



TYPE THEORY

Theory exercises

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Exercise 1

3.1 Singleton type and exercises

3. Show that the rule E-S) is derivable in the type theory T_1 replacing the rule E-S) elimination with the E- N_{1prog}) rule and adding the substitution and weakening rules and the sanitary checks rules set out in the previous sections.

- Rule E-S)

$$\text{E-S)} \frac{t \in N_1 [\Gamma] \quad M(z) \text{ type } [\Gamma, z \in N_1] \quad c \in M(\star) [\Gamma]}{El_{N_1}(t, c) \in M(t) [\Gamma]}$$

- Rule E- N_{1prog})

$$\text{E-}N_{1prog}) \frac{D(w) \text{ type } [\Sigma, w \in N_1] \quad d \in D(\star) [\Sigma]}{El_{N_1}(w, d) \in D(w) [\Sigma, w \in N_1]}$$

Solution

Assuming:

$a_1) t \in N_1 [\Gamma]$

$a_2) M(z) \text{ type } [\Gamma, z \in N_1]$

$a_3) c \in M(\star) [\Gamma]$

The rule E-S) is derivable:

$$\text{E-}N_{1prog}) \frac{\frac{a_2 \quad a_3}{M(z) \text{ type } [\Gamma, z \in N_1] \quad c \in M(\star) [\Gamma]} \quad a_1}{\text{sub-ter)} \frac{El_{N_1}(z, c) \in M(z) [\Gamma, z \in N_1] \quad t \in N_1 [\Gamma]}{El_{N_1}(t, c) \in M(t) [\Gamma]}}$$

Exercise 2

3.2 Natural Numbers Type

3. Define the addition operation using the rules of the natural number type

$$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$$

such that $x + 0 = x \in \text{Nat} [x \in \text{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$El_{\text{Nat}}(y, x, (w, z). \text{succ}(z))$$

Let $\Gamma = x \in \text{Nat}, y \in \text{Nat}$;

$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$ is derivable:

$$\begin{array}{c}
\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{\text{Nat type } [\Gamma, w \in \text{Nat}]}{\Gamma, w \in \text{Nat}, z \in \text{Nat} \text{ cont}} \\
\text{F-c)} \frac{\Gamma, w \in \text{Nat}, z \in \text{Nat} \text{ cont}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{var)} \frac{\Gamma \text{ cont}}{y \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type } [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]}{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{E-Nat)} \frac{}{\text{El}_{\text{Nat}}(y, x, (w, z). \text{succ}(z)) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, y \in \text{Nat}$
- $x \in \text{Nat}, y \in \text{Nat} \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\
\text{F-c)} \frac{\text{Nat type } []}{x \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}]}{x \in \text{Nat}, y \in \text{Nat} \text{ cont}}
\end{array}$$

Correctness

The definition is correct, in fact:

Base case

$$y = 0 \Rightarrow x + y = x + 0 = x$$

This is true, because:

- $x + 0 = \text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z))$
- $\text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z)) = x \in \text{Nat} [x \in \text{Nat}]$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\
\text{F-c)} \frac{\text{Nat type } []}{x \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}]}{x \in \text{Nat}, w \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat}, w \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}, w \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}, w \in \text{Nat}]}{x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat} \text{ cont}} \\
\text{var)} \frac{x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat} \text{ cont}}{z \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{I}_2\text{-Nat)} \frac{\text{succ}(z) \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]}{\text{succ}(z) \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{C}_1\text{-Nat)} \frac{}{\text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z)) = x \in \text{Nat} [x \in \text{Nat}]}
\end{array}$$

Inductive case

$$y = \text{succ}(v) [v \in \text{Nat}] \Rightarrow x + y = x + \text{succ}(v) = \text{succ}(x + v)$$

This is true, because:

- $x + \text{succ}(v) = \text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z))$
- $\text{succ}(x + v) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z)))$
- Let $\Gamma = x \in \text{Nat}, v \in \text{Nat}$;
 $\text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z)) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z))) \in \text{Nat} [\Gamma]$ derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{v \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type} [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat cont}} \quad \text{F-Nat)} \frac{\text{Nat type} [\Gamma, w \in \text{Nat}]}{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}} \quad \text{var)} \frac{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{C}_2\text{-Nat)} \frac{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]}{\text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z)) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z))) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, v \in \text{Nat}$
- $x \in \text{Nat}, v \in \text{Nat cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type} []} \\
\text{F-c)} \frac{\text{Nat type} []}{x \in \text{Nat cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat cont}}{\text{Nat type} [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type} [x \in \text{Nat}]}{x \in \text{Nat}, v \in \text{Nat cont}}
\end{array}$$

