Opgave 2

Exercise (2.6)**.**

Show that we can ensure the commutation relation $[\phi(x), \pi(x')] = i\delta^3(x-x')$ if we take the operator commutation relations to be

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$$
$$[c_{\mathbf{p}}, c^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$$

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Solution. Recall,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + c^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}})$$
$$\pi(x') = \int \frac{d^3p'}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}'}}{2}}(a_{\mathbf{p}'}e^{i\mathbf{p}'\cdot\mathbf{x}'} - c^\dagger_{\mathbf{p}'}e^{-i\mathbf{p}'\cdot\mathbf{x}'})$$

And the following rules for commutators

$$[\alpha A, B] = \alpha[A,B]$$
$$[A, B \pm C] = [A,B] \pm [A,C]$$

Let's take a look at the commutator

$$[\phi(x), \pi(x')] = \left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{ip\cdot x} + c^\dagger_p e^{-ip\cdot x}), \int \frac{d^3p'}{(2\pi)^3} i\sqrt{\frac{\omega_{p'}}{2}}(a^\dagger_{p'}e^{-ip'\cdot x'} - c_{p'}e^{ip'\cdot x'})\right]$$

For simplicity let's introduce $k = ip\cdot x$, $k' = ip'\cdot x'$

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{i}{(2\pi)^6}\right)\int d^3p \int d^3p' \left[a_p e^k + c^\dagger_p e^{-k}, a^\dagger_{p'}e^{-k'} - c_{p'}e^{k'}\right] \\
&= A\left(\left[a_p e^k, a^\dagger_{p'}e^{-k'} - c_{p'}e^{k'}\right] + \left[c^\dagger_p e^{-k}, a^\dagger_{p'}e^{-k'} - c_{p'}e^{k'}\right]\right) \\
&= A\left(\left[a_p e^k, a^\dagger_{p'}e^{-k'}\right] - \left[a_p e^k, c_{p'}e^{k'}\right] + \left[c^\dagger_p e^{-k}, a^\dagger_{p'}e^{-k'}\right] - \left[c^\dagger_p e^{-k}, c_{p'}e^{k'}\right]\right)
\end{aligned}$$

At this point we see that the terms with different operators definitely cancel.

$$= A\left(e^{k-k'}[a_p, a^\dagger_{p'}] - e^{k'-k}[c^\dagger_p, c_{p'}]\right)$$

Before moving on let's quickly look at the commutator,

$$
\begin{aligned}
[c_p, c_{p'}^\dagger]^\dagger &= \left((2\pi)^3\delta^3(p-p')\right)^\dagger = (2\pi)^3\delta^3(p-p') \\
&= [c_p, c_{p'}^\dagger]
\end{aligned}
$$

And,

$$
\begin{aligned}
[c_p, c_{p'}^\dagger]^\dagger &= (c_p c_{p'}^\dagger)^\dagger - (c_{p'}^\dagger c_p)^\dagger \\
&= c_{p'}^{\dagger\dagger} c_p^\dagger - c_p^\dagger c_{p'}^{\dagger\dagger} = c_{p'} c_p^\dagger - c_p^\dagger c_{p'} = [c_{p'}, c_p^\dagger] \\
&= -[c_p^\dagger, c_{p'}] = [c_p, c_{p'}^\dagger]
\end{aligned}
$$

We can then insert this.

$$
\begin{aligned}
&= A\left(e^{k-k'}[a_p, a_{p'}^\dagger] + e^{k'-k}[c_p, c_{p'}^\dagger]\right) \\
&= A\left(e^{k-k'}(2\pi)^3\delta^3(p-p') + e^{k'-k}(2\pi)^3\delta^3(p-p')\right)
\end{aligned}
$$

We pull out the $(2\pi)^3$ and begin evaluating the integrals.

$$
= \frac{1}{2}\left(\frac{i}{(2\pi)^3}\right)\int d^3p\int d^3p'\left(e^{i(p\cdot x - p'\cdot x')}\delta^3(p-p') + e^{i(p'\cdot x' - p\cdot x)}\delta^3(p-p')\right)
$$

The innermost integral just selects the value where the delta function is 0. Therefore, we get $p' = p$.

$$
= \frac{1}{2}\left(\frac{i}{(2\pi)^3}\right)\int d^3p\left(e^{ip\cdot(x-x')} + e^{ip\cdot(x'-x)}\right)
$$

This is just the definition of the delta function

$$
= \frac{1}{2}\left(\frac{i}{(2\pi)^3}\right)\left((2\pi)^3\delta^3(x-x') + (2\pi)^3\delta^3(x'-x)\right)
$$

Recall

$$
\delta(-x) = \delta(x) \implies \delta(x-x') = \delta(-(x-x')) = \delta(x'-x).
$$

So we get,

$$
= \frac{1}{2}i\left(\frac{1}{(2\pi)^3}\right)2\cdot(2\pi)^3\delta^3(x-x') = i\delta^3(x-x')
$$

Exercise (8)**.**

For the field operators, the conserved charge becomes:

$$
Q = i\int d^3x\left(\phi^\dagger(x)\pi^\dagger(x) - \pi(x)\phi(x)\right)
$$

Argue that this expression makes sense when compared to the flow current of a classical scalar field.

