Complex Analysis

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I wrote these set of notes to review some of the concepts from the latter half of the course. The Riemann Mapping Theorem is hard.

Contents

1 Argument Principle & Corollaries

Theorem 1.1 (Argument Principle)

Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles nor zeros on C, then

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz &= (\#\text{ of zeros of } f \text{ inside } C) \\ &\quad - (\#\text{ poles of } f \text{ inside } C)\end{aligned}$$

Proof. Suppose f has a zero of order N at, say, 0. Then we can write

$$f(z) = z^N g(z)$$

for some holomorphic g non-vanishing in a neighborhood of 0. Then

$$\frac{f'(z)}{f(z)} = \frac{Nz^{N-1}g(z) + z^N g'(z)}{z^N g(z)} = \frac{N}{z} + \frac{g'(z)}{g(z)}.$$

Note that the quotient $g'(z)/g(z)$ is holomorphic in a neighborhood of 0. Similarly, if there were a pole of order N at 0, we'd have

$$\frac{f'(z)}{f(z)} = \frac{-N}{z} + \frac{h'(z)}{h(z)}.$$

Now, since f is meromorphic, we have accounted for all the poles of f'/f, which are simple with residues corresponding to the orders of the zeros and poles of f. Therefore, an application of the residue formula gives us the desired result. □

Corollary 1.2 (Rouche's Theorem)

Suppose f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)| \quad \forall z \in C,$$

then f and $f + g$ have the same number of zeros inside the circle C.

Proof. For $t \in [0, 1]$, define

$$f_t(z) = f(z) + tg(z).$$

Let n_t denote the number of zeros of f_t inside the circle, counting multiplicities. In particular, $n_t \in \mathbb{Z}$ for all t. By the Argument Principle, we have

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz.$$

Our method of proof will be to show that n_t is a continuous function of t, hence implying that it must be constant. Denote $h_t(z) = f_t'(z)/f_t(z)$. Since $C \times [0,1]$ is a compact set and $h_t(z)$ is a continuous function of t and z, we may apply the Bounded Convergence Theorem to get

$$\lim_{s \to t} \left| \int_C h_t(z) - h_s(t) dz \right| \leq \lim_{s \to t} \int_C |h_t(z) - h_s(z)| dz = 0.$$

This implies that n_t is continuous, so that $f_0 = f$ and $f_1 = f + g$ have the same number of zeros inside C. $\square$

Corollary 1.3 (Open Mapping Theorem)

If $f : \Omega \to \mathbb{C}$ is holomorphic a nonconstant, then it is an *open mapping*, i.e. it sends open sets to open sets.

Proof. Let $U \subset \Omega$ be an open set, and $w_0 = f(z_0)$ for $z_0 \in U$. Then we must show that for all points w close to w_0, that w also belongs to $f(U)$. Choose $\delta > 0$ sufficiently small so that the disc $D_\delta(z_0)$ is contained in U and $f(z) \neq w_0$ on the circle $C = \partial D_\delta(z_0)$. Note that if we were unable to pick such a δ, then f would have a sequence of zeros limiting to z_0, enforcing $f \equiv 0$, contradicting our nonconstant assumption.

Now, choose $\epsilon > 0$ small enough so that $|f(z) - w_0| > \epsilon$ for $z \in C$ (we can do this since f is continuous, and C is compact). Then for $w \in D_\epsilon(w_0)$, define

$$g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z).$$

We have that F and G are holomorphic on U containing $D_\delta(z_0)$, and that

$$|F(z)| = |f(z) - w_0| > \epsilon > |w_0 - w|,$$

for all $z \in C$. So, by Rouche's Theorem, we see that F and $F + G$ have the same number of zeros inside C. Since F has at least one zero, i.e. $F(z_0) = 0$, there exists a $z \in U$ such that $f(z) = w$, as desired. $\square$

Corollary 1.4 (Maximum Modulus Principle)

If f is a non-constant holomorphic function on an open domain Ω, then f cannot achieve a maximum in Ω. Furthermore, if $\overline{\Omega}$ is compact, and f is continuous on $\overline{\Omega}$, then f achieves its maximum on $\partial\Omega$.

Proof. Suppose f achieves a maximum at $z_0 \in \Omega$. Then since f is an open mapping, we see that the image of a neighborhood of z_0 is open, and in particular contains points z such that $|f(z)| > |f(z_0)|$, a contradiction.

Furthermore, if $\overline{\Omega}$ is compact and f is continuous on $\overline{\Omega}$, then $f(\overline{\Omega})$ is compact, and in particular achieves its maximum somewhere in $\overline{\Omega} \setminus \Omega = \partial\Omega$. $\square$

2 Conformal Mappings

Essentially the homeomorphisms of topology, *conformal mappings* are holomorphic bijections. It will turn out that the inverse mapping is also holomorphic.