Exercise 3

3.2 Natural Numbers Type

4. Define the addition operation using the rules of the natural number type

$$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$$

such that $0 + x = x \in \text{Nat} [x \in \text{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$\text{El}_{\text{Nat}}(x, y, (w, z). \text{succ}(z))$$

Let $\Gamma = x \in \text{Nat}, y \in \text{Nat}$;

$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type} [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{y \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat cont}} \quad \text{F-Nat)} \frac{\text{Nat type} [\Gamma, w \in \text{Nat}]}{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}} \quad \text{var)} \frac{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{E-Nat)} \frac{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]}{\text{El}_{\text{Nat}}(x, y, (w, z). \text{succ}(z)) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, y \in \text{Nat}$
- $x \in \text{Nat}, y \in \text{Nat cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type} []} \\
\text{F-c)} \frac{\text{Nat type} []}{x \in \text{Nat cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat cont}}{\text{Nat type} [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type} [x \in \text{Nat}]}{x \in \text{Nat}, y \in \text{Nat cont}}
\end{array}$$

The definition is correct, in fact:

$$x = 0 \Rightarrow x + y = 0 + y = y$$

- $0 + y = \text{El}_{\text{Nat}}(0, y, (w, z). \text{succ}(z))$
- $\text{El}_{\text{Nat}}(0, y, (w, z). \text{succ}(z)) = y \in \text{Nat} \ [y \in \text{Nat}]$ derivable:

5

Exercise 4

3.2 Natural Numbers Type

6. Define the predecessor operator

$$\mathbf{p}(x) \in \text{Nat} \ [x \in \text{Nat}]$$

such that

$$\mathbf{p}(0) = 0$$

$$\mathbf{p}(\text{succ}(\mathbf{n})) = \mathbf{n}$$

Solution

The predecessor $\mathbf{p}(x)$ can be defined as:

$$\text{El}_{\text{Nat}}(x, 0, (w, z). w)$$

$\mathbf{p}(x) \in \text{Nat} \ [x \in \text{Nat}]$ is derivable:

$$\frac{\begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{var)} \frac{}{x \in \text{Nat } [x \in \text{Nat}]} \\ \text{E-Nat)} \frac{}{x \in \text{Nat } [x \in \text{Nat}]} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat, } w \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}} \\ \text{var)} \frac{}{w \in \text{Nat } [x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat}]} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{I}_1\text{-Nat)} \frac{}{0 \in \text{Nat } [x \in \text{Nat}]} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat, } w \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}} \\ \text{var)} \frac{}{w \in \text{Nat } [x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat}]} \end{array}}{\text{El}_{\text{Nat}}(x, 0, (w, z). w) \in \text{Nat } [x \in \text{Nat}]}$$

Correctness

The definition is correct, in fact:

Base case

$$x = 0 \Rightarrow \mathbf{p}(x) = \mathbf{p}(0) = 0$$

This is true, because:

- $\mathbf{p}(0) = \text{El}_{\text{Nat}}(0, 0, (w, z). w)$
- $\text{El}_{\text{Nat}}(0, 0, (w, z). w) = 0 \in \text{Nat} \ []$ derivable:

$$\frac{\begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{C}_1\text{-Nat)} \frac{}{\text{Nat type } []} \end{array} \quad \begin{array}{c} \text{I}_1\text{-Nat)} \frac{[] \text{ cont}}{0 \in \text{Nat } []} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{w \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [w \in \text{Nat}]} \\ \text{F-c)} \frac{}{w \in \text{Nat, } z \in \text{Nat cont}} \\ \text{var)} \frac{}{w \in \text{Nat } [w \in \text{Nat, } z \in \text{Nat}]} \end{array}}{\text{El}_{\text{Nat}}(0, 0, (w, z). w) = 0 \in \text{Nat } []}$$

Inductive case

$$x = \text{succ}(y) \ [y \in \text{Nat}] \Rightarrow \mathbf{p}(x) = \mathbf{p}(\text{succ}(y)) = y$$

