3 - Dirac Spinors in the Weyl Representation

Here we explore a different representation of the Dirac matrices, γ_μ, that will generate slightly different expressions for the explicit solutions. Let us define the so-called Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

We take the positive energy solutions to have the form

$$\psi_s = u_s(p)e^{-ipx} = (\not{p} + m) \begin{bmatrix} \chi_s \\ 0 \end{bmatrix} e^{-ipx}, \tag{19}$$

and the negative energy ones to be

$$\psi_s = v_s(p)e^{ipx} = (\not{p} - m) \begin{bmatrix} 0 \\ \chi_{-s} \end{bmatrix} e^{ipx}, \tag{20}$$

where χ_s are 2-spinors with quantization axis along the direction of momentum $\mathbf{p}$ with projection $s = \pm 1$.

Exercise (1)**.**

Find the explicit form of the spinors if we insist on the normalization $\overline{\psi}\psi = 2m$ for positive energy solutions and $\overline{\psi}\psi = -2m$ for negative energy solutions.

Solution.

$$
\begin{aligned}
\Psi_s &= A(\not{p}+m)\begin{bmatrix}\chi_s\\0\end{bmatrix}e^{-ipx}\\
\overline{\Psi}\Psi &= \Psi^\dagger\gamma^0\Psi = A^2\Psi^\dagger\gamma^0(\not{p}+m)\begin{bmatrix}\chi_s\\0\end{bmatrix}\\
&= A^2\Psi^\dagger\gamma^0(\not{p}+m\mathbb{I})\begin{bmatrix}\chi_s\\0\end{bmatrix}\\
&= A^2\Psi^\dagger\gamma^0\begin{bmatrix}E+m & -\sigma\cdot\mathbf{p}\\ \sigma\cdot\mathbf{p} & m-E\end{bmatrix}\begin{bmatrix}\chi_s\\0\end{bmatrix}\\
&= A^2\Psi^\dagger\begin{bmatrix}E+m & -\sigma\cdot\mathbf{p}\\ \sigma\cdot\mathbf{p} & E-m\end{bmatrix}\begin{bmatrix}\chi_s\\0\end{bmatrix}\\
&= A^2\Psi^\dagger\begin{bmatrix}(E+m)\chi_s\\ \sigma\cdot\mathbf{p}\chi_s\end{bmatrix} = A^2\left((\not{p}+\mathbb{I})^\dagger\begin{bmatrix}\chi_s\\0\end{bmatrix}\right)\begin{bmatrix}(E+m)\chi_s\\ \sigma\cdot\mathbf{p}\chi_s\end{bmatrix}\\
&= A^2\begin{bmatrix}\chi_s^\dagger & 0\end{bmatrix}\begin{bmatrix}E+m & -\sigma\cdot\mathbf{p}\\ \sigma\cdot\mathbf{p} & m-E\end{bmatrix}^T\begin{bmatrix}(E+m)\chi_s\\ \sigma\cdot\mathbf{p}\chi_s\end{bmatrix}\\
&= A^2\begin{bmatrix}\chi_s^\dagger & 0\end{bmatrix}\begin{bmatrix}E+m & \sigma\cdot\mathbf{p}\\ -\sigma\cdot\mathbf{p} & m-E\end{bmatrix}\begin{bmatrix}(E+m)\chi_s\\ \sigma\cdot\mathbf{p}\chi_s\end{bmatrix}\\
&= \begin{bmatrix}\chi_s^\dagger & 0\end{bmatrix}\begin{bmatrix}(E+m)^2\chi_s+(\sigma\cdot\mathbf{p})^2\chi_s\\ -(\sigma\cdot\mathbf{p})(E+m)\chi_s+(\sigma\cdot\mathbf{p})(m-E)\chi_s\end{bmatrix}\\
&=A^2\left((E+m)^2+(\sigma\cdot\mathbf{p})^2\right) = A^2\left(E^2+m^2+2mE+p^2\right) = A^2\left(2E(E+m)\right) = 2m\\
\Rightarrow A &= \sqrt{\frac{2m}{2E(E+m)}} = \sqrt{\frac{m}{E(E+m)}}
\end{aligned}
$$

Exercise (2).

Find the solutions in the massless limit, $m = 0$. How does the Dirac equation simplify in this limit?

Exercise (3).

The Dirac Hamiltonian operator has the form $H_D = -i\gamma^0\gamma\cdot\nabla + \gamma^0 m$. Introduce the helicity operator, $h = \sigma\cdot\mathbf{p}/|\mathbf{p}|$. Show that the Dirac operator and h commute. Note that the while h looks like a 2×2 matrix due to the σ part, when we apply it to 4-spinors it is understood that it is a 4×4 matrix also (it has an implicit 2×2 matrix multiplied on it). Argue that the spinors, χ_s, defined above are in fact the helicity eigenfunctions.

Exercise (4).

Look the solutions in the massless limit from 2). Determine the helicity of the four solutions when $m = 0$. How is the spin and helicity connected for positive and negative energy solutions?