SIMPLE GROUPS WITH INFINITE VERBAL WIDTH AND THE SAME POSITIVE THEORY AS FREE GROUPS

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ABSTRACT. In this paper we show that there exists an uncountable family of finitely generated simple groups with the same positive theory as any non-abelian free group. In particular, these simple groups have infinite w-verbal width for all non-trivial words w.

1. INTRODUCTION

The notion of a word is fundamental in group theory. All combinatorial group theory is based on it: groups are described in terms of their presentations, elements of groups are represented by words, all algorithmic problems in groups are formulated in terms of words etc. Words can be used to describe prominent classes of groups, such as varieties, e.g. varieties of abelian, nilpotent, solvable groups; and more generally to study word equations and subgroups generated by their solutions, hence the notion of verbal subgroup.

More precisely, a *word* w is simply an element of a free group $F(x_1, \ldots, x_n)$. Given a group G, any word w naturally defines a map $\varphi_w : G^n \to G$ as follows

$$(g_1, \ldots, g_n) \mapsto w(g_1, \ldots, g_n)$$

and a subgroup $w(G) < G$ generated by the image of φ_w, which is called the *verbal subgroup of G* defined by w. The *variety* defined by a word w is the class of groups G for which $w(G)$ is trivial. We will call a word w *trivial* if the verbal subgroup $w(F) = F$, where F is a non-abelian free group.

By definition, every element of the verbal subgroup $w(G)$ is a product of some values of the function w. Hence, a natural question to ask is, what is the smallest number n such that every element of $w(G)$ can be represented as a product of at most n values of w? If such a number n exists, then the word w is said to be of *finite width* in G and G is said to be *w-elliptic*. If G is a finite group, then obviously every word w has finite width. However, we may ask whether or not the width of w is bounded in some class of finite groups (e.g. in all finite groups, in all finite simple groups, in all finite p-groups, etc). If C is a class of groups, then a word w has *uniform width* in C if there exists a function $f(d)$ such that the width of w in $G \in C$ is not more than $f(d)$, where d is the smallest number of generators of G.

There is a deep relation between the uniform verbal width of a class of finite groups C and the topology of the corresponding pro-C group, since verbal subgroups of uniform finite width are closed in profinite topology. This was the key step in the

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proof of Serre for pro-p groups and Nikolov-Segal for the profinite case that every finite index subgroup of a finitely generated pro-p/profinite group is open, see [14].

Not surprisingly, many classical problems in group theory revolve around laws, verbal subgroups and their width. A prominent example is Ore's Conjecture. In 1951 Ore conjectured that every element of every finite non-abelian simple group is a commutator. Much work on this conjecture has been done over the years until its proof has been completed by Liebeck, O'Brien, Shalev and Tiep in [11].

In light of solution of Ore's conjecture, it is natural to ask whether or not there are infinite simple groups with infinite commutator width. In this direction, Barge and Ghys [3, Theorem 4.3] showed that there are simple groups of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ which possess nontrivial homogeneous quasi-morphisms, and thus their commutator width is infinite. Further, existence of finitely generated simple groups of infinite commutator width was proved in [13]. Recently, Caprace and Fujiwara proved that there are finitely presented simple groups for which the space of homogeneous quasi-morphisms is infinite-dimensional, and in particular whose commutator width is infinite [7].

Generalising Ore's conjecture, one is naturally brought to study its analogues for other words. A group G is said to be verbally elliptic if it is w-elliptic for every word w (this terminology was introduced by P. Hall). In this terminology, in [11] the authors not only establish Ore's conjecture but prove a more general result, which we can qualitatively summarized as follows.

Theorem 1.1 (Liebeck, O'Brien, Shalev and Tiep, [11]). *The class of finite simple groups is uniformly verbally elliptic.*

As in the case with Ore's conjecture, in light of Theorem 1.1, it is natural to ask if there are simple groups which are *verbally parabolic*, that is, simple groups in which every verbal subgroup has infinite width.

In this paper we address this question formulated in a more general setting from the model-theoretic perspective: does there exist a finitely generated simple group with trivial positive theory?

It is easy to see that if a group G satisfies a positive sentence (in the language of groups), then so does any quotient of G. On the other hand, all (non-abelian) free groups satisfy the same collection of positive sentences. Hence, we say that G has trivial positive theory if it satisfies the the same positive sentences as the free group.

An important observation for us is that given a word w, the property that the w-verbal subgroup has finite width k can be expressed by a non-trivial positive sentence.

The main result of this paper is the following.

Theorem 1.2. *There exist an uncountable family of finitely generated simple groups with trivial positive theory. In particular, these groups have infinite w-verbal width for all non-trivial words w and so are verbally parabolic.*

Our proof has two main ingredients. On the one hand, we generalise Camm's construction of an uncountable family of simple groups, [6]. These groups have a very simple structure as amalgamated products of two free groups of rank 3 over an infinite index subgroup. On the other hand, we show that the action of these groups on the associated Bass-Serre is weakly acylindrical and conclude from the

authors' previous work, see [8], that this family of groups has trivial positive theory and infinite dimensional second bounded cohomology.

Our examples are not finitely presented, therefore, it is natural to ask the following questions.

Question 1.3.
(1) *Are there finitely presented simple groups with trivial positive theory?*
(2) *Is the positive theory of finitely presented simple groups with infinite commutator width constructed by Caprace and Fujiwara trivial?*
(3) *A well-known family of finitely presented simple groups was discovered by Burger and Mozes, [5]. Is the positive theory of Burger-Mozes groups trivial?*

2. AN UNCOUNTABLE CLASS OF GROUPS WITH TRIVIAL POSITIVE THEORY

In this section we introduce an uncountable family of groups parametrized by $\omega = (\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$, where $\rho, \theta, \tilde{\rho}, \tilde{\theta}$ are permutations of $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$ and ι is a bijection between some sets. These groups are defined as amalgamated products of free groups of rank 3 over an infinite index subgroup, that is $G = F_3 *_{H(\rho,\theta)=\iota(\tilde{H}(\tilde{\rho},\tilde{\theta}))} F_3$. We then show that this class of groups has trivial positive theory.

