A Quick Introduction to the Basics of Astrodynamics

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This quick introduction booklet is dedicated to Dr.habil. Tobias Cornelius Hinse, my supervisor and esteemed co-author, whose profound expertise, unwavering guidance, and passion for astrodynamics have been the cornerstone of this work. His mentorship at the University of Southern Denmark has not only shaped this booklet but also inspired a deeper appreciation for the science of celestial mechanics.

Contents

Disclaimer

Disclaimer This document is intended solely for explanatory and educational purposes. It does not represent original scientific research, but rather a curated and accessible introduction to astrodynamics, based on public academic literature and technical resources.

Chapter 1

Introduction

Astrodynamics, also known as orbital mechanics, is a fascinating and complex field that blends physics, mathematics, and engineering to study the motion of objects in space under gravitational forces. It is the backbone of space exploration, enabling precise predictions and control of spacecraft, satellites, and celestial bodies. Whether designing a communication satellite's orbit, planning a mission to Mars, or calculating the trajectory of an interplanetary probe, astrodynamics provides the tools to navigate the cosmos. For beginners, the subject can seem daunting due to its mathematical rigor and abstract concepts, but with consistent study and practice, it becomes an exciting journey into understanding the mechanics of the universe.

This document serves as a comprehensive yet accessible introduction to astrodynamics, designed to act as a "life vest" for students and enthusiasts embarking on this journey. We will cover foundational concepts, such as Kepler's laws, Newton's laws, reference frames, orbital elements, the two-body problem, perturbations, and orbital maneuvers, with detailed explanations, practical examples, and additional problems to reinforce learning. Each section includes historical context, intuitive explanations, and real-world applications to make the material engaging and relatable.

1.1 Why Study Astrodynamics?

Astrodynamics is critical for modern space exploration and technology. For instance, GPS satellites rely on precise orbital calculations to provide accurate positioning data. Interplanetary missions, like NASA's Voyager or ESA's Rosetta, depend on astrodynamic principles to navigate vast distances with minimal fuel. Understanding these principles not only equips engineers and scientists to design space missions but also fosters a deeper appreciation of the universe's mechanics.

Chapter 2

Kepler's Laws of Planetary Motion

Johannes Kepler's three laws, formulated in the early 17th century, are the cornerstone of astrodynamics. Derived from Danish Astrophysicist, Tycho Brahe's meticulous astronomical observations, these empirical laws describe the motion of planets around the Sun and apply equally to satellites orbiting Earth or other celestial bodies [10]. Kepler's work revolutionized astronomy by challenging the geocentric model and providing a mathematical framework for orbits, which Newton later explained through gravity [3, 2].

1. **First Law (Law of Ellipses)**: Planets orbit the Sun in elliptical paths, with the Sun at one focus. Unlike perfect circles, ellipses are elongated, with two foci. For satellites orbiting Earth, Earth occupies one focus, and the other is empty. This law implies that orbits vary in distance from the central body, affecting speed and energy.

2. **Second Law (Law of Equal Areas)**: A line from the central body to the orbiting object sweeps out equal areas in equal times. This means a satellite moves faster at periapsis (closest point) and slower at apoapsis (farthest point), conserving angular momentum. This law is crucial for predicting a spacecraft's position over time.

3. **Third Law (Law of Periods)**: The square of the orbital period T is proportional to the cube of the semi-major axis a:
$$T^2 = \frac{4\pi^2}{\mu} a^3$$
Here, μ is the gravitational parameter, $\mu = GM$, where G is the gravitational constant ($6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$) and M is the central body's mass (e.g., Earth's mass, $5.972 \times 10^{24}\,\mathrm{kg}$). For Earth, $\mu \approx 3.986\,00 \times 10^5\,\mathrm{km^3\,s^{-2}}$.

Kepler's three laws for planetary motion can be visualised using the Python Inferface as follows in Figure 2.1:

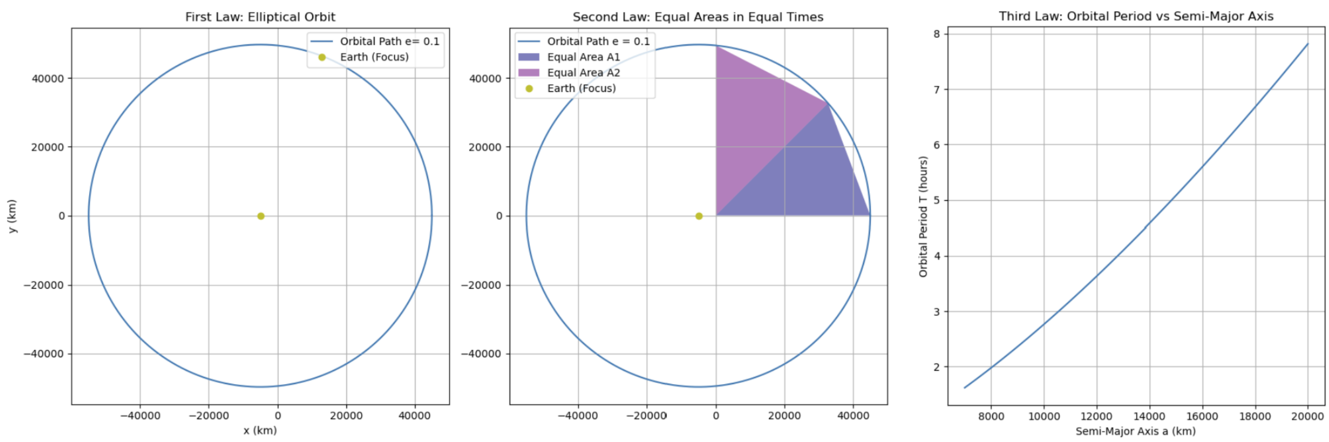

Figure 2.1: Illustration of the Kepler's Laws when $e = 0.1$ using the Python Interface

2.1 Intuitive Explanation

Kepler's First Law tells us that orbits are not perfect circles, which is why satellites experience varying distances from Earth due to the gravitational distortion element called the gravitational perturbation which deviates the movement of a probe from the orbit, but it can be explored later in the chapter. The Second Law reflects conservation of angular momentum: as a satellite approaches periapsis, its speed increases to cover the same "area" of its orbit in the same time. The Third Law connects orbit size to time, showing that larger orbits take longer to complete. These laws are universal, applying to moons, comets, and artificial satellites.

Example 1: Calculate the orbital period of a satellite with a semi-major axis of 1.0000×10^4 km, given Earth's $\mu = 3.986\,00 \times 10^5\,\mathrm{km^3\,s^{-2}}$.$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{(1.0000\times10^4)^3}{3.986\,00\times10^5}} \approx 7.976 \times 10^3\,\mathrm{s} \approx 1.33 \times 10^2\,\mathrm{min}$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{(1.0000\times10^4)^3}{3.986\,00\times10^5}} \approx 7.976 \times 10^3\,\mathrm{s} \approx 1.33 \times 10^2\,\mathrm{min}$ This satellite orbits Earth every 2 hours and 13 minutes, typical for a Low Earth Orbit (LEO) satellite.

Example 2: A geostationary satellite has a semi-major axis of 4.2164×10^4 km. Verify its orbital period matches Earth's rotation period (1 day, 8.6400×10^4 s).

$$T = 2\pi\sqrt{\frac{(4.2164 \times 10^4)^3}{3.986\,00 \times 10^5}} \approx 8.6400 \times 10^4\,\mathrm{s} = 1\ \mathrm{day}$$

This confirms that geostationary satellites remain fixed over the same point on Earth's surface.

Problem 1: A satellite orbits Mars with a semi-major axis of 5.000×10^3 km. Given Mars' $\mu = 4.2828 \times 10^4\,\mathrm{km^3\,s^{-2}}$, calculate its orbital period.

$$T = 2\pi\sqrt{\frac{(5.000 \times 10^3)^3}{4.2828 \times 10^4}} \approx 5.378 \times 10^3\,\mathrm{s} \approx 9.0 \times 10^1\,\mathrm{min}$$

Problem 2: A satellite has an orbital period of 5.400×10^3 s. Calculate its semi-major axis around Earth.

$$a = \left(\frac{T^2\mu}{4\pi^2}\right)^{\frac{1}{3}} = \left(\frac{(5.400 \times 10^3)^2 \cdot 3.986\,00 \times 10^5}{4\pi^2}\right)^{\frac{1}{3}} \approx 7.167 \times 10^3\,\mathrm{km}$$

Chapter 3

Reference Frames

A reference frame is a coordinate system used to measure positions, velocities, and accelerations in space. Choosing an appropriate reference frame is critical in astrodynamics to simplify calculations and ensure accurate trajectory predictions. Different frames serve different purposes, from analyzing orbits to tracking spacecraft from ground stations.

