Fourier Analysis

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1 Fourier Series, Pre-Lebesgue

Definition 1.1 (Fourier Series)**.** Let $f : [a, b] \to \mathbb{R}$ be an integrable function, where $L = b - a$. The n^{th} Fourier coefficient $\hat{f}(n) = a_n$ is defined as

$$a_n := \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x/L} dx, \quad n \in \mathbb{Z}.$$

The *Fourier series* of f is then given by

$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x/L}.$$

Often, we will consider f on the interval $[-\pi, \pi]$, so that our fourier coefficients become

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z},$$

and the Fourier series of f becomes

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

We define the N^{th} partial sum as

$$S_N(f)(x) = \sum_{n=-N}^{N} a_n e^{2\pi i n x/L},$$

and immediately the question of convergence arises. That is, in what sense does $S_N(f) \to f$? Namely, we will explore pointwise, uniform, and mean convergence in the coming sections.

1.1 Convergence at Points of Continuity & Uniform Convergence

First, we deal with the matter of uniqueness: if f and g have the same Fourier coefficients a_n, then is $f = g$ almost everywhere? In particular, this reduces to showing that $f = 0$ almost everywhere given that $a_n = 0$ for all $n \in \mathbb{Z}$. The following result verifies this for continuous functions on the circle.

Theorem 1.2 (Uniqueness of Fourier Series)

Suppose f is a continuous function on the circle with $a_n = 0$ for $n \in \mathbb{Z}$. Then $f = 0$. More generally, if f is piecewise continuous, then $f = 0$ almost everywhere.

Proof. Suppose WLOG that f is real-valued, defined on $[-\pi, \pi]$, and f is continuous at 0. For sake of contradiction, let $f(0) > 0$. The idea is to construct a sequence of trigonometric polynomials $\{p_k\}$ which blow up near zero, so that

$$\int p_k(\theta) f(\theta) d\theta \to \infty,$$

contradicting the fact that these integrals are all zero, since we can decompose each of them into sums of Fourier coefficients, which we have assumed to be zero.

Let $p(\theta) = \cos\theta + \epsilon$, where we will pick the parameter ϵ later. We approach this by picking parameters $0 < \eta < \delta \leq \pi/2$ such that for $|\theta| < \eta$, our p_k blows up; for $\eta < |\theta| < \delta$, the product $p_k(\theta)f(\theta)$ is nonnegative; and for $\delta \leq |\theta| \leq \pi$, the sequence $|p_k(\theta)|$ vanishes.

More precisely, since f is continuous at 0, we can pick $0 < \delta \leq \pi/2$ such that $f(\theta) > 0$ for $|\theta| < \delta$. Then let

$$p(\theta) = \cos\theta + \epsilon,$$

where $\epsilon > 0$ is chosen so small that $|p(\theta)| < 1 - \epsilon/2$ for $\delta \leq |\theta| \leq \pi$. Then, choose $0 < \eta < \delta$ so that $p(\theta) \geq 1 + \epsilon/1$ for $|\theta| < \eta$. Now, let

$$p_k(\theta) = [p(\theta)]^k.$$

Since f is integrable, and thus bounded, we have the bound

$$\left| \int_{\delta \leq |\theta| \leq \pi} p_k(\theta) f(\theta) \right| \leq 2\pi B (1 - \epsilon/2)^k,$$

for some constant B, so that the integral vanishes as $k \to \infty$. On the other hand, by our choice of δ, we know

$$\int_{\eta \leq |\theta| < \delta} p_k(\theta) f(\theta) d\theta \geq 0.$$

Finally, the integral near 0

$$\int_{|\theta| < \eta} p_k(\theta) f(\theta) d\theta \geq 2\eta \left(\frac{f(0)}{2} \right) \left(1 + \frac{\epsilon}{2} \right)^k.$$

It follows that

$$\int_{-\pi}^{\pi} p_k(\theta) f(\theta) d\theta \to \infty,$$

contradiction. Note for complex-valued functions, we can do the usual and split the function into its real and imaginary parts. Hence f must vanish almost everywhere. $\square$

As a corollary, we get our first uniform convergence type result:

Corollary 1.3

Suppose f is a continuous function on the circle and that its Fourier series is absolutely convergent, i.e.

