Assignment 2 - A1: Solving conduction equations using finite differences

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1 Relationship between the conduction velocity and the conductance

In order to estabilish the relationship between the conductance of the cable and the propagation velocity, different simulations based on the 1D finite difference method have been performed, varying independently the values of space discretization dx and of conductance σ, as follows:

- $dx = 0.01cm$, $0.02cm$, $0.025cm$
- $\sigma = 0.0005cm^2/ms$, $0.001cm^2/ms$, $0.0015cm^2/ms$, $0.002cm^2/ms$, $0.0025cm^2/ms$, $0.003cm^2/ms$

Table 1 reports the other parameters used in the simulation.

Table 1: Simulation parameters unchanged through the analysis

Simulation parameters	
Cable Length	4cm
Simulation Time	400ms
Time Discretization	0.02ms
Numerical Method	Implicit Euler

Figure 1 shows double logarithmic plots for the relationships between the conductance (set as an input parameter) and the conduction velocity calculated in post-processing, in three separate graphs that refer to the three separate discretization schemes.

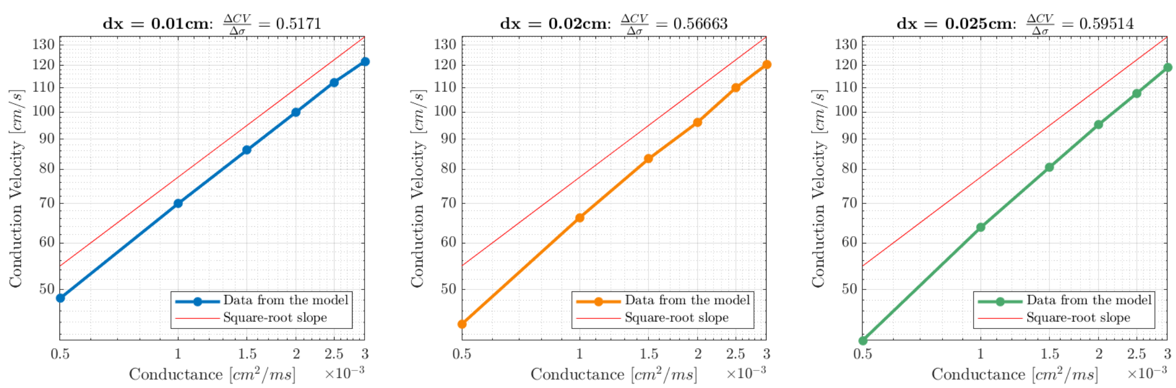

Figure 1: Conduction velocity as a function of conductance for different spatial discretization schemes

The linear trend of the *loglog* plot highlights a power-law relationship between the two variables, in the form $CV \propto \sigma^m$, where $m = \frac{\Delta CV}{\Delta \sigma}$ is the slope of the line in the plot. The approximated slope of such line, in the three cases, is always near the value of 0.5, suggesting an inverse square relationship ($CV \propto \sqrt{\sigma}$). Such correlation is analytically expressed as

$$CV(\sigma) = k\sqrt{\sigma} + b \tag{1}$$

where k and b scale and translate the square-root function to fit the data in the best possible manner and can be obtained with a least-squared-error approach.

The plots in Figure 1 also display a red line that corresponds to the slope of an ideal square-root relationship: it is clear that with an increasingly coarser spatial dicretization scheme the slope approximated from the data is less similar to an ideal square root function. It is therefore concluded that the choice of dx does not influence the relationship between the conductance σ and the Conduction Velocity CV, but given that a coarser discretization scheme produces higher inaccuracies on the calculation of CV in the postprocessing steps, the higher the dx chosen in the model, the more inaccurate the estimation of the power law coefficient would be.

2 Calibrating conductance to achieve a given conduction velocity

The proposed method for calibrating conductance is an iterative one, and it is structured as follows:

1. A reasonable initial value of σ_0 is chosen
2. The corresponding value of $CV_{predicted}$ is calculated running the model with such value of σ as input
3. The midpoint between this value of $CV_{predicted}$ and CV_{target} is calculated and used to obtain a new value of conductance $\sigma_{corrected}$, calculated rearranging the analytical relationship proposed earlier $CV(\sigma) = k1\sqrt{\sigma} + b$ into
$$\sigma(CV) = \left(\frac{CV - b}{k}\right)^2 \tag{2}$$
4. $\sigma_{corrected}$ is then used in point 2 and the method is repeated iteratively, stopping only when the distance between $CV_{predicted}$ and CV_{target} ($48cm/s$ in this case) is less then a threshold set to $0.002 \cdot CV_{target}$

