1.

(a) $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

令 $u = X(x)Y(y)Z(z)$

$X'YZ + yXY'Z + z^2XYZ' = 0 \Rightarrow \frac{X'}{X} + y\frac{Y'}{Y} + z^2\frac{Z'}{Z} = 0$

令 $\frac{X'}{X} = -\lambda$, $X' + \lambda X = 0$

$X(x) = C_1 e^{-\lambda x}$

令 $y\frac{Y'}{Y} = -\mu$, $yY' = -\mu y$

$Y(y) = C_2 y^{-\mu}$

$z^2\frac{Z'}{Z} = \lambda + \mu$

$Z(z) = C_3 e^{\frac{-(\lambda+\mu)}{z}}$

$u(x,y,z) = \sum_{\lambda}\sum_{\mu} C_{\lambda,\mu} e^{-(\lambda x + \frac{\lambda+\mu}{z})} y^{-\mu}$

(b) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = x+y+z$

令 $u = V(x,y,z) + \psi_1(x) + \psi_2(y) + \psi_3(z)$

$$\frac{\partial V}{\partial x} + \psi_1'(x) + \frac{\partial V}{\partial y} + \psi_2'(y) + \frac{\partial V}{\partial z} + \psi_3'(z) = x+y+z$$

$$\begin{cases} \psi_1'(x) = x \\ \psi_2'(y) = y \\ \psi_3'(z) = z \end{cases} \Rightarrow \begin{cases} \psi_1(x) = \frac{1}{2}x^2 + C_1 \\ \psi_2(y) = \frac{1}{2}y^2 + C_2 \\ \psi_3(z) = \frac{1}{2}z^2 + C_3 \end{cases}$$

$V = XYZ$, $X'YZ + XY'Z + XYZ' = 0$

$$\frac{X'}{X} + \frac{Y'}{Y} + \frac{Z'}{Z} = 0$$

令 $\frac{X'}{X} = -\lambda$, $\frac{Y'}{Y} = -\mu$

$X(x) = C_4 e^{-\lambda x}$, $Y(y) = C_5 e^{-\mu y}$

$Z(z) = C_6 e^{(\lambda+\mu)z}$

$$u(x,y,z) = \sum_{\lambda}\sum_{\mu} C_{\lambda,\mu} e^{-\lambda x - \mu y + (\lambda+\mu)z} + \frac{1}{2}(x^2+y^2+z^2)$$

(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial t}$, $0 \le x \le 2,\ 0 \le y \le 2,\ t \ge 0$

$u(0,y,t) = u(2,y,t) = u(x,0,t) = u(x,2,t) = 0$

$u(x,y,0) = (2x - x^2)(2y - y^2)$

令 $u = XYT$

$X''YT + XY''T = XYT' \Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = \dfrac{T'}{T}$

$X(0)YT = X(2)YT = XY(0)T = XY(2)T = 0$

$\Rightarrow X(0) = X(2) = Y(0) = Y(2) = 0$

令 $\begin{cases} \dfrac{X''}{X} = -\lambda = -\alpha^2 \\ \dfrac{Y''}{Y} = -\mu = -\beta^2 \end{cases}$ 且因 $X(0) = X(2) = Y(0) = Y(2) = 0$

$X(x) = C_1 \sin\left(\dfrac{m\pi x}{2}\right)$, $\alpha = m\pi/2$

$Y(y) = C_2 \sin\left(\dfrac{n\pi y}{2}\right)$, $\beta = n\pi/2$

$\dfrac{T'}{T} = -\mu - \lambda \Rightarrow T = C_3 e^{-(\mu+\lambda)t}$

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-\left(\frac{n^2\pi^2 + m^2\pi^2}{4}\right)t} \sin\left(\frac{m\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$$

代入 B.C

$$\sum_m \sum_n A_{m,n} \sin\left(\frac{m\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$$

$$= (2x - x^2)(2y - y^2)$$

$$A_{mn} = \frac{4}{2 \cdot 2} \int_0^2 \int_0^2 (2x - x^2)(2y - y^2) \sin\left(\frac{m\pi x}{2}\right) dx \sin\left(\frac{n\pi y}{2}\right) dy$$

$$= \frac{256}{m^3 n^3 \pi^6} \left(1 - (-1)^m\right)\left(1 - (-1)^n\right)$$

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-\left(\frac{n^2\pi^2 + m^2\pi^2}{4}\right)t} \sin\left(\frac{m\pi x}{2}\right) \sin\left(\frac{n\pi y}{2}\right)$$