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

Then the Fourier series converges uniformly to f:

$$\lim_{N \to \infty} S_N(f) = f.$$

Proof. Note that for $\theta \in [-\pi, \pi]$, we have

$$\left| \sum_{n=-\infty}^{-N} a_n e^{in\theta} + \sum_{n=N}^{\infty} a_n e^{in\theta} \right| \leq \sum_{n=-\infty}^{-N} |a_n| + \sum_{n=N}^{\infty} |a_n| \to 0,$$

since the tail sums of the absolute series must vanish. Then, recall that the uniform limit of a sequence of continuous functions is continuous, so the function

$$g(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n e^{in\theta}$$

is continuous. Now, we can exchange summation with the integral due to uniform convergence to get

$$\begin{aligned} b_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \cdot e^{ik\theta} d\theta \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} a_n e^{i(n-k)\theta} d\theta \\ &= a_k. \end{aligned}$$

Thus, the Fourier coefficients of g are just the Fourier coefficients of f. Now we can apply Theorem 1.2 to deduce that $f = g$, and hence we have uniform convergence of the Fourier series of f. □

It turns out that the smoothness of f is directly related to the decay of the Fourier coefficients, and therefore also the uniform convergence of the Fourier series.

Theorem 1.4 (Uniform Convergence in C^2)
Suppose $f \in C^2([0, 2\pi])$ is a periodic function on the circle. Then $a_n = O(1/n^2)$ as $|n| \to \infty$, so that the Fourier series of f converges absolutely and therefore uniformly to f.

Proof. Using integration by parts twice,

$$\begin{aligned} 2\pi a_n &= \int_0^{2\pi} f(\theta) e^{in\theta} d\theta \\ &= \left[f(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{in} \int_0^{2\pi} f'(\theta) e^{-in\theta} d\theta \\ &= \frac{1}{in} \left[f'(\theta) \cdot \frac{-e^{-in\theta}}{in} \right]_0^{2\pi} + \frac{1}{(in)^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \\ &= \frac{-1}{n^2} \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta. \end{aligned}$$

Now, since the second derivative is continuous on a compact interval, it is bounded, and hence

$$2\pi n^2 |a_n| \leq \left| \int_0^{2\pi} f''(\theta) e^{-in\theta} d\theta \right| \leq \int_0^{2\pi} |f''(\theta)| d\theta \leq C.$$

Thus, $\sum |a_n|$ is dominated by $\sum 1/n^2 < \infty$, and so applying Corollary 1.3 we obtain uniform convergence. $\square$

It turns out uniform convergence can also be shown under the stronger condition of $f \in C^1$. However, our proof fails when we try to modify it to prove that stronger statement.

1.2 Convolution

Definition 1.5 (Convolution)**.** Given two 2π-periodic integrable functions f and g on $\mathbb{R}$, we define their *convolution* $f * g$ on $[-\pi, \pi]$ as

$$(f * g)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(x - t) dt.$$

Actually, convolution is usually defined without the normalizing factor in front. However, for the purpose of this section, we will follow Shakarchi's way of definition.

One motivation for convolutions comes from the relationship between the partial sums of Fourier series and the Dirichlet kernels. In particular, the N^{th} Dirichlet kernel D_N is given by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx},$$

and it turns out that

$$\begin{aligned}
(f * D_N)(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{n=-N}^{N} e^{in(x-y)} \right) dy \\
&= \sum_{n=-N}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx} \\
&= \sum_{n=-N}^{N} a_n e^{inx} \\
&= S_N(f)(x).
\end{aligned}$$

Note the connection to EE: the n^{th} Fourier coefficients are essentially the frequency responses to each complex exponential e^{inx} for a system whose impulse response is given by f.

