EM-HW1 Solutions

(1)-(a): $y''(x)y'(x) = 1\,, y'(0) = 0$

By reduction of order, assume $u(x) = y'(x) \rightarrow u' = y''$

Thus, the original DE can be rewritten as

$$u'u = 1 \ \text{, ICs: } u'(0) = 0$$

$\rightarrow udu = dx \ \rightarrow \frac{1}{2}u^2 = x + c_1$

Apply ICs: $u'(0) = 0$: we obtain $c_1 = 0$

$\frac{1}{2}u^2 = x \rightarrow u^2 = 2x \rightarrow u = \pm\sqrt{2x} \rightarrow y' = \pm\sqrt{2x}$ (Both solutions are satisfied with $y'(0) = 0$)

$\rightarrow dy = \pm\sqrt{2x}dx \rightarrow y = \pm\frac{1}{3}(2x)^{\frac{3}{2}} + c_2$

(1)-(b): $y''(x) = -3y'(x)y^2(x), y(1) = 2^{-\frac{1}{2}}, y'(1) = -2^{-\frac{3}{2}}$

Assume $u(y) = y' \rightarrow u\frac{du}{dy} = y''$

Thus, the original DE can be rewritten as

$$u\frac{du}{dy} = -3uy^2$$

$\rightarrow \frac{du}{dy} = -3y^2 \rightarrow du = -3y^2dy \rightarrow u = -y^3 + c_1 \rightarrow y' = -y^3 + c_1$

Apply ICs: $y(1) = 2^{-\frac{1}{2}}$, $y'(1) = -2^{-\frac{3}{2}}$, we obtain $c_1 = 0 \ \rightarrow y' = -y^3$

$\rightarrow -y^{-3}dy = dx \rightarrow \frac{1}{2}y^{-2} = x + c_2$, and apply ICs: $y(1) = 2^{-\frac{1}{2}}$, we obtain $c_2 = 0$

$\rightarrow \frac{1}{2}y^{-2} = x \rightarrow y = (2x)^{-\frac{1}{2}}$

(1)-(c): $y''(x) = exp\big(y(x)\big),\ y(0) = 0\,, y'(0) = \sqrt{2}$

Assume $u(y) = y' \rightarrow u\frac{du}{dy} = y''$

Thus, the original DE can be rewritten as

$$u\frac{du}{dy} = e^y$$

$\rightarrow udu = e^y dy \rightarrow \frac{1}{2}u^2 = e^y + c_1 \rightarrow \frac{1}{2}(y')^2 = e^y + c_1$

Apply ICs: $y(0) = 0$, $y'(0) = \sqrt{2}$, we obtain $c_1 = 0$ $\rightarrow \frac{1}{2}(y')^2 = e^y$

$\rightarrow y' = \pm\sqrt{2}e^{\frac{y}{2}}$ (Only $y' = \sqrt{2}e^{\frac{y}{2}}$ is satisfied with$y(0) = 0$, $y'(0) = \sqrt{2}$) $\rightarrow y' = \sqrt{2}e^{\frac{y}{2}}$

$\rightarrow e^{-\frac{y}{2}}dy = \sqrt{2}dx \rightarrow -2e^{-\frac{y}{2}} = \sqrt{2}x + c_2$

Apply ICs: $y(0) = 0$, we obtain $c_2 = -2$

$\rightarrow -2e^{-\frac{y}{2}} = \sqrt{2}x - 2 \rightarrow e^{-\frac{y}{2}} = 1 - \frac{x}{\sqrt{2}} \rightarrow y = -2\ln\left|1 - \frac{x}{\sqrt{2}}\right|$

(2)-(a): $\boldsymbol{x^2 \frac{\partial u(x,y)}{\partial x} = y\frac{\partial u(x,y)}{\partial y}}$

Assume $u(x,y) = X(x)Y(y)$ and apply it into the PDE

$x^2\frac{\partial u(x,y)}{\partial x} = y\frac{\partial u(x,y)}{\partial y} \rightarrow x^2X'Y = yXY' \rightarrow \frac{x^2X'}{X} = \frac{yY'}{Y} = -\lambda$

