Contents

1. Intro

1.1. Gennerel info

kan sjekke ut detajer på denne siden https://www.math.ntnu.no/emner/MA8502/2022h/lectures/html/lecture_01.html

1. project 1: Navier stokes
2. project 2: ???

Ikke en tradisjonell øvingstime, man kan bli bedt om å presentere ting som foreleseren har gått gjennom. kanksje man får et topic man skal utforske selv og så vise det til resten av klassen.

1.2. Eksamens form

1. Munlig eksamen
   - Velger et teame man snakker om i 8 min
   - Også blir det stilt spørsmpl om sentrale delet av pensum
2. 2 kode project
3. Seminar om noe man synes er intresannt

Vi kan bruke python julia c++

VI SKAL BRUKE GIT, LETS GOOOO‼!

2. Crash course

FEM er en underkategori av galerkin metoder. Ta utgangspunkt i en sterk formulering $Au = f$ Sterk formulering er en formulering av ligningene hvor man krever at alle de deriverte eksisterer i alle punkt.

Eksempel Possion

$$-\Delta u = f \forall \ x \in \Omega$$
$$u(x) = -1 \forall \ x \in \delta\omega$$

- (kontinuerlig) svak formulering

$$-\int_\Omega \Delta uv = \int_\Omega fv$$

(bruker delvis integrasjon) på Δu og at $\nabla \cdot (\nabla uv) = \Delta uv + \nabla u \nabla v$

$$-\int_{\partial\Omega} \nabla \cdot (\nabla uv) + \int_\Omega \nabla u \nabla v = \int_\Omega fv$$

første ledet forsvinner fordi grensebetingelsen vår er 0 og vi ender opp med.

$$\int_\Omega \nabla u \nabla v = \int_\Omega fv$$

som vi kan formulere som finn $u \in H_0^1(\Omega)\left\{u \mid u \in L^2(\Omega), \nabla u \in \left[L^2(\Omega)\right]^d \ \ \nabla u \mid_{\partial\Omega} = 0\right\} \forall \ v \in V$ (vanlig valg for V er H_0^1)

$$\underbrace{\int_\Omega \nabla u \nabla v}_{\text{a(u,v)}} = \underbrace{\int_\Omega fv}_{\text{l(v)}}$$

3) (Diskrete) svak formulering

$V_0 \to V_{h,0} (\subseteq V_0)$, hvis den dikretiserte versjonen av rommet V er et underom vil det kalles for "conform" vi vil se på tilfeller hvor dette ikke alltid er tilfellet. $v \to v_h$,

vi kan nok engang skrive om ligningen, Finn $u \in V_h$ st $a(u, v_h) = l(v) \forall\ v_h \in V_h$

Siden det finnes en basis for u og v kan vi skrive om u og v til en lineær kombinasjon av disse basisene. $v_h = \sum_{j=0}^{N} V_j \varphi,\ V_{h,0} = span(\{\varphi\})$

skriver om a til å bruke den lineære summen for v

$$a\left(u, \sum_{i=0}^{N} V_i \varphi_i\right) = \sum_{i}^{N} V_i\ l(\varphi_i) \forall\ v \in V$$

siden a er en lineær operator (er bare et integral) så kan vi trekket ut summen og V_i,vi får da $\sum_i V_i a(u, \varphi_i) = \sum_i^N V_i\ l(\varphi_i) \forall\ v \in V$ vi gjentar dette for u

$$\sum_{ij} V_i U_j \underbrace{a(\varphi_j, \varphi_i)}_{A_{ij}} = \underbrace{l(\varphi_i)}_{F_i} \forall\ v \in V$$

siden dette må gjelde for alle V er dette ekvivalent med

$$Au = F$$

basisen (phi) man velger bestemer utsende på A matrisen vår.

2.1. Abstract theory

- propositon (galerkin orthogonalty)

antar følgende, v_h subset v (conform), u solves the continoues weak fomulation, u_h solves the discrete weak fomulation then $a(x, v_h) = l(v_h)$ for x som u og u_h. Dermed hvis man tar en lineær kombinasjon $u - u_h$ vil man få følgende $a(u - u_h, v_h) = 0$

- Cea's lemma (quasi best appoximation property)

same antagelser gjelder her også, ser vi på normen mellom den samme løsningen og approxmiasjonen vår får vi at det er begrensert oven ifra med test funcksjoen v.

$$\|u - u_h\|_a \leq \frac{C}{\alpha} \|u - v_h\|_a$$

(hvor norm_a i possion tilfellet er $\|\nabla(\cdot)\|$). Her kommer c og alpha fra lax milgram.