This is true, because:

- $\mathbf{p}(\text{succ}(y)) = \text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w)$
- $\text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w) = y \in \text{Nat} \ [y \in \text{Nat}]$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c)} \frac{\text{Nat type } []}{y \in \text{Nat cont}} \quad \text{var)} \frac{y \in \text{Nat cont}}{y \in \text{Nat } [y \in \text{Nat}]} \quad \text{C}_2\text{-Nat)} \frac{y \in \text{Nat } [y \in \text{Nat}]}{y \in \text{Nat } [y \in \text{Nat}]} \\
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c)} \frac{\text{Nat type } []}{y \in \text{Nat cont}} \quad \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c)} \frac{\text{Nat type } [y \in \text{Nat}]}{y \in \text{Nat, } w \in \text{Nat cont}} \\
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } [y \in \text{Nat, } w \in \text{Nat}]} \quad \text{F-c)} \frac{\text{Nat type } [y \in \text{Nat, } w \in \text{Nat}]}{y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}} \quad \text{var)} \frac{y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}}{w \in \text{Nat } [y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat}]} \\
\text{I}_1\text{-Nat)} \frac{[] \text{ cont}}{0 \in \text{Nat } []} \quad \text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w) = y \in \text{Nat } [y \in \text{Nat}]
\end{array}$$

Exercise 5

3.6 Martin-Löf's Intensional Propositional Equality

7. Prove that there exists a proof-term **pf** such that.

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, \star, w) [w \in \mathbf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \text{El}_{\mathbf{N}_1}(w, \text{id}(\star)), (x). \text{id}(x)$, such that

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, \star, w) [w \in \mathbf{N}_1]$$

is derivable, in fact $\text{El}_{\mathbf{N}_1}(w, \text{id}(\star)) \in \text{Id}(\mathbf{N}_1, \star, w) [\Gamma]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{w \in \mathbf{N}_1 [\Gamma]} \quad \text{F-S)} \frac{\Gamma \text{ cont}}{\mathbf{N}_1 \text{ type } [\Gamma]} \quad \text{I-S)} \frac{\Gamma \text{ cont}}{\star \in \mathbf{N}_1 [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{w \in \mathbf{N}_1 [\Gamma]} \quad \text{I-S)} \frac{\Gamma \text{ cont}}{\star \in \mathbf{N}_1 [\Gamma]} \\
\text{F-Id)} \frac{\mathbf{N}_1 \text{ type } [\Gamma]}{\text{Id}(\mathbf{N}_1, \star, w) \text{ type } [\Gamma]} \quad \text{I-Id)} \frac{\star \in \mathbf{N}_1 [\Gamma]}{\text{id}(\star) \in \text{Id}(\mathbf{N}_1, \star, \star) [\Gamma]} \\
\text{E-S)} \frac{\Gamma \text{ cont}}{w \in \mathbf{N}_1 [\Gamma]} \quad \text{El}_{\mathbf{N}_1}(w, \text{id}(\star)) \in \text{Id}(\mathbf{N}_1, \star, w) [\Gamma]
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = w \in \mathbf{N}_1$
- $w \in \mathbf{N}_1 \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-S)} \frac{[] \text{ cont}}{\mathbf{N}_1 \text{ type } []} \\
\text{F-c)} \frac{\mathbf{N}_1 \text{ type } []}{w \in \mathbf{N}_1 \text{ cont}}
\end{array}$$

Exercise 6

3.6 Martin-Löf's Intensional Propositional Equality

8. Prove that there exists a proof-term **pf** such that.

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, x, w) [x \in \mathbf{N}_1, w \in \mathbf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \text{El}_{\mathbf{N}_1}(x, \text{El}_{\mathbf{N}_1}(w, \text{id}(\star))), (y). \text{id}(y)$, such that

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, x, w) [x \in \mathbf{N}_1, w \in \mathbf{N}_1]$$

is derivable, in fact $\text{El}_{\mathbf{N}_1}(x, \text{El}_{\mathbf{N}_1}(w, \text{id}(\star))) \in \text{Id}(\mathbf{N}_1, x, w) [\Gamma]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{F-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \\
\text{E-S)} \frac{x \in N_1 [\Gamma]}{\text{F-Id)} \frac{N_1 \text{ type } [\Gamma]}{\text{ld}(N_1, x, w) \text{ type } [\Gamma]} \quad \text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]} \\
\text{El}_{N_1}(x, \text{El}_{N_1}(w, \text{id}(\star))) \in \text{ld}(N_1, x, w) [\Gamma]
\end{array}$$

