Type Theory

Theory exercises

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Exercise 1

3.1 Singleton type and exercises

3. Show that the rule E-S) is derivable in the type theory T_1 replacing the rule E-S) elimination with the E-N_{1prog}) rule and adding the substitution and weakening rules and the sanitary checks rules set out in the previous sections.

- Rule E-S)

$$\text{E-S)}\ \frac{t \in \mathsf{N}_1\ [\Gamma] \qquad M(z)\ \mathit{type}\ [\Gamma, z \in \mathsf{N}_1] \qquad c \in M(\star)\ [\Gamma]}{\mathsf{El}_{\mathsf{N}_1}(t, c) \in M(t)\ [\Gamma]}$$

- Rule E-N_{1prog})

$$\text{E-}\mathsf{N}_{1prog})\ \frac{D(w)\ \mathit{type}\ [\Sigma, w \in \mathsf{N}_1] \qquad d \in D(\star)\ [\Sigma]}{\mathsf{El}_{\mathsf{N}_1}(w, d) \in D(w)\ [\Sigma, w \in \mathsf{N}_1]}$$

Solution

Assuming:
$a_1)\ \ t \in \mathsf{N}_1\ [\Gamma]$
$a_2)\ \ M(z)\ \mathit{type}\ [\Gamma, z \in \mathsf{N}_1]$
$a_3)\ \ c \in M(\star)\ [\Gamma]$

The rule E-S) is derivable:

$$\text{sub-ter)}\ \frac{\text{E-}\mathsf{N}_{1prog})\ \dfrac{\overset{a_2}{M(z)\ \mathit{type}\ [\Gamma, z \in \mathsf{N}_1]} \qquad \overset{a_3}{c \in M(\star)\ [\Gamma]}}{\mathsf{El}_{\mathsf{N}_1}(z, c) \in M(z)\ [\Gamma, z \in \mathsf{N}_1]} \qquad \overset{a_1}{t \in \mathsf{N}_1\ [\Gamma]}}{\mathsf{El}_{\mathsf{N}_1}(t, c) \in M(t)\ [\Gamma]}$$

Exercise 2

3.2 Natural Numbers Type

3. Define the addition operation using the rules of the natural number type

$$x + y \in \mathsf{Nat}\ [x \in \mathsf{Nat}, y \in \mathsf{Nat}]$$

such that $x + 0 = x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$\mathsf{El}_{\mathsf{Nat}}(y, x, (w, z).\,\mathsf{succ}(z))$$

Let $\Gamma = x \in \mathsf{Nat}, y \in \mathsf{Nat}$;
$x + y \in \mathsf{Nat}\ [x \in \mathsf{Nat}, y \in \mathsf{Nat}]$ is derivable:

$$
\begin{prooftree}
\AxiomC{$\Gamma\ cont$}
\LeftLabel{var)}
\UnaryInfC{$y \in \mathsf{Nat}\ [\Gamma]$}
\AxiomC{$\Gamma\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\Gamma]$}
\AxiomC{$\Gamma\ cont$}
\LeftLabel{var)}
\UnaryInfC{$x \in \mathsf{Nat}\ [\Gamma]$}
\AxiomC{$\Gamma\ cont$}
\LeftLabel{F-c)}
\UnaryInfC{$\Gamma, w \in \mathsf{Nat}\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\Gamma, w \in \mathsf{Nat}]$}
\LeftLabel{F-c)}
\UnaryInfC{$\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ cont$}
\LeftLabel{var)}
\UnaryInfC{$z \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]$}
\LeftLabel{I$_2$-Nat)}
\UnaryInfC{$\mathsf{succ}(z) \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]$}
\LeftLabel{E-Nat)}
\QuaternaryInfC{$\mathsf{El}_{\mathsf{Nat}}(y, x, (w,z).\, \mathsf{succ}(z)) \in \mathsf{Nat}\ [\Gamma]$}
\end{prooftree}
$$

Where $\Gamma\ cont$ derivable, because:

- $\Gamma = x \in \mathsf{Nat}, y \in \mathsf{Nat}$
- $x \in \mathsf{Nat}, y \in \mathsf{Nat}\ cont$ derivable:

$$
\begin{prooftree}
\AxiomC{$[\ ]\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\ ]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [x \in \mathsf{Nat}]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}, y \in \mathsf{Nat}\ cont$}
\end{prooftree}
$$

Correctness

The definition is correct, in fact:

Base case

$y = 0 \Rightarrow x + y = x + 0 = x$

This is true, because:

- $x + 0 = \mathsf{El}_{\mathsf{Nat}}(0, x, (w,z).\, \mathsf{succ}(z))$
- $\mathsf{El}_{\mathsf{Nat}}(0, x, (w,z).\, \mathsf{succ}(z)) = x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$ derivable:

$$
\begin{prooftree}
\AxiomC{$[\ ]\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\ ]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [x \in \mathsf{Nat}]$}
\AxiomC{$[\ ]\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\ ]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}\ cont$}
\LeftLabel{var)}
\UnaryInfC{$x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$}
\AxiomC{$[\ ]\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [\ ]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [x \in \mathsf{Nat}]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}, w \in \mathsf{Nat}\ cont$}
\LeftLabel{F-Nat)}
\UnaryInfC{$\mathsf{Nat}\ type\ [x \in \mathsf{Nat}, w \in \mathsf{Nat}]$}
\LeftLabel{F-c)}
\UnaryInfC{$x \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ cont$}
\LeftLabel{var)}
\UnaryInfC{$z \in \mathsf{Nat}\ [x \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]$}
\LeftLabel{I$_2$-Nat)}
\UnaryInfC{$\mathsf{succ}(z) \in \mathsf{Nat}\ [x \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]$}
\LeftLabel{C$_1$-Nat)}
\TrinaryInfC{$\mathsf{El}_{\mathsf{Nat}}(0, x, (w,z).\, \mathsf{succ}(z)) = x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$}
\end{prooftree}
$$

