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Problem 1

Using **method of moments** the result will be $p = 0.1\bar{3}$:

The first population moment of the binomial distribution is

$$E(X) = np = 4p$$

The first sample moment is (m = number of samples)

$$\frac{1}{m}\sum X_i = \frac{8}{15}$$

Set them equal to each other to solve for p

$$p = \frac{8}{4\dot{1}5} = 0.1\bar{3}$$

Using the **method of maximum likelihood** the result will be $p = 0.25$. [1]

$$f(x_1...x_m; p) = L(p) = \prod_{i=1}^{m} \binom{n}{x_i} p_i^x (1-p)p^{n-x_i}$$

$$L(P) = 6\left(\binom{4}{1}p^1(1-p)^3\right) \times 8\left(\binom{4}{0}p^0(1-p)^4\right) \times 1\left(\binom{4}{2}p^2(1-p)^2\right)$$

$$6 \times 8 \times \frac{4!}{3!} \times \frac{4!}{4!} \times \frac{4!}{2!(2)!} \times p^3(1-p)^9 = 1152p^3(1-p)^9$$

$$\ln(L(p)) = 1152\,(3\ln(p) + 9\ln(1-p))$$

$$\frac{d}{dp}L(p) = 1152\left(\frac{3}{p} - \frac{9}{1-p}\right) = 0$$

$$\implies p = \frac{3}{12} = 0.25$$

Comment:

If the business believes that a 10% failure rate is unacceptable, then they should not accept the material for these brake shoes because both methods of point estimation have given results with probability greater than 0.1.

[1] $f(x_1...x_m; p)$ is the notation used in textbook, $L(p)$ is the notation used in class for the joint pmf of the sample.

Problem 2

Exercise 26, on page 360.

a)

Estimates attained using maximum likelihood method for the true average weight is 113.0 grams and standard deviation 3.91 grams.

For normal distribution, the joint PMF of the sample is:

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\sum(x_i-\mu)^2/2\sigma^2}$$

$$\ln(L(\mu, \sigma^2)) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

Take the partial derivatives to get a system of equations

$$\frac{\partial}{\partial\sigma^2}\ln(L(\mu, \sigma^2)) = -\frac{n}{2}\frac{1}{\sigma^2} - \frac{\sum(x_i-\mu)^2}{2(\sigma^2)^2} = 0$$

$$\implies \sigma^2 = \frac{\sum(x_i-\mu)^2}{n} \tag{1}$$

$$\frac{\partial}{\partial\mu}\ln(L(\mu, \sigma^2)) = 0 - \frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\sum(x_i-\mu)^2$$

$$= -\frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\sum(x_i^2 - 2x_i\mu + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\left(\sum x_i^2 - 2n\mu\sum x_i + mu^2\right)$$

$$= -\frac{-2n\sum x_i + 2\mu}{2\sigma^2} = 0$$

$$\implies \mu = \frac{\sum x}{n} = \bar{X} \tag{2}$$

Solve the system of equations (1 and 2) to get the estimators:

$$\hat{\mu} = \bar{X} = 113.0 \qquad \hat{\sigma^2} = \frac{\sum(x_i-\bar{X})^2}{n} = 15.3$$

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Calculations [2]

```
import java.util.*;
import java.lang.Math;

public class Question2 {

  public static void main(String[] args) {
    double[] values = {117.6, 109.5, 111.6, 109.2, 119.1, 110.8};

    double mean = Arrays.stream(values)
      .reduce((a, b) -> a + b).getAsDouble()
      / (double) values.length;

    double variance = Arrays.stream(values)
      .reduce(0.0, (a, b) ->  a + Math.pow(b - mean, 2.0))
      / (double) values.length;

    System.out.println("mean␣is␣" + mean);
    System.out.println("variance␣is␣" + variance);
    System.out.println("standard␣deviation␣is␣" + Math.sqrt(variance));
  }
}
```

[2] Java

b)

It is estimated 95% of bagels weigh less than 120.0 grams.

