# Reasoning with Many-Valued Interval Temporal Logic

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# Introduction

# Motivation

- Let's start with an example:
  - Patient exhibits symptoms of **"depressed mood"** and **"insomnia"**
  - Symptoms:
    - Vary in intensity over time
    - Meet during certain intervals
  - Need to model:
    - Degrees of symptom severity
    - Temporal relationships between symptoms
- **Real-world scenarios involve degrees of "truth", uncertainty, and temporal information**

# Limitations of Classical Logic

- Traditional binary logic is insufficient for modeling such complexities
- **Binary truth values** (true or false)
- Cannot represent partial truths or degrees of certainty

# Many-Valued Logics

- **Handle partial truths and uncertainty**
- Extend beyond the binary truth values of classical logic
- Examples:
  - Fuzzy logics:
    - Łukasiewicz logic
    - Gödel logic
    - Product logic
  - Intuitionistic logic

# Interval Temporal Logic

- Useful for modeling temporal relationships between events
- Focuses on reasoning over time intervals rather than time points
- Uses **Allen's interval relations**:
    - After, Later, Begins, Ends, During, Overlaps.

# Many-Valued Interval Temporal Logic

- **Objective**: Model graded truths over time intervals
- Challenges:
    - Integrating many-valued truth with temporal relations
    - Developing reasoning systems to handle complexity

# Our Contribution

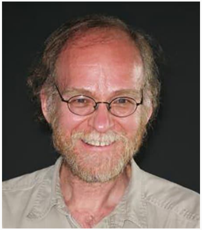
- A **sound and complete tableau system** for many-valued interval temporal logic
- Based on $\mathrm{FL}_{\mathrm{ew}}$-**algebras** to handle graded truth values
- Open-source **implementation** for real-world applicability

# Presentation Overview

# Preliminaries

# Halpern and Shoham's Interval Temporal Logic (HS)

- **HS** is a modal logic for reasoning about **time intervals**
- Uses modalities corresponding to **Allen's interval relations**
- Allows expression of **temporal relationships** between intervals
- Widely used in **temporal reasoning, representation, and planning** within AI



**(a)** Joseph Halpern.

**(b)** Yoav Shoham.

# Allen's Interval Relations

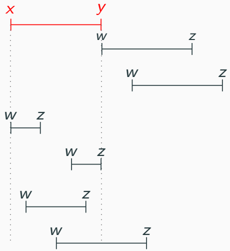
| relation | definition | | | example |
|---|---|---|---|---|
| after | $R_A([x,y],[w,z])$ | $=$ | $=(y,w)$ |  |
| later | $R_L([x,y],[w,z])$ | $=$ | $<(y,w)$ | |
| begins | $R_B([x,y],[w,z])$ | $=$ | $=(x,w) \wedge <(z,y)$ | |
| ends | $R_E([x,y],[w,z])$ | $=$ | $<(x,w) \wedge =(y,z)$ | |
| during | $R_D([x,y],[w,z])$ | $=$ | $<(x,w) \wedge <(z,y)$ | |
| overlaps | $R_O([x,y],[w,z])$ | $=$ | $<(x,w) \wedge <(w,y) \wedge <(y,z)$ | |

**Table 1:** Allen's interval relations.

# Limitations of Classical HS

- HS is based on **classical (binary) logic**
  - Propositions and temporal relations are either **true** or **false**
- **Cannot handle**:
  - **Graded truths** (partial truth values)
  - **Uncertainty** or **imprecision**
- Inadequate for modeling **real-world scenarios**

# $\mathrm{FL_{ew}}$-Algebras

- $\mathrm{FL_{ew}}$-**algebras** [6]

$$\boldsymbol{A} = \langle A, \cap, \cup, \cdot, +, 0, 1 \rangle$$

  are defined over **bounded integral commutative residuated lattices**

  - $A$ is the algebra's **domain**
  - $\langle A, \cap, \cup, 0, 1 \rangle$ represents a **bounded complete lattice** with **upper bound** 1 and **lower bound** 0
  - $\langle A, \preceq \rangle$ corresponds to its **lattice-ordered set** ($\alpha \preceq \beta$ iff $\alpha = \alpha \cap \beta$)
  - $\langle A, \cdot, 1 \rangle$ and $\langle A, +, 0 \rangle$ are **commutative monoids**, namely **t-norm** and **t-co-norm**, with **dual** operations both **monotone** w.r.t. $\preceq$ (if $\gamma \preceq \alpha$ and $\delta \preceq \beta$, then $\gamma \cdot \delta \preceq \alpha \cdot \beta$ and $\gamma + \delta \preceq \alpha + \beta$)
  - We also define an **implication** operation $\hookrightarrow$

$$\alpha \hookrightarrow \beta = \sup\{\gamma \mid \alpha \cdot \gamma \preceq \beta\}$$

- $\boldsymbol{A}$ is a **chain** if $\preceq$ is a total order; **standard** if $A = [0, 1] \subset \mathbb{R}$; **finite** if $A$ is finite. We will focus on **finite $\mathbf{FL}_{ew}$-algebras**.

