Advanced Particle Physics 2025 Problem Set 2

1 The Klein-Gordon Equation

The Klein-Gordon equation is a famous wave equation for scalar (spin zero, i.e. *no* internal degrees of freedom) particles that was actually found by Schrödinger prior to him finding his much more famous non-relativistic equation. Since we are discussing relativistic dynamics here, the most important relation is arguably $E^2 = \boldsymbol{p}^2 + m^2$ and we will now attempt to find a corresponding wave equation when we assume spin zero particles. Let the wave function of the particle be ϕ. Since m is the rest mass, $m^2\phi^2$ is an invariant (Lorentz scalar) and does not change under Lorentz coordinate transformations.

1) Use Einstein's energy momentum relation, $E^2 - \boldsymbol{p}^2 = m^2$, and canonical quantization $E \to i\frac{\partial}{\partial t}$ and $\boldsymbol{p} \to -i\boldsymbol{\nabla}$ to derive the Klein-Gordon equation

$$-\frac{\partial^2 \phi}{\partial t^2} = \left(-\nabla^2 + m^2\right)\phi \tag{1}$$

2) Argue that $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$ and consequently derive the short-hand form

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0. \tag{2}$$

3) Show that $\phi(x) = Ne^{-ip\cdot x}$ is a solution of the Klein-Gordon equation. Here $p = p^\mu = (E, \boldsymbol{p})$, $x = x^\mu = (t, \boldsymbol{x})$, and N is a normalization constant which we will not worry about right now. The short-hand for the four-vector contraction is $p \cdot x = p^\mu x_\mu = p_\mu x^\mu$. Show that there are solutions with both negative and positive energies, $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$.

The Klein-Gordon field is generally complex, just like the non-relativistic wave function from the Schrödinger equation. We can therefore define a current along the same lines as for the non-relativistic case. Let us define a 'probability' density and a current

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t}\phi\right) \qquad \boldsymbol{j} = -i\left(\phi^* \boldsymbol{\nabla}\phi - \phi \boldsymbol{\nabla}\phi^*\right) \tag{3}$$

4) Use the Klein-Gordon equation to derive the conservation law

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4}$$

Define the total charge

$$Q = \int d^3x\rho. \tag{5}$$

When is this charge constant in time?

5) Define the four-vector current, $j^\mu = (\rho, \boldsymbol{j})$ and show that $\partial_\mu j^\mu = 0$.

6) Show that for $\phi(x) = Ne^{-ip\cdot x}$ the current is $j^\mu = 2|N|^2 p^\mu$. Use this to argue that ρ cannot be positive definite.

7) Consider a situation where we have prepared a state with one spin zero particle in a volume V. Show that in this case the normalization may be chosen as $N = 1/\sqrt{2|E|V}$. (Hint: Consider the time-component of the four-current j^μ.) In many context and presentations one consider unit volume, i.e. $V = 1$, and in many places people will just write $N = 1/\sqrt{2|E|}$. Use this information to discuss how the energy of a plane wave is related to whether it represent a particle or an antiparticle.

8) The Lagrangian for the free complex Klein-Gordon field is

$$\mathcal{L}_{KG} = (\partial^\mu\phi)^* \partial_\mu\phi - m^2\phi^*\phi. \tag{6}$$

Use the Euler-Lagrange equation to derive the Klein-Gordon equation from the Lagrangian. (Hint: It can be done by consider the real and imaginary parts of ϕ to be two independent degrees of freedom. However, it is simpler and more elegant to consider ϕ and ϕ^* as the two independent degrees of freedom. You may want to convince yourself that these two approaches are equivalent.)

9) The Hamiltonian for the Klein-Gordon field may be obtained from the definition

$$H = \int d^3x \left[\pi(\boldsymbol{x},t)\dot{\phi}(\boldsymbol{x},t) + \pi(\boldsymbol{x},t)^*\dot{\phi}(\boldsymbol{x},t)^* - \mathcal{L}_{KG}\right] = \int d^3x\mathcal{H}, \tag{7}$$

where $\mathcal{H}$ is the Hamiltonian density. The notation $\dot{\phi} = \frac{\partial\phi}{\partial t}$ is the usual one from mechanics. The canonical momentum, π, belonging to ϕ is also defined in analogy to standard analytical mechanics for continuum theories, i.e. $\pi(\boldsymbol{x},t) = \frac{\partial\mathcal{L}_{KG}}{\partial\dot{\phi}(\boldsymbol{x},t)}$. Show that

$$H = \int d^3x \left(\pi(\boldsymbol{x},t)^*\pi(\boldsymbol{x},t) + \boldsymbol{\nabla}\phi(\boldsymbol{x},t)^* \cdot \boldsymbol{\nabla}\phi(\boldsymbol{x},t) + m^2\phi(\boldsymbol{x},t)^*\phi(\boldsymbol{x},t)\right). \tag{8}$$

10) Show that the Lagrangian is invariant under the transformation $\phi \to e^{i\theta}\phi$, where θ is a real number. Use Noether's theorem to derive the corresponding conserved current. Compare your result to the current in 5) above.

2 Klein-Gordon and Harmonic Oscillators

We now want to quantize the Klein-Gordon field and will use an analogy to the non-relativistic case and the Schrödinger equation. Typically, we have a system with a discrete set of position and momentum variables. In terms of generalized coordinates, we write q_i and p_i, respectively. The canonical quantization rules should be very familiar (in units where $\hbar = 1$)

$$[q_i, p_j] = i\delta_{ij}, \tag{9}$$
$$[q_i, q_j] = [p_i, p_j] = 0. \tag{10}$$

When the wave function is just a function and the canonical quantization rules are used, this is often called *first quantization*. This is to be constrated with so-called *second quantization* where we promote the wave function (ϕ in our case) from a function to an operator. This is what we will do now. Do keep in mind that the commutation rules for q_i and p_i above continue to hold also in second quantization.

