TU Eindhoven, 2WAG0

Problems in Measure theory

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“What I don’t like about measure theory is that you have to say “almost everywhere” almost everywhere”

– Kurt Friedrichs

Disclaimer:

This is a broad selection of exercises for the course *Measure, integration and probability theory*. They are meant to accompany the lecture notes and give you the opportunity to exercise. If you wish to have your solution checked, send it in LaTeX, and we will correct and polish it together, so that it can be featured in this notes in the “Solutions” part.

These collection of exercises are still in progress and they might contain small typos. If you see any or if you think that the statement of the problems is not yet crystal clear, feel free to drop a line. The most efficient way is to send an email to me, a.chiarini@tue.nl. All comments and suggestions will be greatly appreciated.

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Part I.

Problems

1. Warming up

In this section we will review some of the basic set operations which will be much needed in the sequel.

Problem 1.1. Let A, B and C be sets, show that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Problem 1.2. Let A, B be sets, show that $A \cap (A \cup B) = A$.

Problem 1.3. Let $A, B \subseteq \Omega$. We define the symmetric difference to be

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Show that $A \cup B$ is the disjoint union of $A \Delta B$ and $A \cap B$.

Problem 1.4. Let $A, B \subseteq \Omega$, show that

$$\Omega \setminus (A \cup B) = (\Omega \setminus A) \cap (\Omega \setminus B).$$

Problem 1.5. (De Morgan's law) Let I be any index set and let $\{A_i\}_{i \in I} \subseteq 2^{\Omega}$ be a family subsets of Ω. Show that

$$\Omega \setminus \left(\bigcup_{i \in I} A_i \right) = \bigcap_{i \in I} \Omega \setminus A_i.$$

Problem 1.6. Let $f : \Omega \to E$ be some function. Recall that for any $D \subseteq \Omega$ the *image* of D under f is the set

$$f(D) = \{f(x) : \, x \in D\},$$

Let $A, B \subseteq \Omega$. Show that

- ▶ $f(A \cap B) \subseteq f(A) \cap f(B)$,
- ▶ $f(A \cup B) = f(A) \cup f(B)$.

Find an example where $f(A \cap B) \neq f(A) \cap f(B)$. Is it true that $f(\Omega \setminus A) = E \setminus f(A)$?

Problem 1.7. Let $f : \Omega \to E$ be some function. Recall that for any $F \subseteq E$ the *inverse image* of F under f is the set

$$f^{-1}(F) = \{x : \, f(x) \in F\}.$$

Let $H, K \subseteq E$. Show that, taking the inverse image commutes with the set operations:

- ▶ $f^{-1}(H \cap K) = f^{-1}(H) \cap f^{-1}(K)$,

- $f^{-1}(H \cup K) = f^{-1}(H) \cup f^{-1}(K)$,
- $f^{-1}(E \setminus H) = \Omega \setminus f^{-1}(H)$.

Problem 1.8. Let $f : \Omega \to E$ be some function.

- Let $A \subseteq \Omega$. Is it true that $f^{-1}(f(A)) = A$? Provide a proof or a counterexample.
- Let $H \subseteq E$. Is it true that $f(f^{-1}(H)) = H$? Provide a proof or a counterexample.

Problem 1.9. Recall that given a set Ω, 2^Ω denotes the set of all subsets of Ω. Suppose $\Omega = \{0, 1\}$, list all the elements of 2^Ω. What is $|2^\Omega|$, where $|\cdot|$ denotes the number of elements of a set? Suppose that $|\Omega| < \infty$, what is $|2^\Omega|$ in this case?

Problem 1.10. Let $\{a_i\}_{i\in I} \subseteq [0, \infty]$, where I is an arbitrary (index) set. Recall that their sum is defined by

$$\sum_{i\in I} a_i = \sup\Big\{ \sum_{i\in K} a_i : \ K \subseteq I, \ K \text{ finite}\Big\}.$$

Now, suppose that $I = \mathbb{N}$. Show that the above definition agrees with the standard one, that is

$$\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Show that the value of the series does not depend on the ordering of the elements in the sequence. That is, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} a_{\sigma(i)}.$$

Problem 1.11. Let $\{a_i\}_{i\in I} \subseteq [0, \infty)$, where I is an arbitrary (index) set. Suppose that

$$\sum_{i\in I} a_i < \infty.$$

Show that the set $J_n = \{i \in I : \ a_i > 1/n\}$ is finite. Conclude that the set of $i \in I$ such that $a_i > 0$ is at most countable.

Problem 1.12. Let $\{a_i\}_{i\in I} \subseteq (0, \infty)$ be a family of *positive* real numbers, where I is an (index) set with uncountably many elements. Show that

$$\sum_{i\in I} a_i = \infty.$$

Problem 1.13. (*) Let Ω be a non-empty set and $p_\omega \in [0, 1]$, $\omega \in \Omega$ be real numbers such that

$$\sum_{\omega\in\Omega} p_\omega = 1.$$

Define the set function $\mathbb{P} : 2^\Omega \to [0, 1]$ by

$$\mathbb{P}(A) = \sum_{\omega\in A} p_\omega.$$

Show that $\mathbb{P}$ is a measure on 2^Ω.

Problem 1.14. (*) Let $A \subset \mathbb{R}$ be an open set. Show that A is the union of at most countable many intervals. (*Hint:* define for all $x \in A$ the interval $I_x = \bigcup_{I \text{ interval}: x \in I \subseteq A} I$ to be the largest interval contained in A containing x)

Problem 1.15. Let I and J be two index sets and $a_{i,j}$, $i \in I$ and $j \in J$ be non-negative real numbers. Show that

$$\sum_{i \in I} \sum_{j \in J} a_{i,j} = \sum_{j \in J} \sum_{i \in I} a_{i,j}.$$

2. Measurable sets and σ-algebras

Problem 2.1. Show that there is no σ-algebra with an odd number of elements.

Problem 2.2. Let $(\Omega, \mathscr{F})$ be a measurable space and $A, B \in \mathscr{F}$. Show, starting from the definition of σ-algebra, that $A \cup B$, $A \cap B$, $A \setminus B$, and $A \Delta B$ all belong to $\mathscr{F}$.

Problem 2.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $A_1, A_2, \ldots$ be a sequence of sets in $\mathscr{F}$. Define the following sets

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m, \qquad \liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

Show that:

- $(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c$,
- $\liminf_{n\to\infty} A_n \in \mathscr{F}$ and $\limsup_{n\to\infty} A_n \in \mathscr{F}$,
- $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$.
- $\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n$ for infinitely many $n\}$.
- $\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \exists m \in \mathbb{N}$ such that $\omega \in A_n$ for all $n \geq m\}$.

Problem 2.4. Let Ω, E be non-empty, $\mathscr{G}$ a σ-algebra on E and $f : \Omega \to E$. Show that

$$\mathscr{F} = \{f^{-1}(B) : B \in \mathscr{G}\},$$

is a σ-algebra on Ω.

Problem 2.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $(A_n)_{n\in\mathbb{N}}$ a collection of sets in $\mathscr{F}$. Show that:

- There are $(E_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ mutually disjoint such that $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} E_n$.
- There are $(F_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

Problem 2.6. Let Ω be a non-empty set and $\mathscr{F}$ a non-empty collection of subsets of Ω which is closed under taking complements and finite unions (such a collection is called an *algebra*). Show that $\mathscr{F}$ is a σ-algebra if and only if it is closed under countable increasing unions (i.e., if $\{A_n\} \subseteq \mathscr{F}$ and $A_1 \subseteq A_2 \subseteq \ldots$, then $\cup_{n\in\mathbb{N}} A_n \in \mathscr{F}$).

Problem 2.7. (Restriction of σ-algebra) Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Suppose that $A \subseteq \Omega$ is non-empty. Show that

$$\mathscr{F}_A = \{B \cap A : B \in \mathscr{F}\}$$

is a σ-algebra on A.

Problem 2.8. (Extension of σ-algebra) Let $(\Omega, \mathscr{F})$ be a measurable space, and let K be some non-empty set such that $\Omega \cap K = \varnothing$. Define $\overline{\Omega} = \Omega \cup K$ and $\overline{\mathscr{F}} = \sigma(\mathscr{F} \cup K)$ be a σ-algebra on $\overline{\Omega}$. Show that $\overline{\mathscr{F}} = \{A \subseteq \overline{\Omega} : A \cap \Omega \in \mathscr{F}\}$.

Problem 2.9. Let Ω be a infinite non-empty set.

- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.
- Define the collection of sets $\mathscr{F} = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}$. Is $\mathscr{F}$ a σ-algebra? Prove or disprove.

Problem 2.10. Let $\mathscr{F}$ and $\mathscr{G}$ be σ-algebras on Ω. Show that $\mathscr{F} \cap \mathscr{G}$ is a σ-algebra. Prove or disprove whether $\mathscr{F} \cup \mathscr{G}$ is in general a σ-algebra.

Problem 2.11. Let $\mathscr{F}_n$, $n \in \mathbb{N}$ be σ-algebras on Ω such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1}$ for all $n \in \mathbb{N}$ (such a sequence $\{\mathscr{F}_n\}$ is called a *filtration*).

- Show that $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ is an algebra.
- Is $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n$ a σ-algebra? Consider $\Omega = \mathbb{N}$ and $\mathscr{F}_n = \sigma(\{A : A \subseteq \mathbb{N} \cap \{1, \dots n\}\})$.

Problem 2.12. Let $\mathscr{E} \subseteq \mathscr{A}$ be two collections of sets. Show that $\sigma(\mathscr{E}) \subseteq \sigma(\mathscr{A})$.

Problem 2.13. (Product sigma algebra) Let Ω_1 and Ω_2 be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be σ-algebras on Ω_1 and Ω_2 respectively. Consider the *product σ-algebra* on $\Omega_1 \times \Omega_2$

$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}).$$

Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Show that $\mathscr{F}_1 \otimes \mathscr{F}_2$ is generated by $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$.

Problem 2.14. Show that the Borel σ-algebra $\mathscr{B}_{\mathbb{R}^2}$ equals $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$.

Problem 2.15. Show that the Borel σ-algebra on $\mathbb{R}$ is generated by each of the following:

i. the open intervals: $\mathscr{A}_1 = \{(a, b) : a < b\}$,

ii. the closed intervals: $\mathscr{A}_2 = \{[a, b] : a < b\}$,

iii. the half open intervals $\mathscr{A}_3 = \{[a, b) : a < b\}$ or $\mathscr{A}_4 = \{(a, b] : a < b\}$,

iv. the open rays: $\mathscr{A}_5 = \{(a, \infty) : a \in \mathbb{R}\}$ or $\mathscr{A}_6 = \{(-\infty, a) : a \in \mathbb{R}\}$,

v. the closed rays: $\mathcal{A}_7 = \{[a, \infty) : a \in \mathbb{R}\}$ or $\mathcal{A}_8 = \{(-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.16. Recall that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is the extended real line. Also recall that

$$\mathcal{B}_{\overline{\mathcal{B}}} = \{A \subseteq \overline{\mathcal{B}} : A \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$$

is a σ-algebra on $\overline{\mathbb{R}}$. Show that $\mathcal{B}_{\overline{\mathcal{B}}}$ is generated by the family of closed rays $\mathcal{A} = \{[-\infty, a] : a \in \mathbb{R}\}$.

Problem 2.17. Let $\mathcal{F}$ be an infinite σ-algebra.

- ▶ Show that $\mathcal{F}$ contains an infinite sequence of disjoint sets.
- ▶ (*) Show that $\mathrm{Card}(\mathcal{F}) \geq \mathrm{Card}([0, 1])$.
  (*Hint:* think about binary representation of numbers in $[0, 1]$).

Problem 2.18. Show that Λ is a λ-system on Ω if and only if

I. $\Omega \in \Lambda$,

II. if $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$,

III. if $A_1, A_2, \ldots$ is a sequence of subsets in Λ such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then

$$\bigcup_{n \in \mathbb{N}} A_n \in \Lambda.$$

Problem 2.19. Let Λ be a λ-system. Show that $\varnothing \in \Lambda$.

Problem 2.20. Let $\mathcal{A}$ be both a λ-system and a π-system. Show that $\mathcal{A}$ is a σ-algebra.

3. Measures

Problem 3.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ such that $E \subseteq F$, one has $\mu(E) \le \mu(F)$.

Problem 3.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Show that for all $E, F \in \mathscr{F}$ then

$$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F).$$

Conclude that $\mu(E \cup F) \le \mu(E) + \mu(F)$.

Problem 3.3. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, \infty]$ be a set function. Assume that $\mu(\varnothing) = 0$, μ is finitely additive and continuous from below. Show that μ is a measure.

Problem 3.4. Consider a non-empty uncountable set Ω and the σ-algebra

$$\mathscr{F} = \{A \subseteq \Omega : \ A \text{ is countable or } \Omega \setminus A \text{ is countable}\}.$$

We define the set function $\mu : \mathscr{F} \to [0, \infty]$ by $\mu(E) = 0$ if E is countable, and $\mu(E) = 1$ if $\Omega \setminus E$ is countable. Show that μ is a measure on $(\Omega, \mathscr{F})$.

Problem 3.5. Let Ω be an infinite set and $\mathscr{F} = 2^{\Omega}$. Define μ on $\mathscr{F}$ by $\mu(E) = 0$ if E is finite and $\mu(E) = \infty$ if E is not finite. Show that μ is finitely additive but not a measure.

Problem 3.6. Let $(\Omega, \mathscr{F})$ be a measurable space and $\mu : \mathscr{F} \to [0, 1]$ be an additive set function, that is, $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathscr{F}$ such that $A \cap B = \varnothing$. Show that $\mu(\varnothing) = 0$.

Problem 3.7. (Inclusion-exclusion) Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $A_1, A_2, A_3 \in \mathscr{F}$. Show that

$$\begin{aligned}\mu(A_1 \cup A_2 \cup A_3) = \mu(A_1) + \mu(A_2) + \mu(A_3) \\ - \mu(A_1 \cap A_2) - \mu(A_2 \cap A_3) - \mu(A_3 \cap A_1) + \mu(A_1 \cap A_2 \cap A_3).\end{aligned}$$

Let $A_1, \dots, A_n \in \mathscr{F}$. Show that

$$\mu\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| = j}} \mu\Big(\bigcap_{i \in I} A_i\Big).$$

Problem 3.8. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space.

- If $E, F \in \mathscr{F}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

- ▶ Define $\rho(E,F)=\mu(E\Delta F)$ for all $E,F\in\mathscr{F}$. Show that $\rho(E,F)\le\rho(E,G)+\rho(G,F)$ for all $E,F,G\in\mathscr{F}$.

Problem 3.9. If $\mu_1,\dots,\mu_n$ are measures on a measurable space $(\Omega,\mathscr{F})$, and $a_1,\dots,a_n$ are non-negative real numbers, then $\mu:=\sum_{i=1}^n a_i\mu_i$ is a measure on $\mathscr{F}$. Moreover μ is σ-finite if μ_i is σ-finite for all $i=1,\dots,n$.

Problem 3.10. Let $(\Omega,\mathscr{F},\mu)$ be a measure space and $\mathscr{G}\subseteq\mathscr{F}$ be a σ-algebra. Show that $(\Omega,\mathscr{G},\mu)$ is a measure space.

Problem 3.11. Let μ be a finite measure on $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$. Define $F:\mathbb{R}\to\mathbb{R}$ by

$$F(x)=\mu((-\infty,x]),\quad \forall x\in\mathbb{R}.$$

Show that F is non-decreasing, right-continuous, and

$$\lim_{x\to-\infty}F(x)=0,\quad \lim_{x\to\infty}F(x)=\mu(\mathbb{R}).$$

Problem 3.12. Find a measure space $(\Omega,\mathscr{F},\mu)$ and a decreasing sequence $B_1\supseteq B_2\supseteq\ldots\in\mathscr{F}$ such that $\lim_{n\to\infty}\mu(B_n)>\mu(\cap_{n\in\mathbb{N}}B_n)$.

Problem 3.13. Let $(\Omega,\mathscr{F},\mu)$ be a measure space and $A\in\mathscr{F}$. Show that the set function $\mu_A:\mathscr{F}\to[0,\infty]$ defined by $\mu_A(B):=\mu(B\cap A)$ is a measure on $(\Omega,\mathscr{F})$.

Problem 3.14. Let $(\Omega,\mathscr{F})$ be a measure space $\mu:\mathscr{F}\to[0,\infty]$ be a set function which is finitely additive and such that $\mu(\varnothing)=0$. Show that μ is a measure on $(\Omega,\mathscr{F})$ if and only if μ is continuous from below.

Problem 3.15. Let $(\Omega,\mathscr{F})$ be a measure space $\mu:\mathscr{F}\to[0,\infty]$ be a set function which is finitely additive and such that $\mu(\Omega)<\infty$. Show that μ is a measure on $(\Omega,\mathscr{F})$ if and only if μ is continuous from above.