Proposition 2.1

If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for $z \in U$.

Proof. Suppose for sake of contradiction that $f'(z_0) = 0$ for some $z_0 \in U$. Then, considering the Taylor expansion, we may write

$$f(z) - f(z_0) = a(z - z_0)^k + G(z),$$

where $a \neq 0$, $k \geq 2$, and G is holomorphic vanishing at z_0 with order $k + 1$. Now, pick $\delta > 0$ small enough so that

- $f'(z) \neq 0$ for $z \neq z_0$ and $|z - z_0| \leq \delta$.
- $|a\delta^k|/2 > |G(z)|$ for $|z - z_0| = \delta$.

Then set $\epsilon = a\delta^k/2$, so that by Rouche's Theorem,

$$|a(z - z_0)^k| = a\delta^k > \epsilon$$

implies that $a(z - z_0)^k$ and $a(z - z_0)^k + \epsilon$ have the same number of roots in $B_\delta(z_0)$, i.e. they both have two roots. Furthermore, if we let $F(z) = a(z - z_0)^k + \epsilon$, then

$$|F(z)| \geq |a(z - z_0)^k| - |\epsilon| = |a\delta^k|/2 > |G(z)|$$

for all $|z - z_0| = \delta$. So, by another application of Rouche's Theorem, we see that $F(z) + G(z) = f(z) + f(z_0) + \epsilon$ has two roots in $B_\delta(z_0)$. Furthermore, these two roots are distinct, since we chose δ so that $f'(z) \neq 0$ in the δ vicinity. But then $f(z) + f(z_0) + \epsilon$ is not injective, so neither is $f(z)$, a contradiction. □

Corollary 2.2

The inverse of a conformal mapping $f : U \to V$ is holomorphic.

Proof. Let $g = f^{-1}$ denote the inverse of f. Let $w_0 \in V$ and $w \to w_0$. We write $w_0 = f(z_0)$ and $w = f(z)$. Then if $w \neq w_0$, we have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$, we may take the limit and deduce that g is holomorphic at w_0 with

$$g'(w_0) = \frac{1}{f'(g(w_0))}.$$

□

2.1 The Disc & Upper Half Plane

We use $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to denote the unit disc and $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to denote the upper half plane.

Theorem 2.3

There exists a conformal mapping between $\mathbb{H}$ and $\mathbb{D}$.

Proof. Consider the map $F : \mathbb{H} \to \mathbb{D}$ given by

$$F(z) = \frac{i-z}{i+z},$$

as well as the inverse map $G : \mathbb{D} \to \mathbb{H}$ given by

$$G(w) = i\frac{1-w}{1+w}.$$

□

2.2 Automorphisms of the Disc & Upper Half Plane

Lemma 2.4 (Schwarz)

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

(i) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(ii) If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation.

(iii) $|f'(0)| \leq 1$, and if equality holds, then f is a rotation.

Proof. Since $f(0) = 0$, there exists a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ nonvanishing at 0 such that $f(z) = zg(z)$. For $0 < |z| = r < 1$, we have

$$|g(z)| = \left|\frac{f(z)}{z}\right| < \frac{1}{r}.$$

By the Maximum Modulus Principle, we have $|g(z)|\langle\frac{1}{r}$ for $z \in B_r(0)$. Taking $r \to 1$, we get $|g(z) \leq 1$ for $z \in \mathbb{D}$. It follows that $|f(z)| \leq |z|$ in $\mathbb{D}$.

Next, if $|z_0| = |f(z_0)|$, then g achieves its maximum at $z_0 \in \mathbb{D}$. By the Maximum Modulus Principle, we deduce that g is constant, so in particular $f(z) = cz$ for some $c \in \mathbb{C}$. Plugging in the condition at z_0, we get $|c| = 1$. It follows that f is a rotation.

Finally, note that

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z\to 0} g(z),$$

and so $|f'(0)| \leq 1$ since $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. If equality holds then $|g(0)| = 1$, so by the Maximum Modulus Principle, $g(z) = c$ with $|c| = 1$. It follows that f is a rotation. □

We are concerned with automorphisms of the unit disc, i.e. conformal mappings from a set to itself. Two notable ones are rotations,

$$r_\theta : z \mapsto e^{i\theta} z,$$

and

$$\psi_\alpha : z \mapsto \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad \alpha \in \mathbb{D}.$$

It can easily be shown that $\psi_\alpha^2 = I$, and it interchanges α and 0:

$$\psi_\alpha(0) = \alpha \quad \text{and} \quad \psi_\alpha(\alpha) = 0.$$

It turns out that all automorphisms of the disc can be described using these two mappings, as the next theorem suggests.