3.1 Importance of Reference Frames

Reference frames are essential for:

- **Simplifying Calculations**: Inertial frames, like the Earth-Centered Inertial (ECI) frame, align with gravitational forces, making orbital equations easier to solve.
- **Consistency Across Missions**: A consistent frame ensures that position and velocity data remain accurate during different mission phases (e.g., launch, orbit, landing).
- **Task Specialization**: Frames are tailored to specific tasks, such as orbital analysis (inertial frames) or surface operations (body-fixed frames).
- **Coordination with Observations**: Frames like the International Celestial Reference Frame (ICRF) provide a universal standard for precise measurements across the solar system.

3.2 Common Reference Frames

- **Earth-Centered Inertial (ECI)**: The origin is at Earth's center, with axes fixed relative to distant stars (inertial, non-rotating). Ideal for orbital dynamics, as it simplifies gravitational calculations.
- **Earth-Centered, Earth-Fixed (ECEF)**: The origin is at Earth's center, but axes rotate with Earth. Used for ground-based tracking, as observatories and ground stations are fixed on Earth's surface.
- **Orbital Frame**: Axes are aligned with the spacecraft's velocity (along-track), radius vector (radial), and their cross-product (normal). Useful for planning maneuvers, such as orbit corrections.
- **International Celestial Reference Frame (ICRF)**: A quasi-inertial frame defined by distant celestial objects (e.g., quasars). It serves as a solar-system-wide standard for high-precision astrodynamics calculations.

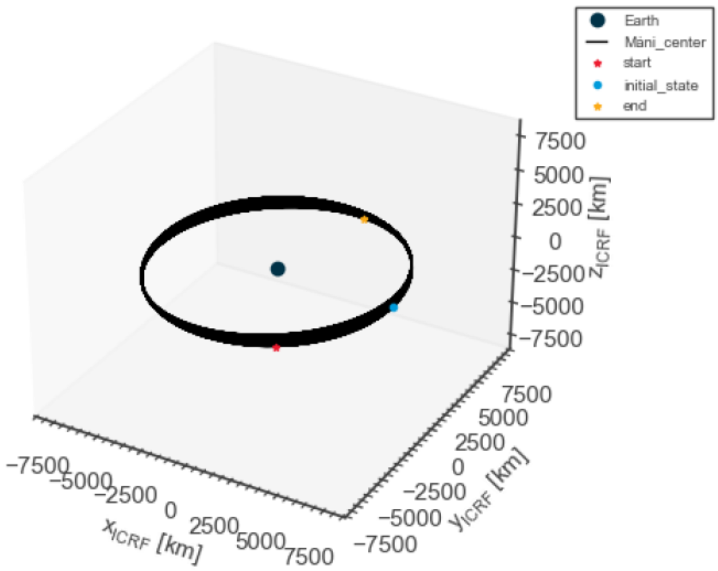

Figure 3.1: Low Earth Orbit (LEO) at 2.00×10^2 km altitude in a circular orbit, referenced in the International Celestial Reference Frame (ICRF).

3.3 Coordinate Transformations

Transformations between frames account for relative motion, such as Earth's rotation. For example, converting from ECI to ECEF requires a rotation matrix based on Earth's angular velocity ($7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$). These transformations are vital for navigation, ensuring that a spacecraft's position is accurately represented in different contexts [7].

Example 1: A satellite's position in ECI is (7.000×10^3, 0, 0) km. After Earth rotates $9.0 \times 10^{1\circ}$ since the reference epoch, estimate its ECEF coordinates, assuming rotation in the xy-plane [7].

$$\mathrm{ECEF} = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} 7000 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -7000 \end{bmatrix} \text{ km}$$

This shows the satellite's position in the rotating ECEF frame after a 90-degree rotation.

Example 2: A spacecraft in ECI has velocity (7.5, 0, 0) $\mathrm{km\,s^{-1}}$ at $t = 0$. After 3.600×10^3 s, calculate its approximate ECEF velocity, assuming rotation about the z-axis.

$$\theta = 7.292 \times 10^{-5}\,\mathrm{rad\,s^{-1}} \cdot 3.600 \times 10^3\,\mathrm{s} \approx 2.62 \times 10^{-1}\,\mathrm{rad} \approx 1.5 \times 10^{1\circ}$$

$$\text{ECEF velocity} = \begin{bmatrix} \cos 15^\circ & \sin 15^\circ \\ -\sin 15^\circ & \cos 15^\circ \end{bmatrix} \begin{bmatrix} 7.5 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 7.24 \\ -1.94 \end{bmatrix} \mathrm{km\,s^{-1}}$$

Problem 1: A satellite's ECI position is (8.000×10^3, 2.000×10^3, 0) km. After Earth rotates $4.5 \times 10^{1\circ}$, calculate its ECEF coordinates.

$$\mathrm{ECEF} = \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} \approx \begin{bmatrix} 7071 \\ -4243 \end{bmatrix} \text{ km}$$

Problem 2: A ground station observes a satellite in ECEF at (6.000×10^3, 0, 0) km. Convert this to ECI coordinates, assuming Earth has rotated $3.0 \times 10^{1\circ}$ since the reference epoch.

$$\mathrm{ECI} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} 6000 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 5196 \\ 3000 \end{bmatrix} \text{ km}$$

3.4 Practical Application

Reference frames are used in mission planning to align calculations with observational data. For example, GPS satellites are tracked in ECI for orbit determination but converted to ECEF for ground-based positioning. The ICRF is used for interplanetary missions, ensuring consistency across vast distances.

Chapter 4

Newton's Laws and Gravity

Isaac Newton's laws of motion and universal gravitation provide the physical foundation for astrodynamics. Developed in the late 17th century, these laws explain why satellites orbit and how spacecraft maneuver, building on Galileo's earlier work on motion and gravity [3].

- **First Law (Inertia)**: An object remains at rest or in uniform motion unless acted upon by an external force. In space, satellites orbit without propulsion due to the absence of significant friction.
- **Second Law**: The force acting on an object equals its mass times its acceleration:

  $$F = ma$$

  This quantifies how gravitational forces accelerate a spacecraft.
- **Third Law**: For every action, there is an equal and opposite reaction:

  $$\sum F_{1-2} = -\sum F_{2-1}$$

  This is evident in rocket propulsion, where exhaust gases propel the rocket forward.

4.1 Historical Context

Galileo Galilei's experiments with inclined planes laid the groundwork for Newton's First Law. By observing a ball rolling down one incline and up another, Galileo noted that as the second incline approached horizontal, the ball traveled farther, suggesting perpetual motion in the absence of friction. This insight, from over 450 years ago, is vividly demonstrated in microgravity environments like the International Space Station. Galileo's discovery that objects fall at the same rate regardless of mass inspired Newton to define force as mass times acceleration, revolutionizing physics.

Newton's law of universal gravitation states:

$$F = G \cdot \frac{m_1 m_2}{r^2}$$

Where:

- F: Gravitational force.
- $G \approx 6.674 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$: Gravitational constant.
- m_1, m_2: Masses of the two bodies.
- r: Distance between their centers.

4.2 Intuitive Explanation

Gravity acts as the centripetal force keeping satellites in orbit. The balance between a satellite's velocity and gravitational pull results in a stable orbit, like a ball swinging on a string. Newton's Third Law explains rocket propulsion: expelling mass backward accelerates the rocket forward, a principle used in all space launches.

Example 1: A 5.00×10^2 kg satellite orbits at 7.000×10^3 km from Earth's center. Calculate the gravitational force, given Earth's mass is 5.972×10^{24} kg.

$$F = \frac{6.674 \times 10^{-11} \cdot 500 \cdot 5.972 \times 10^{24}}{(7 \times 10^6)^2} \approx 4.061 \times 10^3\,\mathrm{N}$$

Example 2: Calculate the acceleration of a 1.000×10^3 kg satellite at 8.000×10^3 km from Earth's center.

$$a = \frac{3.986\,00 \times 10^5}{(8.000 \times 10^3)^2} \approx 6.23\,\mathrm{m\,s^{-2}}$$

Problem 1: A 2.000×10^3 kg spacecraft is 9.000×10^3 km from Earth's center. Calculate the gravitational force.

$$F = \frac{6.674 \times 10^{-11} \cdot 2000 \cdot 5.972 \times 10^{24}}{(9 \times 10^6)^2} \approx 9.854 \times 10^3\,\mathrm{N}$$

Problem 2: A 3.00×10^2 kg satellite experiences a gravitational force of 2.500×10^3 N. Calculate its distance from Earth's center.

$$r = \sqrt{\frac{6.674 \times 10^{-11} \cdot 300 \cdot 5.972 \times 10^{24}}{2.500 \times 10^3}} \approx 7.776 \times 10^3\,\mathrm{km}$$

Chapter 5

Orbital Elements

Orbital elements are six parameters that fully describe a spacecraft's orbit in three-dimensional space. They define the orbit's size, shape, orientation, and the object's position within it, making them essential for mission planning and navigation.

