Languages for Concurrency and Distribution

Exercises

Alberto Lazari

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Index

Exercise C

Lesson 5 – page 7

Theorem : Let $[\![\ \ ]\!]$: CCS-VP $\longrightarrow$ CCS as above.

Then for all CCS-VP programs P

$$\text{(i)} \qquad P \xrightarrow{\alpha} P' \qquad \Longrightarrow \qquad [\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]$$

$$\text{(ii)} \qquad [\![\, P \,]\!] \xrightarrow{\hat{\alpha}} Q \qquad \Longrightarrow \qquad \exists\, P' .\, P \xrightarrow{\alpha} P' \wedge [\![\, P' \,]\!] = Q$$

where

$$\hat{\alpha} = \begin{cases} a_n & \text{if } \alpha = a(n) \\ \overline{a}_n & \text{if } \alpha = \overline{a}(n) \\ \tau & \text{if } \alpha = \tau \end{cases}$$

Proof by structural induction on P

Let's first introduce the rule "ind-(i)", which is derived by the consequence (i) of the inductive hypothesis of the theorem.

$$\text{ind-(i)} \quad \frac{P \xrightarrow{\alpha} P'}{[\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]}$$

Case $a(x).P$

(i) Assuming:

$\pi_1)$ $a(x).P \xrightarrow{a(n)} P\{^n/_x\}$

$\pi_2)$ $[\![\, a(x).P \,]\!] = \sum_{i \in \mathbb{N}} a_i.[\![\, P\{^n/_x\} \,]\!]$

$\Longrightarrow [\![\, a(x).P \,]\!] \xrightarrow{a_n} [\![\, P\{^n/_x\} \,]\!]$, because:

$$\text{SUM} \quad \frac{\text{ACT} \quad \dfrac{}{a_n.[\![\, P\{^n/_x\} \,]\!] \xrightarrow{a_n} [\![\, P\{^n/_x\} \,]\!]}}{\sum_{i \in \mathbb{N}} a_i.[\![\, P\{^n/_x\} \,]\!] \xrightarrow{a_n} [\![\, P\{^n/_x\} \,]\!]} \quad n \in \mathbb{N}$$

(ii) Assuming:

$\pi_1)$ $[\![\, a(x).P \,]\!] \xrightarrow{a_n} [\![\, P\{^n/_x\} \,]\!]$

$\pi_2)$ Let $Q = [\![\, P\{^n/_x\} \,]\!]$

Let $P' = P\{^n/_x\}$

$\Longrightarrow [\![\, P' \,]\!] = Q$ by π_2 and $a(x).P \xrightarrow{a(n)} P'$, because:

$$\text{input} \quad \frac{}{a(x).P \xrightarrow{a(n)} P\{^n/_x\}} \quad n \in \mathbb{N}$$

Case $\overline{a}(e).P$

(i) Assuming:

$\pi_1)$ e evaluates to n

$\pi_2)$ $\overline{a}(e).P \xrightarrow{\overline{a}(n)} P$

$\pi_3)$ $[\![\, \overline{a}(e).P \,]\!] = \overline{a}_n.[\![\, P \,]\!]$

$\implies [\![\, \overline{a}(e).P \,]\!] \xrightarrow{\overline{a}_n} [\![\, P \,]\!]$, because:

$$\text{ACT} \quad \frac{}{\overline{a}_n.[\![\, P \,]\!] \xrightarrow{\overline{a}_n} [\![\, P \,]\!]}$$

(ii) Assuming:

$\pi_1)$ e evaluates to n

$\pi_2)$ $[\![\, \overline{a}(e).P \,]\!] \xrightarrow{\overline{a}_n} [\![\, P \,]\!]$

$\pi_3)$ Let $Q = [\![\, P \,]\!]$

Let $P' = P$

$\implies [\![\, P' \,]\!] = Q$ by π_2 and $\overline{a}(e).P \xrightarrow{\overline{a}(n)} P'$, because:

$$\text{output} \quad \frac{}{\overline{a}(e).P \xrightarrow{\overline{a}(n)} P} \quad \pi_1$$

Case $\tau.P$

(i) Assuming:

$\pi_1)$ $\tau.P \xrightarrow{\tau} P$

$\pi_2)$ $[\![\, \tau.P \,]\!] = \tau.[\![\, P \,]\!]$

$\implies [\![\, \tau.P \,]\!] \xrightarrow{\tau} [\![\, P \,]\!]$, because:

$$\text{ACT} \quad \frac{}{\tau.[\![\, P \,]\!] \xrightarrow{\tau} [\![\, P \,]\!]}$$

(ii) Assuming:

$\pi_1)$ $[\![\, \tau.P \,]\!] \xrightarrow{\tau} [\![\, P \,]\!]$

$\pi_2)$ Let $Q = [\![\, P \,]\!]$

Let $P' = P$

$\implies [\![\, P' \,]\!] = Q$ by π_2 and $\tau.P \xrightarrow{\tau} P'$, because:

$$\text{silent} \quad \frac{}{\tau.P \xrightarrow{\tau} P}$$

$$\boxed{\textbf{Case} \sum_{i \in I} P_i}$$

(i) Assuming:

$\pi_1)$ $\sum_{i \in I} P_i \xrightarrow{\alpha} P'_j$

$\pi_2)$ $\left[\!\!\left[\sum_{i \in I} P_i \right]\!\!\right] = \sum_{i \in I} [\![P_i]\!]$

$\implies \left[\!\!\left[\sum_{i \in I} P_i \right]\!\!\right] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]$, because:

$$\textrm{SUM}\ \dfrac{\textrm{ind-(i)}\ \dfrac{\overset{\pi_3}{P_j \xrightarrow{\alpha} P'_j}}{[\![P_j]\!] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]}}{\sum_{i \in I} [\![P_i]\!] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]}\ j \in I$$

Where π_3 is deduced from the fact that the assumption π_1 can only be derived by the following tree:

$$\textrm{SUM}\ \dfrac{P_j \xrightarrow{\alpha} P'_j}{\sum_{i \in I} P_i \xrightarrow{\alpha} P'_j}\ j \in I$$

(ii) Assuming:

$\pi_1)$ $\left[\!\!\left[\sum_{i \in I} P_i \right]\!\!\right] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]$

$\pi_2)$ Let $Q = [\![P'_j]\!]$

$\pi_3)$ $\left[\!\!\left[\sum_{i \in I} P_i \right]\!\!\right] = \sum_{i \in I} [\![P_i]\!]$

$\implies [\![P_j]\!] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]$ is derivable, because π_1 can only be derived by the following tree:

$$\textrm{SUM}\ \dfrac{[\![P_j]\!] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]}{\sum_{i \in I} [\![P_i]\!] \xrightarrow{\hat{\alpha}} [\![P'_j]\!]}\ j \in I$$

Then, by inductive hypothesis, $\exists\, P'_{\textrm{ind}}.\, P_j \xrightarrow{\alpha} P'_{\textrm{ind}} \wedge P'_{\textrm{ind}} = [\![P'_j]\!]$

- Let $P'_{\textrm{ind}} = P'_j.\, P_j \xrightarrow{\alpha} P'_j$
- Let $P' = P'_j$

$\Longrightarrow [\![\, P' \,]\!] = Q$ by π_2 and $\sum_{i \in I} P_i \xrightarrow{\alpha} P'$, because:

$$\text{SUM}\ \frac{P_j \xrightarrow{\alpha} P'_j}{\sum_{i \in I} P_i \xrightarrow{\alpha} P'_j}\ j \in I$$

Case $P_1 \mid P_2$

For this case there are more sub-cases to consider.

Sub-case $P_1 \mid P_2 \xrightarrow{\alpha} P'_1 \mid P_2$

(i) Assuming:

π_1) $P_1 \mid P_2 \xrightarrow{\alpha} P'_1 \mid P_2$

π_2) $[\![\, P_1 \mid P_2 \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!]$

π_3) $[\![\, P'_1 \mid P_2 \,]\!] = [\![\, P'_1 \,]\!] \mid [\![\, P_2 \,]\!]$

$\Longrightarrow [\![\, P_1 \mid P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \mid P_2 \,]\!]$, because:

$$\text{parallel}\ \frac{\text{ind-(i)}\ \dfrac{\overset{\pi_4}{P_1 \xrightarrow{\alpha} P'_1}}{[\![\, P_1 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \,]\!]}}{[\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \,]\!] \mid [\![\, P_2 \,]\!]}$$

Where π_4 is derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \frac{P_1 \xrightarrow{\alpha} P'_1}{P_1 \mid P_2 \xrightarrow{\alpha} P'_1 \mid P_2}$$

(ii) Assuming:

π_1) $[\![\, P_1 \mid P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \mid P_2 \,]\!]$

π_2) Let $Q = [\![\, P'_1 \mid P_2 \,]\!]$

π_3) $[\![\, P_1 \mid P_2 \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!]$

π_4) $[\![\, P'_1 \mid P_2 \,]\!] = [\![\, P'_1 \,]\!] \mid [\![\, P_2 \,]\!]$

$\Longrightarrow [\![\, P_1 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \,]\!]$ is derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \frac{[\![\, P_1 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \,]\!]}{[\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P'_1 \,]\!] \mid [\![\, P_2 \,]\!]}$$

Then, by inductive hypothesis, $\exists P'_{\text{ind}}.\, P_1 \xrightarrow{\alpha} P'_{\text{ind}} \wedge [\![\, P'_{\text{ind}} \,]\!] = [\![\, P'_1 \,]\!]$

- Let $P'_{\text{ind}} = P'_1.\, P_1 \xrightarrow{\alpha} P'_1$
- Let $P' = P'_1 \mid P_2$

$\Longrightarrow [\![\, P' \,]\!] = Q$ by π_2 and $P_1 \mid P_2 \xrightarrow{\alpha} P'$, because:

$$\text{parallel}\ \frac{P_1 \xrightarrow{\alpha} P_1'}{P_1 \mid P_2 \xrightarrow{\alpha} P_1' \mid P_2}$$

Sub-case $P_1 \mid P_2 \xrightarrow{\alpha} P_1 \mid P_2'$

(i) Assuming:

$\pi_1)$ $P_1 \mid P_2 \xrightarrow{\alpha} P_1 \mid P_2'$

$\pi_2)$ $[\![\, P_1 \mid P_2 \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!]$