Let us start defining the class of groups. Consider two free groups $F = \langle x, y, z \rangle$ and $\tilde{F} = \langle \tilde{x}, \tilde{y}, \tilde{z} \rangle$ and two permutations ρ and θ of the set $\mathbb{Z}^*$. Let $H = H_{\rho,\theta} \leqslant F$ be the subgroup generated by the collection $\mathcal{S}_{\rho,\theta} = \mathcal{V}_{\rho,\theta} \cup \mathcal{W}_{\rho,\theta}$, where $\mathcal{V}_{\rho,\theta} = \{v_j^l \,|\, l \in \mathbb{Z}, j \in \mathbb{Z}^*\}$ and $\mathcal{W}_{\rho,\theta} = \{w_j^l \,|\, l \in \mathbb{Z}, j \in \mathbb{Z}^*\}$ and v_j^l, w_j^l are defined by:

$$v_j^l = (x^j y^{-\rho(j)})^{z^l}$$
$$w_j^l = (x^j z y^{-\theta(j)})^{z^l},$$

see Figure 2. Unless strictly required, we will omit referring to ρ and θ in the notation $\mathcal{S}_{\rho,\theta}$, and so on.

Similarly, consider $\tilde{H} = \tilde{H}_{\tilde{\rho}\tilde{\theta}} \leqslant F$ be the subgroup generated by the collection $\tilde{\mathcal{S}} = \{\tilde{v}_j^l, \tilde{w}_j^l \,|\, l \in \mathbb{Z}, j \in \mathbb{Z}^*\}$ where

$$\tilde{v}_j^l = (\tilde{x}^j \tilde{y}^{-\tilde{\rho}(j)})^{\tilde{z}^l}$$
$$\tilde{w}_j^l = (\tilde{x}^j \tilde{z} \tilde{y}^{-\tilde{\theta}(j)})^{\tilde{z}^l}$$

For the sake of clarity we will abbreviate v_j^0 as v_j and $\tilde{v}_j^0$ as $\tilde{v}_j$, and we will use a similar abbreviation. for w_j^0 and $\tilde{w}_j^0$. Notice that $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are free basis for H and $\tilde{H}$ respectively, so any bijection ι between $\mathcal{S}$ and $\tilde{\mathcal{S}}$ extends to an isomorphism $\iota*$ between H and $\tilde{H}$.

Let $G = G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ be the amalgamated product of F_1 and F_2 with respect to the amalgamation induced by the isomorphism $\iota* : H \to \tilde{H}$.

We will say that a minimal action of a group G on a simplicial tree T (without inversions) is *weakly acylindrical* if it neither fixes a point nor acts 2-transitively on the boundary of T.

Theorem 2.1. *The action of $G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ on the tree dual to its decomposition as an amalgamated product as above is weakly acylindrical.*
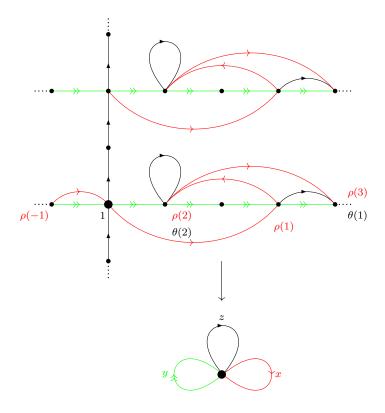
FIGURE 1. Cover associated to the subgroup $H < F(x,y,z)$; in order to encode the bijections ρ and θ, we identify $\mathbb{Z}$ with the vertices of the axis of y, i.e. the vertex corresponding to y^n with n, for instance, $\rho(1) = 3$ and so on.

Proof. The group $G = G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ is the amalgamated product of $F_1 = \langle x, y, x \rangle$ and $F_2 = \langle \tilde{x}, \tilde{y}, \tilde{z} \rangle$ with respect to the amalgamation induced by the isomorphism $\iota* : H \to \tilde{H}$. Consider the element $h = z^{-1}\tilde{z}$. Clearly, h is hyperbolic with translation length 2; furthermore, from the definition of the subgroups H and H', it follows that z and $\tilde{z}$ (and so h) normalize the subgroup $H = \tilde{H}$; indeed H is generated by v_j^l and w_j^l, $l \in \mathbb{Z}, j \in \mathbb{Z}^*$ and by definition, $(v_j^l)^z = v_j^{l+1}$ and $(w_j^l)^z = w_j^{l+1}$, so $H^z = H$ in F_1. Similarly, $\tilde{H}^{\tilde{z}} = \tilde{H}$ and thus $H^h = H$ in G. Since the edge in the tree whose stabliser is H is in the axis of h, the previous fact implies that H is also the pointwise stabiliser of the axis of h. From this we get that h satisfies the property $FS(\frac{1}{2})$: if an element g fixes a segment of length $1 = \frac{1}{2}tl(h)$ of the axis of h, then g belongs to H and so to the pointwise stabiliser of the axis of h. Then, by [8, Lemma 2.10], the element h is weakly $(\frac{1}{2} + 2)$-stable, see [8, Definition 2.9.1], and so is h^n for any $n \in \mathbb{N}\backslash\{0\}$, see [8, Remark 4.1]. Since the action of G on the Bass-Serre tree is irreducible, there exists g such that $\langle g, h \rangle$ also acts irreducibly and $d(A(g), A(h)) < tl(h^n)$, for some $n \in \mathbb{N}$. From [8, Corollary 4.7], it follows that G has weakly N-small cancellation m-tuples, for all $N, m \in \mathbb{N}$

and from [8, Proposition 7.5] we conclude that the action of G on the Bass-Serre tree is weakly acylindrical. □

Corollary 2.2. *For any $(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ as above the group $G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ has trivial positive theory and infinite dimensional second bounded cohomology.*

Proof. From Theorem 2.1 we have that these groups act weakly acylindrically on a tree and so from Theorem 7.6 in [8] we obtain that they have trivial positive theory. On the other hand, from the definition of weakly acylindrical action and the trichotomoy provided in [10, Theorem 1], we conclude that these groups have infinite dimensional second bounded cohomology. □

3. Simplicity of an uncountable subclass of groups

This section is devoted to the proof of the following result:

Proposition 3.1. *There are $2^{\aleph_0}$ mutually non-isomorphic groups of the form $G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ which are simple.*

The strategy follows the lines introduced by Ruth Camm in [6]. The idea is to show that local conditions on elements that all together will imply the simplicity of the group, follow from local restrictions on the set of permutations. We will show that on the one hand, the set of local element conditions imply indeed simplicity of the group, and on the other one, that the local restrictions on the permutations can be consistently satisfied and extended, and that there are in fact, uncountably many parameters $\omega = (\rho, \theta, \tilde{\rho}, \tilde{\omega}, \iota)$ that satisfy them.

