1. FRBL & Strong EPI (July 8, 2021)

Let X and W be independent, with W Gaussian and $Y = X + W$. Suppose V is a random variable such that $X \to Y \to V$ forms a Markov chain. Then we have

$$\lambda h(X) + (1-\lambda)h(W) + I(X;V) \le h(Y) + \lambda I(Y;V). \tag{1}$$

Note that without the terms $I(X;V) \le \lambda I(Y;V)$, this is the vanilla EPI.

The data processing inequality tells us that $I(X;V) \le I(Y;V)$, so for any $0 \le \eta \le \lambda$, to prove (1) it'd suffice to show

$$\lambda h(X) + (1-\lambda)h(W) + (1-\eta)I(X;V) \le h(Y) + (\lambda-\eta)I(Y;V). \tag{2}$$

But this is not generally true. For instance, take $\eta = \lambda$ and consider the extremal cases where X and Y are Gaussian and EPI is met with equality. Then (2) becomes

$$\lambda h(X) + (1-\lambda)h(W) + (1-\lambda)I(X;V) \le h(Y), \tag{3}$$

and we would be forced to have $I(X;V) = 0$, which is clearly not the case. However, one benefit of this form is that it may fit better with entropic form of FRBL. In particular, we can rewrite (1) as

$$(1+\lambda)h(X) + (1-\lambda)h(W) + (1-\lambda)h(V) + \lambda h(Y,V) \le (1+\lambda)h(Y) + h(X,V).$$

However, the term $h(Y,V)$ is difficult to deal with, so instead we may consider (3), which rewrites as

$$h(X) + (1-\lambda)h(W) + (1-\lambda)h(V) + \lambda h(X,V) \le h(Y) + h(X,V).$$

If we let $X = Z_1$, $W = Z_2$, $V = Z_3$, and $(X,V) = Z_4$, then we'd want the following constraints to hold a.s. in our coupling on the right:

$$\begin{aligned} Z_1 &= Z_4^{(1)} \\ Z_3 &= Z_4^{(2)} \\ Z_1 &\perp\!\!\!\perp Z_2. \end{aligned}$$

For each of these, we add the following regularizers (for now assume our r.v.'s are log-concave, so that we can make use of the Cover-Zhang result)):

$$\begin{aligned} &-\alpha[h(Z_1) - h((Z_1 + Z_4^{(1)})/2)] \\ &-\beta[h(Z_3) - h((Z_3 + Z_4^{(2)})/2)] \\ &-\theta[h(Z_1) + h(Z_2) - h(Z_1, Z_2)] \end{aligned}$$

Therefore, our instance of FRBL would look something like this (omitting some constant coefficients):

$$\begin{aligned} &(1+\alpha+\theta)h(Z_1) + (1-\lambda+\theta)h(Z_2) + (1-\lambda+\beta)h(Z_3) + \lambda h(Z_4) \\ &\le \sup_{\Pi(\vec{Z})} [h(\sqrt{\lambda}Z_1 + \sqrt{1-\lambda}Z_2) + h\left(\frac{1}{2}\left(Z_4 + \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix}\right)\right) + \alpha h\left(\frac{Z_1 + Z_4^{(1)}}{2}\right) \\ &+ \beta h\left(\frac{Z_3 + Z_4^{(2)}}{2}\right) + \theta h(Z_1, Z_2)] + C(\lambda, \alpha, \beta, \theta), \end{aligned}$$

where $C(\lambda,\alpha,\beta,\theta)$ is our FRBL constant. I've checked that this is homogeneous and the dimension condition holds, so for any fixed set of parameters $(\lambda,\alpha,\beta,\theta)$ the constant is finite and the inequality is Gaussian extremizable. As $\alpha, \beta, \theta \to \infty$, we'd ideally recover an inequality of the form

$$\lambda h(X) + (1-\lambda)h(W) + (1-\lambda)I(X;V) \le h(Y) + C(\lambda),$$

where $C(\lambda) = \lim_{\alpha,\beta,\theta\to\infty} C(\lambda, \alpha, \beta, \theta)$. I believe this approach of going directly through FRBl has an advantage (over the approach that goes through the ISIT result) of having better control over the slack, since here we are not fixing $h(X_g) = h(X)$, $h(Y_g) = h(Y)$, etc., so the regularizer for enforcing dependence is not bounded strictly below 0 uniformly over all couplings, which makes the min-max quantity involving Gaussian variables $-\infty$.

