Introducing a general framework for many-valued temporal and spatial logic

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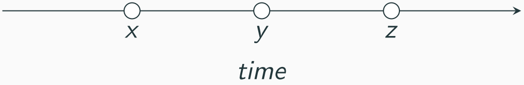
Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can define **relations** between points in time as follows:

$$
\begin{aligned}
F(x, y) &: x < y && \qquad\qquad \textit{future} \\
P(x, y) &: y < x && \qquad\qquad \textit{past} \\
=(x, y) &: x = y && \qquad\qquad \textit{equality}
\end{aligned}
$$



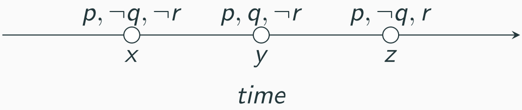
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can associate **events** with points in time:

- p: I am doing a PhD
- q: I am giving this talk
- r: I am having lunch



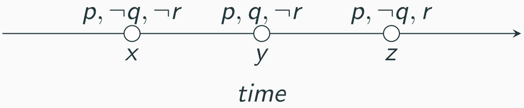
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, some ordered **points** in time ($x < y < z$).

We can **reason** about events in time:

- $p \wedge r$ (is there a point in time where I am both giving this talk and getting lunch?) ✗
- $q \wedge \langle F \rangle r$ (is there a point in time where I am giving this talk and in the future I am getting lunch?) ✓



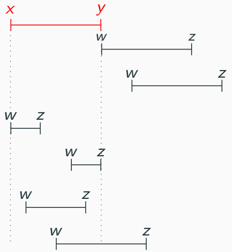
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, time **intervals** $I(x, y) : x < y$.

We can define **relations** between time intervals:

relation	**definition**	**example**
after	$R_A([x,y],[w,z]) \quad = \quad =(y,w)$	
later	$R_L([x,y],[w,z]) \quad = \quad <(y,w)$	
begins	$R_B([x,y],[w,z]) \quad = \quad =(x,w) \wedge <(z,y)$	
ends	$R_E([x,y],[w,z]) \quad = \quad <(x,w) \wedge =(y,z)$	
during	$R_D([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(z,y)$	
overlaps	$R_O([x,y],[w,z]) \quad = \quad <(x,w) \wedge <(w,y) \wedge <(y,z)$	



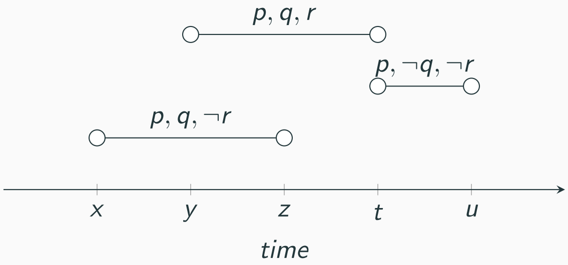
Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, time **intervals** $I(x, y) : x < y$.

We can associate **events** with time intervals and **reason** about them:

- $p \wedge \langle O \rangle r$ (is there an interval of time in which I am giving this talk overlapping with an interval of time in which I am getting lunch?) ✓

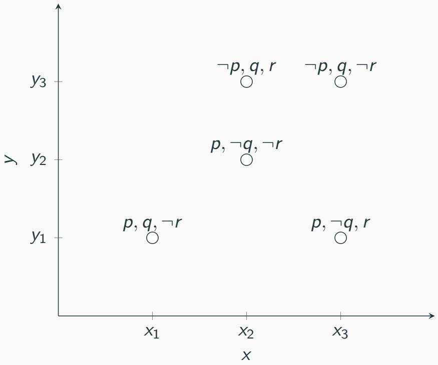


Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, **points** in space $P(x, y)$.

