# Advanced Particle Physics 2025 Problem Set 5

## 1 Matrices of SU(2)

Show that a general Hermitian $2 \times 2$ matrix can be expressed as a linear combination of the identity $I$ and the three Pauli matrices $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$

$$H = \theta I + \boldsymbol{\sigma} \cdot \mathbf{a}, \tag{1}$$

where $\theta$ is a real number and $\mathbf{a} = (a_1, a_2, a_3)$ is a real vector.

Argue that this means that we can write any unitary $2 \times 2$ matrix as

$$U = e^{i\theta} e^{i\boldsymbol{\sigma} \cdot \mathbf{a}}, \tag{2}$$

and realize that this is the natural extension of the $1 \times 1$ unitary $U = e^{i\theta}$.

## 2 Transformation of the gauge fields $\mathbf{A}^\mu$

Consider a *local* SU(2) transformation

$$\psi \to V(x)\psi, \quad \text{where} \quad V(x) = e^{i\mathbf{a}(x)\cdot\boldsymbol{\sigma}}. \tag{3}$$

In order to keep the Lagrangian $\mathcal{L}$ invariant under the transformation we introduce the covariant derivative

$$D_\mu = \partial_\mu - ig\mathbf{A}_\mu \cdot \boldsymbol{\sigma}, \tag{4}$$

since the derivative picks up an extra term

$$\partial_\mu \psi \to V(x)\partial_\mu \psi + (\partial_\mu V(x))\psi. \tag{5}$$

1. Show that the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\not{D} - m\right)\psi, \tag{6}$$

becomes invariant if we assign a transformation rule such that the covariant derivative transform in the same way as the field:

$$D_\mu \psi \to V(x)(D_\mu \psi). \tag{7}$$

2. Starting from the transformation rule in eq. (7) show that the gauge fields transform as

$$\mathbf{A}_\mu \cdot \boldsymbol{\sigma} \to V(x)\left(\mathbf{A}_\mu \cdot \boldsymbol{\sigma} + \frac{i}{g}\partial_\mu\right)V^\dagger(x). \tag{8}$$

## 3 Free gauge fields Lagrangian

I this exercise we wish to show that the free Lagrangian of the gauge fields $\mathbf{A}_\mu$,

$$\mathcal{L}_A = -\frac{1}{16\pi}\mathbf{F}^{\mu\nu} \cdot \mathbf{F}_{\mu\nu}, \tag{9}$$

is invariant under a local gauge transformation. Note that the field tensor in Yang-Mills theory takes the form

$$\mathbf{F}^{\mu\nu} = \partial^\mu \mathbf{A}^\nu - \partial^\nu \mathbf{A}^\mu + 2g(\mathbf{A}^\mu \times \mathbf{A}^\nu). \tag{10}$$

1. Show that the commutator of the covariant derivative is

$$[D_\mu, D_\nu] = -ig\mathbf{F}_{\mu\nu} \cdot \boldsymbol{\sigma}. \tag{11}$$

Hint: You might find the following identity for the Pauli matrices useful: $(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors.

2. Argue that the transformation law of the covariant derivative in eq. (7) implies that

$$[D_\mu, D_\nu]\psi \to V(x)[D_\mu, D_\nu]\psi, \tag{12}$$

and show that this implies that

$$\mathbf{F}_{\mu\nu} \cdot \boldsymbol{\sigma} \to V(x)\mathbf{F}_{\mu\nu} \cdot \boldsymbol{\sigma} V^\dagger(x). \tag{13}$$

3. Using the transformation of eq. (13) to show that

$$\mathrm{Tr}\left[(\mathbf{F}^{\mu\nu} \cdot \boldsymbol{\sigma})(\mathbf{F}_{\mu\nu} \cdot \boldsymbol{\sigma})\right], \tag{14}$$

is invariant.