Om l er begrenset, a er bilinær (linær i både u og v) bounded og coercive eksisterer det en løsning til $Au = F$

Her står det litt om hva det menes med bounded og coercive for a

- bounded

$a(u, v) \leq C\|u\|_a \|v\|_a \forall\ u, v \in V$

- corcivty

$\exists\ \alpha > 0$ st $a(u, u) \geq \|a\|^2 \forall\ v \in V$

2.1.1. Bevis (Cea's lemma)

u og u_h er begge i V, dermed vil også en liære kobo også være det. Vi kan dermed bruke egenskapene nevte over.

$$
\begin{aligned}
\alpha\|u-u_h\|_a^2 \underbrace{\leq}_{\text{coesiverty}} & \; a(u-u_h, u-u_h) \\
& \leq a(u-u_h, u-u_h-v_h+v_h) \\
& = a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{= 0 \text{ bc galerkin orthogonalty}} \\
& = a(u-u_h, u-v-h) \\
& \underbrace{\leq}_{\text{Bounded}} c\|u-u_h\|_a \|u-v_h\|_a
\end{aligned}
$$

Krysning av det kvadrerte leddet og det siste leddet gir oss det resultatet vi ønsker.

Definition 2.1.1.1 (A Finite Element): q A FEM is a tuple (T,P,Σ),
where
- T is a simplex. (polyheadron)
- P is a finite dimentional function space defined on T (polynomials,trigonometric functions)
- $\Sigma = \{\sigma_1, \sigma_2, ..., \sigma_n\}$ is a basis for the dual space of P'

The last one is somewhat confussing but it basicly one usally refers to them as degrees of freedom. Sigma could be the nodal values, it could also be nodal + midpoint values. This depends on what kind of sapce P is. (linear, qudratic). Link to video that does a good job explaing some of this.

3. Advection

$$
\begin{aligned}
\partial_t u + u \cdot \nabla u - v\Delta u + \nabla p = f \\
\nabla \cdot u = 0
\end{aligned}
$$

We start of by lineariztion, and introducing timestepping. This will results in the following

$$
\sigma u + b \cdot \nabla u - v\nabla u + \nabla p = f
$$
$$
\nabla \cdot u = 0
$$

in some cases the impact of the speed (∇u) could be more dominating then the viscity $v\nabla u$. Lets look at a model probelm where this is the case.

(as a side note the first eqaution is a saddle point probelm and one could also look at a model problem for this too. This could be called stokes problem.)

3.1.1. advection-diff-reaction

$$
\sigma u + b\nabla u - v\Delta y = f \text{ and some bc}
$$

3.1.2. stokes problem

$$
u_t = v\frac{\partial^2 u}{\partial v^2}
$$

and some more that i didn't write down.

we will be looking at each of these model problems seperatily and build up to solving the real navir stokes.

(Det som står over skal skrives om. tanken her er å vise at man kan skrive om nativer stokes til flere små oppgaver)

3.2. ADR problem

3.2.1. Model probelm

$$\varepsilon \Delta u + b \nabla u + cu = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

$b : \Omega \to \mathbb{R}^n$ advection, $\varepsilon > 0$ diffusivity, $c : \Omega \to \mathbb{R}$ reaction term