# Relation to Other Algebras

- $\mathrm{FL}_{\mathrm{ew}}$-algebras encompass several known algebras:
  - **Gödel algebras**
  - **MV algebras**
  - **Product algebras** (not in the finite case!)
  - **Heyting algebras**
- **Generalization** allows for unified treatment

$$
\begin{tikzcd}
 & \mathsf{BA} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathsf{G}\ [1] \arrow[d, dash] \arrow[dr, dash] & \Pi\ [5] \arrow[d, dash] & \mathsf{MV}\ [2, 7] \arrow[dl, dash] \\
\mathsf{H}\ [3] \arrow[dr, dash] & \mathsf{BL}\ [4] \arrow[d, dash] & \\
 & \mathsf{FL}_{\mathsf{ew}}\ [6] &
\end{tikzcd}
$$

**Figure 2:** A partial taxonomy of well-known many-valued algebras.

# Example: Simple $\mathrm{FL_{ew}}$-Algebra (G3)

- **Set of truth values**: $A = \{0, \alpha, 1\}$ with $0 \prec \alpha \prec 1$
- **Operations** defined as:
  - **t-norm ($\cdot$)**:
  $$a \cdot b = \min(a, b)$$
  - **t-co-norm ($+$)**:
  $$a + b = \max(a, b)$$
  - **implication ($\hookrightarrow$)**:
  $$a \hookrightarrow b = \begin{cases} 1 & \text{if } a \preceq b \\ b & \text{otherwise} \end{cases}$$
- **Calculation example**:
  - $\alpha \cdot 1 = \min(\alpha, 1) = \alpha$
  - $\alpha + 1 = \max(\alpha, 1) = 1$
  - $\alpha \hookrightarrow 1 = 1$ since $\alpha \prec 1$

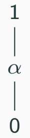



**Figure 3:** Lattice representing the order between the values in the designated $\mathrm{FL_{ew}}$-algebra.

# Many-Valued Halpern and Shoham's Logic (MVHS)

# Syntax of MVHS

- **Propositional letters:** $p, q, r, \ldots$
- **Truth constants:** $\alpha \in A$ (elements of the $\mathrm{FL_{ew}}$-algebra)
- **Logical binary connectives**
  - Conjunction: $\wedge$
  - Disjunction: $\vee$
  - Implication: $\rightarrow$
- **Unary modalities**
  - $\langle X \rangle \varphi$ (there exists an interval related by $X$ where $\varphi$ holds)
  - $[X]\varphi$ (for all intervals related by $X$, $\varphi$ holds)
- **Formulas** are built inductively using these elements

# Many-Valued Linear Orderings and Strict Intervals

- **Many-valued linear orders**

$$\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=} \rangle$$

  - $D$ is the **domain**
  - $\widetilde{<}, \widetilde{=} \colon D \times D \to A$ are two functions mapping pairs of domain values to $A$ of a $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$ satisfying
    1. $\widetilde{=}(x, y) = 1$ iff $x = y$
    2. $\widetilde{=}(x, y) = \widetilde{=}(y, x)$
    3. $\widetilde{<}(x, x) = 0$
    4. $\widetilde{<}(x, z) \succeq \widetilde{<}(x, y) \cdot \widetilde{<}(y, z)$
    5. if $\widetilde{<}(x, y) \succ 0$ and $\widetilde{<}(y, z) \succ 0$, then $\widetilde{<}(x, z) \succ 0$
    6. if $\widetilde{<}(x, y) = 0$ and $\widetilde{<}(y, x) = 0$, then $\widetilde{=}(x, y) = 1$
    7. if $\widetilde{=}(x, y) \succ 0$, then $\widetilde{<}(x, y) \prec 1$

- **Many-valued strict intervals** $\mathbb{I}(\widetilde{\mathbb{D}}) = \{[x, y] \mid \widetilde{<}(x, y) \succ 0\}$

# Many-Valued Allen's Relations

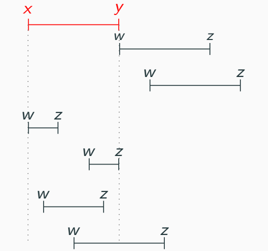
| relation | definition | example |
|---|---|---|
| after | $R_A([x,y],[w,z]) \quad = \quad =(y,w)$ |  |
| later | $R_L([x,y],[w,z]) \quad = \quad <(y,w)$ | |
| begins | $R_B([x,y],[w,z]) \quad = \quad =(x,w) \wedge <(z,y)$ | |
| ends | $R_E([x,y],[w,z]) \quad = \quad <(x,w) \wedge =(y,z)$ | |
| during | $R_D([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(z,y)$ | |
| overlaps | $R_O([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(w,y) \wedge <(y,z)$ | |