2. $0 \le r \le 1$, $0 \le \theta \le \frac{\pi}{3}$

$u(1,\theta) = \sin(6\theta) + \sin(12\theta)$

$u(r,0) = u(r,\frac{\pi}{3}) = 0$, $0 < r < 1$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$\Theta(0) = 0$

$\Theta(\frac{\pi}{3}) = 0$

令 $u = R(r)\Theta(\theta)$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda \Rightarrow \begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$$

① $\lambda = 0$, $\Theta(\theta) = C_1 + C_2\theta$

代入 $u(r,0) = u(r,\frac{\pi}{3}) = 0$, $\Theta(\theta) = 0$, trivial solution

② $\lambda < 0$. $\lambda = -\alpha^2$, $\Theta'' - \alpha^2\Theta = 0$

$\Theta(\theta) = C_3\cosh(\alpha\theta) + C_4\sinh(\alpha\theta)$,

代入 $u(r,0) = u(r,\frac{\pi}{3}) = 0$, $\Theta(\theta) = 0$, trivial solution

③ $\lambda > 0$, $\lambda = \alpha^2$, $\Theta'' + \alpha^2\Theta = 0$

$\Theta(\theta) = C_5\cos(\alpha\theta) + C_6\sin(\alpha\theta)$

代入 $u(r,0) = u(r,\frac{\pi}{3}) = 0$

$\Theta = C_6\sin(3n\theta)$, $\alpha = 3n$, $n = 1, 2, \ldots\ldots$

$r^2R'' + rR' - \lambda R = 0$, $\lambda = \alpha^2$

$m(m-1) + m - \alpha^2 = 0$, $m = \pm\alpha$

$R(r) = C_7 r^{\alpha} + C_8 r^{-\alpha}$, $\alpha = 3, 6, 9 \dots\dots$

$r=0$ 代入, $0^{-\alpha} \to \infty$, 故 $R(r) = C_7 r^{\alpha}$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{3n} \sin(3n\theta)$$

代入 $u(1,\theta) = \sin 6\theta + \sin 12\theta$

$$\sin 6\theta + \sin 12\theta = \sum_{n=1}^{\infty} A_n \sin(3n\theta)$$

$A_2 = A_4 = 1$,

$A_n = 0$ for $n \neq 2, 4$

$$u(r,\theta) = r^6 \sin(6\theta) + r^{12} \sin(12\theta)$$

3. $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0$, $0 \le r \le 1$, $0 \le z \le 2$

$u(1,z) = z$ for $0<z<1$, $u(1,z) = 2-z$ for $1<z<2$

$u(r,0) = u(r,2) = 0$, $0<r<1$

令 $u = R(r)Z(z)$

$R''Z + \frac{1}{r}R'Z + RZ'' = 0$

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = -\lambda \Rightarrow \begin{cases} rR'' + R' + \lambda r R = 0 \\ Z'' - \lambda Z = 0 \end{cases}$$

From $u(r,0) = u(r,2) = 0$, $Z(0) = Z(2) = 0$

if $\lambda \ge 0$, Z 只有 trivial solution

令 $\lambda = -\alpha^2 < 0$, $Z'' + \alpha^2 Z = 0$, $Z(z) = C_1\cos(\alpha z) + C_2\sin(\alpha z)$

代入 $Z(0) = Z(2) = 0$, $Z(z) = C_2 \sin\left(\frac{n\pi z}{2}\right)$, $\alpha = \frac{n\pi}{2}$

$rR'' + R' - \left(\frac{n\pi}{2}\right)^2 rR = 0$

$R(r) = C_3 I_0\left(\frac{n\pi r}{2}\right)$

$$u(r,z) = \sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi r}{2}\right)\sin\left(\frac{n\pi z}{2}\right)$$