We can define **relations** between points in space (*up, right, down, left*), associate **events** with those points, and **reason** about them:

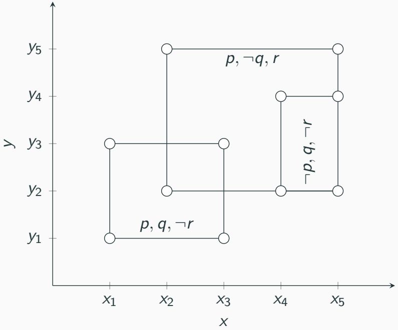


Temporal and spatial logics

Temporal and **spatial** logics are critical in modeling **real-world** scenarios.

Let's consider, for example, **rectangles** $R(I_x, I_y)$, where I_x, I_y are intervals associated to axis x and y respectively.

We can define **relations** between rectangles (*overlapping, disconnected,* $\ldots$), associate **events** with those rectangles, and **reason** about them:



A general framework for temporal and spatial logics

Goal: logical framework for uniform and general treatment of temporal and spatial logics (e.g., fuzzification, manification, reasoning):

- linear temporal logic
- interval temporal logic
- pointwise spatial logic
- spatial logic of topological relations

Solution: multi-modal logic K_n

Multi-modal logics

Multi-modal logics

Definition

Let $\mathcal{A}P$ be a set of propositional letters, $\neg$ and $\vee$ the classical Boolean connectives, and $\{\langle X_1 \rangle, \ldots, \langle X_n \rangle\}$ a finite set of existential modalities. Well-formed *Multi-Modal Logic* K_n[1] formulas are obtained as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X_i \rangle \varphi,$$

for $1 \leq i \leq n$ and $p \in \mathcal{A}P$.

$\wedge$, $\rightarrow$, and $[X_i]$ are derivable in the usual way (e.g., $[X_i]\varphi \equiv \neg\langle X_i \rangle \neg\varphi$).

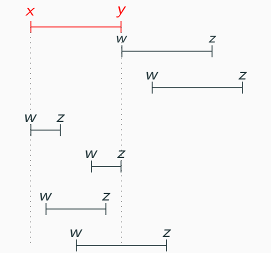
[1] P. Blackburn, M. de Rijke, and Y. Venema. **Modal Logic.** Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001

Example: Halpern and Shoham's Modal Logic of Time Intervals

Let $\mathbb{D} = \langle D, < \rangle$ be a linear order with domain D.

An interval over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x < y$.

There are 12 different binary ordering relations between two intervals:

relation	definition	example
after	$R_A([x,y],[w,z]) \quad = \quad =(y,w)$	
later	$R_L([x,y],[w,z]) \quad = \quad <(y,w)$	
begins	$R_B([x,y],[w,z]) \quad = \quad =(x,w) \wedge <(z,y)$	
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and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

To each relation $R_{X \in \{A,\overline{A},L,\overline{L},B,\overline{B},E,\overline{E},D,\overline{D},O,\overline{O}\}}$ corresponds a modality $\langle X \rangle$.

Multi-modal logics

Definition

Given a non-empty set of *worlds* W, a *Kripke frame* is an object $F = \langle W, R_1 \ldots R_n \rangle$ where each $R_i \subseteq W \times W$ is an *accessibility* relation.

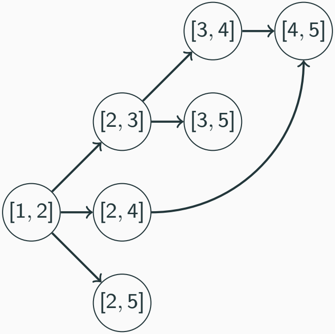

Figure 1: A Kripke frame for the relation R_A (*after*) of Halpern and Shoham's Modal Logic of Time Intervals; each world w_i represents an interval $[x_i, y_i]$.

Multi-modal logics

Definition

A *Kripke structure* (or *model*) is a Kripke frame enriched with a *valuation function* $V : W \to 2^{\mathcal{AP}}$, and it is denoted by $M = \langle F, V \rangle$.

