# The Heat Equation

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In this note, we are concerned with the heat, or sometimes referred to as diffusion, equation:

$$u_t = ku_{xx}. \tag{1}$$

The physical interpretation is that we have some amount of heat is spread out along a rod, and over time it diffuses from regions of high to low heat concentration. First, we will go over the solution to the heat equation on the real line, then follow up with the solution on the half line, which is a simple matter of reflection. After that we'll cover Duhamel's principle applied to the inhomogeneous heat equation. Next we'll use Fourier series in conjunction with separation of variables to solve the problem on any finite interval. Finally, we will go over the energy and maximum principles and use them to prove uniqueness of solutions.

## 1 Diffusion on Whole Line

**Problem 1.1.** We want to solve

$$\begin{aligned} u_t &= ku_{xx} \quad x \in \mathbb{R}, t \in \mathbb{R}^+ \\ u(x,0) &= \varphi(x) \quad x \in \mathbb{R}. \end{aligned} \tag{2}$$

First, we note some invariance properties of the heat equation that will help us in solving for the general solution.

**Proposition 1.2**

Suppose $u(x,t)$ is a solution to (1). Then the following are also solutions to (1):

(i) The translation $u(x-y,t)$.

(ii) Any derivative, e.g. $u_x, u_{xx}, u_t$, etc.

(iii) Any linear combination of solutions.

(iv) The convolution $u \cdot g$ given by

$$(u * g)(x,t) := \int_{-\infty}^{\infty} u(x-y,t)g(y)dy.$$

(v) The dilation $u(\sqrt{a}x, at)$ for any $a > 0$.

*Proof.* Proofs of (i)-(iii), (v) are immediate, through routine uses of the chain rule or via linearity of the differential operators. For example, to prove (v), we apply chain rule to get

$$\begin{aligned}\frac{\partial u(\sqrt{a}x, at)}{\partial t} &= u_t(\sqrt{a}x, at) \cdot a \\ &= k u_{xx}(\sqrt{a}x, at) \cdot \sqrt{a} \cdot \sqrt{a} \\ &= k \frac{\partial^2 u(\sqrt{a}x, at)}{\partial x^2}.\end{aligned}$$

The remaining one follows by Leibniz's rule:

$$\begin{aligned}\frac{\partial^2}{\partial x^2} \int_{\mathbb{R}} u(x-y,t)g(y)dy &= \int_{\mathbb{R}} \frac{\partial^2}{\partial x^2} u(x-y,t)g(y)dy \\ &= \int_{\mathbb{R}} u_{xx}(x-y,t)g(y)dy \\ &= \int_{\mathbb{R}} u_t(x-y,t)g(y)dy \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x-y,t)g(y)dy \\ &= \frac{\partial}{\partial t} \int_{\mathbb{R}} u(x-y,t)g(y)dy,\end{aligned}$$

given that we our particular solution $u$ is already sufficiently smooth. □

**Theorem 1.3**

Given initial condition $u(x,0) = \varphi(x)$, with $\varphi(x) \to 0$ as $|x| \to \infty$, the solution to problem 2 is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)dy.$$

*Proof.* Our method will be to first find a particular solution $Q(x,t)$, and then use the convolution property to construct further solutions that will eventually satisfy the initial condition. Based on the dilation invariance property, we guess that our solution will have the initial condition

$$Q(x,0) = \mathbf{1}_{x>0},$$

along with the general form

$$Q(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right).$$

Here the $x/\sqrt{t}$ is to ensure dilation invariance, and the extra $\sqrt{4k}$ is a normalizing factor that will go into calculations later.