Problem 3.16. Let $(\Omega,\mathscr{F},\mu)$ be a finite measure space. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{F}$. Recall that

$$\limsup_{n\to\infty}A_n=\bigcap_{n\in\mathbb{N}}\bigcup_{m\ge n}A_m,\qquad \liminf_{n\to\infty}A_n=\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}A_m.$$

Show that

$$\limsup_{n\to\infty}\mu(A_n)\le\mu\Big(\limsup_{n\to\infty}A_n\Big),\qquad \liminf_{n\to\infty}\mu(A_n)\ge\mu\Big(\liminf_{n\to\infty}A_n\Big).$$

Problem 3.17. Let $(\Omega,\mathscr{F},\mu)$ be a σ-finite measure space. Let $\mathscr{E}\subseteq\mathscr{F}$ be a π-system such that there exists $E_1,E_2,\ldots\in\mathscr{E}$ such that $E_n\uparrow\Omega$ and $\mu(E_n)<\infty$ for all $n\in\mathbb{N}$. Show that μ is uniquely determined by its values on $\mathscr{E}$.

Problem 3.18. Let $F:\mathbb{R}\to\mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R},\mathscr{B}_{\mathbb{R}})$ associated to F. Show that $\nu_F(\{x\})=F(x)-F(x-)$ where we define

$$F(x-):=\lim_{y\uparrow x}F(y).$$

Conclude that it if F is continuous, then $\nu_F(\mathbb{Q})=0$.

Problem 3.19. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\lambda((a, b]) = b - a$ for all $a < b$. Show that λ is translation invariant, that is $\lambda(A + x) = \lambda(A)$ for all $A \in \mathscr{B}_\mathbb{R}$ and all $x \in \mathbb{R}$, where we write $A + x := \{a + x : a \in A\}$ for the translation of A by x. (*Hint:* a solution can be obtained with the $\pi - \lambda$ theorem.)

Problem 3.20. Let λ be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\lambda((a, b]) = b - a$ for all $a < b$. Show that $\lambda(\tau A) = |\tau|\lambda(A)$ for all $A \in \mathscr{B}_\mathbb{R}$ and all $\tau \neq 0$, where we write $\tau A := \{\tau a : a \in A\}$ for the dilation of A by τ.

Problem 3.21. Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\nu_F((a, b]) = F(b) - F(a)$ for all $a < b$. Show that ν_F is σ-finite.

Problem 3.22. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $(A_i)_{i\in\mathbb{N}} \subseteq \mathscr{F}$ be a sequence such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{i\in\mathbb{N}} A_i) = 1$.

Problem 3.23. Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$. Show that the set $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is at most countable.

4. Null sets, completion and independence

Problem 4.1. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Suppose that $A, N \in \mathscr{F}$ and $\mu(N) = 0$. Show that $\mu(A \cup N) = \mu(A)$.

Problem 4.2. Suppose that $(\Omega, \mathscr{F}, \mu)$ is a measure space and let N be a null set. Show that for all $M \subseteq N$, M is a null set.

Problem 4.3. Let $(N_n)_{n\in\mathbb{N}}$ be null sets in a measure space $(\Omega, \mathscr{F}, \mu)$. Show that $\cup_{n\in\mathbb{N}} N_n$ is a null set.

Problem 4.4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of function. Show that $f_n = 0$ almost everywhere for all $n \in \mathbb{N}$ if and only if almost everywhere $f_n = 0$ for all $n \in \mathbb{N}$. (Careful with the quantifiers!)

Problem 4.5. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability, we say that $A \in \mathscr{F}$ happens almost surely if $\Omega \setminus A$ is a null set for $\mathbb{P}$.

- Show that $A \in \mathscr{F}$ happens almost surely if and only if $\mathbb{P}(A) = 1$.
- Assume now that $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{F}$ is such that A_n happens almost surely for all $n \in \mathbb{N}$. Show that $\mathbb{P}(\cap_{n\in\mathbb{N}} A_n) = 1$.

Problem 4.6. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set and let V be the Vitali set. Show that if $E \subseteq V$, then E is a null-set.

Problem 4.7. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that $\overline{A} \in \overline{\mathscr{F}}$ if and only if there is $A \in \mathscr{F}$ such that $A \Delta \overline{A}$ is a null set.

Problem 4.8. (*) Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $\mathscr{L}(E) > 0$. Show that there exists $N \subseteq E$ not Lebesgue measurable. (Hint: assume first $E \subseteq (0, 1)$ and look at $V \cap E$ where V is the Vitali set.)

Problem 4.9. (The Cantor set) The Lebesgue null sets include not only the countable sets but also many sets having the cardinality of the continuum. The Cantor set C is the set of all $x \in [0, 1]$ that have a base-3 expansion

$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}, \quad \text{with } a_j \in \{0, 2\} \text{ for all } j \in \mathbb{N}.$$

Thus C is obtained from $[0, 1]$ by removing the open middle third $(1/3, 2/3)$, then removing the middle thirds $(1/9, 2/9)$ and $(7/9, 8/9)$ of the remaining intervals and so forth. Show that

- C is compact and with zero Lebesgue measure.

- $\mathrm{Card}(C) = \mathrm{Card}([0,1])$. Hint: consider the so called Cantor function, for $x \in C$, $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$, define
$$f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}, \quad b_j = a_j/2.$$
- (*) Show that C has empty interior and is totally disconnected (that is for all $x < y \in C$ there is $z \in (x,y)$ such that $z \notin C$). Moreover C has no isolated points.

Problem 4.10. Show that for any Lebesgue measurable set $E \subseteq \mathbb{R}$ and any real number $\lambda \in \mathbb{R}$, $\mathscr{L}(E+\lambda) = \mathscr{L}(E)$ and $\mathscr{L}(\lambda E) = |\lambda|\mathscr{L}(E)$.

Problem 4.11. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. We say that two σ-algebras $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{F}$ are independent if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{A}_1, \forall A_2 \in \mathscr{A}_2.$$
Suppose that $\mathscr{E}_1, \mathscr{E}_2$ are π-systems generating $\mathscr{A}_1$ and $\mathscr{A}_2$ respectively. Show that $\mathscr{A}_1$ and $\mathscr{A}_2$ are independent if and only if
$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2), \quad \forall A_1 \in \mathscr{E}_1, \forall A_2 \in \mathscr{E}_2.$$

Problem 4.12. Are the following true of false?

- If A is an open subset of $[0,1]$, then $\mathscr{L}^1(A) = \mathscr{L}^1(\overline{A})$, where $\overline{A}$ is the closure of the set.
- If A is a subset of $[0,1]$ such that $\mathscr{L}^1(\mathrm{int}(A)) = \mathscr{L}^1(\overline{A})$, then A is measurable. Here $\mathrm{int}(A)$ denotes the interior of the set A.

Problem 4.13. Show that if $A \subset [0,1]$ and $\mathscr{L}^1(A) > 0$, then there are x and y in A such that $|x-y|$ is an irrational number.

Problem 4.14. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Show that a set $A \subseteq \Omega$ is a μ-null set if and only if A is a $\overline{\mu}$-null set.

5. Measurable functions

Problem 5.1. Let $(\Omega, \mathscr{F})$ be a measure space. Let $A \subseteq \Omega$ and consider the indicator function $\mathbb{1}_A : \Omega \to \mathbb{R}$, defined by

$$\mathbb{1}_A(\omega) = 1, \text{ if } \omega \in A, \quad \mathbb{1}_A(\omega) = 0, \text{ if } \omega \notin A.$$

Show that $\mathbb{1}_A$ is $(\mathscr{F}, \mathscr{B})$-measurable if and only if $A \in \mathscr{F}$.

Problem 5.2. Let $(\Omega, \mathscr{F})$ be a measure space and $A, B \in \mathscr{F}$. Show that

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B}, \qquad \mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B.$$

Problem 5.3. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and $X, Y : \Omega \to \mathbb{R}$ be random variables. Recall that X and Y are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \qquad \forall A \in \sigma(X), \forall B \in \sigma(Y).$$

Show that X and Y are independent if and only if

$$\mathbb{P}(X \geq s, Y \geq t) = \mathbb{P}(X \geq s)\mathbb{P}(Y \geq t), \qquad \forall s, t \in \mathbb{R}.$$

(Hint: you might need the $\pi - \lambda$ Theorem.)

Problem 5.4. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be $\mathscr{F}$-measurable functions. Show that $\max(f, g)$, $\min(f, g)$ and $|f|$ are measurable.

Problem 5.5. Let $(\Omega, \mathscr{F})$ be a measurable space and $E_1, \ldots, E_n \in \mathscr{F}$ be measurable sets. Show that if $a_1, \ldots, a_n$ are real numbers, then

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}$$

is $\mathscr{F}$-measurable.

Problem 5.6. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$, and $Y = f^{-1}(\mathbb{R})$. Show that f is measurable if and only if $Y \in \mathscr{F}$, $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted to Y.

Problem 5.7. Let $(\Omega, \mathscr{F})$ be a measurable space and f_i, $i \in \mathbb{N}$ be measurable functions from $(\Omega, \mathscr{F})$ to $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = 0\}$$

is measurable, that is, it belongs to $\mathscr{F}$. (Hint: recall $\lim_{n\to\infty} f_n(\omega) = 0$ means that for all $m > 0$ there exists $N > 0$ such that $|f_n(\omega)| \leq 1/m$ for all $n \geq N$. Can you now write the set as countable union and intersections of measurable sets.)

Problem 5.8. Let $(\Omega, \mathscr{F})$ be a measurable space and $f, g : \Omega \to \mathbb{R}$ be two measurable functions. Show that $\{\omega \in \Omega : f(\omega) = g(\omega)\}$ is a measurable set.

Problem 5.9. Let $(\Omega, \mathscr{F})$ be a measurable space and (f_n) be a sequence of $\mathscr{F}$-measurable functions. Show that the set

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\}$$

is measurable.

Problem 5.10. Consider a function $f : \mathbb{R} \to \mathbb{R}$. Show that f is Lebesgue measurable if and only if there exists a Borel measurable function g such that $f \equiv g$ almost everywhere.

Problem 5.11. Show that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel-measurable.

Problem 5.12. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an homeomorphism, that is f is continuous with continuous inverse. Show that f maps Borel measurable sets to Borel measurable sets.

Problem 5.13. Let $(\Omega, \mathscr{F}, \mu)$ be a complete measure space. Show that

- if f is measurable and $f \equiv g$ μ-a.e. then g is measurable.
- if $f_n, n \in \mathbb{N}$ are measurable and $f_n \to f$ μ-a.e. then f is measurable.

Problem 5.14. Let $(\Omega, \mathscr{F})$ be a measurable space and $f : \Omega \to \overline{\mathbb{R}}$ be a function. Define $f^+(\omega) = \max(f(\omega), 0)$ and $f^-(\omega) = -\min(f(\omega), 0)$. Show that f is measurable if and only if f^+ and f^- are measurable.

Problem 5.15. Let $(\mathscr{X}, \mathscr{F})$ and $(\mathscr{Y}, \mathscr{G})$ be two measurable spaces. Let $E \in \mathscr{F} \otimes \mathscr{G}$, and for $y \in \mathscr{Y}$ define the y-section of E to be

$$E_y = \{x \in \mathscr{X} : (x, y) \in E\},$$

Similarly define for $x \in \mathscr{X}$ the x-section of E to be

$$E_x = \{y \in \mathscr{Y} : (x, y) \in E\}.$$

Show that E_y is measurable for all $y \in \mathscr{Y}$.

Problem 5.16. Let $(\Omega, \mathscr{F})$ and $(E, \mathscr{G})$ be measurable spaces, and let $f : \Omega \to E$ be a function. Suppose that $A \in \mathscr{F}$, we say that f is measurable on A if $f^{-1}(B) \cap A \in \mathscr{F}$ for all $B \in \mathscr{G}$. Show that f is measurable on $A \in \mathscr{F}$ if and only if $f|_A$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, where $\mathscr{F}_A = \{A \cap B : B \in \mathscr{F}\}$. Show that if f is $(\mathscr{F}, \mathscr{G})$-measurable, then f is measurable on A for all $A \in \mathscr{F}$.

Problem 5.17. Let $(\Omega, \mathscr{F})$ be a measurable space, $f : \Omega \to \overline{\mathbb{R}}$ and $\mathscr{Y} = f^{-1}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}(\{-\infty\}) \in \mathscr{F}$, $f^{-1}(\{\infty\}) \in \mathscr{F}$, and f is measurable when restricted on $\mathscr{Y}$.

Problem 5.18. If a function $Y_A : A \to E$ is $(\mathscr{F}_A, \mathscr{G})$-measurable, and $p \in E$, then the extension Y defined by

$$Y(\omega) := \begin{cases} Y_A(\omega), & \omega \in A, \\ p, & \omega \notin A, \end{cases}$$

is $(\mathscr{F}, \mathscr{G})$-measurable.

Problem 5.19. Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable?

Problem 5.20. Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere, then f is Lebesgue-measurable.

Problem 5.21. Is the following true of false? If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

Problem 5.22. (Convergence in measure/probability) Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $X_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ be random variables. We say that X_n converges in probability to a random variable X, if for all $\epsilon > 0$

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0, \qquad \text{as } n \to \infty.$$

Show that if X_n converges to a random variable X in probability, then there exists a subsequence of $(X_n)_{n \in \mathbb{N}}$ which converges almost surely to X. (*Hint: use Borell-Cantelli Lemma.)*

6. Integration

Problem 6.1. Consider the measure space $(\mathscr{X}, 2^{\mathscr{X}}, \delta_x)$ where δ_x is the delta measure at $x \in \mathscr{X}$. Let $f : \mathscr{X} \to \overline{\mathbb{R}}$ be a non-negative measurable function. Show that

$$\int_{\mathscr{X}} f \, \mathrm{d}\delta_x = f(x).$$

Problem 6.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $A \in \mathscr{F}$, and $f : \Omega \to \overline{\mathbb{R}}$ be a non-negative function. We say that f is measurable on A if $f|_A$ is $\mathscr{F}_A$-measurable (recall that $\mathscr{F}_A = \{A \cap B : B \in \mathscr{F}\}$). Show that $f\mathbb{1}_A$ is $\mathscr{F}$ measurable and also

$$\int_\Omega f\mathbb{1}_A \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu_A,$$

where μ_A is the restriction of μ to $\mathscr{F}_A$.

Problem 6.3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f : \Omega \to \overline{\mathbb{R}}$ be a non-negative measurable function

- Let $A, B \in \mathscr{F}$ be such that $A \subseteq B$. Show that

$$\int_A f \, \mathrm{d}\mu \leq \int_B f \, \mathrm{d}\mu.$$

- Let $A, B \in \mathscr{F}$ be such that $A \cap B = \varnothing$. Show that

$$\int_{A \cup B} f \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu + \int_B f \, \mathrm{d}\mu.$$

Problem 6.4. Suppose that $(f_n) \subseteq L^+$, $f_n \to f$ pointwise and $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu < \infty$. Show, without the dominated convergence theorem, that for all $A \in \mathscr{F}$

$$\lim_{n\to\infty} \int_A f_n \, \mathrm{d}\mu = \int_A f \, \mathrm{d}\mu.$$

Show that the conclusion might fail if $\int f \, \mathrm{d}\mu = \infty$.

Problem 6.5. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $(\Omega, \overline{\mathscr{F}}, \overline{\mu})$ its completion. Suppose that f is integrable with respect to μ. Show that f is integrable with respect to $\overline{\mu}$ and in particular

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\overline{\mu}.$$

(*Hint: start with simple functions.*)

Problem 6.6. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and suppose that (f_n) is a sequence of measurable functions such that $f_1 \geq f_2 \geq \ldots \geq 0$. Show using the MCT that if $f := \lim_{n\to\infty} f_n$ and $\int_\Omega f_1 < \infty$, then

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

Show that if $\int_\Omega f_1 \, \mathrm{d}\mu = \infty$ the above conclusion might not hold.

Problem 6.7. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and $f \in L^+$. Show that if $\int f \, \mathrm{d}\mu < \infty$, then

$$\lim_{n\to\infty} \int_\Omega f \mathbb{1}_{\{f \geq n\}} \, \mathrm{d}\mu = 0.$$

Problem 6.8. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and assume that $f : \Omega \to \overline{\mathbb{R}}$ is integrable with respect to μ. Show that $|f| < \infty$ μ-almost everywhere.

Problem 6.9. Suppose that f is a Lebesgue-integrable function on $\mathbb{R}$. Show that for all $z \in \mathbb{R}$

$$\int_\mathbb{R} f(x) \, \mathrm{d}\mathscr{L}^1(x) = \int_\mathbb{R} f(x+z) \, \mathrm{d}\mathscr{L}^1(x).$$

Show that for all $\tau \neq 0$

$$\int_\mathbb{R} f\Big(\frac{x}{\tau}\Big) \, \mathrm{d}\mathscr{L}^1(x) = |\tau| \int_\mathbb{R} f(x) \, \mathrm{d}\mathscr{L}^1(x).$$

Problem 6.10. Assume Fatou's lemma and deduce the Monotone Convergence Theorem.