Where:

π_1) $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in N_1, w \in N_1$
- $x \in N_1, w \in N_1 \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-S)} \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c)} \frac{x \in N_1 \text{ cont}}{N_1 \text{ type } [x \in N_1]} \\
\text{F-S)} \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c)} \frac{x \in N_1, w \in N_1 \text{ cont}}{x \in N_1, w \in N_1 \text{ cont}}
\end{array}$$

π_2) $\text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]$ derivable:

$$\begin{array}{c}
\text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{F-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{I-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{I-S)} \frac{\pi_1}{\Gamma \text{ cont}} \\
\text{E-S)} \frac{w \in N_1 [\Gamma]}{\text{F-Id)} \frac{N_1 \text{ type } [\Gamma]}{\text{ld}(N_1, \star, w) \text{ type } [\Gamma]} \quad \text{I-Id)} \frac{\star \in N_1 [\Gamma]}{\text{id}(\star) \in \text{ld}(N_1, \star, \star) [\Gamma]} \\
\text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]
\end{array}$$

Exercise 7

5 How to translate predicative logic with equality into type theory

15. Show that by using the **Propositional Equality with Path Induction**, for any type A and $a \in A$, there exists a proof-term \mathbf{q}

$$\mathbf{q} \in \sum_{z \in \sum_{x \in A} \text{ld}_p(A, a, x)} \forall_{w \in \sum_{x \in A} \text{ld}_p(A, a, x)} \text{ld}_p(\sum_{x \in A} \text{ld}_p(A, a, x), z, w)$$

First, I transform the universal quantifier into a dependent product, in order to be able to derive it in type theory. The original judgment so becomes

$$\sum_{z \in \sum_{x \in A} \text{ld}_p(A, a, x)} \prod_{w \in \sum_{x \in A} \text{ld}_p(A, a, x)} \text{ld}_p(\sum_{x \in A} \text{ld}_p(A, a, x), z, w)$$

I will assume the following rules, that are essential to use the **Propositional Equality with Path Induction**:

$$\begin{array}{c}
\text{F-Id}_p) \frac{A \text{ type } [\Gamma] \quad a \in A [\Gamma] \quad b \in A [\Gamma]}{\text{ld}_p(A, a, b) \text{ type } [\Gamma]} \quad \text{I-Id}_p) \frac{a \in A [\Gamma]}{\text{id}_p(a) \in \text{ld}_p(A, a, a) [\Gamma]} \\
\text{E-Id}_p) \frac{C(y, z) \text{ type } [\Gamma, y \in A, z \in \text{ld}_p(A, a, y)] \quad a \in A [\Gamma] \quad b \in A [\Gamma] \quad p \in \text{ld}_p(A, a, b) [\Gamma] \quad c \in C(a, \text{id}_p(a)) [\Gamma]}{\text{El}_{\text{Id}_p}(p, c) \in C(b, p) [\Gamma]}
\end{array}$$

These are different than the ones written in page 34 of the course notes, in fact a subscript p was added to F-Id) and I-Id), in order to distinguish the ones related to Propositional Equality with Path Induction from the ones defined for Martin-Löf's Propositional Equality. The subscript was also added to the term $\text{id}(x)$, for the same reason ($\text{id}(x) \in \text{ld}(A, x, x)$, but $\text{id}(x) \notin \text{ld}_p(A, x, x)$). Another small change in E-Id_p) was that $z \in \text{ld}_p(A, a, y)$, not $z \in \text{ld}(A, a, y)$.