- **Semi-major axis** (a): Half the longest diameter of an ellipse, defining the orbit's size.
- **Eccentricity** (e): Defines the orbit's shape: $e = 0$ (circular), $0 < e < 1$ (elliptical), $e = 1$ (parabolic), $e > 1$ (hyperbolic).
- **Inclination** (i): Angle between the orbital plane and a reference plane (e.g., Earth's equator).
- **Right ascension of ascending node** (Ω): Angle defining the orbital plane's orientation relative to a reference direction (e.g., vernal equinox).
- **Argument of periapsis** (ω): Angle from the ascending node to periapsis, specifying the closest approach.
- **True anomaly** (θ): Angle from periapsis to the object's current position, indicating its location in the orbit.
- **Gravitational Parameters** (μ): The product of the gravitational constant (G) and the mass M of the celestial body. For two-body problem where m is the mass of the satellite or probe, the gravitational parameter can be written and then applied further as follows:

$$\mu = G(M + m) \approx GM \tag{5.1}$$

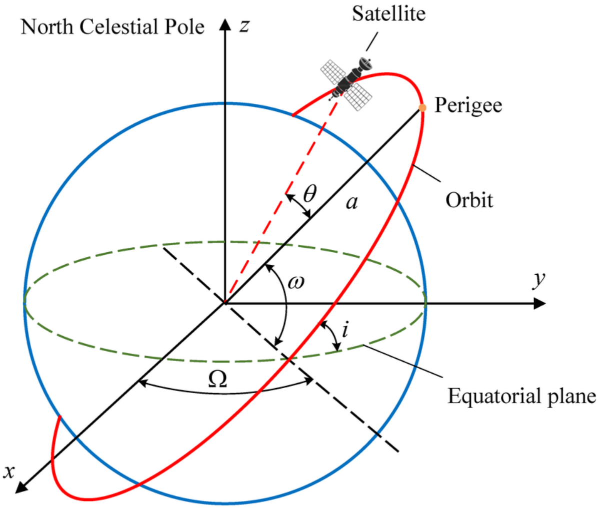

Figure 5.1: Satellite orbiting in a geocentric frame with the representation of orbital elements[1]

5.1 Practical Application

Orbital elements are used to catalog satellites (e.g., Two-Line Element sets for GPS satellites) and plan maneuvers. For example, a polar orbit has $i \approx 9.0 \times 10^{1\circ}$, ideal for Earth observation, while a geostationary orbit has $i \approx 0^\circ$.

Example 1: A satellite has $a = 1.0000 \times 10^4$ km, $e = 0.1$, $\theta = 9.0 \times 10^{1\circ}$. Calculate its radial distance.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{1.0000 \times 10^4 \cdot (1-0.1^2)}{1+0.1\cdot\cos 90^\circ} \approx 9.900 \times 10^3\,\text{km}$$

Example 2: A satellite's position is $(8.000 \times 10^3, 0, 0)$ km and velocity $(0, 7.0, 0)$ $\text{km}\,\text{s}^{-1}$. Calculate a and e.

$$r = 8.000 \times 10^3\,\text{km}, \quad v = 7.0\,\text{km}\,\text{s}^{-1}$$

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \implies 49 = 3.98600 \times 10^5\left(\frac{2}{8.000 \times 10^3} - \frac{1}{a}\right) \implies a \approx 1.0638 \times 10^4\,\text{km}$$

$$h = rv = 8.000 \times 10^3 \cdot 7.0 = 5.6000 \times 10^4\,\text{km}^2\,\text{s}^{-1}$$

$$r = \frac{h^2}{\mu(1+e\cos\theta)} \implies 8.000 \times 10^3 = \frac{(5.6000 \times 10^4)^2}{3.98600 \times 10^5(1+e)} \implies e \approx 0.248$$

Problem 1: A satellite has $a = 1.5000 \times 10^4$ km, $e = 0.15$, $\theta = 1.80 \times 10^{2\circ}$. Calculate its radial distance.

$$r = \frac{1.5000 \times 10^4 \cdot (1-0.15^2)}{1+0.15\cdot\cos 180^\circ} \approx 1.7250 \times 10^4\,\text{km}$$

Problem 2: A satellite's position is $(1.0000 \times 10^4, 0, 0)$ km, velocity $(0, 6.5, 0)$ $\text{km}\,\text{s}^{-1}$. Calculate a, e, and periapsis.

$$a \approx 1.3333 \times 10^4\,\text{km}, \quad e \approx 0.333, \quad r_p = 1.3333 \times 10^4 \cdot (1-0.333) \approx 8.889 \times 10^3\,\text{km}$$

[1]Source: Geocentric Frame Picture from ResearchGate

Chapter 6

Two-Body Problem and Types of Orbits

The two-body problem models the gravitational interaction between two objects (e.g., Earth and a satellite), assuming other influences are negligible. The resulting orbits are conic sections: circular, elliptical, parabolic, or hyperbolic, determined by the eccentricity e [3].

1. **Circular** ($e = 0$): Constant radius, used for geostationary satellites.
2. **Elliptical** ($0 < e < 1$): Varying radius, common for many satellites and planets.
3. **Parabolic** ($e = 1$): Escape trajectory with zero specific energy, seen in some comets.
4. **Hyperbolic** ($e > 1$): Escape trajectory with positive specific energy, used for interplanetary missions.

The Vis-Viva Equation

The vis-viva equation relates velocity, distance, and semi-major axis [3]:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \tag{6.1}$$

Where $\mu \approx 3.986\,00 \times 10^5\,\mathrm{km}^3\,\mathrm{s}^{-2}$ for Earth according to the Gaussian approximation[3]. The specific energy is:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \text{ (elliptical)}, \quad \varepsilon = 0 \text{ (parabolic)}, \quad \varepsilon = \frac{\mu}{2|a|} \text{ (hyperbolic)} \tag{6.2}$$

Derivation:

$$\text{Kinetic energy} = \frac{v^2}{2}, \quad \text{Potential energy} = -\frac{\mu}{r} \tag{6.3}$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} \tag{6.4}$$

Using the vis-viva equation:

$$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{2a} \implies \varepsilon = -\frac{\mu}{2a} \tag{6.5}$$

6.1 Circular Orbits ($e = 0$)

According to Curtis' textbook on the orbital mechanics for engineering students [3], Circular orbits have constant radius and zero eccentricity. The orbital equation is:

$$r = \frac{h^2}{\mu}, \quad h = rv \tag{6.6}$$

The orbital period is:

$$T = \frac{2\pi}{\sqrt{\mu}} r^{\frac{3}{2}} \tag{6.7}$$

The specific energy is:

$$\varepsilon = -\frac{\mu}{2r} \tag{6.8}$$

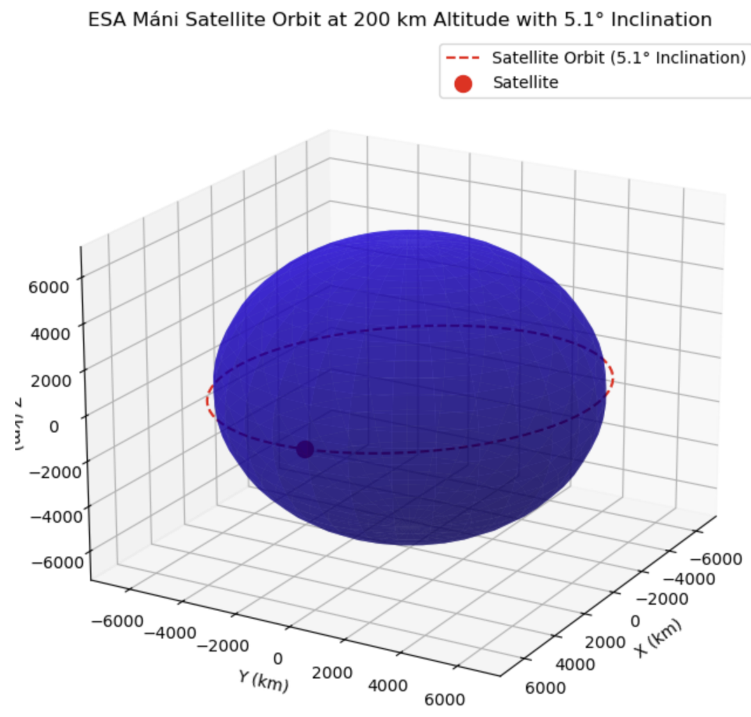

Figure 6.1: ESA Máni Satellite in Low Earth Orbit (LEO) at 2.00×10^2 km altitude in a circular orbit, with equal aspect ratios for x, y, and z axes to accurately represent Earth.