By *edge* group of an amalgamated product of two groups G_1 and G_2 we refer to the amalgamated subgroup, and by *vertex* group we mean either G_1 and G_2.

The type of local conditions that we will use are the following: the normal closure of an element of *length k* contains an element of the edge group (see Section 3.3); the normal closure of an element of the edge group of length k (in the subgroup contains the generators of the vertex group (see Section 3.5). Finally, in Section 3.6, we show that we can concatenate the local conditions consistently to obtain simplicity of the whole group.

Let $\mathcal{P}(X, Y)$ be the collection of partial bijections between two sets X and Y. We write $\mathcal{P}(X) = \mathcal{P}(X, X)$, and denote by $\mathcal{P}^{<\omega}(X, Y)$ the collection of all finite (i.e. with finite support) partial bijections between X and Y. We define $\mathcal{P}^{<\omega}(X)$ similarly.

To prove Proposition 3.1 we prove the existence of partial maps $\rho_{\text{simple}}, \tilde{\rho}_{\text{simple}} \in \mathcal{P}(\mathbb{Z}^*)$ and $\iota_{\text{simple}} \in \mathcal{P}(\mathcal{S}, \tilde{\mathcal{S}})$ such that:

- there are countably many elements in the complement of their domains and images
- for any tuple $\omega = (\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ such that $\rho, \tilde{\rho}$ and ι extend $\rho_{\text{simple}}, \tilde{\rho}_{\text{simple}}, \iota_{\text{simple}}$, respectively, the group $G = G(\omega)$ is simple.

Some basic notation. The proof is rather technical and strongly relies on computations. To facilitate the reading, in this section we collect the notation that we will use throughout the text.

Definition 3.2. *Below X, Y, X' and Y' denote arbitrary sets, unless otherwise specified.*
- *Given $f \in \mathcal{P}(X)$ we let $|f| = im(f) \cup dom(f)$.*

- *Given $f \in \mathcal{P}(X, X')$ and $Y \subset X$, $Y' \subset X'$ we say that f is* independent *from (Y, Y') if $dom(f) \cap Y = im(f) \cap Y' = \varnothing$. If $f \in \mathcal{P}(X)$ then we say that f is* independent *from Y if it is independent from (Y, Y).*
- *Given $X \subset \mathbb{Z}^2$, let $\mathcal{V}(X)$ denotes the collection of all the generators of the form $v_j^l = (x^j y^{-\rho(j)})^{z^l}$, where $(j, l) \in X$. Similarly, $\mathcal{W}(X)$ denotes the collection of elements $w_j^l = (x^j z y^{-\theta(j)})^{z^l}$, $(j, l) \in X$.*
- *$\mathcal{S}(X)$ denotes $\mathcal{V}(X) \cup \mathcal{W}(X)$. We define sets $\tilde{\mathcal{S}}(X)$, $\tilde{\mathcal{V}}(X)$ and $\tilde{\mathcal{W}}(X)$ in a similar fashion.*
- *Whenever X is omitted from the notation introduced in the items above, it will be understood that $X = \mathbb{Z} \times \mathbb{Z}^*$*
- *Given $Z \subset \mathbb{Z}^*$ and $l \in \mathbb{Z}$ let $\mathcal{V}^l(Z) = \mathcal{V}(Z \times \{l\})$ By $\mathcal{V}^l$ we refer to $\mathcal{V}^l(Z)$.*
- *Given, $k, l \in \mathbb{Z}$ the expression $[k, l]$ will stand for the set $\{k, k+1, \ldots l-1, l\}$. We write $\mathcal{S}_N = \mathcal{S}([-N, N] \times [-N, N])$; we define $\tilde{\mathcal{S}}_N$ analogously.*
- *Given $(l, j) \in \mathbb{Z}^2$ let $c_{l,j} = z^l x^j$ and $\tilde{c}_{l,j} = \tilde{z}^l \tilde{x}^j$.*
- *Given $j, M \in \mathbb{Z}$, let $\delta(M, j) = \rho(M+j) - \rho(j)$ and $\tilde{\delta}(M, j) = \tilde{\rho}(M+j) - \tilde{\rho}(j)$.*
- *Given any $(M, M', l) \in \mathbb{Z}^3$, define $d_{M,M'}^l = v_M^l (v_{M'}^l)^{-1}$ and, dually, define $\tilde{d}_{M,M'}^l = \tilde{v}_M^l (\tilde{v}_{M'}^l)^{-1}$.*
- *As defined previously F and $\tilde{F}$ denote the free groups generated by x, y, z and by $\tilde{x}, \tilde{y}, \tilde{z}$, respectively. We also defined $H = \langle \mathcal{S} \rangle \leqslant F$ and $\tilde{H} = \tilde{\mathcal{S}} \leqslant \tilde{F}$.*

As it is customary, given $X \subset G$ we let $\langle\langle X \rangle\rangle_G$, or simply $\langle\langle X \rangle\rangle$, stand for the normal closure of X in G.

Sometimes we will identify a partial map f from a set X to another set Y with the set $\{(x, y) \mid y = f(x), \ x \in dom(f)\}$. This allows us to use expressions such as $f \subset g$ given two partial maps f and g.

Preliminary definitions and calculations in $G(\omega)$. Unless otherwise specified, throughout the rest of the paper we will use $\omega = (\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ to denote an arbitrary tuple where $\rho, \tilde{\rho}, \theta, \tilde{\theta}$ are permutations of $\mathbb{Z}$ and ι is a bijection between $\mathcal{S}_{\rho,\theta}$ and $\tilde{\mathcal{S}}_{\tilde{\rho},\tilde{\theta}}$. We shall use the notation $\mathcal{S}$ and $\tilde{S}$ instead of $\mathcal{S}_{\rho,\theta}$ and $\tilde{\mathcal{S}}_{\tilde{\rho},\tilde{\theta}}$.

