EM_HW4 P12 solutions

Problem 1

$x[n] = [1, 0, 2, 3, -1, 2, 2, 1, 0]$, $h[n] = [2, 2, 1, 1, 0, 0, 0, 0, 0]$

Let $g[n] = x[n] *_c h[n]$

$g[1] = \langle x[n], [2, 0, 0, 0, 0, 0, 1, 1, 2]\rangle = 5$

$g[2] = \langle x[n], [2, 2, 0, 0, 0, 0, 0, 1, 1]\rangle = 3$

$g[3] = \langle x[n], [1, 2, 2, 0, 0, 0, 0, 0, 1]\rangle = 5$

$g[4] = \langle x[n], [1, 1, 2, 2, 0, 0, 0, 0, 0]\rangle = 11$

$g[5] = \langle x[n], [0, 1, 1, 2, 2, 0, 0, 0, 0]\rangle = 6$

$g[6] = \langle x[n], [0, 0, 1, 1, 2, 2, 0, 0, 0]\rangle = 7$

$g[7] = \langle x[n], [0, 0, 0, 1, 1, 2, 2, 0, 0]\rangle = 10$

$g[8] = \langle x[n], [0, 0, 0, 0, 1, 1, 2, 2, 0]\rangle = 7$

$g[9] = \langle x[n], [0, 0, 0, 0, 0, 1, 1, 2, 2]\rangle = 6$

$\therefore g[n] = [5, 3, 5, 11, 6, 7, 10, 7, 6]$

Problem 2

(a)

For $x < 0$, $F_x(x) = 0$. For $x > 2$, $F_x(x) = 1$

For $0 < x < 2$, $F_x(x) = \int_0^x x/2dx = x^2/4$

$\mathbb{E}[x] = \int_0^2 x * x/2dx = 4/3$

(b)

$\mathbb{E}[x^2] = \int_0^2 x^2 * x/2dx = 2$

$variance = \mathbb{E}[x^2] - (\mathbb{E}[x])^2 = 2/9 \quad \therefore std = \sqrt{2}/3$

$skewness = \frac{\mathbb{E}[(x-\mu_x)^3]}{std^3} = \frac{\int_0^2 (x-4/3)^3 x/2dx}{(\sqrt{2}/3)^3} = -\frac{2\sqrt{2}}{5}$

3.

(a) Y is a linear transformation of X and the coefficient is $\frac{1}{5} > 0 \ \rightarrow$ correlation $= 1$

(b)

Mean	$\mathrm{E}[Y] = \mathrm{E}\left[\frac{X}{5} + 5\right] = \frac{1}{5}\mathrm{E}[X] + 5 = \frac{\mu}{5} + 5$
Variance	$\mathrm{Var}[Y] = \mathrm{E}[Y^2] - \mathrm{E}[Y]^2$ $= \mathrm{E}\left[\frac{X^2}{25} + 2X + 25\right] - \left(\frac{\mu^2}{25} + 2\mu + 25\right)$ $= \mathrm{E}\left[\frac{X^2}{25}\right] - \left(\frac{\mu^2}{25}\right) = \frac{1}{25}\mathrm{Var}[X] = \frac{1}{25}v$
Skewness	$s_Y = \mathrm{E}\left[\frac{(Y - \mathrm{E}[Y])^3}{(\sqrt{\mathrm{Var}(Y)})^3}\right] = \mathrm{E}\left[\frac{\frac{1}{125}(X - \mathrm{E}[X])^3}{\frac{1}{125}(\sqrt{\mathrm{Var}(X)})^3}\right] = s_X = s$
Kurtosis	$k_Y = \mathrm{E}\left[\frac{(Y - \mathrm{E}[Y])^4}{(\sqrt{\mathrm{Var}(Y)})^4}\right] = \mathrm{E}\left[\frac{\frac{1}{625}(X - \mathrm{E}[X])^3}{\frac{1}{625}(\sqrt{\mathrm{Var}(X)})^3}\right] = k_X = k$

4.