4. Show that the trace in eq. (14) is equal to $2\mathbf{F}^{\mu\nu} \cdot \mathbf{F}_{\mu\nu}$.

5. Combine 3) and 4) to show that the Lagrangian in eq. (9) is invariant.

## 4 Spontaneous symmetry breaking

Consider a two-particle real scalar Lagrangian

$$\mathcal{L} = (\partial_\mu \phi_1)(\partial^\mu \phi_1) + (\partial_\mu \phi_2)(\partial^\mu \phi_2) - U(\phi_1, \phi_2), \tag{15}$$

where the potential is

$$U(\phi_1, \phi_2) = -\mu^2(\phi_1^2 + \phi_2^2) + \frac{1}{2}\lambda^2(\phi_1^2 + \phi_2^2)^2. \tag{16}$$

Introduce the fields

$$\eta = \phi_1 - \frac{\mu}{\lambda} \qquad \text{and} \qquad \xi = \phi_2, \tag{17}$$

and rewrite the Lagrangian into

$$\begin{aligned}\mathcal{L} =& (\partial_\mu \eta)(\partial^\mu \eta) - 2\mu^2\eta^2 + (\partial_\mu \xi)(\partial^\mu \xi) \\ & - \left[2\mu\lambda(\eta^3 + \eta\xi^2) + \frac{\lambda^2}{2}(\eta^4 + \xi^4 + 2\eta^2\xi^2)\right] + \frac{\mu^4}{2\lambda^2}.\end{aligned} \tag{18}$$

Comment on each of the terms in the new Lagrangian (draw vertices for all interactions), which particle have mass and which does not? Which symmetry have been broken? Could we have broken the symmetry differently?

# 5 The Higgs mechanism

When we broke the symmetry in the previous exercise we obtained the desired mass term, but we also obtain a massless scalar particle (called a Goldstone boson). We want to fix this problem of the emerging Goldstone boson, and we do this by making the Lagrangian locally gauge invariant.

1. Argue that if we introduce the massless gauge field $A^\mu$ and write the two real field into a single complex field $\phi = \phi_1 + i\phi_2$, then the invariant Lagrangian takes the form

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) + \mu^2\phi^*\phi - \frac{1}{2}\lambda^2(\phi^*\phi)^2 - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}, \tag{19}$$

where $D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative.

2. Using the fields in eq. (17) rewrite the Lagrangian into

$$\begin{aligned}\mathcal{L} =& (\partial_\mu \eta)(\partial^\mu \eta) - 2\mu^2\eta^2 + (\partial_\mu \xi)(\partial^\mu \xi) \\ & + \left[-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \left(\frac{\mu}{\lambda}\right)^2 A^\mu A_\mu\right] \\ & + \Big\{2q\left[\eta(\partial_\mu\xi) - \xi(\partial_\mu\eta)\right]A^\mu + 2\frac{\mu}{\lambda}q^2\eta(A^\mu A_\mu) + q^2(\xi^2 + \eta^2)(A^\mu A_\mu) \\ & - 2\lambda\mu(\eta^3 + \eta\xi^2) - \frac{1}{2}\lambda^2(\eta^4 + 2\eta^2\xi^2 + \xi^4)\Big\} \\ & + 2\frac{\mu q}{\lambda}(\partial_\mu\xi)A^\mu + \frac{1}{2}\left(\frac{\mu^2}{\lambda}\right)^2.\end{aligned} \tag{20}$$

Comment on all of the terms in the Lagrangian (draw vertices for all interactions), which particles have mass and which does not? Which term do you find most suspicious?

3. Find a gauge transformation which makes the Goldstone boson, $\xi$ disappear. How does the Lagrangian look after such a transformation, i.e. which terms survive in eq. (20)?

4. How would you interpret the Goldstone boson, $\xi$, knowing that it can be transformed away by chosing the right gauge? Does this mean that we have to chose this particular gauge? How many degrees of freedom does the original Lagrangian in eq. (19) have, and how many degrees of freedom does the final Lagrangian have?