$\rightarrow \begin{cases} x^2X' = -\lambda X \\ yY' = -\lambda Y \end{cases}$

$\rightarrow For \;\; x^2X' = -\lambda X \rightarrow \frac{1}{X}dX = -\lambda x^{-2}dx \rightarrow X = c_1e^{\frac{\lambda}{x}}$

$For \;\; yY' = -\lambda Y \rightarrow \frac{1}{Y}dY = -\lambda\frac{1}{y}dy \rightarrow Y = c_2y^{-\lambda}$

Thus,

$$u(x,y) = X(x)Y(y) = \sum_{\lambda} c_\lambda e^{\frac{\lambda}{x}}y^{-\lambda}$$

(2)-(b):

$$\boldsymbol{\frac{\partial^2 u(x,y)}{\partial x^2} = u(x,y) + \frac{\partial u(x,y)}{\partial y} \;,\;\; 0 < x < 2 \;,\;\; y > 0}$$

$$\boldsymbol{u(0,y) = u(2,y) = 0, u(x,0) = \cos(\pi x)\sin(2\pi x)}$$

Assume $u(x,y) = X(x)Y(y)$ and apply it into the PDE

$\frac{\partial^2 u(x,y)}{\partial x^2} = u(x,y) + \frac{\partial u(x,y)}{\partial y} \rightarrow X''Y = XY + XY' \rightarrow \frac{X''}{X} = 1 + \frac{Y'}{Y} = -\lambda$

$\rightarrow \begin{cases} X'' + \lambda X = 0 \\ Y' + (1+\lambda)Y = 0 \end{cases}$, $u(0,y) = u(2,y) = 0 \rightarrow X(0) = X(2) = 0$

For $X'' + \lambda X = 0$:

<Case 1>: $\lambda = 0$

$X'' = 0 \to X(x) = c_1 x + c_2$ and apply $X(0) = X(2) = 0 \to c_1 = c_2 = 0 \to X = 0$ (Trivial)

<Case 2>: $\lambda < 0$ and let $\lambda = -\beta^2$ $(\beta > 0)$

$X'' - \beta^2 X = 0 \to X(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x)$ and apply $X(0) = X(2) = 0$

$\to c_1 = c_2 = 0 \to X = 0$ (Trivial)

<Case 3>: $\lambda > 0$ and let $\lambda = \beta^2$ $(\beta > 0)$

$X'' + \beta^2 X = 0 \to X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$ and apply $X(0) = X(2) = 0$

$\to X(0) = 0 :\ c_1 = 0 \to X(x) = c_2 \sin(\beta x)$

$\to X(2) = 0 :\ c_2 \sin(2\beta) = 0 \to \sin(2\beta) = 0 \to 2\beta = n\pi$, $n = 1,2,\ldots$

$\to \beta = \frac{n\pi}{2}$ $, \lambda = \beta^2 = \frac{n^2\pi^2}{4}$, $n = 1,2,\ldots$

$\to X(x) = c_2 \sin\left(\frac{n\pi}{2} x\right)$, $n = 1,2,\ldots$

For $Y' + (1+\lambda)Y = 0$:

$\to Y(y) = e^{-(1+\lambda)y} = e^{-\left(1+\frac{n^2\pi^2}{4}\right)y}$, $n = 1,2,\ldots$

Thus, we obtain $u(x,y) = X(x)Y(y)$

$$u(x,y) = \sum_{n=1}^{\infty} C_n\, e^{-\left(1+\frac{n^2\pi^2}{4}\right)y} \sin\left(\frac{n\pi}{2} x\right)$$

Apply the initial condition: $u(x,0) = \cos(\pi x)\sin(2\pi x)$

By the trigonometric rule: $\sin(x)cos(y) = \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)$

$\to u(x,0) = \cos(\pi x)\sin(2\pi x) = \frac{1}{2}\sin(\pi x) + \frac{1}{2}\sin(3\pi x)$ and apply it into $u(x,y)$

$\to \frac{1}{2}\sin(\pi x) + \frac{1}{2}\sin(3\pi x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2} x\right)$ and we obtain

$$\to \begin{cases} C_2 = C_6 = \frac{1}{2} \\ C_n = 0\,, otherwise \end{cases}$$

