Advanced Particle Physics 2025 Problem Set 3

1 The Dirac Equation and Dimensionality

Dirac started by postulating an equation linear in time and space of the form

$$(\boldsymbol{\alpha} \cdot \boldsymbol{p} + m\beta)\psi = i\frac{\partial \psi}{\partial t}, \tag{1}$$

and then proceeded to figure out what the conditions on $\boldsymbol{\alpha}$ and β would have to be to ensure that it obeys the energy-momentum relation of special relativity, i.e. $E^2 = |\boldsymbol{p}|^2 + m^2$.

1) Show that the conditions are

$$\alpha_i \beta + \beta \alpha_i = 0, \quad i = 1, 2, 3 \tag{2}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0, \quad i, j = 1, 2, 3; i \neq j \tag{3}$$

$$\alpha_i^2 = \beta^2 = 1, \quad i = 1, 2, 3. \tag{4}$$

(Hint: Take the square of the operator on both sides of Eq. (1).)

2) Prove that $\alpha_i, i = 1, 2, 3$ and β are all Hermitian (Hint: First find the Hamiltonian for the Dirac particles).

3) Prove that $\mathrm{Tr}(\alpha_i) = \mathrm{Tr}(\beta) = 0$ where Tr is the matrix trace (sum of diagonal entries).

4) Prove that the eigenvalues of α_i and β are all either $+1$ or -1.

5) Prove that the dimensionality of α_i and β is even.

6) Argue that this implies that the dimension of α_i and β must be at least 4.

2 Solutions of the Dirac Equation and Polarization Sums

In this problem we want to study solutions of the Dirac equation and some of their general properties that are extremely useful when calculating matrix elements. We start from the Dirac equation written using the Dirac gamma matrices, γ^μ.

1) Start from Eq. (1) of the previous exercise and define $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha^i$ (or equivalently $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$). Show that the Dirac equation can now be written

$$i\gamma_\mu \partial^\mu \psi = m\psi. \tag{5}$$

In order to advance, we need a concrete representation of the gamma matrices. In this problem we pick the so-called Dirac-Pauli representation where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},\ \text{and}\ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

The fact that the last matrix is called '5' is a relic of older times when γ^0 was typically called γ^4. The commutation relations for the Dirac gamma matrices are

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \tag{7}$$

$$\{\gamma_\mu, \gamma_5\} = 0. \tag{8}$$

Below we will also be use the notation, $\not{p} = \gamma_\mu p^\mu$. As the notation implies, the quantity γ^μ should behave in a way similar to a four-vector, but of course a four-vector with entries that are each 4x4 matrices.

2) Suppose we make a Lorentz transformation on the four-spinor ψ, i.e. $\psi \to \psi' = S(\Lambda)\psi$, where $S(\Lambda)$ is a 4x4 matrix that does a Lorentz transform on a four-spinor with Λ the corresponding change to space-time under the Lorentz transformation. Show that if

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu, \tag{9}$$

then the Dirac equation is Lorentz invariant. This shows that the gamma matrices do transform non-trivially under Lorentz transformations. (Hint: Look at the equation in terms of ψ' and consider how the four-derivative transforms under a Lorentz transformation Λ).

3) Consider now the rest frame where $\boldsymbol{p} = 0$. Show that

$$i\gamma_0 \frac{\partial}{\partial t}\psi = m\psi. \tag{10}$$

Now write the four-component spinor in the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{11}$$

and show that ψ_1 and ψ_2 are solutions with $E = m$, while ψ_3 and ψ_4 are solutions with $E = -m$. Give a physical interpretation of this finding.

4) Consider now the non-zero momentum case and let $p^\mu = (E, \boldsymbol{p})$. We will take $E > 0$ from now on. Show that $\psi = u(p)e^{-ipx}$ and $\psi = v(p)e^{ipx}$, where u and v are 4-spinors, are solutions of the Dirac equation when $(\not{p} - m)u(p) = 0$ and $(\not{p} + m)v(p) = 0$. Notice that since we assume $E > 0$, the $u(p)$ solutions may be called positive energy solutions and the $v(p)$ solutions may be called negative energy solutions.