6.1.1 Practical Application

Understanding the use of a circular orbit is important, especially for the satellites and space stations circulating Earth or other celestial bodies.

Example 1: A 1.000×10^3 kg satellite orbits at 6.778×10^3 km (4.00×10^2 km altitude). Calculate the gravitational force.

$$F = \frac{3.986\,00 \times 10^5 \cdot 1000}{(6.778 \times 10^3)^2} \approx 8.684 \times 10^3\,\mathrm{N}$$

Example 2: Calculate the velocity and orbital period at 7.000×10^3 km.

$$v = \sqrt{\frac{3.986\,00 \times 10^5}{7.000 \times 10^3}} \approx 7.55\,\mathrm{km\,s^{-1}}$$

$$T = \frac{2\pi}{\sqrt{3.986\,00 \times 10^5}} \cdot (7.000 \times 10^3)^{\frac{3}{2}} \approx 5.830 \times 10^3\,\mathrm{s} \approx 9.7 \times 10^1\,\mathrm{min}$$

Problem 1: A satellite orbits at 7.500×10^3 km. Calculate its velocity, orbital period, and specific energy.

$$v \approx 7.29\,\mathrm{km\,s^{-1}}, \quad T \approx 6.318 \times 10^3\,\mathrm{s} \approx 1.05 \times 10^2\,\mathrm{min}, \quad \varepsilon \approx -2.657 \times 10^1\,\mathrm{km^2\,s^{-2}}$$

Problem 2: A 1.200×10^3 kg satellite orbits at 6.800×10^3 km. Calculate the gravitational force and velocity.

$$F \approx 1.0309 \times 10^4\,\mathrm{N}, \quad v \approx 7.66\,\mathrm{km\,s^{-1}}$$

6.2 Elliptical Orbits $(0 < e < 1)$

Elliptical orbits have varying radius, defined by semi-major axis a and eccentricity $0 < e < 1$. The orbital equation is:

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{6.9}$$

The position vector is:

$$r(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \tag{6.10}$$

Solving Kepler's equation, the mean anomaly, $(M = E - e\sin E)$ is required to find the position at time t, using numerical methods like Newton-Raphson [3]:

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n} \tag{6.11}$$

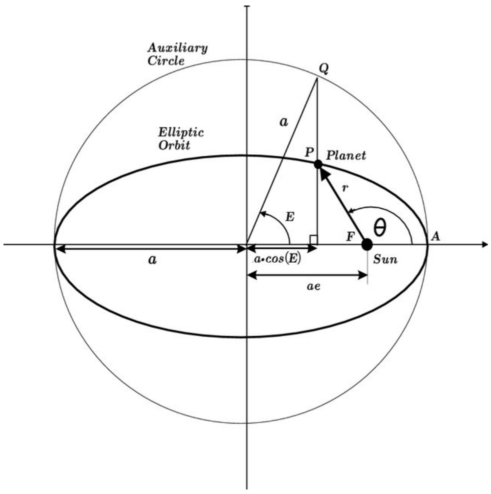

Figure 6.2: Elliptic orbit compared to an auxiliary circle in a Heliocentric frame (Sun-centered frame)[1]

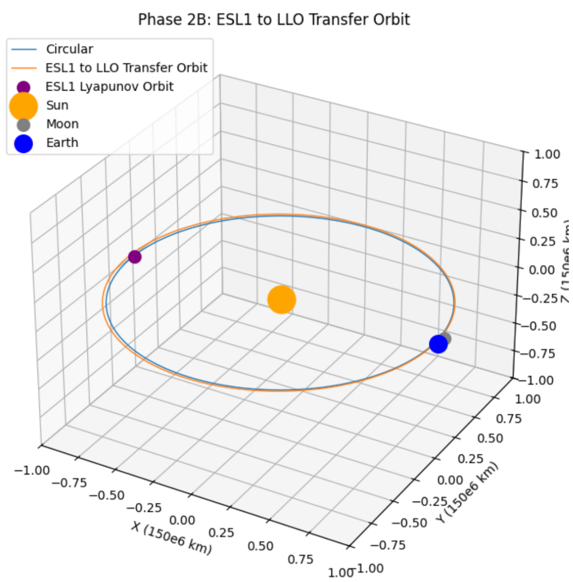

Figure 6.3: Elliptic orbit transfer from ESL1 Lyapunov Orbit to Low Lunar Orbit indicated in an orange line in Heliocentric Frame indicated in orange[3]

[1]Source: Elliptic orbit from ResearchGate

According to Figure 6.2 cited from Gonzalez-Gaxiola and Fernándes-Linares' study on the efficient iterative method for solving the elliptical Kepler's equations, this geometrical representation of an elliptic orbit exemplifies the position of a planet orbiting around the Sun meaning that it is illustrated in a heliocentric frame [5].

In Figure 6.2, this ellipse has a semi-major axis of a and eccentricity denoted as e with the focus point of F. The centre of the ellipse is located at the origin, and the distance from the centre to the focus is ae resulting in the linear eccentricity of ae.

Following this, an auxiliary circle which is also known as as the reference circle is drawn with the radius of a centered at the origin. This circle is used to define the eccentric anomaly E which is a crucial angular parameter in orbital calculations [3].

On the other hand, in Figure 6.3 from the Amarsanaa-Hinse study on the lunar transfer, the transfer is also illustrated in a heliocentric frame sending a probe or satellite from the ESL1 Lyapunov Orbit which is also known as Earth-Sun Lagrange Point-1 Lyapunov Orbit, a periodic orbit that exists within the equilibrium point of celestial bodies called Lagrange Point-1 [3, 1]. While the transfer orbit from ESL1 to Low Lunar Orbit abbreviated by LLO seems circular, however, in order to prove that this transfer orbit in orange is accepted to be an elliptic orbit due to the eccentric value of it by comparing it to the circular orbit in blue colour.

6.2.1 Practical Application

This orbit type is very useful when doing the orbital manoeuvre such as the Hohmann Transfer which is to be explained later in the chapter in a more efficient manner due to the low eccentric value in comparison to parabolic or hyperbolic transfers which are not as useful as elliptic transfers.

Example 1: A satellite has $a = 1.2000 \times 10^4\,\text{km}$, $e = 0.3$. Calculate periapsis, apoapsis, and orbital period.

$$r_p = 1.2000 \times 10^4 \cdot (1 - 0.3) = 8.400 \times 10^3\,\text{km}, \quad r_a = 1.2000 \times 10^4 \cdot (1 + 0.3) = 1.5600 \times 10^4\,\text{km}$$

$$T = 2\pi\sqrt{\frac{(1.2000 \times 10^4)^3}{3.986\,00 \times 10^5}} \approx 9.763 \times 10^3\,\text{s} \approx 1.63 \times 10^2\,\text{min}$$

Example 2: A spacecraft at periapsis ($7.000 \times 10^3\,\text{km}$, $v = 8.5\,\text{km}\,\text{s}^{-1}$). Calculate a, e, and apoapsis.

$$8.5^2 = 3.986\,00 \times 10^5 \left(\frac{2}{7.000 \times 10^3} - \frac{1}{a}\right) \implies a \approx 9.580 \times 10^3\,\text{km}$$

$$h = 7.000 \times 10^3 \cdot 8.5 = 5.9500 \times 10^4\,\text{km}^2\,\text{s}^{-1}$$

$$e = \sqrt{1 - \frac{h^2}{\mu a}} \approx 0.2695, \quad r_a = 9.580 \times 10^3 \cdot (1 + 0.2695) \approx 1.2160 \times 10^4\,\text{km}$$

Problem 1: A satellite has $a = 1.5000 \times 10^4\,\text{km}$, $e = 0.2$. Calculate velocity at $\theta = 9.0 \times 10^{1\circ}$.

$$r = \frac{1.5000 \times 10^4 \cdot (1 - 0.2^2)}{1 + 0.2 \cdot \cos 90^\circ} \approx 1.4400 \times 10^4\,\text{km}$$

$$v \approx 5.14\,\text{km}\,\text{s}^{-1}$$

Problem 2: A spacecraft at apoapsis ($2.0000 \times 10^4\,\text{km}$, $v = 4.0\,\text{km}\,\text{s}^{-1}$). Calculate a, e, and periapsis.

$$a \approx 1.4286 \times 10^4\,\text{km}, \quad e \approx 0.4, \quad r_p \approx 8.571 \times 10^3\,\text{km}$$

6.3 Parabolic Orbits $(e = 1)$

Parabolic orbits are escape trajectories with zero specific energy, often seen in comets or spacecraft leaving a planet's gravity [3]. However, it is important to understand that the parabolic orbit is open meaning that the semi-major axis of a parabolic orbit is as follows:

$$a = \infty \tag{6.12}$$

However, as explained in Halpern's lectures notes on parabolic orbits [6], this product can remain finite meaning that it can be approximated with the following equation of the parabolic conic section as:

$$r = \frac{2q}{1 + \cos\theta} \tag{6.13}$$

where q represents the point of minimum radius which occurs at $\theta = 0$.