Proposition 1.6

Suppose f, g, and h are 2π-periodic integrable functions. Then we have the following properties:

(i) $f * (g + h) = (f * g) + (f * h)$.

(ii) $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{C}$.

(iii) $f * g = g * f$.

(iv) $(f * g) * h = f * (g * h)$.

(v) $f * g$ is continuous.

(vi) The Fourier coefficients of the convolution are the product of the Fourier coefficients of f and g:
$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Properties (i)-(iv) are straightforward. Note that (v) says, surprisingly, that $f * g$ is more “regular” than its operands. Property (vi) in EE terms says that the frequency response of the convolution (which is just the composition of systems by associativity) is just the product of the frequency responses of the individual systems, which makes sense if we think of frequency responses as a sort of eigenvalue for the system. We will postpone the proof of these properties for later when we are equipped the Lebesgue integral, which makes things easier.

1.3 L^2 Convergence

Our goal is to prove mean square convergence of a Fourier series. Before we begin, we need the following two lemmas:

Lemma 1.7 (Approximating Integrable by Continuous)

Suppose f is integrable on the circle and bounded by some constant C. Then there exists a sequence (f_n) of continuous functions on the circle uniformly bounded by C so that
$$\int_{-\pi}^{\pi} |f(x) - f_n(x)| dx \to 0 \quad \text{as } n \to \infty.$$

Proof. Actually, we can prove something stronger. We can relax f to be defined on any interval, finite or infinite. Then since f is integrable, there exists a sequence of step functions (given by partitions) which approximate f. We also know that step functions can be approximated by continuous functions, defined by replacing each neighborhood of discontinuity with a slanted line. It’s easy to show then, using the $\epsilon/2^n$ trick (the step functions have at most countable many discontinuities), that overall the L^1 distance between the step functions and the continuous functions can be made arbitrarily small. □

Before we move on to the next lemma, we need the concept of a "good" kernel. We can think of these as an approximation to the identity, or the "distribution" which represents the dirac delta (a point mass at 0).

Definition 1.8 (Good Kernels)**.** We say a family of kernels (K_n) is good if it satisfies

(i) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1.$$

(ii) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|dx \leq M.$$

(iii) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|dx \to 0, \qquad \text{as } n \to \infty.$$

Proposition 1.9

Let (K_n) be a family of good kernels, and f an integrable function on the circle. Then

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, then the limit is uniform.

Proof. Let $\epsilon > 0$, and f be continuous at x. So choose δ so that $|y| < \delta$ implies $|f(x-y) - f(x)| < \epsilon$. Then though the defining properties of a good kernel, we have

$$\begin{aligned}
|(f * K_n)(x) - f(x)| &= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)(f(x-y) - f(x))dy \right| \\
&\leq \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)||f(x-y) - f(x)|dy \\
&\quad + \frac{1}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)||f(x-y) - f(x)|dy \\
&\leq \frac{\epsilon M}{2\pi} + \frac{B}{2\pi}\int_{\delta \leq |y| \leq \pi} |K_n(y)|dy.
\end{aligned}$$

By the vanishing property (iii), we see that the second term goes to zero, and so

$$(f * K_n)(x) \to f(x).$$

If in addition f is continuous on $[-\pi, \pi]$, then it is uniformly continuous, so in particular our choice of δ works for all $x \in [-\pi, \pi]$, giving us uniform convergence. □

Lemma 1.10 (Fejér's Theorem, Approximating Continuous by Trigonometric)
Let f be a continuous function on the circle $[-\pi, \pi]$. Then f can be uniformly approximated by trigonometric polynomials.

Proof. For a sequence of partial sums (s_n) of some series $\sum z_k$, we call σ_N the N^{th} Cesaro mean or sum of the series, given by

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

If $\lim \sigma_N = \sigma$ exists, then we say that the series $\sum z_k$ is Cesaro summable to σ. In particular, form the N^{th} Cesaro mean of the Fourier series, given by

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N}.$$

Recall that $S_n(f) = f * D_n$, so it follows that

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where we define $F_N(x)$ to be the N^{th} Fejér kernel given by the average of first N Dirichlet kernels.