Another way to potentially view this is in terms of the gap in the DPI. In particular, $C(\lambda)$ is essentially the constant which makes $\lambda I(X; V) + C(\lambda) \le \lambda I(Y; V)$ tight in our setting. In other words, perhaps the inequality

$$I(X; V) \le \lambda I(Y; V) = \lambda I(\sqrt{\lambda} X + \sqrt{1-\lambda} W; V) \tag{4}$$

can be seen as a tight or strong version of DPI (are there any counterexamples to this?). Note that (4) is satisfied at the endpoints $\lambda = 0, 1$, so it's not entirely implausible.

2. FRBL with L^2 term: optimality "only if" direction (July 13)

Theorem 2.1. *Let (c, d, B, Q) satisfy*

$$\sum_{i=1}^{k} c_i D(\mu_i \| \gamma_{E_i}) \geq \inf_{\mu \in \Pi(\mu_1,\ldots,\mu_k)} \left\{ \sum_{j=1}^{m} d_j D(B_j \# \mu \| \gamma_{E^j}) + \frac{1}{2} \int |Qx|^2 d\mu(x) \right\}, \tag{5}$$

where Σ is p.s.d satisfying

$$B_j \Sigma B_j^* = I_{E^j}; \quad \pi_{E_i} \Sigma \pi_{E_i}^* = I_{E_i}; \quad \langle Q^* Q, \Sigma \rangle = 0.$$

Then (c, d, B, Q) satisfies the operator inequality

$$\sum_{j=1}^{m} d_j B_j^* B_j + Q^* Q \leq \sum_{i=1}^{k} c_i \pi_{E_i} \pi_{E_i^*}. \tag{6}$$

Proof. Under gaussian extremizers, we may rewrite the RHS of (5) as

$$\begin{aligned}
&\inf_{K \in \Pi} \frac{1}{2} \left\{ \sum_{j=1}^{m} d_j \left(-\log |B_j K B_j^*| \underbrace{- \dim(E^j) + \mathrm{tr}\,((B_j K B_j^*)^{-1})}_{0} \right) + \mathrm{tr}\,(QKQ^*) \right\} \\
&\geq \inf_{K \in S^+(E_0)} \frac{1}{2} \left\{ \sum_{j=1}^{m} d_j \left(-\log |B_j K B_j^*| - \dim(E^j) + \langle K, \Lambda_c \rangle \right) + \mathrm{tr}\,(QKQ^*) \right\}
\end{aligned}$$

Choose K to be optimal, and Taylor expand the term in the brackets under a pertubation $K + \epsilon A$ to get

$$\begin{aligned}
&\sum_{j=1}^{m} d_j \left(-\log |B_j (K + \epsilon A) B_j^*| - \dim(E^j) + \langle K + \epsilon A, \Lambda_c \rangle \right) + \mathrm{tr}\,(Q(K + \epsilon A)Q^*) \\
&= \sum_{j=1}^{m} d_j (-\log |B_j K B_j^*| - \dim(E^j) + \langle K, \Lambda_c \rangle + \mathrm{tr}\,(QKQ^*) \\
&+ \epsilon \left(-\sum_{j=1}^{m} d_j \langle A, B_j^* (B_j K B_j^*)^{-1} B_j \rangle + \langle A, \Lambda_c \rangle + \langle A, Q^* Q \rangle \right) + o(\epsilon) \\
&\geq \sum_{j=1}^{m} d_j \left(-\log |B_j K B_j^*| - \dim(E^j) + \langle K, \Lambda_c \rangle \right) + \mathrm{tr}\,(QKQ^*).
\end{aligned}$$

Sending $\epsilon \downarrow 0$ and letting A vary enforces (6). □

3. Functional Dual of FRBL Inequality with L2 term

Theorem 3.1. *Assume that* (c, d, B, Q) *satsifies*

$$\sum_{j=1}^{m} d_j B_j^* B_j + Q^* Q \le \sum_{i=1}^{k} c_i \pi_{E_i} \pi_{E_i}^*$$

and there is a Σ *p.s.d. satisfying*

$$B_j \Sigma B_j^* = I_{E^j}; \quad \pi_{E_i} \Sigma \pi_{E_i}^* = I_{E_i}; \quad \langle Q^* Q, \Sigma \rangle = 0.$$

