EM-HW2 Solution

1. (a) : Let $u(x,y,z) = X(x)Y(y)Z(z)$ and apply $u(x,y,z)$ into the PDE

$$X'YZ + yXY'Z + z^2XYZ' = 0$$

Divided by XYZ

$$\frac{X'}{X} + \frac{yY'}{Y} + \frac{z^2Z'}{Z} = 0$$

Let $\frac{X'}{X} = \lambda$, $\frac{yY'}{Y} = \tau$, then $\frac{z^2Z'}{Z} = -\lambda - \tau$, by separation variable:

$$X = c_1 e^{\lambda x} \ , Y = c_2 y^{\tau} \ , Z = c_3 e^{\frac{\lambda+\tau}{z}}$$

Thus,

$$u(x,y,z) = \sum_{\lambda} \sum_{\tau} C_{\lambda,\tau} y^{\tau} e^{\lambda x + \frac{\lambda+\tau}{z}}$$

1. (b): Let $u(x,y,z) = X(x)Y(y)Z(z) + \phi(x) + \phi(y) + \phi(z)$ and apply $u(x,y,z)$ into the PDE.

$$X'YZ + \phi'(x) + XY'Z + \phi'(y) + XYZ' + \phi'(z) = x + y + z$$

Let $\phi'(x) = x \ , \phi'(y) = y \ , \phi'(z) = z$, $X'YZ + XY'Z + XYZ' = 0$

For $\phi'(x) = x \ , \phi'(y) = y \ , \phi'(z) = z$, by separation variable:

$$\phi(x) = \frac{x^2}{2} + c_1 \ , \phi(y) = \frac{y^2}{2} + c_2 \ , \phi(z) = \frac{z^2}{2} + c_3$$

For $X'YZ + XY'Z + XYZ' = 0$, divided by XYZ:

$$\frac{X'}{X} + \frac{Y'}{Y} + \frac{Z'}{Z} = 0$$

Let $\frac{X'}{X} = \lambda$, $\frac{Y'}{Y} = \tau$, $\frac{Z'}{Z} = -(\lambda+\tau)$, by separation variable:

$$X(x) = c_4 e^{\lambda x} \ , Y(y) = c_5 e^{\tau y} \ , Z(z) = c_6 e^{-(\lambda+\tau)z}$$

Thus,

$$X(x)Y(y)Z(z) = \sum_{\lambda}\sum_{\tau} C_{\lambda,\tau} e^{\lambda x + \tau y - (\lambda+\tau)z}$$

The overall solution is

$$u(x,y,z) = \sum_{\lambda}\sum_{\tau} C_{\lambda,\tau} e^{\lambda x + \tau y - (\lambda+\tau)z} + \frac{x^2+y^2+z^2}{2} + c$$

1. (c): Let $u(x,y,t) = X(x)Y(y)T(t)$ and apply $u(x,y,t)$ into the PDE

$$X''YT + XY''T = XYT'$$

Divided by XYT

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{T'}{T}$$

Let $\frac{X''}{X} = -\lambda$, $\frac{Y''}{Y} = -\tau$, $\frac{T'}{T} = -(\lambda+\tau)$

For $\frac{X''}{X} = -\lambda$ and $u(0,y,t) = u(2,y,t) = 0 \rightarrow X(0) = X(2) = 0$:

<Case 1>: $\lambda = 0$

$X'' = 0 \rightarrow X(x) = c_1 x + c_2$ and apply $X(0) = X(2) = 0$

$\rightarrow c_1 = c_2 = 0 \quad \rightarrow X = 0$ (Trivial)

<Case 2>: $\lambda < 0$ and let $\lambda = -\beta^2$ $(\beta > 0)$

$X'' - \beta^2 X = 0 \rightarrow X(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x)$ and apply

$X(0) = X(2) = 0 \rightarrow c_1 = c_2 = 0 \quad \rightarrow X = 0$ (Trivial)

<Case 3>: $\lambda > 0$ and let $\lambda = \beta^2$ $(\beta > 0)$

$X'' + \beta^2 X = 0 \rightarrow X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$ and apply

$X(0) = X(2) = 0 \rightarrow X(0) = 0: \quad c_1 = 0 \quad \rightarrow X(x) = c_2 \sin(\beta x)$

$\rightarrow X(2) = 0: \quad c_2 \sin(2\beta) = 0 \rightarrow \sin(2\beta) = 0 \rightarrow 2\beta = m\pi \quad, m = 1,2,\dots$

$\rightarrow \beta = \frac{m\pi}{2} \quad ,\lambda = \beta^2 = \frac{m^2\pi^2}{4}, \; m = 1,2,\dots$