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Solution. In the classical picture we have:

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right)$$

With Lagrangian:

$$\mathscr{L}_{KG} = (\partial^\mu \phi^*)\partial_\mu \phi - m^2 \phi^* \phi$$

Now, we also know that:

$$\pi = \frac{\partial \mathscr{L}_{KG}}{\partial(\partial_t \phi)} = (\partial^t \phi)^* = \frac{\partial \phi^*}{\partial t}$$

And likewise:

$$\pi^* = \frac{\partial \phi}{\partial t}$$

Translating this we get a nice correspondence between:

$$i\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) \to i\left(\phi^\dagger \pi^\dagger - \pi\phi\right)$$

So it does make sense

Exercise (2.9)**.**

Show that,

$$[H, a_p^\dagger] = \omega_p a_p^\dagger \quad \text{and} \quad [H, a_p] = -\omega_p a_p$$

And assuming $a_p|0\rangle = 0$, show that the full spectrum of energy can be obtained from repeated application of the creation operator.

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Solution. Let's write it out,

$$[H, a_p^\dagger] = \left[\int \frac{d^3 p'}{(2\pi)^3}\omega_{p'}(a_{p'}^\dagger a_{p'} + \frac{1}{2}[a_{p'}, a_{p'}^\dagger] + c_{p'}^\dagger c_{p'} + \frac{1}{2}[c_{p'}, c_{p'}^\dagger], a_p^\dagger\right]$$

We can move out the integrals, and also see that the terms with $c_{p'}^\dagger c_{p'}$ will generate mixed commutators, which we know are zero.

$$
\begin{aligned}
&= \omega_p \int \frac{d^3p'}{(2\pi)^3}[a^\dagger_{p'}a_{p'} + \frac{1}{2}[a_{p'}, a^\dagger_{p'}], a^\dagger_p] \\
&= \omega_p \int \frac{d^3p'}{(2\pi)^3}\left([a^\dagger_{p'}a_{p'}, a^\dagger_p] + \frac{1}{2}[[a_{p'}, a^\dagger_{p'}], a^\dagger_p]\right)
\end{aligned}
$$

Now $[a_{p'}, a^\dagger_{p'}] = \alpha\delta^{(3)}(0)$, i.e. is a fixed quantity, so it commutes with everything.

$$
= \omega_p \int \frac{d^3p'}{(2\pi)^3}[a^\dagger_{p'}a_{p'}, a^\dagger_p]
$$

Let's stop for a while.

$$
\begin{aligned}
[a^\dagger_{p'}a_{p'}, a^\dagger_p] &= -[a^\dagger_p, a^\dagger_{p'}a_{p'}] \\
&= -[a^\dagger_p, a^\dagger_{p'}]a_{p'} - a^\dagger_{p'}[a^\dagger_p, a_{p'}] \\
&= a^\dagger_{p'}[a_{p'}, a^\dagger_p] = a^\dagger_{p'}(2\pi)^3\delta^{(3)}(p'-p)
\end{aligned}
$$

Insert,

$$
= \omega_p \int \frac{d^3p'}{(2\pi)^3}a^\dagger_{p'}(2\pi)^3\delta^{(3)}(p'-p) = \omega_p a^\dagger_p
$$

We now show, the corresponding commutator for the annihilation operator

$$
\begin{aligned}
[H, a_p] &= \int \frac{d^3p}{(2\pi)^3}\omega_p[a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p], a'_p] \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_p([a^\dagger_p a_p, a'_p] + \frac{1}{2}[[a_p, a^\dagger_p], a'_p])
\end{aligned}
$$

Now $[a_p, a^\dagger_p] = (2\pi)^3\delta^{(3)}(0)$. So it commutes with a'_p.

$$
\begin{aligned}
&= -\int \frac{d^3p}{(2\pi)^3}\omega_p[a'_p, a^\dagger_p a_p] \\
&= -\int \frac{d^3p}{(2\pi)^3}\omega_p([a'_p, a^\dagger_p]a_p + a^\dagger_p[a'_p, a_p]) \\
&= -\int \frac{d^3p}{(2\pi)^3}\omega_p(2\pi)^3\delta^{(3)}(p'-p)a'_p \\
&= -\omega'_p a'_p
\end{aligned}
$$