typical assumtion are that

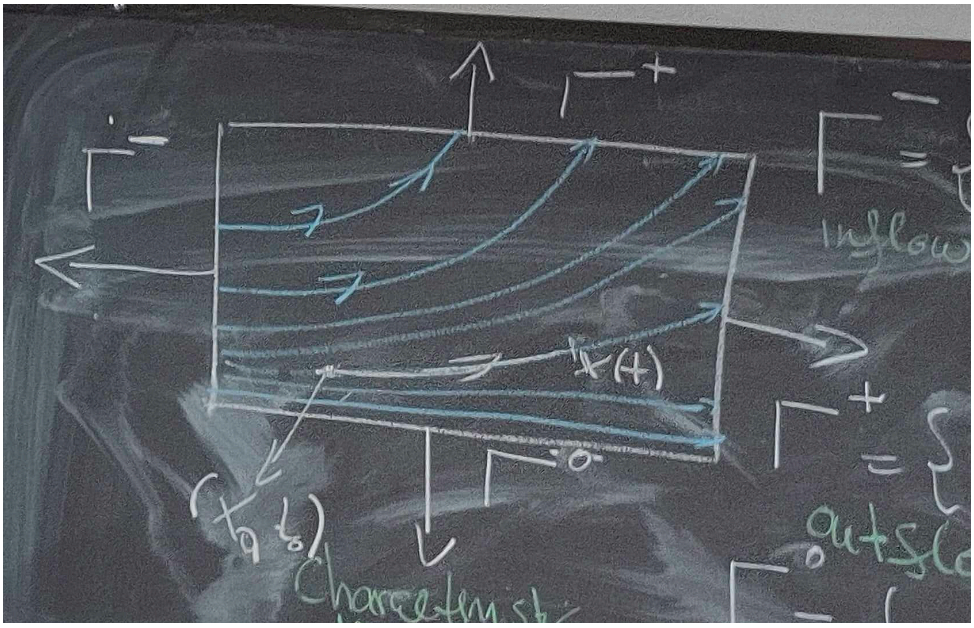
- $\varepsilon \ll \|b\|_{L^\infty(\Omega)}, \; \|c\|_{L^\infty(\Omega)} \ll \|b\|_{L^\infty(\Omega)}$
- div $b = 0$ alteratively, $c(x) - \frac{1}{2}\nabla \cdot b(x) \geq c_0$

As ε gets smaller we can ignore it, we get the reduced probelm. $b \cdot \nabla u + cu = f$ this will introduce a probelm with the bc since we no longer have a second order term. also as epsilon goes to zero the problem becomes similar to a problem that can be solved by the method of characterisics. From the method of characterisics we know that the boundry condtions are only needed on the inflow boundry.

the invered pointing blue curves can be written as. $\Gamma^- = \{x \in \partial\Omega \mid b(x) \cdot n(x) < 0\}$

while the outword going blue curves can be written as $\Gamma^+ = \{x \in \Gamma \mid b(x) \cdot n(x) > 0\}$

$\Gamma^0 = \{x \in \Gamma : b(x) \cdot n(x) = 0$



3.2.2. weak formulation of ADR

Strong form

$$\varepsilon \nabla u + b \Delta u + cu = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

Natrual weak formulation

$$\text{Find } u \in H_0^1(\Omega) \text{ st } \forall \; v \in H_0^1(\Omega)$$
$$\underbrace{a(u, v)}_{(\nabla u, \nabla v)_\Omega + (b \nabla u, v)_\Omega + (cu, v)} = l(v)$$

Lax - milgram

$\|a(u,v)\| \leq c\|u\|_{V_0}\|v\|_{V_0} \forall\ v, u$

$a(u,u) \geq \alpha\|u\|_{V_0}^2 \forall\ y \in V_0$

We are going to use the following norm. $\|v\|_{V_0} = \|\nabla v\|_\Omega \forall\ v \in H_0^1(\Omega)$

- coercivity $u \in H_0^1(\Omega)$

$$a(u,v) = \varepsilon\|\nabla u\|_\Omega^2 + (b\nabla u, u) + (cu, u)$$

using gauss theorem.

$$\int_{\partial u}(buv)\cdot n = \int_\Omega \nabla\cdot(buv) = \int_\Omega \nabla\cdot buv + \int_\Omega b\cdot\nabla uv + \int_\Omega ub\cdot\nabla v$$

moving over some term we get the following

$$(b\nabla u, v)_\Omega = -(u, b\cdot\nabla v)_\Omega - \cancel{(\nabla\cdot bu, v)_\Omega}\cancel{(b\cdot nu, v)_\Gamma}$$

earlyer we assumed that div $b = 0$ aslo $u, v \in H_0^1(\Omega)$ we can therefoe cross the last two terms. This leaves us with

$$(b\nabla u, v)_\Omega = -(u, b\cdot\nabla v)_\Omega$$

and this is skew symmetric $\rightarrow (b\cdot\nabla u, u) = 0$ This is only works if our assumtion of div b = 0 is ture. however in some cases this might not be true.

we will use a trick to rewrite this as a sum of a skrew symetric part and a symetric part.

