# Free gauge fields Lagrangian

In this exercise we wish to show that the free Lagrangian of the gauge fields $\mathbf{A}_\mu$,

$$\mathscr{L}_A = -\frac{1}{16\pi}\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}, \tag{9}$$

is invariant under a local gauge transformation. Note that the field tensor in Yang-Mills theory takes the form

$$\mathbf{F}^{\mu\nu} = \partial^\mu\mathbf{A}^\nu - \partial^\nu\mathbf{A}^\mu + 2g(\mathbf{A}^\mu\times\mathbf{A}^\nu). \tag{10}$$

**Exercise** (1)**.**

Show that the commutator of the covariant derivative is

$$[D_\mu, D_\nu] = -ig\mathbf{F}_{\mu\nu}\cdot\sigma. \tag{11}$$

Hint: You might find the following identity for the Pauli matrices useful: $(\sigma\cdot\mathbf{a})(\sigma\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\sigma\cdot(\mathbf{a}\times\mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors.

Lets start by recalling the covariant derivative,

$$D_\mu = \partial_\mu - ig\mathbf{A}_\mu\cdot\sigma.$$

The following commutator will be useful,

$$\left[\partial_\mu, \mathbf{A}_\mathbf{v}\right].$$

Let's see how it acts on a test function $\Psi$,

$$\begin{aligned}\left[\partial_\mu, \mathbf{A}_\nu\right]\Psi &= \partial_\mu\left(\mathbf{A}_\nu\Psi\right) - \mathbf{A}_\nu\left(\partial_\mu\Psi\right)\\ &= \left(\partial_\mu\mathbf{A}_\nu\right)\Psi + \mathbf{A}_\mathbf{v}\partial_\mu\Psi - \mathbf{A}_\nu\left(\partial_\mu\Psi\right)\\ &= \left(\partial_\mu\mathbf{A}_\nu\right)\Psi.\end{aligned}$$

Therefore,

$$\left[\partial_\mu, \mathbf{A}_\nu\right] = \partial_\mu\mathbf{A}_\nu.$$

Using this, we can calculate the commutator of the covariant derivative,

$$\begin{aligned}\left[D_\mu, D_\nu\right] &= \left[\partial_\mu - ig\mathbf{A}_\mu\cdot\sigma, \partial_\nu - ig\mathbf{A}_\nu\cdot\sigma\right]\\ &= \left[\partial_\mu, \partial_\nu\right] + \left[\partial_\mu, -ig\mathbf{A}_\mathbf{v}\cdot\sigma\right] + \left[-ig\mathbf{A}_\mu\cdot\sigma, \partial_\nu\right] + \left[-ig\mathbf{A}_\mu\cdot\sigma, -ig\mathbf{A}_\nu\cdot\sigma\right]\\ &= -ig\left[\partial_\mu, \mathbf{A}_\mathbf{v}\right]\cdot\sigma - ig\left[\mathbf{A}_\mu, \partial_\nu\right]\cdot\sigma - g^2\left[\mathbf{A}_\mu\cdot\sigma, \mathbf{A}_\nu\cdot\sigma\right]\\ &= -ig\left(\partial_\mu\mathbf{A}_\mathbf{v}\right)\cdot\sigma + ig\left(\partial_\nu\mathbf{A}_\mu\right)\cdot\sigma - g^2\left[\mathbf{A}_\mu\cdot\sigma, \mathbf{A}_\nu\cdot\sigma\right]\\ &= -ig\left(\partial_\mu\mathbf{A}_\mathbf{v} - \partial_\nu\mathbf{A}_\mu\right)\cdot\sigma - g^2\left[\mathbf{A}_\mu\cdot\sigma, \mathbf{A}_\nu\cdot\sigma\right]\\ &.\end{aligned}$$

Now we just need to calculate the final commutator,

$$\begin{aligned}\left[\mathbf{A}_\mu\cdot\sigma, \mathbf{A}_\nu\cdot\sigma\right] &= \left(\mathbf{A}_\mu\cdot\sigma\right)\cdot\left(\mathbf{A}_\nu\cdot\sigma\right) - \left(\mathbf{A}_\nu\cdot\sigma\right)\cdot\left(\mathbf{A}_\mu\cdot\sigma\right)\\ &= \mathbf{A}_\mu\cdot\mathbf{A}_\mathbf{v} + \left(\mathbf{A}_\mu\times\mathbf{A}_\mathbf{v}\right)\cdot i\sigma - \left(\mathbf{A}_\nu\cdot\mathbf{A}_\mu + \left(\mathbf{A}_\mathbf{v}\times\mathbf{A}_\mu\right)\cdot i\sigma\right)\\ &= \left(\mathbf{A}_\mu\times\mathbf{A}_\mathbf{v}\right)\cdot i\sigma - \left(\left(\mathbf{A}_\mathbf{v}\times\mathbf{A}_\mu\right)\cdot i\sigma\right)\\ &= 2\left(\mathbf{A}_\mu\times\mathbf{A}_\nu\right)\cdot i\sigma.\end{aligned}$$

we can insert this,

$$\begin{aligned}
\left[D_\mu, D_\nu\right] &= -ig\left(\partial_\mu \mathbf{A_\nu} - \partial_\nu \mathbf{A}_\mu\right)\cdot\sigma - g^2\left[\mathbf{A}_\mu\cdot\sigma, \mathbf{A}_\nu\cdot\sigma\right] \\
&= -ig\left(\partial_\mu \mathbf{A_\nu} - \partial_\nu \mathbf{A}_\mu\right)\cdot\sigma - g^2 2\left(\mathbf{A}_\mu \times \mathbf{A}_\nu\right)\cdot i\sigma \\
&= -ig\left(\partial_\mu \mathbf{A_\nu} - \partial_\nu \mathbf{A}_\mu + 2g\left(\mathbf{A}_\mu \times \mathbf{A}_\nu\right)\right)\cdot\sigma = -ig\mathbf{F}_{\mu\nu}\cdot\sigma.
\end{aligned}$$