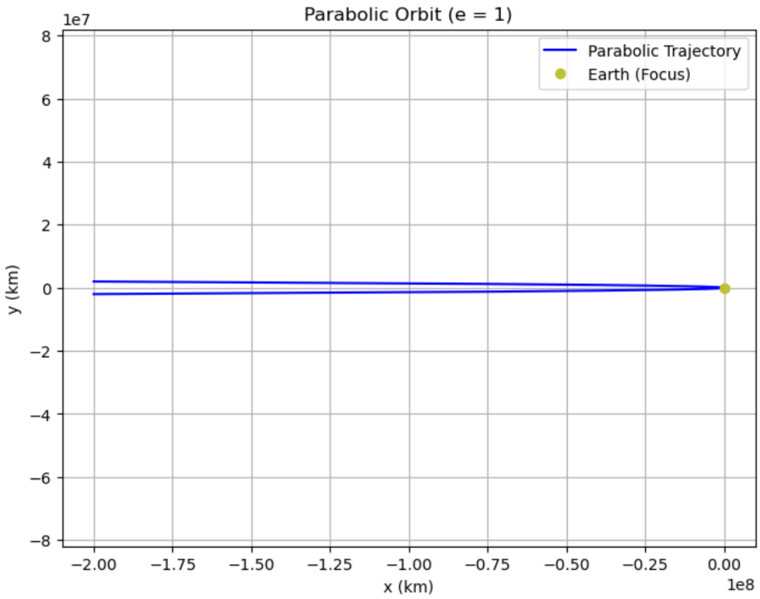

Figure 6.4: Parabolic Orbit of the semi-latus rectum of $r = 10000$km

According to Figure 6.4, it is evident that the parabolic orbit has an open end meaning that the orbiting body such as a satellite reaches to infinity.

Energy:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = 0 \quad \Rightarrow v = \sqrt{\frac{2\mu}{r}} \tag{6.14}$$

The reason why the energy of a parabolic orbit is zero is that the orbiting body reaches an infinite distance from the central body which is different and avoids a closed path like circular or elliptical orbits. This means the parabolic orbit has **just enough** energy to escape the gravitational pull of the central body, but no surplus energy. Therefore, it serves as the boundary between bound orbits (circular and elliptical) and unbound orbits (hyperbolic).

Orbital equation of the semi-latus rectum which defines the size of a parabola[3]:

$$r = \frac{h^2}{\mu} \cdot \frac{1}{1 + \cos\theta} \tag{6.15}$$

6.3.1 Barker's Equation

When a planet, comet or spacecraft is flying along a parabolic orbit, it indicates that it is moving just enough to escape velocity as stated above. This is seen as the special case right between a closed orbit such Earth orbiting around the Sun and an open, hyperbolic escape [6].

When it comes to delving to knowing where that object is in its orbit at a given time, Barker's equation is used.

The Problem

Let us exemplify that we are tracking a comet that is flying by the Sun in a parabolic path. Following this, we know:

- The closest approach to the Sun which is known as **perihelion** happens at a given time denoted as τ.
- We intend to find out how much time has passed since perihelion for a given position in the orbit measured by the angle of true anomaly.

Given this, the Barker equation connects the true anomaly and time since perihelion as the nearest point to the Sun for example denoted as $t - \tau$ as follows:

$$t - \tau = \sqrt{\frac{2r_p^3}{\mu}} \left(\tan\left(\frac{\theta}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\theta}{2}\right) \right) \tag{6.16}$$

where t is the time at the current position in orbit, τ is the time of perihelion, r_p is the perihelion distance, μ is the gravitational parameter of the Sun as stated in the problem above and θ is the true anomaly.

Nota Bene: This is important that the Barker equation is only intended to calculate the time of flight with respect to true anomaly for the parabolic orbit. This means that this equation cannot be used in other orbits.

6.3.2 Practical Application

Example 1: Calculate velocity at periapsis (7.000×10^3 km).

$$v = \sqrt{\frac{2 \cdot 3.986\,00 \times 10^5}{7.000 \times 10^3}} \approx 1.067 \times 10^1 \,\mathrm{km\,s^{-1}}$$

Example 2: A spacecraft on a parabolic orbit has periapsis 6.800×10^3 km. Calculate velocity at 1.0000×10^4 km.

$$v = \sqrt{\frac{2 \cdot 3.986\,00 \times 10^5}{1.0000 \times 10^4}} \approx 8.93 \,\mathrm{km\,s^{-1}}$$

Example 3: Suppose a comet reaches perihelion (closest to the Sun) on the 20th of June 2025, and its minimum distance is $r_p = 1$AU. You want to know when it reaches at $\theta = 85°$

- Compute the following angle:

$$\tan\left(\frac{85°}{2}\right) = 0.91633$$

- Plug the value into Barker's equation:

$$t - \tau = \sqrt{\frac{2(1)^3}{\mu_{\mathrm{Sun}}}} \left(0.91633 + \frac{1}{3} \cdot (0.91633)^3 \right)$$

Problem 1: A comet has periapsis at 6.700×10^3 km. Calculate its velocity at 1.2000×10^4 km.

$$v \approx 8.15 \,\mathrm{km\,s^{-1}}$$

Problem 2: Calculate the angular momentum for a parabolic orbit with periapsis 6.500×10^3 km.

$$h \approx 7.1955 \times 10^4 \,\mathrm{km^2\,s^{-1}}$$

Problem 3: A particle starts at rest from infinity and falls towards a central mass denoted as M under gravity. How long does it take to reach a distance r from the centre?

$$t = \frac{2}{3} \cdot \frac{r^{\frac{3}{2}}}{\sqrt{2\mu}}$$

6.4 Hyperbolic Orbits ($e > 1$)

Hyperbolic orbits are escape trajectories with positive specific energy, used for interplanetary missions or flybys. Unlike circular and elliptic transfers, hyperbolic orbits have excessive escape energy meaning that it can serve as a capturing trajectory as shown in Figure 6.5 and 6.6 below:

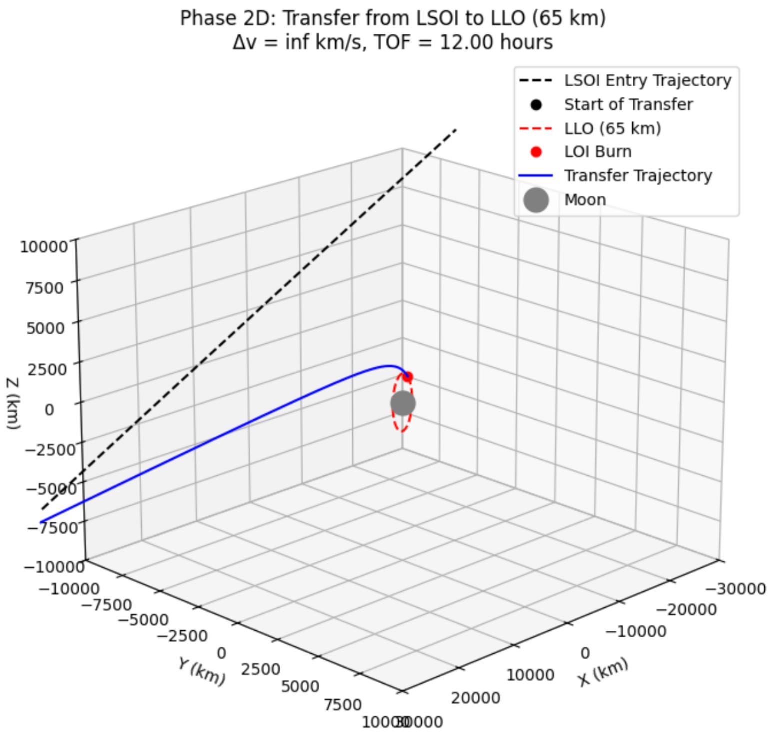

Figure 6.5: ESA Máni Satellite arriving at its operating orbit via a hyperbolic trajectory [1]