We start with an easy observation, which we keep for later:

Lemma 3.3. *Given any tuple $\omega = (\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$, if $\{(s, \iota(s)) \mid s \in \mathcal{S}\} \cap (\mathcal{V} \times \tilde{\mathcal{W}}) \neq \varnothing$ then $G(\omega) = \langle\langle x, y, \tilde{x}, \tilde{y} \rangle\rangle$.*

Proof. Indeed, from the fact that there is some $v_j^l \in \mathcal{V}$ such that $\iota(v_j^l) = \tilde{w}_{j'}^{l'}$ for some $j' \in \mathbb{Z}^*, l' \in \mathbb{Z}$, it follows that $\tilde{z}$ is in the normal closure of $x, y, \tilde{x}, \tilde{y}$. Now, since $\iota(w_j^l)$ is in the normal closure of $\tilde{x}, \tilde{y}, \tilde{z}$ for all j, l, and $\tilde{z}$ is in turn, in the normal closure of $x, y, \tilde{x}, \tilde{y}$, the result follows. $\qquad\square$

Recall that, given $(l, j) \in \mathbb{Z}^2$ we defined $c_{l,j} = z^l x^j$ and $\tilde{c}_{l,j} = \tilde{z}^l \tilde{x}^j$. The following is an easy exercise:

Lemma 3.4. *The set $\{c_{l,j}\}_{(j,l)\in\mathbb{Z}^2} \subset F$ is a system of representatives of the right cosets of H in F. The set $\{\tilde{c}_{l,j}\}_{(j,l)\in\mathbb{Z}^2} \subset \tilde{F}$ is a system of representatives of the right cosets of $\tilde{H}$ in $\tilde{F}$.*

Given $j, M \in \mathbb{Z}$, recall that $\delta(M, j) = \rho(M+j) - \rho(j)$ and $\tilde{\delta}(M, j) = \tilde{\rho}(M+j) - \tilde{\rho}(j)$. A simple calculation yields:

$$(1) \qquad x^M v_j = v_{M+j} y^{\delta(M,j)}, \qquad \tilde{x}^M \tilde{v}_j = \tilde{v}_{M+j} \tilde{y}^{\tilde{\delta}(M,j)}$$

Remark 3.5. *If $\iota(v_m) = \tilde{v}_{m'}$, then the corresponding defining relation of $G(w)$ can be rewritten as $x^{-m}\tilde{x}^{m'} = y^{-\rho(m)}\tilde{y}^{\tilde{\rho}(m')}$.*

The following is just a straightforward calculation.

Lemma 3.6. *In any group $G(\omega)$ we have:*
$$(v_M^{-l})^{-1}c_{l,j} = z^l y^{-\delta(-j,M)}(v_{M-j})^{-1}$$
$$(\tilde{v}_M^{-l})^{-1}\tilde{c}_{l,j} = \tilde{z}^l \tilde{y}^{-\tilde{\delta}(-j,M)}(\tilde{v}_{M-j})^{-1}$$

Given any pair $(M, M', l) \in \mathbb{Z}^3$, recall that $d_{M,M'}^l = v_M^l (v_{M'}^l)^{-1}$ and dually $\tilde{d}_{M,M'}^l = \tilde{v}_M^l (\tilde{v}_{M'}^l)^{-1}$. From Lemma 3.6 we can easily deduce:

Lemma 3.7. *If (M, M', l) and (j, l) satisfy $\delta(-j, M) = \delta(-j, M')$, then*
$$(d_{M,M'}^{-l})^{c_{l,j}} = d_{M-j,M'-j}^0.$$

In particular, $d_{M,M'}^{x^j} = d_{M-j,M'-j}$ A similar property holds in the group $\tilde{H}$ with respect to the function $\tilde{d}$.

Lemma 3.8. *Let a_m stand for the element $x^{-m}\tilde{x}^m$. Suppose we are given non-trivial integers $j, \tilde{j}, M, M' \in \mathbb{Z}^*$. Then the equality $a_M^{v_j} = a_{M'}$ holds if all the following conditions are satisfied:*

(i) $\iota(v_j) = \tilde{v}_{\tilde{j}}$
(ii) $\iota(v_{M+j}) = \tilde{v}_{M+\tilde{j}}$
(iii) $\rho(M') = \delta(M, j)$
(iv) $\tilde{\rho}(M') = \tilde{\delta}(M, \tilde{j})$

Proof. Using (i) and (1) we get:
$$(x^{-M}\tilde{x}^M)^{v_j} = v_j^{-1} x^{-M}\tilde{x}^M v_j =$$
$$= v_j^{-1} x^{-M}\tilde{x}^M \tilde{v}_{\tilde{j}} = y^{-\delta(M,j)}v_{M+j}^{-1}\tilde{v}_{M+\tilde{j}}\tilde{y}^{\tilde{\delta}(M,\tilde{j})}$$

Using (ii) for the first equality below, and then Remark 3.5 together with (iii) and (iv) we get:
$$y^{-\delta(M,j)}v_{M+j}^{-1}\tilde{v}_{M+\tilde{j}}\tilde{y}^{\tilde{\delta}(M,\tilde{j})} = y^{-\delta(M,j)}\tilde{y}^{\tilde{\delta}(M,\tilde{j})} = x^{-M'}\tilde{x}^{M'}.$$

$\square$

From $G\backslash\{1\}$ to $H\backslash\{1\}$. Let $\mathcal{G}$ be the collection of sequences of the form
$$u, c_{l_0,j_0}, \tilde{c}_{l_1,j_1}, \ldots, c_{l_{2k},j_{2k}}, \tilde{c}_{l_{2k+1},j_{2k+1}},$$
where $k \geq 0$, u is a possibly empty element in the free group generated by $\mathcal{S}$ and $(c_i, j_i) \in \mathbb{Z}^2\backslash\{(0,0)\}$ for all $1 \leq 2k+1$.

Remark 3.9. *By Lemma 3.4 and the normal form theorem for amalgamated free products for any element $g \in G(w)$ there exists $u, c_{l_0,j_0}, \ldots, \tilde{c}_{l_{2k+1},j_{2k+1}}$ as above such that $g = u c_{l_0,j_0} \ldots \tilde{c}_{l_{2k+1},j_{2k+1}}$ in $G(\omega)$.*

The following result will be fundamental for the proof of Proposition 3.1.