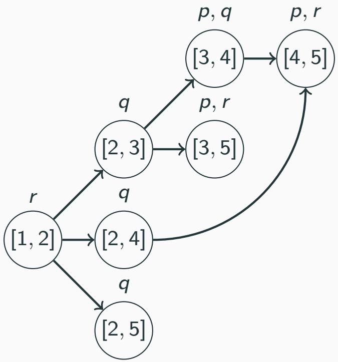

Figure 2: A Kripke structure for the Kripke frame in Fig. 1 and the set of propositional letters $\mathcal{AP} = \{p, q, r\}$; for each world, we represent only the propositional letters which are true in that world.

Multi-modal logics

Definition

Given a well-formed formula φ, we say that φ is *satisfied in M at w*, for some world w, and we denote it by $M, w \Vdash \varphi$, if and only if

$$\begin{array}{lll} M, w \Vdash p & \text{iff} & w \in V(p), \text{ for each } p \in \mathcal{A}P, \\ M, w \Vdash \neg\psi & \text{iff} & M, w \nVdash \psi, \\ M, w \Vdash \psi \vee \xi & \text{iff} & M, w \Vdash \psi \text{ or } M, w \Vdash \xi \\ M, w \Vdash \langle X_i \rangle \psi & \text{iff} & \text{there is } s \text{ s.t. } wR_i s \text{ and } M, s \Vdash \psi. \end{array}$$

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M, w \Vdash \psi \vee \xi & \text{iff} & M, w \Vdash \psi \text{ or } M, w \Vdash \xi \\
M, w \Vdash \langle X_i \rangle \psi & \text{iff} & \text{there is } s \text{ s.t. } wR_is \text{ and } M, s \Vdash \psi.
\end{array}
$$

Definition

A formula φ is *satisfiable* iff there exists a structure and a world in which it is satisfied, and *valid* if it is satisfied at every world in every structure.

A general framework for temporal and spatial logics

Objective: leverage a uniform and general framework, in this case **multi-modal logic** K_n, to define (meta-)algorithms parametrized:

- on the number of **linear orders**
- on the type of **events** (worlds) – points, intervals, rectangles, ...
- on the **relations** between such events – future, past, overlapping, ...

Such framework is suitable for uniform **fuzzification**, **manification**, **reasoning** and **(symbolic) learning**.

Many-valued logics

Motivation

Why fuzzification? – degrees of truth coming from a total order

- Inacuraccies due to **sensing** and **discretization**
- **Unclear boundaries** due to treating numerical values semantically (when is a temperature high/low?)

Why manification? – degrees of truth coming from a partial order

- Better **semantic treatment** of all of the above
- **Many-expert** systems[2]

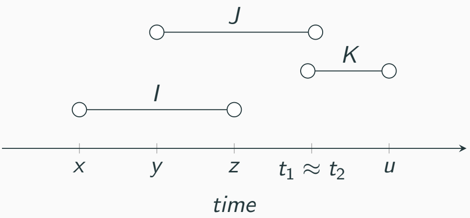
In one word: to get a more accurate representation of the real world

[2] M. Fitting. **"Many-valued modal logics".** In: *Fundamenta Informaticae* 15.3-4 (1991), pp. 235–254

A simple example

Example: consider we are sensing data with photocells, and, analysing their measurements, points t_1 and t_2 appear really close to each other.

What if in real life they were the same point? What if one came after the other? (and in that case, which came first?)



Solution: consider all 3 possibilities (possibly, with different values)

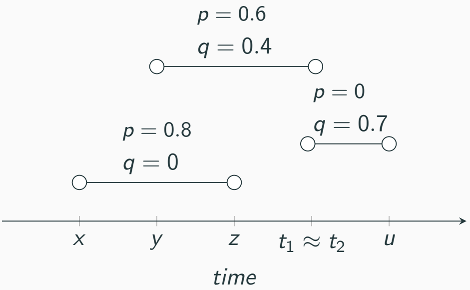
E.g., $\mathit{overlaps}(J, K) = 0.7, \quad \mathit{after}(J, K) = 0.5, \quad \mathit{later}(J, K) = 0.2$

A simple example

And, as we said, events can also be **partially true**.

