



TYPE THEORY

Theory exercises

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II Semester A.Y. 2022-2023

Exercise 1

3.1 Singleton type and exercises

3. Show that the rule E-S) is derivable in the type theory T_1 replacing the rule E-S) elimination with the E- N_{1prog}) rule and adding the substitution and weakening rules and the sanitary checks rules set out in the previous sections.

- Rule E-S)

$$\text{E-S)} \frac{t \in N_1 [\Gamma] \quad M(z) \text{ type } [\Gamma, z \in N_1] \quad c \in M(\star) [\Gamma]}{El_{N_1}(t, c) \in M(t) [\Gamma]}$$

- Rule E- N_{1prog})

$$\text{E-}N_{1prog}) \frac{D(w) \text{ type } [\Sigma, w \in N_1] \quad d \in D(\star) [\Sigma]}{El_{N_1}(w, d) \in D(w) [\Sigma, w \in N_1]}$$

Solution

Assuming:

$a_1) t \in N_1 [\Gamma]$

$a_2) M(z) \text{ type } [\Gamma, z \in N_1]$

$a_3) c \in M(\star) [\Gamma]$

The rule E-S) is derivable:

$$\text{E-}N_{1prog}) \frac{\frac{a_2 \quad a_3}{M(z) \text{ type } [\Gamma, z \in N_1] \quad c \in M(\star) [\Gamma]} \quad a_1}{\text{sub-ter)} \frac{El_{N_1}(z, c) \in M(z) [\Gamma, z \in N_1] \quad t \in N_1 [\Gamma]}{El_{N_1}(t, c) \in M(t) [\Gamma]}}$$

Exercise 2

3.2 Natural Numbers Type

3. Define the addition operation using the rules of the natural number type

$$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$$

such that $x + 0 = x \in \text{Nat} [x \in \text{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$El_{\text{Nat}}(y, x, (w, z). \text{succ}(z))$$

Let $\Gamma = x \in \text{Nat}, y \in \text{Nat}$;

$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$ is derivable:

$$\begin{array}{c}
\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{\text{Nat type } [\Gamma, w \in \text{Nat}]}{\Gamma, w \in \text{Nat}, z \in \text{Nat} \text{ cont}} \\
\text{F-c)} \frac{\Gamma, w \in \text{Nat}, z \in \text{Nat} \text{ cont}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{var)} \frac{\Gamma \text{ cont}}{y \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type } [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]}{\text{succ}(z) \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{E-Nat)} \frac{}{\text{El}_{\text{Nat}}(y, x, (w, z). \text{succ}(z)) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, y \in \text{Nat}$
- $x \in \text{Nat}, y \in \text{Nat} \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\
\text{F-c)} \frac{\text{Nat type } []}{x \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}]}{x \in \text{Nat}, y \in \text{Nat} \text{ cont}}
\end{array}$$

Correctness

The definition is correct, in fact:

Base case

$$y = 0 \Rightarrow x + y = x + 0 = x$$

This is true, because:

- $x + 0 = \text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z))$
- $\text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z)) = x \in \text{Nat} [x \in \text{Nat}]$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\
\text{F-c)} \frac{\text{Nat type } []}{x \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}]}{x \in \text{Nat}, w \in \text{Nat} \text{ cont}} \\
\text{F-Nat)} \frac{x \in \text{Nat}, w \in \text{Nat} \text{ cont}}{\text{Nat type } [x \in \text{Nat}, w \in \text{Nat}]} \\
\text{F-c)} \frac{\text{Nat type } [x \in \text{Nat}, w \in \text{Nat}]}{x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat} \text{ cont}} \\
\text{var)} \frac{x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat} \text{ cont}}{z \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{I}_2\text{-Nat)} \frac{\text{succ}(z) \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]}{\text{succ}(z) \in \text{Nat} [x \in \text{Nat}, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{C}_1\text{-Nat)} \frac{}{\text{El}_{\text{Nat}}(0, x, (w, z). \text{succ}(z)) = x \in \text{Nat} [x \in \text{Nat}]}
\end{array}$$

Inductive case

$$y = \text{succ}(v) [v \in \text{Nat}] \Rightarrow x + y = x + \text{succ}(v) = \text{succ}(x + v)$$

This is true, because:

