Overview of General Topology

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Contents

1 The Framework

Let X be a nonempty set. Then we define a *topology* on X to be a family $\mathcal{T}$ of subsets of X which satisfies:

- Contains $\emptyset$ and X.
- Closed under arbitrary unions.
- Closed under finite intersections.

Members of $\mathcal{T}$ will be called open sets, and their complements will be called closed sets. Note the similarity in definition to a σ-field. Indeed, both can be thought of as sets of "useful" information. In the σ-field case, useful would be measureability, whereas in topology, useful carries a notion of openness.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_1$ is *weaker* (or *coarser*) than $\mathcal{T}_2$, and likewise $\mathcal{T}_2$ is *stronger* (or *finer*) than $\mathcal{T}_1$. Note that the *trivial topology* containing only $\emptyset$ and X is the weakest, and the *discrete topology* where every set is open is the strongest. For any collection $\mathcal{E} \subset 2^X$, there is a unique weakest topology $\mathcal{T}(\mathcal{E})$ on X which contains $\mathcal{E}$. Similar to the notation $\sigma(\cdot)$, we call this the topology *generated* by $\mathcal{E}$. It turns out that $\mathcal{T}(\mathcal{E})$ is just the collection containing $\emptyset$, X, and all unions of finite intersections of members of $\mathcal{E}$.

There is also the concept of a *base* for a topology. A base for $\mathcal{T}$ at $x \in X$ is a family $\mathcal{N} \subset \mathcal{T}$ such that

- $x \in V$ for all $V \in \mathcal{N}$.
- if $U \in \mathcal{T}$ and $x \in U$, then there exists $V \in \mathcal{N}$ such that $x \in V$ and $V \subset U$.

A base for $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ containing a neighborhood base for $\mathcal{T}$ at every point $x \in X$. Essentially, a base is just a collection of open sets which we decide to call neighborhoods. For example, if X is a metric space, the collection of all open balls forms a base.

The notion of convergence of a sequence $\{x_n\}$ in a topological space X is as follows. We say $\{x_n\}$ converges to $x \in X$ if for every neighborhood U of x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$.

Finally, we have the separation axioms. Without assuming any of these, there is only so much we can do with general topology.

- T_0: If $x \neq y$, there is an open set containing x but not y or an open set containing y but not x.
- T_1: If $x \neq y$, there is an open set containing y but not x.
- T_2: If $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$.
- T_3: X is a T_1 space, and for any closed set $A \subset X$ and any $x \in A^c$, there are disjoint open sets U, V with $x \in U$ and $A \subset V$.
- T_4: X is a T_1 space, and for any disjoint closed sets A, B in X there are disjoint open sets U, V with $A \subset U$ and $B \subset V$.

We also call T_2 spaces *Hausdorff*, T_3 spaces *regular*, and T_4 spaces *normal*. Note that $T_0 \subset T_1 \subset T_2 \subset T_3 \subset T_4$. It turns out that most nice spaces are at least Hausdorff, or become Haussdorf after simple modifications. In fact, all metric spaces are normal, and therefore Haussdorf.

2 Continuous Maps

A map $f : X \to Y$ is *continuous* if $f^{-1}(V)$ is open in X for every open $V \subset Y$. This turns out to be equivalent to the $\epsilon - \delta$ definition of continuity for metric spaces.

Let X be any set and $\{f_\alpha : X \to Y_\alpha\}$ be a family of maps from X to some topological spaces Y_α. Then there is a unique weakest topology $\mathcal{T}$ on X which makes all the f_α continuous. Namely, it is the *weak topology* generated by sets of the form $f_\alpha^{-1}(U_\alpha)$, where U_α is open in Y_α.

An important example of this is the *product topology* on $X = \prod_{\alpha \in A} X_\alpha$, the topology generated by the projection maps

$$\pi_\alpha : X \to X_\alpha.$$

A closure property of the product topology is that if each X_α is Hausdorff, then so is X. Indeed, if x and y are distinct points of X, then $\pi_\alpha(x) \neq \pi_\alpha(y)$ for some α. Letting U and V be disjoint neighborhoods of $\pi_\alpha(x)$ and $\pi_a(y)$ in X_α, we note that $\pi_a^{-1}(U)$ and $\pi_a^{-1}(V)$ are disjoint neighborhoods of x and y in X, which implies that X is Hausdorff.