Lemma 3.10. *For any sequence $\sigma \in \mathcal{G}$ of the form $u, c_{l_0,j_0}, \tilde{c}_{l_1,j_1} \ldots \tilde{c}_{l_{2k+1},j_{2k+1}} \in \mathcal{G}$ and for any $N > 0$ there exist maps $\rho^{GH} = \rho^{GH}(N, \sigma)$, $\tilde{\rho}^{GH} = \tilde{\rho}^{GH}(N, \sigma) \in \mathcal{P}^{<\omega}(\mathbb{Z}^*)$ and $\iota^{GH} = \iota^{GH}(N, \sigma) \in \mathcal{P}^{<\omega}(\mathcal{S}, \tilde{\mathcal{S}})$ such that:*

(i) ρ^{GH} and $\tilde{\rho}^{GH}$ are independent from $[-N, N]$ (in the sense of Item 2 of Definition 3.2) and ι^{GH} is independent from $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$.

(ii) If $\rho^{GH} \subset \rho$, $\tilde{\rho}^{GH} \subset \tilde{\rho}$ and $\iota^{GH} \subset \iota$ then in the group $G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ we have $\langle\langle g \rangle\rangle \cap H \neq \{1\}$, where g is the element represented by the product $uc_{l_0, j_0} \tilde{c}_{l_1, j_1} \ldots c_{l_{2i}, j_{2i}} \tilde{c}_{l_{2k+1}, j_{2k+1}}$.

Proof. For sequences of integers $(M_i)_{i=0}^{2k+2}$, $(M_i')_{i=0}^{2k+2}$ consider the following conditions:

(a) $\iota(v_{M_{2i}-j_{2i}}) = \tilde{v}_{M_{2i+1}}^{-l_{2i+1}}$ and $\iota(v_{M_{2i}'-j_{2i}}) = \tilde{v}_{M_{2i+1}'}^{-l_{2i+1}}$ for $0 \leqslant i \leqslant k$

(b) $\iota(v_{M_{2i+2}}^{-l_{2i+2}}) = \tilde{v}_{M_{2i+1}-j_{2i+1}}^0$ and $\iota(v_{M_{2i+2}'}^{-l_{2i+2}}) = \tilde{v}_{M_{2i+1}'-j_{2i+1}}^0$ for $0 \leqslant i \leqslant k$.

(c) $\rho(M_i - j_i) = \rho(M_i) - j_i$ and $\rho(M_i' - j_i) = \rho(M_i') - j_i$ for $0 \leqslant i \leqslant 2k+2$

(d) $\tilde{\rho}(M_i - j_i) = \tilde{\rho}(M_i) - j_i$ and $\tilde{\rho}(M_i' - j_i) = \tilde{\rho}(M_i') - j_i$ for $0 \leqslant i \leqslant 2k+2$

(e) $v_{M_0}^{-l_0}$ and $v_{M_0'}^{-l_0}$ do not appear in the alphabet of the word u

(f) $\max_{0 \leqslant i \leqslant 2k+1} |j_i| < \min_{0 \leqslant i < i' \leqslant 2k+2} |M_i - M_{i'}|, \min_{0 \leqslant i < i' \leqslant 2k+2} |M_i' - M_{i'}'|$

(g) $\max_{0 \leqslant i \leqslant 2k+1} |j_i| < \min_{0 \leqslant i, i' \leqslant 2k+2} \{|M_i - M_{i'}'|\}$. In particular, $M_i \neq M_i'$ for all $i = 0, \ldots, 2k+2$.

We will need the following

Claim 3.11. *If $(M_i)_{i=0}^{2k+2}$, $(M_i')_{i=0}^{2k+2}$, and $\rho, \tilde{\rho}, \iota$ are integer sequences and maps such that the conditions (a)-(d) above hold. Then for $-1 \leqslant i \leqslant k$ we have:*

$$(d_{M_0, M_0'}^{-l_0})^{g_i} = d_{M_{2i+2}, M_{2i+2}'}^{-l_{2i+2}}$$

where g_i is defined as $c_{l_0, j_0} \tilde{c}_{l_1, j_1} \ldots c_{l_{2i}, j_{2i}} \tilde{c}_{l_{2i+1}, j_{2i+1}}$ if $i \geqslant 0$ and g_{-1} is set to be 1.

Indeed, we proceed by descending induction. On the one hand, by Condition (c) above we have

$$\delta(-j_{2i}, M_{2i}) = \rho(-j_{2i} + M_{2i}) - \rho(M_{2i}) = 0 = \rho(-j_{2i} + M_{2i}') - \rho(M_{2i}') = \delta(-j_{2i}, M_{2i}'),$$

and so we can use Lemma 3.7 and Condition (a) to obtain:

$$(d_{M_{2i}, M_{2i}'}^{-l_{2i}})^{c_{l_{2i}, j_{2i}}} = d_{M_{2i}-j_{2i}, M_{2i}'-j_{2i}}^0 = \tilde{d}_{M_{2i+1}, M_{2i+1}'}^{-l_{2i+1}}.$$

A similar argument involving condition (b) and (d) yields:

$$(d_{M_{2i}, M_{2i}'}^{-l_{2i}})^{c_{l_{2i}, j_{2i}} \tilde{c}_{l_{2i+1}, j_{2i+1}}} = (\tilde{d}_{M_{2i+1}, M_{2i+1}'}^{-l_{2i+1}})^{\tilde{c}_{l_{2i+1}, j_{2i+1}}} =$$

$$= \tilde{d}_{M_{2i+1}-j_{2i+1}, M_{2i+1}'-j_{2i+1}}^0 = d_{M_{2i+2}, M_{2i+2}'}^{-l_{2i+2}}$$

This completes the proof of Claim 3.11.

We shall now use Claim 3.11 in order to prove Item (ii) of Lemma 3.10. Suppose that we are given integer sequences $(M_i)_{i=0}^{2k+2}$, $(M_i')_{i=0}^{2k+2}$, as well as $\rho, \tilde{\rho}, \iota$ satisfying the conditions (a) through (g) above, and consider the element $e = (d_{M_0, M_0'}^{-l_0})^{u^{-1}}$. The element $e^g e^{-1} = g^{-1} g^{e^{-1}}$ belongs to the normal closure of g. According to Claim 3.11 and since $ug_k = g$, the element $e^g e^{-1}$ is equal to $e' = d_{M_{2k+2}, M_{2k+2}'}^{l_{2k+2}} e^{-1} \in H$. By Condition (e) neither $v_{M_0}^{-l_0}$ nor $v_{M_0'}^{-l_0}$ appear in u and by Condition (f), $M_i \neq M_i'$ and so it follows that e' is non-trivial and we are done.

