TU EINDHOVEN, 2WAG0

Problems in Measure theory

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"What I don't like about measure theory is that you have to say "almost everywhere" almost everywhere"

– Kurt Friedrichs

Disclaimer:

This is a broad selection of exercises for the course *Measure, integration and probability theory*. They are meant to accompany the lecture notes and give you the opportunity to exercise. If you wish to have your solution checked, send it in LaTeX, and we will correct and polish it together, so that it can be featured in this notes in the "Solutions" part.

These collection of exercises are still in progress and they might contain small typos. If you see any or if you think that the statement of the problems is not yet crystal clear, feel free to drop a line. The most efficient way is to send an email to me, a.chiarini@tue.nl. All comments and suggestions will be greatly appreciated.

Contents

Part I.

Problems

1. Warming up

In this section we will review some of the basic set operations which will be much needed in the sequel.

Problem 1.1. Let A, B and C be sets, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 1.2. Let A, B be sets, show that $A \cap (A \cup B) = A$.

Problem 1.3. Let $A, B \subseteq \Omega$. We define the symmetric difference to be

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that $A \cup B$ is the disjoint union of $A \Delta B$ and $A \cap B$.

Problem 1.4. Let $A, B \subseteq \Omega$, show that

$$\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B).$$

Problem 1.5. (De Morgan's law) Let I be any index set and let $\{A_i\}_{i \in I} \subseteq 2^{\Omega}$ be a family subsets of Ω. Show that

$$\Omega \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \Omega \setminus A_i.$$

Problem 1.6. Let $f : \Omega \to E$ be some function. Recall that for any $D \subseteq \Omega$ the *image* of D under f is the set

$$f(D) = \{f(x) : x \in D\},$$

Let $A, B \subseteq \Omega$. Show that

- $f(A \cap B) \subseteq f(A) \cap f(B)$,
- $f(A \cup B) = f(A) \cup f(B)$.

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$. Is it true that $f(\Omega \setminus A) = E \setminus f(A)$?

Problem 1.7. Let $f : \Omega \to E$ be some function. Recall that for any $F \subseteq E$ the *inverse image* of F under f is the set

$$f^{-1}(F) = \{x : f(x) \in F\}.$$

Let $H, K \subseteq E$. Show that, taking the inverse image commutes with the set operations:

- $f^{-1}(H \cap K) = f^{-1}(H) \cap f^{-1}(K)$,

- $f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$,
- $f^{-1}(E \setminus H) = \Omega \setminus f^{-1}(H)$.

Problem 1.8. Let $f : \Omega \to E$ be some function.

- Let $A \subseteq \Omega$. Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.
- Let $H \subseteq E$. Is it true that $f(f^{-1}(H)) = H$? Provide a proof or a counterexample.

Problem 1.9. Recall that given a set Ω, 2^Ω denotes the set of all subsets of Ω. Suppose $\Omega = \{0, 1\}$, list all the elements of 2^Ω. What is $|2^\Omega|$, where $|\cdot|$ denotes the number of elements of a set? Suppose that $|\Omega| < \infty$, what is $|2^\Omega|$ in this case?

Problem 1.10. Let $\{a_i\}_{i\in I} \subseteq [0, \infty]$, where I is an arbitrary (index) set. Recall that their sum is defined by

$$\sum_{i\in I} a_i = \sup\Big\{\sum_{i\in K} a_i : \ K \subseteq I, \ K \text{ finite}\Big\}.$$

Now, suppose that $I = \mathbb{N}$. Show that the above definition agrees with the standard one, that is

$$\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Show that the value of the series does not depend on the ordering of the elements in the sequence. That is, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

Problem 1.11. Let $\{a_i\}_{i\in I} \subseteq [0, \infty)$, where I is an arbitrary (index) set. Suppose that

$$\sum_{i\in I} a_i < \infty.$$

Show that the set $J_n = \{i \in I : \ a_i > 1/n\}$ is finite. Conclude that the set of $i \in I$ such that $a_i > 0$ is at most countable.

Problem 1.12. Let $\{a_i\}_{i\in I} \subseteq (0, \infty)$ be a family of *positive* real numbers, where I is an (index) set with uncountably many elements. Show that

$$\sum_{i\in I} a_i = \infty.$$

Problem 1.13. (*) Let Ω be a non-empty set and $p_\omega \in [0, 1]$, $\omega \in \Omega$ be real numbers such that

$$\sum_{\omega\in\Omega} p_\omega = 1.$$

Define the set function $\mathbb{P} : 2^\Omega \to [0, 1]$ by

$$\mathbb{P}(A) = \sum_{\omega\in A} p_\omega.$$

Show that $\mathbb{P}$ is a measure on 2^Ω.