Thus,

$$u(x,y) = \frac{1}{2} e^{-(1+\pi^2)y} sin(\pi x) + \frac{1}{2} e^{-(1+9\pi^2)y} sin(3\pi x)$$

(2)-(c):

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0\,,\ \ 0 < x < 1\ \ ,\ \ 0 < y < 1$$

$$u(0,y) = u(1,y) = u(x,0) = 0\ ,\qquad u(x,1) = 1 - 2\left|x - \frac{1}{2}\right|$$

Assume $u(x,y) = X(x)Y(y)$ and apply it into the PDE

$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \to X''Y + XY'' = 0 \to \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$\to \begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$, $u(0,y) = u(1,y) = 0 \to X(0) = X(1) = 0$

For $X'' + \lambda X = 0$:

<Case 1>: $\lambda = 0$

$X'' = 0 \to X(x) = c_1 x + c_2$ and apply $X(0) = X(1) = 0$ $\to c_1 = c_2 = 0$ $\to X = 0$ (Trivial)

<Case 2>: $\lambda < 0$ and let $\lambda = -\beta^2$ $(\beta > 0)$

$X'' - \beta^2 X = 0 \to X(x) = c_1 \cosh(\beta x) + c_2 \sinh(\beta x)$ and apply $X(0) = X(1) = 0$

$\to c_1 = c_2 = 0$ $\to X = 0$ (Trivial)

<Case 3>: $\lambda > 0$ and let $\lambda = \beta^2$ $(\beta > 0)$

$X'' + \beta^2 X = 0 \to X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x)$ and apply $X(0) = X(1) = 0$

$\to X(0) = 0 :\ c_1 = 0$ $\to X(x) = c_2 \sin(\beta x)$

$\to X(1) = 0 :\ c_2 \sin(\beta) = 0 \to \sin(\beta) = 0 \to \beta = n\pi$, $n = 1,2,\ldots$

$\to \beta = n\pi$ $, \lambda = \beta^2 = n^2\pi^2,$ $n = 1,2,\ldots$

$\to X(x) = c_2 \sin(n\pi x)$, $n = 1,2,\ldots$

For $Y'' - \lambda Y = 0$:

$\to \lambda = \beta^2 = n^2\pi^2,$ $n = 1,2,\ldots$ $\to Y'' - n^2\pi^2 Y = 0 \to Y(y) = c_3 \cosh(n\pi y) + c_4 \sinh(n\pi y)$

$u(x,0) = 0 \to Y(0) = 0 \to c_3 = 0$

$\to Y(y) = c_4 \sinh(n\pi y)$, $n = 1,2,\ldots$

Thus, we obtain $u(x,y) = X(x)Y(y)$

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi y)\, sin(n\pi x)$$

The rule of Fourier series:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\ ,\ \ A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

Apply the rule of Fourier series and $u(x,1) = 1 - 2\left|x - \frac{1}{2}\right|$, we obtain

$$1-2\left|x-\frac{1}{2}\right|=\sum_{n=1}^{\infty} C_n \sinh(n\pi)\, sin(n\pi x)$$

$$\rightarrow C_n\sinh(n\pi)=\int_0^1\left(1-2\left|x-\frac{1}{2}\right|\right) sin(n\pi x)dx$$

$$\rightarrow C_n\sinh(n\pi)=\int_0^{\frac{1}{2}} 2x\, sin(n\pi x)dx+\int_{\frac{1}{2}}^{1}(2-2x)\, sin(n\pi x)dx$$

Use integration by parts $\int udv=uv-\int vdu$:

$$\rightarrow C_n\sinh(n\pi)=\int_0^{\frac{1}{2}} 2x\, sin(n\pi x)dx+\int_{\frac{1}{2}}^{1}(2-2x)\, sin(n\pi x)dx$$

$$\rightarrow C_n\sinh(n\pi)=\left[-\frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}+\frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right]+\left[\frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}+\frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\right]=\frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}$$

$$\rightarrow C_n=\frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh(n\pi)}\ ,\quad n=1,2,\ldots$$

Thus,

$$u(x,y)=\sum_{n=1}^{\infty}\frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh(n\pi)}\sinh(n\pi y)\, sin(n\pi x)$$