**Table 2:** Allen's interval relations.

# Many-Valued Allen's Relations

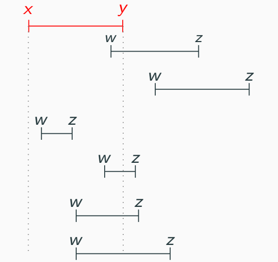
| relation | definition | example |
|---|---|---|
| after | $\widetilde{R}_A([x,y],[w,z]) \quad = \quad \widetilde{=}(y,w)$ |  |
| later | $\widetilde{R}_L([x,y],[w,z]) \quad = \quad \widetilde{<}(y,w)$ | |
| begins | $\widetilde{R}_B([x,y],[w,z]) \quad = \quad \widetilde{=}(x,w) \cdot \widetilde{<}(z,y)$ | |
| ends | $\widetilde{R}_E([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) \cdot \widetilde{=}(y,z)$ | |
| during | $\widetilde{R}_D([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) \cdot \widetilde{<}(z,y)$ | |
| overlaps | $\widetilde{R}_O([x,y],[w,z]) \quad = \quad \widetilde{<}(x,w) \cdot \widetilde{<}(w,y) \cdot \widetilde{<}(y,z)$ | |

**Table 2:** Many-valued Allen's interval relations.

# Semantics of MVHS

- **Many-valued interval models** $\widetilde{M} = \langle \mathbb{I}(\widetilde{\mathbb{D}}), \widetilde{V} \rangle$
  - **Valuation function** $\widetilde{V}$: Assigns truth values from $A$ to formulas at intervals
- **Atoms**:
  - $\widetilde{V}(p, [x, y]) \in A$
  - $\widetilde{V}(\alpha, [x, y]) = \alpha \in A$
- **Logical connectives**:
  - $\widetilde{V}(\varphi \wedge \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) \cdot \widetilde{V}(\psi, [x, y])$
  - $\widetilde{V}(\varphi \vee \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) + \widetilde{V}(\psi, [x, y])$
  - $\widetilde{V}(\varphi \rightarrow \psi, [x, y]) = \widetilde{V}(\varphi, [x, y]) \hookrightarrow \widetilde{V}(\psi, [x, y])$
- **Modalities**:
  - $\widetilde{V}(\langle X \rangle \varphi, [x, y]) = \bigcup_{[w,z] \in \mathbb{I}(\widetilde{\mathbb{D}})} \left( \widetilde{R}_X([x, y], [w, z]) \cdot \widetilde{V}(\varphi, [w, z]) \right)$
  - $\widetilde{V}([X] \varphi, [x, y]) = \bigcap_{[w,z] \in \mathbb{I}(\widetilde{\mathbb{D}})} \left( \widetilde{R}_X([x, y], [w, z]) \hookrightarrow \widetilde{V}(\varphi, [w, z]) \right)$

# Satisfiability and Validity

- A formula $\varphi$ is $\alpha$-**satisfied** at $[x, y]$ in $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, [x, y]) \succeq \alpha$$

- A formula is $\alpha$-**satisfiable** if and only if an interval exists in a multi-valued interval model where is $\alpha$-satisfed
- A formula is $\alpha$-**valid** if and only if it is $\alpha$-satisfiable at every interval in every multi-valued interval model
- A formula is **valid** if and only if it is 1-valid

# Application Example: Medical Diagnosis

- **Scenario**:
  - Patient exhibits symptoms:
    - **"Depressed mood"** ($p$)
    - **"Insomnia"** ($q$)
  - Symptoms vary in intensity over intervals
  - Algebra's domain $A = [0, 1] \subset \mathbb{R}$
  - Pick a t-norm: e.g., **min** (Gödel Logic)
- **Goal**: Determine the degree to which an interval of "depressed mood" meets a period of "insomnia"
- **Formula**:

$$\varphi = p \wedge \langle A \rangle q$$

# Application Example: Medical Diagnosis

- **Assign truth values**:
  - $\widetilde{V}(p, [x, y]) = 0.7$
  - $\widetilde{V}(q, [w, z]) = 0.8$
  - $\widetilde{R}_A([x, y], [w, z]) = \widetilde{=}(y, w) = 0.9$