Problem 1.14. (*) Let $A \subset \mathbb{R}$ be an open set. Show that A is the union of at most countable many intervals. (*Hint:* define for all $x \in A$ the interval $I_x = \bigcup_{I \text{ interval}: x \in I \subseteq A} I$ to be the largest interval contained in A containing x)

Problem 1.15. Let I and J be two index sets and $a_{i,j}$, $i \in I$ and $j \in J$ be non-negative real numbers. Show that

$$\sum_{i \in I} \sum_{j \in J} a_{i,j} = \sum_{j \in J} \sum_{i \in I} a_{i,j}.$$

2. Measurable sets and σ-algebras

Problem 2.1. Show that there is no σ-algebra with an odd number of elements.

Problem 2.2. Let $(\Omega, \mathscr{F})$ be a measurable space and $A, B \in \mathscr{F}$. Show, starting from the definition of σ-algebra, that $A \cup B$, $A \cap B$, $A \setminus B$, and $A \Delta B$ all belong to $\mathscr{F}$.

Problem 2.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $A_1, A_2, \ldots$ be a sequence of sets in $\mathscr{F}$. Define the following sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Show that:

- $(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c$,
- $\liminf_{n\to\infty} A_n \in \mathscr{F}$ and $\limsup_{n\to\infty} A_n \in \mathscr{F}$,
- $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.
- $\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n$ for infinitely many $n\}$.
- $\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \exists m \in \mathbb{N}$ such that $\omega \in A_n$ for all $n \geq m\}$.

Problem 2.4. Let Ω, E be non-empty, $\mathscr{G}$ a σ-algebra on E and $f : \Omega \to E$. Show that

$$\mathscr{F} = \{f^{-1}(B) : B \in \mathscr{G}\},$$

is a σ-algebra on Ω.

Problem 2.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $(A_n)_{n\in\mathbb{N}}$ a collection of sets in $\mathscr{F}$. Show that:

- There are $(E_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ mutually disjoint such that $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} E_n$.
- There are $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

Problem 2.6. Let Ω be a non-empty set and $\mathscr{F}$ a non-empty collection of subsets of Ω which is closed under taking complements and finite unions (such a collection is called an *algebra*). Show that $\mathscr{F}$ is a σ-algebra if and only if it is closed under countable increasing unions (i.e., if $\{A_n\} \subseteq \mathscr{F}$ and $A_1 \subseteq A_2 \subseteq \ldots$, then $\cup_{n\in\mathbb{N}} A_n \in \mathscr{F}$).

Problem 2.7. (Restriction of σ-algebra) Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Suppose that $A \subseteq \Omega$ is non-empty. Show that

$$\mathscr{F}_A = \{B \cap A : B \in \mathscr{F}\}$$

is a σ-algebra on A.

Problem 2.8. (Extension of σ-algebra) Let $(\Omega, \mathscr{F})$ be a measurable space, and let K be some non-empty set such that $\Omega \cap K = \varnothing$. Define $\overline{\Omega} = \Omega \cup K$ and $\overline{\mathscr{F}} = \sigma(\mathscr{F} \cup K)$ be a σ-algebra on $\overline{\Omega}$. Show that $\overline{\mathscr{F}} = \{A \subseteq \overline{\Omega} : A \cap \Omega \in \mathscr{F}\}$.

Problem 2.9. Let Ω be a infinite non-empty set.

- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.
- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.

Problem 2.10. Let $\mathscr{F}$ and $\mathscr{G}$ be σ-algebras on Ω. Show that $\mathscr{F} \cap \mathscr{G}$ is a σ-algebra. Prove or disprove whether $\mathscr{F} \cup \mathscr{G}$ is in general a σ-algebra.

Problem 2.11. Let $\mathscr{F}_n$, $n \in \mathbb{N}$ be σ-algebras on Ω such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for all $n \in \mathbb{N}$ (such a sequence $\{\mathscr{F}_n\}$ is called a *filtration*).

- Show that $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is an algebra.
- Is $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ a σ-algebra? Consider $\Omega = \mathbb{N}$ and $\mathscr{F}_n = \sigma(\{A : A \subseteq \mathbb{N} \cap \{1, \ldots n\}\})$.

Problem 2.12. Let $\mathscr{E} \subseteq \mathscr{A}$ be two collections of sets. Show that $\sigma(\mathscr{E}) \subseteq \sigma(\mathscr{A})$.

Problem 2.13. (Product sigma algebra) Let Ω_1 and Ω_2 be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be σ-algebras on Ω_1 and Ω_2 respectively. Consider the *product σ-algebra* on $\Omega_1 \times \Omega_2$

$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}).$$

Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Show that $\mathscr{F}_1 \otimes \mathscr{F}_2$ is generated by $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$.