The efficiency of the calibrating procedure performed with this method depends on the choice of σ_0. In order to calculate it, the parameters k and b are obtained by fitting the general form of Equation 1 to the data, so that the analytic relationship can be used to calculate optimally

$$\sigma_0 = \sigma(CV_{target}) = \left(\frac{CV_{target} - b}{k}\right)^2 \tag{3}$$

In particular, the analytic model is found to be

$$CV(\sigma) = 2.436 \cdot 10^{-4} \cdot \sqrt{\sigma} - 10.8 \tag{4}$$

and an initial value of $\sigma_0 = 0.5826 \cdot 10^{-3} cm^2/ms$ is set: this choice allows the calibration to converge in just 2 iterations, whereas a different initial value of conductance results in the calibration requiring more steps. Table 2 reports the values of conductance and velocity calculated in each iterations for different choices of σ_0.

The same data is plotted in Figure 2, where it is evident how the optimal initialization of σ_0 makes the calibration process converge faster. The calibrated value of conductance corresponding to a target velocity of $CV_{target} = 48cm/s$ is $\sigma_{target} = 0.5834 \cdot 10^{-3} cm^2/ms$.

The simulation parameters are unchanged from Table 1, in addition to a fixed value of $dx = 0.02cm$.

Table 2: Intermediate values of convergence

	Iter. 1	**Iter. 2**	**Iter. 3**	**Iter. 4**	**Iter. 5**
σ_0 from the analytic function					
CV $[cm/s]$	47.8501	48.0769	-	-	-
$\sigma[cm^2/ms] \cdot 10^{-3}$	0.5826	0.5834	-	-	-
σ_0 from the data (the closer to the left)					
CV $[cm/s]$	43.6706	49.0196	47.6234	48.0769	-
$\sigma[cm^2/ms] \cdot 10^{-3}$	0.5000	0.6043	0.5776	0.5845	-
σ_0 from the data (the closer to the right)					
CV $[cm/s]$	66.2340	43.4815	49.2647	47.6234	48.0769
$\sigma[cm^2/ms] \cdot 10^{-3}$	1.0000	0.4958	0.6052	0.5764	0.5845

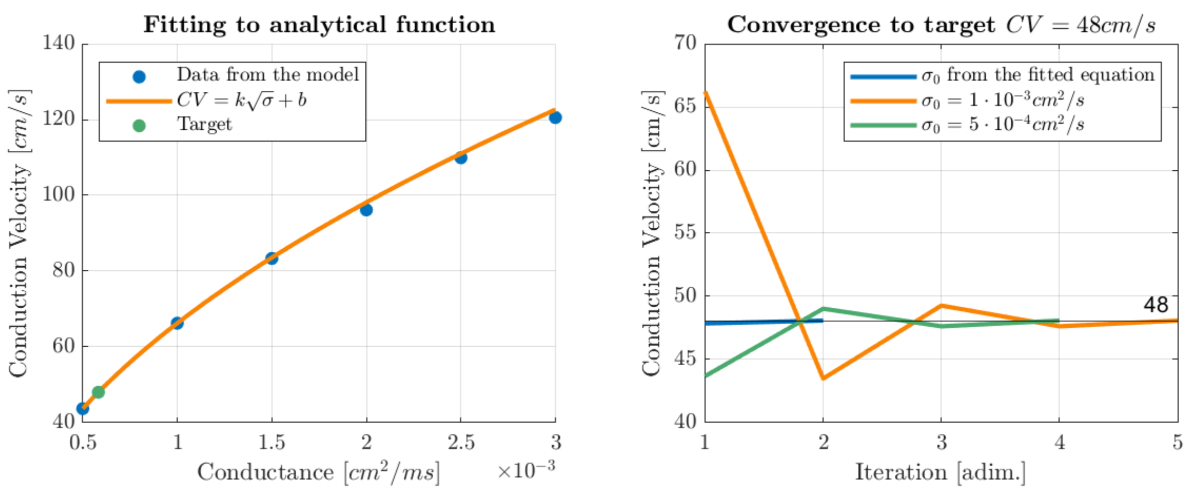

Figure 2: *Left*: The data fitted with the analytic function; *Right*: The values of CV predicted at each iteration

Stability Analysis The integration scheme chosen for the resolution of the differential equations is the Implicit Euler, which is always stable independently of the time step chosen for the calculation. If an Explicit Euler method had been applied instead, the *Modified Wave Number analysis* would impose a stability condition on the temporal discretization, as follows: $\Delta t \leq \frac{\Delta x^2}{2\sigma}$. In this case, the maximum value of dt satisfying the stability condition is $\Delta t_{max} = 0.3428ms$: if the numerical method employed for solving the system of PDEs would have been the Explicit Euler method, the stability condition would have been satisfied with the current employed dt.