Math 218A – Probability Theory

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Notes from first two days got corrupted :(.

Goal: To define uniform/length measure on $B(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$.

Define a pre-measure μ on $\Sigma_{\text{semi}} = \{(a, b] : -\infty \leq a < b \leq \infty\}$. We verified μ satisfies

(i) finite additivity
(ii) countable subadditivity

This was where we stopped last class. Now, the second step is to extend this semi-algbra to an algebra.

Definition 1 (Algebra)**.** An *algebra* $\mathcal{A}$ (or field) on state space Ω is a subset of the power set 2^Ω satisfying

(i) closed under complements
(ii) closed under finite unions and intersections
(iii) $\emptyset, \Omega \in \mathcal{A}$

We claim that given a semialgebra $\mathcal{A}$, the smallest algebra $\mathcal{B}$ containing $\mathcal{A}$ is the set

$$\{\sqcup_{i=1}^k A_i : A_i \in \mathcal{A}\}$$

Proof. It suffices to prove for $C, D \in \mathcal{B}$ that $C^C, C \cap D \in \mathcal{B}$. To see that $C \cap D \in \mathcal{B}$, we use distributive property

$$C \cap D = \sqcup_{i,j} A_i \cap B_j \implies C \cap D \in \mathcal{B}$$

where we let

$$C = \sqcup_{i=1}^k A_i, \qquad D = \sqcup_{j=1}^l B_j.$$

Now, to see that $C^C \in \mathcal{B}$,... $\square$

We now extend the semi-algebra to an algebra.

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Plan:

- complete proof of uniqueness under desirable conditions
- random variables
- integration

Theorem 2 (Dynkin's $\pi - \lambda$ Theorem)

A class of sets $\mathcal{P}$ is called a π-system if closed under intersections. A clsas of sets $\mathcal{L}$ is a λ-system if

1. $\Omega \in \mathcal{L}$
2. $A \subset B \in \mathcal{L}$ implies $B \setminus A \in \mathcal{L}$
3. closed under increasing limits: $(A_n) \in \mathcal{L}$ and $A_n \uparrow A$ implies $A \in \mathcal{L}$

If we have $\mathcal{P} \subset \mathcal{L}$ then $\sigma(P) \subset \mathcal{L}$.

Now, to show uniqueness of extension, we use the "good set principle". Basic strategy is to show every element of $\sigma(\mathcal{P})$ satisfies some nice properties and show that the class of sets satisfying it contains $\mathcal{P}$ and is a λ-system.

Proof. See **A.1** in Durrett. □

2.1 Random Variables

Definition 3 (Measurable Function)**.** Suppose we have (Ω, Σ, μ), and an observable (r.v.) $f : \Omega \to \mathbb{R}$. Then f is measurable (w/r to Σ) if $f^{-1}(B) \in \Sigma \forall B \in \mathcal{B}(\mathbb{R})$.

FIX: $(\Omega_1, \Sigma_1), (\Omega_2, \Sigma_2)$

Lemma 4. If $\Sigma_2 = \Sigma(\mathcal{A})$ then f is measurable if $f^{-1}(A) \in \Sigma_1$ for every $A \in \mathcal{A}$.

Proof. Consider the class of sets $X = \{B \in \Sigma(\mathcal{A}) : f^{-1}(B) \in \Sigma_1\}$. By construction it contains $\mathcal{A}$. So it suffices to show that X is a σ-algebra.

Suppose $B \in X$. Then

$$f^{-1}(B^C) = \left(f^{-1}(B)\right)^C$$

and

$$f^{-1}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} f^{-1}(B_i).$$

Therefore $X = \sigma(\mathcal{A})$. □

Exercise. Show that composition of measurable functions is measurable.

Exercise. Suppose $(f_i)_{i=1}^d$ are measurable functions from $(\Omega, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the tuple $(f_1, \ldots, f_d)$ is a measurable function going into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. (Hint: suffices to verify measurability for the generating sets)

Exercise. Suppose we have $(f_i)_{i=1}^d$ are as above. Then the sum $f_1 + \cdots + f_d$ is measurable. (Hint: express as composition of functions; also show that continuous functions are measurable)

Exercise. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions. Then $\limsup f_n$ and $\liminf f_n$ are measurable. (Hint: show $\sup f_n$ measurable first)

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Consider two measurable spaces (Ω_1, Σ_1) and $(\Omega_2, \Sigma_2) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then a measurable function $f : \Omega_1 \to \mathbb{R}$ will be called a *random variable*.