Problem 6.11. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and assume that f is integrable with respect to μ and g is a measurable function such that $f = g$ μ-almost everywhere. Show that g is integrable with respect to μ and

$$\int_\Omega f \, \mathrm{d}\mu = \int_\Omega g \, \mathrm{d}\mu.$$

Problem 6.12. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and f be integrable with respect to μ. Show that the set $\{\omega \in \Omega : \ f(\omega) \neq 0\}$ is a σ-finite set.

Problem 6.13. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space and f, g be integrable with respect to μ. Then

$$\int_A f \, \mathrm{d}\mu = \int_A g \, \mathrm{d}\mu, \qquad \forall A \in \mathscr{F},$$

if and only if $f = g$ μ-almost everywhere, and if and only if $\int_\Omega |f - g| \, \mathrm{d}\mu = 0$.

Problem 6.14. Let $(\Omega_1, \mathscr{F}_1)$ and $(\Omega_2, \mathscr{F}_2)$ be two measure spaces. Let $f : \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$, $f \geq 0$ be a $\mathscr{F}_1 \otimes \mathscr{F}_2$-measurable function. Show that for all $y \in \Omega_2$ the map $f_y : \Omega_1 \to \overline{\mathbb{R}}$, defined by $f_y(x) = f(x, y)$ for all $x \in \Omega_1$, is $\mathscr{F}_1$-measurable.
(*Hint:* use approximations with simple functions.)

Problem 6.15. Let f be Lebesgue-integrable on the real line. Show that

$$\lim_{n\to\infty} \frac{1}{2n} \int_{[-n,n]} f \,\mathrm{d}\mathscr{L}^1 = 0.$$

Show that the result need not be true if f is not assumed to be integrable on $\mathbb{R}$.

Problem 6.16. Let μ be a measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ concentrated on $\mathbb{Z}$. This means that $\mu(B) = \mu(B \cap \mathbb{Z})$ for all $B \in \mathscr{B}_{\mathbb{R}}$, or equivalently $\mu = \sum_{n\in\mathbb{Z}} \mu_n \delta_n$, where μ_n are non-negative real numbers and δ_n is the Dirac measura at $n \in \mathbb{Z}$. Show that $\mu_n = \mu(\{n\})$ for all $n \in \mathbb{Z}$ and that for all f non-negative and measurable

$$\int_{\mathbb{R}} f \,\mathrm{d}\mu = \sum_{n\in\mathbb{Z}} \mu_n f(n).$$

Problem 6.17. Consider the functions

$$f_1(x) = \begin{cases} +\infty, & \text{if } x = 0, \\ \log|x|, & \text{if } 0 < |x| < 1, \\ 0, & \text{if } |x| \geq 1, \end{cases} \qquad f_2(x) = \begin{cases} \frac{1}{x^2-1}, & \text{if } |x| \neq 0, \\ 20, & \text{if } |x| = 0, \end{cases} \qquad f_3(x) \equiv 1.$$

Determine if these functions are integrable on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ in each of the following two cases, and if possible compute the value of the integral

a) $m = \mathscr{L}^1$ is the Lebesgue measure,

b) m is defined by

$$m(B) = \sum_{n\in\mathbb{Z}} \frac{1}{1+(n+1)^2} \delta_n(B)$$

for all $B \in \mathscr{B}_{\mathbb{R}}$.

Problem 6.18. Let $f, g : [0,1] \to \mathbb{R}$ be two non-negative integrable functions. Show that fg is not necessarily integrable.

Problem 6.19. Let f be integrable on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ with respect to the Lebesgue measure.

a) Show that for all $\epsilon > 0$ there exists a simple function $g = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$ with $A_1, \dots, A_k$ bounded intervals, such that $\int_{\mathbb{R}} |f - g| \,\mathrm{d}\mathscr{L} < \epsilon$.

b) Show that all $\epsilon > 0$ there exists a bounded continuous function h such that $\int_{\mathbb{R}} |f - h| \,\mathrm{d}\mathscr{L} < \epsilon$.

Problem 6.20. Determine the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} \,\mathrm{d}x, \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \,\mathrm{d}x.$$

Problem 6.21. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Let $(f_n)_n$ be a sequence of measurable functions such that $f_n \to f$ uniformly on Ω. Show that if f_n is integrable for all $n \in \mathbb{N}$ then f is also integrable and

$$\lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu.$$

Recall that f_n converges uniformly to f on Ω if $\lim_{n\to\infty} \sup_{\omega\in\Omega} |f_n(\omega) - f(\omega)| = 0$. Show that the result is in general false if $\mu(\Omega) = \infty$.

Problem 6.22. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to the Lebesgue measure. Show that the map $x \mapsto \int_{[-\infty,x]} f \,\mathrm{d}\mathscr{L}$ is continuous in $x \in \mathbb{R}$.

7. Littlewood's Principles

Problem 7.1. Prove that the Lebesgue measure on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$ is both inner regular and outer regular.

Problem 7.2. Let μ be an outer regular measure on $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$. Suppose that μ is *inner regular on open sets*, i.e.

$$\mu(O) = \sup\left(\mu(K) \ : \ K \subset O, \ K \text{ is compact}\right)$$

for every open $O \in \mathscr{B}_{\mathbb{R}^n}$. Show that μ is inner regular on every $B \in \mathscr{B}_{\mathbb{R}^n}$ if $\mu(B) < \infty$.

Problem 7.3. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that converges pointwise a.e. to f and for all $n \in \mathbb{N}$ $|f_n| \leq g$, where g is integrable. Prove that for any $\varepsilon > 0$ there exists $E \in \mathscr{F}$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.

Problem 7.4. Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions such that converges pointwise a.e. to f. Show that there exists a sequence of measurable sets (A_k), such that $\mu\left(\bigcap_k A_k^c\right) = 0$ and such that for any $k \in \mathbb{N}$ (f_n) converges uniformly to f on A_k.

Problem 7.5. Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. Use Egorov's theorem (and not Dominated convergence) to show the following *bounded convergence* theorem. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions such that converges pointwise a.e. to f, and assume that there is $M > 0$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$. Show that

$$\int_\Omega |f_n - f| \, \mathrm{d}\mu \to 0.$$

Problem 7.6. Let $\mathscr{L}_{(0,1)}$ be a restriction of the Lebesgue measure on $\mathbb{R}$ to $(0,1)$. Consider a function $f = \mathbb{1}_{\{(0,1)\cap\mathbb{Q}\}}$. Lusin's theorem asserts that for every $\varepsilon > 0$ there exists a compact set $K \subset (0,1)$ and a continuous function g such that $\mathscr{L}_{(0,1)}\left((0,1) \setminus K\right) < \varepsilon$ and $f = g$ on K. Find such g and K.

8. Product measures and Fubini-Tonelli

9. L^p spaces

Part II.

Solutions

10. Solutions: Warming up

Solution to Problem 1.1 *(by Kempen, S.F.M.)*
a) To be proven: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
" $\subseteq$ " Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$, which means $x \in A$ and ($x \in B$ or $x \in C$).

- If $x \in A$ and $x \in B$, then $x \in A \cap B$ so also $x \in (A \cap B) \cup (A \cap C)$.
- If $x \in A$ and $x \in C$, then $x \in A \cap C$ so also $x \in (A \cap B) \cup (A \cap C)$.

" $\supseteq$ " Let $x \in (A \cap B) \cup (A \cap C)$ then $x \in A \cap B$ or $x \in A \cap C$.

- If $x \in A \cap B$, then $x \in A$ and $x \in B$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.
- If $x \in A \cap C$, then $x \in A$ and $x \in C$ so $x \in B \cup C$ so also $x \in A \cap (B \cup C)$.

b) To be proven: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
" $\subseteq$ " Let $x \in A \cup (B \cap C)$ then $x \in A$ or $x \in B \cap C$, which means $x \in A$ or ($x \in B$ and $x \in C$).

- If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.
- If $x \in B$ and $x \in C$, then $x \in A \cup B$ and $x \in A \cup C$ so also $x \in (A \cup B) \cap (A \cup C)$.

" $\supseteq$ " Let $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup B$ and $x \in A \cup C$, which means ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$).

- If $x \in A$ then definitely $x \in A \cup (B \cap C)$.
- If $x \notin A$ then $x \in B$ and $x \in C$ which means $x \in B \cap C$ so also $x \in A \cup (B \cap C)$.

😄

Solution to Problem 1.2 *(by Kempen, S.F.M.)* To be proven: $A \cap (A \cup B) = A$.
" $\subseteq$ " Let $x \in A \cap (A \cup B)$, then $x \in A$ so we are done.
" $\supseteq$ " Let $x \in A$, then also ($x \in A$ or $x \in B$) is true, therefore $x \in A \cup B$. So $x \in A \cap (A \cup B)$. 😄

Solution to Problem 1.5 *(by Kempen, S.F.M.)* To be proven: $\Omega \setminus \left(\bigcup_{i \in I} A_i\right) = \bigcap_{i \in I} \Omega \setminus A_i$.
" $\subseteq$ " Let $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$ then $x \in \Omega$ and $x \notin \bigcup_{i \in I} A_i$, so for all $i \in I$ holds $x \notin A_i$. Then for all $i \in I$ we have $x \in \Omega \setminus A_i$. Since this is true for any $i \in I$, we can write $x \in \bigcap_{i \in I} \Omega \setminus A_i$.
" $\supseteq$ " Let $x \in \bigcap_{i \in I} \Omega \setminus A_i$ then for all $i \in I$ we have $x \in \Omega \setminus A_i$, so $x \in \Omega$ and $x \notin A_i$. Since this holds for all $i \in I$, we can write $x \notin \bigcup_{i \in I} A_i$ and therefore $x \in \Omega \setminus \left(\bigcup_{i \in I} A_i\right)$. 😄

Solution to Problem 1.8 *(by Kempen, S.F.M.)*
a) The statement $f^{-1}(f(A)) = A$ is not true since f is not assumed to be injective. As a counterexample, take $\Omega = \{0,1\}, E = \{0\}, f(\{0\}) = f(\{1\}) = \{0\}, A = \{0\}$ then $f(A) = \{0\}$ and $f^{-1}(f(A)) = f^{-1}(\{0\}) = \{0,1\} \neq A$.
b) The statement $f(f^{-1}(H)) = H$ is not true since f is not assumed to be surjective. As a counterexample, take $\Omega = \{1\}$, $E = \{1,2\}$, $f(\{1\}) = \{1\}$, $H = \{1,2\}$ then $f^{-1}(H) = f^{-1}(\{1,2\}) = \{1\}$ and $f(f^{-1}(H)) = f(\{1\}) = \{1\} \neq H$. 😄

Solution to Problem 1.10 *(by Beurskens, T.P.J.)* Let $n \in \mathbb{N}$, and define $K = \{1,\dots,n\}$. By definition of the supremum, we then have

$$\sum_{i=1}^{n} a_i = \sum_{i\in K} a_i \leq \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) = \sum_{i\in I} a_i.$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_i \leq \sum_{i\in I} a_i.$$

Next, let $K \subseteq \mathbb{N}$ be finite, so that $K \subset \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Note that $n \geq \sup K$. We get

$$\sum_{i\in K} a_i \leq \sum_{i\in\{1,\dots,n\}} a_i = \sum_{i=1}^{n} a_i \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Since this holds for arbitrary finite K, it holds for all finite K. Thus we get

$$\sum_{i\in I} a_i = \sup\left(\sum_{i\in K} a_i : K \subseteq I, K \text{ finite}\right) \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

Using both inequalities, we see that indeed $\sum_{i\in I} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$.

We are left with showing that the sum $\sum_{i=1}^{\infty} a_i$ does not depend on the ordering of the elements in the sequence (a_i). This follows immediately from the fact that for any index set I

$$\sum_{i\in I} a_i = \sup\Big\{\sum_{i\in K} a_i \,:\, K \subseteq I,\ K \text{ finite}\Big\}$$

is completely blind to any ordering of I, in fact I is possibly not even ordered. To be more precise, if $\sigma : I \to I$ is a bijection, then

$$\begin{aligned}\sum_{i\in I} a_{\sigma(i)} &= \sup\Big\{\sum_{i\in K} a_{\sigma(i)} \,:\, K \subseteq I,\ K \text{ finite}\Big\}\\ &= \sup\Big\{\sum_{i\in\sigma^{-1}(K)} a_i \,:\, \sigma^{-1}(K) \subseteq I,\ \sigma^{-1}(K) \text{ finite}\Big\} = \sum_{i\in I} a_i,\end{aligned}$$

where we used that $K \subseteq I$ is finite if and only if $\sigma^{-1}(K)$ is finite. So, from this observation and the first part of the problem, one has that for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$

$$\sum_{i=1}^{\infty} a_i = \sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} a_{\sigma(i)} = \sum_{i=1}^{\infty} a_{\sigma(i)},$$

where the summation in the middle is with respect the new notion. 😄

Solution to Problem 1.11 *(by Vrămuleț, A. Ș.)*
Note for every $a_i > 0$, there exists $n \in \mathbb{N}$ such that

$$a_i > \frac{1}{n}$$

(choose $n := \max(\lceil a_i \rceil + 1, \left\lceil \frac{1}{a_i} \right\rceil + 1)$).
We can now define

$$A := \{i \in I | a_i > 0\} = \bigcup_{n \geq 1} J_n.$$

Suppose for a contradiction J_n is infinite. Since

$$\sum_{i \in I} a_i = \sum_{i \in I, a_i > 0} a_i,$$

it suffices to show the supremum of (finite) partial sums on RHS is infinity (to derive a contradiction).
Let $n \in \mathbb{N}$. Since J_n is infinite, then for every $N_0 \in \mathbb{N}$ we may choose a finite subset $K \subset I$ with at least N_0 elements, such that

$$\sum_{i \in K} a_i > \frac{N_0}{n}.$$

Since the set of partial sums is unbounded above, then it does not admit a finite supremum.
Hence the supremum is ∞ (over the extended real line).
This contradicts that the arbitrary indexed sum is finite. So each J_n is finite.
Finally, since countable union of finite sets is countable, we conclude A is countable.

😄

Solution to Problem 1.12 *(by Bakker, A.)* Proof by contradiction. Suppose $\sum_{i \in I} a_i < \infty$, then by Problem 1.11 we have that the set I contains at most a countable number of elements i with a_i positive. This, together with the fact that $a_i > 0$ for all $i \in I$, contradicts that there are uncountable many elements in I. Hence $\sum_{i \in I} a_i = \infty$. 😄

Solution to Problem 1.13 *(by Vrămuleț, A. Ș)*
Note by the previous exercise that since the indexed sum over Ω is finite (here, 1), then there are countably many positive (and nonzero) p_ω.
Suffices to show $\mathbb{P}(\emptyset) = 0$ and that $\mathbb{P}$ is countably additive. Then

$$\mathbb{P}(\emptyset) = \sum_{\omega \in \emptyset} p_w = 0.$$

Now let (A_i) be a sequence of pairwise disjoint subsets of Ω. We show

$$\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Let $A := \bigcup_{i=1}^{\infty} A_i$. Since the sequence of sets is pairwise disjoint, then

$$\begin{aligned}
\mathbb{P}(A) &= \sum_{\omega \in A} p_w \\
&= \sup\{\sum_{\omega \in A_0} p_w | \text{ finite} A_0 \subset A\} \\
&= \sup\{\sum_{i=1}^{n} \sum_{\omega \in A_i} p_w | \text{ finite} A_i \subset A_0 \subset A\} \quad \text{(after relabelling)} \\
&= \sum_{i=1}^{\infty} \sum_{\omega \in A_i} p_w \quad \text{(definition of series as limit of the (increasing) sequence of partial sums)} \\
&= \sum_{i=1}^{\infty} \mathbb{P}(A_i).
\end{aligned}$$

Note the suprema above are finite, since the indexed sum over Ω is 1 and we are summing subsets of omega. Hence $\mathbb{P}$ is a measure on $(\Omega, 2^{\Omega})$.

Solution to Problem 1.14 *(by Vrămuleț, A. Ș)*
Let $x \in A$. Then x is an interior point of A. So there exists $r > 0$ such that

$$B(x,r) \subset A, \text{ where } B(x,r) = (x-r, x+r).$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, then there exist rationals p and q such that

$$p \in (x-r, x) \text{ and } q \in (x, x+p).$$

So $(p,q) \subset (x-r, x+r) \subset A$ and $x \in (p,q)$. Note there are countably many such intervals (since $\mathbb{Q}$ is countable, so is $\mathbb{Q} \times \mathbb{Q}$). So A is the union of countably many intervals with rational endpoints of the form (p,q).