$\pi_3)$ $[\![\, P_1 \mid P_2' \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2' \,]\!]$

$\Longrightarrow [\![\, P_1 \mid P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_1 \mid P_2' \,]\!]$, because:

$$\text{parallel}\ \frac{\text{ind-(i)}\ \dfrac{\overset{\pi_4}{P_2 \xrightarrow{\alpha} P_2'}}{[\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_2' \,]\!]}}{[\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_1 \,]\!] \mid [\![\, P_2' \,]\!]}$$

Where π_4 is derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 \mid P_2 \xrightarrow{\alpha} P_1 \mid P_2'}$$

(ii) Assuming:

$\pi_1)$ $[\![\, P_1 \mid P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_1 \mid P_2' \,]\!]$

$\pi_2)$ Let $Q = [\![\, P_1 \mid P_2' \,]\!]$

$\pi_3)$ $[\![\, P_1 \mid P_2 \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!]$

$\pi_4)$ $[\![\, P_1 \mid P_2' \,]\!] = [\![\, P_1 \,]\!] \mid [\![\, P_2' \,]\!]$

$\Longrightarrow [\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_2' \,]\!]$ is derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \frac{[\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_2' \,]\!]}{[\![\, P_1 \,]\!] \mid [\![\, P_2 \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P_1 \,]\!] \mid [\![\, P_2' \,]\!]}$$

Then, by inductive hypothesis, $\exists\, P_{\text{ind}}'.\, P_2 \xrightarrow{\alpha} P_{\text{ind}}' \wedge [\![\, P_{\text{ind}}' \,]\!] = [\![\, P_2' \,]\!]$

- Let $P_{\text{ind}}' = P_2'.\, P_2 \xrightarrow{\alpha} P_2'$
- Let $P' = P_1 \mid P_2'$

$\Longrightarrow [\![\, P' \,]\!] = Q$ by π_2 and $P_1 \mid P_2 \xrightarrow{\alpha} P'$, because:

$$\text{parallel}\ \frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 \mid P_2 \xrightarrow{\alpha} P_1 \mid P_2'}$$

Sub-case $P_1 \mid P_2 \xrightarrow{\tau} P_1' \mid P_2'$

(i) Assuming:

$\pi_1)$ $P_1 \mid P_2 \xrightarrow{\tau} P_1' \mid P_2'$

$\pi_2)$ $[\![P_1 \mid P_2]\!] = [\![P_1]\!] \mid [\![P_2]\!]$

$\pi_3)$ $[\![P_1' \mid P_2']\!] = [\![P_1']\!] \mid [\![P_2']\!]$

$\Longrightarrow [\![P_1 \mid P_2]\!] \xrightarrow{\tau} [\![P_1' \mid P_2']\!]$, because:

$$\text{parallel}\ \dfrac{\text{ind-(i)}\ \dfrac{\overset{\pi_4}{P_1 \xrightarrow{\alpha} P_1'}}{[\![P_1]\!] \xrightarrow{\widehat{\alpha}} [\![P_1']\!]} \qquad \text{ind-(i)}\ \dfrac{\overset{\pi_5}{P_2 \xrightarrow{\overline{\alpha}} P_2'}}{[\![P_2]\!] \xrightarrow{\widehat{\overline{\alpha}}} [\![P_2']\!]}}{[\![P_1]\!] \mid [\![P_2]\!] \xrightarrow{\tau} [\![P_1']\!] \mid [\![P_2']\!]}$$

Where π_4 and π_5 are derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \dfrac{\overset{\pi_4}{P_1 \xrightarrow{\alpha} P_1'} \qquad \overset{\pi_5}{P_2 \xrightarrow{\overline{\alpha}} P_2'}}{P_1 \mid P_2 \xrightarrow{\tau} P_1' \mid P_2'}$$

(ii) Assuming:

$\pi_1)$ $[\![P_1 \mid P_2]\!] \xrightarrow{\tau} [\![P_1' \mid P_2']\!]$

$\pi_2)$ Let $Q = [\![P_1' \mid P_2']\!]$

$\pi_3)$ $[\![P_1 \mid P_2]\!] = [\![P_1]\!] \mid [\![P_2]\!]$

$\pi_4)$ $[\![P_1' \mid P_2']\!] = [\![P_1']\!] \mid [\![P_2']\!]$

$\Longrightarrow [\![P_1]\!] \xrightarrow{\tau} [\![P_1']\!]$ and $[\![P_2]\!] \xrightarrow{\tau} [\![P_2']\!]$ are derivable, because π_1 can only be derived by the following tree:

$$\text{parallel}\ \dfrac{[\![P_1]\!] \xrightarrow{\widehat{\alpha}} [\![P_1']\!] \qquad [\![P_2]\!] \xrightarrow{\widehat{\overline{\alpha}}} [\![P_2']\!]}{[\![P_1]\!] \mid [\![P_2]\!] \xrightarrow{\tau} [\![P_1']\!] \mid [\![P_2']\!]}$$

Then, by inductive hypothesis follows:

1) $\exists P'_{\text{ind}_1}.\, P_1 \xrightarrow{\alpha} P'_{\text{ind}_1} \wedge \left[\!\left[P'_{\text{ind}_1} \right]\!\right] = [\![P_1']\!]$

