6 - Time Ordered Operators and the Klein-Gordon Equation

Consider a real scalar field operator, $\hat{\phi}(x,t)$, in one dimension (x is a single number, not a vector!) for simplicity with mass m. The Klein-Gordon wave equation is assumed to hold for the field operator as well, and it can be written

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\hat{\phi}(x,t) = 0.$$

Exercise (1).

Prove that

$$T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right] = \theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1),$$

where θ is the Heaviside step function.

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Solution. Let's recall the time-ordering operator:

$$T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right] = \begin{cases} \hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) & t_1 > t_2 \\ \hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1) & t_2 > t_1 \end{cases}$$

So it's clearly true!

Exercise (2).

Show that

$$\frac{d}{dx}\theta(x-a) = \delta(x-a)$$

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Solution. Let $x > a$ then

$$\frac{d}{dx}\Theta(x-a) = \frac{d}{dx}1 = 0$$

for $x < a$ then

$$\frac{d}{dx}\Theta(x-a) = \frac{d}{dx}0 = 0$$

We will try to show that the derivative at $x = a$ is infinity

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{f(a+\varepsilon)-f(a-\varepsilon)}{\varepsilon} &= \lim_{\varepsilon\to 0} \frac{\Theta(a+\varepsilon-a)-\Theta(a-\varepsilon-a)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{\Theta(\varepsilon)-\Theta(-\varepsilon)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} = \infty
\end{aligned}$$

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Exercise (3)**.**

Use the previous results and the commutator $\left[\hat{\phi}(x_1,t),\hat{\phi}(x_2,t)\right] = 0$ to show that

$$\frac{\partial}{\partial t_1}\left\{T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]\right\} = \theta(t_1-t_2)\left(\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}\right)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\left(\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}\right),$$

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Solution. For any equal times $\left[\hat{\phi}(x_1,t),\hat{\phi}(x_2,t)\right] = 0$

$$\begin{aligned}
\frac{\partial}{\partial t_1}\left\{T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]\right\} &= \frac{\partial}{\partial t_1}\left(\Theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \Theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1)\right) \\
&= \delta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \Theta(t_1-t_2)\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}\hat{\phi}(x_2,t_2) \\
&+ \delta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1) + \Theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}
\end{aligned}$$

The δ-function is symmetric and the fields commute so we get the desired result.

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Exercise (4)**.**

Using the related commutator $\left[\hat{\phi}(x_1,t),\frac{\partial\hat{\phi}(x_2,t)}{\partial t}\right] = i\delta(x_1-x_2)$ show

$$\frac{\partial^2}{\partial t_1^2}\left\{T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]\right\} = -i\delta(x_1-x_2)\delta(t_1-t_2) + T\left[\frac{\partial^2\hat{\phi}(x_1,t_1)}{\partial t_1^2}\hat{\phi}(x_2,t_2)\right],$$

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Solution.

$$\frac{\partial^2}{\partial t_1^2}\left\{T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right]\right\} = \frac{\partial}{\partial t_1}\left(\Theta(t_1-t_2)\frac{\partial\hat\phi(x_1,t_1)}{\partial t_1}\hat\phi(x_2,t_2)+\Theta(t_2-t_1)\hat\phi(x_2,t_2)\frac{\partial\hat\phi(x_1,t_1)}{\partial t_1}\right)$$
$$= \delta(t_1-t_2)\frac{\partial\hat\phi(x_1,t_1)}{\partial t_1}\hat\phi(x_2,t_2)+\Theta(t_1-t_2)\frac{\partial^2\hat\phi(x_1,t_1)}{\partial t_1^2}\hat\phi(x_2,t_2)$$
$$-\delta(t_2-t_1)\hat\phi(x_2,t_2)\frac{\partial\hat\phi(x_1,t_1)}{\partial t_1}+\Theta(t_2-t_1)\hat\phi(x_2,t_2)\frac{\partial^2\hat\phi(x_1,t_1)}{\partial t_1^2}$$

Now flipping the first term costs a commutator and if we are smart with the sign the terms will cancel. The second part is just the Time-Ordering operator

$$= -i\delta(x_1-x_2)\delta(t_1-t_2)+T\left[\frac{\partial^2\hat\phi(x_1,t_1)}{\partial t_1^2},\hat\phi(x_2,t_2)\right]$$

Exercise (5).

Argue that the above demonstrates that the so-called two-point function, $T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right]$, obeys

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right] = -i\delta(x_1-x_2)\delta(t_1-t_2),$$

which implies that the two-point function is in fact the *Green's function* for the Klein-Gordon equation. The derivation here is easily generalized to three dimensions of space.

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Solution. Argue that the above demonstrates that the so-called two-point function, $T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right]$, obeys

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right] = -i\delta(x_1-x_2)\delta(t_1-t_2).$$

Let's calculate:

$$\left(-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\left[\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)\right] = \Theta(t_1-t_2)\left(-\frac{\partial^2}{\partial x_1^2}+m^2\right)\hat\phi(x_1,t_1)\hat\phi(x_2,t_2)$$
$$+\Theta(t_2-t_1)\left(-\frac{\partial^2}{\partial x_1^2}+m^2\right)\hat\phi(x_2,t_2)\hat\phi(x_1,t_1).$$

Now we know that the $\hat\phi$'s commute, so we can pull them outside:

$$= (\Theta(t_1-t_2)+\Theta(t_2-t_1))\left(-\frac{\partial^2}{\partial x_1^2}+m^2\right)\hat\phi(x_1,t_1)\hat\phi(x_2,t_2).$$

Rearrange a bit:

$$\left(-\frac{\partial^2}{\partial x_1^2}\hat{\phi}(x_1,t_1)+m^2\hat{\phi}(x_1,t_1)\right)(\Theta(t_1-t_2)+\Theta(t_2-t_1))\,\hat{\phi}(x_2,t_2).$$

Let's include the time derivative:

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]=-i\delta(x_1-x_2)\delta(t_1-t_2)+T\left[\frac{\partial^2\hat{\phi}(x_1,t_1)}{\partial t_1^2},\hat{\phi}(x_2,t_2)\right]$$
$$+\left(-\frac{\partial^2}{\partial x_1^2}+m^2\right)\hat{\phi}(x_1,t_1)\,(\Theta(t_1-t_2)+\Theta(t_2-t_1))\,\hat{\phi}(x_2,t_2).$$

Expand the time-ordering operator:

$$=-i\delta(x_1-x_2)\delta(t_1-t_2)+\Theta(t_1-t_2)\frac{\partial^2\hat{\phi}(x_1,t_1)}{\partial t_1^2}\hat{\phi}(x_2,t_2)$$
$$+\Theta(t_2-t_1)\hat{\phi}(x_2,t_2)\frac{\partial^2\hat{\phi}(x_1,t_1)}{\partial t_1^2}+\text{ the rest.}$$

What we see here is that we may commute the differential terms, and we can then collect everything:

$$=-i\delta(x_1-x_2)\delta(t_1-t_2)+\left(\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)\hat{\phi}(x_1,t_1)\right)(\Theta(t_1-t_2)+\Theta(t_2-t_1))\,\hat{\phi}(x_2,t_2).$$

Since $\hat{\phi}$ satisfies the Klein-Gordon equation, the second term is zero. And we get the desired result,

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}+m^2\right)T\left[\hat{\phi}\left(x_1,t_1\right)\hat{\phi}\left(x_2,t_2\right)\right]=-i\delta\left(x_1-x_2\right)\delta\left(t_1-t_2\right).$$