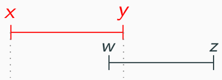



**Figure 4:** y and w are "almost" the same point

- **Then**:

$$\begin{array}{rcl} \widetilde{V}(\varphi, [x, y]) & = & \widetilde{V}(p \wedge \langle A \rangle q, [x, y]) \\ & = & \widetilde{V}(p, [x, y]) \cdot \widetilde{V}(\langle A \rangle q, [x, y]) \\ & = & 0.7 \cdot \widetilde{R}_A([x, y], [w, z]) \cdot \widetilde{V}(q, [w, z]) \\ & = & 0.7 \cdot 0.9 \cdot 0.8 \\ & = & 0.7 \end{array}$$

- **Interpretation**: It is not always the case that a period of "depressed mood" is followed by a period of "insomnia," but we can say that it happens in a non-negligible manner

# Tableau System: Theory and Implementation

# Need for a Tableau System

- **Challenges in reasoning** with MVHS
  - Many-valued truth values increase the complexity
  - Temporal modalities over intervals add to the intricacy
- **Objective**
  - Develop a systematic method for determining **satisfiability** and **validity**
  - Ensure **soundness** and **completeness**
- **Solution: Fitting's style tableau system** adapted for MVHS over **finite** $\mathrm{FL}_{\mathrm{ew}}$-algebras

# Overview of the Tableau Structure

- Tree-like structure with nodes and branches

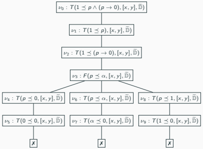



**Figure 5:** A tableau.

- Each node is associated with a **decoration**

$$Q(\beta \preceq \psi, [x, y], \widetilde{\mathbb{D}}) \quad \text{or} \quad Q(\psi \preceq \beta, [x, y], \widetilde{\mathbb{D}})$$

  - $Q$ is a **truth judgment** - either $T$ (true) or $F$ (false)
  - $\beta \in A$ is a truth value from the $\mathrm{FL_{ew}}$-algebra
  - $\psi \in sub(\varphi)$ is a sub-formula of $\varphi$
  - $[x, y]$ is an interval
  - $\widetilde{\mathbb{D}}$ is a many-valued linear order
- Branches represent possible **evaluations** and are associated with a finite many-valued linear order

# Overview of the Tableau Procedure

- **Purpose:** Systematically explore possible valuations to determine:
  - **Satisfiability:** If starting from $T(\alpha \preceq \varphi)$ it finds an open branch (**SAT-tableau**), or
  - **Validity:** If starting from $F(\alpha \preceq \varphi)$ it closes all branches (**VAL-tableau**)
- **Expansion and branching:** Systematically apply expansion rules to generate new nodes
- **Closure:** Close branches that contain contradictions using branch closing rules
- **Termination**
  - If all branches are closed, the formula is unsatisfiable
  - If at least one open branch remains, a satisfying model exists

## Expansion Rules: Reverse

$$(T \succeq)\ \frac{T(\beta \preceq \psi, [x, y], \widetilde{\mathbb{D}})}{F(\psi \preceq \gamma, [x, y], c(B))}$$

where $\beta \neq 0$ and $\gamma$ is any maximal element not above $\beta$, i.e., $\gamma \not\succeq \beta$

$$(F \succeq)\ \frac{F(\beta \preceq \psi, [x, y], \widetilde{\mathbb{D}})}{T(\psi \preceq \gamma_i, [x, y], c(B)) \mid \ldots \mid T(\psi \preceq \gamma_n, [x, y], c(B))}$$

where $\beta \neq 0$ and $\gamma_1, \ldots, \gamma_n$ are all maximal elements not above $\beta$, i.e., $\gamma_1, \ldots, \gamma_n \not\succeq \beta$

**Figure 6:** Reverse rules (1).

# Expansion Rules: Reverse

$$(T \preceq)\ \frac{T(\psi \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{F(\gamma \preceq \psi, [x,y], c(B))}$$

where $\beta \neq 1$ and $\gamma$ is any minimal element not below $\beta$, i.e., $\gamma \npreceq \beta$

$$(F \preceq)\ \frac{F(\psi \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{T(\gamma_i \preceq \psi, [x,y], c(B)) \mid \ldots \mid T(\gamma_i \preceq \psi, [x,y], c(B))}$$

where $\beta \neq 1$ and $\gamma_1, \ldots, \gamma_n$ are all minimal elements not below $\beta$, i.e., $\gamma_1, \ldots, \gamma_n \npreceq \beta$