2) $\exists P'_{\text{ind}_2}.\, P_2 \xrightarrow{\overline{\alpha}} P'_{\text{ind}_2} \wedge \left[\!\left[P'_{\text{ind}_2} \right]\!\right] = [\![P_2']\!]$

Using 1) and 2):

- Let $P'_{\text{ind}_1} = P_1'.\, P_1 \xrightarrow{\alpha} P_1'$
- Let $P'_{\text{ind}_2} = P_2'.\, P_2 \xrightarrow{\overline{\alpha}} P_2'$
- Let $P' = P_1' \mid P_2'$

$\Longrightarrow [\![P']\!] = Q$ by π_2 and $P_1 \mid P_2 \xrightarrow{\alpha} P'$, because:

$$\text{parallel} \quad \frac{P_1 \xrightarrow{\alpha} P_1' \qquad P_2 \xrightarrow{\overline{\alpha}} P_2'}{P_1 \mid P_2 \xrightarrow{\tau} P_1' \mid P_2'}$$

Case $P \setminus L$

(i) Assuming:

$\pi_1)$ $P \setminus L \xrightarrow{\alpha} P' \setminus L$

$\pi_2)$ $[\![P \setminus L]\!] = [\![P]\!] \setminus L'$, where $L' = \{a_n \mid a \in L, n \in \mathbb{N}\}$

$\pi_3)$ $[\![P' \setminus L]\!] = [\![P']\!] \setminus L'$

$\Longrightarrow [\![P \setminus L]\!] \xrightarrow{\hat{\alpha}} [\![P' \setminus L]\!]$, because:

$$\text{RES} \quad \frac{\text{ind-(i)} \quad \dfrac{\overset{\pi_4}{P \xrightarrow{\alpha} P'}}{[\![P]\!] \xrightarrow{\hat{\alpha}} [\![P']\!]}}{[\![P]\!] \setminus L' \xrightarrow{\hat{\alpha}} [\![P']\!] \setminus L'} \quad \hat{\alpha}, \hat{\overline{\alpha}} \notin L'$$

Where π_4 is derivable and $\alpha, \overline{\alpha} \notin L$, because π_1 can only be derived by the following tree:

$$\text{RES} \quad \frac{P \xrightarrow{\alpha} P'}{P \setminus L \xrightarrow{\alpha} P' \setminus L} \quad \alpha, \overline{\alpha} \notin L$$

Note that $\hat{\alpha}, \hat{\overline{\alpha}} \notin L'$ by construction of L', because $\alpha, \overline{\alpha} \notin L$.

(ii) Assuming:

$\pi_1)$ $[\![P \setminus L]\!] \xrightarrow{\hat{\alpha}} [\![P' \setminus L]\!]$

$\pi_2)$ Let $Q = [\![P' \setminus L]\!]$

$\pi_3)$ $[\![P \setminus L]\!] = [\![P]\!] \setminus L'$, where $L' = \{a_n \mid a \in L, n \in \mathbb{N}\}$

$\pi_4)$ $[\![P' \setminus L]\!] = [\![P']\!] \setminus L'$

$\Longrightarrow [\![P]\!] \xrightarrow{\hat{\alpha}} [\![P']\!]$ is derivable and $\hat{\alpha}, \hat{\overline{\alpha}} \notin L'$, because π_1 can only be derived by the following tree:

$$\text{RES} \quad \frac{[\![P]\!] \xrightarrow{\hat{\alpha}} [\![P']\!]}{[\![P]\!] \setminus L' \xrightarrow{\hat{\alpha}} [\![P']\!] \setminus L'} \quad \hat{\alpha}, \hat{\overline{\alpha}} \notin L'$$

Note that $\alpha, \overline{\alpha} \notin L$ by construction of L', because $\hat{\alpha}, \hat{\overline{\alpha}} \notin L'$.

Then, by inductive hypothesis, $\exists\, P'_{\text{ind}}.\, P \xrightarrow{\alpha} P'_{\text{ind}} \wedge [\![P'_{\text{ind}}]\!] = [\![P']\!]$

- Let $P'_{\text{ind}} = P'.\, P \xrightarrow{\alpha} P'$
- Let $P'_{\text{th}} = P' \setminus L$

$\Longrightarrow [\![P'_{\text{th}}]\!] = Q$ by π_2 and $P \setminus L \xrightarrow{\alpha} P'_{\text{th}}$, because:

$$\text{RES} \quad \frac{P \xrightarrow{\alpha} P'}{P \setminus L \xrightarrow{\alpha} P' \setminus L} \quad \alpha, \overline{\alpha} \notin L$$

$\boxed{\textbf{Case } P\,[\,f\,]}$

(i) Assuming:

$\pi_1)$ $P\,[\,f\,] \xrightarrow{f(\alpha)} P'\,[\,f\,]$

$\pi_2)$ $[\![P\,[\,f\,]]\!] = [\![P]\!]\,[\,f'\,]$, where $f' = \lambda a_n.f(a)_n, n \in \mathbb{N}$