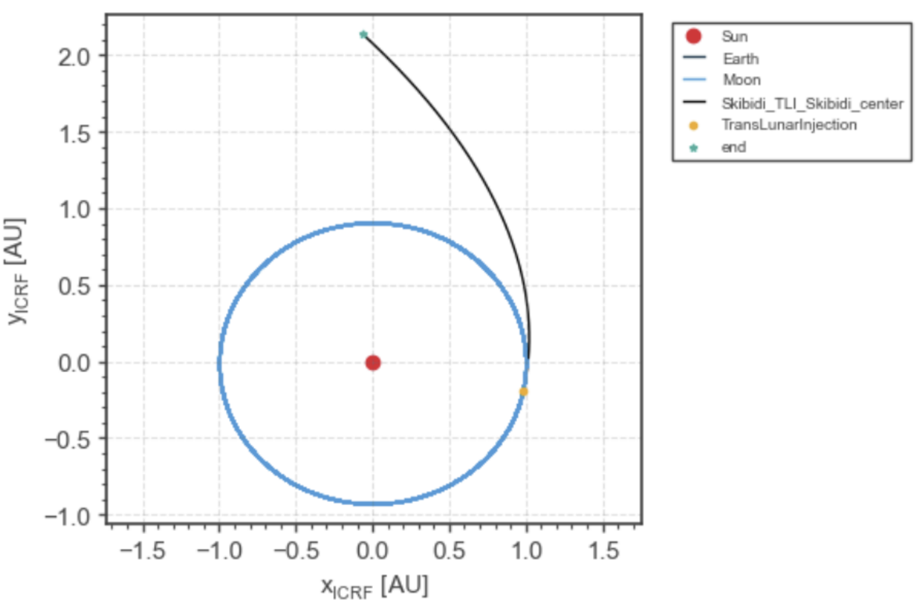

Figure 6.6: Heliocentric plot of a spacecraft performing a Trans-Lunar Injection (TLI), resulting in a hyperbolic escape trajectory from Earth. The Earth follows a circular orbit around the Sun (in blue), while the black curve represents the spacecraft's hyperbolic path relative to the Sun, beginning from the TLI point (yellow) and continuing outward

Energy:

$$\varepsilon = \frac{\mu}{2|a|} \tag{6.17}$$

As for the energy formula derivation, it is also possible to write it with respect to the hyperbolic excess velocity denoted as v_∞ as follows:

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{v_\infty^2}{2} \tag{6.18}$$

Orbital equation:

$$r = \frac{a(e^2-1)}{1+e\cos\theta} \tag{6.19}$$

Specific angular momentum:

$$h = rv_\perp = bv_\infty \tag{6.20}$$

where h is the specific angular momentum, b is the impact parameter which is the perpendicular distance between the incoming asymptote of the hyperbolic trajectory and the central mass depending on the referenced body [3], $v_\perp$ is the perpendicular velocity component at distance r.

Following this, in hyperbolic orbit transfers, the eccentricity is also dependent on the energy and angular momentum as follows:

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}} \tag{6.21}$$

According to Figures 6.5 and 6.6, it is evident that the importance of utilising the hyperbolic orbits for the mission is to escape trajectory for leaving Earth or any celestial body to enter interplanetary or interstellar space as shown in Figure 6.6. As for the Figure 6.5 from the Amarsanaa-Hinse study on the lunar transfer and perturbation analysis [1], the probe can be benefited from hyperbolic trajectory modelling to understand the complex energy exchanges and conditions for lunar orbit captures.

6.4.1 True Anomaly in Hyperbolic Orbits

Here it is important to note that the true anomaly denoted as θ is defined as the angle between the direction of periapsis and the current position of the celestial body or an object, measured at the focus of the ellipse which is evident in Figure 6.2. It is related to the eccentric anomaly E is by:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right) \quad \text{for} \quad 0 < e < 1$$

In hyperbolic transfer, on the other hand, the true anomaly is still the angle from periapsis to the current position, but the motion is **unbounded** similar to parabolic orbits, θ can take values beyond $\pm180°$ or $\pm2\pi$ meaning that $\theta \in (-\infty, \infty)$. It is related to the hyperbolic eccentric anomaly denoted as H by:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right) \quad \text{for} \quad e > 1 \tag{6.22}$$

6.4.2 Practical Application

Example 1: A probe with $e = 1.5$, periapsis 7.000×10^3 km. Calculate hyperbolic excess velocity.

$$v_\infty = \sqrt{\frac{3.986\,00 \times 10^5 \cdot (1.5^2 - 1)}{7.000 \times 10^3}} \approx 6.71\,\mathrm{km\,s^{-1}}$$

Example 2: A spacecraft with $v_\infty = 5\,\mathrm{km\,s^{-1}}$, periapsis 7.000×10^3 km. Calculate e and v_p.

$$e \approx 1.00022, \quad v_p \approx 1.067 \times 10^1\,\mathrm{km\,s^{-1}}$$

Problem 1: A probe has $e = 1.3$, periapsis 7.100×10^3 km. Calculate a and velocity at $\theta = 6.0 \times 10^{1\circ}$.

$$a \approx -1.7391 \times 10^4\,\mathrm{km}, \quad v \approx 9.89\,\mathrm{km\,s^{-1}}$$

Problem 2: A spacecraft has $v_\infty = 7\,\mathrm{km\,s^{-1}}$ $r_p = 7.200 \times 10^3\,\mathrm{km}$ Calculate the eccentricity.

$$e \approx 1.00044$$

6.5 Perturbations

Real orbits deviate from ideal two-body motion due to perturbations:

- **Atmospheric Drag**: Slows satellites in LEO, causing orbital decay.
- **Third-Body Effects**: Gravitational influences from the Moon, Sun, or other planets.
- **Earth's Oblateness (J2 Effect)**: Earth's equatorial bulge causes orbital precession.

6.5.1 Atmospheric Drag

In Low Earth Orbit (LEO), satellites experience atmospheric drag, a force that opposes their motion and slowly causes them to lose altitude requiring additional burn to stay in the orbit. This drag force is critical to model accurately, especially for tracking and predicting satellite orbits. The High Accuracy Satellite Drag Model abbreviated by HASDM uses a physics-based approach to refine these predictions as follows [8]:

The fundamental drag equation of the acceleration due to atmospheric drag denoted as a_D is described by the following formula:

$$a_D = \frac{1}{2} B \rho V^2 \tag{6.23}$$

where a_D is the drag acceleration, B is the ballistic coefficient representing the ratio of the satellite, ρ is the atmospheric density at the satellite's altitude, and V is the velocity of the satellite relative to the atmosphere.

Since the ρ represents the theoretical estimation of the atmospheric drag, it means that in reality, it is barely impractical to know the exact atmospheric density. Theoretical models such as HASDM use a predicted value, ρ_{model} and estimate the ballistic coefficient, B_{model} to match observed satellite motion.

According to Storz's study on the HASDM model [8], it is possible to assume that the observed drag matches the modelled drag as follows:

$$\frac{1}{2} B_{\text{True}} \rho_{\text{True}} V^2 \approx \frac{1}{2} B_{\text{Model}} \rho_{\text{Model}} V^2 \tag{6.24}$$

This simplifies to as follows:

$$B_{\text{True}} \rho_{\text{True}} \approx B_{\text{Model}} \rho_{\text{Model}} \tag{6.25}$$

Through this, this allows us to estimate the true atmospheric density as follows:

$$\rho_{\text{True}} \approx \left(\frac{B_{\text{Model}}}{B_{\text{True}}} \right) \rho_{\text{Model}} \tag{6.26}$$

Example 1: A $5.00 \times 10^2\,\mathrm{kg}$ satellite in LEO experiences a drag force of $1 \times 10^{-2}\,\mathrm{N}$. Calculate its deceleration.

$$a = \frac{1 \times 10^{-2}}{5.00 \times 10^2} = 2 \times 10^{-5}\,\mathrm{m\,s^{-2}}$$

Example 2: Estimate the altitude loss per orbit for a $1.000 \times 10^3\,\mathrm{kg}$ satellite at $3.00 \times 10^2\,\mathrm{km}$ ($r = 6.678 \times 10^3\,\mathrm{km}$) with a drag force of $2 \times 10^{-2}\,\mathrm{N}$, period $5.400 \times 10^3\,\mathrm{s}$.

$$\Delta v = \frac{2 \times 10^{-2}}{1.000 \times 10^3} \cdot 5.400 \times 10^3 \approx 1.08 \times 10^{-1}\,\mathrm{m\,s^{-1}}$$

$$\Delta r \approx \frac{\Delta v}{v} \cdot r \approx 1 \times 10^{-1}\,\mathrm{km}$$

Problem 1: A 8.00×10^2 kg satellite at 4.00×10^2 km experiences 1.5×10^{-2} N drag. Calculate deceleration.

$$a \approx 1.875 \times 10^{-5}\,\mathrm{m\,s^{-2}}$$

Problem 2: Estimate altitude loss per orbit for a 1.500×10^3 kg satellite at 2.50×10^2 km, drag 2.5×10^{-2} N, period 5.300×10^3 s.

$$\Delta r \approx 1.2 \times 10^{-1}\,\mathrm{km}$$

6.5.2 Solar Radiation Perturbation

According to Xu's study on the orbital disturbing effects of the solar radiation[9], solar radiation pressure caused by the solar light, is a disturbing force (scaled by mass or acceleration) acting on a probe's surface which disturbs the orbit of the satellite or planet in the Solar system. This is can be written as follows:

$$f_{\text{solar}} = P_s \frac{\eta}{2\pi c} \frac{A}{m} \frac{1}{r^2} \tag{6.27}$$

where P_s is the solar luminosity $= 3.846 \cdot 10^{26} W$, η is the surface reflectivity ranging between ($0.5 \leq \eta \leq 1$ where 0.5 corresponds to the total absorption of photons and 1 to total reflection).

$$\xi = P_s \frac{\eta}{2\pi c} \frac{A}{m} \tag{6.28}$$

where ξ is the representation of the coefficient part of the solar radiation perturbation force scaled by mass or acceleration.