- $x + \text{succ}(v) = \text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z))$
- $\text{succ}(x + v) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z)))$
- Let $\Gamma = x \in \text{Nat}, v \in \text{Nat}$;
 $\text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z)) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z))) \in \text{Nat} [\Gamma]$ derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{v \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type} [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{var)} \frac{\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat cont}}}{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{C}_2\text{-Nat)} \frac{\quad}{\text{El}_{\text{Nat}}(\text{succ}(v), x, (w, z). \text{succ}(z)) = \text{succ}(\text{El}_{\text{Nat}}(v, x, (w, z). \text{succ}(z))) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, v \in \text{Nat}$
- $x \in \text{Nat}, v \in \text{Nat cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type} []} \\
\text{F-c)} \frac{\quad}{x \in \text{Nat cont}} \\
\text{F-Nat)} \frac{\quad}{\text{Nat type} [x \in \text{Nat}]} \\
\text{F-c)} \frac{\quad}{x \in \text{Nat}, v \in \text{Nat cont}}
\end{array}$$

Exercise 3

3.2 Natural Numbers Type

4. Define the addition operation using the rules of the natural number type

$$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$$

such that $0 + x = x \in \text{Nat} [x \in \text{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$\text{El}_{\text{Nat}}(x, y, (w, z). \text{succ}(z))$$

Let $\Gamma = x \in \text{Nat}, y \in \text{Nat}$;

$x + y \in \text{Nat} [x \in \text{Nat}, y \in \text{Nat}]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{x \in \text{Nat} [\Gamma]} \quad \text{F-Nat)} \frac{\Gamma \text{ cont}}{\text{Nat type} [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{y \in \text{Nat} [\Gamma]} \quad \text{I}_2\text{-Nat)} \frac{\text{var)} \frac{\text{F-c)} \frac{\Gamma \text{ cont}}{\Gamma, w \in \text{Nat cont}}}{\Gamma, w \in \text{Nat}, z \in \text{Nat cont}}}{z \in \text{Nat} [\Gamma, w \in \text{Nat}, z \in \text{Nat}]} \\
\text{E-Nat)} \frac{\quad}{\text{El}_{\text{Nat}}(x, y, (w, z). \text{succ}(z)) \in \text{Nat} [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in \text{Nat}, y \in \text{Nat}$
- $x \in \text{Nat}, y \in \text{Nat cont}$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type} []} \\
\text{F-c)} \frac{\quad}{x \in \text{Nat cont}} \\
\text{F-Nat)} \frac{\quad}{\text{Nat type} [x \in \text{Nat}]} \\
\text{F-c)} \frac{\quad}{x \in \text{Nat}, y \in \text{Nat cont}}
\end{array}$$

The definition is correct, in fact:

$$x = 0 \Rightarrow x + y = 0 + y = y$$

- $0 + y = \text{El}_{\text{Nat}}(0, y, (w, z). \text{succ}(z))$
- $\text{El}_{\text{Nat}}(0, y, (w, z). \text{succ}(z)) = y \in \text{Nat} \ [y \in \text{Nat}]$ derivable:

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Exercise 4

3.2 Natural Numbers Type

6. Define the predecessor operator

$$\mathbf{p}(x) \in \text{Nat} \ [x \in \text{Nat}]$$

such that

$$\mathbf{p}(0) = 0$$

$$\mathbf{p}(\text{succ}(\mathbf{n})) = \mathbf{n}$$

Solution

The predecessor $\mathbf{p}(x)$ can be defined as:

$$\text{El}_{\text{Nat}}(x, 0, (w, z). w)$$

$\mathbf{p}(x) \in \text{Nat} \ [x \in \text{Nat}]$ is derivable:

$$\frac{\begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{var)} \frac{}{x \in \text{Nat } [x \in \text{Nat}]} \\ \text{E-Nat)} \frac{}{x \in \text{Nat } [x \in \text{Nat}]} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [x \in \text{Nat, } w \in \text{Nat}]} \\ \text{F-c)} \frac{}{x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}} \\ \text{var)} \frac{}{w \in \text{Nat } [x \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat}]} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{x \in \text{Nat cont}} \\ \text{I}_1\text{-Nat)} \frac{}{0 \in \text{Nat } [x \in \text{Nat}]} \end{array}}{\text{El}_{\text{Nat}}(x, 0, (w, z). w) \in \text{Nat } [x \in \text{Nat}]}$$

