4 - The Fermion Propagator

The following problem derives the propagator for a fermion. First we have to figure out the Hamiltonian. We start from

$$\mathscr{L} = \overline{\psi}(i\gamma_\mu \partial^\mu - m)\psi, \tag{24}$$

where we use the convenient short-hand $\partial^\mu = \frac{\partial}{\partial x_\mu}$.

Exercise (1).

Show that the corresponding Hamiltonian can be written

$$H = \int d^3x \overline{\psi}(-i\gamma \cdot \nabla + m)\psi, \tag{25}$$

where γ is the 3-vector $(\gamma^1, \gamma^2, \gamma^3)$. Remember that $\partial^\mu = (\partial_t, -\nabla)$.

Exercise (2).

Now consider Heisenberg's equations of motion for the field operator $\psi(\mathbf{x},t)$, i.e.

$$\frac{\partial \psi_\alpha(\mathbf{x},t)}{\partial t} = -i[\psi_\alpha(\mathbf{x},t), H], \tag{26}$$

where α is an index indicating that keep track of the fact that there are four components in the field of Dirac particle. Show that

$$\frac{\partial \psi_\alpha(\mathbf{x},t)}{\partial t} = \left(-\gamma^0 \gamma \cdot \nabla - im\gamma_0\right)_{\alpha\beta} \psi_\beta(\mathbf{x},t), \tag{27}$$

where we sum over the spinor index β (repeated use of spinor indices mean summation from now on). You will need the canonical anti-commutator for fermion fields is

$$\left\{ \psi_\alpha(\mathbf{x}_1,t), \psi^\dagger_\beta(\mathbf{x}_2,t) \right\} = \delta_{\alpha\beta}\delta(\mathbf{x}_1 - \mathbf{x}_2). \tag{28}$$

Exercise (3).

We now define the time-ordering operator for fermionic fields, ψ and $\overline{\psi}$ in the following way

$$T\left\{ \psi_\alpha(\mathbf{x}_1,t_1)\overline{\psi}_\beta(\mathbf{x}_2,t_2) \right\} = \begin{cases} \psi_\alpha(\mathbf{x}_1,t_1)\overline{\psi}_\beta(\mathbf{x}_2,t_2) & \text{for } t_1 > t_2 \\ -\overline{\psi}_\beta(\mathbf{x}_2,t_2)\psi_\alpha(\mathbf{x}_1,t_1) & \text{for } t_2 > t_1 \end{cases}, \tag{29}$$

where this is now a matrix indexed by α and β since the fields are four-component quantities.

Show that

$$\frac{\partial}{\partial t_1} T\left\{\psi_\alpha(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} = \delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2)[\gamma_0]_{\alpha\beta} + T\left\{\frac{\partial\psi_\alpha(\mathbf{x}_1,t_1)}{\partial t_1}\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} \tag{30}$$

Exercise (4)**.**

Now use Eq. (27) to show that

$$\left[i\gamma_\mu\partial^\mu - m\right]_{\alpha\eta} T\left\{\psi_\eta(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} = i\delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2)\delta_{\alpha\beta}, \tag{31}$$

where ∂_μ act on $\mathbf{x}_1$ and t_1 only.

Solution. We shall make use of each the results from the previous exercise in turn. Lets start by expanding the four-vector on the left-hand side

$$[i\gamma_0\partial_0 - \gamma\cdot\nabla - m]_{\alpha\eta} T\left\{\psi_\eta(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} = \left[i\gamma_0\frac{\partial}{\partial t_1} - \gamma\cdot\nabla_{x_1} - m\right]_{\alpha\eta} T\left\{\psi_\eta(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}$$

We can distribute these terms, noting that the $\alpha\eta$ index is equivalent to multiplying by a kronecker delta. Also note that since we are using einstein summation, things commute (not the fields themselves however). We can therefore apply the kronecker delta to the Ψ_η term changing it to an α. This yields,

$$= i\gamma_0\frac{\partial}{\partial t_1}T\left\{\psi_\alpha(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} - \gamma\cdot\nabla_{x_1}T\left\{\psi_\alpha(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} - mT\left\{\psi_\alpha(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}.$$

Let's focus on the first term, which is similar to what we calculated in the previous ecxercise,

$$i\gamma_0\frac{\partial}{\partial t_1}T\left\{\psi_\alpha(\mathbf{x}_1,t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\} = i\gamma_0\left(\delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2)[\gamma_0]_{\alpha\beta} + T\left\{\frac{\partial\psi_\alpha(\mathbf{x}_1,t_1)}{\partial t_1}\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}\right).$$

We can substitute in the dirac-equation, and use the fact that $\gamma_0^2 = 1$,

$$\begin{aligned}
&= i\delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2) + i\gamma_0 T\left\{\left(-\gamma^0\gamma\cdot\nabla - im\gamma_0\right)_{\alpha\eta}\psi_\eta(\mathbf{x_1},t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}\\
&= i\delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2) + i\gamma_0\left(-\gamma^0\gamma\cdot\nabla - im\gamma_0\right)_{\alpha\eta} T\left\{\psi_\eta(\mathbf{x_1},t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}\\
&= i\delta(t_1-t_2)\delta(\mathbf{x}_1-\mathbf{x}_2) + i\left(-\gamma\cdot\nabla - im\right)_{\alpha\eta} T\left\{\psi_\eta(\mathbf{x_1},t_1)\overline{\Psi}_\beta(\mathbf{x}_2,t_2)\right\}
\end{aligned}$$

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