

Problem 6

Exercise 12 on page 348.

For an unbiased estimator

$$E(\hat{\theta}) - \theta = 0 \implies \theta = E(\hat{\theta})$$

For the estimator in the problem

$$\begin{aligned} E(\hat{\sigma}^2) &= E\left(\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}\right) \\ &= \frac{1}{n_1 + n_2 - 2} E((n_1 - 1)S_1^2 + (n_2 - 1)S_2^2) \\ &= \frac{1}{n_1 + n_2 - 2} (E((n_1 - 1)S_1^2) + E((n_2 - 1)S_2^2)) \\ &= \frac{(n_1 - 1)E(S_1^2) + (n_2 - 1)E(S_2^2)}{n_1 + n_2 - 2} \end{aligned}$$

In example 7.6 of the textbook, it is shown that

$$E(S^2) = \sigma^2$$

In the problem it states both types of fertilizer have the same variance

$$E(S_1^2) = E(S_2^2) = \sigma^2$$

Substitute into above to show the estimate is unbiased

$$E(\hat{\sigma}^2) = \frac{(n_1 - 1)\sigma^2 + (n_2 - 1)\sigma^2}{n_1 + n_2 - 2} = \frac{(n_1 - 1 + n_2 - 1)\sigma^2}{n_1 + n_2 - 2} = \frac{(n_1 + n_2 - 2)\sigma^2}{n_1 + n_2 - 2} = \sigma^2$$