Math 202A – Topology & Analysis

Albert Zhang
Professor Marc Rieffel

Fall 2019

Contents

1 August 29

1.1 Overview

First half of the semester: (Point-Set) Topology, "The mathematics of continuity"

Second half of the semester: Measure & Integration

Towards end: Functional Analysis

GSI OH: Evans 1042, MW 12:30-2pm, mitchelltaylor.berkeley.edu

1.2 Metric Spaces

Definition 1 (metric)**.** Let X be a set. A metric on X is a (distance) function $d : X \times X \to \mathbb{R}$ such that

1. $\forall x, d(x,x) = 0$, and if $d(x,y) = 0$ then $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,y) \leq d(x,z) + d(z,y)$

Example 1.1

Some basic metric spaces: $\mathbb{R}$, $\mathbb{R}^n$. Examples of some norms and their corresponding metrics:

- $\|v\|_2 = \left(\sum |v_j|^2\right)^{1/2}$, $d_2(v,w) = \|v-w\|_2$
- $\|v\|_1 = \sum |v_j|$, $d_1(v,w) = \|v-w\|_1$
- $\|v\|_\infty = \max\{|v_j|\}$, etc.
- $\|v\|_p = (\sum |v_j|^p)^{1/p}$, etc.

Consider the metric space (X,d). Let $Y \subset X$. THen the restriction of d to $Y \times Y \subseteq X \times X$ fashions Y as a new metric space with metric d.

Definition 2 (Norm)**.** Let V be a vector space on $\mathbb{R}$ or $\mathbb{C}$. A norm on V is a function $\| \quad \| : V \to \mathbb{R}^+$ satisfying:

1. $\|v\| = 0$ if and only if $v = 0$
2. $\|\alpha v\| = |\alpha| \|v\|$, $\alpha \in \mathbb{R}$ or $\mathbb{C}$
3. $\|v+w\| \leq \|v\| + \|w\|$

From each norm we get a metric on V defined by

$$d(v,w) = \|v-w\|$$

Example 1.2

Consider the vector space $C([0,1])$ of $\mathbb{R}$-valued continuous functions on [0, 1]. We have the following norms:

- $\|f\|_2 = \left(\int |f(t)|^2 dt\right)^{1/2}$
- $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$
- others defined similarly

Definition 3 (Convergence)**.** Let (X,d) be a metric space, $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ converges to a point $x \in X$ if $\forall \epsilon > 0$ there exists an N such that for $n \geq N$ we have $d(x_n, x) < \epsilon$.

Why do we use $\mathbb{R}$ instead of $\mathbb{Q}$? Completeness. This can be formulated in terms of Cauchy sequences.

Definition 4 (Cauchy Sequence)**.** Let (X,d) be a metric space, and let $\{x_k\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if $\forall \epsilon > 0$, there is an N such that for $m, n \geq N$ then $d(x_m, x_n) < \epsilon$.

The problem with the rationals is that you can construct cauchy sequences that don't converge to some $x \in \mathbb{Q}$. But for $\mathbb{R}$ they do.

Definition 5 (Completeness)**.** A metric space X is complete if every Cauchy sequence converges to some point of X.

Therefore, in a complete metric space, we have Cauchy convergence $\iff$ regular convergence. Furthermore, note that any closed subset of $\mathbb{R}^n$ is complete with the Euclidean metric. Going back to the $C([0,1])$ example, the convergence of the $\|f\|_\infty$ norm is just uniform convergence, i.e. the uniform limit of continuous functions is continuous. This is a theorem that'll be proved later.

On the other hand, the L^1 norm $\|f\|_1$ is not complete. For example, consider the sequence of piecewise function that is 1 on $[0, 1/2]$ and 0 on $[1/2 + 1/n, 1]$ with a straight line connecting the two flat lines. It converges to the function that is 1 on $[0, 1/2]$ and 0 on $(1/2, 1]$, but this is not continuous.

Definition 6. Let (X,d) be a metric space, S a subset of X. We say that S is dense in X if every open ball in X contains a point of S. By a completion of X we mean a metric space $(\bar{X}, \bar{d})$ together with a function $j : X \to \bar{X}$ such that j is isometric (preserves distances, i.e. $\bar{d}(j(x), j(y)) = d(x,y)$) and $j(X)$ is dense in $\bar{X}$.

Theorem 7

Every metric space has a completion, and this completion is essentially unique.

2 September 3

Theorem 8

Every metric space has a completion.