Inductive case

$y = \mathsf{succ}(v)\ [v \in \mathsf{Nat}] \Rightarrow x + y = x + \mathsf{succ}(v) = \mathsf{succ}(x + v)$

This is true, because:

- $x + \mathsf{succ}(v) = \mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), x, (w,z).\, \mathsf{succ}(z))$
- $\mathsf{succ}(x + v) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, x, (w,z).\, \mathsf{succ}(z)))$
- Let $\Gamma = x \in \mathsf{Nat}, v \in \mathsf{Nat}$;
  $\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), x, (w,z).\, \mathsf{succ}(z)) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, x, (w,z).\, \mathsf{succ}(z))) \in \mathsf{Nat}\ [\Gamma]$ derivable:

$$
\text{C}_2\text{-Nat)}\ \dfrac{
\text{var)}\ \dfrac{\Gamma\ \mathit{cont}}{v \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{F-Nat)}\ \dfrac{\Gamma\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma]}
\quad
\text{var)}\ \dfrac{\Gamma\ \mathit{cont}}{x \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{I}_2\text{-Nat)}\ \dfrac{\text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\Gamma\ \mathit{cont}}{\Gamma, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma, w \in \mathsf{Nat}]}}{\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{z \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}}{\mathsf{succ}(z) \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{
\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), x, (w, z).\, \mathsf{succ}(z)) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, x, (w, z).\, \mathsf{succ}(z))) \in \mathsf{Nat}\ [\Gamma]
}
$$

Where Γ *cont* derivable, because:

- $\Gamma = x \in \mathsf{Nat}, v \in \mathsf{Nat}$
- $x \in \mathsf{Nat}, v \in \mathsf{Nat}$ *cont* derivable:

$$
\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [x \in \mathsf{Nat}]}}{x \in \mathsf{Nat}, v \in \mathsf{Nat}\ \mathit{cont}}
$$

Exercise 3

> ### 3.2 Natural Numbers Type
>
> 4. Define the addition operation using the rules of the natural number type
>
> $$x + y \in \mathsf{Nat}\ [x \in \mathsf{Nat}, y \in \mathsf{Nat}]$$
>
> such that $0 + x = x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$.

Solution

The addition $x + y$ can be defined as:

$$\mathsf{El}_{\mathsf{Nat}}(x, y, (w, z).\, \mathsf{succ}(z))$$

Let $\Gamma = x \in \mathsf{Nat}, y \in \mathsf{Nat}$;
$x + y \in \mathsf{Nat}\ [x \in \mathsf{Nat}, y \in \mathsf{Nat}]$ is derivable:

$$
\text{E-Nat)}\ \dfrac{
\text{var)}\ \dfrac{\Gamma\ \mathit{cont}}{x \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{F-Nat)}\ \dfrac{\Gamma\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma]}
\quad
\text{var)}\ \dfrac{\Gamma\ \mathit{cont}}{y \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{I}_2\text{-Nat)}\ \dfrac{\text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\Gamma\ \mathit{cont}}{\Gamma, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma, w \in \mathsf{Nat}]}}{\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{z \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}}{\mathsf{succ}(z) \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{
\mathsf{El}_{\mathsf{Nat}}(x, y, (w, z).\, \mathsf{succ}(z)) \in \mathsf{Nat}\ [\Gamma]
}
$$

Where Γ *cont* derivable, because:

- $\Gamma = x \in \mathsf{Nat}, y \in \mathsf{Nat}$
- $x \in \mathsf{Nat}, y \in \mathsf{Nat}$ *cont* derivable:

$$
\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [x \in \mathsf{Nat}]}}{x \in \mathsf{Nat}, y \in \mathsf{Nat}\ \mathit{cont}}
$$

Correctness

The definition is correct, in fact:

Base case

$x = 0 \Rightarrow x + y = 0 + y = y$

Note that the exercise requires that $0 + x = x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$, but that is equivalent to proving that $0 + y = y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]$, by renaming y to x in the latter, and this is true, because:

- $0 + y = \mathsf{El}_{\mathsf{Nat}}(0, y, (w, z).\, \mathsf{succ}(z))$
- $\mathsf{El}_{\mathsf{Nat}}(0, y, (w, z).\, \mathsf{succ}(z)) = y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]$ derivable:

$$
\text{C}_1\text{-Nat)}\,\dfrac{
\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}]}
\quad
\text{var)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]}
\quad
\text{I}_2\text{-Nat)}\,\dfrac{\text{var)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}]}}{y \in \mathsf{Nat}, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}, w \in \mathsf{Nat}]}}{y \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{z \in \mathsf{Nat}\ [y \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}}{\mathsf{succ}(z) \in \mathsf{Nat}\ [y \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{\mathsf{El}_{\mathsf{Nat}}(0, y, (w, z).\, \mathsf{succ}(z)) = y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]}
$$