Proposition 2.1

If X_α and Y are topological spaces and $X = \prod_{\alpha \in A} X_\alpha$, then $f : Y \to X$ is continuous if and only if $\pi_\alpha \circ f$ is continuous for each α.

Proof. If $\pi_\alpha \circ f$ is continuous for each α, then $f^{-1}(\pi_\alpha^{-1}(U_\alpha))$ is open in Y for every open U_α in X_α. Since X is generated by π_α, it follows that f is continuous.

Conversely, if f is continuous, then the composition $\pi_\alpha \circ f$ is continuous since π_α is continuous as well. □

We will be concerned with certain classes of real/complex-valued functions on topological spaces. Let $B(X, \mathbb{F})$ be the space of bounded, $\mathbb{F}$-valued mappings on X. Similarly, we use $C(X, \mathbb{F})$ and $BC(X, \mathbb{F})$ to denote continuous and bounded continuous functions, respectively. Usually we will omit the $\mathbb{F}$ since the difference between $\mathbb{R}$ and $\mathbb{C}$ isn't important.

Note that $B(X)$, $C(X)$, and $BC(X)$ are all vector spaces. Furthermore, we can equip them with the uniform (sometimes called sup, or infinity) norm:

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

This gives us a metric function $\rho(f, g) = \|f - g\|_\infty$, the metric of uniform convergence. It turns out that $B(X)$ and $BC(X)$ are also *Banach spaces*, i.e. complete normed vector spaces, and $C(X)$ is Banach if X is compact.

Theorem 2.2

If X is a topological space, $B(X)$ and $BC(X)$ are Banach spaces. Furthermore $C(X)$ is Banach if X is compact.

Proof. If (f_n) is uniformly Cauchy, then $(f_n(x))$ is Cauchy for each x and so converges. So set $f(x) = \lim_n f_n(x)$ for each x, and note that

$$\|f_n - f_m\|_\infty \to 0,$$

for fixed n, m sufficiently large. Note that for all x,

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty.$$

Therefore,

$$\|f_n - f\|_\infty \leq \|f_n - f_m\|_\infty \to 0,$$

as $n \to \infty$.

Now, to verify that $BC(X)$ is also complete, we just need to show that it is a closed subspace of $B(X)$. Suppose (f_n) is a sequence in $BC(X)$ and $\|f_n - f\|_\infty \to 0$. Clearly f is bounded. Furthermore f is continuous since the f_n are a sequence of continuous functions converging uniformly (use the $\epsilon/3$ argument).

For the final claim, note that continuous on a compact set is bounded, so $C(X)$ where X is compact reduces to $BC(X)$. □

2.1 Two Important Construction Theorems

Theorem 2.3 (Urysohn's Lemma)

Let X be a normal space. If A and B are disjoint closed sets in X, there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof. The idea is to construct a map that adheres to "topographic level curves". Let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rational numbers in $(0,1)$. We wish to construct a family $\{U_r\}_{r\in\Delta}$ of open sets in X such that

$$A \subset U_r \subset B^c \quad \text{and} \quad \overline{U}_r \subset U_s \text{ for } r < s.$$

Since X is normal, there are disjoint open sets V, W such that $A \subset V$ and $B \subset W$. Let $U_{1/2} = V$. Then we have

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset W^c \subset B^c.$$

We construct the rest of the U_r's by induction. Namely, we apply the same argument as above to any pair of consecutive sets in the chain, obtaining a halfway set in between. In this way we may obtain all U_r for $r \in \Delta$. Let $U_1 = X$.

We now construct the desired function f. Define

$$f(x) = \inf\{r : x \in U_r\}.$$

For $x \in A$, clearly $f(x) = 0$, and for $x \in B$, we have $f(x) = 1$. To check continuity, it suffices to check preimages of all open half-lines, for they generate the topology on $\mathbb{R}$. Note that $f(x) < t$ if and only if

$$x \in \cup_{r<t} U_r,$$

which is open. Furthermore, $f(x) > t$ if and only if

$$x \in \cap_{r>t}(\overline{U}_r)^c,$$

which is open. Thus f is continuous. □

Theorem 2.4 (Tietze Extension Theorem)

Let X be a normal space. If A is a closed subset of X and $f \in C(A)$, there exists an extension $F \in C(X)$ such that $F|_A = f$.

