Practical session 4: Constrained Optimization

Optimization Techniques, UPF

May, 2023

In this practice we will solve several optimization problems which are constrained to one or more equalities and inequalities. We will solve them by using both the KKT optimality conditions and the Python function `scipy.optimize.minimize`.

Deadline P101: Friday, June 2nd (at 23:55)

Deadline P102: Tuesday, May 30th (at 23:55)

Grading: The evaluation is based on the report documenting your work (with figures), results, conclusions and the commented code. For example,

- The goal of the lab.
- Summary with your own words of the topic.
- When we use the KKT conditions.
- Conclusions for each exercise.

Nneka Okolo: `nnekamaureen.okolo@upf.edu`

1 Karush-Kuhn-Tucker (KKT) optimally conditions

Let first review some of the related concepts and methods. Consider the smooth functions $f, c_1, \cdots, c_k, d_1, \ldots, d_r : \mathbb{R}^n \to \mathbb{R}$, and the following constrained optimization problem

$$\begin{array}{lll} \min\limits_{x\in\mathbb{R}^n} f(x) & & \\ \text{s.t.} \quad c_1(x) \geq 0, \ldots, c_k(x) \geq 0 & & \text{(inequality constraints)} \\ \text{and} \quad d_1(x) = 0, \ldots, d_r(x) = 0 & & \text{(equality constraints)} \end{array} \tag{1}$$

Suppose that there is a **minimum** x_0 **of (1)**.

It coincides with a **saddle point** (x_0, λ^0, ν^0) **of** $\mathcal{L}(x, \lambda, \nu)$, for

$$\mathcal{L}(x, \lambda, \nu) = f(x) - \sum_{i=1}^{k} \lambda_i c_i(x) - \sum_{j=1}^{r} \nu_j d_j(x)$$

The method of Lagrange Multipliers is used to find the solution for optimization problems constrained to one or more equalities. When our constraints also have inequalities, we need to extend the method to the KKT conditions.

The **Karush-Kuhn-Tucker (KKT) optimality conditions** are

$$
\begin{aligned}
\nabla_x \mathcal{L}(x_0, \lambda^0) &= 0 && && \text{stationarity}\\
c_i(x_0) &\geq 0 && \forall i && \text{primal feasibility}\\
d_j(x_0) &= 0 && \forall j && \text{primal feasibility}\\
\lambda_i^0 &\geq 0 && \forall i && \text{dual feasibility}\\
\lambda_i^0 c_i(x_0) &= 0 && \forall i && \text{complementary slackness}
\end{aligned}
$$

(for $1 \leq i \leq k$ and $1 \leq j \leq r$).

1.1 Toy Problem

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = x_2$, and the problem

$$
\begin{aligned}
&\min f(x_1, x_2)\\
&\text{subject to} \quad 9 \geq (x_1 - 1)^2 + (x_2 - 2)^2\\
&\qquad\qquad\quad\;\; x_2 \leq 0
\end{aligned}
$$

- Sketch the set of constraints of the problem and the level lines of the objective function.
- Guess from your sketch the solution of the problem.
- Find the solution using the function *scipy.optimize.minimize* from Python.
- Write the KKT optimality conditions and check if the minimum satisfy these conditions. Find the solution of the dual variable/s from the KKT conditions.

1.2 Water-filling

This problem arises in Information Theory, in allocating power to a set of n communication channels. Indeed, let us assume that we have a communication network and we would like to maximize the capacity or communication rate of the network. Let a variable x_i represent the capacity or transmitter power allocated to the i-th channel (connecting, e.g., two nodes of the communication network). Let $\alpha_i > 0$ represent a certain level of noise. Now, $\log(\alpha_i + x_i)$ gives the capacity or communication rate of the i-th channel. Assuming that the network has n communication channels, the problem is to allocate a total power of one to the channels, in order to maximize the total communication rate:

$$\sum_{i=1}^{n} \log(\alpha_i + x_i)$$

In other words, the convex optimization problem we have to consider is

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} -\textstyle\sum_{i=0}^{n} \log(\alpha_i + x_i)\\
&\text{s.t.} \qquad x \geq 0\\
&\text{and} \qquad \mathbf{1}^T x - 1 = 0
\end{aligned}
\tag{2}
$$

- Find the solution using the function `scipy.optimize.minimize` from Python of a network with four communication channels and the case (1) where the noise is random and between 0 and 0.25 or (2) the noise is equal in each communication channel. Comment each case.

- Write the KKT optimality conditions and check if the minimum satisfy these conditions for each case. Find the solution of the dual variable/s from the KKT conditions

1.3 Entropy maximization problem

We'll use the simplest version of entropy maximization for this lab. Entropy maximization is an important basic problem in information theory.

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} f(x) = \sum_{i=0}^{n} x_i \log x_i & \\ \text{s.t.} \qquad Ax \leq b & \\ \text{and} \qquad \mathbf{1}^T x - 1 = 0 & \end{array} \tag{3}$$

where $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ and b=$\left[\begin{smallmatrix} 7 \\ 1 \end{smallmatrix}\right]$.

- Find the solution using the function *scipy.optimize.minimize* from Python.
- Write the KKT optimality conditions and check if the minimum satisfy these conditions. Find the solution of the dual variable/s from the KKT conditions.