$$\text{Note that } y + 2x - 1 \text{ should pass } 0 \text{ so that } \delta(y + 2x - 1) \neq 0$$
$$\text{and } 0 < x < 100$$
$$\rightarrow -199 < y < 1$$
$$f_Y = \int_0^{100} \frac{1}{100}\delta(y + 2x - 1)dx$$
$$= \int_0^{200} \frac{1}{200}\delta(y + t - 1)dt = \frac{1}{200} \quad \text{for} \quad -199 < y < 1$$
$$f_X = \int_{-199}^{1} \frac{1}{100}\delta(y + 2x - 1)dy = \frac{1}{100} \quad \text{for} \quad 0 < x < 100$$

uniform distribution:

$$\mu_X = 50, \mu_Y = -99, \sigma_X = \sqrt{\frac{2500}{3}}, \sigma_Y = \sqrt{\frac{10000}{3}}$$

$$\mathrm{Cov}(x,y) = \mathrm{E}[(X-\mu_X)(Y-\mu_Y)] = \mathrm{E}[XY] + 4950$$
$$= \int_0^{100}\int_{-1}^{199} xy\frac{1}{100}\delta(y+2x-1)dy\,dx + 4950$$
$$= \int_0^{100} x(1-2x)\frac{1}{100}dx + 4950 = -\frac{5000}{3}$$
$$\mathrm{Corr}(x,y) = \frac{\mathrm{Cov}(x,y)}{\sigma_X\sigma_Y} = -1$$

5.

$$A\otimes B = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} & a_{13}b_{11} & a_{13}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} & a_{13}b_{21} & a_{13}b_{22} \end{bmatrix}$$
$$B\otimes A = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{11} & a_{13}b_{11} & a_{11}b_{12} & a_{12}b_{12} & a_{13}b_{12} \\ a_{11}b_{21} & a_{12}b_{21} & a_{13}b_{21} & a_{11}b_{22} & a_{12}b_{22} & a_{13}b_{22} \end{bmatrix}$$
$$J_1: 2*2, \text{Permute rows}$$
$$\text{No permutation of rows} \rightarrow J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$J_2: 6*6, \text{Permute columns}$$
$$\text{1st column of } B\otimes A \rightarrow \text{ 1st column of } A\otimes B$$
$$\text{2nd column of } B\otimes A \rightarrow \text{ 3rd column of } A\otimes B$$
$$\text{3rd column of } B\otimes A \rightarrow \text{ 5th column of } A\otimes B$$
$$\text{4th column of } B\otimes A \rightarrow \text{ 2nd column of } A\otimes B$$
$$\text{5th column of } B\otimes A \rightarrow \text{ 4th column of } A\otimes B$$
$$\text{6th column of } B\otimes A \rightarrow \text{ 6th column of } A\otimes B$$
$$\rightarrow \text{Nonzero indices: } (row, col) = (1,1), (2,3), (3,5), (4,2), (5,4), (6,6)$$
$$\rightarrow J_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

6.

$$\|\boldsymbol{v}(\alpha)\|_0 = \text{Number of nonzero elements} = \begin{cases} N, & \text{if } \alpha \neq 0 \\ 1, & \text{if } \alpha = 0 \end{cases}$$
$$\|\boldsymbol{v}(\alpha)\|_1 = \sum_{i=1}^{N} |\alpha^{i-1}| = \begin{cases} \dfrac{1-|\alpha|^N}{1-|\alpha|}, & \text{if } |\alpha| \neq 1 \\ N, & \text{if } |\alpha| = 1 \end{cases}$$

$$\|\boldsymbol{v}(\alpha)\|_2 = \sqrt{\sum_{i=1}^{N} |\alpha^{i-1}|^2} = \begin{cases} \sqrt{\dfrac{1-|\alpha|^{2N}}{1-|\alpha|^2}}, & \text{if } |\alpha| \neq 1 \\ \sqrt{N}, & \text{if } |\alpha| = 1 \end{cases}$$

$$\|\boldsymbol{v}(\alpha)\|_\infty = \max(|\alpha|) = \begin{cases} 1, & \text{if } |\alpha| \leq 1 \\ |\alpha^{N-1}|, & \text{if } |\alpha| = 1 \end{cases}$$

$$\boldsymbol{v}(\alpha)\boldsymbol{v}^T(\alpha) = \begin{bmatrix} 1 & & & & & \\ & \alpha^2 & & & & \\ & & \alpha^4 & & & \\ & & & \ddots & & \\ & & & & \alpha^{2N-4} & \\ & & & & & \alpha^{2N-2} \end{bmatrix}$$

$$\|\boldsymbol{v}(\alpha)\boldsymbol{v}^T(\alpha)\|_F = \mathrm{trace}\big(\boldsymbol{v}(\alpha)\boldsymbol{v}^T(\alpha)\big) = \sum_{i=1}^{N} \left|\alpha^{i-1}\right|^2 = \begin{cases} \dfrac{1-|\alpha|^{2N}}{1-|\alpha|^2}, & \text{if } |\alpha| \neq 1 \\ N, & \text{if } |\alpha| = 1 \end{cases}$$