$\pi_3)$ $[\![P'\,[\,f\,]]\!] = [\![P']\!]\,[\,f'\,]$

For simplicity, let's assume that $f(a(n)) = f(a)$. This allows for a more concise definition of f':

$$f' = \lambda\hat{\alpha}.f\widehat{(\alpha)}$$

$\Longrightarrow [\![P\,[\,f\,]]\!] \xrightarrow{f\widehat{(\alpha)}} [\![P'\,[\,f\,]]\!]$, because:

$$\text{Redirection} \quad \dfrac{\text{ind-(i)} \quad \dfrac{\overset{\pi_4}{P \xrightarrow{\alpha} P'}}{[\![P]\!] \xrightarrow{\hat{\alpha}} [\![P']\!]}}{[\![P]\!]\,[\,f'\,] \xrightarrow{f'(\hat{\alpha})} [\![P']\!]\,[\,f'\,]}$$

Where π_4 is derivable, because π_1 can only be derived by the following tree:

$$\text{Redirection} \quad \frac{P \xrightarrow{\alpha} P'}{P\,[\,f\,] \xrightarrow{f(\alpha)} P'\,[\,f\,]}$$

(ii) Assuming:

$\pi_1)$ $[\![P\,[\,f\,]]\!] \xrightarrow{f\widehat{(\alpha)}} [\![P'\,[\,f\,]]\!]$

$\pi_2)$ Let $Q = [\![P'\,[\,f\,]]\!]$

$\pi_3)$ $[\![P\,[\,f\,]]\!] = [\![P]\!]\,[\,f'\,]$, where $f' = \lambda a_n.f(a)_n, n \in \mathbb{N}$

$\pi_4)$ $[\![P'\,[\,f\,]]\!] = [\![P']\!]\,[\,f'\,]$

For simplicity, let's assume that $f(a(n)) = f(a)$. This allows for a more concise definition of f':

$$f' = \lambda\hat{\alpha}.f\widehat{(\alpha)}$$

$\Longrightarrow [\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]$ is derivable, because π_1 can only be derived by the following tree:

$$\text{Redirection}\quad \frac{[\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]}{[\![\, P \,]\!]\,[\, f' \,] \xrightarrow{f'(\hat{\alpha})} [\![\, P' \,]\!]\,[\, f' \,]}$$

Then, by inductive hypothesis, $\exists\, P'_{\text{ind}}.\, P \xrightarrow{\alpha} P'_{\text{ind}} \wedge [\![\, P'_{\text{ind}} \,]\!] = [\![\, P' \,]\!]$

- Let $P'_{\text{ind}} = P'.\, P \xrightarrow{\alpha} P'$
- Let $P'_{\text{th}} = P'\,[\, f \,]$

$\Longrightarrow [\![\, P'_{\text{th}} \,]\!] = Q$ by π_2 and $P\,[\, f \,] \xrightarrow{\alpha} P'_{\text{th}}$, because:

$$\text{Redirection}\quad \frac{P \xrightarrow{\alpha} P'}{P\,[\, f \,] \xrightarrow{f(\alpha)} P'\,[\, f \,]}$$

Case if b then P

For this case there are two sub-cases to consider.

Sub-case $b = \mathsf{true}$

(i) Assuming:

π_1) $b = \mathsf{true}$

π_2) if b then $P \xrightarrow{\alpha} P'$

π_3) $[\![$ if b then $P \,]\!] = [\![\, P \,]\!]$

$\Longrightarrow [\![$ if b then $P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]$, because:

$$\text{ind-(i)}\quad \frac{\begin{array}{c}\pi_4\\ P \xrightarrow{\alpha} P'\end{array}}{[\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]}$$

Where π_4 is derivable, because π_2 can only be derived by the following tree:

$$\text{conditionals}\quad \frac{P \xrightarrow{\alpha} P'}{\text{if } b \text{ then } P \xrightarrow{\alpha} P'}\quad \pi_1$$

(ii) Assuming:

π_1) $b = \mathsf{true}$

π_2) $[\![$ if b then $P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]$

π_3) Let $Q = [\![\, P' \,]\!]$

π_4) $[\![$ if b then $P \,]\!] = [\![\, P \,]\!]$

$\Longrightarrow [\![\, P \,]\!] \xrightarrow{\hat{\alpha}} [\![\, P' \,]\!]$ is derivable, by π_2 and π_4.

Then, by inductive hypothesis, $\exists\, P'_{\text{ind}}.\, P \xrightarrow{\alpha} P'_{\text{ind}} \wedge [\![\, P'_{\text{ind}} \,]\!] = [\![\, P' \,]\!]$

- Let $P'_{\text{ind}} = P'.\, P \xrightarrow{\alpha} P'$
- Let $P'_{\text{th}} = P'$

$\implies [\![\, P'_{\text{th}} \,]\!] = Q$ by π_3 and if b then $P \xrightarrow{\alpha} P'_{\text{th}}$, because:

$$\text{conditionals}\ \frac{P \xrightarrow{\alpha} P'}{\text{if } b \text{ then } P \xrightarrow{\alpha} P'}\ \pi_1$$

Sub-case $b = \mathsf{false}$

Assuming:

$\pi_1)$ $b = \mathsf{false}$

$\pi_2)$ $[\![\ \text{if } b \text{ then } P \,]\!] = \emptyset$

$\implies$ if b then $P \not\rightarrow$, because there's no rule to derive it and $[\![\ \text{if } b \text{ then } P \,]\!] \not\rightarrow$, by π_2