Problem 2.14. Show that the Borel σ-algebra $\mathscr{B}_{\mathbb{R}^2}$ equals $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$.

Problem 2.15. Show that the Borel σ-algebra on $\mathbb{R}$ is generated by each of the following:

i. the open intervals: $\mathscr{A}_1 = \{(a, b) : a < b\}$,

ii. the closed intervals: $\mathscr{A}_2 = \{[a, b] : a < b\}$,

iii. the half open intervals $\mathscr{A}_3 = \{[a, b) : a < b\}$ or $\mathscr{A}_4 = \{(a, b] : a < b\}$,

iv. the open rays: $\mathscr{A}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,

v. the closed rays: $\mathscr{A}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.16. Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Also recall that

$$\mathscr{B}_{\overline{\mathscr{B}}} = \{A \subseteq \overline{\mathscr{B}} : A \cap \mathbb{R} \in \mathscr{B}_{\mathbb{R}}\}$$

is a σ-algebra on $\overline{\mathbb{R}}$. Show that $\mathscr{B}_{\overline{\mathscr{B}}}$ is generated by the family of closed rays $\mathscr{A} = \{[-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.17. Let $\mathscr{F}$ be an infinite σ-algebra.

- Show that $\mathscr{F}$ contains an infinite sequence of disjoint sets.
- (*) Show that $\mathrm{Card}(\mathscr{F}) \geq \mathrm{Card}([0, 1])$.
  (*Hint:* think about binary representation of numbers in $[0, 1]$).

Problem 2.18. Show that Λ is a λ-system on Ω if and only if

I. $\Omega \in \Lambda$,

II. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,

III. if $A_1, A_2, \ldots$ is a sequence of subsets in Λ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

Problem 2.19. Let Λ be a λ-system. Show that $\varnothing \in \Lambda$.

Problem 2.20. Let $\mathscr{A}$ be both a λ-system and a π-system. Show that $\mathscr{A}$ is a σ-algebra.

3. Measures

Problem 3.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ such that $E \subseteq F$, one has $\mu(E) \leq \mu(F)$.

Problem 3.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ then

$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F).$$

Conclude that $\mu(E \cup F) \leq \mu(E) + \mu(F)$.

Problem 3.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, \infty]$ be a set function. Assume that $\mu(\varnothing) = 0$, μ is finitely additive and continuous from below. Show that μ is a measure.

Problem 3.4. Consider a non-empty uncountable set Ω and the σ-algebra

$$\mathscr{F} = \{A \subseteq \Omega : \ A \text{ is countable or } \Omega \setminus A \text{ is countable}\}.$$

We define the set function $\mu : \mathscr{F} \to [0, \infty]$ by $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if $\Omega \setminus E$ is countable. Show that μ is a measure on $(\Omega, \mathscr{F})$.

Problem 3.5. Let Ω be an infinite set and $\mathscr{F} = 2^{\Omega}$. Define μ on $\mathscr{F}$ by $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is not finite. Show that μ is finitely additive but not a measure.

Problem 3.6. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, 1]$ be an additive set function, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$ such that $A \cap B = \varnothing$. Show that $\mu(\varnothing) = 0$.

Problem 3.7. (Inclusion-exclusion) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $A_1, A_2, A_3 \in \mathscr{F}$. Show that

$$\begin{aligned}\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) \\ - \mu(A_1 \cap A_2) - \mu(A_2 \cap A_3) - \mu(A_3 \cap A_1) + \mu(A_1 \cap A_2 \cap A_3).\end{aligned}$$

Let $A_1, \ldots, A_n \in \mathscr{F}$. Show that

$$\mu\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{j=1}^{n} (-1)^{j-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=j}} \mu\Big(\bigcap_{i \in I} A_i\Big).$$

Problem 3.8. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space.

- If $E, F \in \mathscr{F}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

- Define $\rho(E,F) = \mu(E\Delta F)$ for all $E, F \in \mathscr{F}$. Show that $\rho(E,F) \le \rho(E,G) + \rho(G,F)$ for all $E, F, G \in \mathscr{F}$.

Problem 3.9. If $\mu_1, \dots, \mu_n$ are measures on a measurable space $(\Omega, \mathscr{F})$, and $a_1, \dots, a_n$ are non-negative real numbers, then $\mu := \sum_{i=1}^n a_i \mu_i$ is a measure on $\mathscr{F}$. Moreover μ is σ-finite if μ_i is σ-finite for all $i = 1, \dots, n$.

Problem 3.10. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra. Show that $(\Omega, \mathscr{G}, \mu)$ is a measure space.