Inductive case

$x = \mathsf{succ}(v)\ [v \in \mathsf{Nat}] \Rightarrow x + y = \mathsf{succ}(v) + y = \mathsf{succ}(v + y)$

This is true, because:

- $\mathsf{succ}(v) + y = \mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), y, (w, z).\, \mathsf{succ}(z))$
- $\mathsf{succ}(v + y) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, y, (w, z).\, \mathsf{succ}(z)))$
- Let $\Gamma = v \in \mathsf{Nat}, y \in \mathsf{Nat}$;
  $\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), y, (w, z).\, \mathsf{succ}(z)) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, y, (w, z).\, \mathsf{succ}(z))) \in \mathsf{Nat}\ [\Gamma]$ derivable:

$$
\text{C}_2\text{-Nat)}\,\dfrac{
\text{var)}\,\dfrac{\Gamma\ \mathit{cont}}{v \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{F-Nat)}\,\dfrac{\Gamma\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma]}
\quad
\text{var)}\,\dfrac{\Gamma\ \mathit{cont}}{y \in \mathsf{Nat}\ [\Gamma]}
\quad
\text{I}_2\text{-Nat)}\,\dfrac{\text{var)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\Gamma\ \mathit{cont}}{\Gamma, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [\Gamma, w \in \mathsf{Nat}]}}{\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{z \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}}{\mathsf{succ}(z) \in \mathsf{Nat}\ [\Gamma, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(v), y, (w, z).\, \mathsf{succ}(z)) = \mathsf{succ}(\mathsf{El}_{\mathsf{Nat}}(v, y, (w, z).\, \mathsf{succ}(z))) \in \mathsf{Nat}\ [\Gamma]}
$$

Where $\Gamma\ \mathit{cont}$ derivable, because:

- $\Gamma = v \in \mathsf{Nat}, y \in \mathsf{Nat}$
- $v \in \mathsf{Nat}, y \in \mathsf{Nat}\ \mathit{cont}$ derivable:

$$
\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{v \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [v \in \mathsf{Nat}]}}{v \in \mathsf{Nat}, y \in \mathsf{Nat}\ \mathit{cont}}
$$

Exercise 4

3.2 Natural Numbers Type

6. Define the predecessor operator

$$\mathbf{p}(x) \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$$

such that

$$\mathbf{p}(0) = 0$$
$$\mathbf{p}(\mathsf{succ}(\mathbf{n})) = \mathbf{n}$$

Solution

The predecessor $\mathbf{p}(x)$ can be defined as:

$$\mathsf{El}_{\mathsf{Nat}}(x, 0, (w, z).\, w)$$

$\mathbf{p}(x) \in \mathsf{Nat}\ [x \in \mathsf{Nat}]$ is derivable:

$$\text{E-Nat)}\ \dfrac{
\text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{x \in \mathsf{Nat}\ [x \in \mathsf{Nat}]}
\quad
\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [x \in \mathsf{Nat}]}
\quad
\text{I}_1\text{-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{0 \in \mathsf{Nat}\ [x \in \mathsf{Nat}]}
\quad
\text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{x \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [x \in \mathsf{Nat}]}}{x \in \mathsf{Nat}, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [x \in \mathsf{Nat}, w \in \mathsf{Nat}]}}{x \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{w \in \mathsf{Nat}\ [x \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{\mathsf{El}_{\mathsf{Nat}}(x, 0, (w, z).\, w) \in \mathsf{Nat}\ [x \in \mathsf{Nat}]}$$

Correctness

The definition is correct, in fact:

Base case

$x = 0 \Rightarrow \mathbf{p}(x) = \mathbf{p}(0) = 0$

This is true, because:

- $\mathbf{p}(0) = \mathsf{El}_{\mathsf{Nat}}(0, 0, (w, z).\, w)$
- $\mathsf{El}_{\mathsf{Nat}}(0, 0, (w, z).\, w) = 0 \in \mathsf{Nat}\ [\,]$ derivable:

$$\text{C}_1\text{-Nat)}\ \dfrac{
\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}
\quad
\text{I}_1\text{-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{0 \in \mathsf{Nat}\ [\,]}
\quad
\text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-Nat)}\ \dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [w \in \mathsf{Nat}]}}{w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{w \in \mathsf{Nat}\ [w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{\mathsf{El}_{\mathsf{Nat}}(0, 0, (w, z).\, w) = 0 \in \mathsf{Nat}\ [\,]}$$

Inductive case

$x = \mathsf{succ}(y)\ [y \in \mathsf{Nat}] \Rightarrow \mathbf{p}(x) = \mathbf{p}(\mathsf{succ}(y)) = y$

This is true, because:

- $\mathbf{p}(\mathsf{succ}(y)) = \mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(y), 0, (w, z).\, w)$
- $\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(y), 0, (w, z).\, w) = y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]$ derivable:

$$
\text{C}_2\text{-Nat)}\,\dfrac{
\text{var)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]}
\quad
\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}]}
\quad
\text{I}_1\text{-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{0 \in \mathsf{Nat}\ [\,]}
\quad
\text{var)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{\text{F-c)}\,\dfrac{\text{F-Nat)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{Nat}\ \mathit{type}\ [\,]}}{y \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}]}}{y \in \mathsf{Nat}, w \in \mathsf{Nat}\ \mathit{cont}}}{\mathsf{Nat}\ \mathit{type}\ [y \in \mathsf{Nat}, w \in \mathsf{Nat}]}}{y \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}\ \mathit{cont}}}{w \in \mathsf{Nat}\ [y \in \mathsf{Nat}, w \in \mathsf{Nat}, z \in \mathsf{Nat}]}
}{\mathsf{El}_{\mathsf{Nat}}(\mathsf{succ}(y), 0, (w,z).\, w) = y \in \mathsf{Nat}\ [y \in \mathsf{Nat}]}
$$