$\implies$ vacuously true

Case $K(e_1, \ldots, e_n)$

(i) Assuming:

$\pi_1)$ e_i evaluates to k_i, $\forall\, i \in \{1..n\}$

$\pi_2)$ $K(x_1, \ldots, x_n) \overset{\text{def}}{=} P$

$\pi_3)$ $K(e_1, \ldots, e_n) \xrightarrow{\alpha} P'$

$\pi_4)$ $K_{k_1, \ldots, k_n} \overset{\text{def}}{=} \left[\!\!\left[\, P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \right]\!\!\right]$

$\pi_5)$ $[\![\, K(e_1, \ldots, e_n) \,]\!] = K_{k_1, \ldots, k_n}$

$\implies [\![\, K(e_1, \ldots, e_n) \,]\!] \xrightarrow{\widehat{\alpha}} [\![\, P' \,]\!]$, because:

$$\text{Constant}\ \frac{\text{ind-(i)}\ \dfrac{\overset{\pi_6}{P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \xrightarrow{\alpha} P'}}{\left[\!\!\left[\, P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \right]\!\!\right] \xrightarrow{\widehat{\alpha}} [\![\, P' \,]\!]}}{K_{k_1, \ldots, k_n} \xrightarrow{\widehat{\alpha}} [\![\, P' \,]\!]}\ \pi_4$$

Where π_6 is derivable, because π_3 can only be derived by the following tree:

$$\text{constant}\ \frac{P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \xrightarrow{\alpha} P'}{K(e_1, \ldots, e_n) \xrightarrow{\alpha} P'}\ \pi_1, \pi_2$$

(ii) Assuming:

$\pi_1)$ e_i evaluates to k_i, $\forall\, i \in \{1..n\}$

$\pi_2)$ $K(x_1, \ldots, x_n) \overset{\text{def}}{=} P$

$\pi_3)$ $[\![\, K(e_1, \ldots, e_n) \,]\!] \xrightarrow{\widehat{\alpha}} [\![\, P' \,]\!]$

$\pi_4)$ $K_{k_1,\ldots,k_n} \stackrel{\text{def}}{=} \left[\!\left[P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \right]\!\right]$

$\pi_5)$ Let $Q = [\![P']\!]$

$\pi_6)$ $[\![K(e_1, \ldots, e_n)]\!] = K_{k_1,\ldots,k_n}$

$\Longrightarrow \left[\!\left[P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \right]\!\right] \xrightarrow{\hat{\alpha}} [\![P']\!]$ is derivable, because π_3 can only be derived by the following tree:

$$\text{Constant} \quad \frac{\left[\!\left[P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \right]\!\right] \xrightarrow{\hat{\alpha}} [\![P']\!]}{K_{k_1,\ldots,k_n} \xrightarrow{\hat{\alpha}} [\![P']\!]} \quad \pi_4$$

Then, by inductive hypothesis, $\exists\, P'_{\text{ind}} .\, P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \xrightarrow{\alpha} P'_{\text{ind}} \wedge [\![P'_{\text{ind}}]\!] = [\![P']\!]$

- Let $P'_{\text{ind}} = P' .\, P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \xrightarrow{\alpha} P'$
- Let $P'_{\text{th}} = P'$

$\Longrightarrow [\![P'_{\text{th}}]\!] = Q$ by π_5 and $K(e_1, \ldots, e_n) \xrightarrow{\alpha} P'_{\text{th}}$, because:

$$\text{constant} \quad \frac{P\left\{{}^{k_1}\!/_{x_1}, \ldots, {}^{k_n}\!/_{x_n}\right\} \xrightarrow{\alpha} P'}{K(e_1, \ldots, e_n) \xrightarrow{\alpha} P'} \quad \pi_1, \pi_2$$

$\blacksquare$

Exercise G

Until (strong)

- Let $\eta(\varphi) = [\![\, \varphi \,]\!]_\eta = [\![\, \varphi \,]\!] = \{P \mid P \vDash \varphi\}$
- Let $[\text{Act}]\ S = \left\{P \mid \forall\, \alpha \in \text{Act}\ .\, P \xrightarrow{\alpha} P' \implies P' \in S\right\}$
- Let $\upsilon(\varphi, \psi) = \psi \vee (\varphi \wedge \langle \text{Act} \rangle\ \text{T} \wedge [\text{Act}]\ X)$

Lesson 13 – page 6

- Let $f_{\upsilon(\varphi,\psi)}(S) = [\![\, \upsilon(\varphi, \psi) \,]\!]_{\eta[X \to S]}$
- Let $\text{CC} = \{[P_1\ \ P_2\ \ \dots\ \ P_n] \mid P_1 \longrightarrow P_2 \longrightarrow \dots \longrightarrow P_n \not\longrightarrow\} \subseteq \bigcup_{i \in \mathbb{N}} \text{Proc}^i$

  be the set of all the complete computations of any process

See $\text{CTr}(P)$ (completed traces): lesson 6 – page 4

- Let $\text{CCmp} : \text{Proc} \to 2^{\text{CC}}$ s.t.
  $\text{CCmp}(P) = \{c \in \text{CC} \mid P = c_1\}$ is the set of all the complete computations of P