5) Next show that $u(p) = (\not{p} + m)u(0)$ and $v(p) = (\not{p} - m)v(0)$ are solutions. Here $u(0)$ and $v(0)$ are momentum independent functions. What other quantum number must $u(0)$ and $v(0)$ depend on?

6) Now we write

$$u_s(0) = \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \quad \text{and} \quad v_s(0) = \begin{pmatrix} 0 \\ \chi_{-s} \end{pmatrix}, \tag{12}$$

where χ_s is a 2-spinor spin one-half wave function and $s = \pm 1$ denotes the two orthogonal spin states along some direction in space. Notice the minus sign on the 2-spinor in the $v_s(0)$ part (it is connected to the anticommutation relations of fermion fields). We do not need to specify a quantization axis for the spin at this point and we keep it completely general. Show that

$$u_s(p) = \begin{pmatrix} (E+m)\chi_s \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_s \end{pmatrix} \quad \text{and} \quad v_s(p) = \begin{pmatrix} -\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_{-s} \\ -(E+m)\chi_{-s} \end{pmatrix}, \tag{13}$$

are solutions for positive and negative energy states.

7) We also need to normalize the Dirac 4-spinor solutions appropriately. The four-current for a Dirac spinor field is $\bar{\psi}\gamma^\mu\psi$, where $\bar{\psi} = \psi^\dagger\gamma^0$. Show that the density (which is the zeroth component) is simply $\rho = \psi^\dagger\psi$. Consider

$$\psi = u_s(p)e^{-ipx}, \tag{14}$$

and show that $\int d^3x\rho = 2E(m+E)V$. Argue (without doing more calculations) that if we had taken $v_s(p)$ instead of $u_s(p)$ we get the same result.

8) Show that the properly normalized ($\int d^3x\psi^\dagger\psi = 1$) 4-spinors may be written in the form

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_s}{E+m} \end{pmatrix} \quad \text{and} \quad v_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_{-s}}{E+m} \\ \chi_{-s} \end{pmatrix}, \tag{15}$$

and that this produces the positive and negative energy solutions of the form

$$\psi_+ = \frac{1}{\sqrt{2EV}}u_s(p)e^{-ipx} \quad \text{and} \quad \psi_- = \frac{1}{\sqrt{2EV}}v_s(p)e^{ipx}. \tag{16}$$

Here we have insisted that the total wave functions, ψ_+ and ψ_-, carry normalization factors similar to scalar wave functions ($1/\sqrt{2EV}$) such that all the details of the fermionic nature and the Dirac equation are completely contained in the u and v factors.

9) Using the explicit form of the 4-spinors in Eq. (15) show the important and extremely useful completeness relations

$$\sum_s u_s(p)\overline{u_s(p)} = \not{p} + m \quad \text{and} \quad \sum_s v_s(p)\overline{v_s(p)} = \not{p} - m. \tag{17}$$

3 Dirac Spinors in the Weyl Representation

Here we explore a different representation of the Dirac matrices, γ_μ, that will generate slightly different expressions for the explicit solutions. Let us define the so-called Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \text{and}\, \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{18}$$

We take the positive energy solutions to have the form

$$\psi_s = u_s(p)e^{-ipx} = \left(\not{p} + m\right) \begin{bmatrix} \chi_s \\ 0 \end{bmatrix} e^{-ipx}, \tag{19}$$

and the negative energy ones to be

$$\psi_s = v_s(p)e^{ipx} = \left(\not{p} - m\right) \begin{bmatrix} 0 \\ \chi_{-s} \end{bmatrix} e^{ipx}, \tag{20}$$

where χ_s are 2-spinors with quantization axis along the direction of momentum $\boldsymbol{p}$ with projection $s = \pm 1$.

1) Find the explicit form of the spinors if we insist on the normalization $\overline{\psi}\psi = 2m$ for positive energy solutions and $\overline{\psi}\psi = -2m$ for negative energy solutions.