$$(b\nabla u, v) = \frac{1}{2}(b\nabla u, v)_\Omega + \frac{1}{2}(b\nabla u, v)_\Omega$$

$$= \underbrace{\frac{1}{2}(b\cdot\nabla u, v) - \frac{1}{2}(u, b\cdot\nabla v)}_{\text{skew-symetric}} - \frac{1}{2}(\nabla\cdot bu, v) + \frac{1}{2}\left(b\cdot n, \underbrace{uv}_{0}\right)$$

$$\begin{aligned} a(u,u) &= \varepsilon\|\nabla u\|_\Omega^2 - \frac{1}{2}(\nabla\cdot b, u)_\Omega + (cu, u)_\Omega \\ &\geq \varepsilon\|\nabla u\|_\Omega^2 + \left(\left(c - \frac{1}{2}\nabla\cdot b\right), u\right)_\Omega \\ &\geq \varepsilon\|\nabla u\|^2 + c_0\|u\|_\Omega^2 = (\varepsilon + c_0)\|u\|_\Omega^2 \end{aligned}$$

Boudednes $a(u,v) = \varepsilon(\nabla u, \nabla v)_\Omega + (b\nabla u, v)_\Omega + (cu, v)_\Omega$ use cashy swartz on everything more or less

$$\leq \varepsilon\|\nabla u\|\|\nabla v\| + \|b\|_{L^\infty(\Omega)}\|\nabla u\|_\Omega\|v\|_\Omega + \|c\|_{L^\infty(\Omega)}\|u\|_\Omega\|v\|_\Omega$$

using pointcare on the terms that are not in the energy norm. (used om the onces that are not in Omega)

$$\leq \varepsilon\|\nabla u\|\|\nabla v\| + \|b\|_{L^\infty(\Omega)}\|\nabla u\|_\Omega C_p\|\nabla v\|_\Omega + C_p^2\|c\|_{L^\infty(\Omega)}\|\nabla u\|\|\nabla v\|$$

$$\left(\varepsilon + \|b\|_{L^\infty(\Omega)}C_p + \|c\|_{L^\infty(\Omega)}C_p^2\right)\|\nabla u\|_\Omega\|\nabla v\|_\Omega$$

Cea's $\rightarrow$

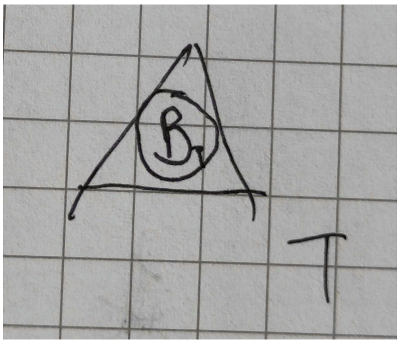
$$\|u-u_h\|_V \leq \frac{C}{\alpha} \inf \|u-v_h\|$$

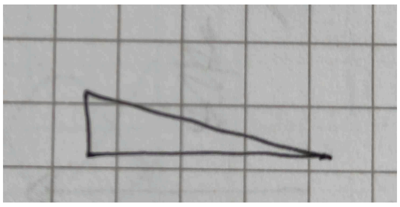
$$\|u-u_h\|_V \leq \varepsilon + \|b\|_{L^\infty(\Omega)} C_p + \|c\|_{L^\infty(\Omega)} C_p^2 = 1 + \frac{\|b\|_{L^\infty(\Omega)} C_p}{\varepsilon} + \frac{\|u\|_{L^\infty(\Omega)} C_p^2}{\varepsilon}$$

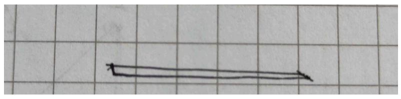
Motivation: we wish to solve fluid dynamical equations. To do this advection will be centrel in propagating inital condtions and sutch.

4. Discontionuse galerking methods

We start of by motivating why DG methods are of intresst. The inital conseption of a dg methodd were infact to handle advection problems. In this class we will use it on $b \cdot \nabla u + cu$. Simplex mesh $\mathcal{T}_h = \{T\}_h$ is sharp regular







The Ball has has radius ρ_T, We wish to aviod elemets that make ρ_t too small such as the two last images. We wish to keep the following ratio $\frac{h_T}{\rho_T} \leq c \forall\ T \in \mathcal{T}_h$

Lemma 4.1 (h-scaled union of trace equality):

T is form regular requries that all T in $\mathcal{T}_h$ is form regular. Then $v \in H^1(T)$ then $\exists\ c$ indepent of T st $\|v\|_{\partial T} \leq \hat{C}\left(h^{-\frac{1}{2}}{}_T \|v\|_T + {h_T}^{\frac{1}{2}} \|\nabla v\|_T\right)$

where $\|v\|_{\partial T} = \left(\int_{\partial T} V^2\right)^{\frac{1}{2}}$

Corollary 4.1:

