SPECTRAL RADIUS OF A RANDOM MATRIX

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1. Introduction

1.1. Setting. Given a collection $\{a_{jk}\}_{j,k\geq 1}$ of i.i.d. complex random variables with $\mathbb{E}[a_{11}] = 0$ and $\mathbb{E}[|a_{11}|^2] = 1$, let

$$A_n = (a_{j,k})_{1\leq j,k\leq n}$$

be the n-th Girko matrix. We will be interested in the normalized collection of random matrices $\frac{1}{\sqrt{n}}A_n$, with characteristic polynomials given by

$$p_n(z) = \det\left(z - \frac{A_n}{\sqrt{n}}\right).$$

The roots of $p_n(z)$ form the spectrum Λ_n of $\frac{1}{\sqrt{n}}A_n$, and the spectral radius of $\frac{1}{\sqrt{n}}A_n$ is defined by

$$\rho_n = \max_{\lambda\in\Lambda_n} |\lambda|.$$

The main result we want to show is

Theorem 1.1 (Spectral radius). *We have* $\lim_{n\to\infty} \rho_n = 1$ *in probability, i.e. for all* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}(|\rho_n - 1| \geq \varepsilon) = 0.$$

We will be giving a detailed exposition of the proof given in [1]. In doing so, we will also be giving detailed proofs of results from [2].

1.2. Proof Intuition. Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, and denote $D = D_1$. The idea of the proof of Theorem 1.1 is to view $p_n(z)$ as a random holomorphic function on $z \in \mathbb{C}\cup\{\infty\}\setminus\overline{D}$ and prove it converges in law to a particular random holomorphic function. In the subsequent section we will make precise the notion of convergence in law.

For convenience, we identify $\mathbb{C}\cup\{\infty\}\setminus\overline{D}$ with D by considering the *reciprocal polynomial* $q_n(z) = z^n p_n(1/z)$. Then for $z \in D$, we have

$$q_n(z) = \det\left(I - z\frac{A_n}{\sqrt{n}}\right).$$

1.3. Convergence in law. Let $H(D)$ be the set of complex analytic functions on D, equipped with the topology of uniform convergence on compact subsets, which we refer to as the *compact-open topology.* If $\{K_j : j \geq 1\}$ is an exhaustion of D by compact sets, then this topology can be induced by the metric

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}.$$

Basic sets of this topology consist of finite intersections of sets of the form

$$B_{f,K,\varepsilon} = \{g \in H(D) : \|f - g\|_K < \varepsilon\}$$

where $f \in H(D)$, $K \subset D$ compact, and $\varepsilon > 0$.

Let (Ω, Σ, P) be the probability space associated to the Girko matrices. Equip $H(D)$ with the Borel σ-field, so that we may treat $q_n(z)$ as a random variable from $\Omega \to H(D)$. Convergence in law in this setting can then be interpreted as follows: $q_n \to q$ in law if for every Borel set $B \in \mathcal{B}(H(D))$,

$$\mathbb{P}(q_n(z) \in B) \to \mathbb{P}(q(z) \in B).$$

2. Proof of Theorem 1.1

The main part of the proof is to show

THEOREM 2.1 (Convergence of reciprocal polynomial). *As $n \to \infty$, $q_n(z)$ converges in law to a random holomorphic function on D of the form*

$$\kappa e^{-F} = \sqrt{1 - z^2 \mathbb{E}[a_{11}^2]}\, e^{-F}$$

where F is a random holomorphic function on D defined by

$$F(z) = \sum_{k=1}^{\infty} X_k \frac{z^k}{\sqrt{k}}$$

where $\{X_k\}_{k\geq 1}$ is a sequence of independent complex Gaussian random variables such that

$$\mathbb{E}[X_k] = 1, \quad \mathbb{E}[|X_k|^2] = 1, \quad \mathbb{E}[X_k^2] = \mathbb{E}[a_{11}^2]^k.$$

Once we've shown this, the main theorem follows more or less by applying definitions of weak convergence. Since $|\mathbb{E}[a_{11}^2]| \leq \mathbb{E}[|a_{11}|^2] = 1$, we have for every $r \in (0, 1)$,

$$\inf_{z \in \overline{D}_r} \left\{ |\kappa(z)| e^{-\mathrm{Re}(F(z))} \right\} = \inf_{z \in \overline{D}_r} |f(z)| > 0.$$