As the Klein-Gordon equation is a wave equation for a field, it has a continuum of degrees of freedom (the value of ϕ at every point in space and in time). The momentum associated with the Klein-Gordon field is also a continous variable and we will denote it by π. We now generalize the canonical commutation relations above and insist that we must have

$$[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{11}$$
$$[\phi(\boldsymbol{x}), \phi(\boldsymbol{x}')] = [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = 0. \tag{12}$$

Note that these are so-called *equal-time commutation relations* since we assume that the fields are evaluated at equal times. This is a trivial observation when working in the Schrödinger picture of quantum mechanics where operators do not depend on time. However, we will often need to switch pictures to the Heisenberg or Interaction picture where operators have a dependence on time. In this case the relations above still hold for equal times.

1) Consider the classical Klein-Gordon field, i.e. where ϕ is a function, and do a Fourier transform on the spatial coordinates

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \phi(\boldsymbol{p}, t). \tag{13}$$

Show that if you require that $\phi(\boldsymbol{x}, t)$ is a real function, then $\phi(\boldsymbol{p}, t)^* = \phi(-\boldsymbol{p}, t)$.

2) By using the Fourier transform, show that the Klein-Gordon equation can be written

$$\left[\frac{\partial^2}{\partial t^2} + \left(\boldsymbol{p}^2 + m^2\right)\right] \phi(\boldsymbol{p}, t) = 0, \tag{14}$$

and convince yourself that this is just the equation of motion for the classical harmonic oscillator with frequency $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. What is the general solution to this equation?

3) Let us focus on a fixed value of $\boldsymbol{p}$, and focus on an example in one dimension for simplicity, i.e. the momentum is just p (*not* to be confused with the four-momentum!). A simple harmonic oscillator has the Hamiltonian

$$H_{HO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2. \tag{15}$$

Introduce the step down and step up operators, a and $a^\dagger$ and define

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right), \tag{16}$$

$$p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right). \tag{17}$$

Show that $[\phi, p] = i$ implies that $[a, a^\dagger] = 1$. Furthermore, show that

$$H_{HO} = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{18}$$

4) The oscillator vacuum state, $|0\rangle$ is define by $a|0\rangle = 0$. Show that it is an eigenstate of $H_{HO}$with eigenvalue $\omega/2$. Show that

$$\left[H_{HO}, a^\dagger\right] = \omega a^\dagger, \tag{19}$$

$$[H_{HO}, a] = -\omega a. \tag{20}$$

Finally, show that the state $|n\rangle := (a^\dagger)^n|0\rangle$ is an eigenstate of H_{HO} with eigenvalue $(n + 1/2)\omega$.

We now proceed by analogy and assert that for each *mode* with momentum $\boldsymbol{p}$ of the Klein-Gordon field there is an associated harmonic oscillator with corresponding operators. The general expansion we will use has the form

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + c^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right). \tag{21}$$

The conjugate momentum field for ϕ can be written down by analogy with Eq. (17) and has the form

$$\pi^\dagger(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - c^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right). \tag{22}$$

The reason that we need a dagger on the left-hand side of the conjugate momentum is seen in question number 6) below. From a more general point of view, this also follows from the fact that when we include the full time-dependence we need the field to satisfy the Heisenberg equations of motion for the free Hamiltonian (see Eq. (34) below).

5) Show that if we insist that $\phi(\boldsymbol{x})$ is real, then $a_{\boldsymbol{p}} = c_{\boldsymbol{p}}$ (Hint: The complex conjugate of fields quite naturally becomes the hermitian conjugate for field operators.). The interpretation in quantum field theory of this result is that real fields correspond to particles that are their own anti-particle.

6) Show that we can ensure the commutation relation $[\phi(\boldsymbol{x}), \pi(\boldsymbol{x}')] = i\delta^3(\boldsymbol{x}-\boldsymbol{x}')$ if we take the operator commutation relations to be

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \tag{23}$$

$$[c_{\boldsymbol{p}}, c^\dagger_{\boldsymbol{p}'}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}'), \tag{24}$$

and assume that all other commutators with a and c vanish (so that $[a_{\boldsymbol{p}}, a_{\boldsymbol{p}'}] = 0$, $[c_{\boldsymbol{p}}, a_{\boldsymbol{p}'}] = 0$, $[c_{\boldsymbol{p}}, c_{\boldsymbol{p}'}] = 0$ for any values).

7) The Hamiltonian is given in Eq. (8). Replacing complex conjugate for numbers with hermitian conjugate for operators, show that the Hamiltonian may now be written in the form

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\boldsymbol{p}} \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{p}}\right] + c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}} + \frac{1}{2}\left[c_{\boldsymbol{p}}, c^\dagger_{\boldsymbol{p}}\right] \right). \tag{25}$$

What values do you get from the two commutator terms and what problems could this create? Typically we will disregard these commutator terms.

8) For the field operators, the conserved charge $Q = \int d^3x\rho$ from Eq. (5) becomes the charge operator

$$Q = i\int d^3x \left(\phi(\boldsymbol{x})^\dagger \pi(\boldsymbol{x})^\dagger - \pi(\boldsymbol{x})\phi(\boldsymbol{x})\right). \tag{26}$$

Argue that this expression makes sense when compared to the four-current of a classical scalar field obeying the Klein-Gordon equation (Hint: First show that for classical fields we have $\pi(\boldsymbol{x}, t) = \frac{\partial\phi(\boldsymbol{x},t)^*}{\partial t}$). Show that the charge operator can be written

$$Q = \int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}}\right), \tag{27}$$

where we have throw away any commutator terms under the integral. Interpret the result in terms of particles and antiparticles.