Exercise 5

3.6 Martin-Löf's Intensional Propositional Equality

7. Prove that there exists a proof-term **pf** such that.

$$\mathbf{pf} \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [w \in \mathsf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)), (x).\, \mathsf{id}(x)$, such that

$$\mathbf{pf} \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [w \in \mathsf{N}_1]$$

is derivable, in fact $\mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)) \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [\Gamma]$ is derivable:

$$
\text{E-S)}\,\dfrac{
\text{var)}\,\dfrac{\Gamma\ \mathit{cont}}{w \in \mathsf{N}_1\ [\Gamma]}
\quad
\text{F-Id)}\,\dfrac{\text{F-S)}\,\dfrac{\Gamma\ \mathit{cont}}{\mathsf{N}_1\ \mathit{type}\ [\Gamma]} \quad \text{I-S)}\,\dfrac{\Gamma\ \mathit{cont}}{\star \in \mathsf{N}_1\ [\Gamma]} \quad \text{var)}\,\dfrac{\Gamma\ \mathit{cont}}{w \in \mathsf{N}_1\ [\Gamma]}}{\mathsf{Id}(\mathsf{N}_1, \star, w)\ \mathit{type}\ [\Gamma]}
\quad
\text{I-Id)}\,\dfrac{\text{I-S)}\,\dfrac{\Gamma\ \mathit{cont}}{\star \in \mathsf{N}_1\ [\Gamma]}}{\mathsf{id}(\star) \in \mathsf{Id}(\mathsf{N}_1, \star, \star)\ [\Gamma]}
}{\mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)) \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [\Gamma]}
$$

Where Γ *cont* derivable, because:

- $\Gamma = w \in \mathsf{N}_1$
- $w \in \mathsf{N}_1$ *cont* derivable:

$$
\text{F-c)}\,\dfrac{\text{F-S)}\,\dfrac{[\,]\ \mathit{cont}}{\mathsf{N}_1\ \mathit{type}\ [\,]}}{w \in \mathsf{N}_1\ \mathit{cont}}
$$

Exercise 6

3.6 Martin-Löf's Intensional Propositional Equality

8. Prove that there exists a proof-term **pf** such that.

$$\mathbf{pf} \in \mathsf{Id}(\mathsf{N}_1, x, w)\ [x \in \mathsf{N}_1, w \in \mathsf{N}_1]$$

is derivable.

Solution

There exists a proof-term $\mathbf{pf} = \mathsf{El}_{\mathsf{N}_1}\big(x, \mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star))\big), (y).\, \mathsf{id}(y)$, such that

$$\mathbf{pf} \in \mathsf{Id}(\mathsf{N}_1, x, w)\ [x \in \mathsf{N}_1, w \in \mathsf{N}_1]$$

is derivable, in fact $\mathsf{El}_{\mathsf{N}_1}\big(x, \mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star))\big) \in \mathsf{Id}(\mathsf{N}_1, x, w)\ [\Gamma]$ is derivable:

$$
\text{E-S)}\ \dfrac{
\text{var)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{x \in \mathsf{N}_1\ [\Gamma]}
\qquad
\text{F-Id)}\ \dfrac{
\text{F-S)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{\mathsf{N}_1\ \mathit{type}\ [\Gamma]}
\quad
\text{var)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{x \in \mathsf{N}_1\ [\Gamma]}
\quad
\text{var)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{w \in \mathsf{N}_1\ [\Gamma]}
}{\mathsf{Id}(\mathsf{N}_1, x, w)\ \mathit{type}\ [\Gamma]}
\qquad
\begin{matrix}\pi_2\\ \mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)) \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [\Gamma]\end{matrix}
}{\mathsf{El}_{\mathsf{N}_1}\left(x, \mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star))\right) \in \mathsf{Id}(\mathsf{N}_1, x, w)\ [\Gamma]}
$$

Where:

π_1) Γ *cont* derivable, because:

- $\Gamma = x \in \mathsf{N}_1, w \in \mathsf{N}_1$
- $x \in \mathsf{N}_1, w \in \mathsf{N}_1$ *cont* derivable:

$$
\text{F-c)}\ \dfrac{\text{F-S)}\ \dfrac{\text{F-c)}\ \dfrac{\text{F-S)}\ \dfrac{[\ ]\ \mathit{cont}}{\mathsf{N}_1\ \mathit{type}\ [\ ]}}{x \in \mathsf{N}_1\ \mathit{cont}}}{\mathsf{N}_1\ \mathit{type}\ [x \in \mathsf{N}_1]}}{x \in \mathsf{N}_1, w \in \mathsf{N}_1\ \mathit{cont}}
$$