Now, note that $\phi(f) = \inf_{z \in \overline{D}_r} |f(z)|$ is a continuous mapping from $H(D)$ to $\mathbb{R}$. Indeed, if $f_n \to f$ in the compact-open topology, then $f_n \to f$ uniformly on $\overline{D}_r$, and so $\phi(f_n) \to \phi(f)$. Therefore, by the convergence in law of $q_n(z)$ and the continuous mapping theorem, we have

$$\inf_{z \in \overline{D}_r} |q_n(z)| \xrightarrow{\text{law}} \inf_{z \in \overline{D}_r} \left\{ |\kappa(z)| e^{-\mathrm{Re}(F(z))} \right\}.$$

It follows then that for every $r \in (0, 1)$,

$$\begin{aligned}
\mathbb{P}\left(\rho_n < \frac{1}{r}\right) &= \mathbb{P}\left(\inf_{|z| \geq \frac{1}{r}} |p_n(z)| > 0\right) \\
&= \mathbb{P}\left(\inf_{z \in \overline{D}_r} |q_n(z)| > 0\right) \to \mathbb{P}\left(\inf_{z \in \overline{D}_r} \left\{ |\kappa(z)| e^{-\mathrm{Re}(F(z))} \right\} > 0\right) = 1.
\end{aligned}$$

Hence for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(\rho_n \geq 1 + \varepsilon) = 0. \tag{1}$$

To get the other half, we proceed by contradiction. That is, suppose there were a subsequence (n_k) such that $\mathbb{P}(\rho_{n_k} < 1-\varepsilon) > \delta > 0$ for all k. Then for test functions φ approximating $\mathbf{1}_{|x|<1-\varepsilon}$, we'd see that

$$\mathbb{P}\left(\int \varphi d\mu_{n_k} \approx 1 > (1-\varepsilon)^2 \approx \int \varphi d\mu\right) > \delta$$

where μ is the circular law. But this contradicts convergence in probability to the circular law, and hence we have

$$\lim_{n\to\infty} \mathbb{P}(\rho_n < 1-\varepsilon) = 0. \tag{2}$$

Putting (1) and (2) together, we get Theorem 1.1.

3. Proof of Theorem 2.1

We can rewrite

$$q_n(z) = 1 + \sum_{k=1}^{n} (-z)^k P_k^{(n)}.$$

where

$$P_k^{(n)} = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} n^{-k/2} \det(A_n(I)), \quad A_n(I) = \{a_{jk}\}_{j,k\in I}.$$

Later we will prove that $\{q_n\}_{n\geq 1}$ is a tight sequence. This will allow us to apply the following theorem.

LEMMA 3.1 (Reduction to convergence of coefficients). *Let $\{f_n\}_{n\geq1}$ be a tight sequence of random elements of $H(D)$, and let us write, for every $n \geq 1$, $f_n(z) = \sum_{k=0}^{\infty} z^k P_k^{(n)}$. If for every $m \geq 0$,*

$$(P_0^{(n)}, \dots, P_m^{(n)}) \xrightarrow[n\to\infty]{law} (P_0, \dots, P_m)$$

for a sequence of scalar random variables $\{P_m\}_{m\geq0}$, then $f = \sum_{k=0}^{\infty} z^k P_k$ is well-defined in $H(D)$ and

$$f_n \xrightarrow{law} \sum_{k=0}^{\infty} z^k P_k.$$

We then reduce further to bounded entries.