Proof. First, we assume $f \in C(A, [a,b])$, and construct an extension $F \in C(X, [a,b])$. Further note that w.l.o.g. we may assume $[a,b] = [0,1]$.

Our strategy will be to construct a series of continuous functions $\sum g_n$ which converge uniformly to our desired extension F, whilst also converging to f on A. In particular, we want (g_n) to satisfy

- $0 \leq g_n \leq 2^{n-1}/3^n$ on X.
- $0 \leq f - \sum_{j=1}^n g_j \leq (2/3)^n$ on A.

We construct them recursively. Starting with g_1, let $B_1 = f^{-1}([0,1/3])$ and $C_1 = f^{-1}([2/3,1])$. These are disjoint closed subsets of A, and since A is closed, they are closed in X. Then we may use Urysohn's lemma to obtain a continuous function $g_1 : X \to [0,1/3]$ with $g_1 = 0$ on B_1 and $g_1 = 1/3$ on C. Clearly this satisfies the two conditions above.

For the recursive step, suppose we've found $g_{1:n-1}$. By the same reasoning, we may construct a $g_n : X \to [0, 2^{n-1}/3^n]$ such that $g_n = 0$ on the set where

$$f - \sum_{j=1}^{n-1} g_j \leq 2^{n-1}/3^n,$$

and $g_n = 2^{n-1}/3^n$ on the set where

$$f - \sum_{j=1}^{n-1} g_j \geq (2/3)^n.$$

Finally, let $F = \sum_{j=1}^{\infty} g_j$. By the M-test, $\sum g_n$ converges uniformly to F, and since each g_n is continuous, we know that F is continuous on X. Furthermore, on A we have $0 \leq f - F \leq (2/3)^n$, so that $f = G$ on A.

Now, to show the statement where $[a,b]$ is replaced with $\mathbb{R}$, we use the transformation:

$$g = \frac{f}{1+|f|}.$$

Note that $g \in C(A, (-1,1))$, so there exists $G \in C(X, [-1,1])$ with $G|_A = g$. Let $B = G^{-1}(\{-1,1\})$, which is disjoint from A. Then by Urysohn's lemma there is an

$h \in C(X, [0, 1])$ with $h = 0$ on B and $h = 1$ on A, so that $hG = G$ on A and $|hG| < 1$ everywhere else, so

$$F = \frac{hG}{1 - |hG|}$$

is the desired extension of f. $\square$

3 Compact Spaces

We say a topological space X is *compact* if every open cover $\{U_\alpha\}_{\alpha \in A}$ has a finite subcover $\{U_\alpha\}_{\alpha \in B}$, where B is a finite subset of A. A subset Y of X is compact if it is compact in the relative topology. We say Y is *precompact* is its closure is compact.

We now list some facts about compact spaces, whose proofs are primarily just set theoretic exercises.

(i) A closed subset of a compact space is compact.
(ii) A compact subset of a Hausdorff space is closed.
(iii) Every compact Hausdorff space is normal.
(iv) If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.
(v) If X is compact and Y is Hausdorff, then any continuous bijection is a homeomorphism.
(vi) If X is compact then every sequence in X has a convergent subsequence.

Turning now to metric spaces, we have the following characterization of compactness:

Theorem 3.1

If E is a subset of a metric space. Then the following are equivalent:

(i) E is compact.
(ii) E is sequentially compact. That is, every sequence in E has a subsequence converging to a point in E.
(iii) E is totally bounded, i.e. given any $\epsilon > 0$, it can be covered by finite ϵ-balls, and E is complete.

Proof. (i) $\implies$ (ii): Suppose there is a sequence (x_n) with no convergent subsequence in E. Then at each $x \in E$, there is a ball B_x which contains at most finitely many of the x_n. But then the open covering $(B_x)_{x \in E}$ cannot have a finite subcovering, for otherwise the sequence (x_n) would be finite.

