Problem 1

$$\det(A-\lambda\mathrm{I}_3)=\det\left(\begin{bmatrix}4-\lambda & 0 & 0\\ 2.48 & 1.52-\lambda & 0.36\\ 0.64 & -0.64 & 2.48\end{bmatrix}\right)$$

$$=(4-\lambda)\left(\left(\frac{38}{25}-\lambda\right)\left(\frac{62}{25}-\lambda\right)+\frac{144}{625}\right)$$

$$\rightarrow (4-\lambda)(\lambda^2-4\lambda+4)=0\rightarrow \lambda=4,2,2$$

$$\text{When } \lambda=2\text{: } \begin{bmatrix}100 & 0 & 0\\ 62 & 38 & 9\\ 16 & -16 & 62\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=50\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$x=0, 4y=3z\rightarrow$ Only one linear independent eigenvector

So the Jordan canonical form of A is $\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$

Problem 2

$$\operatorname{vec}\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}X\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\right)+\operatorname{vec}(X)=\begin{bmatrix}0 & -1 & 8 & 22\end{bmatrix}^{\mathrm{T}}$$

By $\operatorname{vec}(ABC)=(C^{\mathrm{T}}\otimes A)\operatorname{vec}(B)$

$$\rightarrow\left(\begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\mathrm{I}_4\right)\operatorname{vec}(X)=\begin{bmatrix}0 & -1 & 8 & 22\end{bmatrix}^{\mathrm{T}}$$

$$\text{Let } X=\begin{bmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{bmatrix}$$

$$\begin{bmatrix}4 & 6 & 0 & 0\\ 9 & 13 & 0 & 0\\ 0 & 0 & 2 & 2\\ 0 & 0 & 3 & 5\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{21}\\ x_{12}\\ x_{22}\end{bmatrix}=\begin{bmatrix}0\\ -1\\ 8\\ 22\end{bmatrix}$$

$$\begin{cases}2x_{11}+3x_{21}=0\\ 9x_{11}+13x_{21}=-1\\ x_{12}+x_{22}=4\\ 3x_{12}+5_{x22}=22\end{cases}\rightarrow\begin{bmatrix}x_{11}\\ x_{21}\\ x_{12}\\ x_{22}\end{bmatrix}=\begin{bmatrix}-3\\ 2\\ -1\\ 5\end{bmatrix}$$

$$\rightarrow X=\begin{bmatrix}-3 & -1\\ 2 & 5\end{bmatrix}$$

Problem 3

$$\text{Two part:}\, B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$B_1^N = \begin{bmatrix} 1^N & \binom{N}{1} 1^{N-1} & \binom{N}{2} 1^{N-2} \\ 0 & 1^N & \binom{N}{1} 1^{N-1} \\ 0 & 0 & 1^N \end{bmatrix}, B_2^N = \begin{bmatrix} 2^N & \binom{N}{1} 2^{N-1} \\ 0 & 2^N \end{bmatrix}$$

$$B^{10} = \begin{bmatrix} B_1^{10} & \mathbf{0} \\ \mathbf{0} & B_2^{10} \end{bmatrix} = \begin{bmatrix} 1 & 10 & 45 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1024 & 5120 \\ 0 & 0 & 0 & 0 & 1024 \end{bmatrix}$$

Problem 4

4a

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, A^{\mathrm{H}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$AA^{\mathrm{H}} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}, A^{\mathrm{H}}A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\lambda_{v_1} = 3, \lambda_{v_2} = 2 \to \boldsymbol{v_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\to V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\lambda_{u_1} = 3, \lambda_{u_2} = 2, \lambda_{u_3} = 0 \to \boldsymbol{u_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$$\to U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$S=\begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\\ 0 & 0\end{bmatrix}=\begin{bmatrix}\sqrt{3} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}$$

4b

$$A=\begin{bmatrix}1\\ \mathrm{j}\end{bmatrix}, A^{\mathrm{H}}=[1 \quad -\mathrm{j}]$$

$$AA^{\mathrm{H}}=\begin{bmatrix}1 & -\mathrm{j}\\ \mathrm{j} & 1\end{bmatrix}, A^{\mathrm{H}}A=[1]$$

$$\lambda_{u_1}=2, \lambda_{u_2}=0 \rightarrow \boldsymbol{u_1}=\begin{bmatrix}1\\ \mathrm{j}\end{bmatrix}, \boldsymbol{u_2}=\begin{bmatrix}1\\ -\mathrm{j}\end{bmatrix}$$

$$\rightarrow U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{\mathrm{j}}{\sqrt{2}} & -\frac{\mathrm{j}}{\sqrt{2}}\end{bmatrix}$$

$$V=[1]$$

$$S=\begin{bmatrix}\sqrt{\lambda_1}\\ 0\end{bmatrix}=\begin{bmatrix}\sqrt{2}\\ 0\end{bmatrix}$$