If $(f_i), (g_j)$ *satisfy (for all* x*)*

$$\prod_i e^{c_i f_i(x)} \le e^{|Qx|^2} \prod_j e^{d_j g_j(B_j x)}, \tag{7}$$

then

$$\prod_i \left(\mathbb{E}[e^{f_i}]\right)^{c_i} \le \prod_j \left(\mathbb{E}[e^{g_j}]\right)^{d_j}. \tag{8}$$

Proof. Pick μ_i s.t. (by duality)

$$D(\mu_i \| \gamma_i) = \mathbb{E}_{\mu_i}[f_i(X_i)] - \log \mathbb{E}_{\gamma_i}[e^{f_i(X_i)}].$$

Then from (5) and duality we can write

$$\begin{aligned} &\sum_i c_i \left(\mathbb{E}_{\mu_i}[f_i(X_i)] - \log \mathbb{E}_{\gamma_i}[e^{f_i(X_i)}] \right) \\ &\ge \sum_j d_j \left(\mathbb{E}_{\mu}[g_j(B_j X)] - \log \mathbb{E}_{\gamma}[e^{g_j(X)}] \right) + \mathbb{E}_{\mu}[|QX|^2] \end{aligned}$$

Rearranging and applying (7), we get

$$\begin{aligned} 0 &\ge \sum_i c_i \mathbb{E}_{\mu_i}[f_i(X_i)] - \left(\sum_j d_j \mathbb{E}_{\mu}[g_j(B_j X)] + \mathbb{E}_{\mu}[|QX|^2] \right) \\ &\ge \log \left[\frac{\prod \left(\mathbb{E}[e^{f_i}]\right)^{c_i}}{\prod \left(\mathbb{E}[e^{g_j}]\right)^{d_j}} \right]. \end{aligned}$$

□

4. ISIT Result in Relative Entropies (July 15)

In relative entropies, the ISIT result should look something like

$$\inf_{\Pi} \sum_j d_j D(Q_j \# X \| \gamma_j) \le \inf_{\Pi} \sum_r d_j D(Q_j \# \tilde{X} \| \gamma_j), \tag{9}$$

where $\tilde{X}$ is Gaussian with $D(X_i \| \gamma_i) = D(\tilde{X}_i \| \gamma_i)$.

Proof. By FRBL and the entropy constraint, we have

$$\begin{aligned} \inf_{\Pi} \sum_j d_j D(Q_j \# X \| \gamma_j) &\le \sum_i c_i D(X_i \| \gamma_i) + D_g(c, d, Q) \\ &= \sum_i c_i D(\tilde{X}_i \| \gamma_i) + D_g(c, d, Q). \end{aligned}$$

From Lemma 8 and Theorem 6 of the paper, we have equality in

$$\sum_i c_i h(\tilde{X}_i) = \sup_{\Pi} \sum_j d_j h(Q_j \tilde{X}_j) + D_g(c, d, Q),$$

which we can rewrite as (not sure if this is valid)

$$\sum_i c_i D(\tilde{X}_i \| \gamma_i) + C = \inf_{\Pi} \sum_j d_j D(Q_j \# \tilde{X} \| \gamma_j).$$

Combined with above, we obtain (9). $\square$

5. Barthe's Inequality for Gaussian Kernels Implies a Regularized FRBL (July 21)

Consider an inequality of the form

$$\int e^{-P(x)} \prod_{k=1}^{m} f_k^{c_k}(B_k x)dx \geq C \prod_{k=1}^{m} \left(\int f_k\right)^{c_k}, \tag{10}$$

where here P is a p.s.d. quadratic form, and the c_k are allowed to be negative. The inequality is saturated by centered Gaussians. Then we claim the following is true:

Theorem 5.1. *An inequality of the form (10) implies a regularized f.r.b.l. That is,*

$$\left.\begin{array}{ll} \textit{if} & \prod f_i^{c_i}(x_i) \leq e^{|Qx|^2} \prod g_j^{d_j}(B_j x) \\ \textit{then} & \prod \left(\int f_i\right)^{c_i} \leq e^{C_2} \prod \left(\int g_j\right)^{d_j} \end{array}\right\} \tag{11}$$

where C_2 is the best constant under Gaussian extremizers.