代入 B.C

$$\sum_{n=1}^{\infty} A_n I_0\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi z}{2}\right) = \begin{cases} z, & 0<z<1 \\ 2-z, & 1<z<2 \end{cases}$$

$$A_n I_0\left(\frac{n\pi}{2}\right) = \frac{2}{2}\left[\int_0^1 z \sin\left(\frac{n\pi z}{2}\right) dz + \int_1^2 (2-z) \sin\left(\frac{n\pi z}{2}\right) dz\right]$$

$$= \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{8}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)} \sin\left(\frac{n\pi}{2}\right)$$

$$u(r,z) = \sum_{n=1}^{\infty} \frac{8 I_0\left(\frac{n\pi r}{2}\right)}{n^2\pi^2 I_0\left(\frac{n\pi}{2}\right)} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi z}{2}\right)$$

4. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 1$, $x>0$, $t>0$

$u(0,t) = t^2+t$, $u(x,0)=0$

$U(x,s) = \mathcal{L}_{t\to s}\{u(x,t)\}$

$$\frac{dU}{dx} + sU - u(x,0) = \frac{1}{s}$$

$$\frac{dU}{dx} + sU = \frac{1}{s}$$

$$U = \frac{1}{s^2} + Ce^{-sx}$$

$$\mathcal{L}_{t\to s}\{u(0,t)\} = U(0,s) = \frac{2}{s^3} + \frac{1}{s^2}$$

$$= \frac{1}{s^2} + C$$

$$C = \frac{2}{s^3}$$

$$U(x,s) = \frac{1}{s^2} + \frac{2}{s^3}e^{-sx}$$

$$u(x,t) = \mathcal{L}^{-1}_{s\to t}\left\{\frac{1}{s^2} + \frac{2}{s^3}e^{-sx}\right\}$$

$$= t + (t-x)^2 u(t-x)$$

5. $P_0 = 1$, $P_1 = x$, $P_2 = x^2$

(a) $$\phi_0(x) = \frac{1}{\sqrt{\int_0^4 1^2 dx}} = \frac{1}{2}$$

$$g_1 = x - \frac{1}{2}\int_0^4 \frac{1}{2} x \, dx = x - 2$$

$$\phi_1(x) = \frac{x-2}{\sqrt{\int_0^4 (x-2)^2 dx}} = \frac{\sqrt{3}}{4}(x-2)$$

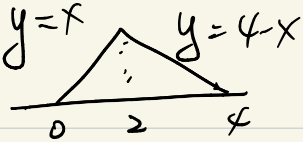
$$g_2 = x^2 - \frac{1}{2}\int_0^4 \frac{1}{2} x^2 dx - \frac{\sqrt{3}}{4}(x-2)\int_0^4 \frac{\sqrt{3}}{4}(x-2)x^2 dx$$

$$= x^2 - \frac{16}{3} - 4x + 8 = x^2 - 4x + \frac{8}{3}$$

$$\|g_2\| = \sqrt{\int_0^4 (x^2 - 4x + \tfrac{8}{3})^2 dx} = \frac{16}{3\sqrt{5}}$$

$$\phi_2(x) = \frac{g_2}{\|g_2\|} = \frac{3\sqrt{5}}{16}x^2 - \frac{3\sqrt{5}}{4}x + \frac{\sqrt{5}}{2}$$

$$\{\phi_0(x), \phi_1(x), \phi_2(x)\} = \left\{\frac{1}{2}, \frac{\sqrt{3}}{4}(x-2), \frac{3\sqrt{5}}{16}x^2 - \frac{3\sqrt{5}}{4}x + \frac{\sqrt{5}}{2}\right\}$$

(b) 

$$\min(x, 4-x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$$

$$c_0 = \int_0^2 x\cdot\frac{1}{2}\,dx + \int_2^4 (4-x)\frac{1}{2}\,dx = 2$$

$$c_1 = \int_0^2 x\cdot\frac{\sqrt{3}}{4}(x-2)\,dx + \int_2^4 (4-x)\frac{\sqrt{3}}{4}(x-2)\,dx = 0$$

$$c_2 = \int_0^2 x\,\phi_2(x)\,dx + \int_2^4 (4-x)\,\phi_2(x)\,dx = -\frac{\sqrt{5}}{2}$$

$$2\cdot\frac{1}{2} - \frac{\sqrt{5}}{2}\left(\frac{3\sqrt{5}}{16}x^2 - 3\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2}\right)$$

$$= 1 - \frac{15}{32}x^2 + \frac{15}{8}x - \frac{5}{4}$$

$$= -\frac{15}{32}x^2 + \frac{15}{8}x - \frac{1}{4}$$ #