Problem 5

$$\text{Expand } M \rightarrow M=\begin{bmatrix}7 & -5\\ -1 & 5\\ 0 & 0\end{bmatrix}$$

$$\text{Rewrite } M \rightarrow M=\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\sqrt{10} & 0\\ 0 & \sqrt{10}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}=USV^{\mathrm{H}}$$

$$\rightarrow \sigma_1=3\sqrt{10}, \sigma_2=\sqrt{10}$$

$$\|M\|_1=\max_j \sum_i |A_{ij}|=10$$

$$\|M\|_2=\sigma_1=3\sqrt{10}$$

$$\|M\|_\infty=\max_i \sum_j |A_{ij}|=12$$

$$\|M\|_*=\sum \sigma=4\sqrt{10}$$

Problem 7

$$A = \begin{bmatrix} 1 & 0 \\ j & j \\ 0 & 2 \end{bmatrix} \text{ full rank} \rightarrow A^{\dagger} = (A^{\mathrm{H}}A)^{-1}A^{\mathrm{H}}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\mathrm{j} & 0 \\ 0 & -\mathrm{j} & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 1 & -\mathrm{j} & 0 \\ 0 & -\mathrm{j} & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} & -\frac{\mathrm{j}4}{9} & -\frac{2}{9} \\ -\frac{1}{9} & -\frac{\mathrm{j}}{9} & \frac{4}{9} \end{bmatrix}$$

$$x_{\mathrm{LS}} = A^{\dagger}b = \begin{bmatrix} \frac{5}{9} & -\frac{\mathrm{j}4}{9} & -\frac{2}{9} \\ -\frac{1}{9} & -\frac{\mathrm{j}}{9} & \frac{4}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7-\mathrm{j}4}{9} \\ \frac{-5-\mathrm{j}}{9} \end{bmatrix}$$

Problem 8

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 0 & 5 \\ 0 & 10 \\ 0 & 20 \end{bmatrix}$$

$$\text{Two part: } A_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 5 \\ 10 \\ 20 \end{bmatrix}, \text{both full rank}$$

$$A_1^{\dagger} = (A^{\mathrm{H}}A)^{-1}A^{\mathrm{H}} = \frac{1}{21}\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$$

$$\text{Note that } A_2 = 5A_1 \rightarrow A_2^{\dagger} = \frac{1}{105}\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}$$

$$A^{\dagger} = \begin{bmatrix} A_1^{\dagger} & \mathbf{0} \\ \mathbf{0} & A_2^{\dagger} \end{bmatrix} = \begin{bmatrix} \frac{1}{21} & \frac{2}{21} & \frac{4}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{105} & \frac{2}{105} & \frac{4}{105} \end{bmatrix}$$

Problem 9

$$\text{Let } \boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a_r + \mathrm{j}a_i \\ b_r + \mathrm{j}b_i \\ c_r + \mathrm{j}c_i \end{bmatrix}, \text{where } a_r, b_r, c_r, a_i, b_i, c_i \in \mathbb{R}$$

$$\text{Plug in } 3 + \begin{bmatrix} 2 & \mathrm{j} & 1 \end{bmatrix} \boldsymbol{x} = 0$$

$$\rightarrow (3 + 2a_r - b_i + c_r) + \mathrm{j}(2a_i + b_r + c_i) = 0$$

$$\rightarrow \begin{cases} 3 + 2a_r - b_i + c_r = 0 \\ 2a_i + b_r + c_i = 0 \end{cases}$$

Minimize $\|\boldsymbol{x}\|_2^2$ means: Find minimal distance from the two plane to origin

Plane 1: $3 + 2a_r - b_i + c_r = 0$

Normal vector: $[2 \quad -1 \quad 1]^{\mathrm{T}}$

$\rightarrow$ the point on plane 1 that leads to minimal distance should be

$[a_r \quad b_i \quad c_r]^{\mathrm{T}} = [2t \quad -t \quad t]^{\mathrm{T}}$, where $t \in \mathbb{R}$

$$\text{Plug in plane 1} \rightarrow 3 + 4t + t + t = 0 \rightarrow t = -\frac{1}{2}$$

$$\rightarrow a_r = -1, b_i = \frac{1}{2}, c_r = -\frac{1}{2}$$

Plane 2: $2a_i + b_r + c_i = 0$

Plane 2 passes origin and 0 must be minimal distance

$\rightarrow a_i = 0, b_r = 0, c_i = 0$

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ \mathrm{j}\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \text{and } \|\boldsymbol{x}\|_2^2 = \frac{3}{2}$$