LEMMA 3.2 (Truncation). *For $M > 0$ define*

$$A_n^{(M)} = \left\{a_{ij}^{(M)}\right\} \quad \textit{where} \quad a_{ij}^{(M)} = a_{ij}\,\mathbf{1}_{|a_{ij}|<M} - \mathbb{E}\left[a_{ij}\,\mathbf{1}_{|a_{ij}|<M}\right]$$

and

$$P_k^{(n,M)} = \sum_{\substack{I\subset\{1,\dots,n\}\\ |I|=k}} n^{-k/2} \det(A_n^{(M)}(I)) \quad \textit{where} \quad A_n^{(M)}(I) = \left\{a_{ij}^{(M)}\right\}_{i,j\in I}.$$

Let $k \geq 1$. If there exists $\left\{\left(Y_1^{(M)} \dots, Y_2^{(M)}\right)\right\}_{M\geq1}$ and a random vector $(Y_1, \dots, Y_k)$ such that for all $M \geq 1$,

$$\left(P_1^{(n,M)}, \dots, P_k^{(n,M)}\right) \xrightarrow{law} \left(Y_1^{(M)} \dots, Y_k^{(M)}\right), \quad \textit{and} \quad \left(Y_1^{(M)} \dots, Y_k^{(M)}\right) \xrightarrow{law} (Y_1, \dots, Y_k),$$

then

$$(P_1^{(n)}, \dots, P_k^{(n)}) \xrightarrow{law} (Y_1, \dots, Y_k).$$

Now note that we can formally write

$$B_n = \ln\left(1 - z\frac{A_n}{\sqrt{n}}\right) = \sum_{k=1}^{\infty}\left(\frac{\mathrm{Tr}A_n}{\sqrt{n}}\right)^k \frac{z^k}{k}$$

where the RHS converges for z small enough. As $\det(e^{B_n}) = e^{\mathrm{Tr}B_n}$, we conclude that

$$q_n(z) = \exp\left(\left(\frac{\mathrm{Tr}A_n}{\sqrt{n}}\right)^k \frac{z^k}{k}\right)$$

Thus, $(P_1^{(n)}, \ldots P_k^{(n)})$ is a polynomial function of $(\frac{\mathrm{Tr}A_n}{n^{1/2}}, \ldots, \frac{\mathrm{Tr}A^k n}{n^{k/2}})$ that does not depend on n. Consequently, we can reduce the convergence in law of $(P_1^{(n)}, \ldots, P_k^{(n)})$ to the convergence in law of the first k moments of a bounded random matrix. The rest of the proof of Theorem 2.1 then involves using combinatorial arguments to study the the asymptotic behavior of these moments.

REMARK 1. Observe that we are studying the asymptotic behavior of random matrices by first fixing k, the number of moments, and then letting $n \to \infty$. The intuitive reason is that we are concerned with the location of zeros of the characteristic polynomial, and these zeros can be detected by a large enough k. We can contrast this approach to the proof of the Bai-Yin theorem, where one first sends $n \to \infty$, and then chooses the number of moments k (often $\sim \log n$) to study.

4. REDUCTION TO COEFFICIENTS

4.1. Tightness of $\{q_n\}$. In order to apply Lemma 3.1, we need to first show that the sequence $\{q_n\}$ is tight. The main idea is that it suffices to show tightness of $\{\|q_n\|_K\}$ for compact sets K, which can then in turn be shown by bounding the L^2 norm of q_n.

LEMMA 4.1. *Let f_n be a sequence of random analytic functions on D. If for all compact sets K, the sequence $\{\|f_n\|_K\}$ is tight, then $\{f_n\}$ is tight.*

PROOF. Let μ_n be the laws of f_n. Fix $\varepsilon > 0$ and let $\{K_j\}_{j\geq 1}$ be an exhaustion of D by compact sets. By tightness of $\{\|f_n\|_{K_j}\}$, we can take an increasing sequence (M_j) such that $\sup_n \mathbb{P}(\|f_n\|_{K_j} > M_j) \leq 2^{-j}\varepsilon$. Then consider the set

$$S = \{h \in H(D) : \|h\|_{K_j} \leq M_j, j \geq 1\}.$$

This is a locally uniformly bounded family and so by Montel's theorem it is precompact in $H(D)$. Therefore

$$\inf_n \mu_n(S) = \inf_n \mathbb{P}(\|f_n\|_{K_j} \leq M_j, j \geq 1) \geq \inf_n \left\{1 - \sum_{j=1}^{\infty} \mathbb{P}(\|f_n\|_{K_j} > M_j)\right\} \geq 1 - \varepsilon,$$

which implies tightness of $\{f_n\}$. □

LEMMA 4.2. *For $K \subset D$ compact there exists $\delta > 0$ such that for all $f \in H(D)$,*

$$\|f\|_K^p \leq (\pi\delta^2)^{-1} \int_{\overline{K}_\delta} |f(z)|^p dm(z)$$

for any $p > 0$, where $\overline{K}_\delta$ is the closure of the δ-neighborhood of K.