**Figure 7:** Reverse rules (2).

# Expansion Rules: Propositional

$$(T\wedge)\ \frac{T(\beta \preceq (\psi \wedge \xi), [x, y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\beta_1 \preceq \psi, [x, y], c(B)) \mid \ldots \mid T(\beta_n \preceq \psi, [x, y], c(B)) \\ T(\gamma_1 \preceq \xi, [x, y], c(B)) \mid \ldots \mid T(\gamma_n \preceq \xi, [x, y], c(B)) \end{array}}$$

where $\beta \neq 0$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \preceq \beta_i \cdot \gamma_i$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \preceq \beta_i' \cdot \gamma_i'$, $\beta_i' \preceq \beta_i$ and $\gamma_i' \preceq \gamma_i$.

$$(F\wedge)\ \frac{F(\beta \preceq (\psi \wedge \xi), [x, y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\psi \preceq \beta_1, [x, y], c(B)) \mid \ldots \mid T(\psi \preceq \beta_n, [x, y], c(B)) \\ T(\xi \preceq \gamma_1, [x, y], c(B)) \mid \ldots \mid T(\xi \preceq \gamma_n, [x, y], c(B)) \end{array}}$$

where $\beta \neq 0$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \not\preceq \beta_i \cdot \gamma_i$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \not\preceq \beta_i' \cdot \gamma_i'$, $\beta_i \preceq \beta_i'$ and $\gamma_i \preceq \gamma_i'$.

**Figure 8:** Propositional rules (1).

# Expansion Rules: Propositional

$$(T\vee)\ \frac{T((\psi \vee \xi) \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\psi \preceq \beta_1, [x,y], c(B)) \mid \ldots \mid T(\psi \preceq \beta_n, [x,y], c(B)) \\ T(\xi \preceq \gamma_1, [x,y], c(B)) \mid \ldots \mid T(\xi \preceq \gamma_n, [x,y], c(B)) \end{array}}$$

where $\beta \neq 1$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta_i + \gamma_i \preceq \beta$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta_i' + \gamma_i' \preceq \beta$, $\beta_i \preceq \beta_i'$ and $\gamma_i \preceq \gamma_i'$.

$$(F\vee)\ \frac{F((\psi \vee \xi) \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\beta_1 \preceq \psi, [x,y], c(B)) \mid \ldots \mid T(\beta_n \preceq \psi, [x,y], c(B)) \\ T(\gamma_1 \preceq \xi, [x,y], c(B)) \mid \ldots \mid T(\gamma_n \preceq \xi, [x,y], c(B)) \end{array}}$$

where $\beta \neq 1$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta_i + \gamma_i \not\preceq \beta$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta_i' + \gamma_i' \not\preceq \beta$, $\beta_i' \preceq \beta_i$ and $\gamma_i' \preceq \gamma_i$.

**Figure 9:** Propositional rules (2).

# Expansion Rules: Propositional

$$(T \hookrightarrow)\ \frac{T(\beta \preceq (\psi \hookrightarrow \xi), [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\psi \preceq \beta_1, [x,y], c(B)) \mid \ldots \mid T(\psi \preceq \beta_n, [x,y], c(B)) \\ T(\gamma_1 \preceq \xi, [x,y], c(B)) \mid \ldots \mid T(\gamma_n \preceq \xi, [x,y], c(B)) \end{array}}$$

where $\beta \neq 0$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \preceq \beta_i \hookrightarrow \gamma_i$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \preceq \beta_i' \hookrightarrow \gamma_i', \beta_i \preceq \beta_i'$ and $\gamma_i' \preceq \gamma_i$.

$$(F \hookrightarrow)\ \frac{F(\beta \preceq (\psi \hookrightarrow \xi), [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T(\beta_1 \preceq \psi, [x,y], c(B)) \mid \ldots \mid T(\beta_n \preceq \psi, [x,y], c(B)) \\ T(\xi \preceq \gamma_1, [x,y], c(B)) \mid \ldots \mid T(\xi \preceq \gamma_n, [x,y], c(B)) \end{array}}$$

where $\beta \neq 0$, $(\beta_i, \gamma_i) \in \boldsymbol{A} \times \boldsymbol{A}$ so that $\beta \not\preceq \beta_i \hookrightarrow \gamma_i$ and there is no other $(\beta_i', \gamma_i') \in \boldsymbol{A} \times \boldsymbol{A}$ such that $\beta \not\preceq \beta_i' \hookrightarrow \gamma_i', \beta_i' \preceq \beta_i$ and $\gamma_i \preceq \gamma_i'$.