Relate values of the standard normal distribution to the bagel population

$$\Phi(z) = P(Z \leqslant z) = P\left(\frac{X-\mu}{\sigma} \leqslant z\right) \implies X \leqslant z \cdot \sigma + \mu$$

From Table A.3 in textbook

$$\Phi(1.65) \approxeq 0.95$$

Using invariance principal, substitute estimate values

$$X \leqslant z \cdot \hat{\sigma} + \hat{\mu} = z \cdot \sqrt{\hat{\sigma^2}} + \bar{X} = 1.65 \cdot \sqrt{15.3} + 113.0 = 119.2$$

c)

The mle of $P(X \leqslant 113.4)$ is 0.5338

$$P(X \leqslant 113.4) = \Phi\left(\frac{113.4-\mu}{\sigma}\right)$$

Substitute the estimates which using the invariance prinipal[3]

$$\Phi\left(\frac{113.4-\bar{\hat{\mu}}}{\hat{\sigma}}\right) = \Phi\left(\frac{113.4-\bar{X}}{\sqrt{\hat{\sigma^2}}}\right) = \Phi\left(\frac{113.4-113.0}{\sqrt{15.3}}\right) = \Phi(0.0957) = 0.5338$$

[3] value from Table A.3 of textbook used

Problem 3

Exercise 30, on page 360.

a)

The mles are

$$\hat{\lambda} = \frac{n}{\sum x_i + n \cdot \min(x_i)} \qquad \hat{\theta} = \min(x_i)$$

The joint PMF is

$$L(\lambda, \theta) = f(x_1, ..., x_n; \lambda, \theta) = (\lambda e^{-\lambda(x_1 - \theta)}).....(\lambda e^{-\lambda(x_n - \theta)}) = \begin{cases} \lambda^n e^{-\lambda \sum x_i - \lambda n \theta}, & x \geqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

$$ln(L(\lambda, \theta)) = \begin{cases} n \ln \lambda - \lambda \sum x_i - \lambda n \theta, & x \geqslant \theta \\ 0 & \text{otherwise} \end{cases}$$

Set the partial derivative to 0 to get an equation for the maximum likelihood

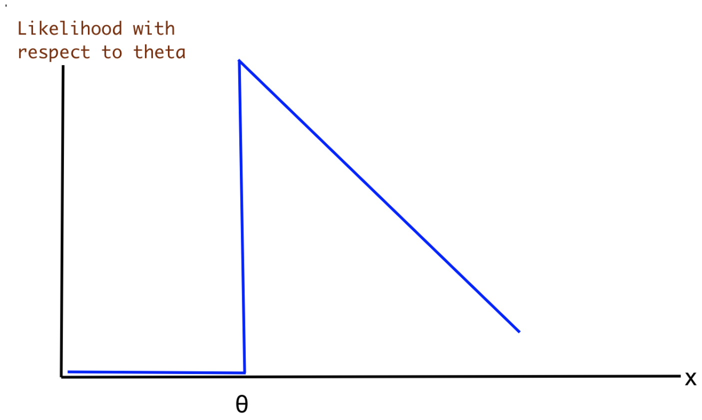
$$\frac{\partial}{\partial \lambda} \left(n \ln \lambda - \lambda \sum x_i - \lambda n \theta \right) = \frac{n}{\lambda} - \sum x_i - n\theta = 0$$

$$\implies \lambda = \frac{n}{\sum x_i + n\theta} \tag{3}$$

Setting the partial derivative with respect to θ would not work well

$$\frac{\partial}{\partial \theta} \left(n \ln \lambda - \lambda \sum x_i - \lambda n \theta \right) = -\lambda n = 0$$

Plotting the likehood with respect to θ, it can be seen that it is maximized by choosing the $\hat{\theta} = \min(x_i)$



Substitute into equation 3 to get the estimator for λ

$$\hat{\lambda} = \frac{n}{\sum x_i + n \cdot \min(x_i)}$$

b)

$$\hat{\theta} = \min(x_i) = 0.64$$

$$\hat{\lambda} = \frac{n}{\sum x_i + n \cdot \min(x_i)} = \frac{10}{55.8 + 10 \times 0.64} = 0.161$$

problem 4

Exercise 4 a, d, e on page 390.

$$\left(\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \ , \ \bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}\right)$$

a)

$$z_{.95/2} = \Phi((1 - .95)/2) = 1.96$$

$$\left(58.3 - 1.96 \cdot \frac{3.0}{\sqrt{25}} \ , \ 58.3 + 1.96 \cdot \frac{3.0}{\sqrt{25}}\right)$$

$$(57.1, 59.5)$$

d)

$$z_{.82/2} = \Phi((1 - .82)/2) = 1.34$$

$$\left(58.3 - 1.34 \cdot \frac{3.0}{\sqrt{100}} \ , \ 58.3 + 1.34 \cdot \frac{3.0}{\sqrt{100}}\right)$$

$$(57.9, 58.7)$$

e)

The sample size would need to be 240.

$$n = \left(2z_{\alpha/2} \cdot \frac{\sigma}{w}\right)^2$$

$$z_{.99/2} = \Phi((1 - .99)/2) = 1.34 = 2.58$$

$$n = \left(2 \times 2.58 \times \frac{3.0}{1.0}\right)^2 = 239.6$$