Solution to Problem 1.15 *(by Vrămuleț, A. Ș)*
Note from before that an indexed sum of positive reals is finite if and only if there are countably many nonzero reals (in the sum). So in the case that at least one of I and J is uncountable, we actually have countable sums.

So the sums can be split as uncountable sum of zeros plus countable sum of positive reals. So in all cases the sums reduce to summing over countable sets. It's known that the result holds (by considering (absolute) convergence of the series)). Note the result also holds when the series converges to ∞ (over the extended real line).

11. Solutions: Measurable sets and σ-algebras

Solution to Problem 2.2 *(by Castella, A.)* By the definition of a σ-algebra, all countable unions of sets in $\mathscr{F}$ are also in $\mathscr{F}$. Thus let create a sequence $(A_n)_{n\in\mathbb{N}}$ such that $A_1 = A$ and $A_i = B$ for all $i > 1$. Clearly

$$\bigcup_{i=1}^{\infty} A_i = A \cup B \in \mathscr{F}.$$

Since A and B are arbitrary, this must hold for all pairs of sets in $\mathscr{F}$. From the fact that $\mathscr{F}$ is a σ-algebra we find that $A^c, B^c \in \mathscr{F}$. As we have proven that unions of pairs are in $\mathscr{F}$, we find that

$$(A^c \cup B^c)^c = A \cap B \in \mathscr{F}.$$

This again holds for all possible pairs of sets in $\mathscr{F}$. Now let us note that

$$A \setminus B = A \cap B^c.$$

It is clear from the properties that we have already proven and the complementation property of σ-algebras, we can conclude that

$$A \setminus B \in \mathscr{F}.$$

The final set, $A\Delta B$ follows by definition. The set is defined by $(A \cup B) \setminus (A \cap B)$. It is clear that this is a composition of the properties we have already proven. Since the sets A and B were arbitrary, we can conclude that

$$A\Delta B \in \mathscr{F}.$$

😃

Solution to Problem 2.3 *(by Castella, A.)*

- Using the definition and standard set theory, we find that

  $$(\limsup_{n\to\infty} A_n)^c = \Big(\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m\Big)^c = \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{m\geq n} A_m\Big)^c = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} (A_m)^c = \liminf_{n\to\infty} A_n^c.$$

  Therefore, the condition holds.

- We know that $\mathscr{F}$ is a σ-algebra, and therefore countable unions and complements are contained in the set. We show that countable intersections follow from those two conditions. By complementation we know that $(A_n)^c$ is in $\mathscr{F}$. Using both conditions in combination we find that
$$\left(\bigcup_{n\in\mathbb{N}}(A_n)^c\right)^c = \bigcap_{n\in\mathbb{N}} A_n \in \mathscr{F}.$$
Thus we find that for all $n \in \mathbb{N}$, we find that
$$\bigcap_{m\geq n} A_m \in \mathscr{F}.$$
Additionally, all countable unions are contained in the set. therefore we find that
$$\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} A_m = \liminf_{n\to\infty} A_n \in \mathscr{F}.$$
Since the sequence is arbitrary, we find that also
$$\liminf_{n\to\infty}(A_n)^c \in \mathscr{F}.$$
By item one we find that
$$(\limsup_{n\to\infty} A_n)^c \in \mathscr{F}.$$
Finally, by the complement property of $\mathscr{F}$ we find that
$$\limsup_{n\to\infty} A_n \in \mathscr{F}.$$

- Let us take an arbitrary $a \in \liminf_{n\to\infty} A_n$. Then we find that there exists $n \in \mathbb{N}$ such that
$$a \in \bigcap_{m\geq n} A_m.$$
Thus we find that for all $m \geq n$, we have
$$a \in A_m.$$
For all $k \in \mathbb{N}$ we can choose $m' \geq \max(k, n)$ such that
$$a \in A_{m'} \subset \bigcup_{m\geq k} A_m.$$
Therefore we find that
$$a \in \bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n} A_m = \limsup_{n\to\infty} A_n.$$
Since a was arbitrary, we now conclude that
$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

► Let us take some $a \in \limsup_{n\to\infty} A_n$. By the definition of intersections we know that

$$a \in \bigcup_{i=1}^{\infty} A_m$$

Therefore there exists $m_0 \in \mathbb{N}$ such that

$$a \in A_{m_0}.$$

Again, by the definition of intersections, we know that for all $n \in \mathbb{N}$ we have

$$a \in \bigcup_{m\geq n} A_m.$$

Therefore, let us choose $n' \geq m_0$, then we know that

$$a \in \bigcup_{m\geq n'} A_m$$

which implies that there exists $m_1 \in \mathbb{N}$ such that

$$a \in A_{m_1}.$$

We find that using a simple induction we can create an increasing sequence $(m_n)_{n\in\mathbb{N}}$ such that for all n, we have

$$a \in A_{m_n}.$$

Thus we conclude that

$$a \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}.$$

Let us now take an arbitrary $b \in \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}$. Clearly, by the definition of the set we can choose a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that

$$b \in A_{n_i}$$

for all i. Since for all i, we know that n_i is integer, it is clear that

$$\lim_{i\to\infty} n_i = \infty.$$

Let us take an arbitrary $k \in \mathbb{N}$. Using the limit we find that there exists i such that $n_i > k$ and therefore

$$b \in A_{n_i} \subset \bigcup_{m\geq k} A_m.$$

Since k was arbitrary, this must hold for all k. Thus we find that

$$b \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m = \limsup_{n\to\infty} A_n.$$

Therefore, we now find that

$$\{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\} \subset \limsup_{n\to\infty} A_n$$

and we can finally conclude that

$$\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \omega \in A_n \text{ for infinitely many } n\}.$$

- Let us take an arbitrary $a \in \liminf_{n\to\infty} A_n$. By the definition of a union, we find that there exists $n \in \mathbb{N}$ such that
$$a \in \bigcap_{m\geq n} A_m.$$
By the definition of intersections, we find that for all $m \geq n$ we have
$$a \in A_m.$$
Clearly this implies that
$$a \in \{\omega \in \Omega : \exists_{m\in\mathbb{N}} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}.$$
Since a was chosen arbitrarily we find that
$$\liminf_{n\to\infty} A_n \subset \{\omega \in \Omega : \exists_{m\in\mathbb{N}} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}.$$
Now let us choose an arbitrary $b \in \{\omega \in \Omega : \exists_{m\in\mathbb{N}} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}$. By the definition, there exists $n' \in \mathbb{N}$ such that
$$b \in A_n$$
for all $n \geq n'$. Therefore we find that
$$b \in \bigcap_{m\geq n'} A_m.$$
By the definition of a union we find that
$$b \in \bigcap_{m\geq n'} A_m \subset \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m = \liminf_{n\to\infty} A_n.$$
Therefore we find that
$$\{\omega \in \Omega : \exists_{m\in\mathbb{N}} \text{ such that } \omega \in A_n \text{ for all } n \geq m\} \subset \liminf_{n\to\infty} A_n.$$
We can now conclude that
$$\liminf_{n\to\infty} A_n = \{\omega \in \Omega : \exists_{m\in\mathbb{N}} \text{ such that } \omega \in A_n \text{ for all } n \geq m\}.$$

😃

Solution to Problem 2.4 *(by Castella, A.)* A σ-algebra $\mathscr{G}$ is defined as a set of subsets such that

- $E \in \mathscr{G}$,
- for all infinite sequences $\{A_i\}$ in $\mathscr{G}$ the union $\bigcup_{i=1}^{\infty} A_i$ also belongs to the set $\mathscr{G}$,
- for all sets A in $\mathscr{G}$, the set A^c belongs to $\mathscr{G}$.

Therefore, we begin by verifying that $\Omega \in \mathscr{F}$. From the definition of f it directly follows that

$$f^{-1}(E) = \Omega.$$

Since $\mathscr{G}$ is a σ-algebra, we find that $E \in \mathscr{G}$. Therefore, we conclude that $\Omega \in \mathscr{F}$.

We now verify the second condition. Let us take an arbitrary infinite sequence $\{A_i\}$ in $\mathscr{F}$. By the definition of $\mathscr{F}$ we know that for all A_i, there exists $B_i \in \mathscr{G}$ such that $A_i = f^{-1}(B_i)$. In order to prove the condition, we notice that

$$\bigcup_{i=1}^{\infty} B_i \in \mathscr{G}.$$

Before we continue with the proof, we need to prove an intermediary. Let us take some family of sets $\{C_\alpha\}_{\alpha \in \mathscr{I}}$ where $\mathscr{I}$ is an index set. We will prove that for some function $g : \Omega \to E$ we have

$$g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) = \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

We begin by showing that $g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha)$. Let us take an arbitrary $a \in g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right)$. Then we find that there exists some $b \in \bigcup_{\alpha \in \mathscr{I}} C_\alpha$ such that $g(a) = b$. This implies that there exists an $\alpha \in \mathscr{I}$ such that $b \in C_\alpha$. From this we find that

$$a \in g^{-1}(C_\alpha) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

Since a was chosen arbitrarily, we can conclude that

$$g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right) \subset \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha).$$

Now we show that the converse is also true. Let us take an arbitrary $a' \in \bigcup_{\alpha \in \mathscr{I}} g^{-1}(C_\alpha)$. Then we find that there must exist some $\alpha' \in \mathscr{I}$ such that $a' \in g^{-1}(C_{\alpha'})$. We can now choose some $b' \in C_{\alpha'}$ such that $g(a') = b'$. It is clear that $b' \in \bigcup_{\alpha \in \mathscr{I}} C_\alpha$ as well. Thus we know that

$$a' \in g^{-1}\left(\bigcup_{\alpha \in \mathscr{I}} C_\alpha\right).$$

Since again, our choice of a' was arbitrary, we can conclude that

$$\bigcup_{\alpha\in\mathscr{I}} g^{-1}(C_\alpha) \subset g^{-1}\left(\bigcup_{\alpha\in\mathscr{I}} C_\alpha\right).$$

These two inequalities clearly imply that

$$g^{-1}\left(\bigcup_{\alpha\in\mathscr{I}} C_\alpha\right) = \bigcup_{\alpha\in\mathscr{I}} g^{-1}(C_\alpha).$$

Using the result that we have just proven we can continue with the proof. For the infinite sequence $\{A_i\}$ we find that

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = \bigcup_{i=1}^{\infty} A_i.$$

By the definition of our set $\mathscr{F}$ we now find that

$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{F}.$$

Therefore, we conclude that the second condition holds for $\mathscr{F}$ as well.

We now prove the third and final condition. Let us take $A \in \mathscr{F}$. By the definition of $\mathscr{F}$ we find that there exists $B \in \mathscr{G}$ such that

$$A = f^{-1}(B).$$

By the definition of a σ-algebra we know that $B^c \in \mathscr{G}$. Additionally, we know that

$$f^{-1}(B^c) = f^{-1}(B)^c = A^c.$$

By using these two facts and the definition of the set $\mathscr{F}$ again, we find that

$$A^c \in \mathscr{F}.$$

This proves that the third condition holds as well.

Since all of the required conditions hold, we can come to the final conclusion that $\mathscr{F}$ is indeed a σ-algebra. 😄

Solution to Problem 2.5 *(by Castella, A.)*

- ▶ Before we begin with the proof, we will prove the intermediary that if $A, B \in \mathscr{F}$, then $A \setminus B \in \mathscr{F}$. We begin by noting that

$$A \setminus B = A \cap (B^c).$$

From this it becomes very clear that the statement is true. We know by the definition of a σ-algebra that $B \in \mathscr{F}$ implies $B^c \in \mathscr{F}$. Additionally, we know that intersections of infinite sequences also belong to the same σ-algebra. We take the sequence $A_1 = A$, $A_i = B^c$ for $i \in \mathbb{N} \setminus \{1\}$. With this we find that $A \cap B^c \in \mathscr{F}$ and therefore

$$A \setminus B \in \mathscr{F}.$$

We now proceed to showing that for all infinite sequences $(A_n)_{n\in\mathbb{N}}$, there exits a mutually disjoint sequence who's union is equal to $\cup_{n\in\mathbb{N}} A_n$. We define the sequence $(E_n)_{n\in\mathbb{N}}$ such that

$$E_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

By the intermediary and since infinite unions, and by the same argument as in the intermediary, also finite unions are contained in the σ-algebra, it is easy to see that

$$E_n \in \mathscr{F}$$

for all $n \in \mathbb{N}$. It is clear from the definition of the sequence that it is mutually disjoint and that its union is equal to the union of $(A_n)_{n\in\mathbb{N}}$.

- We take the same arbitrary sequence $(A_n)_{n\in\mathbb{N}}$ as in the previous item. We define the sequence $(F_n)_{n\in\mathbb{N}}$ by

$$F_n = A_n \cup \left(\bigcup_{i=1}^{n-1} A_i \right) = \bigcup_{i=1}^{n} A_i.$$

We first note that it is clear from the definition that this is an increasing sequence, as each element is the union of A_n and all of its predecessors. As mentioned in the previous item, infinite unions are contained in the σ-algebra as well as finite unions. Therefore we know that

$$F_n \in \mathscr{F}$$

for all $n \in \mathbb{N}$. From the definition of a union we also know that

$$\bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{i} A_j \right) = \bigcup_{i=1}^{\infty} A_n.$$

Thus we have now proven that the sequence $(F_n)_{n\in\mathbb{N}}$ is such that $F_n \subset F_{n+1}$ and $\cup_{n\in\mathbb{N}} A_n = \cup_{n\in\mathbb{N}} F_n$.

😄

Solution to Problem 2.13 *(by Beerens, L.)* Let Ω_1 and Ω_2 be two non-empty sets, and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be σ-algebras on Ω_1 and Ω_2 respectively. We consider the product σ-algebra on $\Omega_1 \times \Omega_2$ given by

$$\mathscr{F}_1 \otimes \mathscr{F}_2 := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}).$$

Suppose that $\mathscr{F}_1$ is generated by $\mathscr{A}_1$ and $\mathscr{F}_2$ is generated by $\mathscr{A}_2$. Let $\mathscr{F} := \mathscr{F}_1 \otimes \mathscr{F}_2$ and

$$\mathscr{S} := \sigma(\{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\}).$$

Let

$$A \in \{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\}.$$

Then $A = A_1 \times A_2$ for some $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. Since $\mathscr{A}_1 \subset \mathscr{F}_1$ and $\mathscr{A}_2 \subset \mathscr{F}_2$, we find that

$$A \in \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}.$$

Therefore,

$$\{A_1 \times A_2 : A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2\} \subset \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\},$$

from which it follows that $\mathscr{S} \subset \mathscr{F}$ (By Problem 2.12).

To prove the converse, suppose that $F \in \{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}$. Then there exist $A_1 \in \mathscr{F}_1$ and $A_2 \in \mathscr{F}_2$ such that $F = A_1 \times A_2 = (A_1 \times \Omega_2) \cap (\Omega_1 \times A_2)$. By definition for all $A \in \mathscr{A}_1$ we have

$$A \times \Omega_2 \in \mathscr{S}.$$

Therefore, the collection $\{A \in \mathscr{F}_1 : A \times \Omega_2 \in \mathscr{S}\}$ is a σ-algebra containing $\mathscr{A}_1$ and contained in $\mathscr{F}_1$. It follows that,

$$\mathscr{F}_1 = \sigma(\mathscr{A}_1) = \{A \in \mathscr{F}_1 : A \times \Omega_2 \in \mathscr{S}\}.$$

Since $A_1 \in \mathscr{F}_1$, it now follows that $A_1 \times \Omega_2 \in \mathscr{S}$. Analogously, $\Omega_1 \times A_2 \in \mathscr{S}$. Since $\mathscr{S}$ is a σ-algebra, it follows that $F = (A_1 \times \Omega_2) \cap (\Omega_1 \times A_2) \in \mathscr{S}$. Thus, $\mathscr{F} \subset \mathscr{S}$, from which we can conclude that $\mathscr{F} = \mathscr{S}$, as was to be shown. 😄

Solution to Problem 2.14 *(by Castella, A.)* In order to prove equality of the sets, we will prove that they are both subsets of one another.