$\|\partial_n V\|_{\partial_T} \leq \hat{C}\left(h^{-\frac{1}{2}} \|\nabla v\|_T + h^{\frac{1}{2}} \|D^2 v\|_T\right)$ where D is a hessian

Lemma 4.2 (inverse estimate):

$\mathcal{T}_h$ sharp regular $\forall\ T \in \{\mathcal{T}_h\}$

$v_h = \{v_h \in L^2(\Omega), v_h|_T \in \mathbb{P}(T)\ \forall\ T \in \mathcal{T}_h\} = \mathbb{P}_k(\mathcal{T}_h)$

$$\|\partial_n u_h\|_{\mathcal{F}_h} \leq C h^{-\frac{1}{2}} \|\nabla u_h\|_{\mathcal{T}_h}$$

since v_h is not in c (not contnouse), the function does not need to be contnouse globally and therefore might look like the following img

For discrete spaces we have the following inequallity $\|\nabla v_h\| \leq Ch^{-1}\|v_h\|$ whis is very similar to poincare inequalty but with the inequlatiy fliped.

5. Symmetric Interior Penalty method (SIP)

Given the poisson problem we wish to solve it using dg. The formulation becomes

$$-\Delta u = f \text{ in } \Omega$$
$$u = u_D \text{ on } \partial\Omega$$

Find $u_h \in v_h \mathbb{P}_k(\mathcal{T}_h) \forall\ v_h \in V_h$ st $a_{h(u_h, v_h)} = l_{h(v_h)}$

we can't just move over a gradient from u to v as we have done before.v_h is not contiuouse over from one elemet to another. Lets assume $u \in H^2(\Omega)$ in additon we now proform integraion by part over each element and not the whole domain.

$$(f, v_h)_\Omega = (-\Delta u, v_h)_\Omega = \sum_{T \in \mathcal{T}_h} -(\partial_n u, v_h)_{\partial T} + \sum_{T \in \mathcal{T}_h} (\nabla u, \nabla v)_T$$

the ∂_n term can be fourther split into two part working on diffrent typs of elements.

Image of more traingles

Image of trainges side by side T+ T-

from the image we see that some facets are so called intiroir and others are boundry facets. We can therefore split the ∂_n term further as sutch

$$\sum_{F \in \mathcal{F}_h^b} -(\partial_n u, v_h)_F - \sum_{F \in \mathcal{F}_h^i} \left(\int_F (\nabla u^+ \cdot n^+ v_h^+ + \nabla u^- \cdot n^- v_h^-) \right)$$

the above term can be somewhat factorized

$$= (\nabla u, \nabla v)_{T_h} - (\partial_n u, v_h)_{\mathcal{F}_h^b} - \sum_{F \in \mathcal{F}_h^i} \left(n_F \cdot \underbrace{(\nabla u^- v_h^- - \nabla u^- v_h^-)}_{[\nabla u, v_h]} \right)$$

We introduce the notion of a mean $\{a\} = \frac{1}{2}(a^+ + a^-)$ this will be equal to a if a is contiuouse. Using the mean in the jump we get this new magic fomula. $[ab] = \{a\}[b] + [a]\{b\}$

with this new fomula we can once again rewrite the function.

$$(\nabla u, \nabla v_h)_{\mathcal{T}_h} - (\partial_n u, v_h)_{\mathcal{F}_h^b} - \sum_{F \in \mathcal{F}_h^i} \int_F n_F \cdot (\{\nabla u\}[v_h] + [\nabla u]\{v_n\})$$

Where $[\nabla u] = 0$ since $u \in H^2$

we now wish to move from u and over the the discete u_h. However our a is no longer bilinear as its symetry has been removed. We fix this by adding term that makes the sum symetric while keeping it consistant.

$$(\nabla u_h, \nabla v_h)_{\mathcal{T}_h} - (\partial_n u_h, v_h)_{\mathcal{F}_h^b} - (\{\partial_n u_h\}[v_h])_{\mathcal{F}_h^i} - (u_h - u_d, \partial_n v_h)_{\mathcal{F}_h^b} = (f, v_h)_\Omega$$

to make $(\{\partial_n u_h\}[v_h])_{\mathcal{F}_h^i}$ sumetric we add $([u_h]\{\partial_n v_h\})_{\mathcal{F}_h^i}$ once agian $[u_h] = 0$ since $u \in H^2$

still this is not stabill. we will add some stabilastion terms.

