10 - LSZ reduction formulas

Calculating S-matrix elements is essential to quantum field theory since they can be used to find cross-section and decay rates. In general one can write the transition of an m-particle in-state to a n particle out-state as

$$S^\dagger_{fi} = \langle \mathbf{q}_1, \ldots, \mathbf{q}_n; \text{out} | \mathbf{p}_1, \ldots, \mathbf{p}_m; \text{in} \rangle. \tag{87}$$

Exercise (1)**.**

Argue that this definition is equivalent to the one in Eq. (52). Note that the above definition is the complex conjugate.

However, this definition is quite abstract, and not immediately useful for practical calculations. We therefore derive the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas, which connects the S matrix element to the n-particle Green's function (or n-point function)

$$\Delta^{(n)}(x_1, \ldots, x_n) = \langle 0 | T\left[\phi(x_1) \ldots \phi(x_n)\right] | 0 \rangle. \tag{88}$$

In order to simplify the calculations we consider first the transition amplitude between two single particle states $(\mathbf{q}_\text{out}|\mathbf{p}_\text{in})$ and then generalize to more particles. We use the field expansion of an real scalar field in Eq. (78).

Exercise (2)**.**

Show that the inversion formulas for the creation and annihilation operators are

$$a^\dagger_\mathbf{p} = \frac{-i}{\sqrt{2}} \int d^3x e^{-ipx} \left[\frac{\pi(x)}{\sqrt{\omega_\mathbf{p}}} + i\sqrt{\omega_\mathbf{p}}\phi(x)\right] = \frac{-i}{\sqrt{2}} \int \frac{d^3x}{\sqrt{\omega_\mathbf{p}}} e^{-ipx} \overleftrightarrow{\partial_0}\, \phi(x), \tag{89a}$$

$$a_\mathbf{p} = \frac{+i}{\sqrt{2}} \int d^3x e^{+ipx} \left[\frac{\pi(x)}{\sqrt{\omega_\mathbf{p}}} - i\sqrt{\omega_\mathbf{p}}\phi(x)\right] = \frac{+i}{\sqrt{2}} \int \frac{d^3x}{\sqrt{\omega_\mathbf{p}}} e^{+ipx} \overleftrightarrow{\partial_0}\, \phi(x), \tag{89b}$$

where we have defined the operator

$$A \overleftrightarrow{\partial_0} B = A(\partial_0 B) - (\partial_0 A)B. \tag{90}$$

Exercise (3)**.**

Argue that we can write

$$\langle \mathbf{q}_\text{out} | \mathbf{p}_\text{in} \rangle = \lim_{x_0 \to -\infty} \left(-i \int d^3x e^{-ipx} \langle \mathbf{q}_\text{out} | \left[\pi(x) + i\omega_\mathbf{p}\phi(x)\right] | 0_\text{in} \rangle \right). \tag{91}$$

Solution. Lets work on the right-hand side,

$$\begin{aligned}
&= \lim_{x_0 \to -\infty} \left(-i \int d^3x e^{-ipx} \langle q_{out} | \left[\pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle \right) \\
&= \lim_{x_0 \to -\infty} \left(-i \langle q_{out} | \left[\int d^3x e^{-ipx} \pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle \right) \\
&= \lim_{x_0 \to -\infty} \left(-i \langle q_{out} | i\sqrt{2\omega_p} a_p^\dagger |0_{in}\rangle \right) = \langle q_{out} \mid p_{in} \rangle .
\end{aligned}$$

Where we have used the inversion fomula for the creation operator and the assumption that,

$$\lim_{x_0 \to -\infty} a_p^\dagger = a_{p_{in}}^\dagger .$$

Exercise (4)**.**

Using the following identity from the fundamental theorem of calculus

$$\lim_{t \to -\infty} F(t) = \lim_{t \to \infty} F(t) - \int_{-\infty}^{\infty} dt \partial_t F(t), \tag{92}$$

show that

$$\langle \mathbf{q}_{\mathrm{out}} | \mathbf{p}_{\mathrm{in}} \rangle = \sqrt{2\omega_{\mathbf{p}}} \sqrt{2\omega_{\mathbf{q}}} \langle 0_{\mathrm{out}} | a_{\mathbf{q}_{\mathrm{out}}} a_{\mathbf{p}_{\mathrm{out}}}^\dagger |0_{\mathrm{in}}\rangle + i \int d^4x \partial_0 e^{-ipx} \langle \mathbf{q}_{\mathrm{out}} | \left[\pi(x) + i\omega_{\mathbf{p}} \phi(x) \right] |0_{\mathrm{in}}\rangle . \tag{93}$$

Let,

$$F(x_0) = -i \int d^3x e^{-ipx} \langle q_{out} | \left[\pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle .$$