2) Find the solutions in the massless limit, $m = 0$. How does the Dirac equation simplify in this limit?

3) The Dirac Hamiltonian operator has the form $H_D = -i\gamma^0\boldsymbol{\gamma} \cdot \nabla + \gamma^0 m$. Introduce the helicity operator, $h = \boldsymbol{\sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$. Show that the Dirac operator and h commute. Note that the while h looks like a 2×2 matrix due to the $\boldsymbol{\sigma}$ part, when we apply it to 4-spinors it is understood that it is a 4×4 matrix also (it has an implicit 2×2 matrix multiplied on it). Argue that the spinors, χ_s, defined above are in fact the helicity eigenfunctions.

4) Look the solutions in the massless limit from 2). Determine the helicity of the four solutions when $m = 0$. How is the spin and helicity connected for positive and negative energy solutions?

5) Show that γ_5 does not commute with the Dirac Hamiltonian operator unless $m = 0$.

6) Now we introduce the so-called chiral projection operators, $P_L = \frac{1}{2}(1 - \gamma_5)$ and $P_R = \frac{1}{2}(1 + \gamma_5)$. These are called the left-handed and right-handed projections respectively. What is the helicity of a massless and left-handed field? What is the helicity of a massless and right-handed field?

7) Consider the spinor $u_s(p)$. Show that to lowest order in m/E

$$P_L u_+(p) = \begin{pmatrix} \frac{m}{\sqrt{2E}}\chi_+ \\ 0 \end{pmatrix} \quad \text{and} \quad , P_L u_-(p) = \begin{pmatrix} \sqrt{2E}\chi_- \\ 0 \end{pmatrix}. \tag{21}$$

Use this to argue that if you have an interaction that only cares about the left-handed component of a fermion field, then $u_+(p)$ states will have suppressed interactions.

8) Do the same calculation for the spinor $v_s(p)$ as in 7). Use this to argue that if an interaction cares only about left-handed components, then $v_-(p)$ states will have suppressed interactions.

9) The weak interaction in the Standard Model has a projection operator onto the left-handed component of the fields involved. Argue that $s = -1$ fermions and $s = +1$ anti-fermions dominate in the weak interaction.

10) Consider the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi. \tag{22}$$

Define the right- and left-handed 4-spinors, $\psi_R = P_R\psi$ and $\psi_L = P_L\psi$. Show that the Lagrangian becomes

$$\mathcal{L} = i\overline{\psi_L}\partial\!\!\!/\psi_L + i\overline{\psi_R}\partial\!\!\!/\psi_R - m\left(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L\right). \tag{23}$$

11) Argue that the presence of mass terms breaks chiral symmetry, i.e. the projections P_L and P_R are mixed.

4 The Fermion Propagator

The following problem derives the propagator for a fermion. First we have to figure out the Hamiltonian. We start from

$$\mathcal{L} = \bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)\psi, \tag{24}$$

where we use the convenient short-hand $\partial^\mu = \frac{\partial}{\partial x_\mu}$.

1) Show that the corresponding Hamiltonian can be written

$$H = \int d^3x \bar{\psi} \left(-i\boldsymbol{\gamma} \cdot \nabla + m \right) \psi, \tag{25}$$

where $\boldsymbol{\gamma}$ is the 3-vector $(\gamma^1, \gamma^2, \gamma^3)$. Remember that $\partial^\mu = (\partial_t, -\nabla)$.