**Figure 10:** Propositional rules (3).

# Expansion Rules: Modalities

$$(T\Box)\ \frac{T(\beta \preceq [X]\psi, [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T((\beta \cdot \gamma_1) \preceq \psi, [z_1, t_1], c(B)) \\ \cdots \\ T((\beta \cdot \gamma_n) \preceq \psi, [z_n, t_n], c(B)) \\ T(\beta \preceq [X]\psi, [x,y], c(B)) \end{array}}$$

where $\gamma_i = \widetilde{R}_X([x,y],[z_i,t_i])$, $[z_i,t_i] \in o(c(B))$,
$\gamma_i \succ 0$, and $\beta \cdot \gamma_i \neq 0$

$$(F\Box)\ \frac{F(\beta \preceq [X]\psi, [x,y], \widetilde{\mathbb{D}})}{F((\beta \cdot \gamma_1) \preceq \psi, [z_1, t_1], c(B)) \mid \ldots \mid F((\beta \cdot \gamma_n) \preceq \psi, [z_n, t_n], c(B))}$$

where $\gamma_i = \widetilde{R}_X([x,y],[z_i,t_i])$,$[z_i,t_i] \in o(c(B)) \cup n(c(B))$,
$\gamma_i \succ 0$, and $\beta \cdot \gamma_i \neq 0$

**Figure 11:** Temporal rules (1).

# Expansion Rules: Modalities

$$(T\Diamond)\ \frac{T(\langle X\rangle\psi \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{\begin{array}{c} T((\psi \preceq (\gamma_1 \hookrightarrow \beta), [z_1, t_1], c(B)) \\ \dots \\ T(\psi \preceq (\gamma_n \hookrightarrow \beta), [z_n, t_n], c(B)) \\ T(\langle X\rangle\psi \preceq \beta, [x,y], c(B)) \end{array}}$$

where $\gamma_i = \widetilde{R}_X([x,y],[z_i,t_i])$, $[z_i,t_i] \in o(c(B))$,

$\gamma_i \succ 0$, and $\gamma_i \hookrightarrow \beta \neq 1$

$$(F\Diamond)\ \frac{F(\langle X\rangle\psi \preceq \beta, [x,y], \widetilde{\mathbb{D}})}{F(\psi \preceq (\gamma_1 \hookrightarrow \beta), [z_1, t_1], c(B)) \mid \dots \mid F(\psi \preceq (\gamma_n \hookrightarrow \beta), [z_n, t_n], c(B))}$$

where $\gamma_i = \widetilde{R}_X([x,y],[z_i,t_i])$,$[z_i,t_i] \in o(c(B)) \cup n(c(B))$,

$\gamma_i \succ 0$, and $\gamma_i \hookrightarrow \beta \neq 1$

**Figure 12:** Temporal rules (2).

## Branch Closing Rules

$$(\text{✗}1)\ \frac{T(\beta \preceq \gamma, [x,y], \widetilde{\mathbb{D}})}{\text{✗}}\quad \text{where } \beta \not\preceq \gamma$$

$$(\text{✗}2)\ \frac{F(\beta \preceq \gamma, [x,y], \widetilde{\mathbb{D}})}{\text{✗}}\quad \text{where } \beta \neq 0,\ \gamma \neq 1, \text{ and } \beta \preceq \gamma$$

$$(\text{✗}3)\ \frac{F(0 \preceq \psi, [x,y], \widetilde{\mathbb{D}})}{\text{✗}}$$

$$(\text{✗}4)\ \frac{F(\psi \preceq 1, [x,y], \widetilde{\mathbb{D}})}{\text{✗}}$$

$$(\text{✗}5)\ \frac{\begin{array}{c} T(\gamma \preceq \psi, [x,y], \widetilde{\mathbb{D}}) \\ F(\beta \preceq \psi, [x,y], \widetilde{\mathbb{D}}) \end{array}}{\text{✗}}\quad \text{where } \beta \preceq \gamma$$

$$(\text{✗}6)\ \frac{Q(\cdot, \cdot, \widetilde{\mathbb{D}})}{\text{✗}}\quad \text{where } \widetilde{\mathbb{D}} \text{ is inconsistent}$$