Proof. Consider the following instance of (10):

$$e^{C_1} \int e^{-|Qx|^2} \prod f_i^{c_i}(x_i) \prod g_j^{-d_j}(B_j x) \geq \prod \left(\int f_i\right)^{c_i} \prod \left(\int g_j\right)^{-d_j}. \tag{12}$$

Let $c_i' = 1 + tc_i$, $d_j' = td_j$, and $Q' = \sqrt{t}Q$. Then by rearranging the hypothesis, raising exponents by t, and integrating we have

$$\begin{aligned} \int e^{-|Q'x|^2} \prod f_i^{c_i'}(x_i) \prod g_j^{-d_j'}(B_j x) &\leq \int \prod f_i^{c_i'}(x_i) \prod f_i^{-tc_i}(x_i) \\ &= \prod \int f_i(x_i). \end{aligned}$$

Then (12) implies

$$\prod \left(\int f_i\right)^{c_i'} \prod \left(\int g_j\right)^{-d_j'} \leq e^{C_{1,t}} \prod \int f_i(x_i),$$

where $C_{1,t} = -C_1$. This can be rewritten as (by rearranging and dividing exponents by t)

$$\prod \left(\int f_i\right)^{c_j} \leq e^{C_{1,t}/t} \prod \left(\int g_j\right)^{d_j}.$$

Therefore, if we can argue that

$$\liminf_{t \to \infty} \frac{C_{1,t}}{t} = C_2,$$

then we'd be done. Indeed, we know that the best constant is given by

$$e^{2C_{1,t}} = \sup \frac{\det\left(Q'^* Q' + \sum_{i=1}^{k} c_i' \pi_{E_i}^* C_i \pi_{E_i} - \sum_j d_j' B_j^* A_j B_j\right)}{\prod_i (\det C_i)^{c_i'} \prod_j (\det A_j)^{-d_j'}},$$

where the supremum is over $C_i \in S^+(E_i)$ and $A_j \in S^+(E^j)$ satisfying

$$Q'^* Q' + \sum_i c_i' \pi_{E_i}^* C_i \pi_{E_i} \geq \sum_j d_j' B_j^* A_j B_j.$$

(something something monotonicity) We may then raise this to the $1/t$ and bound

$$\begin{aligned}
\frac{\det\left(Q'^*Q' + \sum_{i=1}^k c_i'\pi_{E_i}^* C_i \pi_{E_i} - \sum_j d_j' B_j^* A_j B_j\right)^{1/t}}{\prod_i (\det C_i)^{1/t+c_i} \prod_j (\det A_j)^{-d_j}} &\le \frac{\det\left(Q'^*Q' + \sum_{i=1}^k c_i'\pi_{E_i}^* C_i \pi_{E_i}\right)^{1/t}}{\prod_i (\det C_i)^{1/t+c_i} \prod_j (\det A_j)^{-d_j}} \\
&\le \frac{\det\left(t\|Q\|^2 I + \sum_{i=1}^k c_i'\pi_{E_i}^* C_i \pi_{E_i}\right)^{1/t}}{\prod_i (\det C_i)^{1/t+c_i} \prod_j (\det A_j)^{-d_j}} \\
&= \frac{\prod_i (1 + tc_i + t\|Q\|^2)^{\dim(E_i)/t}}{\prod_i (\det C_i)^{c_i} \prod_j (\det A_j)^{-d_j}}
\end{aligned}$$

Therefore

$$\liminf_{t\to\infty} C_{1,t}/t \le \frac{1}{2} \sup \left\{ \sum_j d_j \log\det A_j - \sum_i c_i \log\det C_i \right\}$$

which is precisely the best constant obtained in the second half of (11) under centered gaussians. □

THEOREM 5.2 (Entropic Dual). *The regularized f.r.b.l (11) can be rewritten in entropic form as:*

$$\sum_i c_i h(\mu_i) \le \sup_{\mu\in\Pi(\mu_1,\dots,\mu_n)} \left\{ \sum_j d_j h(B_j\#\mu) - \mathbb{E}_\mu |QX|^2 \right\} + D_g(c,d,B,Q) \tag{13}$$