PROOF. We claim that

$$I(r) = \int_0^{2\pi} |f(z_0 + re^{i\theta})|^p d\theta$$

is a nondecreasing function of r. Without loss of generality, we will prove it for $z_0 = 0$. For $r \in [0, 1)$ define $\varphi : [0, 2\pi] \to \mathbb{C}$ to satisfy $\varphi(\theta)f(re^{i\theta}) = |f(re^{i\theta})|$. Then, extend I to $\mathbb{C}$ by defining

$$F(z) = \int_0^{2\pi} [f(ze^{i\theta})\varphi(\theta)]^p d\theta.$$

By integrating over any closed loop and applying Fubini, we get 0. So by Morera's theorem F_ρ is holomorphic. By the maximum principle,

$$\begin{aligned} I(r) = F(r) &\le \max_{|z|=\rho} |F_\rho(z)| \\ &\le \max_{|z|=\rho} \int |f(ze^{i\theta})|^p d\theta \\ &= \int_0^{2\pi} |f(\rho e^{i\theta})|^p d\theta = I(\rho), \end{aligned}$$

where the third line follows from rotational invariance of the integral. Hence I is nondecreasing.

Now pick $\delta > 0$ small enough so that $\overline{K}_\delta \subset D$. We have

$$|f(z)|^p \le \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta$$

for $0 \le r < \delta$. Therefore, we have

$$\pi\delta^2 |f(z)|^p \le \int_0^\delta r \int_0^{2\pi} |f(z + re^{i\theta})|^p d\theta dr = \int_{|\zeta - z| \le \delta} |f(\zeta)|^p dm(\zeta),$$

and taking supremum both sides over K gives the desired estimate. □

PROPOSITION 4.3. *The sequence* $\{q_n\}_{n\ge1}$ *is tight.*

PROOF. First, we bound $\mathbb{E}[|q_n(z)|^2]$ by a deterministic continuous function of z that is independent of n. Note that for $I, J \subset [n]$,

$$\mathbb{E}\left[\det(A_n(I))\overline{\det A_n(J)}\right] = \begin{cases} 0 & I \ne J, \\ (|I|)! & I = J. \end{cases}$$

Therefore

$$\mathbb{E}\left[P_k^{(n)} P_l^{(n)}\right] = \begin{cases} 0 & k \ne l, \\ n^{-k}\binom{n}{k}\mathbb{E}\left[|\det A_k|^2\right] \le 1 & k = l. \end{cases}$$

So, we have

$$\mathbb{E}[|q_n(z)|^2] \le 1 + \sum_{k=1}^n |z|^{2k}\mathbb{E}[|P_k^{(n)}|^2] \le \frac{1}{1 - |z|^2}.$$

Then, by Lemma 4.2 and Fubini we have

$$\begin{aligned} \mathbb{E}[\|q_n\|_K^2] &\le (\pi\delta^2)^{-1}\mathbb{E}\int_{\overline{K}_\delta} |q_n(z)|^2 dm(z) \\ &= (\pi\delta^2)^{-1}\int_{\overline{K}_\delta} \mathbb{E}|q_n(z)|^2 dm(z) \\ &\le (\pi\delta^2)^{-1}\int_{\overline{K}_\delta} \frac{1}{1 - |z|^2}. \end{aligned}$$

By picking δ so that $\overline{K}_\delta$ is strictly contained in D, this integral is finite. Hence we've shown that

$$\sup_n \mathbb{E}[\|q_n\|_K^2] < \infty.$$

To see that this implies tightness of $\{\|q_n\|_K\}$, suppose otherwise. Then there exists an $\varepsilon > 0$ so that for all M, there is a subsequence n_k for which $\mathbb{P}(\|q_{n_k}\|_K > M) > \varepsilon$. But then $\mathbb{E}[\|q_n\|_K^2] \geq M^2\varepsilon$, and since M is arbitrary we get a contradiction. Tightness of $\{q_n\}$ then follows from Lemma 4.1. □