2) Now consider Heisenberg's equations of motion for the field operator $\psi(\boldsymbol{x}, t)$, i.e.

$$\frac{\partial \psi_\alpha(\boldsymbol{x}, t)}{\partial t} = -i \left[\psi_\alpha(\boldsymbol{x}, t), H \right], \tag{26}$$

where α is an index indicating that keep track of the fact that there are four components in the field of Dirac particle. Show that

$$\frac{\partial \psi_\alpha(\boldsymbol{x}, t)}{\partial t} = \left(-\gamma_0 \boldsymbol{\gamma} \cdot \nabla - im\gamma_0 \right)_{\alpha\beta} \psi_\beta(\boldsymbol{x}, t), \tag{27}$$

where we sum over the spinor index β (repeated use of spinor indices mean summation from now on). You will need the canonical *anti*-commutator for fermion fields is

$$\{ \psi_\alpha(\boldsymbol{x}_1, t), \psi_\beta^\dagger(\boldsymbol{x}_2, t) \} = \delta_{\alpha\beta} \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2). \tag{28}$$

3) We now define the time-ordering operator for fermionic fields, ψ and $\bar{\psi}$ in the following way

$$T \left\{ \psi_\alpha(\boldsymbol{x}_1, t_1) \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \right\} = \begin{cases} \psi_\alpha(\boldsymbol{x}_1, t_1) \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) & \text{for } t_1 > t_2 \\ -\bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \psi_\alpha(\boldsymbol{x}_1, t_1) & \text{for } t_2 > t_1, \end{cases} \tag{29}$$

where this is now a matrix indexed by α and β since the fields are four-component quantities. Show that

$$\begin{aligned} &\frac{\partial}{\partial t_1} T \left\{ \psi_\alpha(\boldsymbol{x}_1, t_1) \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \right\} = \\ &\delta(t_1 - t_2) \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) [\gamma_0]_{\alpha\beta} + T \left\{ \frac{\partial \psi_\alpha(\boldsymbol{x}_1, t_1)}{\partial t_1} \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \right\} \end{aligned} \tag{30}$$

4) Now use Eq. (27) to show that

$$\left[i\gamma_\mu \partial^\mu - m \right]_{\alpha\eta} T \left\{ \psi_\eta(\boldsymbol{x}_1, t_1) \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \right\} = i\delta(t_1 - t_2) \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) \delta_{\alpha\beta}, \tag{31}$$

where ∂_μ act on $\boldsymbol{x}_1$ and t_1 only.

5) Now consider the function

$$G_{F\alpha\beta}(\boldsymbol{x}_1, t_1, \boldsymbol{x}_2, t_2) = \langle 0 | T \left\{ \psi_\alpha(\boldsymbol{x}_1, t_1) \bar{\psi}_\beta(\boldsymbol{x}_2, t_2) \right\} | 0 \rangle. \tag{32}$$

Argue that G_F can only depend on $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $t = t_1 - t_2$. This is the propagator that we are looking for.

6) Take the Fourier transform similar to the exercise above for the scalar fields and show that

$$\left(\not{p} - m\right)_{\alpha\eta} G_{F\eta\beta}(p_\mu) = i\delta_{\alpha\beta}. \tag{33}$$

7) Invert the relation above and show that

$$G_{F\alpha\beta}(p_\mu) = i\frac{(\not{p} + m)_{\alpha\beta}}{p^2 - m^2}, \tag{34}$$

where $p^2 = p^\mu p_\mu$.

8) Argue that a general 4-spinor field can be expanded in field modes as follows

$$\psi_\alpha(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(b_{\boldsymbol{p},\lambda} u(\boldsymbol{p},\lambda)_\alpha e^{-ipx} + d^\dagger_{\boldsymbol{p},\lambda} v(\boldsymbol{p},\lambda)_\alpha e^{ipx}\right), \tag{35}$$

where λ indicates the helicity and $px = p_\mu x^\mu$. $b_{\boldsymbol{p},\lambda}$ and $d_{\boldsymbol{p},\lambda}$ are operators that create fermonic particles and antiparticles respectively with given momentum, $\boldsymbol{p}$, and helicty, λ. Forget about normalization factors coming from momentum space space density.