Propositional Equality with Path Induction

$$\begin{array}{c}
 \text{F-Id} \frac{A \text{ type } [\Gamma] \quad a \in A [\Gamma] \quad b \in A [\Gamma]}{\text{Id}_p(A, a, b) \text{ type } [\Gamma]} \quad \text{I-Id} \frac{a \in A [\Gamma]}{\text{id}(a) \in \text{Id}_p(A, a, a) [\Gamma]} \\
 \text{E-Id}_p \frac{C(y, z) \text{ type } [\Gamma, y \in A, z \in \text{Id}(A, a, y)] \quad a \in A [\Gamma] \quad b \in A [\Gamma] \quad p \in \text{Id}_p(A, a, b) [\Gamma] \quad c \in C(a, \text{id}(a)) [\Gamma]}{\text{El}_{\text{Id}_p}(p, c) \in C(b, p) [\Gamma]}
 \end{array}$$

Page 34 of the course notes

Solution

Assuming:

$a_1) A \text{ type } []$

$a_2) a \in A []$

- Let $\alpha = \langle a, \text{id}_p(a) \rangle$
- Let $\mathbf{q} = \langle \alpha, \lambda w. \text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \rangle$
- Let $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- Let $\psi(z, w) = \text{Id}_p(\phi, z, w)$

$\langle \alpha, \lambda w. \text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \rangle \in \Sigma_{z \in \phi} \Pi_{w \in \phi} \psi(z, w) []$ derivable:

$$\text{I-}\Sigma \frac{\pi_1 \quad \alpha \in \phi [] \quad \text{I-}\Pi \frac{\text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \in \psi(\alpha, w) [w \in \phi] \quad \lambda w. \text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \in \Pi_{w \in \phi} \psi(\alpha, w) []}{\langle \alpha, \lambda w. \text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \rangle \in \Sigma_{z \in \phi} \Pi_{w \in \phi} \psi(z, w) []} \quad \text{F-}\Pi \frac{\pi_3 \quad \psi(z, w) \text{ type } [z \in \phi, w \in \phi] \quad \Pi_{w \in \phi} \psi(z, w) \text{ type } [z \in \phi]}{\psi(z, w) \text{ type } [z \in \phi, w \in \phi]}$$

Where:

$\pi_1) \alpha \in \phi []$ derivable, because:

- $\alpha = \langle a, \text{id}_p(a) \rangle$
- $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- $\langle a, \text{id}_p(a) \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) []$ derivable:

$$\text{I-}\Sigma \frac{a_2 \quad a \in A [] \quad \text{F-Id}_p \frac{a \in A []}{\text{id}_p(a) \in \text{Id}_p(A, a, a) []} \quad \text{ind-ty} \frac{a_1 \quad A \text{ type } [] \quad \text{F-c) } \frac{A \text{ type } []}{x \in A \text{ cont}}}{\text{F-Id}_p \frac{A \text{ type } [x \in A]}{\text{Id}_p(A, a, x) \text{ type } [x \in A]}} \quad \text{ind-ter} \frac{a_2 \quad a \in A [] \quad \text{F-c) } \frac{A \text{ type } []}{x \in A \text{ cont}}}{a \in A [x \in A]} \quad \text{var) } \frac{a_1 \quad A \text{ type } []}{x \in A \text{ cont}}}{\langle a, \text{id}_p(a) \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) []}$$

$\pi_2) \text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \in \psi(\alpha, w) [w \in \phi]$ derivable, because:

- $\psi(\alpha, w) = \text{Id}_p(\phi, \alpha, w)$
- $\text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \in \text{Id}_p(\phi, \alpha, w) [w \in \phi]$ derivable:

$$\text{E-}\Sigma \frac{\text{Id}_p(\phi, \alpha, z) \text{ type } [w \in \phi, z \in \phi] \quad \pi_{2.5} \quad \text{var) } \frac{\pi_{3.1} \quad \phi \text{ type } [] \quad \text{F-c) } \frac{\phi \text{ type } []}{w \in \phi \text{ cont}}}{w \in \phi [w \in \phi]} \quad \pi_{2.1} \quad (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)]}{\text{El}_\Sigma(w, (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha))) \in \text{Id}_p(\phi, \alpha, w) [w \in \phi]}$$

Where:

$\pi_{2.1}) (x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)]$

derivable, because:

- Let $\Gamma = w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)$
- $(x_1, x_2). \text{El}_{\text{Id}_p}(x_2, \text{id}_p(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [\Gamma]$ derivable:

$$\text{E-Id}_p) \frac{\pi_{2,2} \quad \text{var)} \frac{\pi_{2,3} \quad \Gamma \text{ cont}}{a \in A [\Gamma]} \quad \text{var)} \frac{\pi_{2,3} \quad \Gamma \text{ cont}}{x_1 \in A [\Gamma]} \quad \text{var)} \frac{\pi_{2,3} \quad \Gamma \text{ cont}}{x_2 \in \text{Id}_p(\phi, a, x_1) [\Gamma]} \quad \text{ind-te)} \frac{\pi_1 \quad \pi_{2,3} \quad \alpha \in \phi [\] \quad \Gamma \text{ cont}}{\alpha \in \phi [\Gamma]} \quad \text{F-Id}_p)}{\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [\Gamma]}$$