**Exercise** (2)**.**

---

Argue that the transformation law of the covariant derivative in eq. (7) implies that

$$[D_\mu, D_\nu]\psi \to V(x)[D_\mu, D_\nu]\psi, \tag{12}$$

and show that this implies that

$$\mathbf{F}_{\mu\nu}\cdot\sigma \to V(x)\mathbf{F}_{\mu\nu}\cdot\sigma V^\dagger(x). \tag{13}$$

We have the following transformation law,

$$D_\mu\Psi \to V(x)\left(D_\mu\Psi\right).$$

Applying this this to commutator,

$$\left[D_\mu, D_\nu\right]\Psi = D_\mu\left(D_\nu\Psi\right) - D_\nu\left(D_\mu\Psi\right) \to D_\mu\left(V(x) D_\nu\Psi\right) - D_\nu\left(V(x) D_\mu\Psi\right) = V(x)\left[D_\mu, D_\nu\right]\Psi.$$

Lets show that this implies (13). From the previous exercise we know,

$$\left[D_\mu, D_\nu\right] = -ig\mathbf{F}_{\mu\nu}\cdot\sigma.$$

And therefore,

$$\left[D_\mu, D_\nu\right]' = -ig\mathbf{F}'_{\mu\nu}\cdot\sigma.$$

We can insert this,

$$\begin{aligned}
\left(-ig\mathbf{F}'_{\mu\nu}\cdot\sigma\right)\Psi' &= V(x)\left(-ig\mathbf{F}_{\mu\nu}\cdot\sigma\right)\Psi \\
\left(\mathbf{F}'_{\mu\nu}\cdot\sigma\right)\Psi' &= V(x)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right)\Psi \\
\left(\mathbf{F}'_{\mu\nu}\cdot\sigma\right)V(x)\Psi &= V(x)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right)\Psi.
\end{aligned}$$

This will have to hold for any field $\Psi$, so we can equate the operators,

$$\begin{aligned}
\left(\mathbf{F}'_{\mu\nu}\cdot\sigma\right)V(x) &= V(x)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right) \\
\left(\mathbf{F}'_{\mu\nu}\cdot\sigma\right)V(x)V^{-1}(x) &= V(x)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right)V^{-1}(x) \\
\left(\mathbf{F}'_{\mu\nu}\cdot\sigma\right) &= V(x)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right)V(x)^\dagger
\end{aligned}$$

.

**Exercise** (3).

---

Using the transformation of eq. (13) to show that

$$\mathrm{Tr}\left[(\mathbf{F}^{\mu\nu}\cdot\sigma)(\mathbf{F}_{\mu\nu}\cdot\sigma)\right], \tag{14}$$

is invariant.

We show this by applying the transformation, and asserting that the resulting is unchanged,

$$\begin{aligned}\mathrm{Tr}\left[(\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma)\left(\mathbf{F}_{\mu\mathbf{v}}\cdot\sigma\right)\right]' &= \mathrm{Tr}\left[\left(\mathbf{F}^{\mu\mathbf{v}\prime}\cdot\sigma\right)\left(\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma\right)\right] \\ &= \mathrm{Tr}\left[\left(V(x)\,\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma V(x)^{\dagger}\right)\left(V(x)\,\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma V(x)^{\dagger}\right)\right] \\ &= \mathrm{Tr}\left[\left(V(x)\,\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma\right)\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma V(x)^{\dagger}\right] \\ &.\end{aligned}$$

The trace is invariant under circular shifts,

$$\mathrm{Tr}\left[\left(V(x)\,\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma\right)\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma V(x)^{\dagger}\right] = \mathrm{Tr}\left[\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma V(x)^{\dagger}V(x)\right] = \mathrm{Tr}\left[\mathbf{F}^{\mu\mathbf{v}}\cdot\sigma\mathbf{F}_{\mu\mathbf{v}}{}'\cdot\sigma\right].$$

**Exercise** (4).

---

Show that the trace in eq. (14) is equal to $2\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}$.

We do this by expanding the dot-product within the trace.

$$\mathbf{F}^{\mu\nu}\cdot\sigma = \sum_{\alpha}\mathbf{F}^{\mu\nu}_{\alpha}\sigma_{\alpha}.$$

We can insert these,

$$\mathrm{Tr}\left[(\mathbf{F}^{\mu\nu}\cdot\sigma)\left(\mathbf{F}_{\mu\nu}\cdot\sigma\right)\right] = \mathrm{Tr}\left[\left(\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}\right)\mathbb{I}_{2\times2} + i\sigma\cdot\left(\mathbf{F}^{\mu\nu}\times\mathbf{F}_{\mu\nu}\right)\right].$$

Since everything here is index-wise, and we are implicitly performing sums, we can move the the field-strength tensors outside of the trace,

$$= \left(\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}\right)\mathrm{Tr}\left[\mathbb{I}_{2\times2}\right] + i\sum_{k}\mathrm{Tr}\left[\sigma_k\right]\left(\mathbf{F}^{\mu\nu}\times\mathbf{F}_{\mu\nu}\right)_k = 2\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu}.$$

**Exercise** (5).

---

Combine 3) and 4) to show that the Lagrangian in eq. (9) is invariant.