Let (X, d) be a metric space, and $CS(X, d)$ be the set of all Cauchy sequences in (X, d). We want to try to define a distance on $CS(X, d)$. Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy sequences, we claim that the sequence $\{d(x_n, y_n)\}$ in $\mathbb{R}$ is a Cauchy sequence. Set

$$\tilde{d}(\{x_n\}, \{y_n\}) = \lim\{d(x_n, y_n)\}.$$

Recall that we have

$$|d(x, y) - d(x, z)| \leq d(z, y)$$

from the triangle inequality. We have

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &= |d(x_n, y_n) - d(x_n, y_m) + d(x_n, y_m) - d(x_m, y_m)| \\ &\leq |d(x_n, y_n) - d(x_n, y_m)| + |d(x_n, y_m) - d(x_m, y_m)| \\ &\leq d(y_n, y_m) + d(x_n, x_m) \end{aligned}$$

which goes to 0 as n, m go to ∞. We may verify all the properties of a metric except for the one where different pints cannot have a distance of 0 (different cauchy sequences may converge to the same point). Therefore $\tilde{d}$ is a *semi-metric*.

Example 2.1

Consider the function space $C[0, 1]$. Let f_1 be defined as 0 on $[0, 1/2)$ and 1 on $[1/2, 1]$, whereas f_2 be defined as 0 on $[0, 1/2]$ and 1 on $(1/2, 1]$. Then we have

$$\|f_1 - f_2\| = \int |f_1 - f_2| = 0$$

yet $f_1 \neq f_2$. Therefore our $\mathcal{L}^1$ norm is a semi-metric.

So let (X, d) be a semi-metric space. Define an equivalence relation for X by $x \sim y$ if $d(x, y) = 0$. It forms a partition of X. Let $X/\sim$ denote the set of equivalence classes induced by our equivalence relation, and define $\hat{d}$ on $X/\sim$ by

$$\hat{d}([x], [y]) = d(x, y).$$

It is a simple exercise to check that this is well-defined independent of choice of class. Furthermore, $\hat{d}$ is a metric on $X/\sim$.

Now, we may redefine $\tilde{d}$ on $CS(X, d)$ on the set of equivalence classes. The equivalence relation is

$$d(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n) = 0 \implies \{x_n\} \sim \{y_n\}.$$

Embed (X, d) in $CS(X, d)/\sim$ by sending x to a Cauchy sequence $x_n = x$ for all n. That is, $\varphi(x) = \{x_n = x\}$, and

$$\tilde{d}(\varphi(x), \varphi(y)) = \lim d(x_n, y_n) = \lim d(x, y) = d(x, y).$$

So φ is an isometry of X from $CS(X, d) \to CS(X, d)/\sim$. We now want to show that the image of X through $\tilde{\varphi}$ is dense in $CS(X, d)/\sim$. We do this now.

Let $\{x_n\}$ be any Cauchy sequence. Given $\epsilon > 0$, there is an N such that for $m, n \geq N$ we have $d(x_m, x_n) < \epsilon$. Consider $\varphi(x_N)$. Then $\tilde{d}(\{x_n\}, \varphi(x_n)) = \lim_{n\to\infty} d(x_n, x_N) < \epsilon$. Now what's left is to show that $CS(X, d)/\sim$ equipped with $\tilde{d}$ is complete. For each m let $S^m = \{x_n^m\}_{n=1}^{\infty} \in CS(X, d)$. Assume the sequence $\{S^m\}$ is a Cauchy sequence in $CS(X, d)$. For each k find $x_k \in X$ such that $d(\varphi(x_k), S^m) < 1/k$. Then $S = \{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, and as $m \to \infty$, $d(S^m, S) \to 0$.

2.1 Metric Spaces to Topologies

Let (X, d_X) and (Y, d_Y) be metric spaces. Consider a mapping $f : X \to Y$. Let $x_0 \in X$. We say that f is continuous at x_0 if for all $\epsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x) - f(x_0)) < \epsilon$ if $d_X(x - x_0) < \delta$.

Denote by $B(x, r)$ the open ball centered at x with radius r. Then the above statement says that if $x \in B(x_0, \delta) = B$, then $f(x) \in B(f(x_0), \epsilon) = C$. Or in other words, if $x \in B$, then $f(x) \in C$, i.e. $x \in B \subseteq f^{-1}(X)$.

Definition 9. A set $A \subseteq X$ is an open subset (for the given metric d) if for each $x \in A$, there is an open ball about x contained in A.

Reformulating: if f is continuous at all points, then let O be an open set in Y and $x_0 \in f^{-1}(O)$. Then O contains a ball about $f(x_0)$ and for any ball B in O about $f(x_0)$ there is an open ball C about x_0 such that $x_0 \in C \subseteq f^{-1}(B)$, thus $C \subseteq f^{-1}(O)$. That is, $f^{-1}(O)$ is open.

Conversely, let f be any function from X to Y. If it is true that the preimage of any open set O in Y is open in X, then f is continuous. Proof: Let $x_0 \in X$. Given $\epsilon > 0$, let $O = B(f(x_0), \epsilon)$. Then $f^{-1}(O)$ is open in X. So there is a ball $B(x_0, \delta)$ contained in the preimage $f^{-1}(O)$.