- As we know, the set $\mathscr{B}_{\mathbb{R}^2}$ is generated by the π-system of open rectangles. Let us assume that $\mathscr{A}$ is the set of open rectangles in $\mathbb{R}^2$. Let us take some arbitrary $A \in \mathscr{A}$, then there exists $a, b, c, d \in \mathbb{R}$ such that $a < b$, $c < d$, and $A = (a, b) \times (c, d)$. Since the Borel set $\mathscr{B}_{\mathbb{R}}$ is generated by the set of open intervals in $\mathbb{R}$, we find that $(a, b), (c, d) \in \mathscr{B}_{\mathbb{R}}$. From this we immediately find that $(a, b) \times (c, d) \in \mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}}$. Since $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ is the σ-algebra generated by the cross product $\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}}$ we find that

$$\mathscr{B}_{\mathbb{R}^2} \subset \mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}.$$

- We now prove the converse. Let us begin by defining the projections π_1 and π_2 as

$$\pi_1(x, y) = x$$

  and

$$\pi_2(x, y) = y.$$

We will show that these two functions are measurable with respect to $\mathscr{B}_{\mathbb{R}^2}$ and $\mathscr{B}_{\mathbb{R}}$. Let us take an arbitrary $t \in \mathbb{R}$. We will show that the set

$$A = \{(x \times y) \in \mathbb{R}^2 : \pi_1(x, y) < t\}$$

is a Borel set. We find that $\pi_1(x, y) < t$ if and only if $x < t$. Thus we find that

$$A = (-\infty, t) \times \mathbb{R}.$$

Let us define the set B_n as

$$B_n = (-n, t) \times (-n, n)$$

Clearly, for all $n \in \mathbb{N}$, the set B_n is an open rectangle and therefore $B_n \in \mathscr{B}_{\mathbb{R}^2}$. We also find that

$$\bigcup_{i=1}^{\infty} B_n = (-\infty, t) \times \mathbb{R} = A.$$

Since $\mathscr{B}_{\mathbb{R}^2}$ is a σ-algebra, it contains all countable unions of its sets. Therefore, we find that

$$\bigcup_{i=1}^{\infty} B_n = A \in \mathscr{B}_{\mathbb{R}^2}.$$

Thus the set A is indeed a Borel measurable set. Since t was chosen arbitrarily, this holds for all $t \in \mathbb{R}$. Thus π_1 is a measurable function. The proof is analogous for π_2. Since both of these functions are measurable, we know that the preimage of a set in $\mathscr{B}_{\mathbb{R}}$ is in $\mathscr{B}_{\mathbb{R}^2}$. Therefore, for all $A \in \mathscr{B}_{\mathbb{R}}$, we find that

$$A \times \mathbb{R} \in \mathscr{B}_{\mathbb{R}^2},$$

by the measurability of π_1 and

$$\mathbb{R} \times A \in \mathscr{B}_{\mathbb{R}^2},$$

by the measurability of π_2. Let us take arbitrary $A, B \in \mathscr{B}_{\mathbb{R}}$. By our previous result and since $\mathscr{B}_{\mathbb{R}^2}$ is a σ-algebra and thus contains countable intersections, we know that

$$(A \times \mathbb{R}) \cap (\mathbb{R} \times B) = A \times B \in \mathscr{B}_{\mathbb{R}^2}.$$

By the fact that A and B were chosen arbitrarily we find that

$$\mathscr{B}_{\mathbb{R}} \times \mathscr{B}_{\mathbb{R}} \subset \mathscr{B}_{\mathbb{R}^2}.$$

Since $\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}$ is the σ-algebra generated by the Cartesian product of $\mathscr{B}_{\mathbb{R}}$ with itself, we can conclude that

$$\mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}} \subset \mathscr{B}_{\mathbb{R}^2}.$$

Combining both of the results, we arrive at the final conclusion that

$$\mathscr{B}_{\mathbb{R}^2} = \mathscr{B}_{\mathbb{R}} \otimes \mathscr{B}_{\mathbb{R}}.$$

😄

12. Solutions: Measures

Solution to Problem 3.11 *(by Beerens, L.)*

- Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Then

  $$F(x_2) = \mu((-\infty, x_1] \cup (x_1, x_2]) = \mu((-\infty, x_1]) + \mu((x_1, x_2]) \geq \mu((-\infty, x_2]) = F(x_1).$$

  Therefore, F is non-decreasing.

- Consider $\lim_{x \downarrow x_0} F(x)$. This can be written as

  $$\lim_{x \downarrow x_0} \mu((-\infty, x]).$$

  We know that this limit exists, so we can look at a sequence that converges to x_0 from above instead. Let (x_n) be a strictly decreasing sequence in $\mathbb{R}$ such that $x_n \to x_0$. For all $n \in \mathbb{N}$, let $A_n := (-\infty, x_n]$. Then

  $$\lim_{x \downarrow x_0} \mu((-\infty, x]) = \lim_{n \to \infty} \mu((-\infty, x_n]) = \lim_{n \to \infty} \mu(A_n).$$

  Since A_n is a decreasing sequence of measurable sets such that $\mu(A_1)$ is finite, we find that

  $$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu((-\infty, x_0]),$$

  which completes the proof.

- We shall turn towards $\lim_{x \to -\infty} F(x)$. We know that this limit exists, so we can look at a sequence that diverges to $-\infty$ instead. Let the sequence (x_n) be defined by $x_n = -n$. For all $n \in \mathbb{N}$, let $A_n := (-\infty, x_n]$. Then

  $$\lim_{x \downarrow x_0} \mu((-\infty, x]) = \lim_{n \to \infty} \mu((-\infty, x_n]) = \lim_{n \to \infty} \mu(A_n).$$

  Since A_n is a decreasing sequence of measurable sets such that $\mu(A_1)$ is finite, we find that

  $$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu(\varnothing) = 0,$$

  which completes the proof.

- Finally, we consider $\lim_{x\to\infty} F(x)$. We know that this limit exists, so we can look at a sequence that diverges to ∞ instead. Let (x_n) be a sequence in $\mathbb{R}$ defined by $x_n = n$. For all $n \in \mathbb{N}$, let $A_n := (-\infty, x_n]$. Then

  $$\lim_{x\to\infty} F(x) = \lim_{n\to\infty} F(x_n).$$

  Since A_n is an increasing sequence of measurable sets, we find that

  $$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(\mathbb{R}),$$

  which completes the proof.

😄

Solution to Problem 3.14 *(by Beerens, L.)* Let $(\Omega, \mathscr{F})$ be a measure space. Let $\mu : \mathscr{F} \to [0, \infty]$ be a set function which is finitely additive and such that $\mu(\varnothing) = 0$. If μ is a measure, then it is continuous from below, since that is a property of measures. Conversely, suppose that μ is continuous from below. Note that μ assigns to each set $A \in \mathscr{F}$ a nonnegative extended real number $\mu(A)$. It was already assumed that $\mu(\varnothing) = 0$, so we will proceed by proving that μ is σ-additive. Let $A_1, A_2, \ldots$ be a sequence of mutually disjoint elements of $\mathscr{F}$. For all $n \in \mathbb{N}$, let

$$B_n = \bigcup_{i=1}^{n} A_i.$$

Since $\mathscr{F}$ is a σ-algebra, we find that all B_n are sets in $\mathscr{F}$. From finite additivity, we find that for all $n \in \mathbb{N}$ we have

$$\mu(B_n) = \mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i).$$

Note that (B_n) is an increasing sequence of measurable sets in $\mathscr{F}$. By continuity from below it follows that

$$\mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \mu(B_n) = \sum_{i=1}^{\infty} \mu(A_i).$$

Therefore,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus μ is σ-additive and we can conclude that μ is a measure. 😄

Solution to Problem 3.15 *(by Castella, A.)* We note that for a set function μ, continuity from above implies that for all sequences of sets (A_n) such that $A_{i+1} \subset A_i$ and $\mu(A_1) < \infty$ we have

$$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

Let us now choose an arbitrary sequence (B_n) of mutually disjoint sets in $\mathscr{F}$. Let us additionally define the the sequence (C_n) such that $C_j = \bigcup_{i=j}^{\infty} B_i$. By finite additivity of the measure we find that

$$\mu(\Omega) = \mu(C_1) + \mu(C_1^c) < \infty,$$

which, since the measure does not take on negative values, implies that

$$\mu(C_1) < \infty.$$

Additionally, it is clear that the sequence (C_n) is such that $C_{i+1} \subset C_i$. We note that we can also interpret the definition of this sequence as $C_1 = \bigcup_{i=1}^{\infty} B_i$ and $C_j = C_1 \setminus \bigcup_{i=1}^{j-1} B_i$ for $j > 1$. In order to use this property we will quickly show that for all sets $A, B \in \mathscr{F}$ such that $A \subset B$, we have

$$\mu(B \setminus A) = \mu(B) - \mu(A).$$

We find that clearly $B = A \cup (B \setminus A)$. Thus by finite additivity we get that

$$\mu(B) = \mu(A) + \mu(B \setminus A),$$

which proves our property. Thus we can use this and finite additivity to find that for all finite n, we have

$$\mu(C_n) = \mu(C_1) - \mu\left(\bigcup_{i=1}^{n-1} B_i\right) = \mu(C_1) - \sum_{i=1}^{n-1} \mu(B_i).$$

Since we know that the limit exists, we find that

$$\lim_{i\to\infty} \mu(C_i) = \lim_{i\to\infty} \left(\mu(C_1) - \sum_{j=1}^{i-1} \mu(B_j)\right) = \mu(C_1) - \lim_{i\to\infty} \sum_{j=1}^{i-1} \mu(B_j) = \mu(C_1) - \sum_{i=1}^{\infty} \mu(B_i).$$

Applying continuity from above as we stated it at the start of this proof, we find that

$$\lim_{i\to\infty} \mu(C_i) = \mu\left(\bigcap_{i=1}^{\infty} C_i\right) = \mu\left(C_1 \setminus \bigcup_{i=1}^{\infty} B_i\right) = \mu(C_1) - \mu\left(\bigcup_{i=1}^{\infty} B_i\right).$$

Combining both of these results and subtracting $\mu(C_1)$ from both sides, we find that

$$\sum_{i=1}^{\infty} \mu(B_i) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right).$$

Since our sequence (B_n) was chosen as an arbitrary mutually disjoint sequence of sets in $\mathscr{F}$ we find that countable additivity of μ does indeed hold and therefore μ must be a measure. We note that the inverse implication is covered by proposition 3.4.2 of the lecture notes. 😃

Solution to Problem 3.18 *(by Castella, A.)* We will tackle the two parts of the proof separately. We will first show that taking the limit from below will give us the measure of the one element said as proposed by the problem. Then we will use this to show that the set of rational numbers has measure zero if F is continuous.

- Let us assume that ν_F is the unique measure associated to F as given by the question. By definition, we know that
$$\mu((a,b]) = F(b) - F(a).$$
We begin the proof by defining the sequence $(A_n)_{n\in\mathbb{N}}$ such that $A_i = (x - \frac{1}{i}, x]$ for all i. It is clear from its definition that
$$\bigcap_{i=1}^{\infty} A_i = \{x\}.$$
Additionally, by the continuity of the measure proven in proposition 3.4.2 we find that
$$\lim_{i\to\infty} \nu_F(A_i) = \nu_F\left(\bigcap_{i=1}^{\infty} A_i\right) = \nu_F(\{x\}).$$
By the definition of the measure stated above, we also know that
$$\nu_F(A_n) = F(x) - F(x - \frac{1}{n}),$$
for all $n \in \mathbb{N}$. Now assuming that the limit $F(x-)$ exists we find that
$$\lim_{i\to\infty} \nu_F(A_i) = \lim_{i\to\infty}\left(F(x) - F(x - \frac{1}{i})\right) = F(x) - \lim_{i\to\infty} F(x - \frac{1}{i}) = F(x) - F(x-).$$
Combining both of our results we arrive at the conclusion that
$$\nu_F(\{x\}) = F(x) - F(x-).$$

- Assuming that the first part of the proof holds, we find that for all $x \in \mathbb{R}$ we have
$$\nu_F(\{x\}) = F(x) - F(x-).$$
By continuity of the function F however, we find that
$$F(x) = F(x-).$$
This implies that for all one element sets $\{x\}$ we have
$$\nu_F(\{x\}) = 0.$$
We know that the set $\mathbb{Q}$ is countable and thus we can choose a sequence (x_n) such that for all $y \in \mathbb{Q}$, there exists $i \in \mathbb{N}$ such that $x_i = y$. Thus we find that for such a sequence
$$\mathbb{Q} = \bigcup_{i=1}^{\infty} \{x_i\}.$$

However, by countable additivity of the measure ν_F we find that

$$\nu_F(\mathbb{Q}) = \nu_F\left(\bigcup_{i=1}^{\infty}\{x_i\}\right) = \sum_{i=1}^{\infty}\nu_F(\{x_i\}).$$

Since $\{x_n\}$ is ν_F-null for all $n \in \mathbb{N}$, we find that

$$\nu_F(\mathbb{Q}) = \sum_{i=1}^{\infty} 0 = 0.$$

😃

Solution to Problem 3.19 *(by Castella, A.)* We will split the proof of this exercise into 5 individual parts for convenience.

Part I: *Equivalence of the measures on intervals of the form* $(a, b]$.

Let us take a arbitrary values $a, b \in \mathbb{R}$. Let us define the set A as $A = (a, b]$. Then A is a Borel set. We find that for all $x \in \mathbb{R}$, we have $A + x = (a + x, b + x]$ and therefore, by the definition of λ, we get

$$\lambda(A + x) = (b + x) - (a + x) = b - a = \lambda(A).$$

We will now show that we can extend this equivalence to the entire Borel set.

Part II: *Defining the set* $\mathscr{A}_n$

We begin by defining the set E_n as the half open interval

$$E_n = (-n, n]$$

for all $n \in \mathbb{N}$. We will now use the set E_n to define the set of sets on which the two measures are equivalent for the restriction to E_n. We define the set as

$$\mathscr{A}_n = \{A \in \mathscr{B}_{\mathbb{R}} : \lambda(A \cap E_n + x) = \lambda(A \cap E_n) \text{ for all } x \in \mathbb{R}\}.$$

We note that this set contains all the half open intervals in $\mathbb{R}$. This holds for all bounded and unbounded half open intervals.

Part III: *Proving that* $\mathscr{A}_n$ *is a* λ*-system on* $\mathbb{R}$

We will split this into three parts. First we will show that the entire set $\mathbb{R}$ is contained in the set. Then we will show that all complements of sets are contained in $\mathscr{A}_n$ as well. Finally, we will show that all countable unions of mutually disjoint sets belong to it.

- It is clear from its definition that $E_n \subset \mathbb{R}$. Therefore we know that $\mathbb{R} \cap E_n = E_n$. Since E_n is a half open interval, we know that

  $$\lambda(E_n + x) = \lambda(E_n).$$

  Thus we can conclude that $\mathbb{R} \in \mathscr{A}_n$.

- Let us now take an arbitrary set $A \in \mathscr{A}_n$. By the definition of our set $\mathscr{A}_n$ we find that
$$\lambda(A \cap E_n + x) = \lambda(A \cap E_n).$$
Additionally, we note that the measure of E_n is finite since
$$\lambda(E_n) = 2n < \infty.$$
Therefore, the restriction of the measures to E_n is a finite measure. Additionally we note that
$$A^c \cap E_n = E_n \setminus (A \cap E_n).$$
It is clear form the fact that $A \cap E_n \subset E_n$ and since the measures of both sets are finite that we can therefore apply proposition 3.3.1 from the lecture notes. Thus we find that
$$\lambda(A^c \cap E_n + x) = \lambda(E_n + x) - \lambda(A \cap E_n + x).$$
Using what we have already stated and proposition 3.3.1 again we find that
$$\lambda(E_n + x) - \lambda(A \cap E_n + x) = \lambda(E_n) - \lambda(A \cap E_n) = \lambda(A^c \cap E_n).$$
We can now conclude that
$$\lambda(A^c \cap E_n + x) = \lambda(A^c \cap E_n).$$
Therefore we find that $A^c \in \mathscr{A}_n$ as well. Since A was chosen arbitrarily, for all $A \in \mathscr{A}_n$ we have $A^c \in \mathscr{A}_n$.

- We now take an arbitrary sequence $(A_k)_{k\in\mathbb{N}}$ of mutually disjoint sets in $\mathscr{A}_n$. By the disjointedness of the sets and countable additivity of the measures we find that
$$\lambda\left(\bigcup_{i=1}^{\infty}(A_i \cap E_n) + x\right) = \sum_{i=1}^{\infty}\lambda(A_i \cap E_n + x).$$
By the definition of $\mathscr{A}_n$ we find that
$$\lambda(A_k \cap E_n + x) = \lambda(A_k \cap E_n)$$
for all $k \in \mathbb{N}$. Using countable additivity again, we also find that
$$\sum_{i=1}^{\infty}\lambda(A_i \cap E_n + x) = \sum_{i=1}^{\infty}\lambda(A_i \cap E_n) = \lambda\left(\bigcup_{i=1}^{\infty} A_i \cap E_n\right).$$
Combining our results we can conclude that
$$\lambda\left(\bigcup_{i=1}^{\infty} A_i \cap E_n + x\right) = \lambda\left(\bigcup_{i=1}^{\infty} A_i \cap E_n\right)$$
and therefore we find that
$$\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}_n$$
for all sequences of mutually disjoint sets.

Since we have proven all three properties, we can conclude that $\mathscr{A}_n$ is indeed a λ-system.