Lets calculate the two terms,

$$\begin{aligned}
-\int_{-\infty}^{\infty} dt \partial_0 F(x_0) &= -\int_{-\infty}^{\infty} dt \partial_0 - i \int d^3x e^{-ipx} \langle q_{out} | \left[\pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle \\
&= i \int d^4x \partial_0 e^{-ipx} \langle q_{out} | \left[\pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle .
\end{aligned}$$

And the other term,

$$\begin{aligned}
\lim_{x_0 \to -\infty} F(x_0) &= -i \lim_{x_0 \to -\infty} \int d^3x e^{-ipx} \langle q_{out} | \left[\pi(x) + i\omega_p \phi(x) \right] |0_{in}\rangle \\
&= \langle q_{out} | a_{p_{out}}^\dagger \sqrt{2\omega_p} |0_{in}\rangle \\
&= \langle 0_{out} | a_{q_{out}} \sqrt{2\omega_q} a_{p_{out}}^\dagger \sqrt{2\omega_p} |0_{in}\rangle .
\end{aligned}$$

Exercise (5)**.**

We are only interested in interactions. Argue that we can throw away the first term of the above equation.

Exercise (6).

Show that by explicitly performing the time derivative we obtain

$$\langle \mathbf{q}_{\mathrm{out}}|\mathbf{p}_{\mathrm{in}}\rangle = i\int d^4xe^{-ipx}(\partial_\mu\partial^\mu + m^2)\langle \mathbf{q}_{\mathrm{out}}|\phi(x)|0_{\mathrm{in}}\rangle. \tag{94}$$

Solution. lets go ahead and begin,

$$\begin{aligned}\langle \mathbf{q}_{out} \mid \mathbf{p}_{in}\rangle &= i\int d^4x\partial_0 e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[\pi(x) + i\omega_{\mathbf{p}}\phi(x)\right]|0_{\mathrm{in}}\rangle. \\ &= i\int d^4x\left((-i\omega_p)\, e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[\partial_0\phi\,(x) + i\omega_{\mathbf{p}}\phi(x)\right]|0_{\mathrm{in}}\rangle + e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[\partial_0^2\phi\,(x) + i\omega_{\mathbf{p}}\partial_0\phi(x)\right]|0_{\mathrm{in}}\rangle\right).\end{aligned}$$

Collect the terms, and note that the $\partial_0\phi\,(x)$ terms cancel,

$$\begin{aligned} &= i\int d^4x\left(e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[\omega_{\mathbf{p}}^2\phi(x) + \partial_0^2\phi\,(x)\right]|0_{\mathrm{in}}\rangle\right) \\ &= i\int d^4x\left(e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[\left(\mathbf{p}^2 + m^2\right)\phi(x) + \partial_0^2\phi\,(x)\right]|0_{\mathrm{in}}\rangle\right) \\ &= i\int d^4x\left(e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[m^2\phi(x) + \partial_0^2\phi\,(x)\right]|0_{\mathrm{in}}\rangle + \langle \mathbf{q}_{\mathrm{out}}|\left[\left(\mathbf{p}^2e^{-ipx}\right)\phi(x)\right]|0_{\mathrm{in}}\rangle\right) \\ &= i\int d^4x\left(e^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[m^2\phi(x) + \partial_0^2\phi\,(x)\right]|0_{\mathrm{in}}\rangle + \langle \mathbf{q}_{\mathrm{out}}|\left[\left(-\nabla^2e^{-ipx}\right)\phi(x)\right]|0_{\mathrm{in}}\rangle\right) \\ &= i\left(\int d^4xe^{-ipx}\langle \mathbf{q}_{\mathrm{out}}|\left[m^2\phi(x) + \partial_0^2\phi\,(x)\right]|0_{\mathrm{in}}\rangle + \langle \mathbf{q}_{\mathrm{out}}|\left[\int d^4x\left(-\nabla^2e^{-ipx}\right)\phi(x)\right]|0_{\mathrm{in}}\rangle\right).\end{aligned}$$

Lets take a look at integral on the right by itself for a moment, we will be doing integration by parts in three dimensions, to move the laplacian onto the field.

$$\begin{aligned}\int d^4x\left(-\nabla^2e^{-ipx}\right)\phi(x) &= \int dt\int d^3x\left(-\nabla^2e^{-ipx}\right)\phi(x) \\ &= -\int dt\left(\oint_\Omega dS\nabla e^{-ipx}\phi\,(x)\,\hat{n} - \int d^3x\nabla e^{-ipx}\cdot\nabla\phi\,(x)\right) \\ &= \int dt\left(\int d^3x\nabla e^{-ipx}\cdot\nabla\phi\,(x)\right) \\ &= \int dt\left(\oint_\Omega dSe^{-ipx}\nabla\phi\,(x)\,\hat{n} - \int d^3xe^{-ipx}\cdot\nabla^2\phi\,(x)\right) \\ &= -\int dt\left(\int d^3xe^{-ipx}\cdot\nabla^2\phi\,(x)\right) \\ &= -\int d^4xe^{-ipx}\cdot\nabla^2\phi\,(x).\end{aligned}$$

We can insert this, move the integral out again and collect the terms touch $\phi\,(x)$,

$$i\int d^4x\,\langle \mathbf{q}_{out}|\left[\left(m^2 + \partial_0^2 - \nabla^2\right)\phi\,(x)\right]|0_{in}\rangle = i\int d^4x\,\langle \mathbf{q}_{out}|\left[\left(m^2 + \partial_\mu\partial^\mu\right)\phi\,(x)\right]|0_{in}\rangle.$$

Exercise (7).