Given the form of $Q(x,t)$, we may express the PDE $Q_t = kQ_{xx}$ as an ODE via chain rule:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}pg'(p) \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p). \end{aligned}$$

This gives us

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right],$$

which turns into the ODE

$$g'' + 2pg' = 0.$$

By fundamental techniques, we find that the solution is thus given by

$$Q(x,t) = g(p) = c_1 \int_0^p e^{-q^2}dq + c_2.$$

Now, using the initial condition (where $Q(x,0)$ is a step function), we have

$$\begin{array}{rl} \text{If } x > 0, & 1 = \lim_{t\to 0^+} Q(x,t) = c_1 \int_0^\infty e^{-q^2}dq + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2. \\ \text{If } x < 0, & 0 = \lim_{t\to 0^+} Q(x,t) = c_1 \int_0^{-\infty} e^{-q^2}dq + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2. \end{array}$$

From this we deduce that $c_1 = 1/\sqrt{\pi}$ and $c_2 = 1/2$. Thus

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\int_0^{x/\sqrt{4kt}} e^{-q^2}dq.$$

Now, having found $Q$, define $S = \frac{\partial Q}{\partial x}$. It turns out then that, by Leibniz, this is just a family of Gaussian kernels:

$$S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi kt}}e^{-\frac{x^2}{4kt}}, \qquad \text{for } t > 0.$$

Next, we claim that the convolution $S * \varphi$ is the solution to the heat equation with initial condition $\varphi(x)$. To verify this, write

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\varphi(y)dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\varphi(y)dy \\ &= \int_{-\infty}^{\infty} Q(x-y,t)\varphi'(y)dy - Q(x-y,t)\varphi(y)\big|_{y=-\infty}^{y=\infty}. \end{aligned}$$

We will furthermore assume that $\varphi(y)$ vanishes as $|y| \to \infty$. Therefore, we have

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\varphi'(y)dy \\ &= \int_{-\infty}^{x} \varphi'(y)dy \\ &= \varphi(x), \end{aligned}$$

as desired. □

## 2 Diffusion on Half Line

In this section, we are interested in solving the problem on the half line:

$$\begin{aligned} u_t &= ku_{xx} \quad 0 < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \varphi(x) \quad 0 < x < \infty \\ u(0,t) &= 0. \end{aligned}$$

Our solution will involve a reflection principle which is often used to craft solutions to PDEs on the half line from those on the whole line.

**Definition 2.1.** Given a function $\varphi : (0,\infty) \to \mathbb{R}$, define the *odd extension* $\varphi_o : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_o(x) = \begin{cases} \varphi(x) & x > 0, \\ -\varphi(-x) & x < 0, \\ 0 & x = 0. \end{cases}$$

**Theorem 2.2**

The solution to the heat equation on the half line is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}}\varphi(y) - e^{-\frac{(x+y)^2}{4kt}}\varphi(y) \right] dy.$$

*Proof.* Construct the odd extension $\varphi_o$ of the initial condition $\varphi$, and consider the problem

$$\begin{aligned} u_t &= ku_{xx} \quad -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,0) &= \varphi_o(x) \quad -\infty < x < \infty \end{aligned}$$

We know the solution of this is given by Theorem 1.3 to be

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi_o(y)dy.$$

Clearly this satisfies the first two conditions of the half-line problem, so it remains to check the third. We have

$$u(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \varphi_o(y) dy.$$

But note that $e^{-\frac{y^2}{4kt}}$ is an even function of $y$, and $\varphi_o(y)$ is an odd function of $y$, so their product must be an odd function of $y$. And so, the integral evaluates to zero, which gives us our third condition:

$$u(0,t) = 0.$$

Now, to write our solution in terms of $\varphi$ rather than $\varphi_o$, we do a simple change of variables:

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \left[ \int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} \varphi(y) dy - \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} \varphi(-y) dy \right] \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \left[ e^{-\frac{(x-y)^2}{4kt}} \varphi(y) - e^{-\frac{(x+y)^2}{4kt}} \varphi(y) \right] dy. \end{aligned}$$

This is our general formula for the heat equation on the half-line. □

## 3 Diffusion with a Source

Now we are interested in solving the inhomogeneous heat equation on the whole line:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad -\infty < x < \infty, 0 < t < \infty \\ u(x,0) &= \varphi(x) \quad -\infty < x < \infty. \end{aligned}$$

Here $f(x,t)$ is known as a forcing term, and can be thought of a source of heat input. For our purposes, we will assume $f$ to be continuous.