9) Show that

$$[H, a^\dagger_{\boldsymbol{p}}] = \omega_{\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} \quad \text{and} \quad [H, a_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}} a_{\boldsymbol{p}} \tag{28}$$

(with analogous expressions for $c_{\boldsymbol{p}}$). Define the vacuum state, $|0\rangle$ such that $a_{\boldsymbol{p}}|0\rangle = c_{\boldsymbol{p}}|0\rangle$ for all $\boldsymbol{p}$. Argue that the full spectrum of the Hamiltonian can be obtained by applying creation operators, $a^\dagger_{\boldsymbol{p}}$ and $c^\dagger_{\boldsymbol{p}}$, to $|0\rangle$. What is the energy of a state like $a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2} c^\dagger_{\boldsymbol{p}_3}|0\rangle$?

10) Show that $a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle = a^\dagger_{\boldsymbol{p}_2} a^\dagger_{\boldsymbol{p}_1}|0\rangle$ and argue that this implies that these spin zero particles obey Bose-Einstein statistics.

11) Apply $\phi(\boldsymbol{x})$ to the vacuum and show that

$$\phi(\boldsymbol{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} |\boldsymbol{p}\rangle_c, \tag{29}$$

where $|\boldsymbol{p}\rangle_c = \sqrt{2\omega_{\boldsymbol{p}}}\, c^\dagger_{\boldsymbol{p}}|0\rangle$ where the subscript c indicates that we are creating a c particle. The factor $\sqrt{2\omega_{\boldsymbol{p}}}$ is introduced to ensure that the states are normalized in a Lorentz invariant fashion (more precisely, $\langle p|q\rangle = (2\pi)^3 2\omega_{\boldsymbol{p}}\delta^3(\boldsymbol{p}-\boldsymbol{q})$ can be shown to be Lorentz invariant, just consider a boost operation along one direction). Likewise, if we use $\phi^\dagger(\boldsymbol{x})$ we would be creating an a particle. Argue that for small momenta, $\boldsymbol{p}$, $\omega_{\boldsymbol{p}}$ is nearly constant and in that case the above expression is a linear superposition of plane wave states with well-defined momentum which is the Fourier transform of a non-relativistic basis state of position, $|\boldsymbol{x}\rangle$. We thus interpret $\phi(\boldsymbol{x})$ as a field operator that creates a particle at position $\boldsymbol{x}$.

12) Show that $\langle 0|\phi(\boldsymbol{x})|\boldsymbol{p}\rangle = e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$. If we interpret this as the position-space representation of the single-particle wave function of the state $|\boldsymbol{p}\rangle$, then we see that $\langle \boldsymbol{x}|\boldsymbol{p}\rangle \propto e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ is the wave function just as in non-relativistic quantum mechanics.

3 Klein-Gordon Field in Space-Time

The previous problem showed how to quantize the Klein-Gordon field in the Schrödinger picture of quantum mechanics where the operators are independent of time while the state vectors carry all the time-dependence. Here we will consider the (equivalent) Heisenberg picture where the state vectors are time-independent and the operators carry the time-dependence. The definition of an operator in the Heisenberg picture is straighforward

$$\mathcal{O}_H(x) = \mathcal{O}_H(\boldsymbol{x}, t) = e^{iHt}\mathcal{O}_S(\boldsymbol{x})e^{-iHt}, \tag{30}$$

where $\mathcal{O}_S(\boldsymbol{x})$ is an operator in the Schrödinger picture and H is the Hamiltonian operator which we assume has no explicit time-dependence in this problem.

1) Assume that $\mathcal{O}$ does not have any explicit dependence on time t. Derive the

Heisenberg equation of motion

$$i\frac{\partial}{\partial t}\mathcal{O}_H = [\mathcal{O}_H, H]\,. \tag{31}$$

2) The quantized version of the Hamiltonian for the Klein-Gordon field follows from Eq. (8) above and is

$$H = \int d^3x \left(\pi(\boldsymbol{x},t)^\dagger\pi(\boldsymbol{x},t) + \boldsymbol{\nabla}\phi(\boldsymbol{x},t)^\dagger\cdot\boldsymbol{\nabla}\phi(\boldsymbol{x},t) + m^2\phi(\boldsymbol{x},t)^\dagger\phi(\boldsymbol{x},t)\right). \tag{32}$$

Calculate $[\phi(\boldsymbol{x},t), H]$ and show that

$$i\frac{\partial}{\partial t}\phi(\boldsymbol{x},t) = i\pi(\boldsymbol{x},t)^\dagger. \tag{33}$$

You will need the equal-time commutator in Eq. (11) and the fact that all combinations like $[\phi,\pi^\dagger]=0$ vanish.

3) By calculating the commutator $[\pi(x,t),H]$ show that

$$i\frac{\partial}{\partial t}\pi(\boldsymbol{x},t) = -i\left(-\boldsymbol{\nabla}^2 + m^2\right)\phi(\boldsymbol{x},t)^\dagger. \tag{34}$$

(Hint: You will need to do partial integration and throw away a boundary term which we assume vanishes at infinity).

4) Use the results of 2) and 3) to show that the field operator $\phi(\boldsymbol{x},t)$ obeys the Klein-Gordon equation.

Consider now the expansion of the field operator in modes, i.e.

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + c_{\boldsymbol{p}}^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right). \tag{35}$$

We would now like to use the creation and annihilation operators for the modes, $a_{\boldsymbol{p}}$, $a_{\boldsymbol{p}}^\dagger$, $c_{\boldsymbol{p}}$, $c_{\boldsymbol{p}}^\dagger$, to obtain the time-dependence explicitly. Since in the expansion they are the only quantities that are operators, all we need is to determine how they evolve in time in the Heisenberg picture.