(ii) $\implies$ (iii): Suppose E is not totally bounded. Then let $\epsilon > 0$ be such that E cannot be covered by finitely many ϵ-balls. Construct a sequence $(x_n) \subset E$ recursively by first picking $x_1 \in E$ arbitrarily, then picking $x_n \in E \setminus \cup_{j=1}^{n-1} B(\epsilon, x_j)$. Then for all i, j we have $d(x_i, x_j) > \epsilon$, so there can be no convergent subsequence. On the other hand, suppose E is not complete. Then there is a Cauchy sequence (x_n) which cannot have a limit in E. But then it cannot have a convergent subsequence, since limits of Cauchy sequences are unique. Thus by contradiction E must be totally bounded and complete.

(iii) $\implies$ (ii): Let (x_n) be a sequence in E. We may cover E with finitely many balls of radius 2^{-1}, so that there must exist a ball B_1 containing infinitely many of the x_n. Say, $x_n \in B_1$ for $n \in N_1$. Next, repeat the argument for $E \cap B_{k-1}$, to obtain a ball B_k of radius 2^{-k} containing infinitely many of the x_n, for $n \in N_k \subset N_{k-1}$. Then by taking a $x_{n_k} \in B_k$ for each k, we've obtained a Cauchy sequence (x_{n_k}) which converges to a point in E due to completeness. Thus we've found a convergent subsequence of (x_n).

(ii), (iii) $\implies$ (i): Since E can be covered in finitely many ϵ-balls, it suffices to show that if $\{V_\alpha\}_{\alpha \in A}$ is an open cover of E, then there exists an $\epsilon > 0$ such that every ϵ-ball intersecting with E is contained in one of the V_α. Suppose otherwise, that for each $n \in \mathbb{N}$ there is a ball B_n of radius 2^{-n} intersecting E such that B_n is contained in none of the V_α. Pick $x_n \in B_n \cap E$. Then assuming (ii), we may pass to a subsequence (x_{n_k}) and obtain a limit in E, and in particular contained in one of the V_α. But for large enough k, we'd get that $B_{n_k} \subset V_\alpha$ since V_α is open, a contradiction. Thus E must be compact. $\square$

Note that in the special case of $\mathbb{R}^n$, totally bounded is equivalent to bounded, and complete is equivalent to closed, so from the above result we obtain Heine-Borel:

Corollary 3.2 (Heine-Borel)
Let $E \subset \mathbb{R}^n$. Then E is compact if and only if E is closed and bounded.

3.1 Locally Compact Hausdorff Spaces

We say a topological space X is *locally compact* if every point has a compact neighborhood. Here we define a neighborhood of $x \in X$ to be a set $A \subset X$ such that x is contained in the interior of A, which is the largest open set contained in A, given by $A^o = A \setminus \overline{A}$. We will be concerned with locally compact Hausdorff (LCH) spaces.

Lemma 3.3
If X is an LCH space and $K \subset U \subset X$ where K is compact and U is open, there exists a precompact open V such that

$$K \subset V \subset \overline{V} \subset U.$$

Proof. We first show that for a point $x \in U$, there is a compact neighborhood N of x such that $N \subset U$. Note that w.l.o.g. we may assume $\overline{U}$ is compact, for otherwise replace U by $U \cap F^o$, where F is a compact neighborhood of x. Now, since ∂U is compact, and X is Hausdorff, we may construct a collection of open sets which cover both x and ∂U, and then retrieve from this a finite open subcover. In particular, we may obtain open sets $V, W \subset \overline{U}$ with $x \in V$ and $\partial U \subset W$. Furthermore, $\overline{V}$ is a compact subset of $U \setminus W$, and so we may take $N = \overline{V}$.

Now, for each $x \in K$, choose a compact neighborhood N_x of x with $N_x \subset U$ as above. Then $\{N_x^o\}_{x \in K}$ is an open cover of K, so there is a finite open subcover $\{N_{x_j}^o\}_{j=1}^n$. We see that

$$V = \cup_{j=1}^n N_{x_j}^o$$

does the trick. $\square$

Theorem 3.4 (Urysohn's Lemma, LCH Version)
Let X be an LCH space and $K \subset U \subset X$ where K is compact and U is open. There exists $f \in C(X, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.

Proof. Let V be as in Lemma 3.3. Then $\overline{V}$ is normal, so applying Urysohn's Lemma 2.3 we get a function $f \in C(\overline{V}, [0,1])$ with $f = 1$ on K and $f = 0$ on ∂V. We then extend f to X by setting $f = 0$ on $\overline{V}^c$. It's easy to see f remains continuous by checking preimages of closed subsets of $[0,1]$. □

Theorem 3.5 (Tietze Extension Theorem, LCH Version)
Let X be an LCH space and $K \subset X$ where K is compact. If $f \in C(K)$, there exists an extension $F \in C(X)$ such that $F|_K = f$. Morever, F may be constructed to vanish outside a compact set.