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

Now, through some basic trig-bashing, we get that

$$F_N(x) = \frac{\sin^2(Nx/2)}{N \sin^2(x/2)}.$$

Now, we show that the Fejér kernel is a good kernel. Note that $F_N(x) \geq 0$, and by some simple calculations (most of the e^{inx} terms of the Dirichlet kernel cancel out),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F_N(x) dx = 1.$$

Now, for $\delta > 0$, we have that $\sin^2(x/2) \geq \sin^2(\delta/2) > 0$ for $\delta \leq |x| \leq \pi$. Therefore $F_N(x) \leq 1/(N \sin^2(\delta/2))$, so taking $N \to \infty$, gives

$$\int_{\delta \leq |x| \leq \pi} |F_N(x)| dx \to 0.$$

Therefore, by Proposition 1.9, we have the uniform limit

$$\lim_{N \to \infty} \sigma_N(f)(x) = \lim_{N \to \infty} (f * F_N)(x) = f(x).$$

Since the Cesaro means are themselves trigonometric polynomial, the claim follows. $\square$

Equipped with the these lemmas, we are ready to prove convergence in mean of a Fourier series.

Theorem 1.11 (Mean-Square Convergence)
Suppose f is integrable on the circle. Then as $N \to \infty$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 d\theta \to 0.$$

Proof. The idea is to first realize the Fourier sums as projections onto a growing orthonormal family of basis functions. In particular, it's easy to verify the family $\mathcal{B}_N := \{e^{inx}\}_{|n|\leq N}$ is orthonormal with respect to the complex inner product given by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f \cdot \overline{g} dx.$$

Now, note that the Fourier coefficients are just

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \langle f, e^{inx}\rangle.$$

It follows that the partial sums

$$S_N(f)(x) = \sum_{n=-N}^{N} \langle f, e^{inx}\rangle e^{inx}$$

are just projections of f onto the N^{th} hyperplane spanned by the basis $\mathcal{B}_N$. Therefore

$$\|f(x) - S_N(f)(x)\|_2 \leq \|f(x) - \sum_{n=-N}^{N} b_n e^{inx}\|_2,$$

for any set of coefficients $b_n \in \mathbb{C}$. Now, by 1.7 and 1.10, we have that the trigonometric polynomials are dense in the space of continuous functions, which are dense in the space of integrable functions. Therefore, we can approximate any integrable function uniformly by a sequence of trigonometric polynomials p_N formed from the growing bases $\mathcal{B}_N$. Setting $\sum_{n=-N}^{N} b_n e^{inx}$ to this sequence gives us

$$\begin{aligned}\|f(x) - S_N(f)(x)\|_2 &\leq \|f(x) - p_N(x)\|_2 \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - p_N(x)|^2 dx \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \epsilon_N^2 dx,\end{aligned}$$

where $\epsilon_N \to 0$ as $N \to \infty$. This shows that

$$\|f(x) - S_N(f)(x)\|_2 \to 0,$$

as desired. $\square$

Recall that each $S_N(f)(x)$ is the projection of f onto the orthonormal basis $\mathcal{B}_N$. Then we have

$$\|f\|_2^2 = \|f - S_N(f)\|_2^2 + \sum_{n=-N}^{N} |a_n|^2.$$

In particular, if we take $N \to \infty$, then we get Parseval's identity:

Corollary 1.12 (Parseval's Identity)

If f is integrable on the circle, then

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|_2^2.$$

Since $\sum_{n=-\infty}^{\infty} |a_n|^2$ converges, it follows that its terms go to zero, so we get the following result.

Theorem 1.13 (Riemann-Lebesgue Lemma)

If f is integrable on the circle, then $a_n = \hat{f}(n) \to 0$ as $|n| \to \infty$.

From the Riemann-Lebesgue lemma, it turns out we can obtain another condition for pointwise convergence. Its proof we omit for now, as we wait until we have more convenient tools (i.e. Lebesgue integration).