Given a sequence of measurable functions (f_i), we wish to show that

$$\sup f_i, \quad \inf f_i, \quad \limsup f_i, \quad \liminf f_i$$

are all measurable (*Exercise!*)

3.1 Distribution Functions

Given a random variable X on (Ω_1, Σ_1), and a μ on Σ_1, X induces a measure $\hat{\mu}$ on in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\hat{\mu}(B) = \mu(X^{-1}(B))$$

Assume μ is a probability measure. Then $\hat{\mu}$ induces a function F given by

$$F(x) = m\hat{u}((-\infty, x]) = \mathbb{P}(X \leq x)$$

known as the *distribution function*. We have the following properties:

1. F is monotone non-decreasing.
2. Since μ is bounded, F is right-continuous.
3. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$.

Now, the question is: can we retrieve the (unique) random variable from the distribution function satisfying the above properties (1-3)?

Suppose we are given $([0,1], \mathcal{B}([0,1]), \mu)$, where μ is the uniform measure. We want to retrieve $X : [0,1] \to \mathbb{R}$ given $F_X = F$. For $\omega \in (0,1)$. There is a unique x_ω such that $F(x_\omega) = \omega$, where we denote $X(\omega) = x_\omega$. See rest of proof in **Durrett 1.2**.

The moral is that R.V's are in one-to-one correspondence with distribution functions.

3.2 Integrals

Fix a measure space (Ω, Σ, μ), assume that μ is σ-finite. Suppose we have a measurable $f : \Omega \to \mathbb{R}$. We want to assign meaning to

$$\int f d\mu.$$

We do this in steps:

1. Simple functions (linear combinations of indicators):

$$f = \sum_{i=1}^{k} c_i \mathbb{I}_{A_i}, \quad \mu(A_i) < \infty, \quad A_i \text{ disjoint}$$

  2. Bounded functions with support E such that $\mu(E) < \infty$
  3. Nonnegative functions
  4. General functions

1. For simple functions, we define

$$\int f d\mu = \sum_{i=1}^{k} c_i \mu(A_i)$$

We may verify that the following properties are satisfied (*Exercise!*):

(i) $f \geq 0$ a.e. then $\int f \geq 0$.
(ii) $a \in \mathbb{R}$, then $\int af = a \int f$.
(iii) $\int f + g = \int f + \int g$.
(iv) $f \geq g$ a.e. then $\int f \geq \int g$.
(v) $f = g$ a.e then $\int f = \int g$.
(vi) $\left|\int f\right| \leq \int |f|$.

2. For bounded functions with finite support, we define

$$\begin{aligned} \int f &= \sup\left\{\int g : g \leq f, g \text{ is simple}\right\} = a \\ &= \inf\left\{\int f : h \geq f, h \text{ is simple}\right\} = b \end{aligned}$$

We need to show that this is well-defined, and check that (1) and (2) agree for simple functions.

3. For $f \geq 0$, we define

$$\int f = \sup\left\{\int h : 0 \leq h \leq f, \mu(\sup f) < \infty\right\}.$$

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Finishing integrals from last week:

1. Simple functions $\to$ bounded functions $\to$ non-negative functions.
2. From non-negative functions, we may extend to general measurable functions.

Take any measurable f. We say that f is integrablle if $\int |f| < \infty$. We define

$$f = f_+ - f_-,$$

where $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$. Then we define

$$\int f = \int f_+ - \int f_-.$$

Lemma 5. If $f = f_1 - f_2$ where $f_i \geq 0$ and $\int f_i < \infty$ for $i = 1, 2$, then $\int f = \int f_1 - \int f_2$.

Proof. **Lemma 1.4.6** in Durrett. □

Theorem 6 (Bounded Convergence)

Let f_n be defined in E such that $\mu(E) < \infty$ and $|f_n| \leq M$. Suppose $f_n \to f$ a.e. (for every point outside a zero set, $f_n(x) \to f(x)$ pointwise) for some measurable f. Then

$$\int f_n \to \int f.$$

Example 4.1 (Almost Sure Convergence but not in Probability/Measure)

Let $I_n = [-1/n, 1/n]$, and $f_n = \mathbb{I}_{I_n}$, defined on $[-1, 1]$. Move around subsequent intervals...?

Theorem 7 (Fatou's Lemma)

Let $f_n \geq 0$. Then

$$\liminf \int f_n \geq \int \liminf f_n.$$

Example 4.2

If we do not require non-negativity of the f_n, the theorem may not hold. For example, let $f_n = \mathbb{I}_{[-n,n]^C}$. Check that the theorem fails.

Theorem 8 (Monotone Convergence)
If $f_n \geq 0$ and $f_n \uparrow f$ then

$$\int f_n \uparrow \int f$$

Proof. One line proof from Fatou's Lemma. $\square$

Theorem 9 (Dominated Convergence)
If $f_n \to f$ a.e. and $|f| < g$ with $\int g < \infty$, then

$$\int f_n \to \int f.$$