1.

(a) $y''y' = 1$, $y'(0) = 0$

$u = y'$

$\frac{du}{dx} u = 1$

$u\,du = dx$

$\frac{1}{2} u^2 = x + C$

代入 $y'(0) = 0 \Rightarrow 0 = 0 + C$, $C = 0$

$u = \pm\sqrt{2x}$

$y = \int \pm\sqrt{2x}\, dx$

$y = \pm \frac{2\sqrt{2}}{3} x^{3/2} + C_1$ #

(b) $y'' = -3y'y^2$, $y(1) = 2^{-1/2}$, $y'(1) = -2^{-3/2}$

$\frac{dy}{dx} = u$, $\frac{d^2y}{dx^2} = u\frac{du}{dy}$

$u\frac{du}{dy} = -3uy^2$, $du = -3y^2 dy$

$u = -y^3 + C_1$

$\frac{dy}{dx} = -y^3 + C_1$

$x = 1$, $y' = -2^{-3/2}$, $y = 2^{-1/2}$

$-2^{\frac{-3}{2}} = -2^{-3/2} + C_1$, $C_1 = 0$

$\frac{dy}{-y^3} = dx$, $\frac{1}{2}\frac{1}{y^2} = x + C_2$, $\frac{1}{y^2} = 2x + C_3$

代入 $y(1) = 2^{-1/2}$, $2 = 2 + C_3$, $C_3 = 0$

$\underline{y = \frac{1}{\sqrt{2x}}}$ # (負不合)

(c) $y'' = e^{y}$, $y(0) = 0$, $y'(0) = \sqrt{2}$

$u = \dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2} = u\dfrac{du}{dy}$

$u\dfrac{du}{dy} = e^{y}$, $u\,du = e^{y}dy$

$\frac{1}{2}u^2 = e^{y} + C_1$

代入 $x=0$, $y=0$, $u=\sqrt{2}$, $1 = 1 + C_1$, $C_1 = 0$

$u = \sqrt{2}\,e^{y/2}$ （負不合）

$\dfrac{dy}{dx} = \sqrt{2}\,e^{y/2}$, $e^{-y/2}dy = \sqrt{2}\,dx$, $-2e^{-y/2} = \sqrt{2}x + C_2$

代入 $y(0)=0$, $-2 = C_2$, $e^{-y/2} = -\frac{\sqrt{2}}{2}x + 1$

$$\frac{-y}{2} = \ln\left|-\frac{1}{\sqrt{2}}x + 1\right|$$

$$y = -2\ln\left|-\frac{1}{\sqrt{2}}x + 1\right|$$

#

2.

(a) $x^2\frac{\partial u}{\partial x}=y\frac{\partial u}{\partial y}$

令 $u=X(x)Y(y)$

$x^2X'Y=yXY' \Rightarrow \frac{x^2X'}{X}=\frac{yY'}{Y}=-\lambda$

$\begin{cases} x^2X'+\lambda X=0 \\ yY'+\lambda Y=0 \end{cases}$

可解 $X=C_1e^{\frac{\lambda}{x}}$, $Y=C_2y^{-\lambda}$

$u=\sum_{\lambda} Ce^{\frac{\lambda}{x}}y^{-\lambda}$, $C_1C_2=C$, for $\lambda\in R$

(b) $\frac{\partial^2 u}{\partial x^2} = u + \frac{\partial u}{\partial y}$, $0 < x < 2, y > 0$

$u(0,y) = u(2,y) = 0$, $u(x,0) = \cos\pi x \sin(2\pi x)$

令 $u = X(x)Y(y)$

$$X''Y = XY + XY' = X(Y+Y') \Rightarrow \frac{X''}{X} = 1 + \frac{Y'}{Y}$$

令 $\frac{X''}{X} = 1 + \frac{Y'}{Y} = -\lambda$，可解 $Y = C_1 e^{(-\lambda-1)y}$

代入 B.C，$1 = C_1$，$Y = e^{(-\lambda-1)y}$

① $\lambda = 0$，$X'' = 0$，$X = C_2 x + C_3$

代入 B.C，$0 = C_3 = 2C_2 + C_3 \Rightarrow C_2 = C_3 = 0$

$X = 0$ is trivial solution

② $\lambda < 0$，令 $\lambda = -\alpha^2$

$$X'' - \alpha^2 X = 0 \Rightarrow X = C_4 \cosh(\alpha x) + C_5 \sinh(\alpha x)$$