Using the same procedure on the out-state show that we obtain

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = i\int d^4x e^{-ipx}(\Box_x + m^2)$$

$$\left[\sqrt{2\omega_{\mathbf{q}}}\langle 0_{\text{out}}|a_{\mathbf{q}_{\text{in}}}\phi(x)|0_{\text{in}}\rangle + i\int d^4y\partial_{y_0}e^{iqy}\langle 0_{\text{out}}|\left[\pi(y) - i\omega_{\mathbf{q}}\phi(y)\right]\phi(x)|0_{\text{in}}\rangle\right], \tag{95}$$

where we have used $\Box = \partial_\mu\partial^\mu$ with subscript x to denote to which variable the derivation is with respect to.

Solution. We can write,

$$\begin{aligned}\langle \mathbf{q}_{out} \mid \mathbf{p}_{in}\rangle &= \langle 0_{out}|\sqrt{2\omega_q}a_{q_{out}}|\mathbf{p}_{in}\rangle \\ &= \lim_{y_0\to\infty}\sqrt{2\omega_q}\int d^3y\,\langle 0_{out}|\left(\frac{i}{\sqrt{2}}\right)e^{ipy}\left[\frac{\pi(y)}{\sqrt{\omega_p}} - i\sqrt{\omega_p}\phi(y)\right]|\mathbf{p}_{in}\rangle \\ &= \lim_{y_0\to\infty}\int d^3y\, ie^{ipy}\,\langle 0_{out}|\left[\pi(y) - i\omega_p\phi(y)\right]|\mathbf{p}_{in}\rangle\end{aligned}$$

.

We will use the identify from above again,

$$\lim_{x_0\to\infty}F(t) = \lim_{x_0\to-\infty}F(t) + \int_{-\infty}^{\infty}dt\partial_t F(t).$$

And we get,

$$= \sqrt{2\omega_q}\,\langle 0_{out}|a_{\mathbf{q}_{in}}|\mathbf{p}_{in}\rangle + i\int d^4y\,\partial_{y_o}e^{ipy}\,\langle 0_{out}|\left[\pi(y) - i\omega_p\phi(y)\right]|\mathbf{p}_{in}\rangle$$

.

The calculation we did before was essentially just a rewrite of $|\mathbf{p}_{in}\rangle$, so we can do this rewrite again to obtain the desired expression.

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = i\int d^4x e^{-ipx}(\Box_x + m^2)$$

$$\left[\sqrt{2\omega_{\mathbf{q}}}\langle 0_{\text{out}}|a_{\mathbf{q}_{\text{in}}}\phi(x)|0_{\text{in}}\rangle + i\int d^4y\partial_{y_0}e^{iqy}\langle 0_{\text{out}}|\left[\pi(y) - i\omega_{\mathbf{q}}\phi(y)\right]\phi(x)|0_{\text{in}}\rangle\right].$$

Exercise (8)**.**

Argue that we are free to insert a time-ordering product in both terms and argue that this means that we can drop the first term, which leaves

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = i^2\int d^4x d^4y e^{-ipx}(\Box_x + m^2)\partial_{y_0}e^{iqy}\langle 0_{\text{out}}|T\left[(\pi(y) - i\omega_{\mathbf{q}}\phi(y))\,\phi(x)\right]|0_{\text{in}}\rangle. \tag{96}$$

Solution. We want the fields to interact in the right order, so we are free to insert time-orering operators.. Lets first rewrite the left term,

$$\begin{aligned}
\langle 0_{out}|\, a_{\mathbf{q}_{in}}\phi(x)\,|0_{in}\rangle &= \lim_{y_0\to\infty}\langle 0_{out}|\, a_{\mathbf{q}}\phi(x)\,|0_{out}\rangle \\
&= \lim_{y_0\to-\infty}\langle 0_{out}|\, T\left[a_{\mathbf{q}}\phi(x)\right]|0_{out}\rangle \\
&= \lim_{y_0\to-\infty}\langle 0_{out}|\, \theta(y_0-x_0)\, a_{\mathbf{q}}\phi(x) + \theta(x_0-y_0)\,\phi(x)\, a_{\mathbf{q}}\,|0_{out}\rangle \\
&= \lim_{y_0\to-\infty}\langle 0_{out}|\, \theta(y_0-x_0)\, a_{\mathbf{q}}\phi(x)\,|0_{out}\rangle = 0.
\end{aligned}$$