$\pi_{2,2}$) $\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Gamma, y \in A, z \in \text{Id}_p(A, a, y)]$ derivable, because:

- Let $\Sigma = \Gamma, y \in A, z \in \text{Id}_p(A, a, y)$
- $\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Sigma]$ derivable:

$$\text{ind-ty)} \frac{\pi_{3,1} \quad \phi \text{ type } [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\text{F-Id}_p)} \frac{\phi \text{ type } [\Sigma]}{\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Sigma]} \quad \text{ind-ter)} \frac{\pi_1 \quad \alpha \in \phi [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\alpha \in \phi [\Sigma]} \quad \pi_{2,2,2} \quad \langle y, z \rangle \in \phi [\Sigma]$$

Where:

$\pi_{2,2,1}$) $\Sigma \text{ cont}$ derivable, because:

- $\Sigma = \Gamma, y \in A, z \in \text{Id}_p(A, a, y)$
- $\Gamma, y \in A, z \in \text{Id}_p(A, a, y) \text{ cont}$ derivable:

$$\text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,3} \quad \Gamma \text{ cont}}{\text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \text{F-c)} \frac{A \text{ type } [\Gamma]}{\Gamma, y \in A \text{ cont}}}{\text{F-Id}_p)} \frac{A \text{ type } [\Gamma, y \in A]}{\text{F-c)} \frac{\text{Id}_p(A, a, y) \text{ type } [\Gamma, y \in A]}{\Gamma, y \in A, z \in \text{Id}_p(A, a, y) \text{ cont}}} \quad \text{ind-ter)} \frac{a_2 \quad a \in A [\] \quad \pi_{2,3} \quad \Gamma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Gamma]}{\Gamma, y \in A \text{ cont}}} \quad \text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,3} \quad \Gamma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Gamma]}{\Gamma, y \in A \text{ cont}}} \quad \text{var)} \frac{y \in A [\Gamma, y \in A]}{y \in A [\Gamma, y \in A]}$$

$\pi_{2,2,2}$) $\langle y, z \rangle \in \phi [\Sigma]$ derivable, because:

- $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- $\langle y, z \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) [\Sigma]$ derivable:

$$\text{var)} \frac{\pi_{2,2,1} \quad \Sigma \text{ cont}}{y \in A [\Sigma]} \quad \text{var)} \frac{\pi_{2,2,1} \quad \Sigma \text{ cont}}{z \in \text{Id}_p(A, a, y) [\Sigma]} \quad \pi_{2,2,3} \quad \text{Id}_p(A, a, x) [\Sigma, x \in A] \quad \text{I-}\Sigma)}{\langle y, z \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) [\Sigma]}$$

$\pi_{2,2,3}$) $\text{Id}_p(A, a, x) [\Sigma, x \in A]$ derivable:

$$\text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Sigma]}{\Sigma, x \in A \text{ cont}}} \quad \text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Sigma]}{\Sigma, x \in A \text{ cont}}} \quad \text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Sigma]}{\Sigma, x \in A \text{ cont}}} \quad \text{var)} \frac{x \in A [\Sigma, x \in A]}{x \in A [\Sigma, x \in A]} \quad \text{ind-ter)} \frac{a_2 \quad a \in A [\] \quad \pi_{2,2,1} \quad \Sigma \text{ cont}}{\text{F-c)} \frac{A \text{ type } [\Sigma]}{\Sigma, x \in A \text{ cont}}} \quad \text{F-Id}_p)}{\text{Id}_p(A, a, x) [\Sigma, x \in A]}$$