$$a_{h(u_h,v_h)} = (\nabla u, \nabla v_h)_{\mathcal{T}_h} - (\partial_n u_h, v_h)_{\mathcal{F}_h^b} - (\partial_n v_h, u_h)_{\mathcal{F}_h^b} - (\{\partial_n u_h\}[v_h])_{\mathcal{F}_h^i} - (\{\partial_n v_h\}, [u_h])_{\mathcal{F}_h^i}$$
$$+\gamma(h_T^{-1} u_h, v_h)_{\mathcal{F}_h^b} + \gamma(h^{-1}[u_h][v_h])_{\mathcal{F}_h^i} = (f, v)_\Omega - (u_d, \partial_n u_h)_{\mathcal{F}_h^b} + \gamma(h^{-1} u_D v_h)_{\mathcal{F}_h^b}$$

To aboid writting so much we introduce some notation. Since jumps and means only exists on faces. We introduce the following

- $[u_h] = u_h$
- $\{\partial_n u_h\}\partial_n u_h$

resulting in a new formulation

$$(\nabla u, \nabla u)_{\mathcal{T}_h} - (\{\partial_n u_h\}, [v_h])_{\mathcal{F}_h} - ([u_h], \{\partial_n v_h\})_{\mathcal{F}_h} + \gamma(h^{-1}[u_h], [v_h])_{\mathcal{F}_h}$$

SIP is not confiming, meaning $V_h \not\subset H_0^1(\Omega)$. This is bad since galerkin orthogonality requires this. However this is ok since SIP is consistant if $u \in H^2(\Omega)$ and so both u_h and u can solve $a_{h(u,v)} = l_{h(v_h)}$.

let's check if SIP is well posed. looking at the coervity set $u_h = v_h$

$$a(u_h, u_h) = \underbrace{(\nabla u, \nabla u)_{\mathcal{T}_h}}_{\|\nabla u\|^2_{\mathcal{T}_h}} - 2(\{\partial_n u_h\}, [u_h])_{\mathcal{F}_h} + \gamma\underbrace{(h^1[u_h][u_h])_{\mathcal{F}_h}}_{\left\|h^{-\frac{1}{2}}[u_h]\right\|^2_{\mathcal{F}_h}}$$

let the middel term be named I. We wish to rewrite it.

$$I \leq 2\|\{\partial_n u_h\}\|_{\mathcal{F}_h}\|[u_h]\|_{\mathcal{F}_h}$$

using the inverse inequality on first term

$$2\frac{h^{\frac{1}{2}}}{h^{\frac{1}{2}}} c_\xi c \|\{\nabla u_h\}\|_{\mathcal{T}_h}\|[u]\|_{\mathcal{F}_h}$$

using youngs inequalty.

$$2\delta^{\frac{1}{2}} a \frac{b}{\delta^{\frac{1}{2}}} \leq \delta a^2 + \frac{1}{\delta} b^2$$

$$\lesssim \delta c_\xi^2 \left\|h^{\frac{1}{2}} \nabla u_h\right\|^2_{\mathcal{T}_h} + \delta^{-1}\left\|h^{-\frac{1}{2}}[u_h]\right\|$$

inserting this back into

$$a_{h(u_h,u_h)} \gtrsim \|\nabla u_h\|^2_{\mathcal{T}_h} - \delta C_\xi^2 \|\nabla u_h\|^2_{\mathcal{T}_h} - \delta^{-1}\left\|h^{-\frac{1}{2}}[u_h]\right\|^2_{\mathcal{F}_h} + \gamma\left\|h^{\frac{1}{2}}[u_h]\right\|_{\mathcal{F}_h}$$

choose δ st $\left(1 - \delta c_\xi^2\right) \geq \frac{1}{2}$

$$\frac{1}{2}\|\nabla u_h\|^2_{\mathcal{T}_h} + \left(\gamma - \delta^{-1}\right)\left\|h^{-\frac{1}{2}}[u_h]\right\|^2_{\mathcal{F}_h}$$

and γ st $\left(\gamma - \delta^{-1}\right) \geq \frac{\gamma}{2}$

$$\frac{1}{2}\|\nabla u_h\|^2_{\mathcal{T}_h} + \frac{\gamma}{2}\left\|h^{-\frac{1}{2}}[u_h]\right\|^2_{\mathcal{F}_h} =: \|u_h\|^2_{a_h}$$

this is a a mesh dependent norm

Lemma 5.1:

the bilinear $a_h(\cdot,\cdot)$ is coercive on V_h wrt $\|v_h\|_{a_h} := \left(\|\nabla u_h\|_{\mathcal{T}_h}^2 + \gamma\left\|h^{-\frac{1}{2}}[u_h]\right\|_{\mathcal{F}_h}^2\right)^{\frac{1}{2}}$