Problem 3.11. Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \mu((-\infty, x]), \quad \forall x \in \mathbb{R}.$$

Show that F is non-decreasing, right-continuous, and

$$\lim_{x\to-\infty} F(x) = 0, \quad \lim_{x\to\infty} F(x) = \mu(\mathbb{R}).$$

Problem 3.12. Find a measure space $(\Omega, \mathscr{F}, \mu)$ and a decreasing sequence $B_1 \supseteq B_2 \supseteq \ldots \in \mathscr{F}$ such that $\lim_{n\to\infty} \mu(B_n) > \mu(\cap_{n\in\mathbb{N}} B_n)$.

Problem 3.13. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $A \in \mathscr{F}$. Show that the set function $\mu_A : \mathscr{F} \to [0, \infty]$ defined by $\mu_A(B) := \mu(B \cap A)$ is a measure on $(\Omega, \mathscr{F})$.

Problem 3.14. Let $(\Omega, \mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0, \infty]$ be a set function which is finitely additive and such that $\mu(\varnothing) = 0$. Show that μ is a measure on $(\Omega, \mathscr{F})$ if and only if μ is continuous from below.

Problem 3.15. Let $(\Omega, \mathscr{F})$ be a measure space $\mu : \mathscr{F} \to [0, \infty]$ be a set function which is finitely additive and such that $\mu(\Omega) < \infty$. Show that μ is a measure on $(\Omega, \mathscr{F})$ if and only if μ is continuous from above.

Problem 3.16. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. Recall that

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} A_m.$$

Show that

$$\limsup_{n\to\infty} \mu(A_n) \le \mu\Big(\limsup_{n\to\infty} A_n\Big), \qquad \liminf_{n\to\infty} \mu(A_n) \ge \mu\Big(\liminf_{n\to\infty} A_n\Big).$$

Problem 3.17. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $\mathscr{E} \subseteq \mathscr{F}$ be a π-system such that $\sigma(\mathscr{E}) = \mathscr{F}$ and there exists $E_1, E_2, \ldots \in \mathscr{E}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. Show that μ is uniquely determined by its values on $\mathscr{E}$.

Problem 3.18. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ associated to F. Show that $\nu_F(\{x\}) = F(x) - F(x-)$ where we define

$$F(x-) := \lim_{y\uparrow x} F(y).$$

Conclude that it if F is continuous, then $\nu_F(\mathbb{Q}) = 0$.

Problem 3.19. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\lambda((a,b]) = b - a$ for all $a < b$. Show that λ is translation invariant, that is $\lambda(A + x) = \lambda(A)$ for all $A \in \mathscr{B}_\mathbb{R}$ and all $x \in \mathbb{R}$, where we write $A + x := \{a + x : a \in A\}$ for the translation of A by x. (*Hint:* a solution can be obtained with the $\pi - \lambda$ theorem.)

Problem 3.20. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\lambda((a,b]) = b-a$ for all $a < b$. Show that $\lambda(\tau A) = |\tau|\lambda(A)$ for all $A \in \mathscr{B}_\mathbb{R}$ and all $\tau \neq 0$, where we write $\tau A := \{\tau a : a \in A\}$ for the dilation of A by τ.

Problem 3.21. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\nu_F((a,b]) = F(b) - F(a)$ for all $a < b$. Show that ν_F is σ-finite.

Problem 3.22. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $(A_i)_{i\in\mathbb{N}} \subseteq \mathscr{F}$ be a sequence such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{i\in\mathbb{N}} A_i) = 1$.

Problem 3.23. Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$. Show that the set $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is at most countable.

4. Null sets, completion and independence

Problem 4.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Suppose that $A, N \in \mathscr{F}$ and $\mu(N) = 0$. Show that $\mu(A \cup N) = \mu(A)$.

Problem 4.2. Suppose that $(\Omega, \mathscr{F}, \mu)$ is a measure space and let N be a null set. Show that for all $M \subseteq N$, M is a null set.

Problem 4.3. Let $(N_n)_{n\in\mathbb{N}}$ be null sets in a measure space $(\Omega, \mathscr{F}, \mu)$. Show that $\cup_{n\in\mathbb{N}} N_n$ is a null set.

Problem 4.4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of function. Show that $f_n = 0$ almost everywhere for all $n \in \mathbb{N}$ if and only if almost everywhere $f_n = 0$ for all $n \in \mathbb{N}$. (Careful with the quantifiers!)

Problem 4.5. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability, we say that $A \in \mathscr{F}$ happens almost surely if $\Omega \setminus A$ is a null set for $\mathbb{P}$.

- Show that $A \in \mathscr{F}$ happens almost surely if and only if $\mathbb{P}(A) = 1$.
- Assume now that $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ is such that A_n happens almost surely for all $n \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{n\in\mathbb{N}} A_n) = 1$.

