Advanced Particle Physics 2025 Problem Set 4

1 Spin One Particles with Mass

A free massive particle with spin one is represented by a vector field in space-time in a particular way. If we are in the rest frame of the particle (which we can transform into since it has non-zero mass) it must have the same number of degrees of freedom as the spin one systems that we have in non-relativistic quantum mechanics, i.e. it has three polarization states given by the projection of the spin along some fixed axis (typically the z-axis), $S_z = 0, \pm 1$. To fully specify the properties of a free spin one particle we must thus provide the mass, the momentum, and the polarization.

1) While it would naively seem that we could use a three-vector, $\boldsymbol{e}$, as a general polarization vector for a massive spin one field, argue that because we are in relativistic mechanics and must use Lorentz transformations, the the most general form of the polarization vector has to be a four-vector, e_μ. Also, argue that there are three independent such four-vector polarization states.

2) Suppose that we insist on having $e_\mu p^\mu = 0$ and $e_\mu e^\mu = -1$, where p^μ is the four-momentum of our particle. Prove that these relations for e_μ will hold in any inertial frame.

3) Consider the rest frame of the spin one particle and let us take a basis for the three polarization states which is the standard Cartesian basis vectors for a three-dimensional vector space but this time written as four-vectors, i.e.

$$e^\mu(\hat{x}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^\mu(\hat{y}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^\mu(\hat{z}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1}$$

Assume that the three-momentum, $\boldsymbol{p}$, of the particle is along the z-direction. Show that the general polarization vectors in this case become $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(|\boldsymbol{p}|/m, 0, 0, E/m)$.

4) Most often one uses a related but slightly different basis for the polarization states of a massive spin one field, the so-called helicity states which we index

by the integer λ. They have the form

$$e^{\mu}(\lambda=+1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\-i\\0\end{pmatrix},\quad e^{\mu}(\lambda=-1)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix},\quad e^{\mu}(\lambda=0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{2}$$

Show that if we perform a rotation of these polarization states around the z-axis by an angle θ, they are multiplied by a factor $e^{-i\lambda\theta}$. Argue that this implies that λ is the eigenvalue of the quantum mechanical rotation operator, J_z, for each of the states.

5) The helicity is defined as the projection of the spin of a particle onto its momentum, i.e. $\boldsymbol{J}\cdot\boldsymbol{p}$, where $\boldsymbol{J}$ is the spin operator and $\boldsymbol{p}$ is the three-momentum. Argue that if we boost from the rest frame to a frame moving with the particle at velocity $\boldsymbol{p}/E$, then J_z becomes the helicity operator.

6) Show that the polarization states with well-defined helicity for a particle moving along the z-direction are

$$e^{\mu}(\lambda=\pm1)=\mp\left(0,\frac{e_x\pm ie_y}{\sqrt{2}}\right) \tag{3}$$

$$e^{\mu}(\lambda=0)=\frac{1}{m}\left(|\boldsymbol{p}|,0,0,E\right), \tag{4}$$

where e_x and e_y are the standard three-component basis vectors.

7) Suppose someone told you that they wanted a theory with a particle of some general spin $S>1$. Using what you have learned in previous parts of this problem, how would you generalize the procedure to construct polarization states for such a field? Just sketch a procedure, do not embark on detailed calculations.

2 The Photon Polarization and Gauge Invariance

Maxwell's equations for the free photon field which has zero mass can be obtained from the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the antisymmetric field strength tensor for the electromagnetic field which is described by the four-vector A_μ.

1) In the abscence of sources, use the Euler-Lagrange equations to show that the equations of motion are

$$\partial^\mu F_{\mu\nu} = 0. \tag{6}$$

There is a lot of redundancy in the equations when we want to describe photons since we have here four equations (one for each value of ν), but a real photon field has only two independent directions of its polarization. We, therefore, start by choosing the Lorenz gauge condition [1]

$$\partial_\mu A^\mu = 0. \tag{7}$$

2) Show that when imposing the Lorenz condition the equations of motion becomes

$$\partial_\mu \partial^\mu A^\nu = 0, \tag{8}$$

which is just four copies of the Klein-Gordon equation, one for each value of ν. Furthermore, shows that the solutions are therefore simply (disregarding normalization)

$$A^\mu = e^\mu e^{\pm ipx}, \tag{9}$$

with $px = p^\mu x_\mu$ and where e^μ is a polarization vector which is independent of x_μ.

3) Show that the Lorenz gauge condition implies

$$\partial_\mu A^\mu = \pm i p_\mu e^\mu e^{\pm ipx} = 0, \tag{10}$$

and thus $\epsilon p = \epsilon^\mu p_\mu = 0$. This implies that polarization states are orthogonal (transverse) to the momentum direction.