Proof. For each point $x \in K$, there is a compact neighborhood $N_x \subset X$ of x. Since K is compact, we may obtain a finite open covering $C = \cup_{j=1}^n N_{x_j}^o$. Then $\overline{C} = \cup_{j=1}^n N_{x_j}$ is compact, since we may take the union of subcoverings for each N_{x_j}. Set $f = 0$ on ∂C. As $\overline{C}$ is a compact Hausdorff space, it is normal. Furthermore, $K \cup \partial C$ is a closed subset of $\overline{C}$, so by the Tietze Extension Theorem 2.4 we obtain an continuous extension of f to $\overline{C}$. We can then set $F = 0$ on $X \setminus \overline{C}$ as in the previous proof, and we're done. □

We breifly talk about the *one-point compactification* of any noncompact LCH space. Let $(X, \mathcal{T})$ be a noncompact LCH space. We can add a "point at infinity", denoted "∞", so that the space $X^* = X \cup \{\infty\}$ is compact when equipped with the topology $\mathcal{T}^*$ which contains all subsets U of X^* such that either

(i) U is an open subset of X, or
(ii) $\infty \in U$ and U^c is a compact subset of X.

It's easy to verify that the modifed space $(X^*, \mathcal{T}^*)$ is a compact Hausdorff space.

3.2 Product Spaces

Theorem 3.6 (Tychonoff's Theorem)
If each X_α is compact for $\alpha \in A$, then the product topology $\prod_{\alpha \in A} X_\alpha$ is compact.

The proof requires Zorn's Lemma or Axiom of Choice. As such the proof isn't that instructive and is omitted. Instead, we first prove the finite case. Then we prove a version for sequential compactness, which uses a nice diagonalization argument that will appear later in the proof of one of the Arzela-Ascoli theorems.

First, recall the concept of a base for a topology. It's easy to see that an equivalent definition for a base is any collection $\mathcal{B}$ of open sets such that every open set can be expressed as unions of sets in $\mathcal{B}$. We call sets in $\mathcal{B}$ *basic*.

Theorem 3.7 (Tychonoff's Theorem, Finite Case)
If A is a finite set, and X_α is compact for each $\alpha \in A$, then the product topology $\prod_{\alpha \in A} X_\alpha$ is compact.

Proof. It suffices to prove the case where $|A| = 2$, as the rest follows by induction. First, note that in order to verify compactness by checking each open cover, it is sufficient to simply check each basic open cover has a finite subcover.

Since the boxes $\{U \times V : U \in \mathcal{T}_1, V \in \mathcal{T}_2\}$ form a base for the product topology, suppose we have a basic covering of $X_1 \times X_2$ by boxes $(U_i \times V_i)_{i \in I}$. For any $x \in X_1$, this open cover covers $\{x\} \times X_2$, which is compact since X_2 is compact. Therefore, we can cover $\{x\} \times X_2$ by a finite number of open boxes $U_i \times V_i$. Taking the intersection of the U_i, we obtain an open neighborhood U_x of x such that $U_x \times X_2$ is finitely covered. Finally, since X_1 is compact, it only takes finitely many U_x to cover X_1, so we have a finite covering of $X \times Y$. $\square$

Theorem 3.8 (Tychonoff's Theorem, Sequential Version)
If each X_α is sequentially compact for $\alpha \in A$, where A is at most countably infinite, then the product topology $\prod_{\alpha \in A} X_\alpha$ is sequentially compact.

Proof. With the finite case already handled, let's assume A is countably infinite. So, we have

$$X = \prod_{n=1}^{\infty} X_n.$$

Let $(x^{(m)})$ be a sequence in X, so that each $x^{(m)}$ is itself a sequence:

$$x^{(m)} = (x_n^{(m)})_{n=1}^{\infty}.$$

Consider the first coordinate $x_1^{(m)}$. Since X_1 is compact, there is a subsequence

$$(x^{(m_{1,j})})_{j=1}^{\infty}$$

which converges in X_1. Now, suppose we have constructed a nested subsequence that converges on the first i spaces $X_1, \ldots, X_i$. Then we can extract a further subsequence

$$(x^{(m_{i+1,j})})_{j=1}^{\infty}$$

which also converges on X_{i+1} since X_{i+1} is compact.