4.2. Proof of Lemma 3.1. By Prokhorov's theorem, tightness of $\{f_n\}$ implies that every subsequence has a further subsequence which converges weakly to some f whose measure lies in $\mathcal{P}(H(D))$. Recall that for a sequence $\{f_n\}$ topological space, if every subsequence has a further subsequence which converges to the same f, then $f_n \to f$. Combining these facts we see that it suffices to show that any two subsequences which converge must converge to the same weak limit.

Let $\{f_{n_j}\}_{j\geq 1}$ and $\{f_{\tilde{n}_j}\}_{j\geq 1}$ be two subsequences of $\{f_n\}_{n\geq 1}$ that converge weakly to some random analytic functions g and $\tilde{g}$ whose measures lie in $\mathcal{P}(H(D))$. Write

$$g(z) = \sum_{k=0}^{\infty} Q_k z^k, \quad \text{and} \quad \tilde{g}(z) = \sum_{k=0}^{\infty} \tilde{Q}_k z^k,$$

so that $\{Q_k\}$ and $\{\tilde{Q}_k\}$ are two sequences of complex random variables.

Now, consider the coefficient map $T : H(D) \to \mathbb{C}^{\mathbb{Z}_{\geq 0}}$ defined for all $h \in H(D)$ and $k \in \mathbb{Z}_{\geq 0}$ by

$$T(h)_k = \frac{1}{k!}\frac{d^k}{dz^k}h(0).$$

This map is continuous, as from Cauchy's formula we have the bounds

$$|T(h)_k| \leq R^{-k}\|h\|_C,$$

where C is the boundary circle centered at 0 with radius R. From this we see that if $h_n \to h$ in the compact-open topology, then $T(h_n)_k \to T(h)_k$ for every k.

Furthermore this map is injective, with inverse $T^{-1} : \{\{a_k\} \in \mathbb{C}^{\mathbb{Z}_{\geq 0}} : \limsup_{k\to\infty} |a_k|^{1/k} \leq 1\} \to H(D)$ given by

$$(T^{-1}(a))(z) = \sum_{k=0}^{\infty} a_k z^k.$$

The inverse is continuous, for any compact subset of D must be strictly bounded less than 1, and so as $a \to 0$ we can apply the dominated convergence theorem to see that $\sum_{k=0}^{\infty} a_k z^k \to 0$. Therefore T is a homeomorphism.

Then, let

$$T_* : \mathcal{P}(H(D)) \to \mathcal{P}(\mathbb{C}^{\mathbb{Z}_{\geq 0}})$$

be the pushforward map . Since T is a homeomorphism, we get that T_* is injective. Indeed, if $T_*\mu = T_*\nu$, then we have for any Borel set $B \subset \mathcal{B}(H(D))$, $T(B)$ is Borel and so

$$\mu(B) = T_*\mu(T(B)) = T_*\nu(T(B)) = \nu(B).$$

Therefore T preserves convergence in law, and so for every $m \geq 0$,

$$(P_0^{(n_j)}, \dots, P_m^{(n_j)}) \xrightarrow[n\to\infty]{\text{law}} (Q_0, \dots, Q_m),$$
$$(P_0^{(\tilde{n}_j)}, \dots, P_m^{(\tilde{n}_j)}) \xrightarrow[n\to\infty]{\text{law}} (\tilde{Q}_0, \dots, \tilde{Q}_m).$$

By the assumption of Theorem 3.1, it follows that $\{Q_k\}$, $\{\tilde{Q}_k\}$, and $\{P_k\}$ all have the same distribution. Furthermore, injectivity of the pushforward tells us that $g \stackrel{d}{=} \tilde{g}$, and so we conclude that f_n converges in law to f.