EXERCISE : Let's define

$$\text{Until}(\varphi, \psi) = \varphi\ \mathcal{U}\ \psi = \mu X.\, \upsilon(\varphi, \psi) = \mu X.\, \psi \vee (\varphi \wedge \langle \text{Act} \rangle\ \text{T} \wedge [\text{Act}]\ X)$$

$$\text{let} \qquad S_1 = [\![\, \mu X.\, \upsilon(\varphi, \psi) \,]\!]$$

The set of processes for which $\varphi\ \mathcal{U}\ \psi$ is satisfied can be directly expressed as

$$S_2 = \left\{P \mid \forall\, c \in \text{CCmp}(P).\, \exists\, i \in \mathbb{N}.\, \left(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\right)\right\}$$

Are they really the same?

$$S_1 \stackrel{?}{=} S_2$$

$$\to S_1 \subseteq S_2 : \quad S_2 \text{ is a fixpoint of } f_{\upsilon(\varphi,\psi)}$$

$$\to S_2 \subseteq S_1 : \quad \text{by induction on } n \text{ in } f^n_{\upsilon(\varphi,\psi)}(\emptyset)$$

Lesson 13 – page 6

By the definition of the semantics of $[\![\, \mu X.\varphi \,]\!]_\eta$

$$S_1 = [\![\, \mu X.\, \upsilon(\varphi, \psi) \,]\!]_\eta = \text{fix}\left(f_{\upsilon(\varphi,\psi)}\right) \tag{1}$$

$$\boxed{S_1 \subseteq S_2}$$

To prove: $[\![\, \mu X.\, \upsilon(\varphi, \psi) \,]\!]_\eta \subseteq \left\{P \mid \forall\, c \in \text{CCmp}(P).\, \exists\, i \in \mathbb{N}.\, \left(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\right)\right\}$

S_1 is the lfp $\implies$ it is a subset of every fixed point of $f_{\upsilon(\varphi,\psi)}$

(1) $\implies \forall\, S \subseteq \text{Proc}\ .\left(f_{\upsilon(\varphi,\psi)}(S) = S \quad \implies \quad S_1 \subseteq S\right)$

In particular, this holds for S_2:

$$f_{\upsilon(\varphi,\psi)}(S_2) = S_2 \quad \implies \quad S_1 \subseteq S_2 \tag{2}$$

Remember, $f_{\upsilon(\varphi,\psi)}(S) = [\![\, \psi \vee (\varphi \wedge \langle \text{Act} \rangle \ \text{T} \wedge [\text{Act}] \ X) \,]\!]_{\eta[X \to S]}$

S_2 is a fixed point of $f_{\upsilon(\varphi,\psi)}$, in fact:

$S_2 = \big\{ P \mid \forall\, c \in \text{CCmp}(P).\, \exists\, i \in \mathbb{N}.\, \big(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\big) \big\} =$

$P \vDash \psi$, otherwise it has to be that $P \vDash \varphi$ and it does at least a step

All the complete computations of the next steps respect the same property

$= \big\{ P \mid P \vDash \psi \vee \big(P \vDash \varphi \wedge \exists\, P' \in \text{Proc} \ .\, P \longrightarrow P' \wedge \forall\, P \longrightarrow P'.$
$\forall\, c \in \text{CCmp}(P').\, \exists\, i \in \mathbb{N}.\, \big(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\big)\big) \big\}$

Note that

$$\forall\, c \in \text{CCmp}(P').\, \exists\, i \in \mathbb{N}.\, \big(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\big) \quad \Longleftrightarrow \quad P' \in S_2$$

$\Longrightarrow S_2 = \{ P \mid P \vDash \psi \vee (P \vDash \varphi \wedge \exists\, P' \in \text{Proc} \ .\, P \longrightarrow P' \wedge \forall\, P \longrightarrow P'.\, P' \in S_2) \} =$

$= [\![\, \psi \,]\!]_\eta \cup \Big([\![\, \varphi \,]\!]_\eta \cap [\![\, \langle \text{Act} \rangle \ \text{T} \,]\!]_\eta \cap [\![\, [\text{Act}] \ X \,]\!]_{\eta[X \to S_2]} \Big) =$

$= [\![\, \psi \vee (\varphi \wedge \langle \text{Act} \rangle \ \text{T} \wedge [\text{Act}] \ X) \,]\!]_{\eta[X \to S_2]} = f_{\upsilon(\varphi,\psi)}(S_2)$

Which is equivalent to

$$f_{\upsilon(\varphi,\psi)}(S_2) = S_2 \tag{3}$$

From (2):

$$(3) \quad \Longrightarrow \quad S_1 \subseteq S_2$$

■

$\boxed{S_1 \supseteq S_2}$

To prove: $[\![\, \mu X.\, \upsilon(\varphi,\psi) \,]\!]_\eta \supseteq \big\{ P \mid \forall\, c \in \text{CCmp}(P).\, \exists\, i \in \mathbb{N}.\, \big(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\big) \big\}$

Lesson 13 – page 6

For finite state processes it holds that

$$\forall\, n \in N.\, f^n_{\upsilon(\varphi,\psi)}(\emptyset) \subseteq \text{fix}\big(f_{\upsilon(\varphi,\psi)}\big) = S_1 \tag{4}$$