**Figure 13:** Branch closing rules.

# Example: Tableau Construction

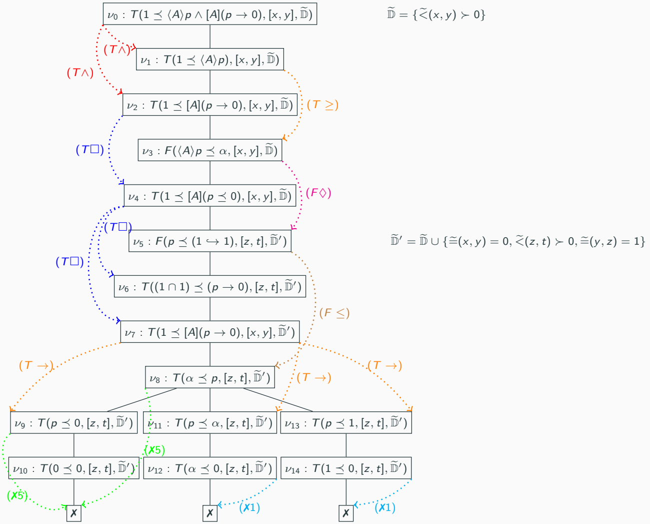



**Figure 14:** Closed branches of the tableau for $\langle A \rangle p \wedge [A](p \to 0)$ and $1 \in G3$.

## Soundness and Completeness over Finite $\mathrm{FL_{ew}}$-Algebras

- **Soundness**
  - If a formula $\varphi$ is $\alpha$-satisfiable, then there exists an opened tableau for $\varphi$ and $\alpha$
  - The rules preserve logical consequence
- **Completeness**
  - If a tableau is opened for $\varphi$ and $\alpha$, then $\varphi$ is $\alpha$-satisfiable.
  - The method explores all necessary valuations
- **Implications**
  - The tableau system is a reliable decision procedure for MVHS over **finite** $\mathrm{FL_{ew}}$-algebras
  - Provides a foundation for automated reasoning in MVHS

# Implementation Overview

- **Programming Language:**
  - **Julia**, chosen for its performance in numerical computations:
    - High-level syntax with efficient execution
    - Strong support for mathematical operations
- **Open-source Advocacy:**
  - **Sole.jl** (**S**ymb**O**lic **LE**arning)[1], a framework for representing, reasoning, and learning from structured and unstructured data
  - **SoleReasoners.jl**, analytic tableau solvers for $\alpha$-sat and $\alpha$-val.
- **Representation of Algebras:**
  - Wrapped in the **ManyValuedLogics** submodule of **SoleLogics.jl**
  - Finite $\mathrm{FL_{ew}}$-algebras defined by specifying:
    - **Domain** (set of truth values $A$)
    - **Truth tables** for $\cap$, $\cup$, $\cdot$, $+$ ($\hookrightarrow$ is derived internally)
    - A one-time check ensures the algebra satisfies the $\mathrm{FL_{ew}}$-axioms.

[1] `https://github.com/aclai-lab/Sole.jl`

# Code Examples: Gödel Algebra (G3)

```
using SoleLogics
using SoleLogics.ManyValuedLogics
using SoleReasoners

α = FiniteTruth("α")
d3 = Vector{FiniteTruth}([⊥, α, ⊤])
# a ∩ b = min{a, b}
∩ = BinaryOperation(d3, Dict{Tuple{FiniteTruth, FiniteTruth}, FiniteTruth}(
    (⊥, ⊥) => ⊥, (⊥, α) => ⊥, (⊥, ⊤) => ⊥,
    (α, ⊥) => ⊥, (α, α) => α, (α, ⊤) => α,
    (⊤, ⊥) => ⊥, (⊤, α) => α, (⊤, ⊤) => ⊤
))
# a ∪ b = max{a, b}
∪ = BinaryOperation(d3, Dict{Tuple{FiniteTruth, FiniteTruth}, FiniteTruth}(
    (⊥, ⊥) => ⊥, (⊥, α) => α, (⊥, ⊤) => ⊤,
    (α, ⊥) => α, (α, α) => α, (α, ⊤) => ⊤,
    (⊤, ⊥) => ⊤, (⊤, α) => ⊤, (⊤, ⊤) => ⊤
))
· = ∩   # In Godel algebras, ∩ and · are the same operator
+ = ∪   # In Godel algebras, ∪ and + are the same operator
G3 = FiniteFLewAlgebra(d3, ∩, ∪, ·, +, ⊥, ⊤)

diamondA = diamond(IA_A)              # ⟨A⟩
boxA = box(IA_A)                      # [A]
p = Atom("p")                         # propositional letter "p"
φ = ∧(diamondA(p), boxA(→(p, ⊥)))    # φ := ⟨A⟩p ∧ [A](p → ⊥)
mvhsalphasat(⊤, φ, G3)                # false
```

**Figure 15:** Evaluation code example for $T(\top \preceq \varphi)$ where $\varphi = \langle A\rangle p \wedge [A](p \rightarrow 0)$ and $\top \in$ G3.

