Math 249 – Algebraic Combinatorics

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Fall 2019

Contents

1 August 28

1.1 Logistics

Seminar on Mondays 12pm at Evans 939; reading seminar on Wednesdays 11am at Evans 748
Office Hours: 11:30am at Evans 859

1.2 Chapter 1 (Sagan)

Question: How to count?

- Answer 1: recurrence relation (ex. fibonacci)
- Answer 2: combinatorial interpretation (ex. tiling w/ dominoes & monominoes)
- Answer 3: explicit formula

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$
$$F_n = \frac{\phi^n - \phi'^n}{\sqrt{5}}$$

- Answer 4: asymptotic formula
- Answer 5: generating function (snake oil method gives second formula, partial fraction decomp gives third form)

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} F_n x^n \\
&= 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \dots \\
F(x) &= \frac{x}{1-(x+x^2)} \\
&= x \sum_{n=0}^{\infty} (x+x^2)^n \\
&= x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} x^{2k} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \right) x^{n+1} \\
F(x) &= \frac{1}{\sqrt{5}} \left(\frac{1}{1-\phi x} - \frac{1}{1-\phi' x} \right)
\end{aligned}$$

1.2.1 Permutations

Denote by Σ_n the set of permutations of $[n]$. Equipping with composition, forms the symmetric group S_n. We use one-line notation:

$$\pi = \pi(1)\pi(2)\dots\pi(n) \in \Sigma_n$$

We also have the cycle decomposition: given $i \in [n]$, let l be the smallest integer such that $\pi^l(i) = i$. Then

$$c = (i, \pi(i), \pi^2(i), \ldots, \pi^{l-1}(i))$$

is a cycle of π. Let $c(n,k)$ be the number of permutations of $[n]$ with k cycles (stirling numbers of first kind). They are given by the recurrence relation

$$c(n,k) = \begin{cases} \delta_{k,0} & \text{if } n = 0 \\ c(n-1,k-1) + (n-1) \cdot c(n-1,k) & \text{otherwise.} \end{cases}$$

Theorem 1

Every $\pi \in \Sigma_n$ has a a unique cycle decomposition (up to order of the cycles and rotation of the cycle elements).

Denote by $T([n])$ the set of trees with vertex set $[n]$, and $G = (V, E)$ a graph with vertex set V, edge set E.

Theorem 2 (Cayley)

The number of trees is $\#T([n]) = n^{n-2}$.

2 August 30

Cayley Trees

Theorem 3 (Cayley)
The number of trees with n distinct vertices is $\#T([n]) = n^{n-2}$.

Before proving the theorem, we need some intermediate definitions/results.

Definition 4 (Birooted Tree)**.** Let (T, a, b) be a tree with two roots $a, b \in [n]$.

Note that if we can show the number of birooted trees on $[n]$ is n^n, the theorem follows.

Proof. Consider the unque path from root a to root b as a permutation on the set of vertices in the path. Draw the directed edges of the cycle decomposition, and orient all other edges toward the two roots. This happens to be in bijection with the number of directed graphs on $[n]$, which is just n^n (check!).

□

Definition 5 (Pruffer Code)**.** Given a tree $T \in T([n])$, we produce a word $w_1, \ldots, w_{n-2}$ where $w_i \in \{1, \ldots, n\}$. We want to construct a bijection between $T([n])$ and the set of words $w_1 \ldots w_{n-2}$.

To determine the word from the tree T, we build a sequence $T_1, \ldots, T_{n-1}$ with $T_1 = T$, and given T_i, let l_i be the (unique) leaf with maximal label and define $T_{i+1} = T_i \setminus l_i$. Then let w_i be the unique vertex in T_i adjacent to l_i.

From the word, we may retrieve the tree as follows. Note that given $w_{1:n-2}$, we have

$$\begin{aligned}
l_1 &= \max[n] \setminus \{w_{1:n-2}\} \\
l_2 &= \max[n] \setminus \{l_1, w_{2:n-2}\} \\
&\vdots \\
l_i &= \max[n] \setminus \{l_{1:i-1}, w_{i:n-2}\}
\end{aligned}$$

From this, we can reconstruct the tree (exercise), proving the number of trees is n^{n-2}. □

Definition 6. The *degree sequence* of a tree is $(d_1, \ldots, d_n)$ where d_i is the degree of vertex i.

We claim that the number of trees with degree sequence $(d_1, \ldots, d_n)$ is the multinomial coefficient $\binom{n-2}{d_1-1, \ldots, d_n-1}$. We can show this through a bijection with the number of Pruffer codes such that $d_i - 1$ of the entries are equal to i.

Proof by Jim Pitman using induction on rooted forests.