Leibniz Rule

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1 Basic Form

Theorem 1.1 (Differentiation Past the Integral at a Point)
At $t = t_0$, we may swap the integration and differentiation operators:

$$\frac{d}{dt}\int_a^b f(x,t)dx = \int_a^b \frac{\partial}{\partial t} f(x,t)dx,$$

in the sense that both sides exist and are equal, provided that $f(x,t)$ and $f_t(x,t)$ exist and are continuous functions of two variables in some neighborhood of t_0 and for all $x \in [a,b]$.

Proof. We compute the derivative

$$\begin{aligned}\frac{d}{dt}\int_a^b f(x,t_0)dx &= \lim_{h\to 0}\frac{\int_a^b f(x,t_0+h)dx - \int_a^b f(x,t_0)dx}{h} \\ &= \lim_{h\to 0}\int_a^b \frac{f(x,t_0+h)-f(x,t_0)}{h}dx. \end{aligned} \tag{1}$$

Now, we must consider under which circumstances we may interchange the limit with the integral. In particular, note that since $f_t(x,t)$ is a continuous function in some neighborhood of t_0, we know that the limit

$$\lim_{h\to 0}\frac{f(x,t_0+h)-f(x,t_0)}{h}$$

exists, and furthermore by the Mean Value theorem, for every fixed $x \in [a,b]$ there exists a $\tau \in [t_0, t_0+h]$ such that

$$\frac{f(x,t_0+h)-f(x,t_0)}{h} = f_t(x,\tau).$$

Continuity of $f_t(x,t_0)$ on the compact interval $[a,b]$ implies that $f_t(x,t_0)$ is bounded. Thus we may obtain a uniform bound on the difference quotient

$$\left|\frac{f(x,t_0+h)-f(x,t_0)}{h}\right| = |f_t(x,\tau)| \le B$$

for every $x \in [a, b]$, independent of the value of h. Now we may apply the Bounded Convergence theorem to see that (1) becomes

$$\int_a^b \lim_{h \to 0} \frac{f(x, t_0 + h) - f(x, t_0)}{h} dx = \int_a^b f_t(x, t_0) dx,$$

as desired. □

Corollary 1.2 (Differentiation Past the Integral on a Set)
The following equality holds for all $t \in \Omega_0$,

$$\frac{d}{dt} \int_a^b f(x, t_0) dx = \int_a^b \frac{\partial}{\partial t} f(x, t_0) dx,$$

provided that $f(x, t)$ and $f_t(x, t)$ exist and are continuous functions of two variables for $(x, t) \in [a, b] \times \Omega$, where Ω is some open set containing Ω_0.

1.1 Examples

Example 1.3. For integers $n \geq 0$, we have Euler's factorial integral formula:

$$\int_0^\infty x^n e^{-x} dx = n!.$$

This can be shown by repeated integration by parts, with the base case of

$$\int_0^\infty e^{-x} dx = 1.$$

However, we will derive it by repeated differentiation. Consider the substitution $x = tu$, for $t > 0$. Then our base integral becomes

$$\int_0^\infty t e^{-tu} du = 1.$$

This becomes

$$\int_0^\infty e^{-tx} dx = \frac{1}{t}.$$

Differentiating under the integral sign, we get

$$\int_0^\infty -x e^{-tx} dx = -\frac{1}{t^2}.$$

Keep doing this, and we get the formula

$$\int_0^\infty x^n e^{-tx} dx = \frac{n!}{t^{n+1}},$$

which gives us, for $t = 1$,

$$\int_0^\infty x^n e^{-x} dx = n!.$$

Note that the key strategy we used was in introducing the dummy variable t, which allowed us to pull out extra factors of x, and then getting rid of t at the end. In essence, to solve the problem it helped to solve a more general problem.

Example 1.4. We are interested in computing the n^{th} moment of a Gaussian, given by

$$\int_{-\infty}^\infty x^n e^{-x^2/2} dx.$$

For $n = 0$, this is well known to be

$$\int_{-\infty}^\infty e^{-x^2/2} dx = \sqrt{2\pi}.$$

We follow the same strategy as in the previous example, using the substitution $x \to \sqrt{t}x$ for $t > 0$ to get

$$\int_{-\infty}^\infty e^{-tx^2/2} dx = \frac{\sqrt{2\pi}}{\sqrt{t}}.$$

Repeated differentiation yields the formula

$$\int_{-\infty}^\infty x^n e^{-tx^2/2} dx = \frac{(n-1)!!}{t^{(n+1)/2}} \sqrt{2\pi},$$

where $n!!$ is understood to be the double factorial of all odd integers up to $n - 1$. Thus letting $t = 1$, we get

$$\int_{-\infty}^\infty x^n e^{-x^2/2} dx = (n-1)!!\sqrt{2\pi}.$$

Exercise 1.5 (Counterexamples)**.** Here is a counterexample:

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

What goes wrong if we try to apply differentiation under the integral using the change of variables $x = ty$? Also consider:

$$f(x,t) = \begin{cases} \frac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0 \text{ or } t \neq 0, \\ 0 & \text{else.} \end{cases}$$

Determine which hypotheses fails for this function when we attempt to use

$$\frac{d}{dt} \int f(x,t) dx = \int \frac{\partial}{\partial t} f(x,t) dx.$$

2 General Form

Theorem 2.1

The following equality holds for...,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx,$$

provided the following conditions hold:

- Both $f(x,t)$ and $f_t(x,t)$ exist and are continuous functions of two variables in some region $[\alpha,\beta]\times(t_1,t_2)$, where $a(t), b(t) \in [\alpha,\beta]$ for $t \in (t_1,t_2)$.
- The functions $a(t)$ and $b(t)$ are continuously differentiable for $t \in [t_1,t_2]$.

Proof. This follows from the Fundamental Theorem of Calculus, the chain rule, and an application of the basic form proved in the previous section (note that all the hypotheses are satisfied). In particular, we write

$$I(t,a(t),b(t)) = \int_{a(t)}^{b(t)} f(x,t)dx.$$

Then we have

$$\begin{aligned}\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx &= \frac{d}{dt} I(t,a(t),b(t)) \\ &= \frac{\partial}{\partial t} I(t,a(t),b(t)) + \frac{\partial}{\partial a} I(t,a(t),b(t))\frac{d}{dt}a(t) + \frac{\partial}{\partial b} I(t,a(t),b(t))\frac{d}{dt}b(t) \\ &= \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx - f(a(t),t)a'(t) + f(b(t),t)b'(t),\end{aligned}$$

as desired. □

Corollary 2.2

Under the hypothesis of Theorem 2.1, we have

$$\frac{d}{dt}\int_{a}^{t} f(x,t)dx = \int_{a}^{t} \frac{\partial}{\partial t} f(x,t)dx + f(t,t).$$

Exercise 2.3. Apply Corollary 2.2 to the integral

$$F(t) = \int_{0}^{t} \frac{\log(1+tx)}{1+x^2}dx.$$