Corollary 1.14 (A Condition for Pointwise Convergence)

Let f be an integrable function on the circle. If it is differentiable or Lipschitz continuous at x_0, then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.

2 Fourier Series

2.1 Tools of Lebesgue

From now on, we will assume familiarity with basic measure theory and the Lebesgue integral. The following tools will be crucial:

Theorem 2.1 (Monotone Convergence)
Suppose $0 \leq f_n \uparrow f$ are measurable functions. Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

The condition $0 \leq f_1$ may be replaced by the condition that $\int f_1^- < \infty$.

Theorem 2.2 (Dominated Convergence)
Suppose $|f_n| < g$, where g is absolutely integrable. If $f_n \to f$, thn

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Lemma 2.3 (Fatou)
If $f_n \geq 0$, then

$$\liminf_{n\to\infty} \int f_n d\mu \geq \int \liminf_{n\to\infty} f_n d\mu.$$

Lemma 2.4 (Borel-Cantelli)
Let $B_n \subset \Omega$ for $n \geq 1$. If $\sum_{n=1}^{\infty} \mu(B_n) < \infty$, then

$$\mu(\omega \in B_n \text{ i.o.}) := \mu(\cap_{n\geq 1} \cup_{m\geq n} B_m) = 0.$$

Theorem 2.5 (Fubini)
Let f be a measurable function defined on the product space $(\Omega, \Sigma_1 \times \Sigma_2, \mu_1 \times \mu_2) := (\Omega_1, \Sigma_1, \mu_1) \times (\Omega_2, \Sigma_2, \mu_2)$. If f is absolutely summable or nonnegative, then

$$\int_{\Omega_1} \int_{\Omega_2} f d\mu_2 d\mu_1 = \int_{\Omega} f d\mu = \int_{\Omega_2} \int_{\Omega_1} f d\mu_1 d\mu_2.$$

2.2 Geometry of L^2([a,b])

In this section, we will develop the basic theory of the L^2 space. We start with some definitions.

Definition 2.6. The space $L^p([a,b])$ is the collection of all functions $f : [a,b] \to \mathbb{C}$ satisfying

$$\|f\|_p := \left(\int |f|^p d\mu\right)^{\frac{1}{p}} < \infty.$$

In this section, we are concerned with the case where $p = 2$.

Definition 2.7. A *Hilbert space* is an inner product space that is complete. For our purposes, we will also assume that it is separable. That is, the space contains a countable dense subset.

Traditional examples of finite dimensional Hilbert spaces include $\mathbb{R}^n$ and $\mathbb{C}^n$. The goal of this section will be to show that L^2 is also a Hilbert space.

First, recall Schwarz's inequality, which says given an inner product space H and $f, g \in H$, we have

$$|\langle f, g\rangle| \leq \|f\|_2\|g\|_2.$$

Equality holds if and only if $f = \alpha g$, where $\alpha \in \mathbb{C}$. From this we also get the triangle inequality,

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Note that these hold in any inner product space, including infinite dimensional ones. So, we can still use finite dimensional geometric intuition on infinite dimensional spaces.

Theorem 2.8

The space $L^2([a,b])$ is a Hilbert space.

Proof. Let $f, g \in L^2([a,b])$. Clearly it is closed under scalar multiplication. Furthermore, we have

$$\int |f+g|^2 \leq \int \left(|f|^2 + 2|f||g| + |g|^2\right) \leq \int 2\left(|f|^2 + |g|^2\right) < \infty,$$

so it is a vector space. Now, the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g\rangle := \int f\overline{g}$$

is well-defined, since

$$2|f\overline{g}| = 2|f||g| \leq |f|^2 + |g|^2,$$

whose integral is finite. It's easy to verify the rest of the properties of an inner product.

We now move on to the meat of the theorem. It turns out that completeness doesn't even hold if we were to define L^2 in terms of the Riemann integral, so already we see an advantage of the Lebesgue integral (in fact, the "L" actually stands for Lebesgue!). $\square$