Correctness

The definition is correct, in fact:

Base case

$$x = 0 \Rightarrow \mathbf{p}(x) = \mathbf{p}(0) = 0$$

This is true, because:

- $\mathbf{p}(0) = \text{El}_{\text{Nat}}(0, 0, (w, z). w)$
- $\text{El}_{\text{Nat}}(0, 0, (w, z). w) = 0 \in \text{Nat} \ []$ derivable:

$$\frac{\begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{C}_1\text{-Nat)} \frac{}{\text{Nat type } []} \end{array} \quad \begin{array}{c} \text{I}_1\text{-Nat)} \frac{[] \text{ cont}}{0 \in \text{Nat } []} \end{array} \quad \begin{array}{c} \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \\ \text{F-c)} \frac{}{w \in \text{Nat cont}} \\ \text{F-Nat)} \frac{}{\text{Nat type } [w \in \text{Nat}]} \\ \text{F-c)} \frac{}{w \in \text{Nat, } z \in \text{Nat cont}} \\ \text{var)} \frac{}{w \in \text{Nat } [w \in \text{Nat, } z \in \text{Nat}]} \end{array}}{\text{El}_{\text{Nat}}(0, 0, (w, z). w) = 0 \in \text{Nat } []}$$

Inductive case

$$x = \text{succ}(y) \ [y \in \text{Nat}] \Rightarrow \mathbf{p}(x) = \mathbf{p}(\text{succ}(y)) = y$$

This is true, because:

- $\mathbf{p}(\text{succ}(y)) = \text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w)$
- $\text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w) = y \in \text{Nat} \ [y \in \text{Nat}]$ derivable:

$$\begin{array}{c}
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c)} \frac{\text{Nat type } []}{y \in \text{Nat cont}} \quad \text{var)} \frac{y \in \text{Nat cont}}{y \in \text{Nat } [y \in \text{Nat}]} \quad \text{C}_2\text{-Nat)} \frac{y \in \text{Nat } [y \in \text{Nat}]}{y \in \text{Nat } [y \in \text{Nat}]} \\
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } []} \quad \text{F-c)} \frac{\text{Nat type } []}{y \in \text{Nat cont}} \quad \text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } [y \in \text{Nat}]} \quad \text{F-c)} \frac{\text{Nat type } [y \in \text{Nat}]}{y \in \text{Nat, } w \in \text{Nat cont}} \\
\text{F-Nat)} \frac{[] \text{ cont}}{\text{Nat type } [y \in \text{Nat, } w \in \text{Nat}]} \quad \text{F-c)} \frac{\text{Nat type } [y \in \text{Nat, } w \in \text{Nat}]}{y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}} \\
\text{var)} \frac{y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat cont}}{w \in \text{Nat } [y \in \text{Nat, } w \in \text{Nat, } z \in \text{Nat}]} \\
\text{I}_1\text{-Nat)} \frac{[] \text{ cont}}{0 \in \text{Nat } []} \\
\text{El}_{\text{Nat}}(\text{succ}(y), 0, (w, z). w) = y \in \text{Nat } [y \in \text{Nat}]
\end{array}$$

Exercise 5

3.6 Martin-Löf's Intensional Propositional Equality

7. Prove that there exists a proof-term \mathbf{pf} such that.

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, \star, w) [w \in \mathbf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \text{El}_{\mathbf{N}_1}(w, \text{id}(\star)), (x). \text{id}(x)$, such that

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, \star, w) [w \in \mathbf{N}_1]$$