We first present the solution and verify that it satisfies the equations above. Afterwards, we'll give a rough sketch of how it was originally derived.

**Theorem 3.1**

The solution to the inhomogeneous heat equation takes the form

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y) dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s) f(y,s) dy ds,$$

where $S$ is the function defined in the proof of Theorem 1.3.

*Proof.* Recall that

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y) dy$$

is the solution to the homogeneous heat equation. So since the differential operators are linear, we may w.l.o.g. solve the simpler problem:

$$u_t - ku_{xx} = f(x,t) \quad -\infty < x < \infty, 0 < t < \infty$$
$$u(x,0) = 0 \quad -\infty < x < \infty.$$

It only remains to check the first equality. Applying Leibniz's integral rule, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y,t-s)f(y,s)dyds \\ &\quad + \lim_{s\to t}\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy.\end{aligned}$$

Now, using the fact that $S(x-y,t-s)$ satisfies the heat equation, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,t-s)f(y,s)dyds \\ &\quad + \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} S(x-y,\epsilon)f(y,t-\epsilon)dy\end{aligned}$$

The first term becomes, by another application of Leibniz's rule,

$$k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds,$$

which is just $k\frac{\partial^2 u}{\partial x^2}$ since we assumed the initial condition was zero. As for the second term, we claim that

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} S(x-y,\epsilon)f(y,t-\epsilon)dy = f(x,t). \tag{3}$$

In Fourier analysis terms, this is just saying that the convolution of a $f$ with a family of "good kernels", which are essentially approximations to the identity, will give back the original function. First recall that

$$S(x-y,\epsilon) = \frac{1}{2\sqrt{\pi kt}}e^{-\frac{x^2}{4kt}}.$$

It is a gaussian, so

$$\int_{-\infty}^{\infty} S(x-y,\epsilon)dy = 1.$$

for all $\epsilon$. Fixing $(x,t)$, and picking $\delta > 0$ small enough so that $|f(y,t-\epsilon) - f(x,t)| < \epsilon'$

for $|x-y|<\delta$, we have

$$\begin{aligned}
|(S_\epsilon * f_{t-\epsilon})(x) - f(x,t)| &= \left|\int_{-\infty}^{\infty} S(x-y,\epsilon)f(y,t-\epsilon)dy - f(x,t)\right| \\
&= \left|\int_{-\infty}^{\infty} [S(x-y,\epsilon)f(y,t-\epsilon) - f(x,t)S(x-y,\epsilon)]dy\right| \\
&\leq \int_{-\infty}^{\infty} |S(x-y,\epsilon)|\cdot|f(y,t-\epsilon)-f(x,t)|dy \\
&= \int_{|y-x|<\delta} |S(x-y,\epsilon)|\cdot|f(y,t-\epsilon)-f(x,t)|dy \\
&\quad + \int_{|y-x|\geq\delta} |S(x-y,\epsilon)|\cdot|f(y,t-\epsilon)-f(x,t)|dy.
\end{aligned}$$

The first term is bounded by $\epsilon'$, and the second term we may pass to the limit and apply DCT to get zero. But we can also take $\epsilon'$ to zero, thus verifying (3). Therefore, we have verified

$$u_t = ku_{xx} + f(x,t).$$

The initial condition is already satisfied, so we are done. □

The way we actually come up with the form in 3.1 is by Duhamel's principle. This is a general method which can be applied to PDEs to construct solutions to inhomogenous versions from the homogeneous version, and can be summarized by the following steps:

1. For every $s>0$, define solutions $u(x,t;s)$ to the PDE with initial condition given by $f(x,s)$.
2. Solve for $u(x,t;s)$ using the solution to the homogeneous problem.
3. Combine the solutions by integrating over $s\in[0,t]$.
4. Verify that the resulting form satisfies the PDE.