Now, it is easy to find tuples $(M_i)_i$ and $(M_i')_i$ satisfying conditions $(e), (f)$ and (g). For fixed values of $\rho(M_i), \tilde{\rho}(M_i), \rho(M_i'), \tilde{\rho}(M_i')$ conditions (a) to (d) can be expressed in terms of $\iota, \rho, \tilde{\rho}$ containing some partial maps ι^{GH}, ρ^{GH}, and $\tilde{\rho}^{GH}$. The

fact that this choice can be made in a way compatible with the bijectivity of ρ follows easily from conditions (f) and (g). As for the bijectivity of ι, we need to show that the elements on the left hand-side of Conditions (a) and (b)

$$\left\{ v_{M_{2i}-j_{2i}}, v_{M'_{2i}-j_{2i}}, v_{M_{2i+2}}^{-l_{2i+2}}, v_{M'_{2i+2}}^{-l_{2i+2}} \mid i = 0, \ldots, k \right\},$$

as well as their images

$$\left\{ \tilde{v}_{M_{2i+1}}^{-l_{2i+1}}, \tilde{v}_{M'_{2i+1}}^{-l_{2i+1}}, \tilde{v}_{M_{2i+1}-j_{2i+1}}, \tilde{v}_{M'_{2i+1}-j_{2i+1}} \mid i = 0, \ldots, k \right\},$$

are pairwise different.

Let us show that $v_{M_{2i'}-j_{2i'}} \neq v_{M_{2i+2}}^{-l_{2i+2}}$, for all $0 \leqslant i, i' \leqslant k$. From condition (f), we have that

$$m_j = \max_{0 \leqslant i \leqslant 2k+1} |j_i| < \min_{0 \leqslant i < i' \leqslant 2k+2} |M_i - M_{i'}|$$

and so if $M_{2i'} - j_{2i'} = M_{2i+2}$, then $i' = i + 1$ and $j_{2i+2} = 0$. Furthermore, if $j_{2i+2} = 0$, then $l_{2i+2} \neq 0$ and hence we have that $v_{M_{2i'}-j_{2i'}} \neq v_{M_{2i+2}}^{-l_{2i+2}}$. On the other hand, if $j_{2i+2} \neq 0$, then the inequality is immediate. Using symmetric arguments, one shows that all elements are pairwise different.

In order to show that the target elements are also pairwise different, that is $\tilde{v}_{M_{2i+1}}^{-l_{2i+1}} \neq \tilde{v}_{M_{2i'+1}-j_{2i'+1}}$ for all $0 \leqslant i, i' \leqslant k$, one uses analogous arguments, using Condition (g) instead of (f).

Independence from $[-N, N]$ and $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$ simply follows by taking M_i, M'_i and their images by ρ and $\tilde{\rho}$ large enough. $\qquad\square$

From $H \backslash \{1\}$ to $V^0 \backslash \{1\}$.

Lemma 3.12. *For any $N > 0$ there exist $\iota^{vert} = \iota^{vert}(N) \in \mathcal{P}^{<\omega}(\mathcal{S}, \tilde{\mathcal{S}})$ and $X = X^{vert}(N) \subset \mathbb{Z}$ such that:*

    (i) ι^{vert} is independent from $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$.
    (ii) $\{(v_j, \tilde{v}_j)\}_{j \in X} \subset \iota^{vert}$
    (iii) If $\iota^{vert} \subset \iota$, then there exists $h \in G(\omega)$ such that $h^{-1}\mathcal{S}_N h \subseteq \langle \mathcal{V}(X \times \{0\}) \rangle$.

Proof. Take for example $X = [2N, 2N + 4N(2N + 1) - 1]$. Define B_N to be $([-N, N]\backslash\{0\}) \times [-N, N]$ and choose any bijection $\mu : B_N \times \{0, 1\} \to X$ and let:

$$\iota^{vert} = \{(v_j^{l+N+1}, \tilde{v}_{\mu(j,l,0)}^1), (w_j^{l+N+1}, \tilde{v}_{\mu(j,l,1)}^1) \mid (j, l) \in B_N\} \cup \{(v_j, \tilde{v}_j) \mid j \in X\}.$$

The relation ι^{vert} is clearly a partial bijection independent from $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$. As h we take the element $z^{N+1}\tilde{z}^{-1}$. We content ourselves with checking condition (iii) on an element of the form v_j^l with $(j, l) \in B_N$. Assume $\iota^{vert} \subset \iota$. Then:

$$(v_j^l)^h = (v_j^l)^{z^{N+1}\tilde{z}^{-1}} = (v_j^{l+N+1})^{\tilde{z}^{-1}} = (\tilde{v}_{\mu(j,l,0)}^1)^{\tilde{z}^{-1}} = \tilde{v}_{\mu(j,l,0)} = v_{\mu(j,l,0)}.$$

$\qquad\square$

Killing $\langle \tilde{x}, \tilde{y} \rangle$. Given $X \subseteq \mathbb{Z}^*$, let $\mathcal{H}(X)$ be the collection of non-empty sequences of the form $u = v_{j_0}^{\epsilon_0} v_{j_1}^{\epsilon_1} \ldots v_{j_k}^{\epsilon_k}$ (v_j is regarded as a letter here) where $\epsilon_k \in \{1, -1\}$, $j_k \in X$, and where $k \geqslant 1$, $j_i = j_{i+1}$ implies $\epsilon_i = \epsilon_{i+1}$. If $X = \mathbb{Z}^*$ then we denote $\mathcal{H}(X)$ simply by $\mathcal{H}$. Let us restate an argument from [6].

Lemma 3.13. *Suppose we are given $N > 0$ and $u = v_{j_0}^{\epsilon_0} v_{j_1}^{\epsilon_1} \dots v_{j_k}^{\epsilon_k} \in \mathcal{H}$ such that* $\max_{0 \leqslant i \leqslant k} |j_i| < N$.