$\rightarrow X(x) = c_2 \sin\left(\frac{m\pi}{2} x\right) \; , \; m = 1,2,\dots$

Similarly available for $\frac{Y''}{Y} = -\tau$

And $\quad u(x,0,t) = u(x,2,t) = 0 \rightarrow Y(0) = Y(2) = 0 \;:$

$\rightarrow \tau = \frac{n^2\pi^2}{4}, \; n = 1,2,\dots$

$\rightarrow Y(y) = c_4 \sin\left(\frac{n\pi}{2} y\right) \; , \; n = 1,2,\dots$

For $\frac{T'}{T} = -(\lambda + \tau)$ and $\lambda = \frac{m^2\pi^2}{4} \; ,\tau = \frac{n^2\pi^2}{4},$ by separation variable:

$$\mathrm{T}(t) = c_6 e^{-\frac{\pi^2}{4}(m^2+n^2)t}$$

Thus,

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n} \sin\left(\frac{m\pi}{2}x\right)\sin\left(\frac{n\pi}{2}y\right) e^{-\frac{\pi^2}{4}(m^2+n^2)t}$$

Apply ICs : $u(x,y,0) = (2x - x^2)(2y - y^2)$ into $u(x,y,t)$

$$(2x - x^2)(2y - y^2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n} \sin\left(\frac{m\pi}{2}x\right)\sin\left(\frac{n\pi}{2}y\right)$$

Because $(2x - x^2)$ *and* $(2y - y^2)$ are un-correlated, we can assume

$C_{m,n} = C_m \cdot C_n$, then use the rule of Fourier series:

$$f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right) \quad , \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

By integration by parts, we can obtain the followings:

$$C_m=\frac{2}{2}\int_0^2(2x-x^2)\sin\left(\frac{m\pi}{2}x\right)dx=\frac{16}{m^3\pi^3}(1-(-1)^m)\ ,m=1,2,\ldots$$

$$C_n=\frac{2}{2}\int_0^2(2y-y^2)\sin\left(\frac{n\pi}{2}y\right)dy=\frac{16}{n^3\pi^3}(1-(-1)^n)\ ,n=1,2,\ldots$$

Thus, $C_{m,n}=C_m\cdot C_n=\frac{256}{m^3n^3\pi^6}(1-(-1)^m)(1-(-1)^n)$

The overall solution is

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{256}{m^3n^3\pi^6}(1-(-1)^m)(1-(-1)^n)\sin\left(\frac{m\pi}{2}x\right)\sin\left(\frac{n\pi}{2}y\right)e^{-\frac{\pi^2}{4}(m^2+n^2)t}$$

2. Let $u(r,\theta)=R(r)\Theta(\theta)$ and apply $u(r,\theta)$ into the following PDE

$$r^2\frac{\partial^2u(r,\theta)}{\partial r^2}+r\frac{\partial u(r,\theta)}{\partial r}+\frac{\partial^2u(r,\theta)}{\partial\theta^2}=0$$

And we obtain

$$r^2R''\Theta+rR'\Theta+R\Theta''=0$$

Divided by Θ :

$$\frac{r^2R''+rR'}{R}+\frac{\Theta''}{\Theta}=0$$

Let $\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda$

For $-\frac{\Theta''}{\Theta}=\lambda$ and BCs: $u(r,0)=u\left(r,\frac{\pi}{3}\right)=0\rightarrow\Theta(0)=\Theta\left(\frac{\pi}{3}\right)=0$

Similarly available from 1-(c):

$$\mathrm{L} = \frac{\pi}{3} \to \lambda = \left(\frac{n\pi}{L}\right)^2 = 9n^2 \ , \Theta(\theta) = c_1 \sin\left(\frac{n\pi}{L}\theta\right) = c_1 \sin(3n\theta),$$

$$n = 1,2, \ldots$$

For $\frac{r^2R''+rR'}{R} = \lambda$:

$$r^2R'' + rR' - \lambda R = 0 \ \ (Cauchy - Euler \ \ equation)$$

Let $R(r) = r^p$ and apply it into Cauchy-Euler equation

$$\mathrm{p}(p-1)r^p + pr^p - \lambda r^p = 0$$

$$p^2 - \lambda = 0 \to p = \sqrt{\lambda} \ , -\sqrt{\lambda} = 3n \ , -3n$$

$$R(r) = c_2 r^{3n} + c_3 r^{-3n} \ , n = 1,2, \ldots, 0 < r < 1$$

Because $0 < r < 1$, $c_3 r^{-3n}$ diverges when $n = 1,2,$ .....