However, we are not finished yet. The Lorenz condition gives us one equation so that naively we would expect that only one degree of freedom is killed, thus leaving three behind and one more than we expect for free photons. What we have to consider now is the more general gauge transformations that will leave the equations of motion unchanged.

4) Show that transformations of the form

$$A'_\mu = A_\mu + \partial_\mu \theta(x), \tag{11}$$

[1] Note that this gauge is correctly termed Lorenz gauge after Danish physicist Ludvig V. Lorenz who introduced it in 1867. It is commonly misattributed to Dutch physicists Hendrik A. Lorentz whose name is associated with Lorentz transformations among other things. This could be a result of the fact that the equation itself is a condition on a *Lorentz* scalar, but this is a misnomer.

where $\theta(x)$ is some scalar function that depends on space and time leave the equations of motion unchanged.

5) Use the Lorenz condition to show that $\theta(x)$ has to obey

$$\partial_\mu \partial^\mu \theta(x) = 0. \tag{12}$$

This is the Klein-Gordon equation for a massless boson. Show that two solutions to this equation are $\theta(x) = Ne^{\pm ipx}$, where N is a constant.

6) We now have

$$A_\mu^{'} = A_\mu \pm iNp_\mu e^{\pm ipx}. \tag{13}$$

or in terms of the polarization vectors we can write

$$e_\mu^{'} = e_\mu \pm iNp_\mu. \tag{14}$$

Show that $p^\mu e_\mu^{'} = 0$, i.e. the polarization states are still transverse to the momentum.

Note that here it is crucial that we have the photon momentum in the solution for $\theta(x)$, i.e. it cannot be an arbitrary solution with some other four-momentum plane wave factor.

7) Show that we may use the constant N to ensure that $e_0^{'} = 0$, that is we make the time component of the polarization vector zero. Show that this implies that

$$e_\mu^{'} p^\mu = -\boldsymbol{e}^{'} \cdot \boldsymbol{p} = 0, \tag{15}$$

so that we indeed get the expected result; photon polarization states that are transverse to the direction of the three-momentum $\boldsymbol{p}$. The two independent polarization basis states may now be chosen in the plane perpendicular to the direction of propagation defined by $\boldsymbol{p}$.

An important point is that any process we consider where there are photons in the initial and/or final state(s) can be used to test gauge invariance. In a given amplitude, $\mathcal{M}_{fi}$, one can replace any in-coming or out-going polarization vector by $e_\mu + Kp_\mu$, where K is a constant and p_μ is the four-momentum of the particular photon in the process. Gauge invariance tells us that this replacement cannot change the final result. This is a very good check of whether one's result for an amplitude such as Compton scattering is consistent. Note that this does not hold for photons that are *internal* lines in a diagram for a given process. They are virtual states and will generally exploit all four polarization states. This can be seen very clearly when deriving the standard Coulomb interaction from second-order diagrams in Quantum ElectroDynamics (QED).

3 Quantum Electrodynamics with Electrons

Consider the Lagrangian for electrons which is the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi, \tag{16}$$

where $\partial\!\!\!/ = \gamma_\mu \partial^\mu$ and γ_μ are the Dirac 4x4 matrices.

1) Consider the transformation $\psi \to e^{-ie\theta(x)}\psi$, where e is the electron charge and $\theta(x)$ is a scalar function that depends on space and time (denoted collectively by the coordinate $x = (\boldsymbol{x}, t)$). Use minimal substitution for a charge $q = -e$ field

$$\partial^\mu \to D^\mu = \partial^\mu - ieA^\mu, \tag{17}$$

in the Dirac Lagrangian and show that we obtain a gauge invariant Lagrangian. What is the necessary transformation law for A^μ?

2) In mathematics, the complex numbers on the unit circle (modulus 1) are denoted collectively by $U(1)$ (the 1x1 unitary matrix group). Explain why it makes sense to call the gauge transformation in 1) and the corresponding theory of Quantum EletroDynamics (QED) a $U(1)$-gauge theory.

3) Show that the interaction Lagrangian we obtain from using the principle of gauge invariance is of the current-vector field form

$$\mathcal{L}_I = e\bar{\psi}\gamma_\mu\psi A^\mu = -J_\mu A^\mu. \tag{18}$$

Show furthermore that $J_\mu(x) = -e\bar{\psi}(x)\gamma_\mu\psi(x)$ is a conserved current in the sense that $\partial^\mu J_\mu(x) = 0$. How does the interaction Lagrangian transform under a gauge transformation? Does this transformation allow the theory to remain gauge invariant? (Hint: Consider the action that is generated by the interaction Lagrangian).