Part IV: *Applying the $\pi - \lambda$ theorem*

Before we start this part of the proof let us note that the set of all half open intervals from below forms a π-system. A π-system requires only finite intersections to be possible. Finite intersections of half open intervals clearly result in half open intervals as well.

As we stated previously, for all n we know that all half open intervals (including unbounded ones) are contained in the λ-system $\mathscr{A}_n$. The $\pi - \lambda$ theorem states that a λ-system generated from a π-system must be a σ-algebra. Therefore, we can conclude that for all n, the set $\mathscr{A}_n$ must be a σ-algebra. Additionally, since it is generated by the half open intervals we find that

$$\mathscr{B}_{\mathbb{R}} \subset \mathscr{A}_n$$

for all n. We also note that from the definition of the set $\mathscr{A}_n$ we find that

$$\mathscr{A}_n \subset \mathscr{B}_{\mathbb{R}}.$$

Thus we can conclude that

$$\mathscr{B}_{\mathbb{R}} = \mathscr{A}_n$$

for all n in the natural numbers.

Part V: *Applying continuity of measures*

Although we have proven that $\mathscr{A}_n$ is equal to the Borel set for all n, this does not imply that the two measures are equivalent on the Borel set. In order to prove this, we must go one step further and apply the continuity of measures.

Let us take an arbitrary set $A \in \mathscr{B}_{\mathbb{R}}$. We define the sequence $(B_n)_{n\in\mathbb{N}}$ such that for all i, the sets are defined=d as $B_i = A \cap E_i$. We note that B_i is clearly an increasing sequence. We now begin by applying continuity of measures to find that

$$\lim_{i\to\infty} \lambda(B_i) = \lambda\left(\bigcup_{i=1}^{\infty} A \cap E_i\right) = \lambda\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \lambda(A \cap \mathbb{R}) = \lambda(A).$$

Similarly for the other measure, we find that

$$\lim_{i\to\infty} \lambda(B_i + x) = \lambda\left(\bigcup_{i=1}^{\infty} A \cap E_i + x\right) = \lambda(A \cap \mathbb{R} + x) = \lambda(A + x).$$

Using the fact that the σ-algebras $\mathscr{A}_n$ are equal for all n, we find that the measures agree on B_i for all i and therefore

$$\lim_{i\to\infty} \lambda(B_i + x) = \lim_{i\to\infty} \lambda(B_i) = \lambda(A),$$

which implies that

$$\lambda(A + x) = \lambda(A).$$

Since we chose the set $A \in \mathscr{B}_{\mathbb{R}}$ arbitrarily, we find that the two measures agree on the entire Borel set and we can therefore conclude the proof. 😄

Solution to Problem 3.20 *(by Castella, A.)* We make a case distinction between positive and negative τ.

- ▶ Let us take an arbitrary $A \in \mathscr{B}_\mathbb{R}$ such that there exist $a, b \in \mathbb{R}$, where $a < b$ and $A = (a, b]$. We take an arbitrary $\tau \in \mathbb{R}^+$. We find that $\tau A = (\tau a, \tau b]$ and therefore, by the definition of λ, we find

$$\lambda(\tau A) = \tau b - \tau a = \tau(b - a) = \tau\lambda(A).$$

By the same argument as in Problem 3.19 we find that $\lambda(\tau A) = \tau\lambda(A)$ holds for all $\tau \in \mathbb{R}^+$ and $A \in \mathscr{B}_\mathbb{R}$.

- ▶ We begin this part of the proof by showing that for all $a, b \in \mathbb{R}$, where $a < b$, we have

$$\lambda([a, b)) = \lambda((a, b]).$$

Let us define the sequence $(A_n)_{n\in\mathbb{N}}$ as $A_n = (a - \frac{1}{n}, b - \frac{1}{n}]$. From the definition, we find that $\lim_n A_n = [a, b)$, since for all $n \in \mathbb{N}$ we have $a \notin A_n$ and $b \in A_n$. Since we can interchange the measure and the limit, we find that

$$\lambda([a, b)) = \lim_n \lambda(A_n) = \lim_n \left(b + \frac{1}{n} - a - \frac{1}{n}\right) = \lim_n (b - a) = (b - a).$$

Therefore, by the definition of the measure we find that

$$\lambda([a, b)) = \lambda((a, b]).$$

We now use this to prove the statement for an arbitrary $\tau \in \mathbb{R}^-$. Let us take an arbitrary $B \in \mathscr{B}_\mathbb{R}$ such that $B = [a, b)$. We find that $\tau A = (\tau b, \tau a]$, since $\tau b < \tau a$. By the definition of λ and the statement we just proved, we find

$$\lambda(\tau B) = \tau a - \tau b = -\tau(b - a) = -\tau\lambda(B).$$

By the same argument as in Problem 3.19 we find that $\lambda(\tau B) = -\tau\lambda(B)$ for all $B \in \mathscr{B}_\mathbb{R}$ and $\tau \in \mathbb{R}^-$.

We can now use these two results to conclude that

$$\lambda(\tau A) = |\tau|\lambda(A),$$

for all $A \in \mathscr{B}_\mathbb{R}$ and $\tau \in \mathbb{R}$. 😃

Solution to Problem 3.21 *(by Beerens, L.)* Let $F : \mathbb{R} \to \mathbb{R}$ be a non-decreasing, right-continuous function. Let ν_F be the unique measure on $(\mathbb{R}, \mathscr{B}_\mathbb{R})$ such that $\nu_F((a, b]) = F(b) - F(a)$ for all $a < b$. For all $i \in \mathbb{Z}$, let

$$A_i := (i, i + 1]$$

and let $\mathscr{A}$ be the collection of all these sets A_i. Note that $\mathscr{A}$ is a countable cover of $\mathbb{R}$. By the assumption we know that for all $i \in \mathbb{Z}$ we have

$$\nu(A_i) = F(i + 1) - F(i).$$

Since F takes values in $\mathbb{R}$, we find that $\nu(A_i)$ is finite for all $i \in \mathbb{Z}$. Therefore, $\mathscr{A}$ is a countable cover of $\mathbb{R}$ that consists of sets with finite measure. Thus we conclude that μ is σ-finite. 😄

Solution to Problem 3.22 *(by Beerens, L.)* Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space. Let $(A_i)_{i\in\mathbb{N}} \subset \mathscr{F}$ be a sequence such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Let

$$A := \bigcup_{i=1}^{\infty} A_i$$

Since we are talking about a probability space, we know that $\mathbb{P}(\Omega) = 1$. Additionally, we have $A_1 \subset A$, which implies that $\mathbb{P}(A) \geq 1$. However, $A \subset \Omega$, from which it now follows that $\mathbb{P}(A) = 1$. We can see that for all $i \in \mathbb{N}$, we have

$$\mathbb{P}(A \setminus A_i) = \mathbb{P}(A) - \mathbb{P}(A_i) = 0.$$

We shall now proceed by proving that

$$A \setminus \bigcup_{i=1}^{\infty}(A \setminus A_i) = \bigcap_{i=1}^{\infty} A_i.$$

Indeed by definition and De Morgan's law

$$A \setminus \bigcup_{i=1}^{\infty}(A \setminus A_i) = A \cap \left(\bigcup_{i=1}^{\infty}(A \cap A_i^c)\right)^c = A \cap \bigcap_{i=1}^{\infty}(A^c \cup A_i) = \bigcap_{i=1}^{\infty} A_i.$$

Hence, we find that

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(A \setminus \bigcup_{i=1}^{\infty}(A \setminus A_i)\right) = \mathbb{P}(A) - \mathbb{P}\left(\bigcup_{i=1}^{\infty}(A \setminus A_i)\right).$$

The countable subadditivity property of measures implies that

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}(A \setminus A_i)\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A \setminus A_i) = 0.$$

Since measures map to $[0, \infty]$, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty}(A \setminus A_i)\right) = 0$$

and therefore

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \mathbb{P}(A) - \mathbb{P}\left(\bigcup_{i=1}^{\infty}(A \setminus A_i)\right) = 1 - 0 = 1,$$

which completes the proof. 😄

Solution to Problem 3.23 *(by Castella, A.)* Let μ be a finite measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Show that the set $\{x \in \mathbb{R} : \mu(\{x\}) > 0\}$ is at most countable. We start the proof by denoting the set

$$S = \{x \in \mathbb{R} : \mu(\{x\}) > 0\}$$

in order to simplify our notation. We now define the sets T_n such that

$$T_n = \{x \in \mathbb{R} : \mu(\{x\} > \frac{1}{n}\},$$

for all $n \in \mathbb{N}$. We will now use a proof by contradiction to show that T_n must be finite for all n. Thus, let us assume the contrary. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in T_n such that for all $i, j \in \mathbb{N}$ we have $x_i \neq x_j$. By countable additivity of the measure μ we find that

$$\mu\left(\bigcup_{i=1}^{\infty}\{x_i\}\right) = \sum_{i=1}^{\infty} \mu(\{x_i\}).$$

By definition of the set T_n we find that

$$\sum_{i=1}^{\infty} \mu(\{x_i\}) > \sum_{i=1}^{\infty} \frac{1}{n} = \infty.$$

This implies that

$$\mu\left(\bigcup_{i=1}^{\infty}\{x_i\}\right) > \infty,$$

which is a contradiction to the finiteness of the measure μ. Thus we can conclude that the sets T_n are indeed finite for all n. Let us now define the set T as

$$T = \bigcup_{i=1}^{\infty} T_n.$$

Since T_n is finite for all n, this implies that T must be at most countable. We will now show that $T = S$. Let us split this into two part. We being by noting that

$$T \subset S$$

follows trivially as T_n is the set of one element sets with measure greater than $\frac{1}{n}$ which is greater than zero, which implies that

$$T_n \subset S.$$

Thus we only need to prove that $S \subset T$. Let us take some arbitrary element $s \in S$. By definition, we find that

$$\mu(\{s\}) > 0.$$

Since it is strictly greater than, we can choose some $n' \in \mathbb{N}$ such that

$$\mu(\{s\}) > \frac{1}{n'}.$$

From this we find that

$$s \in T_{n'} \subset T.$$

Since s was chosen arbitrarily we find that

$$S \subset T.$$

We can now use this to conclude that

$$S = T.$$

Since T is at most countable, we find that S must be at most countable as well. 😄

13. Solutions: Null sets, completion and independence

Solution to Problem 4.7 *(by Castella, A.)*

- Let us first assume there exists a set $\overline{A} \subset \Omega$ and another set $A \in \mathscr{F}$ such that $A \Delta \overline{A}$ is a null set with respect to the measure μ. We first note that

$$A \Delta \overline{A} = (A \setminus \overline{A}) \cup (\overline{A} \setminus A),$$

which implies that

$$A \setminus \overline{A}$$

and

$$\overline{A} \setminus A$$

are both μ-null sets as well. Since they are both μ-null then they must have measure 0 in the completion of the measure, $\overline{\mu}$. Thus both sets are $\overline{\mu}$-measurable and therefore in $\overline{\mathscr{F}}$. We now note that A and $\overline{A} \setminus A$ are disjoint and therefore

$$\overline{\mu}(A \cup (\overline{A} \setminus A)) = \overline{\mu}(A) + \overline{\mu}(\overline{A} \setminus A).$$

Thus the set $A \cup (\overline{A} \setminus A) = A \cup \overline{A}$ is measurable as well. Since $\overline{\mathscr{F}}$ is a σ-algebra, we know that $(A \setminus \overline{A})^c$ is $\overline{\mu}$-measurable as well. The intersection of two measurable sets must be measurable as well and therefore we find that

$$(A \cup \overline{A}) \cap (A \setminus \overline{A})^c = (A \cup \overline{A}) \setminus (A \setminus \overline{A}) = \overline{A}$$

is a $\overline{\mu}$-measurable set. Thus we can conclude that indeed $\overline{A} \in \overline{\mathscr{F}}$.

- We now assume the converse. Let us choose some arbitrary $\overline{A} \in \overline{\mathscr{F}}$. We note that the σ-algebra $\overline{\mathscr{F}}$ is defined as

$$\overline{\mathscr{F}} = \{A \cup U : A \in \mathscr{F}, U \subset \Omega \text{ is a } \mu\text{-null set}\}.$$

Thus there exists $A \in \mathscr{F}$ and $U \subset \Omega$, where U is μ-null that we can assume disjoint from A, such that $\overline{A} = A \cup U$. We find that

$$A \Delta \overline{A} = (A \setminus (A \cup U)) \cup ((A \cup U) \setminus A) = \varnothing \cup U = U.$$

Since U was a μ-null set, we conclude this direction of the proof.

😄

Solution to Problem 4.8 *(by Beerens, L.)* Let $E \subset \mathbb{R}$ such that $\mathscr{L}^1(E) > 0$. For all $i \in \mathbb{Z}$, let

$$E_i = [i, i+1) \cap E.$$

Since (E_i) is a collection of disjoint measurable sets that covers E, we find that

$$\mathscr{L}^1(E) = \sum_{i \in \mathbb{Z}} \mathscr{L}^1(E_i).$$

Thus, there exists $i \in \mathbb{Z}$ such that $\mathscr{L}^1(E_i) > 0$. We shall now consider

$$A = E_i + i = \{a + i : a \in E_i\}.$$

Notice that problem 4.10 implies that $A \subset [0, 1]$ with

$$\mathscr{L}^1(A) = \mathscr{L}^1(E_i) > 0.$$

Let $V \subset [0, 1]$ be a Vitali set, such that for every $a \in A$ the representative that is chosen for the coset $\mathbb{Q} + a$ lies within A. This can be done, since there is always at least one representative within A, which is the element a itself.

Let $B = A \cap V$. Let (q_j) be a sequence of all rational numbers in $[-1, 1]$. For all $k \in \mathbb{N}$, let

$$B_k = B + q_k = \{a + q_k : a \in B\}.$$

We shall now prove three properties of (B_k):

- A property of V is that the difference between two elements of V can never be a rational. Since $B \subset V$, we know that B has the same property. Suppose there exist $j, k \in \mathbb{N}$, where $j \neq k$, such that $B_j \cap B_k \neq \varnothing$. Then let $a \in B_j \cap B_k$. This would imply that there exist $c, d \in B$ such that
$$a = q_j + c = q_k + d.$$
However, this would mean that $q_j - q_k = d - c \in \mathbb{Q}$, which is a contradiction. Therefore, all B_k are disjoint.

- Now suppose that
$$a \in \bigcup_{k=1}^{\infty} B_k.$$
Then there exist $b \in B \subset [0, 1]$ and $q \in \mathbb{Q} \cap [-1, 1]$ such that $a = b + 1$. Therefore $a \in [-1, 2]$ and hence
$$\bigcup_{k=1}^{\infty} B_k \subset [-1, 2].$$

- Let $a \in A$ and consider the coset $\mathbb{Q} + a$. By construction of the particular Vitali set V, we know that there is a representative for $\mathbb{Q} + a$ within B. Let $b \in B$ be this representative. Then there exists $q \in \mathbb{Q}$ such that $q + b = a$. Since $a, b \in [0, 1]$, we know that $q \in [-1, 1]$. Therefore, there exists $k \in \mathbb{N}$ such that $q = q_k$ and thus $a \in B_k$. Hence

$$a \in \bigcup_{k=1}^{\infty} B_k$$

and we can conclude that

$$A \subset \bigcup_{k=1}^{\infty} B_k.$$

Now, we assume that B is measurable. Using the properties that we proved, we find that

$$A \subset \bigcup_{k=1}^{\infty} B_k \subset [-1, 2].$$

Taking the Lebesgue measures of these sets and using that measures are monotonic and σ-additive, we find that

$$\mathscr{L}^1(A) \leq \sum_{k=1}^{\infty} \mathscr{L}^1(B_k) \leq \mathscr{L}^1([-1, 2]).$$

By problem 4.10, we find that for all $k \in \mathbb{N}$ we have $\mathscr{L}^1(B_k) = \mathscr{L}^1(B)$. Therefore,

$$\mathscr{L}^1(A) \leq \sum_{k=1}^{\infty} \mathscr{L}^1(B) \leq 3.$$

Now we can differentiate between two cases. Either $\mathscr{L}^1(B) = 0$ or $\mathscr{L}^1(B) > 0$. In the first case, we find that $\mathscr{L}^1(A) \leq 0$, which is contradicts the fact that $\mathscr{L}^1(A) > 0$. In the second case,

$$\sum_{k=1}^{\infty} \mathscr{L}^1(B)$$

diverges towards infinity, which contradicts that

$$\sum_{k=1}^{\infty} \mathscr{L}^1(B) \leq 3.$$

Hence, we have a contradiction and we can conclude that B is not measurable. Let $N = B + i$. Then $N \subset E_i \subset E$. However, problem 4.10 shows that the Lebesgue measure is translation invariant, which implies that N is not measurable. Therefore, we have found a set $N \subset E$, which is not measurable. 😄

Solution to Problem 4.9 *(by Beerens, L.)* Let C be the Cantor set.