9) We will now impose canonical *anti*-commutation relations

$$\{b_{\boldsymbol{p},\lambda}, b^\dagger_{\boldsymbol{p}',\lambda'}\} = (2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}')\delta_{\lambda,\lambda'} \tag{36}$$

$$\{d_{\boldsymbol{p},\lambda}, d^\dagger_{\boldsymbol{p}',\lambda'}\} = (2\pi)^3\delta(\boldsymbol{p} - \boldsymbol{p}')\delta_{\lambda,\lambda'}, \tag{37}$$

with all other combinations being equal to zero. Show that if you use these relations together with the field expansion in Eq. (35) you recover the anti-commutation relation for a fermion field in Eq. (28).

10) Assume you are given a state with a single fermion of momentum $\boldsymbol{p}$ and helicity λ. Show that if you attempt to put a second fermion with the same momentum $\boldsymbol{p}$ and helicity λ in the state you get zero.

11) Focus now on a fixed helicity and momentum. Define a number operator in the usual way $N^b = b^\dagger b$ (and likewise for d-type particles) and normalize the anti-commutation relations as $\{b, b^\dagger\} = 1$. Show that the eigenvalues of a number operator for fermions have to be either 0 or 1.

12) Insert the expansion of Eq. (35) in Eq. (32) and show that

$$G_{F\alpha\beta}(\boldsymbol{x}_1, t_1, \boldsymbol{x}_2, t_2) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \times$$
$$\left(e^{-ipx} u(\boldsymbol{p},\lambda)_\alpha \bar{u}(\boldsymbol{p},\lambda)_\beta \theta(t) - e^{ipx} v(\boldsymbol{p},\lambda)_\alpha \bar{v}(\boldsymbol{p},\lambda)_\beta \theta(-t)\right), \tag{38}$$

where $t = t_1 - t_2$, $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, and thus $x = x_1 - x_2$.

13) Sum over the polarizations in Eq. (38) and show that

$$G_{F\alpha\beta}(\boldsymbol{x}, t) = \left(i\partial\!\!\!/ + m\right)_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ipx}\theta(t) + e^{ipx}\theta(-t)\right), \tag{39}$$

where $\partial\!\!\!/ = \gamma_\mu \partial^\mu$.

14) Show that for a scalar field, ϕ,

$$\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ipx}\theta(t) + e^{ipx}\theta(-t)\right), \tag{40}$$

where $t = t_1 - t_2$ and $x = x_1 - x_2$.

15) The scalar propagator can be written as in Eq. (40). Show that for a scalar field, ϕ,

$$\langle 0|T\left[\phi(x_1)\phi(x_2)\right]|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ipx}, \tag{41}$$

where m is the mass of the field. Combine this relation with the result in 13) and show that you recover Eq. (34).

5 Fermionic fields and anti-commutation relations

This problem explores how one can derive the fundamental *anticommutation* relations among fermionic creation and annihilation operators by inverting the field expansion. We start from the field expansion for a fermionic (Dirac) field

$$\psi(x) = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(e^{-ip\cdot x} u(p,\lambda) b_{\boldsymbol{p},\lambda} + e^{ip\cdot x} v(p,\lambda) d^\dagger_{\boldsymbol{p},\lambda}\right), \tag{42}$$

where the energy is $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and is assumed positive. λ is the helicity index. Here we will work in the Dirac-Pauli representation where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \text{and } \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{43}$$

1) First, we need to prove some useful relations for Dirac 4-spinors. Show that properly normalized 4-spinors satisfy the relations

$$u(\boldsymbol{p},\lambda)^\dagger u(\boldsymbol{p},\lambda') = 2E_{\boldsymbol{p}}\delta_{\lambda,\lambda'}, \tag{44}$$

$$v(\boldsymbol{p},\lambda)^\dagger v(\boldsymbol{p},\lambda') = 2E_{\boldsymbol{p}}\delta_{\lambda,\lambda'}, \tag{45}$$

$$u(\boldsymbol{p},\lambda)^\dagger v(-\boldsymbol{p},\lambda') = 0, \tag{46}$$

$$v(\boldsymbol{p},\lambda)^\dagger u(-\boldsymbol{p},\lambda') = 0 \tag{47}$$

2) Now show that the nice inversion formulas for the creation and annihilation operators are

$$b_{\boldsymbol{p},\lambda} = \int \frac{d^3x}{\sqrt{2E_{\boldsymbol{p}}}} e^{ip\cdot x} u(\boldsymbol{p},\lambda)^\dagger \psi(x) \tag{48}$$

$$d^\dagger_{\boldsymbol{p},\lambda} = \int \frac{d^3x}{\sqrt{2E_{\boldsymbol{p}}}} e^{-ip\cdot x} v(\boldsymbol{p},\lambda)^\dagger \psi(x) \tag{49}$$