$\pi_{2,3}$) $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)$
- Let $\Delta = w \in \phi, x_1 \in A$
- $\Delta, x_2 \in \text{Id}_p(A, a, x_1) \text{ cont}$ derivable:

$$\text{ind-ty)} \frac{a_1 \quad A \text{ type } [\] \quad \pi_{2,4} \quad \Delta \text{ cont}}{\text{F-Id}_p)} \frac{A \text{ type } [\Delta]}{\text{F-c)} \frac{\text{Id}_p(A, a, x_1) \text{ type } [\Delta]}{\Delta, x_2 \in \text{Id}_p(A, a, x_1) \text{ cont}}} \quad \text{ind-ter)} \frac{a_2 \quad a \in A [\] \quad \pi_{2,4} \quad \Delta \text{ cont}}{a \in A [\Delta]} \quad \text{var)} \frac{\pi_{2,4} \quad \Delta \text{ cont}}{x_1 \in A [\Delta]}$$

$\pi_{2,4}$) $\Delta \text{ cont}$ derivable, because:

- $\Delta = w \in \phi, x_1 \in A \text{ cont}$
- $w \in \phi, x_1 \in A \text{ cont}$ derivable:

$$\text{ind-ty)} \frac{\frac{a_1 \quad \text{F-c)} \frac{\phi \text{ type } []}{w \in \phi \text{ cont}}}{A \text{ type } []} \quad \text{F-c)} \frac{A \text{ type } [w \in \phi]}{w \in \phi, x_1 \in A \text{ cont}}$$

$\pi_{2.5}) \text{ Id}_p(\phi, \alpha, z) \text{ type } [w \in \phi, z \in \phi]$ derivable:

$$\text{ind-ty)} \frac{\frac{\pi_{3.1} \quad \pi_{3.2}}{\phi \text{ type } []} \quad \frac{\pi_{3.2}}{w \in \phi, z \in \phi \text{ cont}}}{\text{F-Id}_p) \frac{\phi \text{ type } [w \in \phi, z \in \phi]}{\text{Id}_p(\phi, \alpha, z) \text{ type } [w \in \phi, z \in \phi]}}$$

Where $w \in \phi, z \in \phi \text{ cont}$ derivable:

$$\text{ind-ty)} \frac{\frac{\pi_{3.1} \quad \text{F-c)} \frac{\phi \text{ type } []}{w \in \phi \text{ cont}}}{\phi \text{ type } []} \quad \text{F-c)} \frac{\phi \text{ type } [w \in \phi]}{w \in \phi, z \in \phi \text{ cont}}$$

$\pi_3) \psi(z, w) \text{ type } [z \in \phi, w \in \phi]$ derivable, because:

- $\psi(z, w) = \text{Id}_p(\phi, z, w)$
- $\text{Id}_p(\phi, z, w) \text{ type } [z \in \phi, w \in \phi]$ derivable:

$$\text{ind-ty)} \frac{\frac{\pi_{3.1} \quad \pi_{3.2}}{\phi \text{ type } []} \quad \frac{\pi_{3.2}}{z \in \phi, w \in \phi \text{ cont}}}{\text{F-Id}_p) \frac{\phi \text{ type } [z \in \phi, w \in \phi]}{\text{Id}_p(\phi, z, w) \text{ type } [z \in \phi, w \in \phi]}}$$

Where:

$\pi_{3.1}) \phi \text{ type } []$ derivable, because:

- $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- $\Sigma_{x \in A} \text{Id}_p(A, a, x) \text{ type } []$ derivable:

$$\text{ind-ty)} \frac{\frac{a_1 \quad \text{F-c)} \frac{A \text{ type } []}{x \in A \text{ cont}}}{A \text{ type } []} \quad \text{ind-ter)} \frac{\frac{a_2 \quad \text{F-c)} \frac{A \text{ type } []}{x \in A \text{ cont}}}{a \in A []} \quad \text{F-c)} \frac{A \text{ type } []}{x \in A \text{ cont}}}{\text{F-Id}_p) \frac{A \text{ type } [x \in A]}{\text{F-}\Sigma) \frac{\text{Id}_p(A, a, x) \text{ type } [x \in A]}{\Sigma_{x \in A} \text{Id}_p(A, a, x) \text{ type } []}}$$

$\pi_{3.2}) z \in \phi, w \in \phi \text{ cont}$ derivable:

$$\text{ind-ty)} \frac{\frac{\pi_{3.1} \quad \text{F-c)} \frac{\phi \text{ type } []}{z \in \phi \text{ cont}}}{\phi \text{ type } []} \quad \text{F-c)} \frac{\phi \text{ type } [z \in \phi]}{z \in \phi, w \in \phi \text{ cont}}$$