1 The Dirac Equation and Dimensionality

Exercise (1).

Dirac started by postulating an equation linear in time and space of the form

$$(\alpha \cdot \mathbf{p} + m\beta)\psi = i\frac{\partial \psi}{\partial t}, \tag{1}$$

and then proceeded to figure out what the conditions on α and β would have to be to ensure that it obeys the energy-momentum relation of special relativity, i.e. $E^2 = |\mathbf{p}|^2 + m^2$.

1. Show that the conditions are

$$\alpha_i\beta + \beta\alpha_i = 0, \quad i = 1,2,3 \tag{1.1}$$

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0, \quad i,j = 1,2,3; i \neq j \tag{1.2}$$

$$\alpha_i^2 = \beta^2 = 1, \quad i = 1,2,3. \tag{1.3}$$

(Hint: Take the square of the operator on both sides of Eq. (1).)

Solution. We use the time-dependent Schrodinger equation,

$$(\alpha \cdot \mathbf{p} + m\beta)\Psi = i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi = E\Psi.$$

This implies that,

$$(\alpha \cdot \mathbf{p} + m\beta)^2 = E^2.$$

Now since we want $E^2 = \|\mathbf{p}\|^2 + m^2$ this leads to some restrictions on α_i and β. Lets by expanding the square,

$$(\alpha \cdot \mathbf{p} + m\beta)^2 = (\alpha \cdot \mathbf{p})^2 + (m\beta)^2 + \alpha \cdot \mathbf{p}m\beta + \beta m\alpha \cdot \mathbf{p}.$$

It seems natural that these cross-terms should dissapear. We can rewrite them as a sum,

$$\alpha \cdot \mathbf{p}m\beta + \beta m\alpha \cdot \mathbf{p} = \sum_i mp_i(\alpha_i\beta + \beta\alpha_i) = 0.$$

Assuming condition (1) it is indeed zero. Lets rewrite the first term,

$$\begin{aligned}
(\alpha \cdot \mathbf{p})^2 &= \sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j = \frac{1}{2}\left(\sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j + p_ip_j\alpha_i\alpha_j\right) \\
&= \frac{1}{2}\left(\sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j + \sum_{i=1}\sum_{j=1} p_ip_j\alpha_i\alpha_j\right) = \frac{1}{2}\left(\sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j + \sum_{i=1}\sum_{j=1} p_jp_i\alpha_i\alpha_j\right) \\
&= \frac{1}{2}\left(\sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j + \sum_{j=1}\sum_{i=1} p_ip_j\alpha_i\alpha_j\right) = \frac{1}{2}\left(\sum_{j=1}\sum_{i=1} p_ip_j(\alpha_i\alpha_j + \alpha_i\alpha_j)\right) \\
&= \frac{1}{2}\left(\sum_{i=1} 2p_i^2\alpha_i^2 + \sum_{j=1,j\neq i}\sum_{i=1} p_ip_j(\alpha_i\alpha_j + \alpha_i\alpha_j)\right) = \sum_i p_i^2 = \mathbf{p}^2.
\end{aligned}$$

2. Prove that $\alpha_i, i = 1,2,3$ and β are all Hermitian. (Hint: First find the Hamiltonian for the Dirac particles).

Solution. The hamiltonian for our system is,

$$\hat{H} = (\alpha \cdot \mathbf{p} + m\beta).$$

We know that hamiltonians are hermitian, so

$$\hat{H} = \hat{H}^\dagger = (\alpha \cdot \mathbf{p} + m\beta)^\dagger = \left(\alpha^\dagger \cdot \mathbf{p} + m\beta^\dagger\right).$$

Which implies $\alpha = \alpha^\dagger$ and $\beta = \beta^\dagger$.

3. Prove that $\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\beta) = 0$ where Tr is the matrix trace (sum of diagonal entries).

Solution. We can use condition (2),

$$\alpha_i\beta + \beta\alpha_i = 0, \quad i = 1,2,3.$$

This implies,

$$\alpha_i = -\beta^{-1}\alpha_i\beta.$$

We also use the fact that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

$$\mathrm{Tr}(\alpha_i) = -\mathrm{Tr}\left(\beta^{-1}(\alpha_i\beta)\right) = -\mathrm{Tr}\left(\beta\beta^{-1}\alpha_i\right) = -\mathrm{Tr}(\alpha_i).$$

Which implies that the trace is zero.

4. Prove that the eigenvalues of α_i and β are all either +1 or -1.

5. Prove that the dimensionality of α_i and β is even.

6. Argue that this implies that the dimension of α_i and β must be at least 4.