6.5.3 Gravitational Perturbation

Gravitational perturbation is typically expressed using a series expansion of scalar spherical harmonics to model the gravitational field of an astronomical body. This approach allows for a more precise and accurate representation of the deviations from a purely spherical mass distribution such as those caused by the Earth's equatorial bulges[3]. These deviations are especially important when modeling high-precision missions such as lunar transfers.

The gravitational potential $U(\vec{r})$ around a celestial body is modeled using a spherical harmonics expansion[4]:

$$U(\vec{r}) = \frac{GM}{r} \left[1 - \sum_{k=2}^{K} \left(\frac{R}{r} \right)^k \sum_{m=0}^{k} \left(C_{km} \cos m\phi + S_{km} \sin m\phi \right) P_{km}(\cos\theta) \right] \tag{6.29}$$

where:

- G is the gravitational constant,
- M is the mass of the central body,
- r is the radial distance from the center,
- R is the reference radius,
- C_{km}, S_{km} are spherical harmonic coefficients,
- k is the radial order
- m is the longitudinal variation
- P_{km} are associated Legendre polynomials,
- θ is co-latitude and ϕ is longitude.

In this booklet, we focus on J_2 and J_4 (the zonal harmonics), which dominate Earth and Moon gravitational irregularities. These perturbations are especially significant for long-duration trajectories and are incorporated into our transfer modeling to ensure accuracy.

Applied Model

To simulate the effects of zonal harmonics on the spacecraft trajectory, we apply the following perturbation potential function[1, 3], and here, it is crucial to see that the longitudinal variation is not considered for the symmetrical rotational perturbation as follows:

$$\Phi_k(i, \omega, \theta) = \frac{\mu}{R} \sum_{k=2}^{\infty} J_k \left(\frac{R}{r(\theta)} \right)^k P_k \left(\sqrt{1 - \sin i \sin(\omega + \theta)} \right) \tag{6.30}$$

Where:

- $\mu = GM$ is the gravitational parameter of the body,
- $r(\theta)$ is the radial distance as a function of true anomaly θ,
- J_k are the zonal harmonic coefficients,
- P_k are the Legendre polynomials,
- i is the inclination, and ω is the argument of perigee.

For the P_k term, in order to simply the Legendre Polynomial at the order of k-th, the Rodrigues' formula is used as follows[3]:

$$P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} (x^2 - 1)^k \quad \text{where} \quad k \in N_0 \quad \text{and} \quad k \geq 2 \tag{6.31}$$

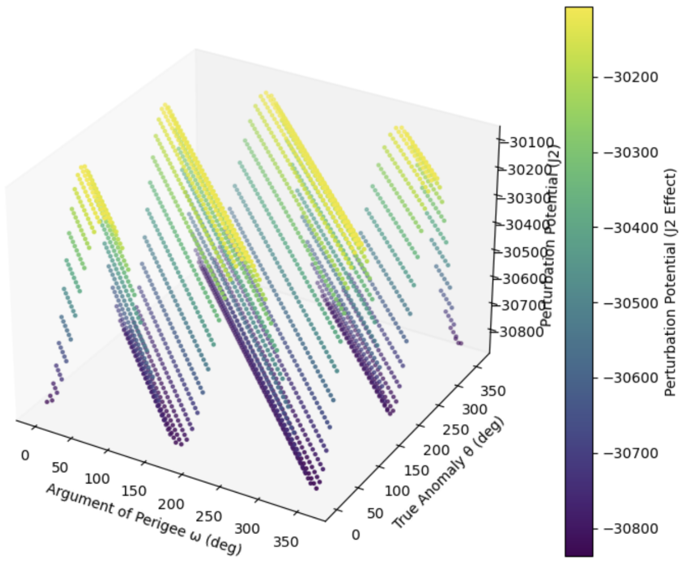

Figure 6.7: Earth's J2 oblateness perturbation effect on a satellite at 2.00×10^2 km altitude [1]

This model enables us to account for the influence of oblateness in both Earth and Moon systems during critical mission phases such as Earth departure and lunar orbit insertion.

However, the applied model of the symmetrical rotational perturbation of a celestial due to k-th order of zonal harmonics is orignally derived from the following equation with respect to the polar angle or longitudinal angle denoted as ϕ in spherical coordinates from the Curtis textbook on the orbital mechanics for engineering students [3]:

$$\Phi_k(\phi) = \frac{\mu}{r} \sum_{k=2}^{\infty} J_k \left(\frac{R}{r} \right)^k P_k(\cos \phi) \tag{6.32}$$

Using the Zwilinger study on the spherical coordinates' trigonometry derivations as stated in Curtis' textbook [3], the following polar angle would be written as follows:

$$\sin \theta = \sin i \sin u \tag{6.33}$$

where i is the inclination angle and u is the argument of latitude.

According to the Gaussian variational equation in celestial mechanics, the variation of the arguments of periapsis and latitude obey the following geometrical relationship of [1, 3]:

$$\omega = u - \theta \tag{6.34}$$

Following this, we get:

$$\sin(\omega + \theta) = \sin u \tag{6.35}$$

$$\cos\phi = \sqrt{1 - \sin i \sin(\omega + \theta)} \tag{6.36}$$

6.6 Orbital Manoeuvres

Orbital maneuvers adjust a spacecraft's trajectory to achieve mission objectives. The following manoeuvres are the most used transfers:

- **Hohmann Transfer**: This transfer is a two-impulsive elliptical transfer between two co-planar circular orbits. The transfer itself consists of an elliptical orbit with a perigee at the inner orbit an an apogee at the outer orbit.
- **Plane Change**: This refers to changing the inclination or orientation of an orbit altering the direction in which a spacecraft is orbiting a planet without necessarily changing its altitude or shape of the orbit.

6.6.1 Hohmann Transfer

In order to further delve into this, according to Curtis' textbook on the orbital mechanics for engineering students, a Hohmann Transfer is a two-burn orbital manoeuvre used to move a spacecraft between two circular orbits of different radii around the same central body (e.g., Earth, Sun). This can be seen as the most efficient and simplest way to change orbits when the initial and target orbits are coplanar meaning that they align in the same plane.

Main phases of the manoeuvre

Assuming that the probe is intended to travel from a lower circular orbit to a higher circular orbit which has more altitude than the lower one. The following phases are considered to perform the Hohmann transfer as follows:

- First burn (Δv_1): At the point in the lower orbit, the spacecraft fires at its engine to increase velocity and enter an elliptical transfer orbit which is also known as the transfer ellipse. The elliptic orbit touches the initial orbit at the periapsis and the target orbit at the apoapsis.
- Coast along transfer orbit: The spacecraft coasts along this elliptical orbit without engine thrust.
- Second Burn (Δv_2): When the spacecraft reaches the apoapsis, it fires its engine again to circularise the orbit at the new altitude, higher orbit by increasing velocity to the circular orbital speed at that specific altitude.

The following figure 6.8 allows us to see how the Hohmann transfer method is performed with the use of the transfer elliptic orbit to reach to the final orbit annotated in a green dotted line.