4) The Dirac field can be expanded in normal modes according to

$$\psi_\alpha(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p},\lambda}u(\boldsymbol{p},\lambda)_\alpha e^{-ipx} + d^\dagger_{\boldsymbol{p},\lambda}v(\boldsymbol{p},\lambda)_\alpha e^{ipx}\right), \tag{19}$$

where λ is the helicity and $px = p_\mu x^\mu$. b and d are operators that create fermionic particles and antiparticles respectively with given momentum, $\boldsymbol{p}$, and helicty, λ. Show that one can write

$$-e\bar{\psi}\gamma_\mu\psi = \sum_{\boldsymbol{p},\boldsymbol{p}',\lambda,\lambda'}\sum_{n=1}^{4} j_\mu^{(n)}\left(p,\lambda,p',\lambda',x\right) = \sum_{n=1}^{4} j_\mu^{(n)}\left(x\right), \tag{20}$$

and find the four terms $j_\mu^{(n)}$ explicitly.

5) Derive the following matrix elements using the current from 4)

$$\langle e^-, \boldsymbol{p}', \lambda' | j_\mu^{(1)}(x) | e^-, \boldsymbol{p}, \lambda \rangle = -e\overline{u}(\boldsymbol{p}', \lambda')\gamma_\mu u(\boldsymbol{p}, \lambda) e^{i(p'-p)x} \tag{21}$$

$$\langle e^+, \boldsymbol{p}', \lambda' | j_\mu^{(2)}(x) | e^+, \boldsymbol{p}, \lambda \rangle = e\overline{v}(\boldsymbol{p}, \lambda)\gamma_\mu v(\boldsymbol{p}', \lambda') e^{i(p'-p)x} \tag{22}$$

$$\langle 0 | j_\mu^{(3)}(x) | e^-, \boldsymbol{p}, \lambda; e^+, \boldsymbol{p}', \lambda' \rangle = -e\overline{v}(\boldsymbol{p}', \lambda')\gamma_\mu u(\boldsymbol{p}, \lambda) e^{-i(p+p')x} \tag{23}$$

$$\langle e^-, \boldsymbol{p}', \lambda'; e^+, \boldsymbol{p}, \lambda | j_\mu^{(4)}(x) | 0 \rangle = -e\overline{u}(\boldsymbol{p}', \lambda')\gamma_\mu v(\boldsymbol{p}, \lambda) e^{i(p+p')x}. \tag{24}$$

Give a physical interpretation of the four terms (you may even like to draw a picture of each term as a subpart of a Feynman diagram).

6) The matrix elements in 5) are called transition currents, $J_\mu^{fi}(x)$. Show explicitly that they are conserved current by applying the gradient operator ∂^μ to each of them.

7) Argue that in QED with interaction Lagrangian Eq. (18), when we calculate physical processes we will always have terms of the form $J_\mu^{fi}(0)\epsilon^\mu(\sigma)$, where $\epsilon^\mu(\sigma)$ is a photon polarization state indexed by σ. Furthermore, argue that then we square the amplitude for a given process we get an expression like

$$|J_\mu^{fi}(0)\epsilon^\mu(\sigma)|^2 = \epsilon^\mu(\sigma)^*\epsilon^\nu(\sigma)J_\mu^{fi}(0)^*J_\nu^{fi}(0). \tag{25}$$

8) Often we need to sum over unobserved polarization states, i.e. we sum over σ

$$\left[\sum_\sigma \epsilon^\mu(\sigma)^*\epsilon^\nu(\sigma)\right] J_\mu^{fi}(0)^*J_\nu^{fi}(0). \tag{26}$$

Assume that the momentum transfer of the current J_μ^{fi} is along the z-direction, i.e. $q = (q_0, 0, 0, q_0)$. Show that for real photons we have

$$\sum_\sigma \epsilon^\mu(\sigma)^*\epsilon^\nu(\sigma) = \delta_1^\mu\delta_1^\nu + \delta_2^\mu\delta_2^\nu, \tag{27}$$

where we use the convention that $\mu = 0$ is the time-direction and $\mu = 1, 2, 3$ the x, y, and z space-directions respectively.

9) Show that for a real photon we have

$$\left[\sum_\sigma \epsilon^\mu(\sigma)^*\epsilon^\nu(\sigma)\right] J_\mu^{fi}(0)^*J_\nu^{fi}(0) = -g^{\mu\nu}J_\mu^{fi}(0)^*J_\nu^{fi}(0). \tag{28}$$

10) Under what circumstances does the result of 9) imply that

$$\sum_\sigma \epsilon^\mu(\sigma)^*\epsilon^\nu(\sigma) = -g^{\mu\nu}. \tag{29}$$

What additional terms could arise in the case of real photons where $q^2 = 0$? What about the case of virtual photons where $q^2 \neq 0$?