Problem 4.6. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and let V be the Vitali set. Show that if $E \subseteq V$, then E is a null-set.

Problem 4.7. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that $\overline{A} \in \overline{\mathscr{F}}$ if and only if there is $A \in \mathscr{F}$ such that $A \Delta \overline{A}$ is a null set.

Problem 4.8. (*) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $\mathscr{L}(E) > 0$. Show that there exists $N \subseteq E$ not Lebesgue measurable. (Hint: assume first $E \subseteq (0, 1)$ and look at $V \cap E$ where V is the Vitali set.)

Problem 4.9. (The Cantor set) The Lebesgue null sets include not only the countable sets but also many sets having the cardinality of the continuum. The Cantor set C is the set of all $x \in [0, 1]$ that have a base-3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{with } a_j \in \{0, 2\} \text{ for all } j \in \mathbb{N}.$$

Thus C is obtained from $[0, 1]$ by removing the open middle third $(1/3, 2/3)$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals and so forth. Show that

- C is compact and with zero Lebesgue measure.

- $\mathrm{Card}(C) = \mathrm{Card}([0,1])$. Hint: consider the so called Cantor function, for $x \in C$, $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$, define
$$f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}, \quad b_j = a_j/2.$$

- (*) Show that C has empty interior and is totally disconnected (that is for all $x < y \in C$ there is $z \in (x,y)$ such that $z \notin C$). Moreover C has no isolated points.

Problem 4.10. Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$ and any real number $\lambda \in \mathbb{R}$, $\mathscr{L}(E+\lambda) = \mathscr{L}(E)$ and $\mathscr{L}(\lambda E) = |\lambda|\mathscr{L}(E)$.

Problem 4.11. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. We say that two σ-algebras $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{F}$ are independent if

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{A}_1, \forall A_2 \in \mathscr{A}_2.$$

Suppose that $\mathscr{E}_1, \mathscr{E}_2$ are π-systems generating $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Show that $\mathscr{A}_1$ and $\mathscr{A}_2$ are independent if and only if

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{E}_1, \forall A_2 \in \mathscr{E}_2.$$

Problem 4.12. Are the following true of false?

- If A is an open subset of $[0,1]$, then $\mathscr{L}^1(A) = \mathscr{L}^1(\overline{A})$, where $\overline{A}$ is the closure of the set.

- If A is a subset of $[0,1]$ such that $\mathscr{L}^1(\mathrm{int}(A)) = \mathscr{L}^1(\overline{A})$, then A is measurable. Here $\mathrm{int}(A)$ denotes the interior of the set A.

Problem 4.13. Show that if $A \subset [0,1]$ and $\mathscr{L}^1(A) > 0$, then there are x and y in A such that $|x-y|$ is an irrational number.

Problem 4.14. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that a set $A \subseteq \Omega$ is a μ-null set if and only if A is a $\overline{\mu}$-null set.

5. Measurable functions

Problem 5.1. Let $(\Omega, \mathscr{F})$ be a measure space. Let $A \subseteq \Omega$ and consider the indicator function $\mathbb{1}_A : \Omega \to \mathbb{R}$, defined by

$$\mathbb{1}_A(\omega) = 1, \text{ if } \omega \in A, \quad \mathbb{1}_A(\omega) = 0, \text{ if } \omega \notin A.$$

Show that $\mathbb{1}_A$ is $(\mathscr{F}, \mathscr{B})$-measurable if and only if $A \in \mathscr{F}$.

Problem 5.2. Let $(\Omega, \mathscr{F})$ be a measure space and $A, B \in \mathscr{F}$. Show that

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B}, \qquad \mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B.$$

Problem 5.3. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $X, Y : \Omega \to \mathbb{R}$ be random variables. Recall that X and Y are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \qquad \forall A \in \sigma(X), \forall B \in \sigma(Y).$$

Show that X and Y are independent if and only if

$$\mathbb{P}(X \geq s, Y \geq t) = \mathbb{P}(X \geq s)\mathbb{P}(Y \geq t), \qquad \forall s, t \in \mathbb{R}.$$

(Hint: you might need the $\pi - \lambda$ Theorem.)

Problem 5.4. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be $\mathscr{F}$-measurable functions. Show that $\max(f, g)$, $\min(f, g)$ and $|f|$ are measurable.

Problem 5.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $E_1, \dots, E_n \in \mathscr{F}$ be measurable sets. Show that if $a_1, \dots, a_n$ are real numbers, then

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}$$

is $\mathscr{F}$-measurable.