PROOF. By duality and regularization, we may write

$$\begin{aligned}
&\sup_{\mu\in\Pi} \left\{ \sum_j d_j h(B_j\#\mu) - \mathbb{E}_\mu |QX|^2 \right\} \\
&= \sup_{\mu\in\Pi} \sum_j d_j \left(\inf_{g_j} \log \int e^{g_j(x)}dx - \mathbb{E}_\mu[g_j(B_jX)] \right) - \mathbb{E}_\mu |QX|^2 \\
&= \sup_{\mu\in S^+} \inf_{f_i,g_j} \left[\sum_j d_j \left(\log \int e^{g_j(x)}dx - \mathbb{E}_\mu[g_j(B_jX)] \right) + \sum_i c_i \left(\mathbb{E}_\mu[f_i(\pi_i X)] - \int f_i d\mu_i \right) \right] - \mathbb{E}_\mu |QX|^2
\end{aligned}$$

After checking the conditions of Sion's minimax theorem (TODO), we'd get

$$\begin{aligned}
\inf_{f_i,g_j} &\left\{ \sum_j d_j \log \int e^{g_j(x)}dx - \sum_i c_i \int f_i d\mu_i \right. \\
&\left. + \sup_{\mu\in S^+} \left[\sum_i c_i \mathbb{E}_\mu[f_i(\pi_i X)] - \sum_j d_j \mathbb{E}_\mu[g_j(B_jX)] - \mathbb{E}_\mu |QX|^2 \right] \right\}.
\end{aligned}$$

Given the operator inequality, the term in the sup vanishes, and we're left with

$$\inf_{f_i} \left\{ \sum_j d_j \log \int e^{g_j(x)}dx - \sum_i c_i \int f_i d\mu_i \right\} = \sum_i c_i h(X_i),$$

as desired. □

6. Finiteness of constants (July 29)

In the case where $Q = 0$, it can be seen that Barthe's constant is finite whenever the f.r.b.l. constant is. Indeed, if we rewrite Barthe's Theorem 8.9 in our setting, finiteness is equivalent to satisfying, for every product form subspace V,

(i) $\dim(V) \geq \sum_i c_i \dim(\pi_i V) - \sum_j d_j(B_j V)$
(ii) $\dim(E_0) - \dim(V) \leq \sum_i c_i [\dim(E_i) - \dim(\pi_i V)] - \sum_j d_j \left[\dim(E^j) - \dim(B_j V)\right]$

Recall that in our proof of (11), we took $c_i' = 1 + tc_i$ and $d_j' = td_j$. Thus we can see that these conditions are equivalent to those in Theorem 1.27 of the f.r.b.l. paper. Furthermore, we already know that f.r.b.l. is Gaussian extremizabile whenever Barthe's result is.

Now our question is: how does the case for general Q translate to the f.r.b.l. setting? Due to a scaling argument, the constant in (13) can blow up for general Q, and this agrees with Barthe's results, as any $Q > 0$ would violate the degeneracy condition (which is sufficient for Gaussian extremizability) in Barthe's paper:

$$\dim(E_0) \geq s^+(Q) + \sum_i \dim(E_i).$$

So the only two ways around this are to either assume $Q \leq 0$, or to extend E_0 to a larger space, i.e. we'd instead assume that $E_0 \supset \oplus_i E_i$.

7. Equivalence through Geometric Settings (August 5)

To further understand this problem, we can also make use of the fact that the vanilla settings are equivalent, up to some linear isomorphisms, to the geometric settings, in both f.r.b.l. and Barthe's result. Ideally, we'd have (for $Q = 0$, we already know this should be possible; for general Q, we'd want some regularized form of f.r.b.l. such as (11) on the left)

$$\text{f.r.b.l.} \leftrightarrow \text{geometric f.r.b.l.} \overset{?}{\leftrightarrow} \text{geometric Barthe} \leftrightarrow \text{Barthe}$$

The first arrow is given by Theorem 1.23 of the f.r.b.l. paper, and the third by remark 4.6 in Barthe's paper. We attempt to give on direction of the second arrow in a bit but see that some things break.