3) Let us now assume that $\psi(x)$ describes a fermionic field and that we want canonical anti-commutation relations, i.e. at equal time

$$\{\psi_\alpha(\boldsymbol{x},t), \psi^\dagger_\beta(\boldsymbol{x}',t)\} = \delta_{\alpha,\beta}\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{50}$$

$$\{\psi_\alpha(\boldsymbol{x},t), \psi_\beta(\boldsymbol{x}',t)\} = 0, \tag{51}$$

$$\{\psi^\dagger_\alpha(\boldsymbol{x},t), \psi^\dagger_\beta(\boldsymbol{x}',t)\} = 0. \tag{52}$$

Show that these relations imply that

$$\left\{b_{\boldsymbol{p},\lambda}, b^\dagger_{\boldsymbol{p}',\lambda'}\right\} = (2\pi)^3\delta_{\lambda,\lambda'}\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{53}$$

$$\left\{d_{\boldsymbol{p},\lambda}, d^\dagger_{\boldsymbol{p}',\lambda'}\right\} = (2\pi)^3\delta_{\lambda,\lambda'}\delta(\boldsymbol{p}-\boldsymbol{p}'), \tag{54}$$

$$\left\{b_{\boldsymbol{p},\lambda}, b_{\boldsymbol{p}',\lambda'}\right\} = 0, \tag{55}$$

$$\left\{d_{\boldsymbol{p},\lambda}, d_{\boldsymbol{p}',\lambda'}\right\} = 0, \tag{56}$$

$$\left\{b_{\boldsymbol{p},\lambda}, d_{\boldsymbol{p}',\lambda'}\right\} = 0, \tag{57}$$

$$\left\{b_{\boldsymbol{p},\lambda}, d^\dagger_{\boldsymbol{p}',\lambda'}\right\} = 0. \tag{58}$$

In this last part it is an advantage to write the Ψ operators and the u and v factors in component form. For instance, $u(p,\lambda)^\dagger\Psi(x) = \sum_{i=1}^4 (u(p,\lambda)^\dagger)_i \Psi(x)_i$ etc.

6 The Helicity Operator

In this problem we explore the helicity as an operator, first in terms of the spinors, and then in terms of the quantized fields. The gamma matrices in the Dirac-Pauli representation are

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \text{and}\, \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{59}$$

The solutions of the Dirac equation have the form

$$u_s(p) = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_s}{E+m} \end{pmatrix} \quad \text{and} \quad v_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi_{-s}}{E+m} \\ \chi_{-s} \end{pmatrix}, \tag{60}$$

for positive and negative energy, respectively. Here χ_s are 2-spinors with quantization axis along the direction of momentum $\boldsymbol{p}$ with projection $s = \pm 1$. Note the change of sign in the 2-spinor on the negative energy solution, $v_s(p)$.

The helicity operator can be defined through the spin operators appropriately generalized to four-spinors. We define

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \tag{61}$$

and the helicity operator can then be written $h = \boldsymbol{\Sigma} \cdot \boldsymbol{p}/|\boldsymbol{p}|$.

1) First we consider the problem of defining a proper four-vector for spin, s^μ. Since it is basically a spatial vector, we will normalize it such that $s^\mu s_\mu = -1$. As usual with such a (spin) polarization vector, a particularly convenient choice is orthogonal to the four-momentum, i.e. $s^\mu p_\mu = 0$. Argue that in the rest frame of the particle (assuming it has non-zero mass), we may take $s^\mu = (0, \hat{\boldsymbol{s}})$, where $\hat{\boldsymbol{s}}$ is any unit three-vector.