(2)-(d): $\boldsymbol{(x+1)\frac{\partial u(x,y)}{\partial x}=\frac{\partial u(x,y)}{\partial y}+\cos(y)}$

Assume $u(x,y)=v(x,y)+\phi(y)$ and apply it into the PDE

$$(x+1)\frac{\partial u(x,y)}{\partial x}=\frac{\partial u(x,y)}{\partial y}+\cos(y)\rightarrow(x+1)\frac{\partial v(x,y)}{\partial x}=\frac{\partial v(x,y)}{\partial y}+\phi'(y)+\cos(y)$$

Let $\phi'(y)+\cos(y)=0\ \rightarrow\ \phi(y)=-\sin(y)$

And we obtain the new PDE $(x+1)\frac{\partial v(x,y)}{\partial x}=\frac{\partial v(x,y)}{\partial y}$

Let $v(x,y)=X(x)Y(y)$ and apply it into the new PDE

$\rightarrow(x+1)X'Y=XY'\rightarrow(x+1)\frac{X'}{X}=\frac{Y'}{Y}=\lambda$ (Also you can assume $-\lambda$)

$$\rightarrow\begin{cases}X'-\frac{\lambda}{(x+1)}X=0\\ Y'-\lambda Y=0\end{cases}$$

For $X'-\frac{\lambda}{(x+1)}X=0$:

$\rightarrow\frac{dX}{X}=\frac{\lambda}{(x+1)}dx\rightarrow X(x)=c_1(x+1)^{\lambda}$

For $Y' - \lambda Y = 0$:

$\rightarrow \frac{dY}{Y} = \lambda dy \rightarrow Y(y) = c_2 e^{\lambda y}$

Thus, we obtain $v(x,y) = X(x)Y(y)$

$$v(x,y) = \sum_{\lambda} C_{\lambda}(x+1)^{\lambda} e^{\lambda y}$$

Finally, we obtain $u(x,y) = v(x,y) + \phi(y)$

$$u(x,y) = \sum_{\lambda} C_{\lambda}(x+1)^{\lambda} e^{\lambda y} - \sin(y)$$

(3) :

```
close; clear all; clc;
% Parameter and array settings:
h = 0.01; y0 = 0; x = 0:h:10; y_euler = zeros(1,length(x)); y_im_euler = zeros(1,length(x)); y_RK4 = zeros(1,length(x));
for i =1:1:length(x)-1
    % Euler's method:
    y_euler(i+1) = y_euler(i) + h*5*cos((-1/5)*abs(x(i)*y_euler(i)));
    % Improved Euler's method:
    y_im_euler_star = y_im_euler(i) + h*5*cos((-1/5)*abs(x(i)*y_im_euler(i)));
    y_im_euler(i+1) = y_im_euler(i) + h*(5*cos((-1/5)*abs(x(i)*y_im_euler(i)))+5*cos((-1/5)*abs(x(i+1)*y_im_euler_star)))/2;
    % RK4 method:
    k1 = 5*cos((-1/5)*abs(x(i)*y_RK4(i)));
    k2 = 5*cos((-1/5)*abs((x(i)+0.5*h)*(y_RK4(i)+0.5*h*k1)));
    k3 = 5*cos((-1/5)*abs((x(i)+0.5*h)*(y_RK4(i)+0.5*h*k2)));
    k4 = 5*cos((-1/5)*abs((x(i)+h)*(y_RK4(i)+h*k3)));
    y_RK4(i+1) = y_RK4(i) + (h/6)*(k1+2*k2+2*k3+k4);
end
% Plot:
figure()
subplot(311)
plot(x,y_euler,'LineWidth',2);grid on;xlabel('x','FontSize',16);ylabel('y(x)','FontSize',16);title('Euler method','FontSize',16);
subplot(312)
plot(x,y_im_euler,'LineWidth',2);grid on;xlabel('x','FontSize',16);ylabel('y(x)','FontSize',16);title('Improved Euler method','FontSize',16);
subplot(313)
plot(x,y_RK4,'LineWidth',2);grid on;xlabel('x','FontSize',16);ylabel('y(x)','FontSize',16);title('RK4 method','FontSize',16);
```