We can now move on to showing that the creation operator generates the different states. Let's assume that the groundstate exists. Then it should be an eigenstate of the Hamiltonian.

$$
\hat{H}|0\rangle = E_0|0\rangle
$$

We can then apply the creation operator.

$$
(\hat{H}a^\dagger_p)|0\rangle = (a^\dagger_p\hat{H} + [\hat{H}, a^\dagger_p])|0\rangle = a^\dagger_p\hat{H}|0\rangle + \omega_p a^\dagger_p|0\rangle = a^\dagger_p(E_0 + \omega_p)|0\rangle
$$

Lets find out what E_0 is. We apply the hamiltonian operator to the ground state.

$$
\begin{aligned}
\hat{H}|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\omega_p \left(a_p^\dagger a_p|0\rangle + \frac{1}{2}[a_p, a_p^\dagger]|0\rangle + c_p^\dagger c_p|0\rangle + \frac{1}{2}[c_p, c_p^\dagger]|0\rangle \right) \\
&= \int \frac{d^3p}{(2\pi)^3}\omega_p \left(\frac{1}{2}[a_p, a_p^\dagger] + \frac{1}{2}[c_p, c_p^\dagger]|0\rangle \right) \\
&= \int \frac{d^3p}{(2\pi)^3} 2 \cdot \frac{1}{2}(2\pi)^3\delta^{(3)}(0)|0\rangle \\
&= \omega_0|0\rangle
\end{aligned}
$$

Exercise (2.10)**.**

Show that $a^\dagger_{\mathbf{p}_1}a^\dagger_{\mathbf{p}_2}|0\rangle = a^\dagger_{\mathbf{p}_2}a^\dagger_{\mathbf{p}_1}|0\rangle$ and argue that this implies that these spin zero particles obey Bose-Einstein statistics.

Exercise (2.11)**.**

Apply $\phi(\mathbf{x})$ to the vacuum and show that

$$
\phi(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle_c, . \tag{29}
$$

where $|\mathbf{p}\rangle_c = \sqrt{2\omega_p}c^\dagger_{\mathbf{p}}|0\rangle$ where the subscript c indicates that we are creating a c particle. The factor $\sqrt{2\omega_p}$ is introduced to ensure that the states are normalized in a Lorentz invariant fashion (more precisely, $\langle\mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 2\omega_p\delta^3(\mathbf{p}-\mathbf{q})$ can be shown to be Lorentz invariant, just consider a boost operation along one direction). Likewise, if we use $\phi^\dagger(\mathbf{x})$ we would be creating an a particle. Argue that for small momenta, $\mathbf{p}$, ω_p is nearly constant and in that case the above expression is a linear superposition of plane wave states with well-defined momentum which is the Fourier transform of a non-relativistic basis state of position, $\mathbf{x}$. We thus interpret $\phi(\mathbf{x})$ as a field operator that creates a particle at position $\mathbf{x}$.

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Solution.

$$
\begin{aligned}
\phi(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{i\mathbf{p}\cdot\mathbf{x}} + c_p^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}e^{-i\mathbf{p}\cdot\mathbf{x}}c_p^\dagger|0\rangle
\end{aligned}
$$

Now inserting $|\mathbf{p}\rangle_c = \sqrt{2\omega_p}c_p^\dagger|0\rangle$, we get:

$$
= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle_c
$$

Exercise (2.12).

12) Show that $\langle 0|\phi(\mathbf{x})|\mathbf{p}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}$. If we interpret this as the position-space representation of the single-particle wave function of the state $|\mathbf{p}\rangle$, then we see that $\langle \mathbf{x}|\mathbf{p}\rangle \propto e^{i\mathbf{p}\cdot\mathbf{x}}$ is the wave function just as in non-relativistic quantum mechanics.

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Solution.

$$
\begin{aligned}
\langle 0|\phi(x)|p\rangle &= \left(\langle p|\phi(x)^\dagger|0\rangle\right)^\dagger \\
&= \left(\langle p|\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{p'}}}e^{i\mathbf{p}'\cdot\mathbf{x}}|p'\rangle\right)^\dagger \\
&= \left(\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{p'}}}e^{i\mathbf{p}'\cdot\mathbf{x}}\langle p|p'\rangle\right)^\dagger \\
&= \left(\int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{p'}}}e^{i\mathbf{p}'\cdot\mathbf{x}}(2\pi)^3 2\omega_{p'}\delta^3(\mathbf{p}'-\mathbf{p})\right)^\dagger \\
&= \left(e^{i\mathbf{p}\cdot\mathbf{x}}\right)^\dagger = e^{-i\mathbf{p}\cdot\mathbf{x}}
\end{aligned}
$$