π_2) $\mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)) \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [\Gamma]$ derivable:

$$
\text{E-S)}\ \dfrac{
\text{var)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{w \in \mathsf{N}_1\ [\Gamma]}
\qquad
\text{F-Id)}\ \dfrac{
\text{F-S)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{\mathsf{N}_1\ \mathit{type}\ [\Gamma]}
\quad
\text{I-S)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{\star \in \mathsf{N}_1\ [\Gamma]}
\quad
\text{var)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{w \in \mathsf{N}_1\ [\Gamma]}
}{\mathsf{Id}(\mathsf{N}_1, \star, w)\ \mathit{type}\ [\Gamma]}
\qquad
\text{I-Id)}\ \dfrac{\text{I-S)}\ \dfrac{\begin{matrix}\pi_1\\ \Gamma\ \mathit{cont}\end{matrix}}{\star \in \mathsf{N}_1\ [\Gamma]}}{\mathsf{id}(\star) \in \mathsf{Id}(\mathsf{N}_1, \star, \star)\ [\Gamma]}
}{\mathsf{El}_{\mathsf{N}_1}(w, \mathsf{id}(\star)) \in \mathsf{Id}(\mathsf{N}_1, \star, w)\ [\Gamma]}
$$

Exercise 7

5 How to translate predicative logic with equality into type theory

15. Show that by using the **Propositional Equality with Path Induction**, for any type A and $a \in A$, there exists a proof-term $\mathbf{q}$

$$
\mathbf{q} \in \Sigma_{z \in \Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A,a,x)} \forall_{w \in \Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A,a,x)} \mathsf{Id}_{\mathsf{p}}\left(\Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A, a, x), z, w\right)
$$

First, I transform the universal quantifier into a dependent product, in order to be able to derive it in type theory. The original judgment so becomes

$$
\Sigma_{z \in \Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A,a,x)} \Pi_{w \in \Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A,a,x)} \mathsf{Id}_{\mathsf{p}}\left(\Sigma_{x \in A} \mathsf{Id}_{\mathsf{p}}(A, a, x), z, w\right)
$$

I will assume the following rules, that are essential to use the **Propositional Equality with Path Induction**:

$$
\text{F-Id}_{\mathsf{p}})\ \dfrac{A\ \mathit{type}\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad b \in A\ [\Gamma]}{\mathsf{Id}_{\mathsf{p}}(A, a, b)\ \mathit{type}\ [\Gamma]}
\qquad
\text{I-Id}_{\mathsf{p}})\ \dfrac{a \in A\ [\Gamma]}{\mathsf{id}_{\mathsf{p}}(a) \in \mathsf{Id}_{\mathsf{p}}(A, a, a)\ [\Gamma]}
$$

$$
\text{E-Id}_{\mathsf{p}})\ \dfrac{C(y, z)\ \mathit{type}\ [\Gamma, y \in A, z \in \mathsf{Id}_{\mathsf{p}}(A, a, y)] \qquad a \in A\ [\Gamma] \qquad b \in A\ [\Gamma] \qquad p \in \mathsf{Id}_{\mathsf{p}}(A, a, b)\ [\Gamma] \qquad c \in C(a, \mathsf{id}_{\mathsf{p}}(a))\ [\Gamma]}{\mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}(p, c) \in C(b, p)\ [\Gamma]}
$$

These are different than the ones written in page 34 of the course notes, in fact a subscript p was added to F-Id) and I-Id), in order to distinguish the ones related to Propositional Equality with Path Induction from the ones defined for Martin-Löf's Propositional Equality. The subscript was also added to the term $\mathsf{id}(x)$, for the same reason ($\mathsf{id}(x) \in \mathsf{Id}(A, x, x)$, but $\mathsf{id}(x) \notin \mathsf{Id}_{\mathsf{p}}(A, x, x)$). Another small change in E-Id_{p}) was that $z \in \mathsf{Id}_{\mathsf{p}}(A, a, y)$, not $z \in \mathsf{Id}(A, a, y)$.

Propositional Equality with Path Induction

$$\text{F-Id)}\ \dfrac{A\ \mathit{type}\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad b \in A\ [\Gamma]}{\mathsf{Id}_{\mathsf{p}}(A, a, b)\ \mathit{type}\ [\Gamma]} \qquad \text{I-Id)}\ \dfrac{a \in A\ [\Gamma]}{\mathsf{id}(a) \in \mathsf{Id}_{\mathsf{p}}(A, a, a)\ [\Gamma]}$$

$$\text{E-}\mathsf{Id}_{\mathsf{p}})\ \dfrac{C(y, z)\ \mathit{type}\ [\Gamma, y \in A, z \in \mathsf{Id}(A, a, y)] \qquad a \in A\ [\Gamma] \qquad b \in A\ [\Gamma] \qquad p \in \mathsf{Id}_{\mathsf{p}}(A, a, b)\ [\Gamma] \qquad c \in C(a, \mathsf{id}(a))\ [\Gamma]}{\mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}(p, c) \in C(b, p)\ [\Gamma]}$$

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Solution

Assuming:

$a_1)$ $A\ \mathit{type}\ [\ ]$

$a_2)$ $a \in A\ [\ ]$

- Let $\alpha = \langle a, \mathsf{id}_{\mathsf{p}}(a) \rangle$
- Let $\mathbf{q} = \langle \alpha, \lambda w.\ \mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \rangle$
- Let $\phi = \Sigma_{x \in A}\ \mathsf{Id}_{\mathsf{p}}(A, a, x)$
- Let $\psi(z, w) = \mathsf{Id}_{\mathsf{p}}(\phi, z, w)$