E.g., consider two photocells also tell the probability of a **false positive**:

- p: 1^{st} photocell actually saw something (avg. accuracy on interval)
- q: 2^{nd} photocell actually saw something (avg. accuracy on interval)



Hypothesis: maybe something is moving from one area to the other?

$\mathrm{FL_{ew}}$-**algebras** encompass several known many-valued algebras:

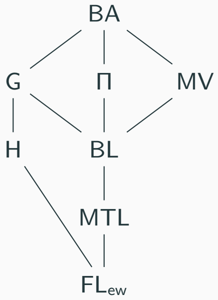


Figure 3: A partial taxonomy of well-known many-valued algebras, namely: Boolean algebra (BA), Gödel algebras (G), Product algebras (Π), MV-algebras (MV), Heyting algebras (H), Basic Fuzzy Logic algebras (BL), Monoidal t-norm logic algebras (MTL), and $\mathrm{FL_{ew}}$-algebras ($\mathrm{FL_{ew}}$).

Definition

$\mathrm{FL_{ew}}$-**algebras**[3]

$$\mathcal{A} = \langle A, \cap, \cup, \cdot, 0, 1 \rangle$$

are defined over **bounded commutative residuated lattices**, where:

- $\langle A, \cap, \cup, 0, 1 \rangle$ represents a **bounded complete lattice**
- $\langle A, \cdot, 1 \rangle$ is a **commutative monoid**
- We can define an **implication** $\hookrightarrow$ (the residuation property holds)

$$\alpha \hookrightarrow \beta = \mathsf{sup}\{\gamma \mid \alpha \cdot \gamma \preceq \beta\}$$

$\mathcal{A}$ is a **chain** if $\langle A, \preceq \rangle$ is a total order, **finite** if A is finite.

[3] Hiroakira Ono and Yuichi Komori. **"Logics without the contraction rule".** In: *The Journal of Symbolic Logic* 50.1 (1985), pp. 169–201

Many-valued multi-modal logics

Many-valued multi-modal logics

Definition

Let $\mathcal{A}P$ be a set of propositional letters and $\mathcal{A}$ a complete $\mathrm{FL_{ew}}$-algebra. The well-formed formulas of the *Multi-Modal Logic* $\mathrm{FL_{ew}}$-K_n are obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X_i \rangle \varphi \mid [X_i]\varphi,$$

for $1 \leq i \leq n$, $p \in \mathcal{A}P$, and $\alpha \in \mathcal{A}$.

In $\mathrm{FL_{ew}}$-algebras, negation is typical defined as $\neg\varphi \equiv \varphi \rightarrow 0$.

However, the double negation axiom ($\neg\neg\varphi \equiv \varphi$) is **not** always vaild.

Hence, $\mathrm{FL_{ew}}$-K_n requires an explicit inclusion of all Boolean operators, as well as the universal version of every modality.

Many-valued multi-modal logics

Definition

Given a non-empty set of *worlds* W and a complete $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$, an $\mathrm{FL_{ew}}$-*Kripke frame* is an object $\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle$, where each $\widetilde{R}_i : (W \times W) \to \mathcal{A}$ is an *accessibility* function.

Many-valued multi-modal logics

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Given a non-empty set of *worlds* W and a complete $\mathrm{FL_{ew}}$-algebra $\mathcal{A}$, an $\mathrm{FL_{ew}}$-*Kripke frame* is an object $\widetilde{F} = \langle W, \widetilde{R}_1 \ldots, \widetilde{R}_n \rangle$, where each $\widetilde{R}_i : (W \times W) \to \mathcal{A}$ is an *accessibility* function.

Definition

An $\mathrm{FL_{ew}}$-*Kripke structure* (or *model*) is an $\mathrm{FL_{ew}}$-Kripke frame enriched with a *valuation function* $\widetilde{V} : (W \times \mathcal{A}P) \to \mathcal{A}$, and it is denoted by $\widetilde{M} = \langle \widetilde{F}, \widetilde{V} \rangle$.

