2 - Solutions of the Dirac Equations and the Polarization Sum

Exercise (4).

Consider now the non-zero momentum case and let $p^\mu = (E, \mathbf{p})$. We will take $E > 0$ from now on. Show that $\psi = u(p)e^{-ipx}$ and $\psi = v(p)e^{ipx}$, where u and v are 4-spinors, are solutions of the Dirac equation when $(\not{p} - m)u(p) = 0$ and $(\not{p} + m)v(p) = 0$. Notice that since we assume $E > 0$, the $u(p)$ solutions may be called positive energy solutions and the $v(p)$ solutions may be called negative energy solutions.

Exercise (5).

Next show that $u(p) = (\not{p} + m)u(0)$ and $v(p) = (\not{p} - m)v(0)$ are solutions. Here $u(0)$ and $v(0)$ are momentum independent functions. What other quantum number must $u(0)$ and $v(0)$ depend on?

Exercise (6).

Now we write

$$u_s(0) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \quad \text{and} \quad v_s(0) = \begin{pmatrix} 0 \\ \chi_{-s} \end{pmatrix}, \tag{12}$$

where χ_s is a 2-spinor spin one-half wave function and $s = \pm 1$ denotes the two orthogonal spin states along some direction in space. Notice the minus sign on the 2-spinor in the $v_s(0)$ part (it is connected to the anticommutation relations of fermion fields). We do not need to specify a quantization axis for the spin at this point and we keep it completely general. Show that

$$u_s(p) = \begin{pmatrix} (E+m)\chi_s \\ \sigma \cdot \mathbf{p}\chi_s \end{pmatrix} \quad \text{and} \quad v_s(p) = \begin{pmatrix} -\sigma \cdot \mathbf{p}\chi_{-s} \\ -(E+m)\chi_{-s} \end{pmatrix}, \tag{13}$$

are solutions for positive and negative energy states.

Exercise (7).

We also need to normalize the Dirac 4-spinor solutions appropriately. The four-current for a Dirac spinor field is $\psi\gamma^\mu\psi$, where $\overline{\psi} = \psi^\dagger\gamma^0$. Show that the density (which is the zeroth component) is simply $\rho = \psi^\dagger\psi$. Consider

$$\psi = u_s(p)e^{-ipx}, \tag{14}$$

and show that $\int d^3x\rho = 2E(m+E)V$. Argue (without doing more calculations) that if we had taken $v_s(p)$ instead of $u_s(p)$ we get the same result.

Exercise (8).

Show that the properly normalized $\left(\int d^3x\psi^\dagger\psi = 1\right)$ 4-spinors may be written in the form

$$u_s(p) = \sqrt{E+m}\begin{pmatrix}\chi_s \\ \frac{\sigma\cdot\mathbf{p}\chi_s}{E+m}\end{pmatrix} \quad \text{and} \quad v_s(p) = \sqrt{E+m}\begin{pmatrix}\frac{\sigma\cdot\mathbf{p}\chi_{-s}}{E+m} \\ \chi_{-s}\end{pmatrix}, \tag{15}$$

and that this produces the positive and negative energy solutions of the form

$$\psi_+ = \frac{1}{\sqrt{2EV}}u_s(p)e^{-ipx} \quad \text{and} \quad \psi_- = \frac{1}{\sqrt{2EV}}v_s(p)e^{ipx}. \tag{16}$$

Here we have insisted that the total wave functions, ψ_+ and ψ_-, carry normalization factors similar to scalar wave functions $(1/\sqrt{2EV})$ such that all the details of the fermionic nature and the Dirac equation are completely contained in the u and v factors.

Exercise (9).

Using the explicit form of the 4-spinors in Eq. (15) show the important and extremely useful completeness relations

$$\sum_s u_s(p)\overline{u_s(p)} = \not{p}+m \quad \text{and} \quad \sum_s v_s(p)\overline{v_s(p)} = \not{p}-m. \tag{17}$$