Problem 5.6. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$, and $Y = f^{-1}(\mathbb{R})$. Show that f is measurable if and only if $Y \in \mathscr{F}$, $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted to Y.

Problem 5.7. Let $(\Omega, \mathscr{F})$ be a measurable space and f_i, $i \in \mathbb{N}$ be measurable functions from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = 0\}$$

is measurable, that is, it belongs to $\mathscr{F}$. (Hint: recall $\lim_{n\to\infty} f_n(\omega) = 0$ means that for all $m > 0$ there exists $N > 0$ such that $|f_n(\omega)| \leq 1/m$ for all $n \geq N$. Can you now write the set as countable union and intersections of measurable sets.)

Problem 5.8. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be two measurable functions. Show that $\{\omega \in \Omega : f(\omega) = g(\omega)\}$ is a measurable set.

Problem 5.9. Let $(\Omega, \mathscr{F})$ be a measurable space and (f_n) be a sequence of $\mathscr{F}$-measurable functions. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$$

is measurable.

Problem 5.10. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Show that f is Lebesgue measurable if and only if there exists a Borel measurable function g such that $f \equiv g$ almost everywhere.

Problem 5.11. Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel-measurable.

Problem 5.12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an homeomorphism, that is f is continuous with continuous inverse. Show that f maps Borel measurable sets to Borel measurable sets.

Problem 5.13. Let $(\Omega, \mathscr{F}, \mu)$ be a complete measure space. Show that

- if f is measurable and $f \equiv g$ μ-a.e. then g is measurable.
- if $f_n, n \in \mathbb{N}$ are measurable and $f_n \to f$ μ-a.e. then f is measurable.

Problem 5.14. Let $(\Omega, \mathscr{F})$ be a measurable space and $f : \Omega \to \overline{\mathbb{R}}$ be a function. Define $f^+(\omega) = \max(f(\omega), 0)$ and $f^-(\omega) = -\min(f(\omega), 0)$. Show that f is measurable if and only if f^+ and f^- are measurable.

Problem 5.15. Let $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{G})$ be two measurable spaces. Let $E \in \mathscr{F} \otimes \mathscr{G}$, and for $y \in \mathscr{Y}$ define the y-section of E to be

$$E_y = \{x \in \mathscr{X} : (x, y) \in E\},$$

Similarly define for $x \in \mathscr{X}$ the x-section of E to be

$$E_x = \{y \in \mathscr{Y} : (x, y) \in E\}.$$

Show that E_y is measurable for all $y \in \mathscr{Y}$.

Problem 5.16. Let $(\Omega, \mathscr{F})$ and $(E, \mathscr{G})$ be measurable spaces, and let $f : \Omega \to E$ be a function. Suppose that $A \in \mathscr{F}$, we say that f is measurable on A if $f^{-1}(B) \cap A \in \mathscr{F}$ for all $B \in \mathscr{G}$. Show that f is measurable on $A \in \mathscr{F}$ if and only if $f|_A$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, where $\mathscr{F}_A = \{A \cap B : B \in \mathscr{F}\}$. Show that if f is $(\mathscr{F}, \mathscr{G})$-measurable, then f is measurable on A for all $A \in \mathscr{F}$.

Problem 5.17. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$ and $\mathscr{Y} = f^{-1}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted on $\mathscr{Y}$.

Problem 5.18. If a function $Y_A : A \to E$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, and $p \in E$, then the extension Y defined by

$$Y(\omega) := \begin{cases} Y_A(\omega), & \omega \in A, \\ p, & \omega \notin A, \end{cases}$$

is $(\mathscr{F}, \mathscr{G})$-measurable.

Problem 5.19. Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable?

Problem 5.20. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere, then f is Lebesgue-measurable.

Problem 5.21. Is the following true of false? If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

Problem 5.22. (Convergence in measure/probability) Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ be random variables. We say that X_n converges in probability to a random variable X, if for all $\epsilon > 0$

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \qquad \text{as } n \to \infty.$$

Show that if X_n converges to a random variable X in probability, then there exists a subsequence of $(X_n)_{n\in\mathbb{N}}$ which converges almost surely to X. (*Hint: use Borell-Cantelli Lemma.)*

6. Integration

Problem 6.1. Consider the measure space $(\mathscr{X}, 2^{\mathscr{X}}, \delta_x)$ where δ_x is the delta measure at $x \in \mathscr{X}$. Let $f : \mathscr{X} \to \overline{\mathbb{R}}$ be a non-negative measurable function. Show that

$$\int_{\mathscr{X}} f \,\mathrm{d}\delta_x = f(x).$$

Problem 6.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $A \in \mathscr{F}$, and $f : \Omega \to \overline{\mathbb{R}}$ be a non-negative function. We say that f is measurable on A if $f|_A$ is $\mathscr{F}_A$-measurable (recall that $\mathscr{F}_A = \{A \cap B : B \in \mathscr{F}\}$). Show that $f\mathbb{1}_A$ is $\mathscr{F}$ measurable and also

$$\int_{\Omega} f\mathbb{1}_A \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu_A,$$

where μ_A is the restriction of μ to $\mathscr{F}_A$.