Then there exist $\rho^{x\tilde{x}} = \rho^{x\tilde{x}}(N, u)$, $\tilde{\rho}^{x\tilde{x}} = \tilde{\rho}^{x\tilde{x}}(N, u) \in \mathcal{P}(\mathbb{Z})$, $\iota^{x\tilde{x}} = \iota^{x\tilde{x}}(N, u) \in \mathcal{P}(\mathcal{S}, \tilde{\mathcal{S}})$ such that:

(I) *The maps $\rho^{x\tilde{x}}$ and $\tilde{\rho}^{x\tilde{x}}$ are independent from $[-N, N]$, and $\iota^{x\tilde{x}}$ is independent from $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$.*

(II) *If $Id_{\{j_i\}_{i=0}^k} \cup \rho^{x\tilde{x}} \subset \rho$, $Id_{\{j_i\}_{i=0}^k} \cup \tilde{\rho}^{x\tilde{x}} \subset \tilde{\rho}$, and $\{(v_{j_i}, \tilde{v}_{j_i})\}_{1 \leqslant i \leqslant k} \cup \iota^{x\tilde{x}} \subset \iota$, then in $G(\omega)$ we have $\tilde{x}, \tilde{y} \in \langle\langle u \rangle\rangle$.*

Proof. Let $M_i = 2(i+1)N$ $(i = 0, \dots, k)$, $C = 4kN$. We let $\rho^{x\tilde{x}}, \tilde{\rho}^{x\tilde{x}}, \iota^{x\tilde{x}}$ be defined by the conditions:

(i) $\rho(M_i) = \tilde{\rho}(M_i) = M_i$ for all $0 \leqslant i \leqslant k$,

(ii) $\rho(C) = \tilde{\rho}(C) = C$, $\rho(C + 1) = C + 1$, $\tilde{\rho}(C + 1) = C + 2$,

(iii) $\rho(2C) = \tilde{\rho}(2C) = 2C$, $\rho(2C + 1) = 2C + 1$, $\tilde{\rho}(2C + 1) = 2C + 3$

(iv) $\rho(3C) = 3C$, $\rho(3C + 2) = 3C + 1$,

(v) $\rho(4C) = 4C$, $\rho(4C + 3) = 4C + 1$

(vi) $\iota(v_k) = \tilde{v}_k$ for $k \in \{C, C + 1, 2C, 2C + 1\}$.

(vii) If $\epsilon_i = 1$, then:

(a) $\rho(M_i + j_i) = M_{i+1} + j_i$

(b) $\tilde{\rho}(M_i + j_i) = M_{i+1} + j_i$

(c) $\iota(v_{M_i+j_i}) = \tilde{v}_{M_i+j_i}$

(viii) If $\epsilon_i = -1$, then:

(a) $\rho(M_{i+1} + j_i) = M_i + j_i$

(b) $\tilde{\rho}(M_{i+1} + j_i) = M_i + j_i$

(c) $\iota(v_{M_{i+1}+j_i}) = \tilde{v}_{M_i+j_i}$

Notice that this implies (using $\rho(j_i) = \tilde{\rho}(j_i) = j_i$ as a consequence of Item II in the statement of the lemma) that $\delta(M_i, j_i) = M_{i+1} = \rho(M_{i+1}) = \tilde{\rho}(M_{i+1})$ in case $\epsilon_i = 1$ and $\delta(M_{i+1}, j_i) = M_i = \rho(M_i) = \tilde{\rho}(M_i)$ in case $\epsilon_i = -1$.

By virtue of Lemma 3.8 together with $\rho(j_i) = \tilde{\rho}(j_i) = j_i$ we have $(x^{-M_i}\tilde{x}^{M_i})^{v_{j_i}^{\epsilon_i}} = x^{-M_{i+1}}\tilde{x}^{M_{i+1}}$ for all $0 \leqslant i \leqslant k-1$, from which it follows that $(x^{-M_0}\tilde{x}^{M_0})^u = x^{-M_k}\tilde{x}^{M_k}$. And thus $x^{-m}\tilde{x}^m = x^{-(M_k-M_0)}\tilde{x}^{M_k-M_0} = [x^{-M_0}\tilde{x}^{M_0}, u]^{\tilde{x}^{-M_0}} \in \langle\langle u \rangle\rangle$, where $m = 2kN$ since $M_i = 2(i+1)N$.

It follows that $x^C = \tilde{x}^C \mod \langle\langle u \rangle\rangle$. Now, the equalities $\rho(C) = \tilde{\rho}(C) = C$ and $\iota(v_C) = \tilde{v}_C$ imply that also $y^C = \tilde{y}^C \mod \langle\langle u \rangle\rangle$, since $x^C y^{-\rho(C)} = \tilde{x}^C \tilde{y}^{-\tilde{\rho}(C)}$. Applying both to the equality $x^{C+1}y^{-C-1} = \tilde{x}^{C+1}\tilde{y}^{-C-2}$ yields $xy^{-1} = \tilde{x}\tilde{y}^{-2} \mod \langle\langle u \rangle\rangle$. An analogous argument using (iii) yields $xy^{-1} = \tilde{x}\tilde{y}^{-3} \mod \langle\langle u \rangle\rangle$. It follows that $\tilde{y} \in \langle\langle u \rangle\rangle$. In a similar fashion clauses (iv) and (v) yield $\tilde{x} \in \langle\langle u \rangle\rangle$.

Define now $M_i' = M_i + j_i$, $M_i'' = M_{i+1} + j_i$, and let $\mathcal{M} = \{M_i\}_{0 \leqslant i \leqslant k+1}$ $\mathcal{M}' = \{M_i'\}_{0 \leqslant i \leqslant k}$ and $\mathcal{M}'' = \{M_i''\}_{0 \leqslant i \leqslant k}$. From the facts that $\max_{0 \leqslant i \leqslant k} |j_i| < N$ (by hypothesis) and $j_i \neq 0$ for all i it follows that the integers in $\mathcal{M}$ are pairwise distinct, and similarly for $\mathcal{M}'$ and $\mathcal{M}''$. Moreover $\mathcal{M} \cap (\mathcal{M}' \cup \mathcal{M}'') = \varnothing$.