So, we need a solution which converges :

$$R(r) = c_2 r^{3n} \ , n = 1,2, \ldots, 0 < r < 1$$

Thus,

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n r^{3n} \sin(3n\theta)$$

Apply $\mathrm{u}(1,\theta) = \sin(6\theta) + \sin(12\theta)$ into $u(r,\theta)$:

$$\sin(6\theta) + \sin(12\theta) = \sum_{n=1}^{\infty} C_n \sin(3n\theta)$$

We obtain $C_2 = C_4 = 1$, $C_n = 0 \ , n \neq 2, 4$

The overall solution is

$$u(r,\theta) = r^6 \sin(6\theta) + r^{12} \sin(12\theta)$$

3. Let $u(r,z) = R(r)Z(z)$ and apply $\mathrm{u}(r,z)$ into the following PDE

$$\frac{\partial^2 u(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z)}{\partial r} + \frac{\partial^2 u(r,z)}{\partial z^2} = 0$$

And we obtain

$$R''Z + \frac{1}{r}R'Z + RZ'' = 0$$

Divided by Z :

$$\frac{R'' + \frac{1}{r}R'}{R} + \frac{Z''}{Z} = 0$$

Let $\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \lambda$

For $-\frac{Z''}{Z} = \lambda$ and BCs: $u(r,0) = u(r,2) = 0 \rightarrow \mathrm{Z}(0) = \mathrm{Z}(2) = 0$

Similarly available from 1-(c):

$$\mathrm{L} = 2 \rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{4} \quad , Z(z) = c_1 \sin\left(\frac{n\pi}{L}z\right) = c_1 \sin\left(\frac{n\pi}{2}z\right),$$

$$n = 1,2, \ldots$$

For $\frac{R'' + \frac{1}{r}R'}{R} = \lambda$:

$$r^2R'' + rR' - \lambda r^2 R = 0$$

Apply $= \frac{n^2\pi^2}{4}$:

$$r^2R'' + rR' - \frac{n^2\pi^2 r^2}{4}R = 0$$

Use the following solutions (Modifed Bessel function PDE):

$$r^2R'' + rR' - (\alpha^2 r^2 + \nu^2)R = 0 \quad \rightarrow R(r) = c_1 I_\nu(\alpha r) + c_2 K_\nu(\alpha r)$$

For $\alpha = \frac{n\pi}{2}$ and $\nu = 0$ and $K_\nu(r=0) \rightarrow \infty$ $(Diverges)$ $, 0 < r < 1$

$$\mathrm{R}(r) = c_3 I_0\left(\frac{n\pi}{2}r\right) \quad , n = 1,2, \ldots$$

Thus,

$$u(r,z)=\sum_{n=1}^{\infty} C_n I_0\left(\frac{n\pi}{2}r\right)\sin\left(\frac{n\pi}{2}z\right)$$

Use ICs:

$$\mathrm{u}(1,\mathrm{z})=\begin{cases}\mathrm{z}, & 0<\mathrm{z}<1\\ 2-\mathrm{z}, & 1<\mathrm{z}<2\end{cases}$$

$$\mathrm{u}(1,\mathrm{z})=\sum_{n=1}^{\infty} C_n I_0\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}z\right)$$

Use the rule of Fourier series:

$$f(x)=\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) \quad , \quad A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Using integration by parts and we obtain

$$\begin{aligned}C_n I_0\left(\frac{n\pi}{2}\right)&=\int_0^2 \mathrm{u}(1,\mathrm{z})\sin\left(\frac{n\pi}{2}z\right)dz\\ &=\int_0^1 \mathrm{z}\sin\left(\frac{n\pi}{2}z\right)dz+\int_1^2 (2-z)\sin\left(\frac{n\pi}{2}z\right)dz\\ &=\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\end{aligned}$$

We obtain

$$C_n=\frac{8}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)}\sin\left(\frac{n\pi}{2}\right) \quad , n=1,2,\ldots$$

The overall solution is

$$u(r,z)=\sum_{n=1}^{\infty}\frac{8}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)}\sin\left(\frac{n\pi}{2}\right) I_0\left(\frac{n\pi}{2}r\right)\sin\left(\frac{n\pi}{2}z\right)$$

4. Use the following rule of Laplace transform $(t \to s)$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(t=0)$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \quad ,\mathrm{n} > -1 \ \text{ and } \ \mathrm{n} \in \mathbb{Z}$$