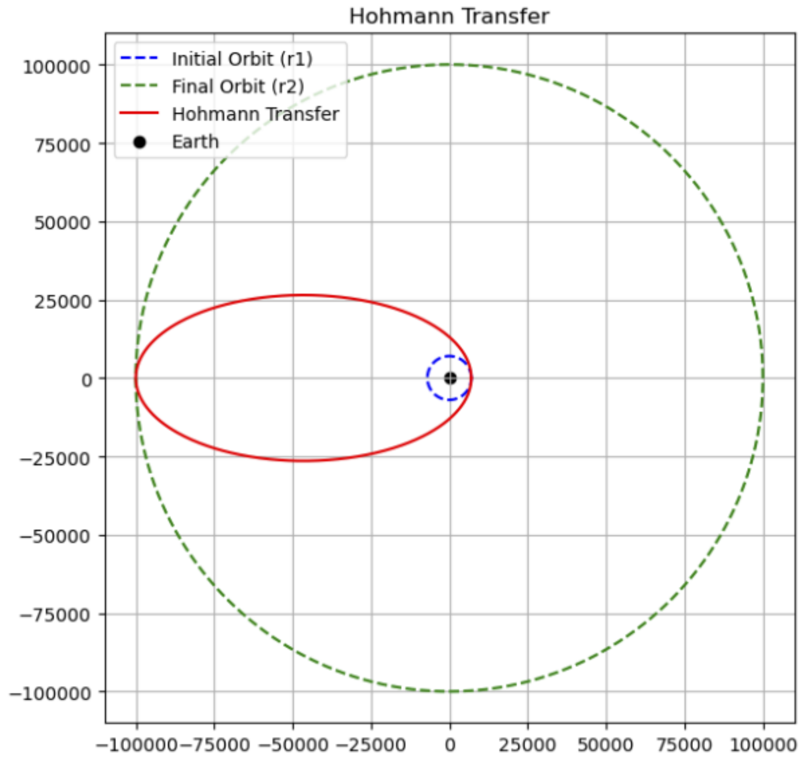

Figure 6.8: Hohmann Transfer for a probe in the geocentric frame

Velocity at Key Points and Delta-V calculations

Velocity at Key Points

- Velocity in initial orbit:

$$v_1 = \sqrt{\frac{\mu}{r_1}} \tag{6.37}$$

- Velocity in transfer orbit at periapsis (First Burn):

$$v_p = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} \quad \text{where} \quad a = \frac{r_1 + r_2}{2} \tag{6.38}$$

- Velocity in transfer orbit at apoapsis (Second Burn):

$$v_a = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} \tag{6.39}$$

- Velocity in final orbit:

$$v_2 = \sqrt{\frac{\mu}{r_2}} \tag{6.40}$$

Δv Calculations

Calculating the differences in speed in different phases is crucial and determines how efficient the transfer is:

- First burn (to enter transfer ellipse):

$$\Delta v_1 = v_p - v_1 \tag{6.41}$$

- Second burn (to circularise at final orbit):

$$\Delta v_2 = v_2 - v_a \tag{6.42}$$

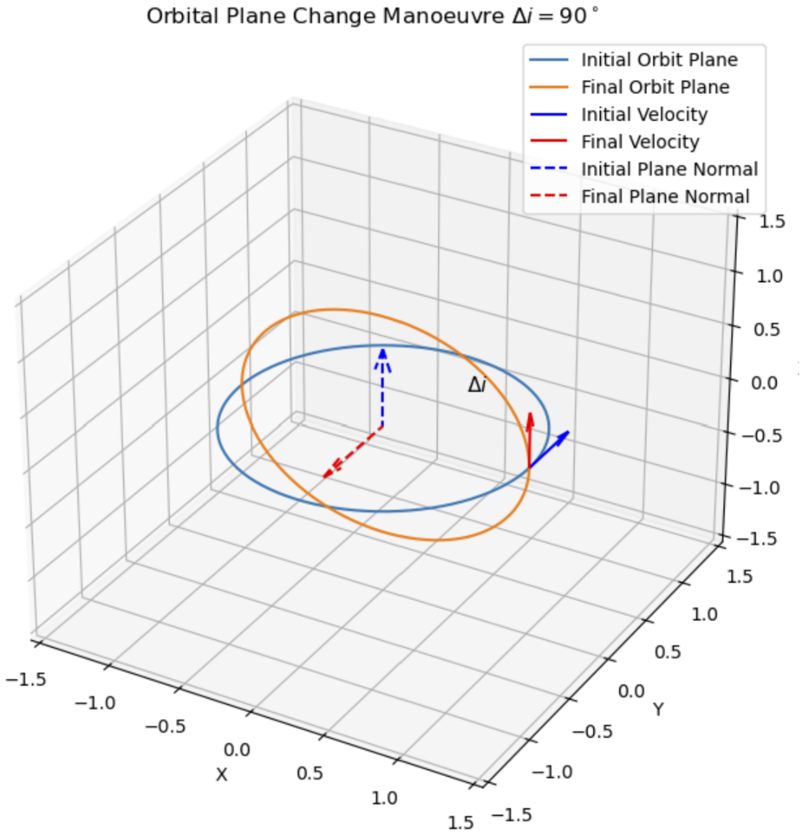

Figure 6.9: Orbital Plane Change with the change in inclination angle of 90 degrees on Python Interface

- Total Δv for transfer:

$$\Delta v_{total} = \Delta v_1 + \Delta v_2 \tag{6.43}$$

6.6.2 Plane Changes

A plane change is an orbital manoeuvre that alters the orientation of a spacecraft's orbit including its inclination or direction of travel. This does not affect the size or shape of the orbit, only the angle of the orbital path in space.

Main procedure of the plane changes

To change the orbital plane, the spacecraft must apply a normal thrust such as velocity change perpendicular to its current direction of motion [1, 3].

- Inclination Change:
  - It is performed at the ascending or descending node where two orbital planes intersect [3].
  - It changes the angle between the orbit and the equator in the geocentric frame for example.
- RAAN Change (Right Ascension of the Ascending Node):
  - This alters the orientation of the orbital plane around the Earth.
  - This is common when it comes to adjusting for time-of-day or longitude shifts.

Energy Cost of Plane Changes

Plane changes are intensive for the fuel-consumption especially in Low Earth Orbit (LEO) where speeds are highest. The required Δv for a pure inclination change is:

$$\Delta v = 2v \sin\left(\frac{\Delta i}{2}\right) \tag{6.44}$$

where v is the orbital speed and Δi is the change in inclination.

6.6.3 Exemplary Questions

Example 1: Calculate delta-v for a Hohmann transfer from 7.000×10^3 km to 1.0000×10^4 km.

$$a_t = \frac{7.000 \times 10^3 + 1.0000 \times 10^4}{2} = 8.500 \times 10^3\,\mathrm{km}$$

$$\Delta v_1 = \sqrt{3.986\,00 \times 10^5 \left(\frac{2}{7.000 \times 10^3} - \frac{1}{8.500 \times 10^3}\right)} - \sqrt{\frac{3.986\,00 \times 10^5}{7.000 \times 10^3}} \approx 1.47\,\mathrm{km\,s^{-1}}$$

$$\Delta v_2 = \sqrt{\frac{3.986\,00 \times 10^5}{1.0000 \times 10^4}} - \sqrt{3.986\,00 \times 10^5 \left(\frac{2}{1.0000 \times 10^4} - \frac{1}{8.500 \times 10^3}\right)} \approx 9.7 \times 10^{-1}\,\mathrm{km\,s^{-1}}$$

$$\Delta v_{\mathrm{total}} \approx 2.44\,\mathrm{km\,s^{-1}}$$

Example 2: Calculate delta-v for a $3.0 \times 10^{1\circ}$ plane change at 8.000×10^3 km.

$$\Delta v = 2 \cdot \sqrt{\frac{3.986\,00 \times 10^5}{8.000 \times 10^3}} \cdot \sin\left(\frac{30}{2}\right) \approx 3.86\,\mathrm{km\,s^{-1}}$$

Problem 1: Calculate delta-v for a Hohmann transfer from 6.800×10^3 km to 1.2000×10^4 km.

$$\Delta v_{\mathrm{total}} \approx 2.89\,\mathrm{km\,s^{-1}}$$

Problem 2: Calculate delta-v for a $4.5 \times 10^{1\circ}$ plane change at 9.000×10^3 km.

$$\Delta v \approx 5.27\,\mathrm{km\,s^{-1}}$$

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Glossary

apoapsis The point in an orbit where the object is farthest from the central body.. 5, 17, 25

astrodynamics The study of the motion of objects in space under gravitational forces, also known as orbital mechanics.. 3–5, 7, 10

eccentricity A parameter describing the shape of an orbit, with $e = 0$ for circular, $0 < e < 1$ for elliptical, $e = 1$ for parabolic, and $e > 1$ for hyperbolic orbits, and this parameter is dimensionless.. 14, 16

gravitational parameter The product of the gravitational constant and the sum of masses of two bodies, denoted $\mu = G(M + m)$, approximated as $\mu \approx GM$ when the orbiting mass m is much smaller than the central mass M.. 5

orbital period The time required for an object to complete one full orbit, denoted T.. 5, 6, 15, 17

periapsis The point in an orbit where the object is closest to the central body.. 5, 6, 12, 13, 17, 19, 21, 25

perturbation Perturbation in space engineering refers to any external or internal influence that causes a spacecraft or celestial body to deviate from its idealised orbit as predicted by two-body Newtonian mechanics. These deviations are small but can accumulate over time, requiring correction or accounting in mission and orbital analysis.. 22

reference frame A coordinate system used to measure positions, velocities, and accelerations of objects.. 7

semi-major axis Half the longest diameter of an elliptical orbit, denoted a, which defines the orbit's size.. 5, 6, 14, 16