5. Proof of Lemma 3.2

LEMMA 5.1. *Suppose for every $1 \le i \le j$ there exists a sequence $\{C_{j,M}\}_{M \ge 1}$ that goes to zero such that*

$$\mathbb{E}\left[\left|P_i^{(n,M)} - P_i^{(n)}\right|^2\right] \le C_{j,M}$$

for every $n, M \ge 1$. Then we have

$$(P_1^{(n)}, \ldots, P_j^{(n)}) \xrightarrow{law} (Y_1, \ldots, Y_j)\,.$$

PROOF. Let μ_n denote the probability measure on $\mathbb{C}^j$ induced by $(P_1^{(n)}, \ldots, P_j^{(n)})$ and let $BX(M)$ denote the box of side length $2M$ centered at the origin and aligned with the axes. Given a test function $\varphi \in C_0(X)$, it suffices to show that

$$\lim_{M\to\infty} \lim_{n\to\infty} \int \varphi \mathbf{1}_{BX(M)} d\mu_n = \lim_{n\to\infty} \lim_{M\to\infty} \int \varphi \mathbf{1}_{BX(M)} d\mu_n$$

since we know

$$\left(P_1^{(n,M)}, \ldots, P_j^{(n,M)}\right) \xrightarrow{\text{law}} \left(Y_1^{(M)} \ldots, Y_j^{(M)}\right), \quad \text{and} \quad \left(Y_1^{(M)} \ldots, Y_j^{(M)}\right) \xrightarrow{\text{law}} (Y_1, \ldots, Y_j)\,.$$

However, observe that

$$\begin{aligned}
\left|\int \varphi d\mu_n - \int \varphi \mathbf{1}_{BX(M)} d\mu_n\right| &\le \sup(\varphi)\mu_n(\mathbb{C}^j \setminus BX(M)) \\
&\le \sup(\varphi)\frac{1}{M^2}\int_{\mathbb{C}^j\setminus BX(M)} |X|^2 d\mu_n. \\
&\le \sup(\varphi)\frac{C_M}{M^2}
\end{aligned}$$

Thus the function

$$n \mapsto \int \varphi \mathbf{1}_{BX(M)} d\mu_n$$

converges uniformly as $M \to \infty$, allowing us to interchange the desired limits. $\square$

As a consequence of Lemma 5.1, it suffices to show that for each $k \ge 1$ there exists a sequence $\{C_M\}_{M\ge1}$ that goes to zero that

$$\mathbb{E}\left[\left|P_k^{(n,M)} - P_k^{(n)}\right|^2\right] \le C_M$$

for every n, M. But

$$\begin{aligned}
\mathbb{E}\left[\left|P_k^{(n,k)} - P_k^{(n)}\right|^2\right] &= n^{-k} \sum_{\substack{I\subset\{1,\ldots,n\}\\ |I|=k}} \mathbb{E}\left[|\det(A_n(I))^{(M)} - \det(A_n(I))|^2\right] \\
&= n^{(-k)}\binom{n}{k}\mathbb{E}\left[\left|a_{11}^{(M)}\cdots a_{1k}^{(M)} - a_{11}\cdots a_{1k}\right|^2\right] k! \\
&\le \mathbb{E}\left[\left|a_{11}^{(M)}\cdots a_{1k}^{(M)} - a_{11}\cdots a_{1k}\right|^2\right]
\end{aligned}$$

which goes to zero as $M \to \infty$.

6. Appendix

Here we present the proofs of two theorems that were used earlier. Note the similarities in their statements. Indeed, both proofs use a diagonalization argument. However, the topology of uniform convergence on compact sets is stronger than that of weak convergence, and so the first will require more work to show. The extra step required is to show equicontinuity. For a family $\mathcal{F}$, we say it is equicontinuous on a compact set K if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for $z, w \in K$ with $|z - w| < \delta$, we have $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.

REMARK 1. Both the compact-open topology and the topology of weak convergence are metrizable. We already gave a metric for the former; the latter can be equipped with the Levy metric:

$$d(F, G) = \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \text{ for all } x\}$$

Hence both theorems can be restated in terms of subsequential convergence.

