1. $A = \begin{bmatrix} 4 & 0 & 0 \\ 62/25 & 38/25 & 9/25 \\ 16/25 & -16/25 & 62/25 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 2, 2, 4$

$\lambda = 4$, eigenvector $= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\lambda = 2$, eigenvector $= \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = V_1$

$(A - \lambda I) V_2 = V_1$, $\begin{bmatrix} 0 \\ 2 \\ 11 \end{bmatrix} = V_2$

$$J = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

2. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} + X = \begin{bmatrix} 0 & 8 \\ -1 & 22 \end{bmatrix}$

$$\mathrm{vec}(ABC) = (C^T \otimes A)\,\mathrm{vec}(B)$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 8 \\ 22 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & 6 & 0 & 0 \\ 9 & 12 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 4 \end{bmatrix} + I \right) \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 8 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & 0 & 0 \\ 9 & 13 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 3 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 8 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -6.5 & 3 & 0 & 0 \\ 4.5 & -2 & 0 & 0 \\ 0 & 0 & 1.25 & -0.5 \\ 0 & 0 & -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 8 \\ 22 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 5 \end{bmatrix}$$

$$X = \begin{bmatrix} -3 & -1 \\ 2 & 5 \end{bmatrix}$$

#

3. $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\det(B-\lambda I) = 0$, $\lambda = 1, 1, 1, 2, 2$

$\lambda = 1$, eigenvector $= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = V_1$

$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$(B-I)V_2 = V_1$, $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$(B-I)V_3 = V_2$, $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 2$, eigenvector $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = V_4$

$(B-2I)V_5 = V_4$, $V_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$B = VJV^{-1}$, $J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$B^{10} = VJ^{10}V^{-1} = J^{10}$

$$J_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_1^{10} = \begin{bmatrix} 1 & 10 & 45 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad J_2^{10} = \begin{bmatrix} 1024 & 5120 \\ 0 & 1024 \end{bmatrix}$$

$$B^{10} = \begin{bmatrix} 1 & 10 & 45 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1024 & 5120 \\ 0 & 0 & 0 & 0 & 1024 \end{bmatrix}$$

故

4. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$

(a)

$$AA^H = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix} V_2 = 0$$

$\det(AA^H - \lambda I) = 0$, $\lambda = 0, 2, 3$

$$\lambda_1 = 3, \quad u_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\lambda_2 = 2, \quad u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\lambda_3 = 0, \quad u_3 = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}, \quad \Sigma\Sigma^H = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$

$$V_1 = \frac{A^H u_1}{\sigma_1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$V_2 = \frac{A^H u_2}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 \\ j \end{bmatrix}$, $A^H = \begin{bmatrix} 1 & -j \end{bmatrix}$

$$AA^H = \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$$

$\det(AA^H - \lambda I) = 0$, $\lambda = 2, 0$

$$\lambda_1 = 2, \quad u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = 0, \quad u_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{j}{\sqrt{2}} \end{bmatrix}$$

σ_1

0

$\sigma_1^H\ 0$

$$\Sigma = \begin{bmatrix} \sigma_1 \\ 0 \end{bmatrix}, \quad \Sigma\Sigma^H = \begin{bmatrix} \sigma_1\sigma_1^H & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}$

$\sigma_1 = \sqrt{2}$

$$V_1 = \frac{A^H u_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -j \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{j}{\sqrt{2}} & -\frac{j}{\sqrt{2}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 \end{bmatrix}$$

5. $$M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ -1 & 5 \\ 0 & 0 \end{bmatrix}$$

$$\|M\|_1 = \max\{8, 10\} = \underline{10} \ \#$$

$$M = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\sigma = \begin{bmatrix} 3\sqrt{10} \\ \sqrt{10} \end{bmatrix}$$

$$\|M\|_2 = \|\sigma\|_\infty = \underline{3\sqrt{10}} \ \#$$

$$\|M\|_\infty = \max\{12, 6, 0\} = \underline{12} \ \#$$

$$\|M\|_* = \|\sigma\|_1 = \underline{4\sqrt{10}} \ \#$$

7. $A = \begin{bmatrix} 1 & 0 \\ j & j \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$X_{LS} = (A^H A)^{-1} A^H b$

$A^H = \begin{bmatrix} 1 & -j & 0 \\ 0 & -j & 2 \end{bmatrix}$, $A^H A = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$, $(A^H A)^{-1} = \begin{bmatrix} \frac{5}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{2}{9} \end{bmatrix}$

$A^H b = \begin{bmatrix} 1-j \\ -2-j \end{bmatrix}$

$(A^H A)^{-1} (A^H b) = \underline{\begin{bmatrix} \frac{7-4j}{9} \\ \frac{-5-j}{9} \end{bmatrix}}$ #

8. $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 4 & 0 \\ 0 & 5 \\ 0 & 10 \\ 0 & 20 \end{bmatrix}$

$\operatorname{rank}(A) = 2 = N$

$A^{+} = (A^H A)^{-1} A^H \quad , \quad A^H = \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 10 & 20 \end{bmatrix}$

$A^H A = \begin{bmatrix} 21 & 0 \\ 0 & 525 \end{bmatrix} \quad , \quad (A^H A)^{-1} = \begin{bmatrix} \frac{1}{21} & 0 \\ 0 & \frac{1}{525} \end{bmatrix}$

$(A^H A)^{-1} A^H = \begin{bmatrix} \frac{1}{21} & \frac{2}{21} & \frac{4}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{105} & \frac{2}{105} & \frac{4}{105} \end{bmatrix}$

9. $[2 \quad \bar{j} \quad 1]\, x = -3$

$$A = [2 \quad \bar{j} \quad 1], \quad A^H = \begin{bmatrix} 2 \\ -j \\ 1 \end{bmatrix}$$

$$AA^H = [6]$$

$$\det(AA^H - \lambda I) = 0, \quad \lambda = 6$$

$$u_1 = [1]$$

$$\Sigma = [\sigma_1 \quad 0 \quad 0], \quad \Sigma\Sigma^H = [\sigma_1^2], \quad \sigma_1 = \sqrt{6}$$

$$V_1 = \frac{A^H u_1}{\sigma_1} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -j/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$A^+ = [V_1 \quad V_2 \quad V_3] \begin{bmatrix} \sigma_1^{-1} \\ 0 \\ 0 \end{bmatrix} [1]^H = \frac{1}{6} \begin{bmatrix} 2 \\ -j \\ 1 \end{bmatrix}$$

$$X_{LS} = A^+ b = \frac{1}{2} \begin{bmatrix} -2 \\ j \\ -1 \end{bmatrix}$$