Theorem 7.1 (1.23). *A datum (c, d, B) is Gaussian extremizable if and only if it is 'equivalent' to a geometric datum (c, d, B'), i.e. there exists linear isomorphisms $C : E_0 \to E_0$ and $C_j : E^j \to E^j$ such that $B_j' = C_j^{-1} B_j C^{-1}$ for each j.*

Theorem 7.2 (4.6). *A datum (c, B, Q) is Gaussian extremizable if and only if it is 'equivalent' to a geometric datum $(c, \tilde{B}, \tilde{Q})$ in the same sense as above.*

Proof. Using the notation in Barthe, we set

$$A = Q + \sum_{k=1}^{m} c_k B_k^* A_k B_k > 0,$$

where A_k are the extremizers satisfying (Theorem 4.5)

$$A_k^{-1} = B_k A^{-1} B_k^* \text{ for all } k = 1, \ldots, m \text{ for which } c_k \neq 0.$$

Then let $\tilde{Q} = A^{-1/2}QA^{-1/2}$ and $\tilde{B} = A_k^{1/2}B_kA^{-1/2}$. We can check that these satisfy the geometric conditions:

$$\begin{aligned}
\tilde{B}_k\tilde{B}_k^* &= A_k^{1/2}B_kA^{-1/2}A^{-1/2}B_k^*A_k^{1/2} \\
&= A_k^{1/2}(A_k^{-1})A_k^{1/2} = I_{H_k}, \\
\tilde{Q} + \sum_{k=1}^m c_k\tilde{B}_k^*\tilde{B}_k &= A^{-1/2}QA^{-1/2} + \sum_{k=1}^m A^{-1/2}B_k^*A_kB_kA^{-1/2} \\
&= A^{-1/2}\underbrace{\left(Q + \sum_{k=1}^m c_kB_k^*A_kB_k\right)}_{A}A^{-1/2} = I_H.
\end{aligned}$$

Thus by Barthe's 4.5, (c, B, Q) is equivalent to the geometric datum $(c, \tilde{B}, \tilde{Q})$ with constant 1.

The converse follows by setting A_k and A according to the linear isomorphisms and then repeating the same computations as above. □

Now, we can try to go from Barthe to geometric Barthe to geometric f.r.b.l. to f.r.b.l. That is, suppose we have a f.r.b.l. datum (c, d, B, Q). From before, we know that this is a limiting case of the Barthe datum $(c', -d', B, Q') = (1 + tc, -td, B, tQ)$. Now let

$$A^{(t)} := Q' + \sum_{k=1}^m c_k'B_k^*A_k^{(t)}B_k, \tag{14}$$

where the $A_k^{(t)}$'s satisfy

$$(A_k^{(t)})^{-1} = B_k(A^{(t)})^{-1}B_k^* \text{ for all } k = 1, \dots, m \text{ for which } c_k' \neq 0. \tag{15}$$

Then the Barthe datum

$$(\underbrace{1 + tc}_{c'}, \underbrace{td}_{d'}, \underbrace{(A_k^{(t)})^{1/2}B_k(A^{(t)})^{-1/2}}_{\tilde{B}_k^{(t)}}, \underbrace{(A^{(t)})^{-1/2}(tQ)(A^{(t)})^{-1/2}}_{\tilde{Q}^{(t)}})$$

satisfies the geometric conditions. But now the proof of (11) no longer goes through. In particular, we'd like to be able to say that our original (c, d, B, Q) is equivalent to the geometric f.r.b.l. datum

$$(c, d, (A_k^{(t)})^{1/2}B_k(A^{(t)})^{-1/2}, (A^{(t)})^{-1/2}Q(A^{(t)})^{-1/2}).$$

However, a priori we don't know how $A_k^{(t)}$ and $A^{(t)}$ behave as $t \to \infty$. So we'd need a better analysis of these maps, perhaps through their defining equations (14) and (15).