# Code Examples: Heyting Algebra (H4)

```
using SoleLogics
using SoleLogics.ManyValuedLogics
using SoleReasoners

α = FiniteTruth("α")
β = FiniteTruth("β")
d4 = Vector{FiniteTruth}([⊥, α, β, ⊤])
∩ = BinaryOperation(d4, Dict{Tuple{FiniteTruth, FiniteTruth}, FiniteTruth}(
        (⊥, ⊥) => ⊥, (⊥, α) => ⊥, (⊥, β) => ⊥, (⊥, ⊤) => ⊥,
        (α, ⊥) => ⊥, (α, α) => α, (α, β) => ⊥, (α, ⊤) => α,
        (β, ⊥) => ⊥, (β, α) => ⊥, (β, β) => β, (β, ⊤) => β,
        (⊤, ⊥) => ⊥, (⊤, α) => α, (⊤, β) => β, (⊤, ⊤) => ⊤
    ))
∪ = BinaryOperation(d4, Dict{Tuple{FiniteTruth, FiniteTruth}, FiniteTruth}(
        (⊥, ⊥) => ⊥, (⊥, α) => α, (⊥, β) => β, (⊥, ⊤) => ⊤,
        (α, ⊥) => α, (α, α) => α, (α, β) => ⊤, (α, ⊤) => ⊤,
        (β, ⊥) => β, (β, α) => ⊤, (β, β) => β, (β, ⊤) => ⊤,
        (⊤, ⊥) => ⊤, (⊤, α) => ⊤, (⊤, β) => ⊤, (⊤, ⊤) => ⊤
    ))
· = ∩   # In Heyting algebras, ∩ and · are the same operator
+ = ∪   # In Heyting algebras, ∪ and + are the same operator
H4 = FiniteFLewAlgebra(d4, ∩, ∪, ·, +, ⊥, ⊤)

diamondA = diamond(IA_A)            # ⟨A⟩
boxA = box(IA_A)                    # [A]
p = Atom("p")                       # propositional letter "p"
φ = ∧(diamondA(p), boxA(→(p, ⊥)))   # φ := ⟨A⟩p ∧ [A](p → ⊥)
mvhsalphasat(⊤, φ, H4)              # false
```

**Figure 16:** Evaluation code example for $T(\top \preceq \varphi)$ where $\varphi = \langle A \rangle p \wedge [A](p \to 0)$ and $\top \in$ H4.

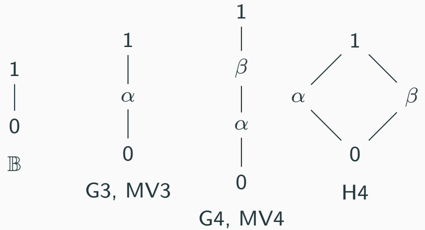
# Key Challenges and Solutions

- Computational complexity increases with:
  - **Size of the algebra**: More truth values to consider
  - **Complexity of the formula**: More nodes and branches
- **Optimization techniques:**
  - Implemented **priority queues** to manage node expansion efficiently
  - **Parallelization**: Expanded independent branches using multi-core processors
- **Pruning strategies:** Periodically clean priority queues to remove expanded or closed nodes
- **Efficient data structures:**
  - Designed compact representations for nodes and branches
  - Minimized memory usage to handle large tableaux

# Experiments and Results

# Experiments and Results

- Six representative finite $\mathrm{FL_{ew}}$-algebras:




- G3 and MV3 (resp. G4 and MV4) differ because of the t-norm but share the same lattice structure
- Each algebra tested on 500 random formulas with heights up to 5 (i.e., 32 symbols)
- $\alpha \succ 0$ chosen randomly
- Branch priority policy kept random

- Impact of using different $\mathrm{FL}_{\mathrm{ew}}$-Algebras
- All tests were conducted on a machine equipped with 2 Intel Xeon Gold 28-Core CPUs and 224GB of RAM
- Timeout of 30 seconds

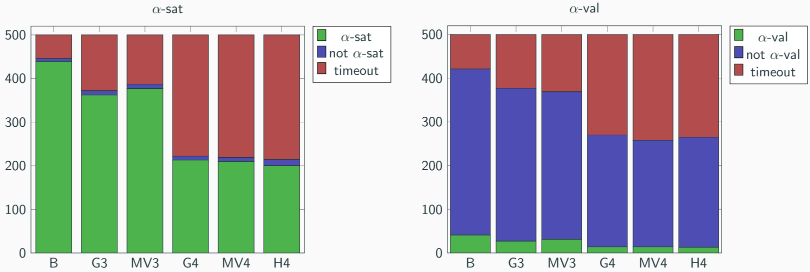



**Figure 17:** Results on common many-valued algebras for formulas of height up to 5 with a timeout of 30 seconds.

# Conclusions and Future Work

# Conclusions and Future Work

- Presented a **customizable and flexible framework** for many-valued interval temporal logic
- Developed a **sound and complete tableau system** for MVHS
- Ready-to-use **open-source implementation** with user-definable finite $\mathrm{FL}_{\mathrm{ew}}$-algebras[2]
- **Tested** the tableau system over different **finite $\mathrm{FL}_{\mathrm{ew}}$-algebras**
- Future work:
    - Support for **Many-Valued Interval Spatial Logic**
    - Real-world applications (**Many-Expert Decision Tree Learning**)

[2] `https://github.com/aclai-lab/Sole.jl`

**Thank you for the attention!**

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