Problem 6.3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be a non-negative measurable function

- Let $A, B \in \mathscr{F}$ be such that $A \subseteq B$. Show that

$$\int_A f \,\mathrm{d}\mu \leq \int_B f \,\mathrm{d}\mu.$$

- Let $A, B \in \mathscr{F}$ be such that $A \cap B = \varnothing$. Show that

$$\int_{A \cup B} f \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu + \int_B f \,\mathrm{d}\mu.$$

Problem 6.4. Suppose that $(f_n) \subseteq L^+$, $f_n \to f$ pointwise and $\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu < \infty$. Show, without the dominated convergence theorem, that for all $A \in \mathscr{F}$

$$\lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu = \int_A f \,\mathrm{d}\mu.$$

Show that the conclusion might fail if $\int f \,\mathrm{d}\mu = \infty$.

Problem 6.5. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Suppose that f is integrable with respect to μ. Show that f is integrable with respect to $\overline{\mu}$ and in particular

$$\int_{\Omega} f \,\mathrm{d}\mu = \int_{\Omega} f \,\mathrm{d}\overline{\mu}.$$

(*Hint: start with simple functions.*)

Problem 6.6. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose that (f_n) is a sequence of measurable functions such that $f_1 \geq f_2 \geq \ldots \geq 0$. Show using the MCT that if $f := \lim_{n\to\infty} f_n$ and $\int_\Omega f_1 < \infty$, then

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

Show that if $\int_\Omega f_1 \, \mathrm{d}\mu = \infty$ the above conclusion might not hold.

Problem 6.7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f \in L^+$. Show that if $\int f \, \mathrm{d}\mu < \infty$, then

$$\lim_{n\to\infty} \int_\Omega f \mathbb{1}_{\{f \geq n\}} \, \mathrm{d}\mu = 0.$$

Problem 6.8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and assume that $f : \Omega \to \overline{\mathbb{R}}$ is integrable with respect to μ. Show that $|f| < \infty$ μ-almost everywhere.

Problem 6.9. Suppose that f is a Lebesgue-integrable function on $\mathbb{R}$. Show that for all $z \in \mathbb{R}$

$$\int_\mathbb{R} f(x) \, \mathrm{d}\mathcal{L}^1(x) = \int_\mathbb{R} f(x+z) \, \mathrm{d}\mathcal{L}^1(x).$$

Show that for all $\tau \neq 0$

$$\int_\mathbb{R} f\Big(\frac{x}{\tau}\Big) \, \mathrm{d}\mathcal{L}^1(x) = |\tau| \int_\mathbb{R} f(x) \, \mathrm{d}\mathcal{L}^1(x).$$

Problem 6.10. Assume Fatou's lemma and deduce the Monotone Convergence Theorem.

Problem 6.11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and assume that f is integrable with respect to μ and g is a measurable function such that $f = g$ μ-almost everywhere. Show that g is integrable with respect to μ and

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega g \, \mathrm{d}\mu.$$

Problem 6.12. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f be integrable with respect to μ. Show that the set $\{\omega \in \Omega : \; f(\omega) \neq 0\}$ is a σ-finite set.

Problem 6.13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and f, g be integrable with respect to μ. Then

$$\int_A f \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu, \qquad \forall A \in \mathcal{F},$$

if and only if $f = g$ μ-almost everywhere, and if and only if $\int_\Omega |f - g| \, \mathrm{d}\mu = 0$.

Problem 6.14. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measure spaces. Let $f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$, $f \geq 0$ be a $\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable function. Show that for all $y \in \Omega_2$ the map $f_y : \Omega_1 \to \overline{\mathbb{R}}$, defined by $f_y(x) = f(x, y)$ for all $x \in \Omega_1$, is $\mathcal{F}_1$-measurable.
(*Hint:* use approximations with simple functions.)

Problem 6.15. Let f be Lebesgue-integrable on the real line. Show that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{[-n,n]} f \, \mathrm{d}\mathscr{L}^1 = 0.$$

Show that the result need not be true if f is not assumed to be integrable on $\mathbb{R}$.