8. Characterization of Extremizers (Valdimarsson)

8.1. Proof of BL by Heat Semigroup. Let (B, c) be a geometric Brascamp-Lieb datum. Assume that $B_j : H \to H_j$ is the orthogonal projection (they remove this restriction later). Let $f_j \in \mathcal{S}(H_j)$ be nonnegative and let $g_j(x,t)$ for $t \geq 0$ be solutions to the IVP

$$\begin{aligned} g_j(x,0) &= f_j(B_j x) \\ \frac{\partial}{\partial t} g_j(x,t) &= \Delta g_j(x,t) \quad t > 0. \end{aligned}$$

By plugging things in, we can see that

$$g_j(x,t) = \int_H \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} f_j(B_j y) dy$$

is a solution to the IVP above.

Define the product

$$F(x,t) = \prod_{j=1}^{m} g_j^{c_j}(x,t).$$

Let $T : H \to \oplus_j H_j$ be the map

$$\begin{bmatrix} \dots & \sqrt{c_1} B_1 & \dots \\ & \vdots & \\ \dots & \sqrt{c_m} B_m & \dots \end{bmatrix}$$

and $A(x) \in \oplus_j H_j$ be the vector

$$\begin{bmatrix} \sqrt{c_1} \nabla h_1(B_1 x) \\ \vdots \\ \sqrt{c_m} \nabla h_m(B_m x) \end{bmatrix}$$

Note that $T^*T = \sum_j p_j B_j^* B_j = \mathrm{Id}_H$ by geometricity. In their calculations, they have

$$\frac{\partial}{\partial t} \int F(x,t) dx = \int F(x,t) \Xi(x,t) dx$$

where

$$\Xi(x) = \langle (I - P) A(x), A(x) \rangle.$$

Here $P = TT^* = T(T^*T)^{-1}T^*$ is the projection onto the image of T. In particular, we have that $\Xi \succeq 0$ and we obtain a monotonicity result:

$$\frac{\partial}{\partial t} \int F(x,t) dx \geq 0.$$

Furthermore, by DCT we have

$$\lim_{t \to \infty} \int F(x,t) dx = \prod_{j=1}^{m} \left(\int_{H_j} f_j \right)^{c_j},$$

and by regularity of the solution g_j, we have

$$\lim_{t \to 0} F(x,t) = \prod_{j=1}^{m} f_j^{c_j}(B_j x)$$

Putting these together, we have

$$\int_H \prod_{j=1}^m f_j^{c_j}(B_jx)dx = \int_H F(x,0)dx \le \int_H F(x,t)dx$$

for all $t > 0$. Taking the r.h.s. to ∞ we get the desired inequality

$$\int_H \prod_{j=1}^m f_j^{c_j}(B_jx)dx \le \prod_{j=1}^m \left(\int_{H_j} f_j\right)^{c_j}$$

8.2. Structure. We start with some definitions. A subspace V of H is *critical* if $V \neq \{0\}, H$ and

$$\dim(V) = \sum_j c_j \dim(B_jV).$$

A subspace K of H is said to be *independent* if $K \neq \{0\}$ and

$$K = \cap_{j=1}^m H_j^a,$$

where $H_j^a = H_j$ or $H^\perp$. It turns out that any independent subspace is also critical. Note that any two independent subspaces are orthogonal to each other. Thus we have the *independent decomposition* of H, given by

$$H = K_{\text{ind}} \bigoplus K_{\text{dep}} = \left(\bigoplus_{k=1}^{k_0} K_k\right) K_{\text{dep}},$$

where $\{K_k : k = 1, \ldots, k_0\}$ is an enumeration of the independent subspaces of H and K_{dep} is the orthogonal complement of K_{ind}.

8.3. Characterization. For now assume that (B, c) is geometric datum, and let $\oplus_{k=0}^{k_0} K_k \oplus K_{\text{dep}}$ be the independent decomposition of H. Then Valdimarsson states the following result:

THEOREM 8.1 (Characterization). *Suppose (f_j) are extremizers for the datum (B, c). Then there exists a critical decomposition*

$$H = \underbrace{\left(\oplus_{k=0}^{k_0} K_k\right)}_{K_{ind}} \oplus \underbrace{\left(\oplus_{k=k_0+1}^{k_1} K_k\right)}_{K_{dep}}$$

with integrable $u_k : H_k \to \mathbb{R}$, $k = 1, \ldots, k_0$, positive constants C_j, d_k, and an element $b \in K_{dep}$ such that

$$f_j(x) = C_j \prod_{k=1}^{k_0} u_k(P_{j,k}B_j^*x) \prod_{k=k_0+1}^{k_1} e^{-d_k\langle P_{j,k}B_j^*x, P_{j,k}(B_j^*x+b)\rangle}, \tag{16}$$

where $P_{j,k}$ is the orthogonal projection from H onto $H_j \cap K_k$. Conversely, all functions of this form are extremizers for (B, c).