5) Use the commutator relations like those in Eq. (28) to deduce the relations

$$H^n a_{\boldsymbol{p}} = a_{\boldsymbol{p}}\left(H-\omega_{\boldsymbol{p}}\right)^n \tag{36}$$

for any integer n. (Hint: Start with $n=1$.) Deduce an analogous relation for $a_{\boldsymbol{p}}^\dagger$ where the minus becomes a plus on the right-hand side. Argue that identical relations can be deduced for the $c_{\boldsymbol{p}}$ and $c_{\boldsymbol{p}}^\dagger$ operators.

6) Use the result of 5) to show that

$$e^{iHt}a_{\boldsymbol{p}}e^{-iHt} = a_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t} \quad \text{and} \quad e^{iHt}a_{\boldsymbol{p}}^\dagger e^{-iHt} = a_{\boldsymbol{p}}^\dagger e^{i\omega_{\boldsymbol{p}}t} \tag{37}$$

with identical relations for the c operators.

7) Show that the field operator in the Heisenberg picture can be written in the elegant form

$$\phi(\boldsymbol{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}} e^{-ip\cdot x} + c^\dagger_{\boldsymbol{p}} e^{ip\cdot x} \right), \tag{38}$$

where $p = p^\mu = (\omega_{\boldsymbol{p}}, \boldsymbol{p})$, $x = x^\mu = (t, \boldsymbol{x})$, and $p \cdot x$ denotes the contraction of the two four-vectors, i.e. $p \cdot x = p^\mu x_\mu$.

Notice how the field operator in the Heisenberg picture in Eq. (38) is an expansion in modes that correspond to the solutions of the free Klein-Gordon equation (plane waves with argument $p \cdot x$). This is a very important result when doing perturbation calculations in quantum field theories.

8) The energy $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2} > 0$ is positive by definition. Argue that if the exponential factors in Eq. (38) are interpreted as single-particle wave functions, they would correspond to states with positive and negative energies respectively. (Hint: Consider the time-evolution (phase) factor for a stationary state in the Schrödinger equation.)

9) Combine the notion of particles and anti-particles with the positive and negative energy solutions in the expansion of the field operator, and argue (following Richard Feynman) that the destruction of a particle with four-momentum p^μ (and thus positive energy) is equivalent to the creation of an anti-particle with four-momentum $-p^\mu$ (and negative energy), and vice versa.

Feynman pushed the analogy a bit further by stating that negative energy solutions corresponds to *positive energy* anti-particles propagating backwards in time. However, when using the most common momentum-space approach to diagramatic perturbation theory this distinction and interpretation is not important, so in modern presentations this is typically either not discussed or only briefly mentioned.

10) Define the three-momentum operator

$$\boldsymbol{P} = -\int \mathrm{d}^3 x \left[\pi(\boldsymbol{x}, t)^\dagger \left(\nabla\phi(\boldsymbol{x}, t)\right)^\dagger + \left(\boldsymbol{\nabla}\phi(\boldsymbol{x}, t)\right) \pi(\boldsymbol{x}, t) \right]. \tag{39}$$

The operator has been properly symmetrized since we work with complex fields. Show that

$$\boldsymbol{P} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \boldsymbol{p} \, \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + c^\dagger_{\boldsymbol{p}} c_{\boldsymbol{p}} \right), \tag{40}$$

where $\boldsymbol{p}$ is the three-momentum.

11) Show that $\left[\boldsymbol{P}, a_{\boldsymbol{p}}^{\dagger}\right] = \boldsymbol{p} a_{\boldsymbol{p}}^{\dagger}$ and $[\boldsymbol{P}, a_{\boldsymbol{p}}] = -\boldsymbol{p} a_{\boldsymbol{p}}$. Use these relations to show that

$$\boldsymbol{P}^n a_{\boldsymbol{p}}^{\dagger} = a_{\boldsymbol{p}}^{\dagger} \left(\boldsymbol{P} + \boldsymbol{p}\right)^n \quad \text{and} \quad \boldsymbol{P}^n a_{\boldsymbol{p}} = a_{\boldsymbol{p}} \left(\boldsymbol{P} - \boldsymbol{p}\right)^n, \tag{41}$$

for any integer n. Finally, derive the translation identities

$$e^{-i\boldsymbol{P}\cdot\boldsymbol{x}} a_{\boldsymbol{p}} e^{i\boldsymbol{P}\cdot\boldsymbol{x}} = a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \quad \text{and} \quad e^{-i\boldsymbol{P}\cdot\boldsymbol{x}} a_{\boldsymbol{p}}^{\dagger} e^{i\boldsymbol{P}\cdot\boldsymbol{x}} = a_{\boldsymbol{p}}^{\dagger} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}. \tag{42}$$

Argue that identical relations hold for the $c_{\boldsymbol{p}}$ and $c_{\boldsymbol{p}}^{\dagger}$ operators.

12) Define the four-momentum operator $P^{\mu} = (H, \boldsymbol{P})$. Using the relations above show that the ϕ field can be translated in space and time by

$$\phi(\boldsymbol{x}, t) = e^{iP^{\mu}x_{\mu}} \phi(0,0) e^{-iP^{\mu}x_{\mu}}, \tag{43}$$

where $x^{\mu} = (t, \boldsymbol{x})$.

