Problem 2

Exercise 26, on page 360.

a)

Estimates attained using maximum likelihood method for the true average weight is 113.0 grams and standard deviation 3.91 grams.

For normal distribution, the joint PMF of the sample is:

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\sum(x_i-\mu)^2/2\sigma^2}$$

$$\ln(L(\mu, \sigma^2)) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \mu)^2$$

Take the partial derivatives to get a system of equations

$$\frac{\partial}{\partial\sigma^2}\ln(L(\mu, \sigma^2)) = -\frac{n}{2}\frac{1}{\sigma^2} - \frac{\sum(x_i-\mu)^2}{2(\sigma^2)^2} = 0$$

$$\implies \sigma^2 = \frac{\sum(x_i-\mu)^2}{n} \tag{1}$$

$$\frac{\partial}{\partial\mu}\ln(L(\mu, \sigma^2)) = 0 - \frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\sum(x_i-\mu)^2$$

$$= -\frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\sum(x_i^2 - 2x_i\mu + \mu^2)$$

$$= -\frac{1}{2\sigma^2}\cdot\frac{\partial}{\partial\mu}\left(\sum x_i^2 - 2n\mu\sum x_i + mu^2\right)$$

$$= -\frac{-2n\sum x_i + 2\mu}{2\sigma^2} = 0$$

$$\implies \mu = \frac{\sum x}{n} = \bar{X} \tag{2}$$

Solve the system of equations (1 and 2) to get the estimators:

$$\hat{\mu} = \bar{X} = 113.0 \qquad \hat{\sigma^2} = \frac{\sum(x_i - \bar{X})^2}{n} = 15.3$$

Calculations [2]

```
import java.util.*;
import java.lang.Math;

public class Question2 {

  public static void main(String[] args) {
    double[] values = {117.6, 109.5, 111.6, 109.2, 119.1, 110.8};

    double mean = Arrays.stream(values)
      .reduce((a, b) -> a + b).getAsDouble()
      / (double) values.length;

    double variance = Arrays.stream(values)
      .reduce(0.0, (a, b) ->  a + Math.pow(b - mean, 2.0))
      / (double) values.length;

    System.out.println("mean␣is␣" + mean);
    System.out.println("variance␣is␣" + variance);
    System.out.println("standard␣deviation␣is␣" + Math.sqrt(variance));
  }
}
```

[2] Java

b)

It is estimated 95% of bagels weigh less than 120.0 grams.

Relate values of the standard normal distribution to the bagel population

$$\Phi(z) = P(Z \leqslant z) = P\left(\frac{X-\mu}{\sigma} \leqslant z\right) \implies X \leqslant z \cdot \sigma + \mu$$

From Table A.3 in textbook

$$\Phi(1.65) \approxeq 0.95$$

Using invariance principal, substitute estimate values

$$X \leqslant z \cdot \hat{\sigma} + \hat{\mu} = z \cdot \sqrt{\hat{\sigma^2}} + \bar{X} = 1.65 \cdot \sqrt{15.3} + 113.0 = 119.2$$

c)

The mle of $P(X \leqslant 113.4)$ is 0.5338

$$P(X \leqslant 113.4) = \Phi\left(\frac{113.4-\mu}{\sigma}\right)$$

Substitute the estimates which using the invariance prinipal[3]

$$\Phi\left(\frac{113.4-\bar{\hat{\mu}}}{\hat{\sigma}}\right) = \Phi\left(\frac{113.4-\bar{X}}{\sqrt{\hat{\sigma^2}}}\right) = \Phi\left(\frac{113.4-113.0}{\sqrt{15.3}}\right) = \Phi(0.0957) = 0.5338$$

[3] value from Table A.3 of textbook used