► For any $A \subset \mathbb{R}$ and $\lambda, b \in \mathbb{R}$, we use the following notation:

$$\lambda A = \{\lambda a : a \in A\}, \qquad A + b = \{a + b : a \in A\}.$$

Let

$$C_0 = [0, 1]$$

and for $n \in \mathbb{N}$

$$C_n := \left(\frac{1}{3} C_{n-1}\right) \cup \left(\frac{1}{3} C_{n-1} + \frac{2}{3}\right).$$

Then

$$C = \bigcap_{i=0}^{\infty} C_i.$$

By this definition of the cantor set, it is clear that $C \subset [0, 1]$. Hence, C is bounded.

Note that C_0 is closed. Suppose for some $k \in \mathbb{N}$ that C_{k-1} is closed. Then let (x_n) be a sequence in $C_{k-1}/3$ and for all $n \in \mathbb{N}$ let $y_n = 3 \cdot x_n$. Then (y_n) is a sequence in C_{k-1} and therefore converges to some point $y \in C_{k-1}$. However, this implies that $x_n \to y/3 \in C_{k-1}/3$. Thus, we find that $C_{k-1}/3$ is closed. Similarly, we can find that $C_{k-1}/3 + 2/3$ is closed too. By the definition of C_k, we now find that C_k is closed too. By induction, all C_n are closed. Therefore C is the intersection of closed sets and hence is a closed set itself.

Now we found that C is a closed and bounded subset of $\mathbb{R}$ and hence is compact.

Note that C and all C_n are Borel sets. Since (C_n) is a decreasing sequence with $\mathscr{L}^1(C_0) = 1 < \infty$, we find that

$$\mathscr{L}^1(C) = \lim_{n \to \infty} \mathscr{L}^1(C_n)$$

Notice that for all $n \in \mathbb{N}$, we have that $\frac{1}{3} C_{n-1}$ and $\frac{1}{3} C_{n-1} + \frac{2}{3}$ are disjoint. Thus, we can use problems 3.19 and 3.20 to find that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mathscr{L}^1(C_n) &= \mathscr{L}^1\left(\frac{1}{3} C_{n-1}\right) + \mathscr{L}^1\left(\frac{1}{3} C_{n-1} + \frac{2}{3}\right) \\
&= \frac{1}{3} \mathscr{L}^1(C_{n-1}) + \mathscr{L}^1\left(\frac{1}{3} C_{n-1}\right) \\
&= \frac{2}{3} \mathscr{L}^1(C_{n-1})
\end{aligned}$$

In addition, we have $\mathscr{L}^1(C_0) = 1$. Therefore $\mathscr{L}^1(C_n) = \left(\frac{2}{3}\right)^n$. It follows that

$$\mathscr{L}^1(C) = \lim_{n \to \infty} \mathscr{L}^1(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.$$

We conclude that the Cantor set is compact and with zero Lebesgue measure.

- We will prove that
$$\text{Card}(C) = \text{Card}([0,1]).$$
Since $C \subset [0,1]$, we know that
$$\text{Card}(C) \leq \text{Card}([0,1]).$$
Since the cardinality of the domain of a surjective function is greater than or equal to the cardinality of its codomain, we shall construct a surjective function $f : C \to [0,1]$. Let $x \in C$. Then there is a unique sequence (a_j) in $\{0,2\}$ such that
$$x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}.$$
Uniqueness follows from the fact that a tail of trailing 2's implies that there is an alternative expansion that does not repeat, but ends in a 1, which is a sequence that is not considered here. Other sequences are obviously unique.

Let g be a function that maps each element $x \in C$ to the corresponding sequence $g(x) := (a_j)$. Define $f : C \to [0,1]$ by
$$f(x) = \sum_{j=1}^{\infty} \frac{g(x)_j}{2^{j+1}}.$$
First, we shall prove that f is surjective. Let $y \in [0,1]$ with binary expansion
$$y = \sum_{j=1}^{\infty} \frac{b_j}{2^j},$$
where (b_j) is a sequence in $\{0,1\}$. Let
$$x = \sum_{j=1}^{\infty} \frac{2 \cdot b_j}{3^j}.$$
Since $2 \cdot (b_j)$ is a sequence in $\{0,2\}$, we find that $x \in C$. Clearly, $f(x) = y$. Thus, we find that f is a surjection. Thus we can finally conclude that
$$\text{Card}(C) = \text{Card}([0,1]).$$

- First, we shall show that int $C = \varnothing$. Suppose that int $C \neq \varnothing$. Let $x \in \text{int}C$. Then for some $\delta > 0$, we have
$$(x - \delta, x + \delta) \in C.$$
We can pick $n \in \mathbb{N}$ such that
$$\frac{1}{3^n} < 2\delta.$$

Then
$$(x-\delta, x+\delta) \in C \subset C_n.$$
However, C_n is a disjoint union of closed intervals of length 3^{-n}, which leads us to a contradiction. Therefore, we can conclude that $\operatorname{int} C = \varnothing$.

Secondly, we shall show that C is totally disconnected. Let $x, y \in C$ such that $x < y$. Let $k \in \mathbb{N}$ such that for $i = 1, \ldots, k-1$ we have
$$g(x)_i = g(y)_i$$
and
$$g(x)_k \neq g(y)_k.$$
Then $g(x)_k = 0$ and $g(y)_k = 2$. For $i = 1, \ldots, k-1$, let $a_i = g(x)_i$. Then
$$x \leq \sum_{j=1}^{k-1} \frac{a_j}{3^j} + 3^{-k} < \sum_{j=1}^{k-1} \frac{a_j}{3^j} + 3^{-k} + 3^{-k-1} < y.$$
As the ternary representation of
$$\sum_{j=1}^{k-1} \frac{a_j}{3^j} + 3^{-k} + 3^{-k-1}$$
ends in two 1's, it does not have an alternative representation without 1's. Thus we can conclude that C is totally disconnected.

Finally, we shall show that C has no isolated points. Let $x \in C$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that
$$\frac{1}{3^n} < \epsilon.$$
Note that after each step from C_k to C_{k+1}, the two boundary points of all the disjoint closed intervals that make up C_k remain in the set. Therefore, these are also in C. We now consider C_n. We know that C_n is the disjoint union of closed intervals of length 3^{-n}. Thus, x lies in such an interval. Let $[a, b] \subset C_n$ be this interval. Then $[a, b] \subset B(x, \epsilon)$. As noted before, we have $a, b \in C$. Since at least one of these two is unequal to x, we have found at least one element $y \in C \setminus \{x\}$ such that $|x - y| < \epsilon$. Thus we can conclude that C has no isolated points.

😄

Solution to Problem 4.10 *(by Castella, A.)* We provide two proofs for the exercise. Let us first recall exercises 3.19 and 3.20. We will refer to the exercises in both proofs and use the result that we attained from them. The first proof is based on the current course content and uses Theorem 5.2.1 to extend the identities to the set of Lebesgue measurable sets. The second proof however, uses the Carathéodory extension theorem. We include this second proof in order to showcase an easier method of proving the identities. Additionally, the second proof only assumes that the identities hold on the half open intervals and does not need the full result of exercises 3.19 and 3.20, which prove the identities for all Borel measurable sets.

- Theorem 5.2.1 of the lectures notes states that the set of Lebesgue measurable sets, denoted by $\overline{\mathscr{B}}^{\mathscr{L}}$, consists of all elements of the form $E \cup U$ where E is a Borel set and $U \in \mathbb{R}^n$ is a Borel-null set. Additionally, we note that the Lebesgue measure is complete, and thus all null sets have measure zero. From this we know that

$$\mathscr{L}(U) = 0.$$

Now let us choose such an element $E \cup U \in \overline{\mathscr{B}}^{\mathscr{L}}$ arbitrarily. We note that

$$U \setminus E \subset U,$$

and is therefore also a Borel-null set. Thus we can assume, without loss of generality, that E and U are disjoint. We now note that the Lebesgue measure $\mathscr{L}$ is equivalent to the measure λ given in exercises 3.19 and 3.20 on all Borel measurable sets. Thus we find that since E is a Borel set, we have

$$\mathscr{L}(E + x) = \mathscr{L}(E)$$

and

$$\mathscr{L}(\tau E) = |\tau| \mathscr{L}(E).$$

From the fact that U is a null set we know that there exists a set $B \in \mathscr{B}$ such that $U \subset B$ and

$$\mathscr{L}(B) = 0.$$

We note that since B is a Borel set then, as we proved in exercises 3.19 and 3.20, both identities must hold. Thus we find that τB and $B + x$ are sets of measure zero as well. It is clear from definition that $U + x \subset B + x$ and $\tau U \subset \tau B$. Therefore, $U + x$ and τU are Borel-null sets as well. Thus we find that since $\mathscr{L}$ is the completion of the measure λ, we have

$$\mathscr{L}(U + x) = \mathscr{L}(\tau U) = 0.$$

Combining this result with the last one and the fact that since $\mathscr{L}$ is a measure it is therefore finitely additive, we find that

$$\mathscr{L}(E \cup U + x) = \mathscr{L}(E + x) + \mathscr{L}(U + x) = \mathscr{L}(E)$$

and

$$\mathscr{L}(\tau E \cup U) = \mathscr{L}(\tau E) + \mathscr{L}(\tau U) = |\tau| \mathscr{L}(E).$$

Finally, we note that, since the measure of U is zero and U is disjoint from E, we have

$$\mathscr{L}(E) = \mathscr{L}(E) + \mathscr{L}(U) = \mathscr{L}(E \cup U).$$

We can therefore conclude that the identities

$$\mathscr{L}(E \cup U + x) = \mathscr{L}(E \cup U)$$

and

$$\mathscr{L}(\tau E \cup U) = |\tau| \mathscr{L}(E \cup U)$$

hold for arbitrary $E \cup U \in \overline{\mathscr{B}}^{\mathscr{L}}$. Thus the equivalence of the measures is established for all Lebesgue measurable sets.

► The Lebesgue measure $\mathscr{L}$ is defined on the ring of half open intervals $(a, b]$. On this ring the measure is defined as

$$\mathscr{L}((a, b]) = b - a.$$

This is clearly equivalent to the measure λ defined in exercises 3.19 and 3.20. In these exercises we proved the equivalences

$$\lambda(A + x) = \lambda(A)$$

and

$$\lambda(\tau A) = |\tau|\lambda(A)$$

on the entire ring. Clearly any measure is a pre-measure as well. Thus $\mathscr{L}$ is a pre-measure on the half open intervals. Additionally, we note that the half open intervals form a ring. Thus they also form a semi-ring. Finally, we show that the pre-measure is also σ-finite. It is defined on the real numbers thus we need to find a countable cover for $\mathbb{R}$ of finite measure sets. Let us take the set

$$\mathscr{A} = \{(n, n + 1] : n \in \mathbb{Z}\}.$$

It is clear that

$$\bigcup_{A \in \mathscr{A}} A = \mathbb{R}.$$

We also note that

$$\mathscr{L}(A) = n - n + 1 = 1 < \infty$$

for all $A \in \mathscr{A}$. Thus we have found a countable cover of finite-measure sets. We can conclude that $\mathscr{L}$ is indeed σ-finite. We now find that, by the Carathéodory extension theorem, this implies that the extension is unique. And thus we find that the extension of both pairs of measures must be the same respectively. We thus find that

$$\mathscr{L}(A + x) = \mathscr{L}(A)$$

and

$$\mathscr{L}(\tau A) = |\tau|\mathscr{L}(A)$$

for all Lebesgue measurable sets A.

😃

14. Solutions: Measurable functions

Solution to Problem 5.3 *(by Kempen, S.F.M.)*
"$\Rightarrow$" Let $s, t \in \mathbb{R}$ and define $A = \{X \geq s\}, B = \{Y \geq t\}$. We have

$$A = \{X \geq s\} = \{\omega \in \Omega : X(\omega) \geq s\} = X^{-1}([s, \infty)).$$

By definition $\sigma(X)$ is the smallest σ-algebra such that $X^{-1}(B) \in \sigma(X)$ for all $B \in \mathscr{B}_\mathbb{R}$ and $[s, \infty) \in \mathscr{B}_\mathbb{R}$ so surely $A \in \sigma(X)$. In a similar way it can be shown that $B \in \sigma(Y)$. Now

$$\mathbb{P}(X \geq s, Y \geq t) = \mathbb{P}(\{X \geq s\} \cap \{Y \geq t\}) = \mathbb{P}(A \cap B)$$

and by our assumption

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(\{X \geq s\}) \cdot \mathbb{P}(\{Y \geq t\}) = \mathbb{P}(X \geq s) \cdot \mathbb{P}(Y \geq t).$$

"$\Leftarrow$" For the first part of the proof, let us define

$$\Lambda_s = \{\ B \in \sigma(Y) : \mathbb{P}(X \geq s, B) = \mathbb{P}(X \geq s)\mathbb{P}(B)\ \}.$$

Λ_s is a collection of sets in $\sigma(Y)$ so $\Lambda_s \subseteq \sigma(Y)$. Furthermore we can show that Λ_s is a λ-system.

1. We have $\Omega \in \Lambda_s$ since

$$\begin{aligned}\mathbb{P}(X \geq s, \Omega) &= \mathbb{P}(\{X \geq s\} \cap \Omega) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \geq s\} \cap \Omega) \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \geq s\}) \\ &= \mathbb{P}(X \geq s) \cdot 1 = \mathbb{P}(X \geq s)\mathbb{P}(\Omega).\end{aligned}$$

2. Let $A \in \Lambda_s$ then

$$\begin{aligned}\mathbb{P}(X \geq s, \Omega \setminus A) &= \mathbb{P}(\{X \geq s\} \cap \Omega \setminus A) \\ &= \mathbb{P}(\{\omega \in \Omega : X(\omega) \geq s\} \cap \Omega \setminus A) \\ &= \mathbb{P}(\{\omega \in \Omega \setminus A : X(\omega) \geq s\}) \\ &= \mathbb{P}(X \geq s) - \mathbb{P}(\{X \geq s\} \cap A) \\ &= \mathbb{P}(X \geq s)(1 - \mathbb{P}(A)) \quad \text{(since } A \in \Lambda_s\text{)} \\ &= \mathbb{P}(X \geq s)\mathbb{P}(\Omega \setminus A).\end{aligned}$$

3. Let $(A_i) \subset \Lambda_s$ be a sequence of mutually disjoint sets then by the σ-additivity of the measure we have

$$\begin{aligned}
\mathbb{P}(X \geq s, \bigcup_{i=1}^{\infty} A_i) &= \mathbb{P}(\{X \geq s\} \cap \bigcup_{i=1}^{\infty} A_i) \\
&= \mathbb{P}(\{\omega \in \left(\Omega \cap \bigcup_{i=1}^{\infty} A_i\right) : X(\omega) \geq s\}) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(\{\omega \in \Omega \cap A_i : X(\omega) \geq s\}) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(\{X \geq s\} \cap A_i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(\{X \geq s\})\mathbb{P}(A_i) \\
&= \mathbb{P}(X \geq s)\mathbb{P}(\bigcup_{i=1}^{\infty} A_i).
\end{aligned}$$

So we conclude that Λ_s is a λ-system. By assumption, we also have $\{Y \geq t\} \subseteq \Lambda_s$ for all $t \in \mathbb{R}$. Next we define

$$\mathscr{E} = \{\{Y \geq t\}, t \in \mathbb{R}\}$$

and show that this is a π-system. Let $s, r \in \mathbb{R}$ then $\{Y \geq s\}, \{Y \geq r\} \in \mathscr{E}$ and

$$\{Y \geq s\} \cap \{Y \geq r\} = \{Y \geq \min(s, r)\} \in \mathscr{E},$$

since the minimum of two real numbers is just a real number. Thus $\mathscr{E}$ is a π-system.
Note $\sigma(Y) \subseteq \sigma(\mathscr{E})$ by definition of $\sigma(Y)$ and next to that $\mathscr{E} \subseteq \Lambda_s$ by our assumption. We can now apply the π-λ theorem, which tells us

$$\sigma(Y) \subseteq \sigma(\mathscr{E}) \overset{\pi-\lambda}{\subseteq} \Lambda_s \subseteq \sigma(Y),$$

so in fact we have $\Lambda_s = \sigma(Y)$. Thus we can conclude that $\mathbb{P}(X \geq s, B) = \mathbb{P}(X \geq s)\mathbb{P}(B)$ holds for all $B \in \sigma(Y)$.
For the second part of the proof, define

$$\mathscr{H} = \{A \in \sigma(X) : \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\ \forall B \in \sigma(Y)\}.$$

We will show that $\mathscr{H}$ is a λ-system. Let $B \in \sigma(Y)$.