Definition

Given a well-formed formula φ, we compute its *value in* $\widetilde{M}$ *at* w, for some $w \in W$, by extending $\widetilde{V}$ to formulas, as follows:

$$
\begin{array}{lcl}
\widetilde{V}(\alpha, w) & = & \alpha, \\
\widetilde{V}(\varphi \wedge \psi, w) & = & \widetilde{V}(\varphi, w) \cdot \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \vee \psi, w) & = & \widetilde{V}(\varphi, w) \cup \widetilde{V}(\psi, w), \\
\widetilde{V}(\varphi \rightarrow \psi, w) & = & \widetilde{V}(\varphi, w) \hookrightarrow \widetilde{V}(\psi, w), \\
\widetilde{V}(\langle X_i \rangle \varphi, w) & = & \bigcup \{\widetilde{R}_i(w, s) \cdot \widetilde{V}(\varphi, s)\}, \\
\widetilde{V}([X_i]\varphi, w) & = & \bigcap \{\widetilde{R}_i(w, s) \hookrightarrow \widetilde{V}(\varphi, s)\}.
\end{array}
$$

Many-valued multi-modal logics

Definition

A formula φ of $\mathrm{FL_{ew}}$-K_n is α-*satisfied* at world w in an $\mathrm{FL_{ew}}$-Kripke structure $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, w) \succeq \alpha.$$

Many-valued multi-modal logics

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A formula φ of $\mathrm{FL_{ew}}$-K_n is α-*satisfied* at world w in an $\mathrm{FL_{ew}}$-Kripke structure $\widetilde{M}$ if and only if

$$\widetilde{V}(\varphi, w) \succeq \alpha.$$

Definition

A formula φ is α-*satisfiable* if and only if there exists a structure and a world in which it is α-satisfied, and it is *satisfiable* if it is α-satisfiable for some $\alpha \in \mathcal{A}$, $\alpha \succ 0$; respectively, a formula is α-*valid* if it is α-satisfied at every world in every model, and *valid* if it is 1-valid.

Example: Halpern and Shoham's Modal Logic of Time Intervals

Definition

Let $\mathcal{A} = \langle A, \cap, \cup, \cdot, 0, 1\rangle$ a complete $\mathrm{FL}_{\mathrm{ew}}$-algebra.

An $\mathrm{FL}_{\mathrm{ew}}$-*linear order* is a structure of the type

$$\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{=}\rangle,$$

where D is a *domain* enriched with two functions $\widetilde{<}, \widetilde{=} \colon D \times D \to A$, for which the following conditions apply for every $x, y, z \in D$:

$$\begin{array}{l}
\widetilde{=}(x, y) = 1 \text{ iff } x = y, \\
\widetilde{=}(x, y) = \widetilde{=}(y, x), \\
\widetilde{<}(x, x) = 0, \\
\widetilde{<}(x, z) \succeq \widetilde{<}(x, y) \cdot \widetilde{<}(y, z), \\
\text{if } \widetilde{<}(x, y) \succ 0 \text{ and } \widetilde{<}(y, z) \succ 0, \text{ then } \widetilde{<}(x, z) \succ 0, \\
\text{if } \widetilde{<}(x, y) = 0 \text{ and } \widetilde{<}(y, x) = 0, \text{ then } \widetilde{=}(x, y) = 1, \\
\text{if } \widetilde{=}(x, y) \succ 0, \text{ then } \widetilde{<}(x, y) \prec 1.
\end{array}$$

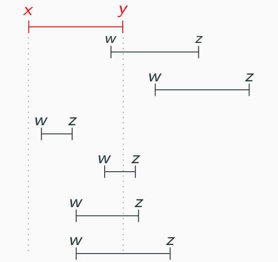
Example: Halpern and Shoham's Modal Logic of Time Intervals

Let $\widetilde{\mathbb{D}} = \langle D, \widetilde{<}, \widetilde{\cong} \rangle$ be a $\mathrm{FL}_{\mathrm{ew}}$-linear order with domain D.

An interval over $\widetilde{\mathbb{D}}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x \widetilde{<} y$.