4 The Time Evolution Operator and the S-matrix

Consider the Schrödinger equation which we write

$$i\frac{\partial}{\partial t}|\Psi, t\rangle_S = H|\Psi, t\rangle_S, \tag{44}$$

where H is the Hamiltonian and $|\Psi, t\rangle$ is the state at time t. The subscript S refers to the Schrödinger picture where operators are time-independent (except for explicit time-dependent terms) and states are time-dependent. Let us furthermore split the Hamiltonian into a free Hamiltonian (containing typically kinetic energy and mass terms) and an interacting part that contains the interactions of different particles, i.e. $H = H_F + H_{I,S}$. Here the notation $H_{I,S}$ means the interaction part of the Hamiltonian in the Schrödinger picture. Define the state

$$|\Psi, t\rangle_I = e^{iH_F t}|\Psi, t\rangle_S, \tag{45}$$

which is called the interaction picture state and has subscript I, and also define the interaction picture operators

$$O_I = e^{iH_F t} O_S e^{-iH_F t}, \tag{46}$$

where O_S is a Schrödinger picture operator (typically time-independent).

1) Show that

$$i\frac{\partial}{\partial t}|\Psi, t\rangle_I = H_I|\Psi, t\rangle_I, \tag{47}$$

where $H_I = e^{iH_F t} H_{I,S} e^{-iH_F t}$, and show that

$$\frac{d}{dt} O_I = -i[O_I, H_F], \tag{48}$$

where you assume no explicit time-dependence in O_S. From now on H_I will denote the interaction part of the Hamiltonian in the interaction picture.

2) Introduce the time evolution operator, $U(t, t_0)$, that evolves states from time t_0 to time t, i.e. $|\Psi, t\rangle_I = U(t, t_0)|\Psi, t_0\rangle_I$. Clearly $U(t_0, t_0) = 1$. Show that

$$i\frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0). \tag{49}$$

Note the explicit time-dependence on $H_I(t)$!

3) Show that

$$U(t, t_0) = 1 - i \int_{t_0}^{t} dt_1 H_I(t_1) U(t_1, t_0), \tag{50}$$

and

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \ldots, \tag{51}$$

where $\ldots$ denote higher-order terms containing three or more factors of $H_I(t)$.

4) Now define the S-matrix for the process from initial state, i, to final state, f, i.e. $i \to f$, by

$$S_{fi} = \lim_{t_0 \to -\infty} \lim_{t \to \infty} \langle f|U(t, t_0)|i\rangle. \tag{52}$$

Give a physical interpretation of this matrix element given what you know about $U(t, t_0)$ and relate it to how experiments are done. What have we assumed about the states $|f\rangle$ and $|i\rangle$? If you calculate S_{fi} in the Schrödinger picture will it be the same result?

5) Show that the first order contribution to S_{fi} can be written

$$S_{fi}^{(1)} = \delta_{fi} - i \int d^4x \langle f|\mathcal{H}|i\rangle, \tag{53}$$

where $H_I = \int d^3x \mathcal{H}$. $\mathcal{H}$ is called the Hamiltonian density. Since it contains only interaction terms, it differs from the Lagrangian density only by a sign (remember the basic idea that $L = T - V$ while $H = T + V$).

If we assume that the interactions conserve energy and momentum, we have the communtation relation $[P^\mu, H] = 0$, where P^μ is the total energy and momentum operator. This operator acts on plane wave as $P^\mu|k\rangle = k^\mu|k\rangle$. It can also be used to generate finite translations in space and time by application of $e^{iP^\mu a_\mu}$, where a^μ is some space-time vector.

6) Show that $[P^\mu, H] = 0$ implies that $[e^{iP^\mu a_\mu}, H] = 0$.

7) The Hamiltonian density depends on space-time coordinates in general, $\mathcal{H}(x)$. Argue we can use translation operators to write $\mathcal{H}(x) = e^{iP^\mu x_\mu}\mathcal{H}(0)e^{-iP^\mu x_\mu}$ (Hint: Consider how $\mathcal{H}(x)$ looks when written in terms of quantum field operators and use the properties of the fields under translations in space and time).

8) Show that this implies that we can isolate the space-time dependence of $S_{fi}^{(1)}$ through

$$\langle f|\mathcal{H}(x)|i\rangle = \langle f|\mathcal{H}(0)|i\rangle e^{i(p_f^\mu - p_i^\mu)x_\mu}, \tag{54}$$

where p_f and p_i are the total four-momenta of final and initial states respectively.

Show that we now obtain

$$S_{fi}^{(1)} = \delta_{fi} - i(2\pi)^4\langle f|\mathcal{H}(0)|i\rangle\delta^4(p_f - p_i). \tag{55}$$

9) Consider now the second order contribution to the S-matrix. Show that it can we written in the form

$$S_{fi}^{(2)} = (-i)^2 \int d^4x_1 \int d^4x_2 \theta(t_1 - t_2)\langle f|\mathcal{H}(x_1)\mathcal{H}(x_2)|i\rangle, \tag{56}$$

where $x_1^\mu = (t_1, \boldsymbol{x}_1)$ and $x_2^\mu = (t_2, \boldsymbol{x}_2)$, while $\theta(x)$ is the Heaviside step function which is $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Finally, show that we can write this in the form

$$S_{fi}^{(2)} = (-i)^2(2\pi)^4\delta^4(p_f - p_i) \int d^4x\theta(t)\langle f|\mathcal{H}(x)\mathcal{H}(0)|i\rangle, \tag{57}$$

where $x = x_1 - x_2$.

5 The Time Ordering Operator

Consider the expansion of the quantum mechanical time evolution operator in the interaction picture

$$U(t, t_0) = 1 - i\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \ldots, \tag{58}$$

where $t_0 < t$.

1) Remembering that H_I is an operator, argue that the operators in the third term in Eq. (58) is correctly ordered as it stands.