Problem 6.16. Let μ be a measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ concentrated on $\mathbb{Z}$. This means that $\mu(B) = \mu(B \cap \mathbb{Z})$ for all $B \in \mathscr{B}_{\mathbb{R}}$, or equivalently $\mu = \sum_{n\in\mathbb{Z}} \mu_n \delta_n$, where μ_n are non-negative real numbers and δ_n is the Dirac measura at $n \in \mathbb{Z}$. Show that $\mu_n = \mu(\{n\})$ for all $n \in \mathbb{Z}$ and that for all f non-negative and measurable

$$\int_{\mathbb{R}} f \, \mathrm{d}\mu = \sum_{n\in\mathbb{Z}} \mu_n f(n).$$

Problem 6.17. Consider the functions

$$f_1(x) = \begin{cases} +\infty, & \text{if } x = 0, \\ \log|x|, & \text{if } 0 < |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{x^2-1}, & \text{if } |x| \neq 0, \\ 20, & \text{if } |x| = 0, \end{cases} \qquad f_3(x) \equiv 1.$$

Determine if these functions are integrable on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ in each of the following two cases, and if possible compute the value of the integral

a) $m = \mathscr{L}^1$ is the Lebesgue measure,

b) m is defined by

$$m(B) = \sum_{n\in\mathbb{Z}} \frac{1}{1+(n+1)^2} \delta_n(B)$$

for all $B \in \mathscr{B}_{\mathbb{R}}$.

Problem 6.18. Let $f, g : [0,1] \to \mathbb{R}$ be two non-negative integrable functions. Show that fg is not necessarily integrable.

Problem 6.19. Let f be integrable on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ with respect to the Lebesgue measure.

a) Show that for all $\epsilon > 0$ there exists a simple function $g = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$ with $A_1, \ldots, A_k$ bounded intervals, such that $\int_{\mathbb{R}} |f - g| \, \mathrm{d}\mathscr{L} < \epsilon$.

b) Show that all $\epsilon > 0$ there exists a bounded continuous function h such that $\int_{\mathbb{R}} |f-h| \, \mathrm{d}\mathscr{L} < \epsilon$.

Problem 6.20. Determine the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} \, \mathrm{d}x, \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, \mathrm{d}x.$$

Problem 6.21. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $(f_n)_n$ be a sequence of measurable functions such that $f_n \to f$ uniformly on Ω. Show that if f_n is integrable for all $n \in \mathbb{N}$ then f is also integrable and

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu.$$

Recall that f_n converges uniformly to f on Ω if $\lim_{n\to\infty} \sup_{\omega\in\Omega} |f_n(\omega) - f(\omega)| = 0$. Show that the result is in general false if $\mu(\Omega) = \infty$.

Problem 6.22. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. Show that the map $x \mapsto \int_{[-\infty,x]} f \,\mathrm{d}\mathscr{L}$ is continuous in $x \in \mathbb{R}$.

7. Littlewood's Principles

Problem 7.1. Prove that the Lebesgue measure on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$ is both inner regular and outer regular.

Problem 7.2. Let μ be an outer regular measure on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$. Suppose that μ is *inner regular on open sets*, i.e.

$$\mu(O) = \sup \left(\mu(K) \ : \ K \subset O, \ K \text{ is compact}\right)$$

for every open $O \in \mathscr{B}_{\mathbb{R}^n}$. Show that μ is inner regular on every $B \in \mathscr{B}_{\mathbb{R}^n}$ if $\mu(B) < \infty$.

Problem 7.3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that converges pointwise a.e. to f and for all $n \in \mathbb{N}$ $|f_n| \leq g$, where g is integrable. Prove that for any $\varepsilon > 0$ there exists $E \in \mathscr{F}$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Problem 7.4. Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions such that converges pointwise a.e. to f. Show that there exists a sequence of measurable sets (A_k), such that $\mu\left(\bigcap_k A_k^c\right) = 0$ and such that for any $k \in \mathbb{N}$ (f_n) converges uniformly to f on A_k.

Problem 7.5. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Use Egorov's theorem (and not Dominated convergence) to show the following *bounded convergence* theorem. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions such that converges pointwise a.e. to f, and assume that there is $M > 0$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$. Show that

$$\int_\Omega |f_n - f| \, \mathrm{d}\mu \to 0.$$

Problem 7.6. Let $\mathscr{L}_{(0,1)}$ be a restriction of the Lebesgue measure on $\mathbb{R}$ to $(0,1)$. Consider a function $f = \mathbb{1}_{\{(0,1)\cap\mathbb{Q}\}}$. Lusin's theorem asserts that for every $\varepsilon > 0$ there exists a compact set $K \subset (0,1)$ and a continuous function g such that $\mathscr{L}_{(0,1)}\left((0,1) \setminus K\right) < \varepsilon$ and $f = g$ on K. Find such g and K.

8. Product measures and Fubini-Tonelli

9. L^p spaces