There are 12 different binary ordering relations between two intervals:

relation	definition	example
after	$\widetilde{R}_A([x,y],[w,z]) = \widetilde{\cong}(y,w)$	
later	$\widetilde{R}_L([x,y],[w,z]) = \widetilde{<}(y,w)$	
begins	$\widetilde{R}_B([x,y],[w,z]) = \widetilde{\cong}(x,w) \cdot \widetilde{<}(z,y)$	
ends	$\widetilde{R}_E([x,y],[w,z]) = \widetilde{<}(x,w) \cdot \widetilde{\cong}(y,z)$	
during	$\widetilde{R}_D([x,y],[w,z]) = \widetilde{<}(x,w) \cdot \widetilde{<}(z,y)$	
overlaps	$\widetilde{R}_O([x,y],[w,z]) = \widetilde{<}(x,w) \cdot \widetilde{<}(w,y) \cdot \widetilde{<}(y,z)$	



and their inverse $R_{\overline{X}} = R_X^{-1}$ for each $X \in \{A, L, B, E, D, O\}$.

To each relation $R_{X \in \{A,\overline{A},L,\overline{L},B,\overline{B},E,\overline{E},D,\overline{D},O,\overline{O}\}}$ corresponds a modality $\langle X \rangle$.

Example: Halpern and Shoham's Modal Logic of Time Intervals

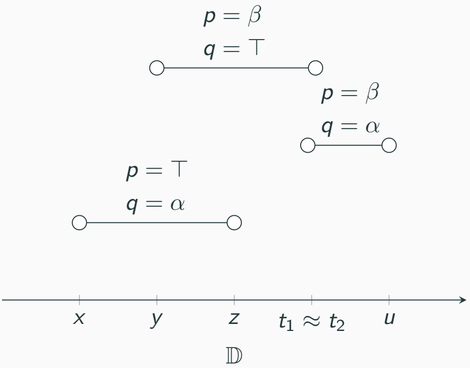


Figure 4: An interval model with six points and thirty intervals, where t_1 and t_2 are slightly apart. We consider an FL_{ew}-algebra with 4 values $\bot \prec \alpha \prec \beta \prec \top$ and $\cap$ as t-norm, and that $R_O([y, t_1], [t_2, u]) = R_A([y, t_1], [t_2, u]) = \alpha$. In this model, $\langle O \rangle (p \wedge q) \succeq \alpha$ at interval I_2.

Applications

Reasoning with many-valued multi-modal logic

We implemented a reasoning system, based on analytic tableau technique, first restricted to many-valued interval temporal logic, introduced in:

- Guillermo Badia, Carles Noguera, Alberto Paparella, Guido Sciavicco, and Ionel Eduard Stan. **"Fitting's Style Many-Valued Interval Temporal Logic Tableau System: Theory and Implementation".** In: *31st International Symposium on Temporal Representation and Reasoning (TIME 2024)*. 2024, 7:1–7:16
- Guillermo Badia, Carles Noguera, Alberto Paparella, Guido Sciavicco, and Ionel Eduard Stan. **"Reasoning with many-valued interval temporal logic".** In: *The Bulletin of Symbolic Logic* 31.2 (2025), pp. 356–358

and later generalized to many-valued multi-modal logic (**W.I.P.!**)

The reasoning system has been implemented in the **Julia** programming language as part of **SOLE**, a framework for learning, reasoning and post hoc analysis, and is provided through the **SoleReasoners.jl** package.

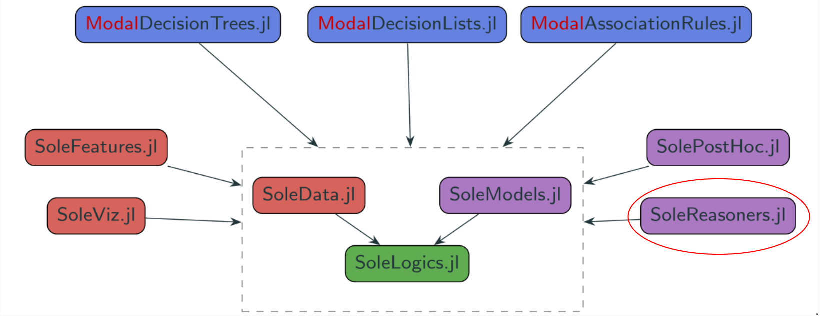


Figure 5: Symbolic Oriented Learning Environment (SOLE) ecosystem.