2) Show that for any operator, $A(t)$, we have

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 A(t_1)A(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\left[A(t_1)A(t_2)\right], \tag{59}$$

where we have defined the *time-ordering operator* by

$$T\left[A(t_1)A(t_2)\right] = \begin{cases} A(t_1)A(t_2) & \text{for } t_1 > t_2 \\ A(t_2)A(t_1) & \text{for } t_2 > t_1 \end{cases} \tag{60}$$

3) Write $U(t, t_0)$ from Eq. (58) using the result of problem 2).

More generally, the time-ordering operator with n terms orders the terms according to the ordering of their time argument, i.e.

$$T\left[A_1(t_1)A_2(t_2)A_3(t_3)\dots A_n(t_n)\right] = A_1(t_1)A_2(t_2)A_3(t_3)\dots A_n(t_n) \tag{61}$$

if $t_1 > t_2 > t_3 > \dots > t_n$, and it reorders if some other order is applied. For example

$$T\left[A_1(t_1)A_2(t_2)A_3(t_3)\right] = A_2(t_2)A_1(t_1)A_3(t_3) \tag{62}$$

if $t_2 > t_1 > t_3$. Basically, it ensures that the operator with the earliest time acts first on the state to the right. This is what we want when discussing $U(t, t_0)$.

4) Prove the general expansion

$$U(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n T\left[H_I(t_1)\dots H_I(t_n)\right]. \tag{63}$$

(Hint: Look at a particular term of nth order and prove that

$$\begin{aligned} &\int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n T\left[H_I(t_1)\dots H_I(t_n)\right] = \\ &n! \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1)H_I(t_2)\dots H_I(t_n) \end{aligned} \tag{64}$$

by splitting the integration region according to the ordering of $t_1, t_2, \dots, t_n$.)

6 Time-ordered Operators and the Klein-Gordon Equation

Consider a real scalar field *operator*, $\hat{\phi}(x, t)$, in one dimension (x is a single number, not a vector!) for simplicity with mass m. The Klein-Gordon wave equation is assumed to hold for the field operator as well, and it can be written

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\hat{\phi}(x,t) = 0. \tag{65}$$

1) Prove that

$$\begin{aligned} &T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right] = \\ &\theta(t_1-t_2)\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\hat{\phi}(x_1,t_1), \end{aligned} \tag{66}$$

where θ is the Heaviside step function.

2) Show that

$$\frac{d}{dx}\theta(x-a) = \delta(x-a) \tag{67}$$

3) Use the previous results and the commutator $[\hat{\phi}(x_1,t), \hat{\phi}(x_2,t)] = 0$ to show that

$$\begin{aligned} &\frac{\partial}{\partial t_1}\left\{T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]\right\} = \\ &\theta(t_1-t_2)\left(\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}\right)\hat{\phi}(x_2,t_2) + \theta(t_2-t_1)\hat{\phi}(x_2,t_2)\left(\frac{\partial\hat{\phi}(x_1,t_1)}{\partial t_1}\right), \end{aligned} \tag{68}$$

4) Using the related commutator $[\hat{\phi}(x_1,t), \frac{\partial\hat{\phi}(x_2,t)}{\partial t}] = i\delta(x_1-x_2)$ show

$$\begin{aligned} &\frac{\partial^2}{\partial t_1^2}\left\{T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]\right\} = \\ &-i\delta(x_1-x_2)\delta(t_1-t_2) + T\left[\frac{\partial^2\hat{\phi}(x_1,t_1)}{\partial t_1^2}\hat{\phi}(x_2,t_2)\right], \end{aligned} \tag{69}$$

5) Argue that the above demonstrates that the so-called *two-point* function, $T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right]$, obeys

$$\left(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2\right)T\left[\hat{\phi}(x_1,t_1)\hat{\phi}(x_2,t_2)\right] = -i\delta(x_1-x_2)\delta(t_1-t_2), \tag{70}$$

which implies that the two-point function is in fact the *Green's function* for the Klein-Gordon equation. The derivation here is easily generalized to three dimensions of space.

7 Green's Function for Scalar Fields in Momentum Space

Let us now consider what happens to Eq. (70) if we switch to momentum space. To avoid confusion, here we use the following Fourier transforms and definitions

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} f(k) \tag{71}$$

$$f(k) = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \tag{72}$$

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}, \tag{73}$$

where we use the (slightly sloppy) convention of using the same symbol f for both the function $f(x)$ and its transform $f(k)$.

1) Consider $\Delta(x_1, t_1; x_2, t_2) := \langle 0| T \left[\hat{\phi}(x_1, t_1) \hat{\phi}(x_2, t_2) \right] |0\rangle$ from Eq. (70) but now we evaluate it in the vacuum ($|0\rangle$). This means that we now have a number and not an operator to deal with. This function is called the *propagator*. Argue that

$$\left(\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial x_1^2} + m^2 \right) \Delta(x_1, t_1; x_2, t_2) = -i\delta(x_1 - x_2)\delta(t_1 - t_2). \tag{74}$$

2) Use Eq. (74) to argue $\Delta(x_1, t_1; x_2, t_2)$ can *only* depend on the differences of time and space, i.e. $\Delta = \Delta(x_1 - x_2, t_1 - t_2)$. Thus we have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2 \right) \Delta(x, t) = -i\delta(x)\delta(t), \tag{75}$$

where $x = x_1 - x_2$ and $t = t_1 - t_2$.

3) Do the Fourier transform to derive the momentum-space expression

$$\Delta(k, \omega) = \frac{i}{\omega^2 - k^2 - m^2}, \tag{76}$$

Generalize to three spatial dimensions and express the result using 4-vectors.

8 Scalar Fields and Causality

Let us consider an expectation value of two fields in the vacuum

$$D(x, y) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{77}$$

where x and y are four-vectors (we suppress the space-time indices for simplicity). We use the field expansion for a real field, ϕ, which has the form

$$\phi = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \left[e^{-ipx} a_{\boldsymbol{p}} + e^{ipx} a^\dagger_{\boldsymbol{p}} \right], \tag{78}$$

where $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$.