Lemma 5.2 (cea's lemma for SIP):

$$\|u-u_h\|_{a_h} \leq \frac{c}{\alpha}\inf\|u-v_h\|_{a_h}$$

Idea $u-u_h = \underbrace{u-v_h}_{e_h} + \underbrace{v_h-u_h}_{V_h}$, where $v_h = \pi_h u$

will now use discreete coervity on $v_h - u := e_h,\ u - v_h := e_\pi$

$$\begin{aligned}
\frac{1}{2}\|u_h-v_h\|_{a_h}^2 &\leq a(u_h-v_h,e_h)\\
&\underbrace{\leq}_{u\in H^2}\ a(u_h-v_h+u-u,e_h)\\
&= \underbrace{a(u_h-u,e_h)}_{=0 \text{ since } u\in H^2} + a(u-v_h,e_h)\\
&\leq a(u-v_h,e_h)\\
&= (\nabla(u-v_h),\nabla e_h)_{\mathcal{T}_h} - (\{\partial_n(u-v_h)\},[e_h])_{\mathcal{F}_h}\\
&-(\{\partial_n e_h\},[u-u_h])_{\mathcal{F}_h} + \gamma(h^{-1}[u-u_h],[e_h])_{\mathcal{F}_h}\\
&= (\nabla(u-v_h),\nabla e_h)_{\mathcal{T}_h} - \left(h^{\frac{1}{2}}\{\partial_n(u-v_h)\},h^{-\frac{1}{2}}[e_h]\right)_{\mathcal{F}_h}\\
&-\left(h^{\frac{1}{2}}\{\partial_n e_h\},h^{-\frac{1}{2}}[u-u_h]\right)_{\mathcal{F}_h} + \gamma(h^{-1}[u-u_h],[e_h])_{\mathcal{F}_h}
\end{aligned}$$

since we have $\left\|h^{\frac{1}{2}}\{\partial_n v\}\right\|_{\mathcal{F}_h}^2 \leq C\|\nabla v\|_{\mathcal{T}_h}^2$

$$\begin{aligned}
\frac{1}{2}\|u_h-v_h\|_{a_h}^2 &\underbrace{\leq}_{\text{CS}} \|\nabla(u-v_h)\|_{\mathcal{T}_h}\|\nabla e_h\|_{\mathcal{T}_h} - \left\|h^{\frac{1}{2}}\{\partial_n(u-v_h)\}\right\|_{\mathcal{F}_h}\left\|h^{-\frac{1}{2}}[e_h]\right\|_{\mathcal{F}_h}\\
&-\left\|h^{\frac{1}{2}}\{\partial_n e_h\}\right\|_{\mathcal{F}_h}\left\|h^{-\frac{1}{2}}[u-u_h]\right\|_{\mathcal{F}_h} + \gamma\left\|h^{-\frac{1}{2}}[u-u_h]\right\|_{\mathcal{F}_h}\left\|h^{-\frac{1}{2}}[e_h]\right\|_{\mathcal{F}_h}\\
&\text{rearanging terms so that we can constuct } \|v\|_{a_h} := \|\nabla v\| + \gamma\left\|h^{-\frac{1}{2}}[v]\right\|
\end{aligned}$$

5.1. Sprsømål til andre

- er inverse estimaten jeg har skrevet riktig
- hvor får man tak i gammaen for å lage $\|e_h\|_{a_h}$ da den trengs for å lage a_{h*}
- hva skjer med $h^{-\frac{1}{2}}$

should add intermidet step here

further rearanging the terms we can get the following norm $\|v\|_{a_{h*}^2} := \|v\|_{a_h^2} + \left\|h^{\frac{1}{2}}\left\{\partial_{n_v}\right\}\right\|_{\mathcal{T}_h}^2$