Performance results

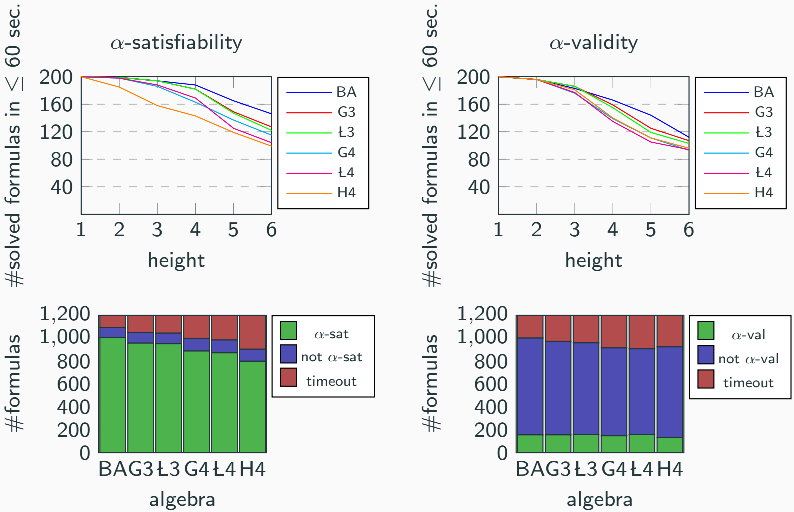


Figure 6: Performance for **many-valued linear temporal logic**.

Performance results

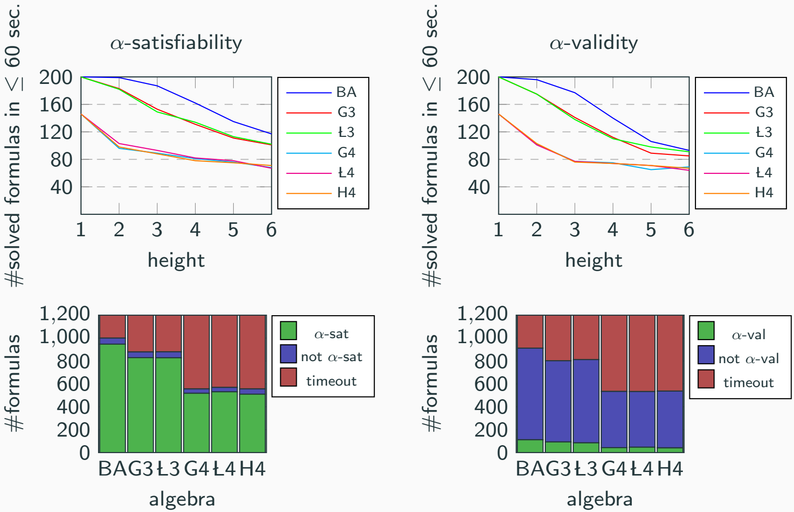


Figure 7: Performance for **many-valued interval temporal logic**.