Let's walk through these steps for the heat equation. Note that we've already done step 4. Consider intermediate function $u(x,t;s)$ which satisfy

$$\begin{aligned}
u_t(x,t;s) - ku_{xx}(x,t;s) = 0 \quad & -\infty<x<\infty, \quad t>s \\
u(x,s;s) = f(x,s) \quad & -\infty<x<\infty.
\end{aligned}$$

By a translation, the solution is given by the homogeneous solution, so we have

$$u(x,t;s) = \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy.$$

Integrating over $[0,t]$, this gives us the inhomogeneous solution

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy.$$

# 4 Diffusion on a Finite Interval

We are interested in solutions to the homogeneous Dirichlet and Neumann initial boundary value problems. The overall idea is to use separation of variables to obtain sequences of particular solutions $u_n(x,t)$, and then take an infinite linear combination of them to obtain the general solution. This infinite sum turns out to be closely related to the Fourier series of certain functions, so we will be using some convergence theory from Fourier analysis to establish our solutions.

**Problem 4.1** (Homogenous Dirichlet IBVP)**.** We want to solve

$$\begin{aligned} u_t &= ku_{xx}, \quad 0 < x < \pi, t > 0 \qquad (4) \\ u(0,t) &= 0 = u(\pi,t), \quad t > 0 \\ u(x,0) &= f(x), \quad 0 < x < \pi \end{aligned}$$

**Theorem 4.2**

The solution to the above problem is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left( \frac{2}{\pi} \int_0^{\pi} f(y) \sin(ny) dy \right) \sin(nx) e^{-n^2 kt}.$$

*Proof.* We proceed by separation of variables. Suppose for some $A(x)$ and $B(t)$, we have

$$u(x,t) = A(x)B(t).$$

Plugging this into (4) we get

$$A(x)B'(t) = kA''(x)B(t).$$

Then for some constant $\lambda$, we have

$$-\lambda = \frac{A''(x)}{A(x)} = \frac{B'(t)}{kB(t)}.$$

We will see why we chose $-\lambda$ later. This gives us a system of ODEs:

$$\begin{aligned} A''(x) + \lambda A(x) &= 0 \\ B'(t) + \lambda k B(t) &= 0. \end{aligned}$$

The first has solutions

$$\lambda_n = n^2, \quad A_n(x) = a_1 \sin(nx) + a_2 \cos(nx).$$

Plugging in the boundary conditions, we see that if $A_n(0) = a_2 \neq 0$, then $B(t) = 0$ for all $t > 0$. But this is a trivial case that we don't really care about, so suppose $a_2 = 0$.

Furthermore, the constant $a_1 \neq 0$ doesn't really matter, as we will be able to absorb them into the Fourier coefficients later. So assume $a_1 = 1$. Our family of solutions for $A$ becomes

$$\lambda_n = n^2, \quad A_n(x) = \sin(nx).$$

Similarly, the solutions for $B$ have the form

$$\lambda_n = n^2, \quad B_n(t) = e^{-n^2kt}.$$

Therefore we have the family of solutions

$$u_n(x,t) = A_n(x)B_n(t) = \sin(nx)e^{-n^2kt}.$$

Since our PDE is linear, we would expect that

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)e^{-n^2kt} \tag{5}$$

is a solution. It turns out that the series converges uniformly, enabling us to differentiate term by term and actually claim that it solves the heat equation. We will prove this in a bit.

