there exist d algebraically independent (over k) elements $t_1, \ldots, t_d \in R$ such that $k[t_1, \ldots, t_d] \leq R$ and R is module-finite over $k[t_1, \ldots, t_d]$. Note as well that F is a finite extension of $k[t_1, \ldots, t_d]$.

Suppose that d=0. Since R is Noetherian, this is equivalent to saying that R is Artinian (see Proposition 9.1 of [7]). All Artinian domains are fields, and thus R=F is a finite field extension of k (see Proposition 8.30 of [2]). This completes the proof of the theorem when R is an integral domain with Krull dimension 0, and so the proof of the last part of the theorem is complete.

Assume now that $d \geq 1$. As done in the proof of Theorem 4.9, we may assume that d = 1 by factoring out the second last ideal in a maximal ascending chain of prime ideals of R. Then, as noted above, F is a finite extension of $k(t_1)$, and thus it is a function field over k. Let \overline{R} be the integral closure of R in F. By Theorem 4.6.3 of [13], \overline{R} is module-finite over R. Hence $(\overline{R}; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$ by Lemma 4.1.

The ring \overline{R} is integrally closed and its field of fractions is F. Moreover, \overline{R} is module-finite over R, and so it is finitely generated as a k-algebra. Hence by Proposition 2.9 and Lemma 2.10 we have that $\overline{R} = O_{F,S}$ for some finite set S of primes of F. Since $k[t_1] \leq R \leq \overline{R}$, it follows that S must contain all the poles of t (see Remark 2.8). In particular, S is nonempty. This and Theorem 6.2 complete the proof of the theorem. \square

²We remark that the condition of S being nonempty is implicit in [39, 40] (otherwise $O_{K,\emptyset}$ coincides with the field of constants of K).

Below we obtain a similar result for the case when R is non-unitary, though in this case one must consider R with the language of k-algebras, and not just with the language of rings. Notice also that one does not have such a transparent classification in terms of Krull dimension.

Corollary 6.3. Let R be a finitely generated associative commutative non-unitary k-algebra. Assume that $R^2 \neq 0$. Then either the first or the second conclusion of Theorem 5.7 hold for $(R; \mathcal{L}_{alg})$.

Proof. By Item 2 of Lemma 5.11 there exists a nonzero finitely generated k-algebra of scalars L such that $(L; \mathcal{L}_{alg})$ (in particular $(L; \mathcal{L}_{ring})$) is e-interpretable in $(R; \mathcal{L}_{alg})$. This same lemma states that if R^2 is nonzero then L is nonzero as well. Now the result follows by Theorem 6.1 and transitivity of e-interpretations.

Similarly as done in Section 5, a combination of Theorem 6.1 and our main result for bilinear maps (Theorem 3.9) yields the following analogue of Lemma 5.1. Note that the full power of the previous Theorem 6.1 is not exploited here, since the aforementioned result for bilinear maps always gives finite-dimensional k-algebras, hence of Krull dimension 0 by Noether normalization.

Lemma 6.4. Let $f: A \times B \to C$ be a k-bilinear map between finite-dimensional k-modules, with $Im(f) \neq 0$. Suppose that $(A, B, C; f, \mathcal{L}_{group})$ is e-interpretable in a structure \mathcal{M} . Then there exists a finite field extension F of k that is e-interpretable in \mathcal{M} in the language of rings (with constants), and $\mathcal{D}(F; \mathcal{L}_{ring})$ is reducible to \mathcal{M} .

Proof. By Theorem 3.9, there exists a nonzero finite-dimensional k-algebra of scalars R such that $(R; \mathcal{L}_{ring})$ is e-interpretable in $(A, B, C; f, \mathcal{L}_{group})$, and thus in \mathcal{M} . Since R is finite-dimensional, the Krull dimension of R is necessarily 0 (by Noether normalization), and the lemma now follows by Theorem 6.1.

The following can be proved in the exact same way as Theorems 5.2, 5.7 and Corollary 5.9, using Lemma 6.4 instead of Lemma 5.1. Below, R may be non-associative, non-commutative, and non-unitary. See Subsection 5.2 for an explanation of the terminology used here.

Theorem 6.5. Let R be a finitely generated algebra over a field k. Suppose that R satisfies one of the following:

- 1. R has finite dimension over k, and $R^2 \neq 0$.
- 2. R is associative or right-normed-generated, and $(R/I_n)^2 \neq 0$ for some $n \geq 1$.
- 3. R is a simply graded Lie algebra and $[R/I_n, R/I_n] \neq 0$ for some $n \geq 1$.

Then there exists a finite field extension K of k such that $(K; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$, and $\mathcal{D}(K; \mathcal{L}_{ring})$ is reducible to $\mathcal{D}(R; \mathcal{L}_{ring})$.

Proof. The three-sorted structure $(R, R, R; \mathcal{L}_{group}, \cdot)$ is e-interpretable in $(R; \mathcal{L}_{ring})$, where as usual \cdot denotes the ring multiplication of R. Hence Item 1 follows from Lemma 6.4. If R is associative or right-normed-generated, then $(R/I_n; \mathcal{L}_{ring})$ is e-interpretable in $(R; \mathcal{L}_{ring})$ by Lemma 5.6, and so Item 2 follows by such lemma and by Item 1 (and by transitivity of e-interpretations). Item 3 is a particular case of Item 2, given that any simply graded Lie algebra is right-normed-generated due to Lemma 5.8.

Hence if k is a field all of whose finite extensions have undecidable Diophantine problem, then $\mathcal{D}(R; \mathcal{L}_{ring})$ is undecidable. This occurs, for example, if k is a global function field [8, 39].

7 References

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