Performance results

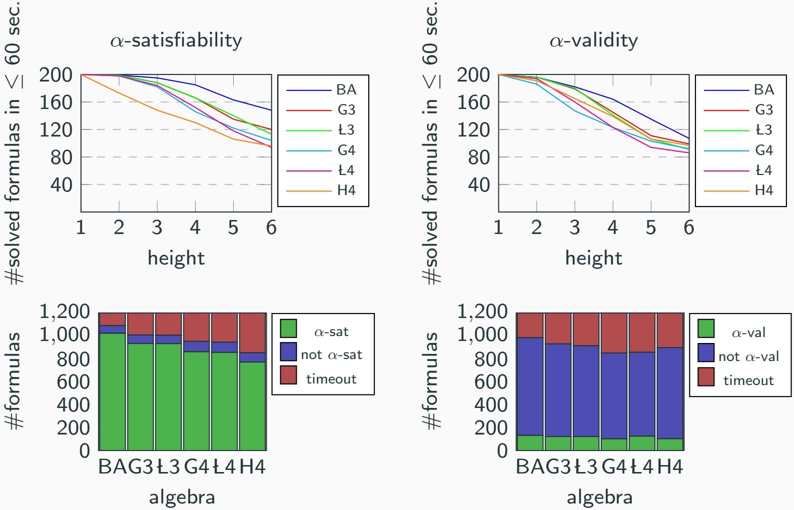


Figure 8: Performance for **many-valued pointwise spatial logic**.

Performance results

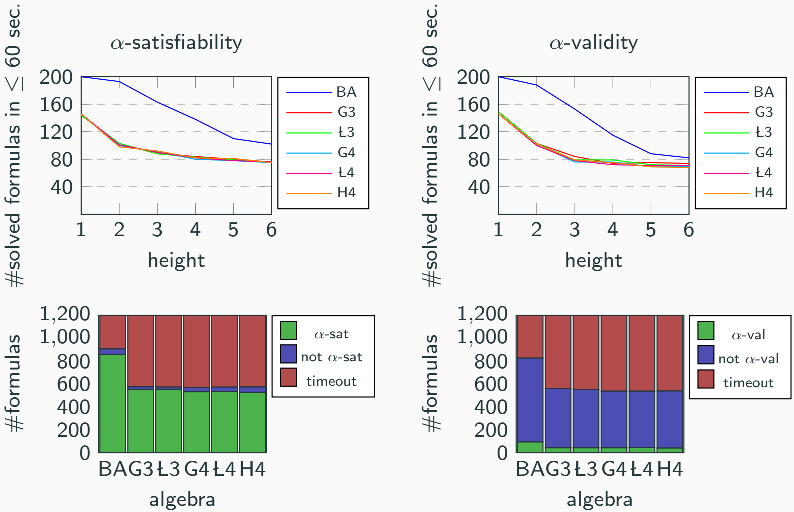


Figure 9: Performance for **many-valued spatial logic of topological relations**.

Reasoning with many-valued multi-modal logic

One could also translate the satisfiability problem for many-valued modal logic to a two-sort first-order logic and leverage an SMT solver (e.g., $Z3$)

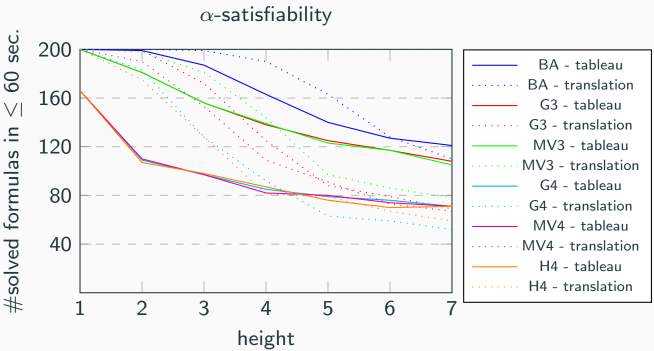


Figure 10: Tableau vs translation performance for solving α-satisfiability for *many-valued Halpern and Shoham's interval temporal logic*: how many formulas can be computed within a 60-second timeout over 1400 formulas (200 for each eight from 1 to 7) for algebras BA, G3, MV3, G4, MV4, H4.

Learning with many-valued multi-modal logic

Generalization of classical symbolic learning tehniques to the many-valued case:

- Guillermo Badia, Carles Noguera, Alberto Paparella, Guido Sciavicco, and Ionel Eduard Stan. **"Many-Expert Decision Trees".** In: *Artificial Intelligence and Formal Verification, Logic, Automata, and Synthesis (OVERLAY 2024)*. 2024, pp. 97–102, also presented at **Brisbane Logic Workshop 2024**

Thank you for your attention!

Questions?