$\langle \alpha, \lambda w.\ \mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \rangle \in \Sigma_{z \in \phi}\ \Pi_{w \in \phi}\ \psi(z, w)\ [\ ]$ derivable:

$$\text{I-}\Sigma)\ \dfrac{\genfrac{}{}{0pt}{}{\pi_1}{\alpha \in \phi\ [\ ]} \qquad \text{I-}\Pi)\ \dfrac{\genfrac{}{}{0pt}{}{\pi_2}{\mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \in \psi(\alpha, w)\ [w \in \phi]}}{\lambda w.\ \mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \in \Pi_{w \in \phi}\ \psi(\alpha, w)\ [\ ]} \qquad \text{F-}\Pi)\ \dfrac{\genfrac{}{}{0pt}{}{\pi_3}{\psi(z, w)\ \mathit{type}\ [z \in \phi, w \in \phi]}}{\Pi_{w \in \phi}\ \psi(z, w)\ \mathit{type}\ [z \in \phi]}}{\langle \alpha, \lambda w.\ \mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \rangle \in \Sigma_{z \in \phi}\ \Pi_{w \in \phi}\ \psi(z, w)\ [\ ]}$$

Where:

$\pi_1)$ $\alpha \in \phi\ [\ ]$ derivable, because:

- $\alpha = \langle a, \mathsf{id}_{\mathsf{p}}(a) \rangle$
- $\phi = \Sigma_{x \in A}\ \mathsf{Id}_{\mathsf{p}}(A, a, x)$
- $\langle a, \mathsf{id}_{\mathsf{p}}(a) \rangle \in \Sigma_{x \in A}\ \mathsf{Id}_{\mathsf{p}}(A, a, x)\ [\ ]$ derivable:

$$\text{I-}\Sigma)\ \dfrac{\genfrac{}{}{0pt}{}{a_2}{a \in A\ [\ ]} \qquad \text{F-}\mathsf{Id}_{\mathsf{p}})\ \dfrac{\genfrac{}{}{0pt}{}{a_2}{a \in A\ [\ ]}}{\mathsf{id}_{\mathsf{p}}(a) \in \mathsf{Id}_{\mathsf{p}}(A, a, a)\ [\ ]} \qquad \text{F-}\mathsf{Id}_{\mathsf{p}})\ \dfrac{\text{ind-ty)}\ \dfrac{\genfrac{}{}{0pt}{}{a_1}{A\ \mathit{type}\ [\ ]} \quad \text{F-c)}\ \dfrac{\genfrac{}{}{0pt}{}{a_1}{A\ \mathit{type}\ [\ ]}}{x \in A\ \mathit{cont}}}{A\ \mathit{type}\ [x \in A]} \qquad \text{ind-ter)}\ \dfrac{\genfrac{}{}{0pt}{}{a_2}{a \in A\ [\ ]} \quad \text{F-c)}\ \dfrac{\genfrac{}{}{0pt}{}{a_1}{A\ \mathit{type}\ [\ ]}}{x \in A\ \mathit{cont}}}{a \in A\ [x \in A]} \qquad \text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\genfrac{}{}{0pt}{}{a_1}{A\ \mathit{type}\ [\ ]}}{x \in A\ \mathit{cont}}}{x \in A\ [x \in A]}}{\mathsf{Id}_{\mathsf{p}}(A, a, x)\ \mathit{type}\ [x \in A]}}{\langle a, \mathsf{id}_{\mathsf{p}}(a) \rangle \in \Sigma_{x \in A}\ \mathsf{Id}_{\mathsf{p}}(A, a, x)\ [\ ]}$$

$\pi_2)$ $\mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \in \psi(\alpha, w)\ [w \in \phi]$ derivable, because:

- $\psi(\alpha, w) = \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, w)$
- $\mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \in \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, w)\ [w \in \phi]$ derivable:

$$\text{E-}\Sigma)\ \dfrac{\genfrac{}{}{0pt}{}{\pi_{2.5}}{\mathsf{Id}_{\mathsf{p}}(\phi, \alpha, z)\ \mathit{type}\ [w \in \phi, z \in \phi]} \qquad \text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\genfrac{}{}{0pt}{}{\pi_{3.1}}{\phi\ \mathit{type}\ [\ ]}}{w \in \phi\ \mathit{cont}}}{w \in \phi\ [w \in \phi]} \qquad \genfrac{}{}{0pt}{}{\pi_{2.1}}{\mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big) \in \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, \langle x_1, x_2 \rangle)\ \big[w \in \phi, x_1 \in A, x_2 \in \mathsf{Id}_{\mathsf{p}}(A, a, x_1)\big]}}{\mathsf{El}_{\Sigma}\Big(w, (x_1, x_2).\ \mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big)\Big) \in \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, w)\ [w \in \phi]}$$

Where:

$\pi_{2.1})$ $\mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big) \in \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, \langle x_1, x_2 \rangle)\ \big[w \in \phi, x_1 \in A, x_2 \in \mathsf{Id}_{\mathsf{p}}(A, a, x_1)\big]$ derivable, because:

- Let $\Gamma = w \in \phi, x_1 \in A, x_2 \in \mathsf{Id}_{\mathsf{p}}(A, a, x_1)$
- $\mathsf{El}_{\mathsf{Id}_{\mathsf{p}}}\big(x_2, \mathsf{id}_{\mathsf{p}}(\alpha)\big) \in \mathsf{Id}_{\mathsf{p}}(\phi, \alpha, \langle x_1, x_2 \rangle)\ [\Gamma]$ derivable:

$$\text{E-}\mathsf{Id_p})\ \dfrac{\pi_{2.2} \quad \text{var})\ \dfrac{\overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{a \in A\ [\Gamma]} \quad \text{var})\ \dfrac{\overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{x_1 \in A\ [\Gamma]} \quad \text{var})\ \dfrac{\overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{x_2 \in \mathsf{Id_p}(\phi, a, x_1)\ [\Gamma]} \quad \text{F-}\mathsf{Id_p})\ \dfrac{\text{ind-te})\ \dfrac{\overset{\pi_1}{\alpha \in \phi\ [\,]} \quad \overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{\alpha \in \phi\ [\Gamma]}}{\mathsf{id_p}(\alpha) \in \mathsf{Id_p}(\phi, \alpha, \alpha)\ [\Gamma]}}{\mathsf{El}_{\mathsf{Id_p}}(x_2, \mathsf{id_p}(\alpha)) \in \mathsf{Id_p}(\phi, \alpha, \langle x_1, x_2\rangle)\ [\Gamma]}$$

$\pi_{2.2}$) $\mathsf{Id_p}(\phi, \alpha, \langle y, z\rangle)\ \mathit{type}\ \big[\Gamma, y \in A, z \in \mathsf{Id_p}(A, a, y)\big]$ derivable, because:

- Let $\Sigma = \Gamma, y \in A, z \in \mathsf{Id_p}(A, a, y)$
- $\mathsf{Id_p}(\phi, \alpha, \langle y, z\rangle)\ \mathit{type}\ [\Sigma]$ derivable:

$$\text{F-}\mathsf{Id_p})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{\pi_{3.1}}{\phi\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{\phi\ \mathit{type}\ [\Sigma]} \quad \text{ind-ter})\ \dfrac{\overset{\pi_1}{\alpha \in \phi\ [\,]} \quad \overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{\alpha \in \phi\ [\Sigma]} \quad \overset{\pi_{2.2.2}}{\langle y, z\rangle \in \phi\ [\Sigma]}}{\mathsf{Id_p}(\phi, \alpha, \langle y, z\rangle)\ \mathit{type}\ [\Sigma]}$$

Where:

$\pi_{2.2.1}$) $\Sigma\ \mathit{cont}$ derivable, because:

- $\Sigma = \Gamma, y \in A, z \in \mathsf{Id_p}(A, a, y)$
- $\Gamma, y \in A, z \in \mathsf{Id_p}(A, a, y)\ \mathit{cont}$ derivable:

$$\text{F-c})\ \dfrac{\text{F-}\mathsf{Id_p})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{A\ \mathit{type}\ [\Gamma]}}{\Gamma, y \in A\ \mathit{cont}}}{A\ \mathit{type}\ [\Gamma, y \in A]} \quad \text{ind-ter})\ \dfrac{\overset{a_2}{a \in A\ [\,]} \quad \text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{A\ \mathit{type}\ [\Gamma]}}{\Gamma, y \in A\ \mathit{cont}}}{a \in A\ [\Gamma, y \in A]} \quad \text{var})\ \dfrac{\text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.3}}{\Gamma\ \mathit{cont}}}{A\ \mathit{type}\ [\Gamma]}}{\Gamma, y \in A\ \mathit{cont}}}{y \in A\ [\Gamma, y \in A]}}{\mathsf{Id_p}(A, a, y)\ \mathit{type}\ [\Gamma, y \in A]}}{\Gamma, y \in A, z \in \mathsf{Id_p}(A, a, y)\ \mathit{cont}}$$

$\pi_{2.2.2}$) $\langle y, z\rangle \in \phi\ [\Sigma]$ derivable, because:

- $\phi = \Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)$
- $\langle y, z\rangle \in \Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)\ [\Sigma]$ derivable:

$$\text{I-}\Sigma)\ \dfrac{\text{var})\ \dfrac{\overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{y \in A\ [\Sigma]} \quad \text{var})\ \dfrac{\overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{z \in \mathsf{Id_p}(A, a, y)\ [\Sigma]} \quad \overset{\pi_{2.2.3}}{\mathsf{Id_p}(A, a, x)\ [\Sigma, x \in A]}}{\langle y, z\rangle \in \Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)\ [\Sigma]}$$

$\pi_{2.2.3}$) $\mathsf{Id_p}(A, a, x)\ [\Sigma, x \in A]$ derivable:

$$\text{F-}\mathsf{Id_p})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{A\ \mathit{type}\ [\Sigma]}}{\Sigma, x \in A\ \mathit{cont}}}{A\ \mathit{type}\ [\Sigma, x \in A]} \quad \text{ind-ter})\ \dfrac{\overset{a_2}{a \in A\ [\,]} \quad \text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{A\ \mathit{type}\ [\Sigma]}}{\Sigma, x \in A\ \mathit{cont}}}{a \in A\ [\Sigma, x \in A]} \quad \text{var})\ \dfrac{\text{F-c})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.2.1}}{\Sigma\ \mathit{cont}}}{A\ \mathit{type}\ [\Sigma]}}{\Sigma, x \in A\ \mathit{cont}}}{x \in A\ [\Sigma, x \in A]}}{\mathsf{Id_p}(A, a, x)\ [\Sigma, x \in A]}$$

$\pi_{2.3}$) $\Gamma\ \mathit{cont}$ derivable, because:

- $\Gamma = w \in \phi, x_1 \in A, x_2 \in \mathsf{Id_p}(A, a, x_1)$
- Let $\Delta = w \in \phi, x_1 \in A$
- $\Delta, x_2 \in \mathsf{Id_p}(A, a, x_1)\ \mathit{cont}$ derivable:

$$\text{F-c})\ \dfrac{\text{F-}\mathsf{Id_p})\ \dfrac{\text{ind-ty})\ \dfrac{\overset{a_1}{A\ \mathit{type}\ [\,]} \quad \overset{\pi_{2.4}}{\Delta\ \mathit{cont}}}{A\ \mathit{type}\ [\Delta]} \quad \text{ind-ter})\ \dfrac{\overset{a_2}{a \in A\ [\,]} \quad \overset{\pi_{2.4}}{\Delta\ \mathit{cont}}}{a \in A\ [\Delta]} \quad \text{var})\ \dfrac{\overset{\pi_{2.4}}{\Delta\ \mathit{cont}}}{x_1 \in A\ [\Delta]}}{\mathsf{Id_p}(A, a, x_1)\ \mathit{type}\ [\Delta]}}{\Delta, x_2 \in \mathsf{Id_p}(A, a, x_1)\ \mathit{cont}}$$

$\pi_{2.4}$) $\Delta\ \mathit{cont}$ derivable, because:

- $\Delta = w \in \phi, x_1 \in A\ cont$
- $w \in \phi, x_1 \in A\ cont$ derivable:

$$\text{F-c)}\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{a_1}{A\ type\ [\,]} \qquad \text{F-c)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]}}{w \in \phi\ cont}}{A\ type\ [w \in \phi]}}{w \in \phi, x_1 \in A\ cont}$$

$\pi_{2.5})$ $\mathsf{Id_p}(\phi, \alpha, z)\ type\ [w \in \phi, z \in \phi]$ derivable:

$$\text{F-Id}_{\text{p}})\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]} \quad \overset{\pi_{3.2}}{w \in \phi, z \in \phi\ cont}}{\phi\ type\ [w \in \phi, z \in \phi]} \qquad \text{ind-ter)}\ \dfrac{\overset{\pi_1}{\alpha \in \phi\ [\,]} \quad \overset{\pi_{3.2}}{w \in \phi, z \in \phi\ cont}}{\alpha \in \phi\ [w \in \phi, z \in \phi]} \qquad \text{var)}\ \dfrac{\overset{\pi_{3.2}}{w \in \phi, z \in \phi\ cont}}{z \in \phi\ [w \in \phi, z \in \phi]}}{\mathsf{Id_p}(\phi, \alpha, z)\ type\ [w \in \phi, z \in \phi]}$$

Where $w \in \phi, z \in \phi\ cont$ derivable:

$$\text{F-c)}\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]} \qquad \text{F-c)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]}}{w \in \phi\ cont}}{\phi\ type\ [w \in \phi]}}{w \in \phi, z \in \phi\ cont}$$

$\pi_3)$ $\psi(z, w)\ type\ [z \in \phi, w \in \phi]$ derivable, because:

- $\psi(z, w) = \mathsf{Id_p}(\phi, z, w)$
- $\mathsf{Id_p}(\phi, z, w)\ type\ [z \in \phi, w \in \phi]$ derivable:

$$\text{F-Id}_{\text{p}})\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]} \quad \overset{\pi_{3.2}}{z \in \phi, w \in \phi\ cont}}{\phi\ type\ [z \in \phi, w \in \phi]} \qquad \text{var)}\ \dfrac{\overset{\pi_{3.2}}{z \in \phi, w \in \phi\ cont}}{z \in \phi\ [z \in \phi, w \in \phi]} \qquad \text{var)}\ \dfrac{\overset{\pi_{3.2}}{z \in \phi, w \in \phi\ cont}}{w \in \phi\ [z \in \phi, w \in \phi]}}{\mathsf{Id_p}(\phi, z, w)\ type\ [z \in \phi, w \in \phi]}$$

Where:

$\pi_{3.1})$ $\phi\ type\ [\,]$ derivable, because:

- $\phi = \Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)$
- $\Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)\ type\ [\,]$ derivable:

$$\text{F-}\Sigma)\ \dfrac{\text{F-Id}_{\text{p}})\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{a_1}{A\ type\ [\,]} \quad \text{F-c)}\ \dfrac{\overset{a_1}{A\ type\ [\,]}}{x \in A\ cont}}{A\ type\ [x \in A]} \qquad \text{ind-ter)}\ \dfrac{\overset{a_2}{a \in A\ [\,]} \quad \text{F-c)}\ \dfrac{\overset{a_1}{A\ type\ [\,]}}{x \in A\ cont}}{a \in A\ [x \in A]} \qquad \text{var)}\ \dfrac{\text{F-c)}\ \dfrac{\overset{a_1}{A\ type\ [\,]}}{x \in A\ cont}}{x \in A\ [x \in A]}}{\mathsf{Id_p}(A, a, x)\ type\ [x \in A]}}{\Sigma_{x \in A}\ \mathsf{Id_p}(A, a, x)\ type\ [\,]}$$

$\pi_{3.2})$ $z \in \phi, w \in \phi\ cont$ derivable:

$$\text{F-c)}\ \dfrac{\text{ind-ty)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]} \qquad \text{F-c)}\ \dfrac{\overset{\pi_{3.1}}{\phi\ type\ [\,]}}{z \in \phi\ cont}}{\phi\ type\ [z \in \phi]}}{z \in \phi, w \in \phi\ cont}$$