First suppose that (f_j) are Schwartz. Since they are extremizers, we must have

$$\frac{\partial}{\partial t}\int F(x,t)dx = 0.$$

From before, this implies that $\langle (I - P)A, A\rangle = 0$, and in particular that $PA = A$. Recall that $P = TT^* = T(T^*T)^{-1}T^*$ is the projection onto the image of T. Therefore there must exist a map $\beta : H \to H$ so that

$$A(x) = T\beta(x).$$

By reading off the rows we get

$$\nabla h_j(B_j x) = B_j \beta(x), \tag{17}$$

and hence

$$\nabla \log F(x) = \sum_j p_j B_j^* \nabla h_j(B_j x) = \sum_j p_j B_j^* B_j \beta(x) = \beta(x)$$

where the last equality follows from geometricity.

Let $b_j = B_j^* e_j$, where $e_j \in H_j$. Then we have

$$\langle \nabla \log F(x), b_j \rangle = \langle \beta(x), b_j \rangle = \langle B_j \beta(x), e_j \rangle = \langle \nabla h_j(B_j x), e_j \rangle.$$

Now if we let $b_j^\perp = B_j^{\perp *} e_j^\perp$ where $e_j^\perp \in H_j^\perp$ and we differentiate this with respect to $b_j^\perp$, we get

$$D^2(\log F)(b_j, b_j^\perp) = 0. \tag{18}$$

This restricts $\log F$ to be of the form $u_j(B_j x) + u_j^\perp(B_j^\perp x)$. In particular for any j, j' we can write

$$\log F = u_j(B_j x) + u_j^\perp(B_j^\perp x). \tag{19}$$

At this point, we may use (19) to prove two lemmas which will be useful in determining the extremizers.

Lemma 8.2. *There exists functions u_{K_k} and $U_{K_{dep}}$ such that*

$$\log F = \left(\sum_{k=1}^{k_0} u_{K_k}(P_{K_k} x) \right) + u_{K_{dep}}(P_{K_{dep}} x).$$

Proof. It suffices to show that the second derivative of $\log F$ w.r.t. any pair of vectors from different components of the independent decomposition is zero.

If the two vectors come from two distinct independent subspaces K_k and $K_{k'}$, then there must be a j such that $K_k \subset H_j$ and $K_{k'} \subset H_j^\perp$. Then the result follows immediately from (18).

Now suppose one vector $b_1 \in K_k$ and another vector $b_2 \in K_{\text{dep}}$. Since

$$K_{\text{dep}} \subset K_k^\perp = \sum_j H_j^{a\perp},$$

b_2 can be written as some linear combination of vectors in $H_j^{a\perp}$. Then we have $D^2(\log F)(b_1, b_j^{a\perp})$ for any $b_j^{a\perp} \in H_j^{a\perp}$. The result then follows by linearity. □

Lemma 8.3. *If a space H has no independent subspaces, then any extremizer must be Gaussian.*

Proof. [Sketch] They show that $\nabla \log F$ has at most linear growth in each x_j. In particular, if $\nabla \log F$ is a linear polynomial, then (17) together with $\nabla \log F = \beta$ implies that each f_j is Gaussian.

To show that $\nabla \log F$ is a polynomial, it suffices to show that the Fourier transform is a distribution supported only at the origin. Considering (19), they compute the Fourier transform of functions of the form $u(B_j x)$ and show that it is supported in H_j. It then follows that the transform of $\nabla \log F$ is supported in $H_j \cup H_j^\perp$, and in particular supported on

$$\cap_j (H_j \cup H_j^\perp).$$

Since H has no independent subspaces, we know this intersection is just $\{0\}$. □

By Lemma 8.3 we see that the $u_{K_{\text{dep}}}$ appearing in Lemma 8.2 is Gaussian. After some more massaging we may obtain the form in (16). Finally, they do some more work to get rid of the restriction to Schwarz functions as well as geometricity.