Suppose the equality in (5) holds. Then we may plug in the initial conditions with $t = 0$ to get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx).$$

This implies that $c_n$ are just the Fourier sine coefficients of $f$, so

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx.$$

Putting this into (5) gives

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{\pi}\int_0^{\pi} f(y)\sin(ny)dy\right)\sin(nx)e^{-n^2kt}. \tag{6}$$

So, with this formula in mind, it remains to show:

1. The initial condition $f$ is equal to its Fourier sine series expansion:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx).$$

2. The series representations of $u$, $u_t$, $u_x$, and $u_{xx}$ converge uniformly so that we may differentiate the series term by term to deduce that the PDE (4) holds.

If we assume $f$ is sufficiently well-behaved then we get the first point. Usually in practice we only ever deal with these nicely behaved classes of functions. In particular, we will assume that $f \in C^2$ and $|f| < M$. Then it suffices to show

$$\sum_{n=1}^{\infty} |c_n| < \infty,$$

for this would give us uniform convergence of the Fourier series (one can prove this using M-test).

Applying integration by parts twice, we get

$$\begin{aligned}
\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx &= -\frac{2}{\pi}\left(f(x)\cdot\frac{\cos(nx)}{n}\right)\Bigg|_{x=0}^{\pi} \\
&\quad + \frac{2}{\pi}\left(f'(x)\cdot\frac{\sin(nx)}{n^2}\right)\Bigg|_{x=0}^{\pi} \\
&\quad - \frac{1}{n^2}\cdot\frac{2}{\pi}\int_0^{\pi} f''(x)\sin(nx)dx \\
&= -\frac{1}{n^2}\cdot\frac{2}{\pi}\int_0^{\pi} f''(x)\sin(nx)dx.
\end{aligned}$$

Since $f \in C^2$ on $[0, \pi]$, it is bounded by $|f''| < M_2$, so we have

$$|c_n| \leq \frac{1}{n^2}\cdot\frac{2}{\pi}\int_0^{\pi} |f''(x)\sin(nx)|dx \leq \frac{2M_2}{n^2}.$$

It follows by a comparison test that $\sum |c_n| < \infty$, so the Fourier series converges uniformly to the initial condition $f(x) = u(x, 0)$.

As for the the second point, we only prove the case for $u_{xx}$, as the proofs of the others follow in a similar spirit. The series representation of $u_{xx}$ is given by

$$-\sum_{n=1}^{\infty} c_n n^2 \sin(nx) e^{-n^2kt}.$$

Since $|f| < M$, we have

$$|c_n| \leq \frac{2}{\pi}\int_0^{\pi} |f(x)\sin(nx)|dx \leq 2M,$$

Furthermore, for all $t \geq t_0 > 0$, it's clear that $\sum n^2 e^{-n^2kt} < \infty$, so by an M-test we see that the series converges uniformly on all sets of the form

$$(x, t) \in D := [0, \pi] \times [t_0, \infty).$$

But then for any point $(x, t) \in [0, \pi] \times (0, \infty)$, we may pick $t_0$ small enough so as to allow us to differentiate the series term by term. This implies that the series representations satisfy

$$u_t = ku_{xx},$$

finishing the proof. $\square$

# 5 The Energy Principle

The energy principle allows us to establish heat conservation and uniqueness properties of the heat equation. It is also more generally used for other PDE such as the wave equation for similar purposes.

We define the physical heat energy function as

$$H(t) = \int u(x,t)dx.$$

If we have a $C^2$ solution $u(x,t)$ of the heat equation on a finite interval $[a,b]$, then we have by Leibniz,

$$H'(t) = \int_a^b u_t(x,t)dx = \int_a^b ku_{xx}(x,t)dx = ku_x(x,t)|_a^b,$$

so if the the heat flux, $-ku_x$ through the ends are the same, then the heat energy is constant. In other words, if the amount of heat entering and leaving either of the ends are equal, then heat energy is conserved through time.

If instead we have a solution on $\mathbb{R}$, it's natural to assume that $u_x$ vanishes for $|x| \to \infty$, which will give us the same result.