Let $R_n = \big\{ P \mid \forall\, c \in \text{CCmp}(P).\, \exists\, i \leqslant n.\, \big(c_i \vDash \psi \wedge \forall\, j < i.\, c_j \vDash \varphi\big) \big\}$ s.t. $\lim_{n \to \infty} R_n = S_2$

Assuming

$$\forall\, n \in \mathbb{N}.\, R_n \subseteq f^n_{\upsilon(\varphi,\psi)}(\emptyset) \tag{5}$$

By (4) and (5)

$$\forall\, n \in \mathbb{N}.\, R_n \subseteq f^n_{\upsilon(\varphi,\psi)}(\emptyset) \subseteq S_1 \quad \Longrightarrow \quad \forall\, n \in \mathbb{N}.\, R_n \subseteq S_1 \quad \Longrightarrow \quad S_2 \subseteq S_1$$

Let's prove (5) by induction on $n \in \mathbb{N}$:

- Case $n = 1$

Let's prove $R_1 \subseteq f_{\upsilon(\varphi,\psi)}(\varnothing)$

$$R_1 = \{P \mid \forall\, c \in \mathrm{CCmp}(P).\, \exists\, i \leqslant 1.\, (c_i \vDash \psi \land \forall\, j < i.\, c_j \vDash \varphi)\} =$$

$$= \{P \mid \forall\, c \in \mathrm{CCmp}(P).\, c_1 \vDash \psi \land \forall\, j < 1.\, c_j \vDash \varphi\} =$$

$$= \{P \mid \forall\, c \in \mathrm{CCmp}(P).\, c_1 \vDash \psi\} = \{P \mid P \vDash \psi\} = [\![\, \psi \,]\!]_\eta$$

$$f_{\upsilon(\varphi,\psi)}(\varnothing) = [\![\, \upsilon(\varphi,\psi) \,]\!]_{\eta[X \to \varnothing]} = [\![\, \psi \lor (\varphi \land \langle \mathrm{Act} \rangle\ \mathrm{T} \land [\mathrm{Act}]\ X) \,]\!]_{\eta[X \to \varnothing]} =$$

$$= [\![\, \psi \,]\!]_\eta \cup [\![\, \varphi \land \langle \mathrm{Act} \rangle\ \mathrm{T} \land [\mathrm{Act}]\ X \,]\!]_{\eta[X \to \varnothing]} \ \supseteq\ [\![\, \psi \,]\!]_\eta = R_1$$

- Case $n \Longrightarrow n + 1$

Assuming

π_1) $R_n \subseteq f^n_{\upsilon(\varphi,\psi)}(\varnothing)$

Let's prove $R_{n+1} \subseteq f^{n+1}_{\upsilon(\varphi,\psi)}(\varnothing)$

$$R_{n+1} = \{P \mid \forall\, c \in \mathrm{CCmp}(P).\, \exists\, i \leqslant n + 1.\, (c_i \vDash \psi \land \forall\, j < i.\, c_j \vDash \varphi)\} =$$

$$= \{P \mid P \vDash \psi \lor (P \vDash \varphi \land \exists\, P' \in \mathrm{Proc}\ .\, P \longrightarrow P' \land \forall\, P \longrightarrow P'.\, \forall\, c \in \mathrm{CCmp}(P').\, \exists\, i \leqslant n.\, (c_i \vDash \psi \land \forall\, j < i.\, c_j \vDash \varphi))\}$$

Note that

$$\forall\, c \in \mathrm{CCmp}(P').\, \exists\, i \leqslant n.\, (c_i \vDash \psi \land \forall\, j < i.\, c_j \vDash \varphi) \quad \Longleftrightarrow \quad P' \in R_n$$

$$\Longrightarrow R_{n+1} = \{P \mid P \vDash \psi \lor (P \vDash \varphi \land \exists\, P' \in \mathrm{Proc}\ .\, P \longrightarrow P' \land \forall\, P \longrightarrow P'.\, P' \in R_n)\} =$$

$$= [\![\, \psi \,]\!]_\eta \cup \left([\![\, \varphi \,]\!]_\eta \cap [\![\, \langle \mathrm{Act} \rangle\ \mathrm{T} \,]\!]_\eta \cap [\![\, [\mathrm{Act}]\ X \,]\!]_{\eta[X \to R_n]}\right) =$$

$$= [\![\, \psi \lor (\varphi \land \langle \mathrm{Act} \rangle\ \mathrm{T} \land [\mathrm{Act}]\ X) \,]\!]_{\eta[X \to R_n]} = f_{\upsilon(\varphi,\psi)}(R_n)$$

Because $f_{\upsilon(\varphi,\psi)}$ is monotone, by π_1 (inductive hypothesis)

$$R_n \subseteq f^n_{\upsilon(\varphi,\psi)}(\varnothing) \implies f_{\upsilon(\varphi,\psi)}(R_n) \subseteq f_{\upsilon(\varphi,\psi)}\left(f^n_{\upsilon(\varphi,\psi)}(\varnothing)\right) = f^{n+1}_{\upsilon(\varphi,\psi)}(\varnothing)$$

$$\Longrightarrow R_{n+1} \subseteq f^{n+1}_{\upsilon(\varphi,\psi)}(\varnothing)$$

■