11) Consider the scattering of electrons on muons, $e^- + \mu^- \to e^- + \mu^-$. Draw the second-order Feynman diagram for this process.

12) Show that the S-matrix for the electron-muon scattering process can be written

$$S_{fi}^{(2)} = (-i)^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 J_\mu^{e^-}(x_1) J_\nu^{\mu^-}(x_2) \langle 0| T\left[A^\mu(x_1) A^\nu(x_2)\right] |0\rangle. \tag{30}$$

Write the electron and muon currents explicitly using the transition currents above. Make sure you label all quantities needed to specify the initial and final states properly.

13) By using the analogy to the Klein-Gordon propagator, argue that we can use the following form of the photon propagator

$$G^{\mu\nu}(q) = \int \mathrm{d}^4x e^{iqx} \langle 0| T\left[A^\mu(x) A^\nu(0)\right] |0\rangle = \frac{-ig^{\mu\nu}}{q^2} \tag{31}$$

in the expression for $S_{fi}^{(2)}$.

14) Define as usual $S_{fi}^{(2)} = -iM_{fi}(2\pi)^4 \delta(p_f - p_i)$. Show that

$$-iM_{fi} = \left(-iJ_\mu^{e^-}(0)\right) \frac{-ig^{\mu\nu}}{q^2} \left(-iJ_\nu^{\mu^-}(0)\right). \tag{32}$$

Express the momentum transfer q in terms of the initial and final state momenta.

15) Use all the information gathered in this exercise to write down a set of Feynman rules for QED at tree level (where there are no loops in diagrams), i.e. figure out what factors are associated with initial and final state particles/antiparticles, what factors are associated with vertices, and what factors are associated with propagators.

4 Compton Scattering in Scalar Electrodynamics

Consider the interaction of a charged (complex) scalar, ϕ, and an electromagnetic field, A^μ. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi + \mathcal{L}_I, \tag{33}$$

where the interaction term is

$$\mathcal{L}_I = -ie\left(\phi^\dagger \partial_\mu \phi - (\partial_\mu \phi)^\dagger \phi\right) A^\mu + e^2 \phi^\dagger \phi A_\mu A^\mu \tag{34}$$

1) Show that the Lagrangian $\mathcal{L}$ is invariant under the *global* gauge transformation

$$\phi(x) \to e^{i\alpha}\phi(x), \tag{35}$$

where α is a real number. Use Nöther's theorem to find the conserved current associated with this global symmetry.

2) Show that the Lagrangian $\mathcal{L}$ is invariant under the *local* gauge transformation

$$\phi(x) \to e^{ie\theta(x)}\phi(x), \qquad A_\mu \to A_\mu - \partial_\mu\theta(x), \tag{36}$$

where $\theta(x)$ is a scalar function of space and time.

3) Show that we can write the Lagrangian in the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi, \tag{37}$$

where $D^\mu = \partial^\mu + ieA^\mu$ is the so-called covariant derivative. Replacing ∂^μ by D^μ is known as 'minimal substitution' and is a well-known procedure in non-relativistic mechanics for obtaining the interaction Hamiltonian for the electron with the electromagnetic field from the non-interacting Hamiltonian.

4) Assume that you have a photon with four-momentum k and a mass m particle with four-momentum p in the initial state, while the photon has k' and the massive particle has p' four-momentum in the final state. Show that $p^\mu k_\mu - p^\mu k'_\mu - k^\mu k'_\mu = 0$. Evaluate this expression in the rest frame of the in-coming massive particle and deduce the Compton relation

$$\frac{1}{E_{k'}} - \frac{1}{E_k} = \frac{1}{m}(1-\cos\theta), \tag{38}$$

where θ is the angle between the incoming photon and outgoing photon, while $E_{k'}$ and E_k are the photon energies.

5) Draw second-order Feynman diagrams for Compton scattering of a scalar particle, $\phi + \gamma \to \phi + \gamma$. (Hint: There are three distinct types of diagrams.)

6) Work out the amplitudes $\mathcal{M}_{fi}$ for each diagram. Be aware that if you use Feynman rules to do this task, you have to be careful to consider combinatorial symmetry factors that may arise.

7) Since Compton scattering has real photons in the initial/final states, we can check gauge invariance by using the transformation $\epsilon_\mu \to \epsilon_\mu + Np_\mu$, where N is a constant. Show that you get a gauge invariant result for the second order process and that all three types of diagrams must be included to achieve this gauge invariance.

8) Calculate the cross section averaged over all photon polarization states in the relativistic limit.