Exercise (9)**.**

Do the derivative with respect to time y^0, as in part 6), to arrive at

$$\langle \mathbf{q}_{\text{out}}|\mathbf{p}_{\text{in}}\rangle = i^2\int d^4x d^4y e^{-i(px-qy)}(\Box_x+m^2)(\Box_y+m^2)\langle 0_{\text{out}}|T\left[\phi(y)\phi(x)\right]|0_{\text{in}}\rangle. \tag{97}$$

Solution. We start by splitting up the time-ordering operator,

$$\langle \mathbf{q}_{out} \mid \mathbf{p}_{in}\rangle = i^2\int d^4x d^4y e^{ipx}\left(\Box_x+m^2\right)\partial_{y_0}e^{iqy}\,\langle 0_{out}|\, T\left[\pi(y)\,\phi(y)\right] - i\omega_q T\left[\phi(y)\,\phi(x)\right]|0_{in}\rangle\,.$$

For now we will just look at the y-part of the integral, we'll leave out the bra-ket, and just do the derivation, and then fetch the integral when we need it.

$$\begin{aligned}
&= \partial_{y_0}e^{iqy}(T\left[\partial_{y_0}\phi(y)\,\phi(y)\right] - i\omega_q T\left[\phi(y)\,\phi(x)\right]) \\
&= \partial_{y_0}e^{iqy}(\partial_{y_0}T\left[\phi(y)\,\phi(y)\right] - i\omega_q T\left[\phi(y)\,\phi(x)\right]) \\
&= \partial_{y_0}e^{iqy}\partial_{y_0}T\left[\phi(y)\,\phi(y)\right] - \partial_{y_0}e^{iqy}i\omega_q T\left[\phi(y)\,\phi(x)\right] \\
&= i\omega_q e^{iqy}\partial_{y_0}T\left[\phi(y)\,\phi(y)\right] + e^{iqy}\partial_{y_0}^2 T\left[\phi(y)\,\phi(x)\right] \\
&- i\omega_q e^{iqy}\partial_{y_0}T\left[\phi(y)\,\phi(y)\right] + \omega_q^2 e^{iqy}T\left[\phi(y)\,\phi(x)\right] \\
&= e^{iqy}\partial_{y_0}^2 T\left[\phi(y)\,\phi(x)\right] + \omega_q^2 e^{iqy}T\left[\phi(y)\,\phi(x)\right] \\
&= e^{iqy}\partial_{y_0}^2 T\left[\phi(y)\,\phi(x)\right] + \left(p^2+m^2\right)e^{iqy}T\left[\phi(y)\,\phi(x)\right] \\
&= e^{iqy}\partial_{y_0}^2 T\left[\phi(y)\,\phi(x)\right] + \left(-\nabla^2+m^2\right)e^{iqy}T\left[\phi(y)\,\phi(x)\right]
\end{aligned}$$

.

At this point we have something that looks very amenable, we can do integration by parts to move things around, and we'll end up with what we need.

Exercise (10)**.**

Using the method above argue that for a general transition, not involving spectators, the S

matrix element is

$$S_{fi} = \langle \mathbf{q}_1, \ldots, \mathbf{q}_n; \text{out} | \mathbf{p}_1, \ldots, \mathbf{p}_m; \text{in} \rangle = i^{n+m} \int \prod_{i=1}^{m} d^4 x_i e^{-ip_i x_i} (\Box_{x_i} + m^2)$$
$$\int \prod_{j=1}^{n} d^4 y_j e^{ip_j y_j} (\Box_{y_j} + m^2) \langle 0_{\text{out}} | T \left[\phi(y_1) \ldots \phi(y_n) \phi(x_1) \ldots \phi(x_m) \right] | 0_{\text{in}} \rangle . \quad (98)$$

Exercise (11)**.**

Consider the Fourier transform of the n-particle Green's function

$$\Delta^{(n)}(k_1, \ldots, k_n) = \int d^4 x_1 \ldots d^4 x_n e^{i(k_1 x_1 + \cdots + k_n x_n)} \langle 0 | T \left[\phi(x_1) \ldots \phi(x_n) \right] | 0 \rangle , \quad (99)$$

and show that the LSZ reduction formula in momentum space takes the form

$$S_{fi} = (-i)^{n+m} \prod_{i=1}^{m} (p_i^2 - m^2) \prod_{j=1}^{n} (q_j^2 - m^2) \Delta^{(n+m)}(-p_1, \ldots, -p_n, q_1, \ldots, q_m) . \quad (100)$$

Up to a phase factor.