2) Argue that since we are interested in helicity, a good choice is $\hat{\boldsymbol{s}} = \boldsymbol{p}/|\boldsymbol{p}|$. Show that if we boost from the rest frame to the frame where the particle has momentum $\boldsymbol{p}$, then the spin four-vector becomes

$$s^\mu = \left(\frac{|\boldsymbol{p}|}{m}, \frac{E}{m} \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \right). \tag{62}$$

3) Show that $\boldsymbol{\Sigma} = \gamma^5 \gamma^0 \boldsymbol{\gamma}$ and prove the useful relation

$$\gamma^5 \not{s} = \boldsymbol{\Sigma} \cdot \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \frac{\not{p}}{m}. \tag{63}$$

4) Show that $u_s(p)$ and $v_s(p)$ are eigenstates of $\gamma^5 \not{s}$, i.e.

$$\gamma^5 \not{s} u_s(p) = s u_s(p) \tag{64}$$
$$\gamma^5 \not{s} v_s(p) = s v_s(p). \tag{65}$$

5) Combining all the elements above show that they are also helicity eigenstates, i.e.

$$h u_s(p) = s u_s(p) \tag{66}$$
$$h v_s(p) = -s v_s(p). \tag{67}$$

Note the sign change on the negative energy solutions, it will be important momentarily. (Hint: The general relations $(\not{p} - m) u_s(p) = 0$ and $(\not{p} + m) v_s(p) = 0$ are very useful here.)

Now we are ready to consider the quantized fermionic fields, Eq. (42). To get to the helicity operator in second quantization, we are faced with the a problem. What we would like to do is to take the projection of the spin onto the direction of propagation. Intuitively, this is of the form $\boldsymbol{\Sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|$. Unfortunately, this can not be done by considering $\hat{\boldsymbol{\Sigma}}\cdot\hat{\boldsymbol{P}}$ and we have to be more careful. So let us consider the spin operator in second quantization projected along some direction given by a unit vector $\hat{\boldsymbol{n}}$, i.e. consider the following expression

$$\int d^3x\psi^\dagger(x)\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\psi(x). \tag{68}$$

6) Show that in terms of the number operators for particles and antiparticles, one finds

$$\begin{aligned}\int d^3x\psi^\dagger(x)\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}\psi(x) = \sum_{s,s'}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\big[&u_s(p)^\dagger\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}u_{s'}(p)b^\dagger_{\boldsymbol{p},s}b_{\boldsymbol{p},s'}\\ +u_s(p)^\dagger\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}v_{s'}(-p)e^{i2E_{\boldsymbol{p}}t}b^\dagger_{\boldsymbol{p},s}d^\dagger_{-\boldsymbol{p},s'} &+ v_s(p)^\dagger\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}u_{s'}(-p)e^{-i2E_{\boldsymbol{p}}t}d_{\boldsymbol{p},s}b_{-\boldsymbol{p},s'}\\ +v_s(p)^\dagger\boldsymbol{\Sigma}\cdot\hat{\boldsymbol{n}}v_{s'}(p)d_{\boldsymbol{p},s}d^\dagger_{\boldsymbol{p},s'}\Big]&. \end{aligned}\tag{69}$$

Now set $\hat{\boldsymbol{n}} = \boldsymbol{p}/|\boldsymbol{p}|$ and show that the helicity operator in second quantization is given by

$$\hat{h} = \sum_s\int\frac{d^3p}{(2\pi)^3}\left[sb^\dagger_{\boldsymbol{p},s}b_{\boldsymbol{p},s} + sd^\dagger_{\boldsymbol{p},s}d_{\boldsymbol{p},s}\right]. \tag{70}$$

7) Argue that if you had used χ_s instead of χ_{-s} in $v_s(p)$ in Eq. (60) above, then your second quantized helicity operator in Eq. (70) be inconsistent as it would have the wrong sign of helicity on the negative energy states.