On the other hand, the only way we could possibly have $M_{i'}' = M_i''$ is if $i' = i+1$, $\epsilon_i = 1$, $\epsilon_{i'} = -1$ and $j_i = j_{i'}$, which cannot occur by assumption. Since C is bigger than any of the M_i or M_i', it easily follows that the list of conditions above is consistent, giving partial bijections $\rho^{x\tilde{x}}, \tilde{\rho}^{x\tilde{x}}, \iota^{x\tilde{x}}$ as in the statement. That the latter are independent from $[-N, N]$ and $(\mathcal{S}_N, \tilde{\mathcal{S}}_N)$ is clear from the definition of M_i. □

Remark 3.14. *The condition $\{(v_{j_i}, \tilde{v}_{j_i})\}_{1 \leqslant i \leqslant k} \cup \iota^{x\tilde{x}} \subset \iota$ in Item II of Lemma 3.13 could be replaced by the weaker requirement that $\iota(v_{j_i}) \in \tilde{\mathcal{V}}$ and the condition that $\rho(j_i) = \tilde{\rho}(j_i) = j_i$ could be removed altogether, at the expense of complicating the calculations.*

Proof of Proposition 3.1. We construct a sequence of positive integers $N_0 < N_1 < \cdots < N_k < N_{k+1} \ldots$ and an increasing sequences of partial bijections

$$(\rho_k)_{k \geqslant 0}, (\tilde{\rho}_k)_{k \geqslant 0} \subset \mathcal{P}([-N_k, N_k]), \quad (\iota_k)_{k \geqslant 0} \subset \mathcal{P}(\mathcal{S}_{N_k}, \tilde{\mathcal{S}}_{N_k}),$$

a sequence of disjoint sets $(X_k)_{k \geqslant 1}$ and a sequence of elements $(u_k^*)_{k \geqslant 1} \subset \mathcal{H}$ as follows.

To begin with, let $\rho_0 = \tilde{\rho}_0 = \varnothing$, $\iota_0 = \{(v_1, \tilde{w}_1), (w_1, \tilde{v}_1)\}$ and $N_0 = 2$. Let $(s_n)_{n \geqslant 0}$ be an enumeration of the set $\mathcal{G}$ and $(w_n)_{n \geqslant 0}$ be an enumeration of $\mathcal{H}$.

Assume now that $k > 1$ and $\rho_{k-1}, \tilde{\rho}_{k-1}, \iota_{k-1}, X_{k-1}, N_{k-1}$ have already been constructed.

Consider the partial maps given by Lemma 3.10, $\rho^{GH} = \rho^{GH}(N_{k-1}, s_{k-1})$, $\tilde{\rho}^{GH} = \tilde{\rho}^{GH}(N_{k-1}, s_{k-1})$, and $\iota^{GH} = \iota^{GH}(N_{k-1}, s_{k-1})$. Let $N' > N_{k-1}$ be such that $\rho^{GH}, \tilde{\rho}^{GH} \in \mathcal{P}^{<\omega}([-N', N'])$ and $\iota^{GH} \in \mathcal{P}^{<\omega}(\mathcal{S}_{N'}, \tilde{\mathcal{S}}_{N'})$. Furthermore, let $\iota^{vert}(N')$ and $X_k := X^{vert}(N')$ be the output from Lemma 3.12 and set $\tilde{\rho}^{vert} = Id_{X_k}$. Let $u_k^* = w_l$ where l is the minimum index l such that

$$w_l \in \cup_{j \leqslant k} \mathcal{H}(X_j) \backslash \{u_j^* \mid 0 \leqslant j < k\},$$

Finally, consider the maps $\rho^{x\tilde{x}} = \rho^{x\tilde{x}}(N', u_k^*)$, $\rho^{x\tilde{x}} = \rho^{x\tilde{x}}(N', u_k^*)$, $\iota^{x\tilde{x}} = \iota^{x\tilde{x}}(N', u_k^*)$ given by Lemma 3.13 and let

$$\rho_k = \rho_{k-1} \cup \rho^{GH} \cup \rho^{vert} \cup \rho^{x\tilde{x}}$$
$$\tilde{\rho}_k = \tilde{\rho}_{k-1} \cup \tilde{\rho}^{GH} \cup \tilde{\rho}^{vert} \cup \tilde{\rho}^{x\tilde{x}}$$
$$\iota_k = \iota_{k-1} \cup \iota^{GH} \cup \iota^{vert} \cup \iota^{x\tilde{x}}$$

Choose $N_k > N'$ such that $\rho_k, \tilde{\rho}_k \in \mathcal{P}([-N_k, N_k])$ and $\iota_k \in \mathcal{P}(\mathcal{S}_{N_k}, \tilde{\mathcal{S}}_{N_k})$ hold.

To conclude, let

$$\rho_{\text{simple}} = \bigcup_{k \geqslant 0} \rho_k, \quad \tilde{\rho}_{\text{simple}} = \bigcup_{k \geqslant 0} \tilde{\rho}_k, \quad \iota_{\text{simple}} = \bigcup_{k > 0} \iota_k$$

Claim 3.15. *$\rho_{simple} \subset \rho$, $\tilde{\rho}_{simple} \subset \tilde{\rho}$, $\iota_{simple} \subset \iota$ implies that $G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ is simple.*

Proof. Let $G = G(\rho, \theta, \tilde{\rho}, \tilde{\theta}, \iota)$ as above and g a non-trivial element of G and let $N_g = \langle\langle g \rangle\rangle$. Let us say that g is represented by $s_{k_0} \in \mathcal{G}$ for some $k_0 > 0$ (see Remark 3.9).

Lemma 3.10 implies the existence of some element $h \in (H \cap N_g)\backslash\{1\}$. Now pick some $k_1 > 0$ such that $h \in \langle \mathcal{S}_{N_{k_1}} \rangle$. By Lemma 3.12 we must have $h' \in N_g$ for some $h' \in \mathcal{H}(X_{k_1})\backslash\{1\}$.

Now, there must be some $k_2 \geqslant k_1$ such that $h' = u_{k_2}^*$ Since $Id_{X_{k_1}} \subset \rho, \tilde{\rho}$, and $(v_j, \tilde{v}_j)_{j \in X_{k_1}} \subset \iota$, by the third step in the construction of $\rho_{k_2}, \tilde{\rho}_{k_2}, \iota_{k_2}$ and Lemma 3.13 we have $\tilde{x}, \tilde{y} \in N_g$. It follows that also $x, y \in N$. The fact that $(v_1, \tilde{w}_1), (w, \tilde{v}) \in \iota_{\text{simple}}$ then implies that $z, \tilde{z} \in N$. $\square$

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