1. We have $\Omega \in \mathscr{H}$ since $\mathbb{P}(\Omega \cap B) = \mathbb{P}(B) = \mathbb{P}(B)\mathbb{P}(\Omega)$.

2. Let $A \in \mathscr{H}$ then

$$\begin{aligned}
\mathbb{P}(B \cap (\Omega \setminus A)) &= \mathbb{P}(B \setminus (A \cap B)) \\
&= \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad \text{(by exclusion)} \\
&= \mathbb{P}(B)(1 - \mathbb{P}(A)) \quad \text{(since } A \in \mathscr{H}\text{)} \\
&= \mathbb{P}(B)\mathbb{P}(\Omega \setminus A).
\end{aligned}$$

3. Let $(A_i) \subset \mathscr{H}$ be a sequence of mutually disjoint sets then by the σ-additivity of the measure $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ and

$$\begin{aligned}\mathbb{P}(B \cap \left(\bigcup_{i=1}^{\infty} A_i\right)) &= \mathbb{P}(\bigcup_{i=1}^{\infty}(B \cap A_i)) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(B \cap A_i) \quad (\sigma\text{-add.}) \\ &= \sum_{i=1}^{\infty} \mathbb{P}(B)\mathbb{P}(A_i) \\ &= \mathbb{P}(B)\mathbb{P}(\bigcup_{i=1}^{\infty} A_i).\end{aligned}$$

So $\mathscr{H}$ is a λ-system.
Let $\mathscr{G} = \{\{X \geq s\} : s \in \mathbb{R}\}$. It can be shown that $\mathscr{G}$ is a π-system in a similar way as we did for $\mathscr{E}$ and, as before, $\sigma(\mathscr{G}) \subseteq \sigma(X)$ (by definition of $\sigma(X)$ and $\mathscr{G} \in \mathscr{H}$ (by assumption). Applying the π-λ theorem we obtain

$$\sigma(X) \subseteq \sigma(\mathscr{G}) \overset{\pi-\lambda}{\subseteq} \mathscr{H} \subseteq \sigma(X),$$

which implies $\mathscr{H} = \sigma(X)$. Thus now we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\ \forall A \in \sigma(X), B \in \sigma(Y).$$

😃

Solution to Problem 5.4 *(by Castella, A.)*

► We begin by showing that $\max(f, g)$ is measurable. It is clear that for all $t \in \mathbb{R}$ we have

$$\max(f, g) \leq t \implies f \leq t \wedge g \leq t.$$

Thus let us take such an arbitrary t. We find that the equality

$$\{\omega \in \Omega : \max(f, g)(\omega) \leq t\} = \{\omega \in \Omega : f(\omega) \leq t \wedge g(\omega) \leq t\}$$

directly follows. Additionally, we note that this set is the intersection of the sets

$$\{\omega \in \Omega : f(\omega) \leq t\}$$

and

$$\{\omega \in \Omega : g(\omega) \leq t\}.$$

Since f and g are both measurable functions, we know that the two above mentioned sets are measurable. Additionally, since $\mathscr{F}$ is a σ-algebra, we know that the intersection of the two sets must be in $\mathscr{F}$ as well. Thus we conclude that the set

$$\{\omega \in \Omega : \max(f, g)(\omega) \leq t\}$$

is indeed measurable. Since t was arbitrary, this holds for all $t \in \mathbb{R}$. Thus we can conclude that $\max(f, g)$ is indeed a measurable function.

- We will now prove that $\min(f,g)$ is a measurable function as well. We note that $-f$ and $-g$ are clearly measurable functions, since

$$\{\omega \in \Omega : f(\omega) \leq t\} = \{\omega \in \Omega : -f(\omega) > -t\}$$

holds for all $t \in \mathbb{R}$. The same applies to any measurable function. Thus the measurablility of $-g$ and $-f$ directly follows. By what we proven in the previous item we find that $\max(-f,-g)$ must be measurable. This implies that the function

$$-\max(-f,-g)$$

must be measurable as well. Finally we note that

$$\min(f,g) = -\max(-f,-g),$$

which implies that $\min(f,g)$ is a measurable function.

- Since the function 0 is clearly measurable and f and g were arbitrary measurable functions, we know that

$$-\min(f,0)$$

and

$$\max(f,0)$$

must be measurable as well. Let us now prove an intermediary. We will show that $f+g$ is measurable. We note that for some arbitrary $t \in \mathbb{R}$, we have that $(f+g)(\omega) < t$ if and only if there exists a rational number r such that $f(\omega) < r$ and $g(\omega) < t-r$. Therefore we find that

$$\{\omega \in \Omega : (f+g)(\omega) < t\} = \bigcup_{r \in \mathbb{Q}} \left(\{\omega \in \Omega : f(\omega) < r\} \cap \{\omega \in \Omega : g(\omega) < t-r\} \right).$$

Since the intersection of two measurable sets must be measurable, each term in the union is measurable. Since countable unions of measurable sets must be measurable, we conclude that the set

$$\{\omega \in \Omega : (f+g)(\omega) < t\}$$

is indeed measurable. Since t was arbitrary, we have proven the intermediary that $f+g$ is measurable. Thus we can now use this result to show that

$$\max(f,0) - \min(f,0)$$

is measurable. Since

$$|f| = \max(f,0) - \min(f,0)$$

we conclude that the function $|f|$ is indeed measurable.

😄

Solution to Problem 5.7 *(by Castella, A.)* We will present two different proofs for this problem. The first proof will make us of the hint and properties of measurable sets. The second proof however, will use of the result from Problem 5.8 rather than the hint.

▶ Let us begin by denoting the set

$$A = \{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = 0\}.$$

We will now show that this set is equivalent to the set denoted by

$$B = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{m}\}.$$

We will prove this equivalence in the standard way, by choosing an element ω from either set and showing that it is contained in the other. We split this into two parts.

- Let us take some $\omega \in A$. We note that for all $m \geq 1$, and therefore also all $k \in \mathbb{N}$ (where $k \leq m$), there exists some $N' \in \mathbb{N}$ such that

  $$\forall_{n\geq N} : |f_n(\omega)| < \frac{1}{m} \leq \frac{1}{k}.$$

  Thus it is clear that, for all $k \in \mathbb{N}$, we have

  $$\omega \in \bigcap_{n=N'}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}$$

  and therefore also

  $$\omega \in \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}.$$

  Since we stated that this holds for all $k \in \mathbb{N}$, it is clear that

  $$\omega \in B.$$

  We therefore conclude this direction of the proof.

- Let us take some $\omega \in B$. From the definition of our set, we can derive that this implies that for all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

  $$\forall_{n\geq N} : |f_n(\omega)| < \frac{1}{k}.$$

  Let us thus take some arbitrary $m > 0$. We know that for all such m there exists some $k \in \mathbb{N}$ such that $k \geq m$ which implies that $\frac{1}{k} \leq \frac{1}{m}$. Since there always exists $N \in \mathbb{N}$ such that the above holds, we find that

  $$\forall_{n\geq N} : |f_n(\omega)| < \frac{1}{k} \leq \frac{1}{m}.$$

  Following the hint, this implies that

  $$\lim_{n\to\infty} f_n(\omega) = 0.$$

  Thus we find that

  $$\omega \in A.$$

Having proven both directions, we can conclude that the sets A and B are equal. We can now use this to prove the measurability of A by proving the measurability of B instead. However, in order to do this, we must first prove the measurability of the set

$$\{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}$$

for all $k \in \mathbb{N}$. We begin by noting that from the measurability of f_n and the fact that $\frac{1}{k} \in \mathbb{R}$ it follows that

$$\{\omega \in \Omega : f_n(\omega) < \frac{1}{k}\}$$

and

$$\{\omega \in \Omega : f_n(\omega) > -\frac{1}{k}\}$$

are measurable sets. The intersection of these two sets is equal to

$$\{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}.$$

Since the intersection of two measurable sets must be measurable (as they form a σ-algebra), we find that this set is measurable. We now note that all countable intersections of measurable sets must be measurable. Thus the set

$$\bigcap_{n=N}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}$$

is measurable for all $N, k \in \mathbb{N}$. By the same countability argument for unions we find that the set

$$\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}$$

and therefore also the set

$$\bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{\omega \in \Omega : |f_n(\omega)| < \frac{1}{k}\}$$

must be measurable. Thus we conclude that B is measurable and therefore A must be measurable.

- We begin this proof by proving the measurability of the functions $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$. In order to do this we first prove the measurability of the functions $\sup_{n\geq k} f_n$ and $\inf_{n\geq k} f_n$ for all $k \in \mathbb{N}$. Let us take an arbitrary $t \in \mathbb{R}$. By the definition of the supremum and the infimum we find that

$$\{\omega \in \Omega : \sup_{n\geq k} f_n(\omega) \leq t\} = \bigcap_{n=k}^{\infty} \{\omega \in \Omega : f_n(\omega) \leq t\}$$

and

$$\{\omega \in \Omega : \inf_{n \geq k} f_n(\omega) \leq t\} = \bigcup_{n=k}^{\infty} \{\omega \in \Omega : f_n(\omega) \leq t\}.$$

Since f_n is measurable for all n, we find that the above sets must be measurable. Since t was arbitrary, we find that this holds for all $t \in \mathbb{R}$ and therefore $\sup_{n \geq k} f_n$ and $\inf_{n \geq k} f_n$ are measurable functions. We note that k was arbitrary as well. Thus let us define the sequences $g_k = \inf_{n \geq k} f_n$ and $h_k = \sup_{n \geq k} f_n$. These are two sequences of measurable sets just like f_n and thus we find that, setting $k = 1$, the functions

$$\inf_k h_k$$

and

$$\sup_k g_k$$

are measurable. Finally, we know that

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_n$$

and

$$\liminf_n f_n = \sup_n \inf_{k \geq n} f_n,$$

which implies the measurability of the functions $\limsup_n f_n$ and $\liminf_n f_n$. We can now use this result combined with the result from Problem 5.8 to prove the exercise. Using problem 5.8, and the measurability of the 0 function, we find that

$$\{\omega \in \Omega : \liminf_n f_n(\omega) = 0\}$$

and

$$\{\omega \in \Omega : \limsup_n f_n(\omega) = 0\}$$

are measurable sets. Finally, we know that

$$\limsup_{n \to \infty} f_n(\omega) = \limsup_{n \to \infty} f_n(\omega) = \lim_{n \to \infty} f_n(\omega)$$

for all ω where the limit exists. From this we find that

$$\{\omega \in \Omega : \lim_{n \to \infty} f_n(\omega) = 0\} = \{\omega \in \Omega : \liminf_{n \to \infty} f_n(\omega) = 0\} \cap \{\omega \in \Omega : \limsup_{n \to \infty} f_n(\omega) = 0\}.$$

Since the intersection of measurable sets must be measurable, we now conclude the proof.

😄

Solution to Problem 5.8 *(by Castella, A.)* We begin the proof by showing that the set

$$\{\omega \in \Omega : f(\omega) < g(\omega)\}$$

is measurable. We know that ω is in this set if and only if there exists some $r \in \mathbb{Q}$ such that

$$f(\omega) < r < g(\omega).$$

Thus we find that

$$\{\omega \in \Omega : f(\omega) < g(\omega)\} = \bigcup_{r \in \mathbb{Q}} \{\omega \in \Omega : f(\omega) < r\} \cap \{\omega \in \Omega : g(\omega) > r\}.$$

Since $\mathbb{Q}$ is countable and both the sets

$$\{\omega \in \Omega : f(\omega) < r\}$$

and

$$\{\omega \in \Omega : g(\omega) > r\}$$

are measurable, we find that the set $\{\omega \in \Omega : f(\omega) < g(\omega)\}$ is measurable as well. Since g and f are arbitrary measurable functions, we now that if we flip the functions, then the new set is still measurable. Additionally, a σ-algebra contains the complement of all of its sets and thus we find that

$$\{\omega \in \Omega : g(\omega) < f(\omega)\}^c = \{\omega \in \Omega : f(\omega) \leq g(\omega)\}$$

is measurable as well. By the same argument, the set

$$\{\omega \in \Omega : g(\omega) \leq f(\omega)\}$$

is measurable too. Finally, since finite intersections of measurable sets are measurable, we find that the set

$$\{\omega \in \Omega : g(\omega) = f(\omega)\} = \{\omega \in \Omega : f(\omega) \leq g(\omega)\} \cap \{\omega \in \Omega : g(\omega) \leq f(\omega)\}$$

is measurable. We thus conclude the proof. 😄

Solution to Problem 5.14 *(by Castella, A.)* We have already proven in problem 5.4 that if f is measurable, then $\max(f, 0)$ and $\min(f, 0)$ are measurable as well. Thus we only need to prove the other direction of the implication. Let us assume that f^+ and f^- are measurable. Then, by intermediaries proven in problem 5.4, we know that $f^+ + f^-$ is measurable as well. It is clear that

$$f = f^+ + f^-$$

and therefore we can already conclude that f is measurable. Thus we have proven both directions of the implication. ======= 😄

Solution to Problem 5.9 *(by Beerens, L.)* Let $(\Omega, \mathcal{F})$ be a measurable space and (f_n) be a sequence of $\mathcal{F}$-measurable functions. We know that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are $\mathcal{F}$-measurable functions. For all $\omega \in \Omega$ we have

$$\lim_{n\to\infty} f_n(\omega) \text{ exists } \iff \limsup_{n\to\infty} f_n(\omega) = \liminf_{n\to\infty} f_n(\omega).$$

From problem 5.8 it now follows that

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) \text{ exists}\} = \{\omega \in \Omega : \limsup_{n\to\infty} f_n(\omega) = \liminf_{n\to\infty} f_n(\omega)\}$$

is measurable. 😃

Solution to Problem 5.19 *(by Beerens, L.)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is a non-negative function such that $\sqrt{f}$ is measurable. It is known that the product of two measurable functions is measurable. Therefore, $\sqrt{f} \cdot \sqrt{f} = f$ is measurable. Thus, there does not exist such non-measurable f. Since it was not proven in the notes that the product of two measurable functions is measurable, we shall give a quick proof here.

Let $f, g : (\Omega, \mathscr{F}) \to \mathbb{R}$ be $\mathscr{F}$-measurable. First, we show that f^2 is measurable. For all $t \in \mathbb{R}$, we have

$$\{\omega \in \Omega : f^2(\omega) \geq t\} = \{\omega \in \Omega : |f(\omega)| \geq \sqrt{\max\{0, t\}}\},$$

which is measurable by the measurability of $|f|$. Therefore, f^2 is measurable. In the same way, the square of any measurable function is measurable, including g^2 and $(f + g)^2$. But $(f + g)^2 = f^2 + g^2 + 2fg$, so $fg = 1/2((f + g)^2 - f^2 - g^2)$ and hence is measurable. 😃

Solution to Problem 5.20 *(by Beerens, L.)* Suppose $f : (\mathbb{R}, \overline{\mathscr{B}}^{\mathscr{L}}) \to (\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ is continuous almost everywhere. Then f is continuous at ω for all $\omega \in \mathbb{R} \setminus U$ and not continuous at ω for all $\omega \in U$, for some set $U \in \overline{\mathscr{B}}^{\mathscr{L}}$ such that $\mathscr{L}^1(U) = 0$. Let $A = \mathbb{R} \setminus U$. Then $f|_A$ is continuous and therefore measurable. Since U has measure zero, we know that all its subsets are measurable, as the Lebesgue measure is a complete measure. Therefore, $f|_U$ is a measurable function. Now, we shall prove that f is measurable. Let $B \in \mathscr{B}_{\mathbb{R}}$. Since $f|_A$ and $f|_U$ are measurable, we find that there exists $C, D \in \overline{\mathscr{B}}^{\mathscr{L}}$ such that

$$f|_A^{-1}(B) = C \cap A$$

and

$$f|_U^{-1}(B) = D \cap U.$$

Therefore,

$$f^{-1}(B) = f|_A^{-1}(B) \cup f|_U^{-1}(B) = (C \cap A) \cup (D \cap U) \in \overline{\mathscr{B}}^{\mathscr{L}}.$$

Thus, we can conclude that f is Lebesgue measurable. 😃

Solution to Problem 5.21 *(by Beerens, L.)* Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then for all $a \in \mathbb{R}$, the limit

$$\lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$

exists. This limit is $f'(a)$. Therefore,

$$\lim_{n\to\infty} n\left(f\left(a + \frac{1}{n}\right) - f(a)\right) = f'(a).$$

For all $n \in \mathbb{N}$, let

$$f_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right).$$

Then for all $a \in \mathbb{R}$ we have $\lim_{n\to\infty} f_n(a) = f'(a)$. Therefore (f_n) converges to f' pointwise and we can denote f' by

$$f' = \limsup_{n\to\infty} f_n.$$

Note that $f(x+1/n)$ is just a shifted version of f and is therefore measurable. It is clear that f_n is simply a linear combination of two measurable functions and hence is measurable itself. Thus, (f_n) is a sequence of measurable functions and we can conclude that f' is measurable. 😄