代入 B.C，$0 = C_4 = C_5 \sinh(2\alpha) \Rightarrow C_4 = C_5 = 0$

$X = 0$ is trivial solution

③ $\lambda > 0$，令 $\lambda = \alpha^2$

$$X'' + \alpha^2 X = 0 \Rightarrow X = C_6 \cos(\alpha x) + C_7 \sin(\alpha x)$$

代入 B.C，$C_6 = 0$，$C_7 \sin(2\alpha) = 0$

$$\alpha = \frac{n\pi}{2}, \quad \lambda = \frac{n^2\pi^2}{4}, \quad n = 1, 2, 3 \dots$$

$$u_n = C_7 e^{\left(-\frac{n^2\pi^2}{4} - 1\right)y} \sin\left(\frac{n\pi}{2}x\right), \quad n = 1, 2, 3, \dots$$

$$u = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n^2\pi^2}{4} + 1\right)y} \sin\left(\frac{n\pi}{2}x\right), \quad n = 1, 2, 3 \dots$$

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right) = \cos(\pi x)\sin(2\pi x)$$

$$= \frac{1}{2}(\sin 3\pi x + \sin \pi x)$$

$$A_2 = A_6 = \frac{1}{2}, \quad \text{for } n \neq 2, 6, \ A_n = 0$$

$$u = \frac{1}{2}\left(e^{-(\pi^2+1)y} \sin(\pi x) + e^{-(9\pi^2+1)y} \sin(3\pi x)\right)$$

此

(c) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, $0<x<1$, $0<y<1$

$u(0,y)=u(1,y)=u(x,0)=0$, $u(x,1)=1-2|x-\frac{1}{2}|$

令 $u=XY$

$X''Y+XY''=0 \Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}$

令 $\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda \Rightarrow \begin{cases} X''+\lambda X=0 \\ Y''-\lambda Y=0 \end{cases}$

由 B.C , $X(0)Y(y)=X(1)Y(y)=X(x)Y(0)=0$

$\Rightarrow X(0)=0$, $X(1)=0$, $Y(0)=0$

① $\lambda=0$, $X''=0 \Rightarrow X=C_1+C_2x$, 代入 $X(0)=X(1)=0$

$C_1=C_2=0$, $X=0$ is trivial solution

② $\lambda<0$, 令 $\lambda=-\alpha^2$, $X''-\alpha^2X=0$

$X=C_3\cosh(\alpha x)+C_4\sinh(\alpha x)$, 代入 $X(0)=X(1)=0$

$C_3=0$, $0=C_4\sinh\alpha=0$, $C_4=0$

$X=0$ is trivial solution

③ $\lambda>0$，令 $\lambda=\alpha^2$，$X''+\alpha^2X=0$

$X=C_5\cos(\alpha x)+C_6\sin(\alpha x)$，代入 $X(0)=X(1)=0$

$C_5=0$，$C_6\sin\alpha=0$

$\alpha=n\pi$，$\lambda=n^2\pi^2$，$n=1,2,3\ldots$

$Y''-n^2\pi^2Y=0$，$Y=C_7\cosh(n\pi y)+C_8\sinh(n\pi y)$

代入 $Y(0)=0$，$C_7=0$

$$u(x,y)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\sinh(n\pi y)$$

代入 B.C，$1-2|x-\frac{1}{2}|=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\sinh(n\pi)$