is derivable, in fact $\text{El}_{\mathbf{N}_1}(w, \text{id}(\star)) \in \text{Id}(\mathbf{N}_1, \star, w) [\Gamma]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\Gamma \text{ cont}}{w \in \mathbf{N}_1 [\Gamma]} \quad \text{F-S)} \frac{\Gamma \text{ cont}}{\mathbf{N}_1 \text{ type } [\Gamma]} \quad \text{I-S)} \frac{\Gamma \text{ cont}}{\star \in \mathbf{N}_1 [\Gamma]} \quad \text{var)} \frac{\Gamma \text{ cont}}{w \in \mathbf{N}_1 [\Gamma]} \quad \text{I-S)} \frac{\Gamma \text{ cont}}{\star \in \mathbf{N}_1 [\Gamma]} \\
\text{F-Id)} \frac{\mathbf{N}_1 \text{ type } [\Gamma]}{\text{Id}(\mathbf{N}_1, \star, w) \text{ type } [\Gamma]} \quad \text{I-Id)} \frac{\star \in \mathbf{N}_1 [\Gamma]}{\text{id}(\star) \in \text{Id}(\mathbf{N}_1, \star, \star) [\Gamma]} \\
\text{E-S)} \frac{w \in \mathbf{N}_1 [\Gamma] \quad \text{Id}(\mathbf{N}_1, \star, w) \text{ type } [\Gamma] \quad \text{id}(\star) \in \text{Id}(\mathbf{N}_1, \star, \star) [\Gamma]}{\text{El}_{\mathbf{N}_1}(w, \text{id}(\star)) \in \text{Id}(\mathbf{N}_1, \star, w) [\Gamma]}
\end{array}$$

Where $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = w \in \mathbf{N}_1$
- $w \in \mathbf{N}_1 \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-S)} \frac{[] \text{ cont}}{\mathbf{N}_1 \text{ type } []} \\
\text{F-c)} \frac{\mathbf{N}_1 \text{ type } []}{w \in \mathbf{N}_1 \text{ cont}}
\end{array}$$

Exercise 6

3.6 Martin-Löf's Intensional Propositional Equality

8. Prove that there exists a proof-term \mathbf{pf} such that.

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, x, w) [x \in \mathbf{N}_1, w \in \mathbf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \text{El}_{\mathbf{N}_1}(x, \text{El}_{\mathbf{N}_1}(w, \text{id}(\star))), (y). \text{id}(y)$, such that

$$\mathbf{pf} \in \text{Id}(\mathbf{N}_1, x, w) [x \in \mathbf{N}_1, w \in \mathbf{N}_1]$$

is derivable, in fact $\text{El}_{\mathbf{N}_1}(x, \text{El}_{\mathbf{N}_1}(w, \text{id}(\star))) \in \text{Id}(\mathbf{N}_1, x, w) [\Gamma]$ is derivable:

$$\begin{array}{c}
\text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{F-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \\
\text{E-S)} \frac{x \in N_1 [\Gamma]}{\text{F-Id)} \frac{N_1 \text{ type } [\Gamma]}{x \in N_1 [\Gamma]}} \quad \text{ld}(N_1, x, w) \text{ type } [\Gamma] \quad \text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]} \\
\hline
\text{El}_{N_1}(x, \text{El}_{N_1}(w, \text{id}(\star))) \in \text{ld}(N_1, x, w) [\Gamma]
\end{array}$$

Where:

π_1) $\Gamma \text{ cont}$ derivable, because:

- $\Gamma = x \in N_1, w \in N_1$
- $x \in N_1, w \in N_1 \text{ cont}$ derivable:

$$\begin{array}{c}
\text{F-S)} \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c)} \frac{x \in N_1 \text{ cont}}{N_1 \text{ type } [x \in N_1]} \\
\text{F-S)} \frac{[] \text{ cont}}{N_1 \text{ type } []} \\
\text{F-c)} \frac{x \in N_1, w \in N_1 \text{ cont}}{x \in N_1, w \in N_1 \text{ cont}}
\end{array}$$

π_2) $\text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]$ derivable:

$$\begin{array}{c}
\text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{F-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{I-S)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{var)} \frac{\pi_1}{\Gamma \text{ cont}} \quad \text{I-S)} \frac{\pi_1}{\Gamma \text{ cont}} \\
\text{E-S)} \frac{w \in N_1 [\Gamma]}{\text{F-Id)} \frac{N_1 \text{ type } [\Gamma]}{\star \in N_1 [\Gamma]}} \quad \text{ld}(N_1, \star, w) \text{ type } [\Gamma] \quad \text{I-Id)} \frac{\text{id}(\star) \in \text{ld}(N_1, \star, \star) [\Gamma]}{\text{El}_{N_1}(w, \text{id}(\star)) \in \text{ld}(N_1, \star, w) [\Gamma]}
\end{array}$$