Do the Laplace transform for the equation from t domain to s domain:

$$\mathcal{L}\left[\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right] = \mathcal{L}[1]$$

$$\to \frac{\mathrm{d}U(x,s)}{\mathrm{dx}} + sU(x,s) - u(x,t=0) = \frac{1}{s}$$

Apply ICs: $u(x,t=0) = 0$ and obtain

$$\frac{\mathrm{d}U(x,s)}{\mathrm{dx}} + sU(x,s) = \frac{1}{s}$$

Use the 1st order linear equation and integration factor to solve:

$$U(x,s) = ce^{-sx} + \frac{1}{s^2}$$

Apply the Laplace transform into $\mathrm{u}(0,t) = t^2 + t$

$$\mathcal{L}[\mathrm{u}(0,t)] = U(0,s) = \frac{2}{s^3} + \frac{1}{s^2}$$

Apply $U(0,s)$ into $U(x,s)$ and we obtain

$$\frac{2}{s^3} + \frac{1}{s^2} = c + \frac{1}{s^2} \quad \to c = \frac{2}{s^3} \quad \to \quad U(x,s) = \frac{2}{s^3}e^{-sx} + \frac{1}{s^2}$$

Use the inverse Laplace transform and finally obtain the overall solution

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\} = (t-x)^2 u(t-x) + t$$

EM_HW2 P5 solution

Problem 5

(a)

Let the orthonormal set be $\{v_1, v_2, v_3\}$.

$v_1 = \frac{1}{\sqrt{\int_0^4 1dx}} = 1/2.$

Let $a_2 = x - \left(\int_0^4 x/2dx\right)\frac{1}{2} = x - 2.$

$v_2 = \frac{a_2}{\sqrt{\int_0^4 a_2a_2dx}} = \frac{x-2}{4/\sqrt{3}} = \frac{\sqrt{3}}{2}(\frac{x}{2} - 1).$

$$\begin{aligned}\text{Let } a_3 &= x^2 - \left(\int_0^4 x^2/2dx\right)\frac{1}{2} - \left(\int_0^4 (\frac{\sqrt{3}x^3}{4} - \frac{\sqrt{3}}{2}x^2)dx\right)\frac{\sqrt{3}}{2}(\frac{x}{2} - 1)\\ &= (x-2)^2 - 4/3\end{aligned}$$

$v_3 = \frac{a_3}{\sqrt{\int_0^4 a_3a_3dx}} = \frac{\sqrt{5}}{4}(3(\frac{x}{2} - 1)^2 - 1)$

(b)

$$\begin{aligned}q(x) = &\left(\int_0^4 \min(x, 4-x)\frac{1}{2}dx\right)\frac{1}{2} +\\ &\left(\int_0^4 \min(x, 4-x)\frac{\sqrt{3}}{2}(\frac{x}{2} - 1)dx\right)\frac{\sqrt{3}}{2}(\frac{x}{2} - 1) +\\ &\left(\int_0^4 \min(x, 4-x)\frac{\sqrt{5}}{4}(3(\frac{x}{2} - 1)^2 - 1)dx\right)\frac{\sqrt{5}}{4}(3(\frac{x}{2} - 1)^2 - 1)\\ &= 2 * \frac{1}{2} + 0 * \frac{\sqrt{3}}{2}(\frac{x}{2} - 1) - \frac{\sqrt{5}}{2}\frac{\sqrt{5}}{4}(3(\frac{x}{2} - 1)^2 - 1)\\ &= 1 - \frac{5}{8}(3(\frac{x}{2} - 1)^2 - 1)\end{aligned}$$

[Another Method]

Let $q(x) = a + bx + cx^2$

$E = \int_0^4 (\min(x, 4-x) - q(x))^2 dx$
$\quad = \int_0^2 (x - q(x))^2 dx + \int_2^4 (4 - x - q(x))^2 dx$
$\quad = \int_0^2 (a + (b-1)x + cx^2)^2 dx + \int_2^4 ((a-4) + (b-1)x + cx^2)^2 dx$

We want to minimize the approximation error E:

$\frac{\partial E}{\partial a} = \int_0^2 2(a + (b-1)x + cx^2)dx + \int_2^4 2((a-4) + (b+1)x + cx^2)dx$
$\quad = 8a + 16b + \frac{128}{3}c - 8$

$\frac{\partial E}{\partial b} = \int_0^2 2(a + (b-1)x + cx^2)xdx + \int_2^4 2((a-4) + (b+1)x + cx^2)xdx$
$\quad = 16a + \frac{128}{3}b + 128c - 16$

$\frac{\partial E}{\partial c} = \int_0^2 2(a + (b-1)x + cx^2)x^2dx + \int_2^4 2((a-4) + (b+1)x + cx^2)x^2dx$
$\quad = \frac{128}{3}a + 128b + \frac{2048}{5}c - \frac{112}{3}$

Let $\frac{\partial E}{\partial a} = 0$; $\frac{\partial E}{\partial b} = 0$; $\frac{\partial E}{\partial c} = 0 \Rightarrow a = \frac{-1}{4}$; $b = \frac{15}{8}$; $\frac{-15}{32}$

$\therefore q(x) = -\frac{1}{4} + \frac{15}{8}x - \frac{15}{32}x^2$