$\Rightarrow A_n\sinh(n\pi)$，$n=1,2\ldots$ 是 $u(x,1)$ 的 Fourier sine series 的 coefficient

$$A_n\sinh(n\pi)=2\int_0^1(1-2|x-\tfrac{1}{2}|)\sin(n\pi x)\,dx$$

$$2\left[\int_0^{1/2}2x\cdot\sin(n\pi x)\,dx+\int_{1/2}^{1}(2-2x)\sin(n\pi x)\,dx\right]$$

$$2\left[\left(-\frac{2x}{n\pi}\cos(n\pi x)+\frac{2}{n^2\pi^2}\sin(n\pi x)\right)\Big|_0^{1/2}\right. +$$

$$\left.\left[(-2+2x)\frac{1}{n\pi}\cos(n\pi x)-\frac{2}{n^2\pi^2}\sin(n\pi x)\right]\Big|_{1/2}^{1}\right]$$

$$=2\left[-\frac{1}{n\pi}\cos\left(\frac{1}{2}n\pi\right)+\frac{2}{n^2\pi^2}\sin\left(\frac{1}{2}n\pi\right)\right.$$

$$\left.+\frac{1}{n\pi}\cos\left(\frac{1}{2}n\pi\right)+\frac{2}{n^2\pi^2}\sin\left(\frac{1}{2}n\pi\right)\right]$$

$$=\frac{8}{n^2\pi^2}\sin\left(\frac{1}{2}n\pi\right)=A_n\sinh(n\pi)$$

$$\boxed{u(x,y)=\sum_{n=1}^{\infty}A_n\sinh(n\pi y)\sin(n\pi x)\qquad A_n=\frac{1}{\sinh(n\pi)}\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),\ n=1,2,3\ldots}$$

#

(d) $(x+1)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + \cos y$

令 $u = v + \psi(y)$

$$(x+1)\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + \psi' + \cos y$$

$$\psi' + \cos y = 0 \Rightarrow \psi = -\sin y + C_1$$

$$V = XY, \quad (x+1)X'Y = XY', \quad \frac{(x+1)X'}{X} = \frac{Y'}{Y} = -\lambda$$

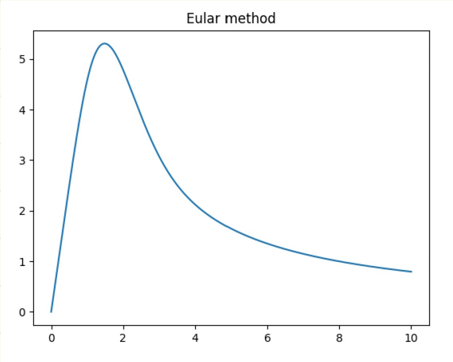
$$\begin{cases} (x+1)X' + \lambda X = 0 \\ Y' + \lambda Y = 0 \end{cases} \Rightarrow \begin{cases} X = C_2 (x+1)^{-\lambda} \\ Y = C_3 e^{-\lambda y} \end{cases}$$

$$V = C_4 (x+1)^{-\lambda} e^{-\lambda y}$$

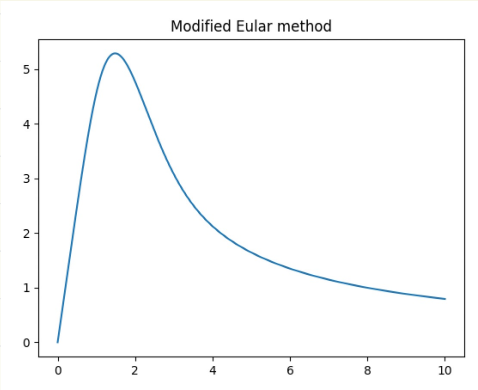
$$u = \sum_{\lambda} C_\lambda (x+1)^{-\lambda} e^{-\lambda y} - \sin y$$

3.

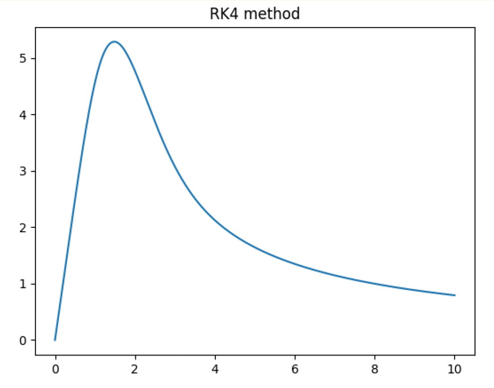
(a)


Eular method
0
1
2
3
4
5
0
2
4
6
8
10

(b)


Modified Eular method
0
1
2
3
4
5
0
2
4
6
8
10

(c)


RK4 method
0
1
2
3
4
5
0
2
4
6
8
10