$$\frac{1}{2}\|u_h - v_h\|_{a_h}^2 \leq C\|u - v_h\|_{a_{h*}} \|e_h\|_{a_{h*}}$$

we have that $\|.\|_{a_{h*}} \sim \|.\|_{a_h}$

$$\frac{1}{2}\|u_h - v_h\|_{a_h}^{\cancel{2}} \leq \hat{C}\|u - v_h\|_{a_{h*}} \cancel{\|e_h\|_{a_h}}$$

$$\frac{1}{2}\|u_h - v_h\|_{a_h} \leq \hat{C}\|u - v_h\|_{a_{h*}}$$

5.2. Apiori error estimate for SIP

$$\|v_h\|_{a_h} = \left(\|\nabla v\|_{\mathcal{T}_h}^2 + \gamma\left(h^{-\frac{1}{2}}[v_h]\right)_{\mathcal{F}_h}^2\right)^{\frac{1}{2}} \forall\ v \in \mathbb{P}_k(\mathcal{T}_h)$$

$$\|v_h\|_{a_h} = \left(\|\nabla v\|_{\mathcal{T}_h}^2 + \gamma\left(h^{-\frac{1}{2}}[v_h]\right)_{\mathcal{F}_h}^2\right)^{\frac{1}{2}} \forall\ v \in \mathbb{P}_k(\mathcal{T}_h)$$

6. temp

hei hei

rewview

we have the following asummtions

- $\|b\|_{1_\infty} < \infty$
- $\inf c(x) - \frac{1}{3}\nabla b(x) \geq c_0 \geq 0$

integration by part rule $(b \cdot \nabla v, w)_\Omega = -(v, b \cdot \nabla w)_\Omega - (\nabla \cdot bv, w)_\Omega + (b \cdot nv, w)_\Gamma$

The abouve roule also holds in the following case

$(b \cdot \nabla v, w)_{\mathcal{T}_h} = -(v, b \cdot \nabla w)_{\mathcal{T}_h} - (\nabla \cdot bv, w)_{\mathcal{T}_h} + (b \cdot nv, w)_{\partial\mathcal{T}_h}$

using the fact that $a(v, v) \geq c_0\|v\|_\Gamma^2 + \frac{1}{2}\|b \cdot n\|$ MANGLER INFO OM NORMEN

Derivation of the central flux formulation we wish to replace o

$V \to V_h = \mathbb{P}_k(\mathcal{T}_h)$

we take $v_h, w_h \in \mathbb{P}_k(\mathcal{T}_h)$ here we drop the h, but the formulation is for the dicrete case

$$(b \cdot \nabla v, w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \left(-(v, b \cdot \nabla w)_{\mathcal{T}_h} - (\nabla \cdot bv, w)_{\mathcal{T}_h}\right) + (b \cdot nv, w)_{\partial\mathcal{T}_h}$$

image of trinagles here

$$= -(v, b \cdot \nabla w)_{\mathcal{T}_h} - (\nabla \cdot bv, w)_{\mathcal{T}_h} + (b \cdot nv, w)_{\mathcal{F}_h^b} \sum_{F \in \mathcal{F}_h^i} ((b \cdot n^+ v^+, w^+) + (b \cdot n^- v^-, w^-))$$

$$= I + \mathrm{II} + \sum_{F \in \mathcal{F}_h^i} (b \cdot_f [vw])_{\mathcal{F}_h}$$

$$(b \cdot \nabla v, w)_\Omega = -(v, b \cdot \nabla w)_\Omega - (\nabla \cdot bv, w)_\Omega + (b \cdot nv, w)_\Gamma$$
$$+ (b \cdot n_F\{v\}[w])_{\mathcal{F}_h^i} + (b \cdot n_F\{w\}[v])_{\mathcal{F}_h^i}$$

the coereivty idea was

$$(b \cdot \nabla v, w)_\Omega = \frac{1}{2}(b \cdot \nabla v, w)_\Omega + \frac{1}{2}(b \nabla v, w)_\Omega$$

$$= \frac{1}{2}(b \cdot \nabla v, w)_\Omega - \frac{1}{2}(v, b \cdot \nabla w)_\Omega - \frac{1}{2}(\nabla \cdot bv, w)_\Omega + \frac{1}{2}(b \cdot nv, w)_\Gamma + (b \cdot n_F \{v\}, [w])_{\mathcal{F}_h^i} + (b \cdot n_F [v], \{w\})_{\mathcal{F}_h^i} = *$$

how to design a_h ?

$a_{h(v,w)} = (b \cdot v, w)_\Omega + (cv, w)_\Omega - (b - \cdot v, w)_{\Gamma^-}$