LEMMA 6.1 (Diagonal argument). *Let $\{f_n\}$ be a sequence of functions, and $\{x_j\}$ be a countable sequence of points. If for every j, the set $\{f_n(x_j)\}$ is sequentially compact, then there exists a subsequence of $\{f_n\}$ which converges at every x_j.*

PROOF. Choose a subsequence $f_{n_1(k)}$ that converges on x_1. Then, recursively pick a further subsequence $f_{n_j(k)}$ from $f_{n_{j-1}(k)}$ which converges on x_j. Then the diagonal subsequence $f_{n(k)} := f_{n_k(k)}$ converges on all the x_j. □

THEOREM 6.2 (Montel). *Let $\mathcal{F}$ be a family of holomorphic functions on D. If $\mathcal{F}$ is uniformly bounded on compact sets, then $\mathcal{F}$ is normal, i.e. it is precompact in the compact-open topology on $H(D)$.*

PROOF. First, we show that $\mathcal{F}$ is equicontinuous on all compact subsets of D. This relies on the nature of holomorphic functions. Let K be compact and choose $r > 0$ small enough so that $D_{3r}(z) \subset D$ for all $z \in K$. Let $z, w \in K$ with $|z - w| < r$. Then from Cauchy's integral formula, we have

$$\begin{aligned} |f(z) - f(w)| &= \left| \frac{1}{2\pi i} \int_{\partial D_{2r}(w)} \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right] d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D_{2r}(w)} |f(\zeta)| \frac{|z - w|}{|\zeta - z||\zeta - w|} d\zeta \\ &\leq C|z - w| \end{aligned}$$

for some constant C. Hence $\mathcal{F}$ is equicontinuous on all compact sets.

Now, from the diagonal argument, we can find a subsequence $\{g_n\}$ of $\{f_n\} \subset \mathcal{F}$ which converges at every rational $q_n \in D$. Let K be any compact subset. Given $\varepsilon > 0$, let δ be as in the definition of equicontinuity, and let $\{D_\delta(w_1), \ldots, D_\delta(w_J)\}$ be a finite δ-net of K. There exists N large so that for $n, m > N$,

$$|g_m(w_j) - g_n(w_j)| < \varepsilon \quad \text{for all } j = 1, 2, \ldots, J.$$

Then if $z \in K$, $z \in D_\delta(w_j)$ for some $1 \leq j \leq J$ and so

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| < 3\varepsilon.$$

Therefore g_n converges uniformly in K. Finally, let $\{K_j\}$ be an exhaustion of D. Applying the diagonal argument one more time, with the K_j acting as points and convergence being interpreted as uniform convergence on these points, we obtain a subsequence which converges uniformly on all compact subsets. As the compact-open topology is metrizable, this is the same as being precompact. □

Theorem 6.3 (Prokhorov, forward direction). *Let* $\{\mu_n\}$ *be a collection of probability measures. If* $\{\mu_n\}$ *is tight then it is precompact in the space of probability measures equipped with the topology of weak convergence.*

Proof. Let F_n be the associated sequence of distribution functions, so $F_n(x) \in [0,1]$ for all x. By the diagonalization argument, tere exists a subsequence F_{n_k} that converges on all the rationals to some G. Define

$$F(x) = \inf\{G(q) : q \in \mathbb{Q}, q > x\}.$$

It is easy to verify that F is a right continuous, nondecreasing function. The extra condition of tightness enforces that it is the distribution function of a probability measure. Let M_ε be as in the definition of tightness. If $r < -M_\varepsilon$ and $s > M_\varepsilon$ are continuity points of F, then

$$\begin{aligned} 1 - F(s) + F(r) &= \lim_{k\to\infty} 1 - F_{n_k}(s) + F_{n_k}(r) \\ &\le \limsup_{n\to\infty} 1 - F_n(M_\varepsilon) + F_n(-M_\varepsilon) \le \varepsilon. \end{aligned}$$

Thus $\limsup_{x\to\infty} 1 - F(x) + F(-x) \le \varepsilon$, and so F must be the distribution function of a probability measure. □

References

[1] Charles Bordenave, Djalil Chafaï, and David García-Zelada. Convergence of the spectral radius of a random matrix through its characteristic polynomial, 2020.

[2] Tomoyuki Shirai. Limit theorems for random analytic functions and their zeros.