Exercise 7

5 How to translate predicative logic with equality into type theory

15. Show that by using the **Propositional Equality with Path Induction**, for any type A and $a \in A$, there exists a proof-term \mathbf{q}

$$\mathbf{q} \in \sum_{z \in \sum_{x \in A} \text{Id}_p(A, a, x)} \forall_{w \in \sum_{x \in A} \text{Id}_p(A, a, x)} \text{Id}_p(\sum_{x \in A} \text{Id}_p(A, a, x), z, w)$$

First, I transform the universal quantifier into a dependent product, in order to be able to derive it in type theory. The original judgment so becomes

$$\sum_{z \in \sum_{x \in A} \text{Id}_p(A, a, x)} \prod_{w \in \sum_{x \in A} \text{Id}_p(A, a, x)} \text{Id}_p(\sum_{x \in A} \text{Id}_p(A, a, x), z, w)$$

Solution

Assuming:

a_1) $A \text{ type } []$

a_2) $a \in A []$

- Let $\alpha = \langle a, \text{id}(a) \rangle$
- Let $\mathbf{q} = \langle \alpha, \lambda w. \text{El}_{\Sigma} \left(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \right) \rangle$
- Let $\phi = \sum_{x \in A} \text{Id}_p(A, a, x)$
- Let $\psi(z) = \text{Id}_p(\phi, z, w)$

$\langle \alpha, \lambda w. \text{El}_{\Sigma} \left(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \right) \rangle \in \sum_{z \in \phi} \prod_{w \in \phi} \psi(z) []$ derivable:

$$\text{I-}\Sigma) \frac{\frac{\pi_1}{\alpha \in \phi []} \quad \text{I-II) } \frac{\text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \in \psi(\alpha) [w \in \phi]}{\lambda w. \text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \in \Pi_{w \in \phi} \psi(\alpha) []} \quad \text{F-II) } \frac{\frac{\pi_3}{\psi(z) \text{ type } [z \in \phi, w \in \phi]} \quad \Pi_{w \in \phi} \psi(z) \text{ type } [z \in \phi]}{\langle \alpha, \lambda w. \text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \rangle \in \Sigma_{z \in \phi} \Pi_{w \in \phi} \psi(z) []}$$

Where:

$\pi_1) \alpha \in \phi []$ derivable, because:

- $\alpha = \langle a, \text{id}(a) \rangle$
- $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- $\langle a, \text{id}(a) \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) []$ derivable:

$$\text{I-}\Sigma) \frac{\frac{a_2}{a \in A []} \quad \text{I-Id) } \frac{a_2}{\text{id}(a) \in \text{Id}_p(A, a, a) []} \quad \text{ind-ty) } \frac{\frac{a_1}{A \text{ type } []} \quad \text{F-c) } \frac{a_1}{x \in A \text{ cont}} \quad \text{F-c) } \frac{a_1}{x \in A \text{ cont}} \quad \text{F-c) } \frac{a_1}{x \in A \text{ cont}}}{\text{F-Id) } \frac{A \text{ type } [x \in A]}{\text{Id}_p(A, a, x) \text{ type } [x \in A]}} \quad \langle a, \text{id}(a) \rangle \in \Sigma_{x \in A} \text{Id}_p(A, a, x) []$$

$\pi_2) \text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \in \psi(\alpha) [w \in \phi]$ derivable, because:

- $\psi(\alpha) = \text{Id}_p(\phi, \alpha, w)$
- $\text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \in \text{Id}_p(\phi, \alpha, w) [w \in \phi]$ derivable:

$$\text{E-}\Sigma) \frac{\frac{\pi_3}{\text{Id}_p(\phi, z, w) \text{ type } [w \in \phi, z \in \phi]} \quad \text{F-c) } \frac{\frac{\pi_{3.1}}{\phi \text{ type } []}}{w \in \phi \text{ cont}} \quad \text{var) } \frac{w \in \phi [w \in \phi]}{w \in \phi [w \in \phi]} \quad \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)]}{\text{El}_\Sigma(w, \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha))) \in \text{Id}_p(\phi, \alpha, w) [w \in \phi]}$$

Where:

$\pi_{2.1}) \text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)]$ derivable, because:

- Let $\Gamma = w \in \phi, x_1 \in A, x_2 \in \text{Id}_p(A, a, x_1)$
- $\text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [\Gamma]$ derivable:

$$\text{E-Id}_p) \frac{\frac{\pi_{2.2}}{a \in A [\Gamma]} \quad \text{var) } \frac{\frac{\pi_{2.3}}{\Gamma \text{ cont}}}{a \in A [\Gamma]} \quad \text{var) } \frac{\frac{\pi_{2.3}}{\Gamma \text{ cont}}}{x_1 \in A [\Gamma]} \quad \text{var) } \frac{\frac{\pi_{2.3}}{\Gamma \text{ cont}}}{x_2 \in \text{Id}_p(\phi, a, x_1) [\Gamma]} \quad \text{I-Id) } \frac{\frac{\pi_1}{\alpha \in \phi []} \quad \frac{\pi_{2.3}}{\Gamma \text{ cont}}}{\text{id}(\alpha) \in \text{Id}_p(\phi, \alpha, \alpha) [\Gamma]}}{\text{El}_{\text{Id}_p}(x_2, \text{id}(\alpha)) \in \text{Id}_p(\phi, \alpha, \langle x_1, x_2 \rangle) [\Gamma]}$$

$\pi_{2.2}) \text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Gamma, y \in A, z \in \text{Id}_p(A, a, y)]$ derivable, because:

- Let $\Sigma = \Gamma, y \in A, z \in \text{Id}_p(A, a, y)$
- $\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Sigma]$ derivable:

$$\text{F-Id) } \frac{\text{ind-ty) } \frac{\frac{\pi_{3.1}}{\phi \text{ type } []} \quad \frac{\pi_{2.2.1}}{\Sigma \text{ cont}}}{\phi \text{ type } [\Sigma]} \quad \text{ind-ter) } \frac{\frac{\pi_1}{\alpha \in \phi []} \quad \frac{\pi_{2.2.1}}{\Sigma \text{ cont}}}{\alpha \in \phi [\Sigma]} \quad \frac{\pi_{2.2.2}}{\langle y, z \rangle \in \phi [\Sigma]}}{\text{Id}_p(\phi, \alpha, \langle y, z \rangle) \text{ type } [\Sigma]}$$

Where:

$\pi_{2.2.1}) \Sigma \text{ cont}$ derivable, because:

- $\Sigma = \Gamma, y \in A, z \in \text{Id}_p(A, a, y)$
- $\Gamma, y \in A, z \in \text{Id}_p(A, a, y) \text{ cont}$ derivable:

$\pi_{3.1}$) $\phi \text{ type } []$ derivable, because:

- $\phi = \Sigma_{x \in A} \text{Id}_p(A, a, x)$
- $\Sigma_{x \in A} \text{Id}_p(A, a, x) \text{ type } []$ derivable:

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} a_1 \\ \text{ind-ty)} \frac{A \text{ type } []}{A \text{ type } [x \in A]} \end{array} & \begin{array}{c} a_1 \\ \text{F-c)} \frac{A \text{ type } []}{x \in A \text{ cont}} \end{array} & \begin{array}{c} a_1 \\ \text{F-c)} \frac{A \text{ type } []}{x \in A \text{ cont}} \end{array} \\
 \text{F-l)} \frac{A \text{ type } [x \in A]}{A \text{ type } [x \in A]} & \text{var)} \frac{x \in A \text{ cont}}{a \in A [x \in A]} & \text{var)} \frac{x \in A \text{ cont}}{x \in A [x \in A]}
 \end{array} \\
 \hline
 \text{F-}\Sigma) \frac{\text{Id}_p(A, a, x) \text{ type } [x \in A]}{\Sigma_{x \in A} \text{Id}_p(A, a, x) \text{ type } []}
 \end{array}$$

$\pi_{3.2}$) $z \in \phi, w \in \phi \text{ cont}$ derivable:

$$\begin{array}{c}
 \begin{array}{c} \pi_{3.1} \\ \text{ind-ty)} \frac{\phi \text{ type } []}{\phi \text{ type } [z \in \phi]} \end{array} \\
 \text{F-c)} \frac{\phi \text{ type } [z \in \phi]}{z \in \phi, w \in \phi \text{ cont}}
 \end{array}$$