11 - Interacting Scalar Fields

Exercise (2)**.**

Find the first order matrix element M_{fi} for the decay process $\phi_3 \to \phi_1 + \phi_2$

These are my thoughts, the first order matrix element is essentially,

$$M_{fi}^{(1)} = \langle f| H_{int}(0) |i\rangle .$$

Where H is the interaction part of the hamiltonian density,

$$L_{int} = g\phi_1\phi_2\phi_3 = -H_{int}.$$

My intuition is that my states are given in the following way,

$$\begin{aligned}\langle f| &= \sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\,\langle 0_{out}|\, a_{q_1} b_{q_2}\\ |i\rangle &= \sqrt{2\omega_{q_3}}\,c^\dagger_{q_3}\,|0_{in}\rangle .\end{aligned}$$

The field expansions at 0 of the real scalar fields are given in the following way,

$$\phi_1(0) = \int \frac{d^3k_1}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{k_1}}}\left(a_{k_1} + a^\dagger_{k_1}\right).$$

We can insert all of this and begin calculating the first order matrix element. We will start pulling the ω_{q_i},

$$\begin{aligned}M^1_{fi} &= -g\sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\sqrt{2\omega_{q_3}}\times\\ &\langle 0|\, a_{q_1} b_{q_2}\int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\frac{1}{\sqrt{2\omega_{k_1}}}\frac{1}{\sqrt{2\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}}}\left(a_{k_1}+a^\dagger_{k_1}\right)\left(b_{k_2}+b^\dagger_{k_2}\right)\left(c_{k_3}+c^\dagger_{k_3}\right)c^\dagger_{q_3}\,|0\rangle\\ &= -g\sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\sqrt{2\omega_{q_3}}\times\\ &\int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\frac{1}{\sqrt{2\omega_{k_1}}}\frac{1}{\sqrt{2\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}}}\langle 0|\, a_{q_1} b_{q_2}\left(a_{k_1}+a^\dagger_{k_1}\right)\left(b_{k_2}+b^\dagger_{k_2}\right)\left(c_{k_3}+c^\dagger_{k_3}\right)c^\dagger_{q_3}\,|0\rangle .\end{aligned}$$

Here we should remind ourselves that, $[a,b] = 0$. The operators corresponding to different fields commute. In our bracket, we end up with 8 terms. 7 of them end up giving zero, corresponding to those that have either a or b annihilation operators on the right or c creation operators on the left. Therefore, the only term that survives is,

$$M^1_{fi} = -g\sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\sqrt{2\omega_{q_3}}\times\int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\frac{1}{\sqrt{2\omega_{k_1}}}\frac{1}{\sqrt{2\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}}}\langle 0|\, a_{q_1} b_{q_2} a^\dagger_{k_1} b^\dagger_{k_1} c_{k_1} c^\dagger_{q_3}\,|0\rangle .$$

To get rid of the last terms we flip the pairs of commutators using the canonical commutation relations,

$$\left[a_{q_1}, a^\dagger_{k_1}\right] = (2\pi)^3\,\delta^3(q_1 - k_1).$$

These delta functions can be pulled out of the bracket and we up with $\langle 0 \mid 0\rangle = 1$

$$\begin{aligned}M^1_{fi} &= -g\sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\sqrt{2\omega_{q_3}}\times\\ &\int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}\frac{1}{\sqrt{2\omega_{k_1}}}\frac{1}{\sqrt{2\omega_{k_2}}}\frac{1}{\sqrt{2\omega_{k_3}}}(2\pi)^3\,\delta^3(q_1-k_1)\,(2\pi)^3\,\delta^3(q_2-k_2)\,(2\pi)^3\,\delta^3(q_3-k_3)\\ &= -g\sqrt{2\omega_{q_1}}\sqrt{2\omega_{q_2}}\sqrt{2\omega_{q_3}}\frac{1}{2\sqrt{\omega_{q_1}}}\frac{1}{2\sqrt{\omega_{q_2}}}\frac{1}{2\sqrt{\omega_{q_3}}} = -g.\end{aligned}$$

Exercise (3).

Find the second order matrix element $M_{fi}^{(2)}$ for the decay process $\phi_1 + \phi_2 \to \phi_1 + \phi_2$