8 - Scalar Fields and Causality

Let us consider an expectation value of two fields in the vacuum

$$D(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle,$$

where x and y are four-vectors (we suppress the space-time indices for simplicity). We use the field expansion for a real field, ϕ, which has the form

$$\phi = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[e^{-ipx}a_\mathbf{p} + e^{ipx}a_\mathbf{p}^\dagger\right],$$

where $\omega_\mathbf{p} = \sqrt{\mathbf{p}^2+m^2}$.

Exercise (1)**.**

Show that

$$D(x,y) = D(x-y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}}e^{-ip(x-y)},$$

and argue that this quantity is a Lorentz scalar, i.e., it is invariant under Lorentz transformations.

Lets plug it in and see what happens. We can pull the integrals and the constants out,

$$D(x,y) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}\langle 0|\left(e^{-ipx}a_\mathbf{p} + e^{ipx}a_\mathbf{p}^\dagger\right)\left(e^{-iqy}a_\mathbf{q} + e^{iqy}a_\mathbf{q}^\dagger\right)|0\rangle.$$

Recall that

$$\langle 0|a_\mathbf{p}^\dagger = 0 \quad , \quad a_\mathbf{p}|0 = 0.$$

So mulitplying the parenthesis out, we only have a single term that survives

$$\begin{aligned}
D(x,y) &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}\langle 0|a_\mathbf{p}a_\mathbf{q}^\dagger e^{-i(px-qy)}|0\rangle \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}\langle 0|a_\mathbf{q}^\dagger a_\mathbf{p} + \delta^3(p-q)|0\rangle e^{-i(px-qy)} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}\delta^3(p-q)e^{-i(px-qy)} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}}e^{-ip(x-y)}.
\end{aligned}$$

Exercise (2).

Show that

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x).$$

This is just computation, so lets go ahead and do that.

$$\begin{aligned}
[\phi(x), \phi(y)] &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{q}}} \left[e^{-ipx} a_\mathbf{p} + e^{ipx} a_\mathbf{p}^\dagger, e^{-iqy} a_\mathbf{q} + e^{iqy} a_\mathbf{q}^\dagger\right] \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{q}}} \left(\left[e^{-ipx} a_\mathbf{p}, e^{iqy} a_\mathbf{q}^\dagger\right] + \left[e^{ipx} a_\mathbf{p}^\dagger, e^{-iqy} a_\mathbf{q}\right]\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{q}}} \left(\left[a_\mathbf{p}, a_\mathbf{q}^\dagger\right] e^{-i(px-qy)} + \left[a_\mathbf{p}^\dagger, a_\mathbf{q}\right] e^{-i(qy-px)}\right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} \left(e^{-ip(x-y)} + e^{-ip(y-x)}\right).
\end{aligned}$$

Exercise (3).

Show that if $(x-y)^2 < 0$ then $[\phi(x), \phi(y)] = 0$. (Hint: First argue that in this case a Lorentz transformation can be found that takes $(x_0 - y_0)$ to $-(x_0 - y_0) = (y_0 - x_0)$.

When we spatial separation we can find a reference where the events happen simultaneously. Lets call this transform L_0. Applying this to the four vector $x - y$, we get

$$L_0(x-y) = \begin{bmatrix} 0 \\ \vec{x}' - \vec{y}' \end{bmatrix} = x' - y'.$$

Since the lorentz transform is linear,

$$L_0(y-x) = -L_0(x-y) = \begin{bmatrix} 0 \\ \vec{y}' - \vec{x}' \end{bmatrix} = y' - x'.$$

Now, since D is lorentz invariant we have that,

$$D(x-y) = D\left(x' - y'\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-i\mathbf{p}\left(\vec{x}' - \vec{y}'\right)}.$$

Now lets a look at the commutator in the transformed frame,

$$D\left(x' - y'\right) - D\left(y' - x'\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-i\mathbf{p}\left(\vec{x}' - \vec{y}'\right)} - \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{q}} e^{-i\mathbf{q}\left(\vec{y}' - \vec{x}'\right)}.$$

We can do a substitution $\mathbf{q} \to -\mathbf{p}$, and noting that $\omega_{-\mathbf{p}} = \omega_\mathbf{p}$, we end up with the same integrals with the signs flipped,

$$= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-i\mathbf{p}\left(\vec{x}' - \vec{y}'\right)} - \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-i\mathbf{p}\left(\vec{x}' - \vec{y}'\right)} = 0.$$

Exercise (4)**.**

Argue that you have now shown that causality is preserved for scalar fields.