Theorem 2.5

If f is an automorphism of the disc, then there is $\theta \in [0, 2\pi)$ and $\alpha \in \mathbb{D}$ such that

$$f(z) = r_\theta(\psi_\alpha(z)) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Proof. If $f \in \text{aut}(\mathbb{D})$, then there is a unique $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Consider the mapping $g = f \circ \psi_\alpha$. Schwarz Lemma gives

$$|g(z)| \leq |z| \tag{1}$$

for all $z \in \mathbb{D}$. Note further that $g^{-1}(0) = \psi_\alpha^{-1}(f^{-1}(0)) = 0$, so applying Schwarz Lemma again to g^{-1}, we have

$$|g^{-1}(w)| \leq |w| \tag{2}$$

for all $w \in \mathbb{D}$. Combining (1) and (2), we have

$$|z| = |g^{-1}(g(z))| \leq |g(z)| \leq |z|,$$

which implies $|g(z)| = |z|$ for all $z \in \mathbb{D}$. From (ii) of Schwarz Lemma, we see that $g(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi)$. Using the fact that $\psi_\alpha^2 = I$, we see that

$$f(z) = f(\psi_\alpha^2(z)) = r_\theta(\psi_\alpha(z)) = e^{i\theta} \frac{\alpha - z}{1 - \overline{\alpha} z},$$

as desired. □

Corollary 2.6

If f is an automorphism of the disc which fixes 0, then f is a rotation.

With a complete description of the automorphisms of the disc, we may also describe those of the upper half plane. This is done by conjugation, with mapping

$$\Gamma : \mathrm{aut}(\mathbb{D}) \to \mathrm{aut}(\mathbb{H}),$$

where $\Gamma(\varphi) = F^{-1} \circ \varphi \circ F$, where $F : \mathbb{H} \to \mathbb{D}$ is the conformal map from before. It's easy to verify that Γ is a group isomorphism, and compute that elements of $\mathrm{aut}(\mathbb{H})$ consist of the *fractional linear transformations*:

$$f_M : z \mapsto \frac{az+b}{cz+d}.$$

There is a close correspondence between maps f_M and matrices $M \in SL_2(\mathbb{R})$, the special linear group, given by

$$SL_2(\mathbb{R}) = \left\{ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} and \det(M) = 1 \right\}.$$

Theorem 2.7

The automorphisms of $\mathbb{H}$ can be characterized by

$$\mathrm{aut}(\mathbb{H}) = \{f_M : M \in SL_2(\mathbb{R})\}.$$

2.3 The Riemann Mapping Theorem

We are now concerned with the problem of whether there exists a conformal map $F : \Omega \to \mathbb{D}$, given some open set Ω. First note that necessary conditions for existence are:

- Ω must be proper, i.e. cannot be all of $\mathbb{C}$ and cannot be empty.
- Ω must be connected.
- Ω must be simply connected.

It turns out that these conditions are also sufficient for existence:

Theorem 2.8 (Riemann Mapping Theorem)

Suppose Ω is proper, connected, and simply connected. If $z_0 \in \Omega$, then there is a unique conformal map $F : \Omega \to \mathbb{D}$ such that

$$F(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0.$$

Corollary 2.9

Any two proper, connected, and simply connected open sets Ω_1 and Ω_2 of $\mathbb{C}$ are conformally equivalent.

Proof of Uniqueness. Suppose F and G are conformal maps satisfying the conditions of Theorem refrmt. Then $H = F \circ G^{-1}$ is an automorphism of the disc fixing zero, so $H(z) = e^{i\theta} z$. But since $H'(0) = F'(G^{-1}(0)) \cdot \frac{1}{G'(G^{-1}(0))} > 0$, we must have $e^{i\theta} = 1$. Thus $F = G$. $\square$

The proof for existence is much more long-winded. The main idea is to consider all injective holomorphic maps $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$. Among these pick an f whose image maps surjectively onto all of $\mathbb{D}$, by making $f'(z_0)$ as large as possible. The chosen f will be obtained as a functional limit, which is what the next theorem deals with.

We say a family $\mathcal{F}$ of holomorphic functions on Ω is *normal* if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on every compact subset of Ω (compare this to the definition of a T_4 space in topology). The family $\mathcal{F}$ is said to be *equicontinuous* on a compact set K if given $\epsilon > 0$, there is a $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, we have $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$. This condition can be thought of as uniform continuity, uniformly in the family.

Theorem 2.10 (Montel)

If $\mathcal{F}$ is a family of holomorphic functions on Ω that are uniformly bounded on every compact subset of Ω, then:

(i) $\mathcal{F}$ is equicontinuous on every compact subset of Ω.

(i) $\mathcal{F}$ is a normal family.