On the other hand, we also have the mathematical energy

$$E(t) = \int \frac{1}{2}u^2(x,t)dx,$$

which can be thought of as an $L^2$ norm of sorts. To analyze the change of energy in time, we write

$$\begin{aligned} E'(t) &= \frac{d}{dt}\int_a^b \frac{1}{2}u^2dx \\ &= \int_a^b uu_t dx \\ &= \int_a^b kuu_{xx}dx \\ &= kuu_x|_a^b - k\int_a^b u_x^2 dx. \end{aligned}$$

now if we have either homogeneous Dirichlet or Neumann boundary conditions, the first term vanishes, and we're left with

$$E'(t) \le 0,$$

so the total energy is decreasing over time. In particular we have $E(t) \le E(0)$ for all $t > 0$, which will allow us to prove uniqueness of any $C^2$ solution. Indeed, suppose $u_1$ and $u_2$ are $C^2$ solutions of the initial boundary value problem:

$$\begin{aligned} u_t - ku_{xx} = f(x,t) \quad 0 < x < L, \quad t > 0 \; u(0,t) &= g(t) \quad t > 0 \\ u(L,t) = h(t) \quad t > 0 \\ u(x,0) = \varphi(x) \quad 0 < x < L. \end{aligned}$$

Then $u = u_1 - u_2$ satisfies the above but with $f \equiv g \equiv h \equiv \varphi \equiv 0$. Thus we'd have

$$0 \le \int_0^L u^2(x,t)dx \le \int_0^L u^2(x,0)dx = 0,$$

which implies $u \equiv 0$, or $u_1 \equiv u_2$.

# 6 The Maximum Principle

Another tool that will allow us to deduce uniqueness is the maximum principle, which states that if $u(x,t)$ satisfies the heat equation on some rectangle

$$(x,t) \in R = [0,L] \times [0,T],$$

then $u$ achives its maximum somewhere on the bottom or lateral sides of $R$, i.e. $x = 0$, $x = L$, or $t = 0$. The proof of this is reminiscent of the proof for the maximum modulus principle from complex analysis, which is just a more general statement for holomorphic functions on opens sets of the complex plane.

The proof goes like this. Suppose $u(x,t)$ satisfies the heat equation on the rectangle $R$. Let $M$ be the maximum value of $u(x,t)$ on the three sides $t = 0$, $x - 0$, and $x = L$. Define a new funtion $v(x,t) = u(x,t) + \epsilon x^2$ for any $\epsilon > 0$. It satisfies

$$v_t - kv_{xx} = u_t - ku_{xx} - 2k\epsilon = -2k\epsilon < 0. \tag{7}$$

Suppose $v$ achieves a maximum on the interior of $R$ at some point $(x_0,t_0)$. Then $v_t = 0$ and $v_{xx} \le 0$ at $(x_0,t_0)$, but this contradicts (7). So, suppose instead $v$ achieves a maximum on the boundary $\partial R$, and in particular on the top edge, $t = T$, at some point $(x_0,t_0)$. Then $v_t \ge 0$ and $v_{xx} \le 0$ at $(x_0,t_0)$, but this contradicts (7) again.

Therefore $v$ achieves its maximum somewhere on the bottom edge or lateral sides, so in particular we have

$$v(x,t) \le M + \epsilon L^2.$$

But this implies

$$u(x,t) \le M + \epsilon(L^2 - x^2),$$

and we can take $\epsilon \to 0$ to see that $u(x,t) \le M$ on $R$, which is what we wanted to prove.

We may now prove uniqueness using the maximum principle, in essentially the same fashion as we did using the energy principle. Suppose $u_1, u_2$ are two solutions of the initial boundary value problem. Then $u = u_1 - u_2$ satisfies the problem with zero initial and boundary conditions, so by the maximum principle, $u(x,t) \le 0$ for all $(x,t)$. By a symmetric argument, we may obtain $u(x,t) \ge 0$, so that together this implies $u \equiv 0$, or that $u_1 \equiv u_2$. Thus the solutions to the heat equations are unique.