1) Show that

$$D(x,y) = D(x-y) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-ip(x-y)}, \tag{79}$$

and argue that this quantity is a Lorentz scalar, i.e. it is invariant under Lorentz transformations.

2) Show that

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x). \tag{80}$$

3) Show that if $(x-y)^2 < 0$ then $[\phi(x), \phi(y)] = 0$. (Hint: First argue that in this case a Lorentz transformation can be found that takes $(x_0 - y_0)$ to $-(x_0 - y_0) = y_0 - x_0$).

4) Argue that you have now shown that causality is preserved for scalar fields.

9 Propagators and the Hidden Causality Term

When deriving the propagators using the equation of motion approach we obtain some very straightforward expressions for the Green's function or two-point function. However, that derivation tends to hide the fact that there is a causality requirement in the time-ordering (operator with smallest time acts to the right first and so on). Here we will look at this in more detail.

1) Show that the vacuum expectation value of the two-point time-ordered product of scalar fields can be written in the form

$$\begin{aligned}\langle 0 | T\left[\phi(x_1)\phi(x_2)\right] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \Big[&\theta(t_1 - t_2) e^{-i\omega_p(t_1 - t_2)} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_1 - \boldsymbol{x}_2)} \\ &+ \theta(t_2 - t_1) e^{-i\omega_p(t_2 - t_1)} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}_2 - \boldsymbol{x}_1)} \Big],\end{aligned} \tag{81}$$

where $\omega_p = \sqrt{\boldsymbol{p}^2 + m^2}$.

2) Use contour integration to show that one can represent the Heaviside step function in the following form

$$\theta(t) = i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z + i\epsilon}, \tag{82}$$

where ϵ is an infinitesimal small positive number, i.e. $\epsilon \to 0^+$. Now show that

$$\theta(t) e^{-i\omega_p t} = i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z - (\omega_p - i\epsilon)}, \tag{83}$$

3) Use the result of 1) and 2) to show that

$$\langle 0| T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_p} \left[\frac{e^{-ip\cdot(x_1-x_2)}}{p_0 - (\omega_p - i\epsilon)} + \frac{e^{ip\cdot(x_1-x_2)}}{p_0 - (\omega_p - i\epsilon)} \right], \tag{84}$$

where the four-vector is $p = (p_0, \boldsymbol{p})$ as usual.

4) Substitute the four-vector $p \to -p$ in the second term in the expression in 3) and show that

$$\langle 0| T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)} \frac{i}{p_0^2 - (\omega_p - i\epsilon)^2}, \tag{85}$$

where the four-vector is $p = (p_0, \boldsymbol{p})$ as usual.

5) Finally, argue that this result can be expressed as

$$\langle 0| T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)} \frac{i}{p^2 - m^2 + i\epsilon}, \tag{86}$$

since ϵ is infinitesimal. So the propagator really comes with a positive imaginary part which we usually do not write or care to much about since it drops out of most calculations. But in principle it is always present.

10 LSZ reduction formulas

Calculating S-matrix elements is essential to quantum field theory since they can be used to find cross-section and decay rates. In general one can write the transition of an m-particle *in*-state to a n particle *out*-state as

$$S_{fi}^\dagger = \langle \mathbf{q}_1, \ldots, \mathbf{q}_n; \mathrm{out}|\mathbf{p}_1, \ldots, \mathbf{p}_m; \mathrm{in}\rangle. \tag{87}$$

1) Argue that this definition is equivalent to the one in Eq. (52). Note that the above definition is the complex conjugate.

However, this definition is quite abstract, and not immediately useful for practical calculations. We therefore derive the Lehman-Symanzik-Zimmermann (LSZ) reduction formulas, which connects the S matrix element to the n-particle Green's function (or n-point function)

$$\Delta^{(n)}(x_1, \ldots, x_n) = \langle 0|T\left[\phi(x_1)\cdots\phi(x_n)\right]|0\rangle. \tag{88}$$

In order to simplify the calculations we consider first the transition amplitude between two single particle states $\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle$ and then generalize to more particles. We use the field expansion of an real scalar field in Eq. (78).

2) Show that the inversion formulas for the creation and annihilation operators are

$$a_{\mathbf{p}}^{\dagger} = \frac{-i}{\sqrt{2}} \int d^3x e^{-ipx}\left[\frac{\pi(x)}{\sqrt{\omega_{\mathbf{p}}}} + i\sqrt{\omega_{\mathbf{p}}}\phi(x)\right] = \frac{-i}{\sqrt{2}} \int \frac{d^3x}{\sqrt{\omega_{\mathbf{p}}}} e^{-ipx}\overleftrightarrow{\partial_0}\phi(x), \tag{89a}$$

$$a_{\mathbf{p}} = \frac{+i}{\sqrt{2}} \int d^3x e^{+ipx}\left[\frac{\pi(x)}{\sqrt{\omega_{\mathbf{p}}}} - i\sqrt{\omega_{\mathbf{p}}}\phi(x)\right] = \frac{+i}{\sqrt{2}} \int \frac{d^3x}{\sqrt{\omega_{\mathbf{p}}}} e^{+ipx}\overleftrightarrow{\partial_0}\phi(x), \tag{89b}$$

where we have defined the operator

$$A\overleftrightarrow{\partial_0}B = A(\partial_0 B) - (\partial_0 A)B. \tag{90}$$

3) Argue that we can write

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = \lim_{x_0\to-\infty}\left(-i\int d^3x e^{-ipx}\langle \mathbf{q}_{\text{out}}|\left[\pi(x) + i\omega_{\mathbf{p}}\phi(x)\right]|0_{\text{in}}\rangle\right). \tag{91}$$