Here is the trick: consider the diagonal subsequence

$$(x^{(m_{j,j})})_{j=1}^{\infty}.$$

It's easy to see that this converges in X_n for each n. It follows that this diagonal sequence converges in X, and hence X is compact. $\square$

3.3 Arzelà-Ascoli Theorems

A family $\mathcal{F} \subset C(X)$ is called *equicontinuous at* $x \in X$ if for every $\epsilon > 0$ there is a neighborhood U_x of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U_x$ and all $f \in \mathcal{F}$. The family $\mathcal{F}$ is called *equicontinuous* if it is equicontinuous at each $x \in X$. If X is a metric space, then we call $\mathcal{F}$ *uniformly equicontinuous* if the neighborhood U does not depend on x, i.e. there is a uniform $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $|x - y| < \delta$, for every $f \in \mathcal{F}$ and $x \in X$.

Also, $\mathcal{F}$ is said to be *pointwise bounded* if $\{f(x) : f \in \mathcal{F}\}$ is bounded for each $x \in X$.

Theorem 3.9 (Arzelà-Ascoli I)

Let X be a compact Hausdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded family in $C(X)$, then $\mathcal{F}$ is totally bounded, and thus the closure of $\mathcal{F}$ in $C(X)$ is compact.

Proof. Since $\mathcal{F}$ is equicontinuous, for each $x \in X$ there is a neighborhood U_x of x such that

$$|f(x) - f(y)| < \frac{\epsilon}{4}$$

for all $f \in \mathcal{F}$ and $y \in U_x$. Since X is compact, there is a finite subcovering $\cup_{j=1}^{n} U_{x_j}$ of X. By pointwise boundedness,

$$\{f(x_j) : f \in \mathcal{F}, 1 \leq j \leq n\}$$

is a bounded subset of $\mathbb{C}$, so there is a finite set $\{z_1, \ldots, z_m\} \subset \mathbb{C}$ that is $\frac{\epsilon}{4}$-dense in it. Denote $A = \{x_1, \ldots, x_n\}$ and $B = \{z_1, \ldots, z_m\}$. Then B^A, the set of functions from A to B, is finite. For each $\phi \in B^A$, define

$$\mathcal{F}_\phi = \{f \in \mathcal{F} : |f(x_j) - \phi(x_j)| < \frac{\epsilon}{4}, 1 \leq j \leq n\}.$$

Then $\cup_{\phi \in B^A} \mathcal{F}_\phi = \mathcal{F}$. Furthermore, note that for any $f, g \in \mathcal{F}_\phi$, and $x \in X$, we can pick j so that $x \in U_{x_j}$, to get

$$\begin{aligned}
|f(x) - g(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\
&\leq \frac{\epsilon}{4} + |f(x_j) - \phi(x_j)| + |g(x_j) - \phi(x_j)| + \frac{\epsilon}{4} \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Therefore, each $\mathcal{F}_\phi$ has diameter at most ϵ, which implies that $\mathcal{F}$ is totally bounded. Since X is compact, $C(X) = BC(X)$, which is complete. Then the closure of a totally bounded set is totally bounded and complete. Thus $\mathcal{F}$ is precompact in $C(X)$. $\square$

Corollary 3.10 (Special Case)

If $f_n : X \to \mathbb{R}^n$ is a sequence of functions from a compact metric space X to $\mathbb{R}^n$ which are equicontinuous and pointwise bounded, then there is a subsequence f_{n_k} which converges uniformly to a limit $f \in BC(X)$.

Another variant is for σ-*compact* LCH spaces. We say a space X is σ-compact if it is the countable union of compact sets.

Theorem 3.11 (Arzelà-Ascoli II)

Let X be a σ-compact LCH space. If (f_n) is an equicontinuous, pointwise bounded sequence in $C(X)$, then there is a subsequence (f_{n_k}) converging to some $f \in C(X)$ uniformly on compact sets.

Proof. Use a diagonalization argument as in the proof of the sequential Tychonoff theorem. $\square$