Theorem 10 (Properties of Open Sets)

For the collection of open sets of a metric space, we have

1. An arbitrary union of open sets is open
2. A finite intersection of open sets is open
3. X and $\emptyset$ are open

These properties are what we want to define a *topology*.

3 September 5

Definition 11 (Topology)**.** Let X be a set. By a *topology* of X we mean a collection $\mathcal{T}$ of subsets of X, usually known as the open sets, satisfying

i. $\mathcal{T}$ is closed under arbitrary unions
ii. $\mathcal{T}$ is closed under finite intersections
iii. $X, \emptyset \in \mathcal{T}$

There are some topologies which do not come from metric spaces (i.e. it is a broader class of possibilities).

Definition 12. Let $\mathcal{T}$ be a topology for X. Then a set $A \subseteq X$ is said to be *closed* if A^C is in $\mathcal{T}$, i.e. the complement is open.

Properties of closed sets:

i. arbitrary intersection of closed sets is closed
ii. finite union of closed sets is closed
iii. X and $\emptyset$ are closed

Definition 13. Let $A \subseteq X$. By the *closure* $\bar{A}$ of A we mean the smallest closed set which contains A, i.e. the intersection of all closed sets that contain A. By the *interior* of A we mean the biggest open set contained in A, i.e. the union of all open sets that are contained in A. Let C be a closed set and $A \subseteq C$. We say that A is *dense* in C if $\bar{A} = C$.

Definition 14. Let X be a set, and $S \subset 2^X$. The smallest topology containing S is said to be the topology generated by S, and S is called a *sub-base* for that topology.

Note: If $\mathcal{C}$ is a collection of topologies for X, then $\cap\{\mathcal{T} \in \mathcal{C}\}$ is a topology for X.

We usually require the union of all sets in this sub-base to equal X.

Example 3.1

Let X be a set, and $\mathcal{D}$ be the collection of all subsets of X. then $\mathcal{D}$ is called the *discrete topology* for X. It is given by the metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Note: it is the largest possible topology on X.

Example 3.2

Let X be a set, then the smallest topology on X is given by $\mathcal{T} = \{X, \emptyset\}$, known as the *indiscrete topology.*

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ are topologies on X, we say the former is smaller, coarser, or weaker than the latter, which is bigger, finer, or stronger.

Definition 15. A collection S of subsets of X is a *base* for a topology if the set of all arbitrary unions of S is a topology.

For S to be a base, it must have the property that:

All finite intersections of sets in S must be a union of elements of S.

If S is any collection of subsets of X, then the collection of all finite intersections of elements of S is a base for a topology.

Definition 16. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and $f : X \to Y$ be a function. We say that f is continuous if

$$\forall O \in \mathcal{T}_Y, f^{-1}(O) \in \mathcal{T}_X$$

Let $f : X \to Y$ and $\{A_i\}$ be a collection of subsets of Y. Then

(i) $f^{-}(\cup A_i) = \cup f^{-1}(A_i)$
(ii) $f^{-1}(\cap A_i) = \cap f^{-1}(A_i)$
(iii) If $A, B \subseteq Y$, then $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$

Theorem 17

Given the topological spaces in the definition above and $f : XtoY$. Then to verify that f is continuous it suffices to check that $f^{-1}(A) \in \mathcal{T}_X$ for all $A \in S$ where S is a sub-base for $\mathcal{T}_Y$.

3.1 New Topologies From Old

Let X be a set, and $(X_i, \mathcal{T}_i)$ be a collection of topological spaces. Let there be functions $f_i : X_i \to X$, and $\mathcal{T}$ be the retrospect topology on X such that all the f_i's are continuous. Given i_0, f_{i_0}. If $A \subseteq X$ then if A is to be open, we must have $f_{i_0}^{-1}(A) \in \mathcal{T}_{i_0}$.

Let $S_{i_0} = \{A \subseteq X : f_{i_0}^{-1}(A) \in \mathcal{T}_{i_0}\}$. It is the strongest topology of X making f_{i_0} continuous. Then the strongest topology making all the f_i continuous is the intersection of all the S_{i_0}.

Let $(X, \mathcal{T})$ be a topological space, Y a set, and $f : X \to Y$. Then $\{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\}$ is the strongest topology making f continuous. We usually want f to be onto Y. This topology is the *quotient topology* determined by f.

Let X, Y be sets, with $f : X \to Y$ onto. Define an equivalence relation $\sim$ on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Each class in X is called a fiber over the point $y \in Y$. A set $B \subseteq X$ is *saturated* for this equivalence relation if whenever $xinB$ and $x' \sim x$, then $x' \in B$.

The open sets of the quotient topology on Y for f are in one to one correspondence with the saturated open subsets of X.