4) Using the following identity from the fundamental theorem of calculus

$$\lim_{t\to-\infty} F(t) = \lim_{t\to\infty} F(t) - \int_{-\infty}^{\infty} dt\partial_t F(t), \tag{92}$$

show that

$$\begin{aligned}\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle =& \sqrt{2\omega_{\mathbf{p}}}\sqrt{2\omega_{\mathbf{q}}}\langle 0_{\text{out}}|a_{\mathbf{q}_{\text{out}}}a^{\dagger}_{\mathbf{p}_{\text{out}}}|0_{\text{in}}\rangle \\ &+ i\int d^4x\partial_{x^0}e^{-ipx}\langle \mathbf{q}_{\text{out}}|\left[\pi(x) + i\omega_{\mathbf{p}}\phi(x)\right]|0_{\text{in}}\rangle.\end{aligned} \tag{93}$$

5) We are only interested in interactions. Argue that we can throw away the first term of the above equation.

6) Show that by explicitly performing the time derivative we obtain

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = i\int d^4x e^{-ipx}\left(\partial_\mu\partial^\mu + m^2\right)\langle \mathbf{q}_{\text{out}}|\phi(x)|0_{\text{in}}\rangle. \tag{94}$$

Hint: Perform integration by parts over the spatial coordinates and argue throwing away the boundary term.

7) Using the same procedure on the *out*-state show that we obtain

$$\begin{aligned}\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle =& i\int d^4x e^{-ipx}\left(\Box_x + m^2\right)\Bigg[\sqrt{2\omega_{\mathbf{q}}}\langle 0_{\text{out}}|a_{\mathbf{q}_{\text{in}}}\phi(x)|0_{\text{in}}\rangle \\ &+ i\int d^4y \partial_{y^0} e^{iqy}\langle 0_{\text{out}}|\left[\pi(y) - i\omega_{\mathbf{q}}\phi(y)\right]\phi(x)|0_{\text{in}}\rangle\Bigg],\end{aligned} \tag{95}$$

where we have used $\Box = \partial_\mu\partial^\mu$ with subscript x to denote to which variable the derivation is with respect to.

8) Argue that we are free to insert a time-ordering product in both terms and argue that this means that we can drop the first term, which leaves

$$\begin{aligned}\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle =& i^2\int d^4x d^4y e^{-ipx}\left(\Box_x + m^2\right) \\ &\times \partial_{y^0} e^{iqy}\langle 0_{\text{out}}|T\left[(\pi(y) - i\omega_{\mathbf{q}}\phi(y))\phi(x)\right]|0_{\text{in}}\rangle.\end{aligned} \tag{96}$$

9) Do the derivative with respect to time y^0, as in part 6), to arrive at

$$\begin{aligned}\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle =& i^2\int d^4x d^4y e^{-i(px-qy)}\left(\Box_x + m^2\right) \\ &\times \left(\Box_y + m^2\right)\langle 0_{\text{out}}|T\left[\phi(y)\phi(x)\right]|0_{\text{in}}\rangle.\end{aligned} \tag{97}$$

10) Using the method above argue that for a general transition, *not* involving spectators, the S matrix element is

$$\begin{aligned}S_{fi} =& \langle \mathbf{q}_1, \ldots, \mathbf{q}_n; \text{out}|\mathbf{p}_1, \ldots, \mathbf{p}_m; \text{in}\rangle \\ =& i^{n+m}\int \prod_{i=1}^m d^4x_i e^{-ip_ix_i}\left(\Box_{x_i} + m^2\right)\int \prod_{j=1}^n d^4y_j e^{ip_jy_j}\left(\Box_{y_j} + m^2\right) \\ &\times \langle 0_{\text{out}}|T\left[\phi(y_1)\cdots\phi(y_n)\phi(x_1)\cdots\phi(x_m)\right]|0_{\text{in}}\rangle.\end{aligned} \tag{98}$$

11) Consider the Fourier transform of the n-particle Green's function

$$\Delta^{(n)}(k_1, \ldots, k_n) = \int d^4x_1\cdots d^4x_n e^{i(k_1x_1+\cdots+k_nx_n)}\langle 0|T\left[\phi(x_1)\cdots\phi(x_n)\right]|0\rangle, \tag{99}$$

and show that the LSZ reduction formula in momentum space takes the form

$$S_{fi} = (-i)^{n+m}\prod_{i=1}^m (p_i^2 - m^2)\prod_{j=1}^n (q_j^2 - m^2)\Delta^{(n+m)}(-p_1, \ldots, -p_n, q_1, \ldots, q_m). \tag{100}$$

Up to a phase factor.

11 Interacting Scalar Fields

In this problem we study three different uncharged scalar fields, i.e. three real fields ϕ_1, ϕ_2, and ϕ_3, with masses m_1, m_2, and m_3. We will assume that $m_3 \ll m_2 \ll m_1$.

1) What kind of particles are these fields? Which equations of motion (wave equation) do the fields obey when they are non-interacting?

Now introduce an interacting term $\mathcal{L}_{\text{int}} = g\phi_1\phi_2\phi_3$, where g is a coupling constant.

2) Find the first order matrix element, M_{fi}, for the decay process $\phi_3 \to \phi_1 + \phi_2$.

3) Find the the second order matrix element, M_{fi}, for the scattering process $\phi_1 + \phi_2 \to \phi_1 + \phi_2$. (Hint: There are two different diagrams that contribute).

4) Use the results of 3) to deduce Feynman rules for getting an amplitude from a diagram without having to go through the calculations in 3).

5) Find the non-relativistic limit of the matrix element in 3).